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Refined counting of necklaces in one-loop $\mathcal{N} = 4$ SYM

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ABSTRACT: We compute the grand partition function of $\mathcal{N} = 4$ SYM at one-loop in the $SU(2)$ sector with general chemical potentials, extending the results of Pólya's theorem. We make use of finite group theory, applicable to all orders of perturbative $1/N_c$ expansion. We show that only the planar terms contribute to the grand partition function, which is therefore equal to the grand partition function of an ensemble of $XXX_{\frac{1}{2}}$ spin chains. We discuss how Hagedorn temperature changes on the complex plane of chemical potentials.

KEYWORDS: AdS-CFT Correspondence, Lattice Integrable Models, Supersymmetry and Duality

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1 Introduction

The $\mathcal{N} = 4$ super Yang-Mills theory (SYM) has attracted a lot of attention owing to its simple and profound structure. Besides being the primary example of the AdS/CFT correspondence [1], this theory is believed to be integrable in the planar limit [2]. The integrability enables us to predict various observables at any values of the 't Hooft coupling; see [3] for a review.

As a parallel development, alternative methods have been developed to uncover the non-planar structure of $\mathcal{N} = 4$ SYM with the gauge group $U(N_c)$ or $SU(N_c)$, based on finite group theory [4]. New bases of gauge-invariant operators have been discovered, which diagonalize the tree-level two-point functions at finite N_c [5–8]; see [9] for a review.

With numerous approaches at hand to study individual operators, let us ask questions complementary to the above line of development. We reconsider the statistical property of $\mathcal{N} = 4$ SYM, namely the grand partition function including perturbative $1/N_c$ corrections.

In [10], the tree-level partition function of $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$ was computed to investigate its phase space structure. The free energy has the expansion $F = N_c^2 F_0 + F_1 + \dots$, where $F_0 = 0$ in the confined or low-temperature phase, and $F_0 > 0$ in the deconfined or high-temperature phase. In the confined phase, the density of states increases exponentially as the energy increases, leading to the singularity of the partition function at a finite temperature. There the vacuum undergoes the so-called Hagedorn transition to the deconfined phase [11, 12]. In the dual supergravity, it is argued that a thermal scalar in $AdS_5 \times S^5$ becomes tachyonic at a finite temperature, and condensates into the AdS blackhole [13].

Below we consider the grand partition function of $\mathcal{N} = 4$ SYM in the low-temperature phase at one-loop. It amounts to summing up the one-loop anomalous dimensions of all gauge-invariant operators. The problem simplifies a lot by noticing that we do not need to take an eigenbasis of the dilatation operator to compute the trace. The main problem is how to take the trace efficiently in the general setup. The one-loop partition function without chemical potential has been obtained by using Pólya enumeration theorem in [14]. The Hagedorn transition in the pp-wave/BMN limit was studied in [15, 16]. The grand partition function with a one-parameter family of chemical potential was given in [17], and the phase space near the critical chemical potentials was studied in [18].

However, at one-loop the Pólya-type formulae are known only for single-variable cases, which makes it difficult to obtain the grand partition function with general chemical potentials. In this paper, we incorporate the fully general chemical potentials in the $SU(2)$ sector, by using finite group theory which is valid to all orders of perturbative $1/N_c$ expansion.¹

In the planar limit, the dilatation operator of $\mathcal{N} = 4$ SYM at one-loop in the $SU(2)$ sector is the Hamiltonian of $XXX_{\frac{1}{2}}$ spin chain. The (canonical) partition function of the spin chain can be computed by using string hypothesis in the thermodynamic limit. Our work provides alternative derivation based on the microscopic counting of states, if we take care of subtle differences to be discussed in section 3.4.

Main results. Let us briefly summarize the main results of this paper. We consider the grand partition function of $\mathcal{N} = 4$ SYM in the confined phase, with the gauge group $U(N_c)$ up to one-loop in the $SU(2)$ sector. The theory is put on $\mathbb{R} \times S^3$, where \mathbb{R} is the radial

¹A similar quantity was computed in [19, 20], that is the grand partition function with general chemical potential at one-loop in the $SU(2)$ sector including $O(N_c^2)$ term. This result involves a multi-dimensional integral, and less explicit than our counting formula. Another formula was obtained in [21], namely the trace of the one-loop dimension over the product of two fundamental representations of $\mathfrak{psu}(2, 2|4)$. This quantity is a building block in the Pólya-type formula, but not identical to the grand partition function.

direction of \mathbb{R}^4 and the radius of S^3 is set to unity. We write the grand partition function as

$$\mathcal{Z}(\beta, \vec{\omega}) = \text{tr} \left(e^{-\beta \mathfrak{D} + \sum_i \omega_i J_i} \right), \quad \mathfrak{D} = \mathfrak{D}_0 + \lambda \mathfrak{D}_2 + \dots, \quad (1.1)$$

where \mathfrak{D} is the dilatation operator and J_i are the R-charges. We take the trace over all gauge-invariant local operators in the SU(2) sector, made out of complex scalars $\{W, Z\}$. The grand partition function (1.1) has the weak coupling expansion

$$\mathcal{Z}(\beta, x, y) = Z_0^{\text{MT}}(x, y) - 2\lambda\beta Z_2^{\text{MT}}(x, y) + O(\lambda^2), \quad \lambda = \frac{N_c g_{\text{YM}}^2}{16\pi^2}, \quad (1.2)$$

where the operators $W^m Z^n$ are weighted by $x^m y^n$. We apply finite group theory to compute (the perturbative part of) Z_0^{MT} and Z_2^{MT} . It will turn out that only the planar term contributes at one-loop in this setup, even though our methods are valid to all orders of perturbative $1/N_c$ corrections.

On top of $1/N_c$ corrections, there are non-perturbative corrections coming from finite N_c constraints. At finite N_c , certain combinations of operators vanishes when their canonical dimension of an operator exceeds N_c . Let us write the exact grand partition function as

$$\mathcal{Z}_{N_c}^{\text{exact}}(\beta, x, y) = \sum_{m,n \geq 0} \mathcal{N}_{m,n}^{(p)}(N_c) x^m y^n - \sum_{\substack{m,n \geq 0 \\ m+n > N_c}} \mathcal{N}_{m,n}^{(np)}(N_c) x^m y^n, \quad (1.3)$$

where the second sum represents the subtraction at finite N_c . If we set $x = y = e^{-\beta}$, then the second sum is of order $e^{-\beta N_c}$ at large β , which are non-perturbative in view of $1/N_c$ expansion. Our methods capture the first sum in (1.3), which may contain the corrections of order $1/N_c^\ell$ at ℓ -loop. To simplify the notation, we write e.g. $Z_0^{\text{MT}}(x, y)$ instead of $Z_0^{\text{MT}(p)}(x, y)$ throughout the paper.

We obtain two expressions of Z_2^{MT} in section 3, which we call Partition form and Totient form. The Z_2^{MT} in Partition form is written as a sum over partitions of operator length L , $r = [1^{r_1}, 2^{r_2}, \dots, L^{r_L}]$:

$$Z_2^{\text{MT}}(x, y) = N_c \sum_{L=0}^{\infty} \sum_{r+L} \prod_{k=1}^{\infty} (x^k + y^k)^{r_k} \left\{ L - \sum_{a=1}^L \theta_{>}(r_a) - \sum_{a=1}^{L/2} a(r_a + 1) \theta_{>}(r_{2a}) - 2 \sum_{a < b}^L \theta_{>}(L + 1 - a - b) \theta_{>}(r_a) \theta_{>}(r_b) - \sum_{a=1}^{L/2} \theta_{>}(r_a - 1) \right\}, \quad (1.4)$$

where $\theta_{>}(x) = 1$ for $x > 1$, and vanishes otherwise. The Z_2^{MT} in Totient form involves Euler's totient function $\text{Tot}(d)$, which counts the number of relatively prime positive integers less than d :

$$Z_2^{\text{MT}}(x, y) = N_c \prod_{h=1}^{\infty} \frac{1}{1 - x^h - y^h} \times \left[\sum_{k=1}^{\infty} 2 \left(\sum_{d=1}^{\infty} \text{Tot}(d) \frac{x^{kd} y^{kd}}{1 - x^{kd} - y^{kd}} - \sum_{L=2}^{\infty} \sum_{m=1}^{L-1} x^{km} y^{k(L-m)} \delta(\text{gcd}(m, L), 1) \right) \right]. \quad (1.5)$$

The equivalence of two results can be checked by expanding both series around the zero temperature, corresponding to the limit of small x, y . When the gauge group is $SU(N_c)$, an overall factor of $(1-x)(1-y)$ should be multiplied.

It is straightforward to compute the Hagedorn temperature by using Totient form (1.5). The Hagedorn temperature $T_H(\lambda)$ in the $SU(2)$ sector is a function of the chemical potentials $(\omega_1, \omega_2) = (\log x + \frac{1}{T}, \log y + \frac{1}{T})$, which is given by

$$T_H(\lambda) = \begin{cases} \frac{1}{\log(e^{\omega_1} + e^{\omega_2})} \left[1 + \frac{4\lambda e^{\omega_1 + \omega_2}}{(e^{\omega_1} + e^{\omega_2})^2} \right] \\ \qquad \qquad \qquad (e^{\omega_1} > 0 \text{ and } e^{\omega_2} > 0, e^{\omega_1} + e^{\omega_2} \geq 1), \\ \frac{2}{\log(e^{2\omega_1} + e^{2\omega_2})} \left[1 + \frac{4\lambda e^{2\omega_1 + 2\omega_2}}{(e^{2\omega_1} + e^{2\omega_2})^2} \right] \\ \qquad \qquad \qquad (e^{\omega_1} < 0 \text{ or } e^{\omega_2} < 0, e^{2\omega_1} + e^{2\omega_2} \geq 1), \end{cases} \quad (1.6)$$

which is valid at large N_c , due to the second sum in (1.3). Roughly said, the first line represents the deconfinement of W and Z , whereas the second line that of W^2 and Z^2 . This result shows that the complex chemical potentials change the location of the Hagedorn transition, as will be discussed further in section 4.

2 Tree-level counting

We introduce two methods of computing the generating function of the number of gauge-invariant operators in $\mathcal{N} = 4$ SYM with $U(N_c)$ gauge group. This generating function is equal to the grand partition function at tree-level.

2.1 Permutation basis of gauge-invariant operators

General gauge-invariant local operators of $\mathcal{N} = 4$ SYM can be specified by an element of permutation group. We introduce the elementary fields of $\mathcal{N} = 4$ SYM

$$\mathcal{W}^A = \{ \nabla^s \Phi^I, \nabla^s F, \nabla^s \bar{F}, \nabla^s \psi, \nabla^s \bar{\psi} \}, \quad (s \geq 0), \quad (2.1)$$

and their polynomials

$$\begin{aligned} \mathcal{O}_\alpha^{A_1 \dots A_L} &= \text{tr}_L [\alpha \mathcal{W}^{A_1} \mathcal{W}^{A_2} \dots \mathcal{W}^{A_L}], \\ &= \sum_{a_1, a_2, \dots, a_L=1}^{N_c} (\mathcal{W}^{A_1})_{a_\alpha(1)}^{a_1} (\mathcal{W}^{A_2})_{a_\alpha(2)}^{a_2} \dots (\mathcal{W}^{A_L})_{a_\alpha(L)}^{a_L}, \quad (\alpha \in S_L). \end{aligned} \quad (2.2)$$

There is an equivalence relation coming from the relabeling $(a_i, A_i) \rightarrow (a_{\gamma(i)}, A_{\gamma(i)})$,

$$\mathcal{O}_\alpha^{A_1 \dots A_L} = \mathcal{O}_{\gamma \alpha \gamma^{-1}}^{A_{\gamma(1)}, \dots, A_{\gamma(L)}}, \quad (\forall \gamma \in S_L). \quad (2.3)$$

Each of the equivalence class uniquely specifies a gauge-invariant operator.

In the SU(2) sector, we restrict \mathcal{W}^A to the elementary fields to a pair of complex scalars $\{W, Z\}$. We use the gauge degrees of freedom (2.3) to set

$$\mathcal{O}_\alpha = \text{tr}_L(\alpha W^m Z^n), \quad \mathcal{W}^{A_i} = \begin{cases} W & (1 \leq i \leq m) \\ Z & (m+1 \leq i \leq m+n). \end{cases} \quad (2.4)$$

The residual gauge degrees of freedom is

$$\mathcal{O}_\alpha = \mathcal{O}_{\gamma\alpha\gamma^{-1}}, \quad (\forall \gamma \in S_m \times S_n). \quad (2.5)$$

This equivalence class uniquely specifies a gauge-invariant operator in the SU(2) sector. The number of such multi-trace operators at a fixed (m, n) is given by summing over all solutions of (2.5),²

$$N_{m,n}^{\text{MT}} = \frac{1}{m!n!} \sum_{\alpha \in S_{m+n}} \sum_{\gamma \in S_m \times S_n} \delta_{m+n}(\alpha^{-1}\gamma\alpha\gamma^{-1}), \quad (2.6)$$

where

$$\delta_L(\sigma) = \begin{cases} 1 & (\sigma = 1 \in S_L) \\ 0 & (\text{otherwise}) \end{cases} \quad (2.7)$$

The generating function of the number of multi-trace operators is defined by

$$Z_0^{\text{MT}}(x, y) = \sum_{m,n=0}^{\infty} N_{m,n} x^m y^n, \quad (2.8)$$

which will be computed below.

2.2 Sum over partitions

First we evaluate $N_{m,n}^{\text{MT}}$ in (2.6). Let us write $\gamma = \gamma_W \cdot \gamma_Z$ with $\gamma_W \in S_m$ and $\gamma_Z \in S_n$. Suppose that γ_W, γ_Z have the cycle structure $p \vdash m, q \vdash n$, respectively.

$$p \vdash m \quad \Leftrightarrow \quad p = [1^{p_1}, 2^{p_2}, \dots, m^{p_m}], \quad \sum_{k=1}^m k p_k = m. \quad (2.9)$$

We also define

$$r_k = p_k + q_k, \quad r \vdash m+n. \quad (2.10)$$

We look for the general solutions of the condition $\alpha^{-1}\gamma\alpha\gamma^{-1} = 1$ for a fixed γ , which we call stabilizer $\text{Stab}(\gamma)$. Let us parametrize γ by

$$\gamma = \prod_{k=1}^{m+n} \prod_{h=1}^{r_k} (g_{h,1}^{(k)} \cdots g_{h,k}^{(k)}), \quad (2.11)$$

²This formula is called Burnside's lemma. The number of operators is related to the large N_c limit of the tree-level two-point functions [22].

where $(g_{h,1}^{(k)} \dots g_{h,k}^{(k)})$ is the cyclic permutation defined in appendix A. The identity (A.4) gives

$$\alpha^{-1}\gamma\alpha = \prod_{k=1}^{m+n} \prod_{h=1}^{r_k} (\alpha(g_{h,1}^{(k)}) \dots \alpha(g_{h,k}^{(k)})), \quad (2.12)$$

and the stabilizer condition is solved by

$$\alpha(g_{h,k}^{(k)}) = g_{\sigma(h),\tau_h(k)}^{(k)}, \quad \sigma \in S_{r_k}, \quad \tau_h \in \mathbb{Z}_k, \quad (h = 1, 2, \dots, r_k). \quad (2.13)$$

Thus, for each γ , α should belong to the direct product of the wreath product groups,

$$\alpha \in \prod_{k=1}^{m+n} S_{r_k}[\mathbb{Z}_k] \equiv \text{Stab}(\gamma), \quad |\text{Stab}(\gamma)| = \prod_{k=1}^{m+n} k^{r_k} r_k!. \quad (2.14)$$

The symbol $|G|$ means the order of the group G .

The number of permutations in S_m with the cycle structure $p \vdash m$ is given by the orbit-stabilizer theorem,

$$|T_p| = \frac{|S_m|}{\prod_{k=1}^m |S_{p_k}[\mathbb{Z}_k]|} = \frac{m!}{\prod_k k^{p_k} p_k!}. \quad (2.15)$$

We can rewrite $N_{m,n}^{\text{MT}}$ in (2.6) as

$$N_{m,n}^{\text{MT}} = \frac{1}{m! n!} \sum_{\substack{p \vdash m \\ q \vdash n}} |T_p| |T_q| |\text{Stab}(\gamma)_{\gamma \in T_p \times T_q}| = \sum_{\substack{p \vdash m \\ q \vdash n}} \prod_k \frac{(p_k + q_k)!}{p_k! q_k!}. \quad (2.16)$$

Consider the generating function (2.8). The double sum $\sum_m \sum_{p \vdash m}$ can be transformed to an infinite product $\prod_{k=1}^{\infty} \sum_{p_k=0}^{\infty}$, and thus

$$Z_0^{\text{MT}}(x, y) = \prod_{k=1}^{\infty} \sum_{p_k=0}^{\infty} \sum_{q_k=0}^{\infty} \frac{(p_k + q_k)!}{p_k! q_k!} x^{kp_k} y^{kq_k} = \prod_{k=1}^{\infty} \sum_{r_k=0}^{\infty} (x^k + y^k)^{r_k} = \prod_{k=1}^{\infty} \frac{1}{1 - x^k - y^k}. \quad (2.17)$$

The first few terms read

$$Z_0^{\text{MT}}(x, y) = 1 + (x + y) + 2(x^2 + xy + y^2) + (3x^3 + 4x^2y + 4xy^2 + 3y^3) \\ + (5x^4 + 7x^3y + 10x^2y^2 + 7xy^3 + 5y^4) + \dots \quad (2.18)$$

The series gives the number of multi-trace operators in the $\text{SU}(2)$ sector of $\mathcal{N} = 4$ SYM with $\text{U}(N_c)$ gauge group. For $\text{SU}(N_c)$ theories, we subtract the terms with $p_1 > 0$ or $q_1 > 0$ in (2.17), which gives

$$\tilde{Z}_0^{\text{MT}}(x, y) = (1 - x)(1 - y) \prod_{k=1}^{\infty} \frac{1}{1 - x^k - y^k}. \quad (2.19)$$

At finite N_c , fewer terms contribute to the generating function (2.8), which modifies the expansion (2.18). The precise expression will be reviewed in section 2.5. Our formula (2.17) is valid up to the order $x^m y^n$ with $m + n \leq N_c$.

2.3 Power enumeration theorem

We review another derivation of the tree-level generating function based on Pólya Enumeration Theorem [10].

Consider a single-trace operator with length p . We define the domain $D = \{1, 2, \dots, p\}$ and the range $R = \{Z, W\}$. A single-trace operator is graphically equivalent to a necklace, that is the map $D \rightarrow R$ modulo the action of the cyclic group \mathbb{Z}_p acting on D ,

$$\text{Single-trace operator} \leftrightarrow \text{Necklace} = \text{Map}(\mathbb{Z}_p \backslash D \rightarrow R). \quad (2.20)$$

We associate the weights in R by $c(x, y) = x + y$, where $x^m y^n$ corresponds to the operator $W^m Z^n$. Then, Pólya Enumeration Theorem says that the generating function of the number of graphs (2.20) is given by

$$Z_0^{\text{ST}}(x, y) = \sum_p Z_{\mathbb{Z}_p}(c(x, y), c(x^2, y^2), \dots, c(x^p, y^p)), \quad (2.21)$$

Here $Z_{\mathbb{Z}_p}(s_1, s_2, \dots, s_p)$ is the cycle index of the cyclic group,

$$Z_{\mathbb{Z}_p}(s_1, s_2, \dots, s_p) = \frac{1}{p} \sum_{h|p} \text{Tot}(h) s_h^{p/h}, \quad (2.22)$$

where we take a sum over h such that p/h is a positive integer, and $\text{Tot}(h)$ is Euler's totient function defined by

$$\text{Tot}(h) = \sum_{d=1}^h \delta(\gcd(d, h), 1). \quad (2.23)$$

By combining (2.21) with (2.22) and writing $p = hs$, we get

$$Z_0^{\text{ST}}(x, y) = \sum_h \sum_s \text{Tot}(h) \frac{(x^h + y^h)^s}{hs} = - \sum_h \frac{\text{Tot}(h)}{h} \log(1 - x^h - y^h). \quad (2.24)$$

The first few terms read

$$\begin{aligned} Z_0^{\text{ST}}(x, y) &= (x + y) + (x^2 + xy + y^2) + (x^3 + x^2y + xy^2 + y^3) \\ &\quad + (x^4 + x^3y + 2x^2y^2 + xy^3 + y^4) + (x^5 + x^4y + 2x^3y^2 + 2x^2y^3 + xy^4 + y^5) \\ &\quad + \dots \end{aligned} \quad (2.25)$$

The generating function of multi-trace operator is given by the plethystic exponential of the single-trace generating function,

$$Z_0^{\text{MT}}(x, y) = \exp\left(\sum_{m=1} Z_0^{\text{ST}}(x^m, y^m)\right) = \prod_{d=1} \frac{1}{1 - x^d - y^d}, \quad (2.26)$$

where we used $\sum_{j|d} \text{Tot}(j) = d$. This result agrees with (2.17). For $\text{SU}(N_c)$ theories, we subtract the $p = 1$ term in (2.21),

$$\tilde{Z}_0^{\text{MT}}(x, y) = \exp\left(-\sum_{n=1}^{\infty} \frac{x^n + y^n}{n}\right) \prod_{k=1}^{\infty} \frac{1}{1 - x^k - y^k}. \quad (2.27)$$

in agreement with (2.19).

2.4 Counting single-traces

For later purposes, we rederive the generating function of the number of single-trace operators by counting the solutions (2.6) under the constraint $\alpha \in \mathbb{Z}_L$, with $L = m + n$.

We will obtain

$$N_{m,n}^{\text{ST}} = \begin{cases} \sum_{\substack{d=1 \\ d|m, d|n}}^L \frac{(L/d)!}{(m/d)!(n/d)!} \frac{\text{Tot}(d)}{L} & (m \neq 0, L) \\ 1 & (m = 0, L), \end{cases} \quad (2.28)$$

which is derived as follows. Suppose m and n are divisible by a positive integer d ,

$$(m, n, L) = (dm', dn', d\ell), \quad m' + n' = \ell, \quad (d = 1, 2, \dots, L). \quad (2.29)$$

The upper bound of d is L if $mn = 0$, and $\text{Min}(m, n)$ otherwise. Choose $\tilde{\mu} \in \mathbb{Z}_d^\ell = \mathbb{Z}_d^{m'} \times \mathbb{Z}_d^{n'}$ from $S_m \times S_n$ and write $\alpha \in \mathbb{Z}_L$ as

$$\alpha = \left(a_1 \dots a_\ell \tilde{\mu}^\kappa(a_1) \dots \tilde{\mu}^\kappa(a_\ell) \tilde{\mu}^{2\kappa}(a_1) \dots \tilde{\mu}^{2\kappa}(a_\ell) \dots \tilde{\mu}^{(d-1)\kappa}(a_1) \dots \tilde{\mu}^{(d-1)\kappa}(a_\ell) \right), \quad (2.30)$$

$$1 \leq \kappa < d, \quad \text{gcd}(\kappa, d) = 1.$$

This set of $(\tilde{\mu}, \kappa, \alpha)$ is the general solution to the conditions

$$\alpha = \tilde{\mu} \alpha \tilde{\mu}^{-1} \quad \text{and} \quad \alpha \in \mathbb{Z}_L. \quad (2.31)$$

Let us parametrize $\mu = \tilde{\mu}$ as

$$\tilde{\mu} = \begin{bmatrix} (\tilde{m}_{11} \tilde{m}_{12} \dots \tilde{m}_{1d}) \\ \vdots \\ (\tilde{m}_{m'+1,1} \tilde{m}_{m'+1,2} \dots \tilde{m}_{m'+1,d}) \\ \vdots \\ (\tilde{m}_{\ell 1} \tilde{m}_{\ell 2} \dots \tilde{m}_{\ell d}) \end{bmatrix} \in \mathbb{Z}_d^{m'} \times \mathbb{Z}_d^{n'}. \quad (2.32)$$

The number of possible $\tilde{\mu}$ chosen from $S_{dm'} \times S_{dn'}$ is³

$$\begin{cases} \frac{(dm')!}{d^{m'}} \frac{(dn')!}{d^{n'}} & (m', n' > 0) \\ (\ell - 1)! & (m' n' = 0). \end{cases} \quad (2.33)$$

For each $\tilde{\mu}$, we sum over α as parametrized in (2.30). For this purpose we identify $\{a_1, \dots, a_\ell\}$ with some of $\{\tilde{m}_{hk}\}$ in (2.32). In order to avoid double counting, we fix $a_1 = \tilde{m}_{11}$ and choose

$$a_h = \tilde{m}_{\sigma(h)k_h} \quad (\sigma \in S_{\ell-1}, 1 \leq k_h \leq d) \quad \text{for each } 2 \leq h \leq d, \quad (2.34)$$

³The condition $\tilde{\mu} \in \mathbb{Z}_d^{m'} \times \mathbb{Z}_d^{n'}$ is equivalent to $\tilde{\mu} \in T_{[dm']} \times T_{[dn']}$, and the order of the latter group is given by the orbit-stabilizer theorem (2.15).

The number of choices of $a_2 \dots a_\ell$ is⁴

$$d^{\ell-1} (\ell - 1)! . \tag{2.35}$$

The number of possible κ is $\text{Tot}(d)$. Thus, the number of possible α , divided by $|S_m \times S_n|$ is

$$\frac{1}{|S_m \times S_n|} \sum_{\mu \in S_m \times S_n} \sum_{\alpha \in T_{[m+n]}} \delta_{m+n} (\alpha^{-1} \mu \alpha \mu^{-1}) = \sum_{\substack{d=1 \\ d|m, d|L}}^L \frac{(m' + n')!}{m'! n'!} \frac{\text{Tot}(d)}{L}, \tag{2.36}$$

which is (2.28). This result is formally correct when $m'n' = 0$ thanks to $\sum_{d|L} \text{Tot}(d) = L$.

Therefore, the tree-level generating function is given by

$$Z_0^{\text{ST}}(x, y) = \sum_L \sum_{m=0}^L \sum_{\substack{d=1 \\ d|m, d|L}}^L x^m y^n \frac{(L/d)!}{(m/d)!((L-m)/d)!} \frac{\text{Tot}(d)}{L}. \tag{2.37}$$

To simplify it, we apply the formulae

$$\begin{aligned} \sum_L \sum_{m=0}^L \sum_{\substack{d=1 \\ d|m, d|L}}^{L-1} f_d(m, L-m) &= \sum_{d=1}^{\infty} \sum_L \sum_{m=0}^L f_d(dm, d(L-m)), \\ \sum_{m=0}^L x^{dm} y^{d(L-m)} \frac{L!}{m!(L-m)!} &= (x^d + y^d)^L, \end{aligned} \tag{2.38}$$

to obtain

$$Z_0^{\text{ST}}(x, y) = \sum_{d=1}^{\infty} \sum_L \frac{\text{Tot}(d)}{d} \frac{(x^d + y^d)^L}{L}, \tag{2.39}$$

which is (2.24).

Consider an example. If $(m, n) = (3, 3)$ and $d = 3$, we find

$$S_3 \times S_3 \supset \mathbb{Z}_3^1 \times \mathbb{Z}_3^1 \ni \tilde{\mu} = \left\{ \left[\begin{array}{c} (123) \\ (456) \end{array} \right], \left[\begin{array}{c} (132) \\ (456) \end{array} \right], \left[\begin{array}{c} (123) \\ (465) \end{array} \right], \left[\begin{array}{c} (132) \\ (465) \end{array} \right] \right\}, \tag{2.40}$$

which is consistent with $\{3!/(3^1 1!)\}^2 = 4$ in (2.33). For each $\tilde{\mu}$ we generate

$$\alpha = (a_1 a_2 \tilde{\mu}^\kappa(a_1) \tilde{\mu}^\kappa(a_2) \tilde{\mu}^{2\kappa}(a_1) \tilde{\mu}^{2\kappa}(a_2)), \quad (\kappa = 1, 2). \tag{2.41}$$

The possible choices of (a_1, a_2) are $(1, 4), (1, 5), (1, 6)$. Thus

$$\frac{1}{|S_3 \times S_3|} \sum_{\mu \in \mathbb{Z}_3 \times \mathbb{Z}_3} \sum_{\alpha \in T_{[6]}} \delta_6 (\alpha^{-1} \mu \alpha \mu^{-1}) = \frac{4 \times 2 \times 3}{3!^2} = \frac{2}{3}, \tag{2.42}$$

which agrees with (2.36).

⁴In other words, we remove the redundancy coming from the overall translation of $\alpha \in \mathbb{Z}_L$.

2.5 Partition function at finite N_c

The exact tree-level partition function of $\mathcal{N} = 4$ SYM at finite N_c can be computed precisely in various methods. We briefly review these arguments, to understand the finite N_c corrections in (1.3).

A straightforward method is to compute the grand partition function is to evaluate the path integral of $\mathcal{N} = 4$ SYM action [12]. Let a_0 be the zero-mode of the gauge field A_0 and $U = \exp(i\beta a_0)$. The grand partition function in the complete $PSU(2, 2|4)$ sector is

$$\mathcal{Z}_{N_c}^{\text{complete}}(w) = \int dU_{\text{SU}(N_c)} \exp\left(\frac{1}{n} \sum_{n=1}^{\infty} \{\zeta_B(w^n) + (-1)^{n+1} \zeta_F(w^n)\} \text{tr}_{\text{adj}}(U^n)\right) \quad (2.43)$$

where $\zeta_B(w), \zeta_F(w)$ are functions of chemical potentials, and $dU_{\text{SU}(N_c)}$ is the $SU(N_c)$ Haar measure.⁵

The complete partition function (2.43) can be reduced to the one in the $SU(2)$ sector by setting

$$\zeta_B(w^n) = x^n + y^n, \quad \zeta_F(w^n) = 0, \quad (2.44)$$

which gives $\mathcal{Z}_{N_c}^{\text{exact}}(\beta, x, y)$ in (1.3). It turns out that the resulting expression is identical to the Molien-Weyl formula which gives the Hilbert-Poincaré series of $GL(N_c)$ invariants [23]. For example, the Molien-Weyl formula for the gauge group $U(N_c)$ with q variables, corresponding to the $SU(q)$ sector, can be written as⁶

$$\begin{aligned} \mathcal{Z}_{N_c}^{\text{SU}(q)}(x_1, \dots, x_q) &= \frac{1}{(2\pi i)^{N_c-1}} \frac{1}{\prod_{i=1}^q (1-x_i)^n} \oint_U \frac{dt_1}{t_1} \dots \oint_U \frac{dt_{N_c-1}}{t_{N_c-1}} \prod_{r=1}^{n-1} \prod_{k=1}^r \frac{\chi_{k,r}^+(1, t)}{\phi_{k,r}(x, t)}, \\ \chi_{k,r}^{\pm}(\alpha, t) &= 1 - \alpha \prod_{j=k}^r t_j^{\pm 1}, \quad \phi_{k,r}(x, t) = \prod_{\ell=1}^q \chi_{k,r}^+(x_{\ell}, t) \chi_{k,r}^-(x_{\ell}, t), \end{aligned} \quad (2.45)$$

where U is the counterclockwise contour of unit radius.⁷ It can also be written as [5, 26]

$$\mathcal{Z}_{N_c}^{\text{SU}(q)}(x_1, \dots, x_q) = \sum_{L=0} \sum_{R \vdash L} \sum_{\substack{\Lambda \vdash L \\ \text{Row}(\Lambda) \leq q}} C(R, R, \Lambda) s_{\Lambda}(x_1, \dots, x_q), \quad (2.46)$$

where $C(R, R, \Lambda)$ is the Clebsch-Gordan multiplicity defined by $R \otimes R = \bigoplus_{\Lambda} C(R, R, \Lambda)$ as S_L -modules, and $s_{\Lambda}(x_1, \dots, x_q)$ is the Schur polynomial. We sum over the partitions Λ having at most q rows, since Λ is related to the $SU(q)$ global symmetry. At finite N_c we should sum over the partitions R having at most N_c rows in (2.46).

One can evaluate the formula (2.45) or (2.46) explicitly when N_c and q are small. At $(N_c, q) = (2, 2)$ we obtain

$$\mathcal{Z}_{N_c}^{\text{SU}(2)}(x, y) = (Z_0^{\text{MT}})_{N_c=2}^{\text{exact}}(x, y) = \frac{1}{(1-xy)} \prod_{k=1}^2 \frac{1}{(1-x^k)(1-y^k)}, \quad (2.47)$$

in agreement with [27] for $q = 2$. The formula also reproduces the $q > 2$ cases in [18].

⁵This is the result for $SU(N_c)$ gauge group. For $U(N_c)$, we replace $\text{tr}_{\text{adj}}(U^n)$ by $\text{tr}_{\text{adj}}(U^n) + 1$. Recall that the $U(1)$ part of $\mathcal{N} = 4$ SYM is free since all interactions are of commutator type.

⁶The explicit form of the Molien-Weyl formula depends on the choice of basis of the (adjoint) representation of $U(N_c)$. The convention of [24] is used here for efficient evaluation.

⁷This formula is elaborated further as the highest weight generating function [25].

3 One-loop counting

We compute the sum of anomalous dimensions at one-loop in the SU(2) sector in two ways, which we call Partition form and Totient form. The corresponding generating function gives the partition function at one-loop.

3.1 Mixing matrix

The dilatation operator of $\mathcal{N} = 4$ SYM is given by [28, 29],

$$\mathfrak{D} = \sum_{n=0} \lambda^n \mathfrak{D}_{2n} = \text{tr}(W\check{W} + Z\check{Z}) - \frac{2\lambda}{N_c} : \text{tr}[W, Z][\check{W}, \check{Z}] : + O(\lambda^2). \quad (3.1)$$

Let $\mathcal{H}_{m,n}$ be the Hilbert space of all gauge-invariant operators in the SU(2) sector with the R-charges (m, n) . We define the mixing matrix as

$$\mathfrak{D}_2 \mathcal{O}_\alpha \equiv \frac{2}{N_c} (M_2)_\alpha^\beta \mathcal{O}_\beta. \quad (3.2)$$

On the permutation basis introduced in section 2.1, the mixing matrix inside $\mathcal{H}_{m,n}$ takes the form [30]

$$\begin{aligned} (M_2)_\alpha^\beta &= \sum_{i \neq j}^L \left[\delta_L([\beta^{-1}][\alpha][i\alpha(j)]) - \delta_L([\beta^{-1}](ij)[\alpha](ij)[i\alpha(j)]) \right], \\ &= \frac{1}{m!n!} \sum_{i \neq j}^L \sum_{\mu \in S_m \times S_n} \delta_L\left(\mu\beta^{-1}\mu^{-1}\left\{\alpha - (ij)\alpha(ij)\right\}[i\alpha(j)]\right). \end{aligned} \quad (3.3)$$

where we introduced the notation $L = m + n$,

$$[ij] = \begin{cases} (ij) & (i \neq j) \\ N_c & (i = j), \end{cases} \quad (3.4)$$

and denoted the equivalence class by

$$[\alpha] = \frac{1}{|S_m \times S_n|} \sum_{\gamma \in S_m \times S_n} \gamma\alpha\gamma^{-1}. \quad (3.5)$$

We will evaluate the sum of one-loop dimensions at a fixed (m, n) ,

$$\langle M_2 \rangle_{m,n} = \sum_{\alpha, \beta \in \mathcal{H}_{m,n}} (M_2)_\alpha^\beta \delta_\beta^\alpha. \quad (3.6)$$

Since the gauge-invariant operator is in one-to-one correspondence with the equivalence class (3.5), we can rewrite the sum (3.6) as

$$\langle M_2 \rangle_{m,n} = \frac{1}{(m!n!)^2} \sum_{\alpha, \beta \in S_L} \sum_{\gamma_1, \gamma_2 \in S_m \times S_n} (M_2)_{\gamma_1\alpha\gamma_1^{-1}\gamma_2\beta\gamma_2^{-1}} \delta_L(\gamma_2\beta^{-1}\gamma_2^{-1}\gamma_1\alpha\gamma_1^{-1}). \quad (3.7)$$

According to (3.3), the mixing matrix is invariant inside the same conjugacy class. By writing $\gamma_2^{-1}\gamma_1 \equiv \gamma$, we find

$$\begin{aligned}
 \langle M_2 \rangle_{m,n} &= \frac{1}{m!n!} \sum_{\alpha, \beta \in S_L} \sum_{\gamma \in S_m \times S_n} (M_2)_\alpha^\beta \delta_L(\beta^{-1}\gamma\alpha\gamma^{-1}), \\
 &= \frac{1}{m!n!} \sum_{\alpha \in S_L} \sum_{\gamma \in S_m \times S_n} (M_2)_\alpha^{\gamma\alpha\gamma^{-1}}, \\
 &= \sum_{\alpha \in S_L} (M_2)_\alpha^\alpha, \\
 &= \frac{1}{m!n!} \sum_{i \neq j} \sum_{\alpha \in S_L} \sum_{\mu \in S_m \times S_n} \delta_L(\mu\alpha^{-1}\mu^{-1}\{\alpha - (ij)\alpha(ij)\} \llbracket i\alpha(j) \rrbracket).
 \end{aligned} \tag{3.8}$$

Let us inspect the argument of the δ -function. Recall that any permutation can be decomposed into the product of transpositions, like $(1234) = (34)(23)(12)$. The number of transpositions defines the parity of a permutation, which is conserved at any orders of perturbative $1/N_c$ expansion.⁸ In particular, odd powers of transpositions cannot become the identity, and only the planar term $i = \alpha(j)$ contributes in (3.8). Thus,

$$\begin{aligned}
 \langle M_2 \rangle_{m,n} &= \frac{N_c}{m!n!} \sum_{i \neq j} \sum_{\alpha \in S_L} \sum_{\mu \in S_m \times S_n} \delta_L(i\alpha(j)) \\
 &\quad \times \left\{ \delta_L(\mu\alpha^{-1}\mu^{-1}\alpha) - \delta_L(\mu\alpha^{-1}\mu^{-1}(ij)\alpha(ij)) \right\}, \tag{3.9}
 \end{aligned}$$

where $L = m + n$. The generating function of the sum of one-loop dimensions is defined by

$$Z_2^{\text{MT}}(x, y) \equiv \sum_{m,n=0}^{\infty} \langle M_2 \rangle_{m,n} x^m y^n. \tag{3.10}$$

3.2 Partition form

We evaluate the sum of dimensions (3.9) by generalizing the methods used in section 2.2.

3.2.1 First term

Consider the first term of (3.9),

$$\langle M_2 \rangle_{m,n}^{(1st)} = \frac{N_c}{m!n!} \sum_{i \neq j} \sum_{\alpha \in S_L} \sum_{\mu \in S_m \times S_n} \delta_L(i\alpha(j)) \delta_L(\mu\alpha^{-1}\mu^{-1}\alpha). \tag{3.11}$$

We denote the cycle type of μ by $p \vdash m$, $q \vdash n$ and define $r_k = p_k + q_k$. We parametrize μ by $\mu = \prod_{k=1}^{m+n} \prod_{h=1}^{r_k} (m_{h,1}^{(k)} m_{h,2}^{(k)} \dots m_{h,k}^{(k)})$ as in (2.11). The condition $\mu\alpha^{-1}\mu^{-1}\alpha = 1$ imposes that α should belong to the stabilizer of μ .

Suppose that i and j are part of the cycle of μ of length- a and length- b , respectively. There are $L(L-1)$ choices of $\{i, j\}$, which can be written as

$$L(L-1) = \sum_{a,b=1}^L ab r_a r_b - \sum_{a=1}^L ar_a = \sum_{a \neq b} ab r_a r_b + \sum_a ar_a (ar_a - 1). \tag{3.12}$$

⁸Conversely said, finite N_c constraints mix permutations with different parity.

From (2.13) we see that $\alpha \in \text{Stab}(\mu)$ permutes $\{1, 2, \dots, L\}$ only among those having the same cycle length in μ . Thus, the condition $i = \alpha(j)$ results in $a = b$, so we neglect the terms $a \neq b$ in (3.12).

Define the number of solutions of the two δ -functions in (3.11) for a given (i, j, μ) by

$$N_{sol}(a, \mu) = \sum_{\alpha \in S_L} \delta(i\alpha(j)) \delta_L(\mu^{-1}\alpha^{-1}\mu\alpha) \Big|_{i,j \in \text{length-}a \text{ cycle}}. \quad (3.13)$$

This can be rewritten as

$$N_{sol}(a, \mu) = \sum_{\alpha \in \text{Stab}(\mu)} \delta(i\alpha(j)) \Big|_{i,j \in \text{length-}a \text{ cycle}}, \quad \text{Stab}(\mu) = \prod_{k=1}^L S_{r_k}[\mathbb{Z}_k]. \quad (3.14)$$

If we introduce $\alpha = \alpha_0(ij)$, then

$$N_{sol}(a, \mu) = \sum_{\alpha_0 \in \text{Stab}(\mu)} \delta(i\alpha_0(i)) \Big|_{i \in \text{length-}a \text{ cycle}}. \quad (3.15)$$

Since the group $\text{Stab}(\mu)$ acts transitively on $S_{r_a}[\mathbb{Z}_a] \subset S_{ar_a}$, the isotropy group satisfies the property

$$(S_{r_a}[\mathbb{Z}_a])^{(i)} \equiv \left\{ g(i) = i \mid g \in S_{r_a}[\mathbb{Z}_a] \right\}, \quad \left| (S_{r_a}[\mathbb{Z}_a])^{(i)} \right| = \frac{1}{ar_a} \left| S_{r_a}[\mathbb{Z}_a] \right|. \quad (3.16)$$

Thus, the number of solutions in (3.11) for a given $\mu \in T_p \times T_q$ is

$$\begin{aligned} N_{sol}(p, q) &\equiv \sum_{i \neq j}^L \sum_{\alpha \in S_L} \delta(i\alpha(j)) \delta_L(\mu^{-1}\alpha^{-1}\mu\alpha) \\ &= \sum_{a=1}^L ar_a (ar_a - 1) N_{sol}(a, \mu) \\ &= \sum_{a=1}^L \theta_{>}(r_a) (ar_a - 1) \left| \text{Stab}(\mu) \right|_{\mu \in T_p \times T_q}, \end{aligned} \quad (3.17)$$

where

$$\theta_{>}(x) = \begin{cases} 1 & (x > 0) \\ 0 & (x \leq 0). \end{cases} \quad (3.18)$$

Proceeding as in (2.17), we obtain the generating function for the first term.

$$\begin{aligned} Z_2^{(1st)}(x, y) &= N_c \sum_{m, n} \sum_{p \vdash m, q \vdash n} |T_p| |T_q| \frac{x^m y^n}{m! n!} N_{sol}(p, q) \\ &= N_c \sum_{m, n} \sum_{p \vdash m, q \vdash n} \prod_{k=1}^L (x^k)^{p_k} (y^k)^{q_k} \frac{(p_k + q_k)!}{p_k! q_k!} \sum_a \theta_{>}(r_a) (ar_a - 1) \\ &= N_c \sum_{L=0}^{\infty} \sum_{r \vdash L} \prod_{k=1}^L (x^k + y^k)^{r_k} \left(L - \sum_{a=1}^L \theta_{>}(r_a) \right). \end{aligned} \quad (3.19)$$

j	1	2	3	4	5	6	7	8	9	10	11
$\alpha(j)$		1^{144}				3^{48}				9^{96}	
		2^{144}				4^{48}				10^{96}	
						5^{48}				11^{96}	
						6^{48}					
						7^{48}					
						8^{48}					

Table 1. Table of $\alpha(j)$ for $1 \leq j \leq 11$ and $\alpha \in S_2[\mathbb{Z}_1] \cdot S_3[\mathbb{Z}_2] \cdot S_1[\mathbb{Z}_3]$. The symbol a^b means that the number a appears b times.

Consider an example. Let μ be $(1)(2)(3,4)(5,6)(7,8)(9,10,11)$, having $r = [1^2, 2^3, 3^1] \vdash 11$. Then $\text{Stab}(\mu) = S_2[\mathbb{Z}_1] \cdot S_3[\mathbb{Z}_2] \cdot S_1[\mathbb{Z}_3]$, which has the order $2 \times 48 \times 3 = 244$. Choose $i \neq j$ from $\{1, 2, \dots, 11\}$. The list $\{\alpha(j) \mid \alpha \in \text{Stab}(\mu)\}$ for all j is summarized in table 1. The list shows that the number of solutions to $i = \alpha(j)$ is precisely given by the formula (3.16), like

$$(i, j) = (1, 2), \quad \#_{\text{sol}} = 144 = \left| 1 \cdot S_3[\mathbb{Z}_2] \cdot S_1[\mathbb{Z}_3] \right| \quad (3.20)$$

$$(i, j) = (3, 4), (3, 5), \dots, (7, 8), \quad \#_{\text{sol}} = 48 = \left| S_2[\mathbb{Z}_1] \cdot S_2[\mathbb{Z}_2] \cdot S_1[\mathbb{Z}_3] \right| \quad (3.21)$$

$$(i, j) = (9, 10), (10, 11), (9, 11), \quad \#_{\text{sol}} = 96 = \left| S_2[\mathbb{Z}_1] \cdot S_3[\mathbb{Z}_2] \cdot 1 \right|. \quad (3.22)$$

3.2.2 One-loop generating function

We will analyze the second term of the mixing matrix in appendix B.1. Here we summarize the results by combining (3.19) and (B.17).

The generating function of the sum of one-loop dimensions over all multi-trace operators in the $SU(2)$ sector is given by

$$\begin{aligned} Z_2^{\text{MT}}(x, y) &= N_c \sum_{L=0}^{\infty} \sum_{r \vdash L} \prod_{k=1}^{\infty} (x^k + y^k)^{r_k} \left\{ L - \sum_{a=1}^L \theta_{>}(r_a) - \Theta(r) \right\}, \\ &= N_c \sum_{L=0}^{\infty} \sum_{r \vdash L} \prod_{k=1}^{\infty} (x^k + y^k)^{r_k} \left\{ L - \sum_{a=1}^L \theta_{>}(r_a) - \sum_{a=1}^{L/2} a(r_a + 1) \theta_{>}(r_{2a}) \right. \\ &\quad \left. - 2 \sum_{a < b}^L \theta_{>}(L + 1 - a - b) \theta_{>}(r_a) \theta_{>}(r_b) - \sum_{a=1}^{L/2} \theta_{>}(r_a - 1) \right\}. \end{aligned} \quad (3.23)$$

The summand can be negative for some $r \vdash L$, though the sum becomes non-negative if we sum over all $r \vdash L$.⁹ The first few terms read

$$\begin{aligned} \frac{Z_2^{\text{MT}}(x, y)}{N_c} &= 6x^2y^2 + (10x^3y^2 + 10x^2y^3) + (26x^4y^2 + 36x^3y^3 + 26x^2y^4) \\ &\quad + (44x^5y^2 + 84x^4y^3 + 84x^3y^4 + 44x^2y^5) \end{aligned}$$

⁹Negative terms are needed to kill the coefficients of BPS terms.

$$\begin{aligned}
 &+ (84x^6y^2 + 176x^5y^3 + 254x^4y^4 + 176x^3y^5 + 84x^2y^6) \\
 &+ (134x^7y^2 + 348x^6y^3 + 548x^5y^4 + 548x^4y^5 + 348x^3y^6 + 134x^2y^7) + \dots
 \end{aligned}
 \tag{3.24}$$

The term $6x^2y^2$ is responsible for the one-loop dimensions of the SU(2) Konishi descendant, which is $3N_c g_{\text{YM}}^2/(4\pi^2)$.

3.3 Totient form

We compute the generating function of the sum of one-loop dimensions in another way. First, we compute the one-loop generating function for single-trace operators by imposing $\alpha \in \mathbb{Z}_L$, as done in section 2.4. Then, we conjecture the generating function for multi-traces, by writing the plethystic exponential of the single-trace results.

3.3.1 First term

Let $d \geq 1$ be a divisor of m and n as in (2.29). Specify the cycle type of μ to $p = [d^{m'}]$ and $q = [d^{n'}]$ and α to $[L]$ simultaneously. Consider the first term of the one-loop mixing matrix:

$$\langle M_2 \rangle_{m,n}^{(1st)} = \frac{N_c}{m!n!} \sum_{i \neq j}^L \sum_{\alpha \in T_{[L]}} \sum_{\substack{d=1 \\ d|m, d|L}}^L \sum_{\mu \in T_{[d^{m'}]} \times T_{[d^{n'}]}} \delta(i\alpha(j)) \delta_L(\alpha^{-1}\mu\alpha\mu^{-1}). \tag{3.25}$$

The number of solutions to the stabilizer condition $\alpha = \mu\alpha\mu^{-1}$ is given by (2.36). For each $(j, \mu = \tilde{\mu})$ and α given by (2.30), there is only one $i \in \{1, 2, \dots, L\}$ satisfying $i = \alpha(j)$. Hence the sum over i, j gives a factor of L , leading to

$$\langle M_2 \rangle_{m,n}^{(1st)} = N_c \sum_{\substack{d=1 \\ d|m, d|L}}^L \frac{(m' + n')!}{m'!n'!} \text{Tot}(d). \tag{3.26}$$

Note that $d = L$ is possible only when $mn = 0$.

3.3.2 One-loop generating function

The second term of the one-loop mixing will be computed in appendix B.2. By combining the results (3.26) and (B.45), we obtain the one-loop generating function for single-trace operators as¹⁰

$$\begin{aligned}
 Z_2^{\text{ST}}(x, y) = N_c \sum_{L=2}^{\infty} \sum_{m=0}^L x^m y^n &\left(\sum_{\substack{d=1 \\ d|m, d|L}}^L \frac{(L/d)!}{(m/d)!(n/d)!} \text{Tot}(d) \right. \\
 &- (1 - \delta(mn, 0)) 2 \delta(\text{gcd}(m, n), 1) - \delta(mn, 0) \text{Tot}(L) \\
 &\left. - \sum_{\substack{d=1 \\ d|m, d|L}}^{L-1} \text{Tot}(d) \left\{ \frac{(L/d-2)!}{(m/d-2)!(n/d)!} + \frac{(L/d-2)!}{(m/d)!(n/d-2)!} \right\} \right), \tag{3.27}
 \end{aligned}$$

¹⁰The sum over L begins with $L = 2$, because $L = 1$ are BPS operators.

with $n = L - m$. On the first line, $d = L$ is possible only if $mn = 0$. This contribution is cancelled exactly by the last term on the second line. It follows that

$$\begin{aligned} \frac{Z_2^{\text{ST}}(x, y)}{N_c} &= -2 \sum_{L=2}^{\infty} \sum_{m=1}^{L-1} x^m y^n \delta(\text{gcd}(m, n), 1) \\ &+ \sum_{L=2}^{\infty} \sum_{m=0}^L x^m y^n \sum_{\substack{d=1 \\ d|m, d|L}}^{L-1} \frac{(L/d)!}{(m/d)!(n/d)!} \text{Tot}(d) \\ &\quad \times \left\{ 1 - \frac{m(m/d-1)}{L(L/d-1)} - \frac{n(n/d-1)}{L(L/d-1)} \right\}. \end{aligned} \tag{3.28}$$

We apply the formula (2.38) and simplify the second line as

$$\begin{aligned} \frac{Z_2^{\text{ST}}(x, y)}{N_c} &= -2 \sum_{L=2}^{\infty} \sum_{m=1}^{L-1} x^m y^{L-m} \delta(\text{gcd}(m, L), 1) \\ &+ \sum_{d=1}^{\infty} \text{Tot}(d) \sum_{L=2}^{\infty} \sum_{m=0}^L x^{dm} y^{d(L-m)} \frac{L!}{m!(L-m)!} \\ &\quad \times \left\{ 1 - \frac{m(m-1)}{L(L-1)} - \frac{(L-m)(L-m-1)}{L(L-1)} \right\}, \\ &= -2 \sum_{L=2}^{\infty} \sum_{m=1}^{L-1} x^m y^{L-m} \delta(\text{gcd}(m, L), 1) + 2 \sum_{d=1}^{\infty} \text{Tot}(d) \sum_{L=2}^{\infty} x^d y^d (x^d + y^d)^{L-2}, \\ &= -2 \sum_{L=2}^{\infty} \sum_{m=1}^{L-1} x^m y^{L-m} \delta(\text{gcd}(m, L), 1) + 2 \sum_{d=1}^{\infty} \text{Tot}(d) \frac{x^d y^d}{1 - x^d - y^d}. \end{aligned} \tag{3.29}$$

The first few terms are

$$\begin{aligned} \frac{Z_2^{\text{ST}}(x, y)}{N_c} &= 6x^2 y^2 + 4x^2 y^2 (x + y) + 2x^2 y^2 (5x^2 + 8xy + 5y^2) \\ &\quad + 2x^2 y^2 (4x^3 + 9x^2 y + 9xy^2 + 4y^3) \\ &\quad + 2x^2 y^2 (7x^4 + 14x^3 y + 24x^2 y^2 + 14xy^3 + 7y^4) + \dots \end{aligned} \tag{3.31}$$

In the degeneration limit $x = y = z$, the generation function (3.30) becomes

$$\begin{aligned} \frac{Z_2^{\text{ST}}(z)}{N_c} &= -2 \sum_{L=2}^{\infty} z^L \text{Tot}(L) + \sum_{d=1}^{\infty} \text{Tot}(d) \frac{2z^{2d}}{1 - 2z^d}, \\ &= 2 \left\{ z - \sum_{L=1}^{\infty} \text{Tot}(L) z^L \frac{1 - 3z^L}{1 - 2z^L} \right\}, \end{aligned} \tag{3.32}$$

in perfect agreement with [14].

We conjecture that the one-loop generating function for multi-traces is given by the plethystic exponential of the single-trace results (3.30):

$$\begin{aligned}
 Z_2^{\text{MT}}(x, y) &= Z_0^{\text{MT}}(x, y) \sum_{k=1}^{\infty} Z_2^{\text{ST}}(x^k, y^k), \\
 &= 2N_c \prod_{h=1}^{\infty} \frac{1}{1 - x^h - y^h} \sum_{k=1}^{\infty} \left(\sum_{d=1}^{\infty} \text{Tot}(d) \frac{x^{kd} y^{kd}}{1 - x^{kd} - y^{kd}} \right. \\
 &\quad \left. - \sum_{L=2}^{\infty} \sum_{m=1}^{L-1} x^{km} y^{k(L-m)} \delta(\text{gcd}(m, L), 1) \right).
 \end{aligned}
 \tag{3.33}$$

One can check that its expansion in small x, y agrees with (3.24). The first line is generalization of the single-variable case discussed in [14].

3.4 Comparison with Bethe Ansatz

We compare our results with the prediction of Bethe Ansatz Equations (BAEs) for $\text{XXX}_{\frac{1}{2}}$ spin chain. The single-trace operators of $\mathcal{N} = 4$ SYM in the $\text{SU}(2)$ sector correspond to the level-matched and physical solutions of BAE. The Bethe roots of the level-matched solutions satisfy

$$\mathbb{Q}(i/2) = \mathbb{Q}(-i/2), \quad \mathbb{Q}(v) = \prod_j (v - u_j),
 \tag{3.34}$$

and in the physical solutions the second \mathbb{Q} -function $\tilde{\mathbb{Q}}(v)$ must be a polynomial in v , as clarified in [31, 32].

A solution of BAEs is called regular if no Bethe roots are located at infinity. Regular BAE solutions correspond to the $\text{SU}(2)$ highest weight states. If we denote a state with $W^m Z^n$ by $|m, n\rangle$, the highest weight states satisfy

$$\mathbf{S}_- |m, n\rangle_{\text{HWS}} = 0, \quad \mathbf{S}_+ |m, n\rangle = |m + 1, n - 1\rangle,
 \tag{3.35}$$

where $\{\mathbf{S}_{\pm}, \mathbf{S}_3\}$ are the $\text{SU}(2)$ generators. We also need $m \geq n$ to count the BAE solutions correctly.

We should include exceptional solutions whose energy superficially diverges due to the Bethe roots at $v = \pm i/2$. In such cases, we should regularize the BAEs by introducing twists and by carefully taking the zero-twist limit. The results are shown in table 2.

A bit of arithmetic is needed to compare the two sets of numbers in table 2. First, consider the second row. At $(m, n) = (3, 2)$ we have $10 = 4 + 6$, where 6 comes from the multi-trace $(m, n) = (2, 2) + (1, 0)$. At $(m, n) = (4, 2)$, we have

$$\begin{aligned}
 26 &= 10 + (4 + 6 + 6), \\
 \text{Multi-traces } (4, 2) &= \{(3, 2) + (1, 0), (2, 2) + (2, 0), (2, 2) + (1, 0) + (1, 0)\}
 \end{aligned}
 \tag{3.36}$$

Next, consider the third row. At $(m, n) = (3, 3)$ we have

$$\begin{aligned}
 36 &= 6 + (10) + (2 \times 6 + 4 + 4) \\
 \text{Descendants } (4, 2) &\rightarrow (3, 3) \\
 \text{Multi-traces } (3, 3) &= \{(2, 2) + (1, 1) \dots, (3, 2) + (0, 1), (2, 3) + (1, 0)\},
 \end{aligned}
 \tag{3.37}$$

$n \backslash m$	1	2	3	4	5
1	0	0	0	0	0
2		6	10	26	44
3			36	84	176

$n \backslash m$	1	2	3	4	5
1	0	0	0	0	0
2		6	4	10	8
3			6	10	14

Table 2. Average SU(2) one-loop dimensions for $W^m Z^n$ with $m \geq n$, in the unit of $N_c g_{\text{YM}}^2 / (8\pi^2)$. Left table shows the sum over all $U(N_c)$ multi-trace operators, and Right table shows the sum over all single-trace SU(2) highest weight states.

where ... means other possible partitions, namely (1, 1) or (1, 0) + (0, 1). At $(m, n) = (4, 3)$ we have

$$\begin{aligned}
 84 &= 10 + (8 + 10) + (10 + 6 + 2 \times 4 + 2 \times 4 + 4 \times 6) \\
 \text{Descendants} & \quad \{(5, 2) \rightarrow (4, 3), (4, 2) \rightarrow (3, 3) + (1, 0)\} \\
 \text{Multi-traces} & \quad \{(4, 2) + (0, 1), (3, 3) + (1, 0), (3, 2) + (1, 1) \dots, \\
 & \quad (2, 3) + (1, 1) \dots, (2, 2) + (2, 1) \dots \}.
 \end{aligned}
 \tag{3.38}$$

The (canonical) partition function of $\text{XXX}_{\frac{1}{2}}$ spin chain of length L has been computed in [33]. He also showed that the partition function gives the character of $\mathfrak{su}(2)$ affine Kac-Moody algebra at level one in the large L limit, as conjectured in [34]. Their analysis slightly differs from ours in three points: we compute $\langle M_2 \rangle$ rather than $\langle e^{\beta M_2} \rangle$, sum over the level-matched states, and consider the grand partition function by summing over L .¹¹

4 Hagedorn transition

We compute the grand partition function of $\mathcal{N} = 4$ SYM in the SU(2) sector at one-loop at large N_c based on the above results. Then we determine the Hagedorn temperature of the $\mathcal{N} = 4$ SYM in the SU(2) sector, namely the smallest temperature $T \geq 0$ at which the grand partition function diverges, as in (1.6). We will see that the Hagedorn temperature has numerous branches depending on the value of the chemical potential on the complex plane.

4.1 Grand partition function

Consider the grand partition function of $\mathcal{N} = 4$ SYM in (1.1),

$$\mathcal{Z}(\beta, \vec{\omega}) = \text{tr} \left(e^{-\beta \mathfrak{D} + \sum_i \omega_i J_i} \right), \quad \mathfrak{D} = \mathfrak{D}_0 + \lambda \mathfrak{D}_2 + \dots
 \tag{4.1}$$

The trace is taken over the Hilbert space of all gauge-invariant operators in the SU(2) sector. The partition function has the weak coupling expansion

$$\begin{aligned}
 \mathcal{Z}(\beta, \vec{\omega}) &= \text{tr} \left(e^{-\beta \mathfrak{D}_0 + \sum_i \omega_i J_i} - \lambda \beta \mathfrak{D}_2 e^{-\beta \mathfrak{D}_0 + \sum_i \omega_i J_i} + \dots \right) \\
 &= Z_0^{\text{MT}}(x, y) - \frac{2\lambda}{N_c} \beta Z_2^{\text{MT}}(x, y) + \dots,
 \end{aligned}
 \tag{4.2}$$

¹¹The level-matching condition may be included by introducing another chemical potential coupled to the total momentum.

where we used (3.2). We assign the R-charge J_i to complex scalars as

$$Z : J_i = \delta_{i1}, \quad W : J_i = \delta_{i2}, \quad N_Z = \frac{\mathfrak{D}_0 + J_1 - J_2}{2}, \quad N_W = \frac{\mathfrak{D}_0 - J_1 + J_2}{2}. \quad (4.3)$$

The partition function (1.1) depends on $(\beta, \omega_1, \omega_2)$, whereas the generating functions in (4.2) depends on (x, y) . The two sets of variables are related by¹²

$$x = e^{-\beta + \omega_1}, \quad y = e^{-\beta + \omega_2}, \quad w = e^{-\beta}, \quad \beta = \frac{1}{T}. \quad (4.4)$$

In particular, the low-temperature expansion corresponds to the expansion in small x, y . The computation below is valid at large N_c , due to the non-perturbative corrections in (1.3).

4.2 Hagedorn temperature

We introduce

$$x \equiv e^{-\beta} \tilde{x}, \quad y \equiv e^{-\beta} \tilde{y}, \quad (4.5)$$

and vary T at a fixed (\tilde{x}, \tilde{y}) . The tree-level generating function (2.17) has simple poles at

$$T_* = \frac{k}{\log(\tilde{x}^k + \tilde{y}^k)}, \quad (k = 1, 2, \dots), \quad (4.6)$$

and the one-loop generating function (3.33) has double poles at the same location. By using

$$(\tilde{x} + \tilde{y}) - (\tilde{x}^k + \tilde{y}^k)^{1/k} = \frac{(\tilde{x} + \tilde{y})^k - (\tilde{x}^k + \tilde{y}^k)}{\sum_{j=0}^{k-1} (\tilde{x} + \tilde{y})^{k-1-j} (\tilde{x}^k + \tilde{y}^k)^{j/k}} \geq 0, \quad (4.7)$$

$$\text{for } (\tilde{x}, \tilde{y}) \in \mathcal{R}_+ = \{\tilde{x} \geq 0 \text{ and } \tilde{y} \geq 0, \tilde{x} + \tilde{y} \geq 1\},$$

The term $k = 1$ gives the smallest value of T_* inside the region \mathcal{R}_+ . The condition $\tilde{x} + \tilde{y} \geq 1$ in \mathcal{R}_+ guarantees $T_* \geq 0$.

We assume that the Hagedorn temperature and the partition function are expanded as

$$T_H(\lambda) = T_* [1 + t_1 \lambda + O(\lambda^2)],$$

$$\mathcal{Z}(\beta, \vec{\omega}) = \frac{c}{T - T_H(\lambda)} = \frac{c}{T - T_*} \left[1 + \frac{\lambda T_* t_1}{T - T_*} + O(\lambda^2) \right]. \quad (4.8)$$

Let us expand the partition function (4.2) around the pole (4.6) with $k = 1$ and compare the result with the above expansion. We find that

$$T_H(\lambda) = \frac{1}{\log(\tilde{x} + \tilde{y})} \left[1 + \frac{4\lambda\tilde{x}\tilde{y}}{(\tilde{x} + \tilde{y})^2} \right], \quad (\tilde{x}, \tilde{y}) \in \mathcal{R}_+. \quad (4.9)$$

Consider the region outside \mathcal{R}_+ . When we cross the line $\tilde{x} + \tilde{y} = 1$, then T_* becomes negative for all k . As we approach $\tilde{x} \rightarrow 0$ keeping $\tilde{x} + \tilde{y} \geq 1$, all simple poles in (4.6)

¹²The tree-level grand partition (1.1) is invariant under the simultaneous shift of $(\beta, \omega_1, \omega_2)$ by ν . This redundancy is broken at one-loop.

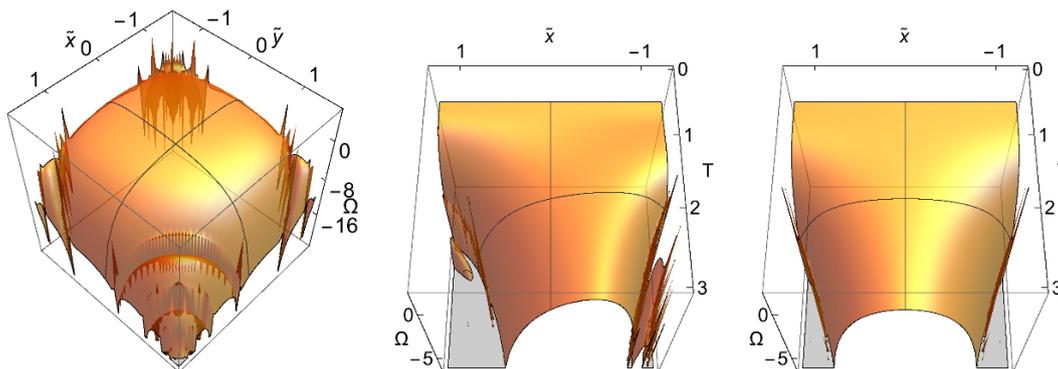


Figure 1. Plots of $\text{Re } \Omega$ at $\lambda = 0.05$. The left figure shows the plot with $(T = 5/3, \tilde{x}, \tilde{y})$, the middle figure with $(T, \tilde{x} = \tilde{y})$, and the right figure with $(T, \tilde{x} = -\tilde{y})$.

accumulate at $T_* = 1/\log(\tilde{y})$. Let us take either \tilde{x} and \tilde{y} negative, where the chemical potentials (4.4) are shifted by πi . When $|\tilde{x}|, |\tilde{y}|$ are large enough, the pole (4.6) with $k = 2$ becomes the closest to the origin among those giving $T_* > 0$. Thus, in the region

$$\mathcal{R}_- = \{ \tilde{x} \leq 0 \text{ or } \tilde{y} \leq 0, \tilde{x}^2 + \tilde{y}^2 \geq 1 \}, \quad (4.10)$$

the one-loop Hagedorn temperature is given by

$$T_H(\lambda) = \frac{2}{\log(\tilde{x}^2 + \tilde{y}^2)} \left[1 + \frac{4\lambda\tilde{x}^2\tilde{y}^2}{(\tilde{x}^2 + \tilde{y}^2)^2} \right], \quad (\tilde{x}, \tilde{y}) \in \mathcal{R}_-. \quad (4.11)$$

More generally, if we put $(\tilde{x}, \tilde{y}) \in \mathbb{C}^2$ on

$$\text{Arg } \tilde{x} = \frac{2\pi}{p_1}, \quad \text{Arg } \tilde{y} = \frac{2\pi}{p_2}, \quad p = \text{lcm}(p_1, p_2), \quad (p_1, p_2 \in \mathbb{Z}_{\geq 1}, p \ll N_c^2), \quad (4.12)$$

the Hagedorn temperature is given by the pole (4.6) at $k = p$. When $p = O(N_c^2)$, the Hagedorn transition may take place around $1/\log|\tilde{x} + \tilde{y}|$, because the free energy becomes $O(N_c^2)$ without hitting the pole.

In figure 1, the plots of the grand partition function $\Omega \equiv -T \log \mathcal{Z}$ are shown as a function of $(T, \tilde{x}, \tilde{y})$. The left figure at a fixed T shows that the singularity of Ω is associated with the boundary of \mathcal{R}_{\pm} . By comparing the middle figure $(T, \tilde{x} = \tilde{y})$ and the right figure $(T, \tilde{x} = -\tilde{y})$, we find that the former is not invariant under the flip $\tilde{x} \leftrightarrow -\tilde{x}$, whereas the latter is invariant. This pattern is consistent with \mathcal{R}_{\pm} .

For comparison with the literature, we vary T at a fixed $(\tilde{\omega}_1 = \omega_1/\beta, \xi = y/x)$. It follows that

$$T_H(\lambda) = \begin{cases} \frac{1 - \tilde{\omega}_1}{\log(1 + \xi)} \left[1 + \frac{4\lambda\xi}{(1 + \xi)^2(1 - \tilde{\omega}_1)} + \dots \right] & (\tilde{x}, \tilde{y}) \in \mathcal{R}_+, \\ \frac{2(1 - \tilde{\omega}_1)}{\log(1 + \xi^2)} \left[1 + \frac{4\lambda\xi^2}{(1 + \xi^2)^2(1 - \tilde{\omega}_1)} + \dots \right] & (\tilde{x}, \tilde{y}) \in \mathcal{R}_-. \end{cases} \quad (4.13)$$

The first line agrees with [14] when $\xi = 1, \tilde{\omega}_1 = 0$, and with [17] when $\xi = 1$.¹³

¹³Note that (1.2) of [14] is the Hagedorn temperature of the entire $\mathcal{N} = 4$ SYM. The Hagedorn temperature in the $\text{SU}(2)$ sector can be found e.g. in [35].

Let us make a few remarks on the Hagedorn transition.

First, the physical partition function should not diverge. This means that the system turns into the deconfined phase around the Hagedorn temperature, when the free energy becomes $O(N_c^2)$. In order to inspect the details of the phase transition, we need to evaluate (2.43) in the large N_c limit as in [10]. In the SU(2) sector, it is expected that the system is described by $N_c^2 + 1$ harmonic oscillators around the Hagedorn temperature [18].

Second, the parameter region $\tilde{x} < 0$ or $\tilde{y} < 0$ can be interpreted as the insertion of the number operator $(-1)^{N_Z}$ or $(-1)^{N_W}$ to the grand partition function (1.1), which makes Z or W effectively a fermion. The pole at $k = 1$ disappears when the scalar becomes fermionic. The pole at $k = 2$ still contributes to the divergence because Z^2 or W^2 are bosonic. Similarly, when the transition takes place at $k = p$ as in (4.12), Z^p, W^p are effectively bosonic. This pattern indicates that only effective bosons form a condensate inside which $U(N_c)$ degrees of freedom are liberated from the confinement.

Third, the grand partition function at finite N_c is a smooth function of the temperature, and no transition should happen [11]. This can be checked by evaluating $\mathcal{Z}_{N_c}^{\text{SU}(2)}(x, y)$ in (2.45). For example, $\mathcal{Z}_{N_c=2}^{\text{SU}(2)}(x, y)$ with $x = y = e^{-\beta}$ in (2.47) is regular for any $\beta > 0$. More generally, it is conjectured that the denominator of $\mathcal{Z}_{N_c}^{\text{SU}(2)}(x, y)$ at any $N_c < \infty$ is always a product of the factors $(1 - x^a y^b)$ for some integers $a, b \geq 0$ [24]. Hence, the Hagedorn temperature is infinite at finite N_c .

5 Conclusion and outlook

In this paper, we computed the grand partition function of $\mathcal{N} = 4$ SYM in the SU(2) sector at one-loop by making use of finite group theory. Only the planar terms contribute in this setup, though our result is valid to all orders of perturbative $1/N_c$ expansion. We derived two expressions for the one-loop generating function, called Partition form and Totient form. Based on Totient form we computed the Hagedorn temperature which depends on general values of the chemical potentials. We argued how the Hagedorn temperature changes when the chemical potentials are complex.

As future directions of research, one can consider the grand partition functions in more general situations, such as finite N_c corrections at one-loop, larger sectors of $\mathcal{N} = 4$ SYM, or higher order in λ in the SU(2) sector. It is also interesting to study superconformal field theories other than $\mathcal{N} = 4$ SYM, such as β -deformed and γ -deformed theories [36], ABJM model [37], theories with 16 supercharges [38], and quiver gauge theories [39]. Our counting methods should be applicable to integrable models like q -deformed Hubbard model [40], which is a generalization of XXZ spin chain.

Another topic is to develop group-theoretical techniques to study multi-point functions. It is well known that the OPE limit of four-point functions in $\mathcal{N} = 4$ SYM yields the sum of the anomalous dimensions of intermediate operators, weighted by the square of OPE coefficients. Such objects have been studied by conformal bootstrap [41] and integrability methods [42–44]. The effects of $1/N_c$ corrections in such a limit is worth further investigation.

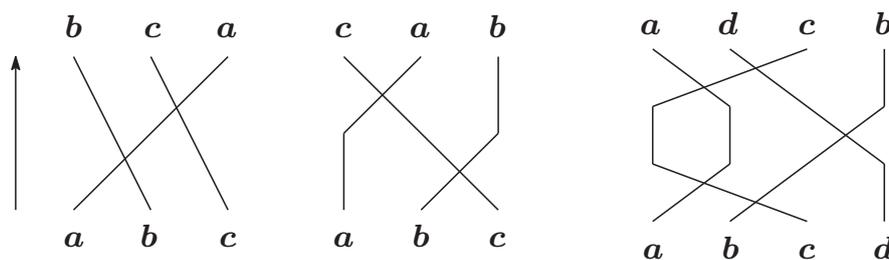


Figure 2. Permutations represented as diagrams. Left figure shows $(123) \in S_3$, Middle figure shows $(12)(23) = (132)$, and Right figure shows $(123)(34)(132) = (24)$.

Acknowledgments

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A Notation

A permutation cycle is denoted by $(i_1 i_2 \dots i_\ell)$. We define the action of permutation by keeping track of the position of a list. Algebraically, it means

$$\sigma : \{v(1), \dots, v(n)\} \mapsto \{v'(1), \dots, v'(n)\}, \quad v'(n) = v(\sigma(n)). \quad (\text{A.1})$$

For example, we have

$$\sigma = (123) : v = \{a, b, c\} \mapsto v' = \{b, c, a\}, \quad (\text{A.2})$$

as shown in figure 2. As corollaries, we find

$$\begin{aligned} (12) \cdot (23) &= (132), & (132) \cdot \{1\} &= (23) \cdot ((12) \cdot \{1\}) = (23) \cdot \{2\} = \{3\}, \\ (123) \cdot (34) \cdot (132) &= (24), \end{aligned} \quad (\text{A.3})$$

and in general

$$\alpha \cdot \beta(n) = \beta(\alpha(n)), \quad \alpha^{-1}(ij)\alpha = (\alpha(i)\alpha(j)). \quad (\text{A.4})$$

In *Mathematica*, `Permute[]` gives $v'(n) = v(\sigma(n))$. For example,

$$\begin{aligned} \text{Permute}[\{a, b, c\}, \text{Cycles}[\{\{1, 2, 3\}\}]] &= \{c, a, b\} \\ \text{PermutationReplace}[\{1, 2, 4\}, \text{Cycles}[\{\{1, 2, 3\}\}]] &= \{2, 3, 4\} \\ \text{PermutationProduct}[\text{Cycles}[\{\{3, 4\}\}], \text{Cycles}[\{\{1, 2, 3\}\}]] &= \text{Cycles}[\{\{1, 2, 3, 4\}\}] \end{aligned} \quad (\text{A.5})$$

B Details of derivation

The second term of (3.9) can be rewritten as

$$\langle M_2 \rangle_{m,n}^{(2\text{nd})} = \frac{N_c}{m! n!} \sum_{\alpha} \sum_{i \neq j=1}^L \sum_{\mu \in S_m \times S_n} \delta(i\alpha(j)) \delta_L(\mu_0 \alpha^{-1} \mu_0^{-1} \alpha), \quad \mu_0 = (ij)\mu. \quad (\text{B.1})$$

This quantity will be computed below. The result is called Partition form when we sum over $\alpha \in S_L$, and Totient form when $\alpha \in \mathbb{Z}_L$.

B.1 Second term in partition form

We evaluate $\langle M_2 \rangle_{m,n}^{(2\text{nd})}$ in the following steps:

- 1) Choose $\mu \in T_p \times T_q \subset S_m \times S_n$
- 2) Generate $\mu_0 = (ij)\mu$ by summing over (i, j)
- 3) Solve the two δ -function constraints simultaneously

1) We denote the cycle type of μ by $r \vdash L$. We have $r_k = p_k + q_k$ for $1 \leq k \leq L$, and

$$\sum_{\mu \in S_m \times S_n} f(\mu) = \sum_{p \vdash m} \sum_{q \vdash n} |T_p| |T_q| f([\mu]). \tag{B.2}$$

2) The cycle type of μ_0 depends on how i and j appear inside μ . Let us parametrize the permutation cycles of μ which contain i, j by $(x_1 \dots x_a)$. We write $(y_1 \dots y_b)$ if i, j belong to different cycles, including the case $a = b$. Then

$$(ij)\mu \sim \begin{cases} (ij)(x_1 \dots x_{l-1} i x_{l+1} \dots x_{a-1} j) = (x_1 \dots x_{l-1} i)(x_{l+1} \dots x_{a-1} j) \\ (ij)(x_1 \dots x_{a-1} i)(y_1 \dots y_{b-1} j) = (x_1 \dots x_{a-1} i y_1 \dots y_{b-1} j) \end{cases} \tag{B.3}$$

Thus, the transposition (ij) relates the cycle type of μ and μ_0 as

$$\{r_l, r_{a-l}, r_a\} \rightarrow \{r_l + 1, r_{a-l} + 1, r_a - 1\}, \quad \text{if } j = \mu^{a-l}(i) \quad (1 \leq l \leq a - 1), \tag{B.4}$$

$$\{r_a, r_b, r_{a+b}\} \rightarrow \{r_a - 1, r_b - 1, r_{a+b} + 1\}, \quad \text{if } j \neq \mu^m(i) \quad (\forall m \in \mathbb{Z}), \tag{B.5}$$

$$\{r_a, r_{2a}\} \rightarrow \{r_a - 2, r_{2a} + 1\}, \quad \text{if } j \neq \mu^m(i) \quad (\forall m \in \mathbb{Z}). \tag{B.6}$$

Let us count the number of (i, j) corresponding to each line of (B.3). As for the first line, i.e. splitting, we choose a cycle of length a and split it into $l + (a - l)$, for $1 \leq l \leq a - 1$. There are $a r_a$ ways to choose i , and the choice of j is unique for a given (i, l) . As for the second line, i.e. joining, we choose two different cycles of length a and b . If $a \neq b$, there are $ab r_a r_b$ ways to identify i, j . And if $a = b$, there are $a^2 r_a (r_a - 1)$ ways. In total, we have¹⁴

$$\begin{aligned} & \sum_{a=1}^L \sum_{l=1}^{a-1} a r_a + \sum_{a \neq b=1}^L ab r_a r_b + \sum_{a=1}^L a^2 r_a (r_a - 1), \\ & = \sum_{a=1}^L a(a-1) r_a + \sum_{a,b=1}^L ab r_a r_b - \sum_{a=1}^L a^2 r_a \\ & = L(L-1). \end{aligned} \tag{B.7}$$

¹⁴We used $\sum_a r_a = L$ and $r_c = 0$ for $c > L$.

Thus, we replace the sum over (i, j) in (B.1) by the sums over a, b, l shown above. Schematically, the sum of dimensions is given by

$$\langle M_2 \rangle_{m,n}^{(2\text{nd})} = [r_a \rightarrow r_l + r_{a-l}]_{\text{split}} + [r_a + r_b \rightarrow r_{a+b}]_{\text{join}} + [2r_a \rightarrow r_{2a}]_{\text{join}}. \quad (\text{B.8})$$

- 3) We label all possible choices of (i, j) by $\zeta = 1, 2, \dots, L(L-1)$, and denote the cycle type of $\mu_0^{(\zeta)}$ by $\bar{r}^{(\zeta)}$. For each $\bar{r}^{(\zeta)}$ we choose α as

$$\alpha \in \text{Stab}(\mu_0^{(\zeta)}) = \prod_{k=1}^L S_{\bar{r}_k^{(\zeta)}}[\mathbb{Z}_k], \quad (\text{B.9})$$

to solve the δ -function constraint (B.1). The ζ belongs to either of the two groups in (B.3).

Next, we solve the planarity condition $i = \alpha(j)$ for the three cases (B.4)–(B.6). Recall that i and j belong to the cycle of the same length in μ_0 to solve the conditions $i = \alpha(j)$ and $\alpha \in \text{Stab}(\mu_0)$, as discussed in section 3.2.1.

As for the splitting case (B.4), only the process $r_{2l} \rightarrow r_l + r_l$ can solve the condition $i = \alpha(j)$. The stabilizer of μ_0 is

$$\text{Stab}(\mu_0)_{\text{split}} = S_{r_l+2}[\mathbb{Z}_l] \cdot S_{r_{2l}-1}[\mathbb{Z}_{2l}] \cdot \prod_{k \neq l, 2l} S_{r_k}[\mathbb{Z}_k]. \quad (\text{B.10})$$

There are $2l r_{2l}$ choices of i, j , including the interchange $i \leftrightarrow j$.¹⁵ In order to solve $i = \alpha(j)$ when i, j appear in the cycles of length l , we write

$$\alpha = (ij)\alpha_0, \quad \alpha_0(i) = i. \quad (\text{B.11})$$

Here α_0 freezes the cycle of i but not of j , which restricts $\text{Stab}(\mu_0)_{\text{split}}$ down to

$$\alpha_0 \in \text{Stab}'(\mu_0)_{\text{split}} = S_{r_l+1}[\mathbb{Z}_l] \cdot S_{r_{2l}-1}[\mathbb{Z}_{2l}] \cdot \prod_{k \neq l, 2l} S_{r_k}[\mathbb{Z}_k]. \quad (\text{B.12})$$

The number of solutions in the splitting process is given by

$$N'_{\text{sol}}(\mu)_{\text{split}} = \sum_{l=1}^{L/2} \theta_{>}(r_{2l}) 2l r_{2l} \left| \text{Stab}'(\mu_0)_{\text{split}} \right| = \sum_{l=1}^{L/2} \theta_{>}(r_{2l}) l (r_l + 1) \left| \text{Stab}(\mu) \right|. \quad (\text{B.13})$$

As for the joining cases (B.5) and (B.6), any a, b can solve the condition $i = \alpha(j)$. The stabilizer of μ_0 is

$$\text{Stab}(\mu_0)_{\text{join}} = \begin{cases} S_{r_a-1}[\mathbb{Z}_a] \cdot S_{r_b-1}[\mathbb{Z}_b] \cdot S_{r_{a+b}+1}[\mathbb{Z}_{a+b}] \prod_{k \neq a, b, a+b} S_{r_k}[\mathbb{Z}_k] & (a \neq b) \\ S_{r_a-2}[\mathbb{Z}_a] \cdot S_{r_{2a}+1}[\mathbb{Z}_{2a}] \cdot \prod_{k \neq a, 2a} S_{r_k}[\mathbb{Z}_k] & (a = b). \end{cases} \quad (\text{B.14})$$

¹⁵The interchange $i \leftrightarrow j$ does not change the transposition (ij) , but we need to sum over all $i \neq j$ in (B.1).

For $a \neq b$ there are $ab r_a r_b$ choices of i, j , and for $a = b$ there are $a^2 r_a (r_a - 1)$ choices, including the interchange $i \leftrightarrow j$. In order to solve $i = \alpha(j)$ when i, j appear in the cycle of length $a + b$, we restrict $\text{Stab}(\mu_0)_{\text{join}}$ down to

$$\alpha_0 \in \text{Stab}'(\mu_0)_{\text{join}} = \begin{cases} S_{r_{a-1}}[\mathbb{Z}_a] \cdot S_{r_{b-1}}[\mathbb{Z}_b] \cdot \prod_{k \neq a, b} S_{r_k}[\mathbb{Z}_k] & (a \neq b) \\ S_{r_{a-2}}[\mathbb{Z}_a] \cdot \prod_{k \neq a} S_{r_k}[\mathbb{Z}_k] & (a = b). \end{cases} \quad (\text{B.15})$$

The number of solutions in the joining process is given by

$$\begin{aligned} N'_{\text{sol}}(\mu)_{\text{join}} &= \sum_{\substack{a \neq b \\ a+b \leq L}} \theta_{>}(r_a) \theta_{>}(r_b) ab r_a r_b \left| \text{Stab}'(\mu_0)_{\text{join}}^{a \neq b} \right| \\ &\quad + \sum_{a=1}^{L/2} \theta_{>}(r_a - 1) a^2 r_a (r_a - 1) \left| \text{Stab}'(\mu_0)_{\text{join}}^{a=b} \right| \\ &= \left\{ \sum_{\substack{a \neq b \\ a+b \leq L}} \theta_{>}(r_a) \theta_{>}(r_b) + \sum_{a=1}^{L/2} \theta_{>}(r_a - 1) \right\} \left| \text{Stab}(\mu) \right|. \end{aligned} \quad (\text{B.16})$$

4) In total, we have

$$\begin{aligned} N'_{\text{sol}} &\equiv N'_{\text{sol}}(\mu)_{\text{split}} + N'_{\text{sol}}(\mu)_{\text{join}} = \Theta(r) \left| \prod_k S_{r_k}[\mathbb{Z}_k] \right|, \\ \Theta(r) &\equiv \sum_{a=1}^{L/2} a (r_a + 1) \theta_{>}(r_{2a}) + \sum_{a < b}^L 2 \theta_{>}(L + 1 - a - b) \theta_{>}(r_a) \theta_{>}(r_b) \\ &\quad + \sum_{a=1}^{L/2} \theta_{>}(r_a - 1), \end{aligned}$$

where we removed the constraint $a + b \leq L$ by inserting $\theta_{>}(L + 1 - a - b)$. Following (2.15)–(2.17), we obtain the generating function for the second term as

$$\begin{aligned} Z_2^{\text{SU}(2), (2\text{nd})}(x, y) &= N_c \sum_{m, n} \sum_{p \vdash m, q \vdash n} |T_p| |T_q| \frac{x^m y^n}{m! n!} N_{\text{sol}}, \\ &= N_c \sum_{L=0}^{\infty} \sum_{r \vdash L} \prod_{k=1}^{\infty} (x^k + y^k)^{r_k} \Theta(r). \end{aligned} \quad (\text{B.17})$$

Consider an example. Let μ be $(1)(23)(456)$, having $r = [1^1, 2^1, 3^1] \vdash 6$. Then $\text{Stab}(\mu) = \mathbb{Z}_1 \cdot \mathbb{Z}_2 \cdot \mathbb{Z}_3$, which has the order $1 \times 2 \times 3 = 6$. The possible splitting process is

$$\mu_0 = (23)\mu = (32)\mu = (1)(2)(3)(456), \quad \text{Stab}(\mu_0) = S_3[\mathbb{Z}_1] \cdot \mathbb{Z}_3. \quad (\text{B.18})$$

We have $(i, j) = (2, 3)$ or $(3, 2)$. The list of $\{\alpha(j) \mid \alpha \in \text{Stab}(\mu_0)\}$ is $\{1^6, 2^6, 3^6\}$ for $j = 2, 3$, in the notation of table 1. Thus, the number of solutions is

$$N'_{\text{sol}}(\mu)_{\text{split}} = 6 \times 2 = 2 \left| \text{Stab}(\mu) \right|, \quad (\text{B.19})$$

which agrees with (B.13). The possible joining processes are

$$\mu_0 = \begin{cases} (123)(456), \dots & (ij) = (12), (13) \\ (23)(1456), \dots & (ij) = (14), (15), (16) \\ (1)(23456), \dots & (ij) = (24), (25), (26), (34), (35), (36), \end{cases} \quad (\text{B.20})$$

and their stabilizers are

$$\text{Stab}(\mu_0) = \begin{cases} S_2[\mathbb{Z}_3] & \bar{r} = [3^2] \\ \mathbb{Z}_2 \cdot \mathbb{Z}_4 & \bar{r} = [2^1, 4^1] \\ \mathbb{Z}_1 \cdot \mathbb{Z}_5 & \bar{r} = [1^1, 5^1]. \end{cases} \quad (\text{B.21})$$

The lists of $\alpha(j)$ is

$$j = 2, \quad \alpha(j) = \{1^3, 2^3, 3^3, 4^3, 5^3, 6^3\}, \quad \bar{r} = [3^2] \quad (\text{B.22})$$

$$j = 4, \quad \alpha(j) = \{1^2, 4^2, 5^2, 6^2\}, \quad \bar{r} = [2^1, 4^1] \quad (\text{B.23})$$

$$j = 4, \quad \alpha(j) = \{2^1, 3^1, 4^1, 5^1, 6^1\}, \quad \bar{r} = [1^1, 5^1]. \quad (\text{B.24})$$

Thus, the number of solutions is

$$N'_{\text{sol}}(\mu)_{\text{join}} = 3 \times 4 + 2 \times 6 + 1 \times 12 = 36 = 6 \left| \text{Stab}(\mu) \right|, \quad (\text{B.25})$$

which agrees with (B.16).

B.2 Second term in totient form

We evaluate $\langle M_2 \rangle_{m,n}^{(2\text{nd})}$ in the following steps:

- 1) Choose $\mu \in T_p \times T_q \subset S_m \times S_n$
- 2) Generate $\mu = (ij) \mu_0$ from μ_0 in \mathbb{Z}_d^ℓ
- 3) Generate $\mu_0 = (ij) \mu$ by reverting the last step
- 4) Solve the two δ -function constraints simultaneously

- 1) We denote the cycle type of $\mu \in S_m \times S_n$ by $p \vdash m$ and $q \vdash n$. Let m, n be divisible by $d \geq 1$ as in (2.29).
- 2) Given $\mu_0 \in \mathbb{Z}_d^\ell$ parametrized as in (2.32), we generate $\mu = (ij) \mu_0$. We classify two cases depending on whether i, j belong to the same or different cycles of μ_0 ,

$$\begin{aligned} \text{Same :} & \quad (i, j) = (\tilde{m}_{kh}, \tilde{m}_{kd}), & (1 \leq k \leq \ell, 2 \leq h \leq d) \\ \text{Different :} & \quad (i, j) = (\tilde{m}_{kd}, \tilde{m}_{k'd}), & (1 \leq k \neq k' \leq \ell) \end{aligned} \quad (\text{B.26})$$

which correspond to

$$(ij)(\tilde{m}_{k1} \dots \tilde{m}_{kd}) \rightarrow (\tilde{m}_{k1} \dots \tilde{m}_{k,h-1} i)(\tilde{m}_{k,h+1} \dots \tilde{m}_{k,d-1} j), \quad (\text{B.27})$$

$$(ij)(\tilde{m}_{k1} \dots \tilde{m}_{kd})(\tilde{m}_{k'1} \dots \tilde{m}_{k'd}) \rightarrow (\tilde{m}_{k1} \dots \tilde{m}_{kd} i \tilde{m}_{k'1} \dots \tilde{m}_{k'd-1} j). \quad (\text{B.28})$$

In terms of cycle types of μ_0 and μ , these processes can be written as

$$\begin{aligned} \text{Same: } & [d^\ell] \rightarrow [h, d-h, d^{\ell-1}] \\ \text{Different: } & [d^\ell] \rightarrow [d^{\ell-2}, 2d] \end{aligned} \tag{B.29}$$

We assume $\ell = 1$ in the Same case and $\ell \geq 2$ in the Different case, because we will see later that other cases do not contribute. These conditions are equivalent to $d = L$ and $d < L$, respectively.

- 3) We revert the above argument, and generate $\mu_0 \in \mathbb{Z}_d^\ell$ from $\mu \in S_m \times S_n$. For the Same case, we choose

$$\mu \in \begin{cases} \mathbb{Z}_m \times \mathbb{Z}_n, & (mn \neq 0) \\ \mathbb{Z}_h \times \mathbb{Z}_{L-h} & (mn = 0), \end{cases} \quad \mu_0 \in \mathbb{Z}_L. \tag{B.30}$$

First, consider the case $mn \neq 0$. There are $(m-1)!(n-1)!$ ways to choose $\mathbb{Z}_m \times \mathbb{Z}_n$ from $S_m \times S_n$. Then we choose (i, j) from $(\mathbb{Z}_m, \mathbb{Z}_n)$ or $(\mathbb{Z}_n, \mathbb{Z}_m)$. Using the formula

$$(m, L)(1, \dots, m)(m+1, \dots, L) = (1, \dots, m, m+1, \dots, L) \in \mathbb{Z}_L, \tag{B.31}$$

we generate $(ij)\mu = \mu_0 \in \mathbb{Z}_L$. There are $2mn$ ways to choose (i, j) , giving us the multiplicity

$$(m-1)!(n-1)! \cdot 2mn = 2m!n!. \tag{B.32}$$

For any choices we obtain

$$j = \begin{cases} \mu_0^n(i) & (\text{if } m+1 \leq j \leq L) \\ \mu_0^m(i) & (\text{if } 1 \leq j \leq n). \end{cases} \tag{B.33}$$

Next, consider the case $mn = 0$. The number of choices of $\mu \in \mathbb{Z}_h \times \mathbb{Z}_{L-h}$ from S_L is

$$\frac{L!}{h(L-h)} \left(1 \leq h < \frac{L}{2}\right), \quad \frac{L!}{2(L/2)^2} \left(h = \frac{L}{2}\right), \tag{B.34}$$

and there are $2h(L-h)$ ways to choose (i, j) for any h , giving the multiplicity¹⁶

$$\frac{L!}{2h(L-h)} \cdot 2h(L-h) = L!, \quad (h = 1, 2, \dots, L-1). \tag{B.35}$$

For any choices, we have $j = \mu_0^h(i)$ or $\mu_0^{L-h}(i)$.

For the Different case, we set

$$\mu \in \left(\mathbb{Z}_d^{m'-2} \times \mathbb{Z}_{2d}\right) \times \mathbb{Z}_d^{n'} \quad \text{or} \quad \mathbb{Z}_d^{m'} \times \left(\mathbb{Z}_d^{n'-2} \times \mathbb{Z}_{2d}\right), \quad \mu_0 \in \mathbb{Z}_d^{m'+n'} = \mathbb{Z}_d^\ell. \tag{B.36}$$

The case with $mn = 0$ is allowed. The number of choices of $\left(\mathbb{Z}_d^{m'-2} \times \mathbb{Z}_{2d}\right) \times \mathbb{Z}_d^{n'}$ from $S_m \times S_n$ is

$$\frac{|S_m \times S_n|}{|\text{Stab}(\mathbb{Z}_d^{m'-2} \times \mathbb{Z}_{2d}) \times \text{Stab}(\mathbb{Z}_d^{n'})|} = \frac{m!n!}{d^{m'+n'-2} (2d) (m'-2)! (n')!}. \tag{B.37}$$

¹⁶We added an extra factor of 1/2 by extending the summation range of h .

The other case $m' \leftrightarrow n'$ can be treated similarly. Then we choose (i, j) from \mathbb{Z}_{2d} and use the formula

$$(m_d m_{2d})(m_1 m_2 \dots m_{2d}) = (m_1 m_2 \dots m_d)(m_{d+1} m_{d+2} \dots m_{2d}) \in \mathbb{Z}_d^2, \quad (\text{B.38})$$

to generate $\mu_0 \in \mathbb{Z}_d^\ell$. There are $2d$ ways to choose such (i, j) from \mathbb{Z}_{2d} , which gives the multiplicity

$$\frac{m!n!}{d^{m'+n'-2} (2d) (m'-2)! (n')!} (2d) = \frac{m!n!}{d^{\ell-2} (m'-2)! (n')!} \quad (\text{B.39})$$

4) Given (i, j, μ_0) , we construct α via (2.30). This equation is repeated below:

$$\alpha = \left(a_1 \dots a_\ell \tilde{\mu}^\kappa(a_1) \dots \tilde{\mu}^\kappa(a_\ell) \tilde{\mu}^{2\kappa}(a_1) \dots \tilde{\mu}^{2\kappa}(a_\ell) \dots \tilde{\mu}^{(d-1)\kappa}(a_1) \dots \tilde{\mu}^{(d-1)\kappa}(a_\ell) \right),$$

$$1 \leq \kappa < d, \quad \text{gcd}(\kappa, d) = 1. \quad (\text{B.40})$$

Now we solve $i = \alpha(j)$, keeping in mind that $(a_1, a_2, \dots, a_\ell)$ in (B.40) belong to the different cycles of $\tilde{\mu} \equiv \mu_0$.

For the Same case, namely when i, j belong to the same cycle of μ_0 , the equation $i = \alpha(j)$ has a solution only if

$$\ell = 1, \quad d = L, \quad i = \tilde{\mu}^\kappa(j) \quad \Leftrightarrow \quad \alpha = \tilde{\mu}^\kappa. \quad (\text{B.41})$$

From (B.33) we find $\kappa = m$ or n for $mn \neq 0$, and $\kappa = h, L - h$ for $mn = 0$.

For the Different case, the general solution is

$$j = \tilde{\mu}^{\omega\kappa}(a_\xi), \quad i = \tilde{\mu}^{\omega\kappa}(a_{\xi+1}), \quad \text{or} \quad j = \tilde{\mu}^{\omega\kappa}(a_\ell), \quad i = \tilde{\mu}^{(\omega+1)\kappa}(a_1),$$

$$\ell \geq 2, \quad d \leq L/2, \quad (1 \leq \xi \leq \ell - 1, \quad 0 \leq \omega \leq d - 1). \quad (\text{B.42})$$

We use the overall translation of $\alpha \in \mathbb{Z}_L$ to fix $j = a_1$ and $i = a_2$, for both solutions in (B.42). Then α is given by

$$\alpha = (j \ i \ a_3 \dots a_\ell \tilde{\mu}(j) \tilde{\mu}(i) \dots \tilde{\mu}^\kappa(a_\ell) \dots \tilde{\mu}^{(d-1)\kappa}(j) \dots \tilde{\mu}^{(d-1)\kappa}(a_\ell)) \quad (\text{B.43})$$

The number of choices of $a_3 \dots a_\ell$ or $a_2 \dots a_{\ell-1}$ is

$$d^{\ell-2} (\ell - 2)! \quad (\text{B.44})$$

5) We summarize the above calculation. The second term of the sum of dimensions consists of two parts:

$$\langle M_2 \rangle_{m,n}^{(2\text{nd})} = \frac{N_c}{m!n!} \left(N_{m,n}^{\text{Same}} + N_{m,n}^{\text{Different}} \right). \quad (\text{B.45})$$

The number of solutions in the splitting case is

$$N_{m,n}^{\text{Same}} = \begin{cases} 2 m!n! \delta(\text{gcd}(m, n), 1), & (mn \neq 0) \\ L! \text{Tot}(L) & (mn = 0). \end{cases} \quad (\text{B.46})$$

where we used (2.23) in the last line. The number of solutions in the joining case is¹⁷

$$\begin{aligned}
 N_{m,n}^{\text{Different}} &= \sum_{\substack{d=1 \\ d|m, d|n}}^{L-1} \frac{m!n!}{d^{\ell-2} (m'-2)! (n')!} d^{\ell-2} (\ell-2)! \text{Tot}(d) + (m' \leftrightarrow n'), \\
 &= \sum_{\substack{d=1 \\ d|m, d|n}}^{L-1} m!n! \text{Tot}(d) \left\{ \frac{(L/d-2)!}{(m/d-2)!(n/d)!} + \frac{(L/d-2)!}{(m/d)!(n/d-2)!} \right\}.
 \end{aligned}
 \tag{B.47}$$

Consider some examples. Suppose $(m, n) = (7, 2)$ and consider the Same case,

$$\mu \in \mathbb{Z}_7 \times \mathbb{Z}_2, \quad \mu_0 \in \mathbb{Z}_9.
 \tag{B.48}$$

There are $6! \times 1! = 720$ ways to choose such μ from $S_7 \times S_2$. Then we choose (i, j) from two cycles, like

$$(ij)\mu = (79)(1234567)(89) = (123456789) = \mu_0 \in \mathbb{Z}_9.
 \tag{B.49}$$

There are $2 \times 7 \times 2 = 28$ ways to choose (i, j) , and all of them satisfy $j = \mu_0^2(i)$ or $j = \mu_0^7(i)$. The solution of $\mu_0 \alpha^{-1} \mu_0^{-1} \alpha = 1$ is

$$\alpha = (a \mu_0^\kappa(a) \mu_0^{2\kappa}(a) \dots \mu_0^{8\kappa}(a)), \quad \text{gcd}(\kappa, 9) = 1.
 \tag{B.50}$$

The condition $i = \alpha(j)$ requires $\kappa = 2, 7$. Therefore, the number of Same solutions is

$$720 \times 28 = 20160 = 7! \times 2! \times 2,
 \tag{B.51}$$

in agreement with (B.46).

If $\mu \in S_6 \times S_3$, then there is no contribution from the Same case due to $\text{gcd}(3, 6) \neq 1$. In the Different case we can generate $\mu_0 \in \mathbb{Z}_3^3$. There are $5! \times 2! = 240$ ways to choose $\mu \in \mathbb{Z}_6 \times \mathbb{Z}_3$ from $S_6 \times S_3$. We choose (i, j) from \mathbb{Z}_6 , like

$$(ij)\mu = (36)(123456)(789) = (123)(456)(789) = \mu_0 \in \mathbb{Z}_3^3.
 \tag{B.52}$$

There are 6 choices of (i, j) . The solution of $\mu_0 \alpha^{-1} \mu_0^{-1} \alpha = 1$ can be written as

$$\alpha = (a_1 a_2 a_3 \mu_0^\kappa(a_1) \mu_0^\kappa(a_2) \mu_0^\kappa(a_3) \dots \mu_0^{2\kappa}(a_3)), \quad \kappa \in \{1, 2\}.
 \tag{B.53}$$

We look for the solutions of $i = \alpha(j)$. By overall translation we put

$$(i, j) = (a_2, a_1).
 \tag{B.54}$$

If $(i, j) = (3, 6)$, there are 3 ways to choose a_3 from (789). The multiplicity for other (i, j) is identical. Therefore, the number of Different solutions is

$$240 \times 6 \times 3 \times \text{Tot}(3) = 6! \times 3! \times 2,
 \tag{B.55}$$

which agrees with (B.47).

¹⁷If $m' \leq 1$ or $n' \leq 1$, one of the two terms in (B.47) vanishes due to $(-1)! = \infty$.

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