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Comment

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Numerical experiments show that the above conjecture fails to hold for the largest zero $x_{n1}(\lambda) := x_{n1}^{(\lambda)}$ when λ is small and n is large enough. We provide two arguments in support of our statement.

The first one is as follows. Observe that $x_{n1}(-1/2) = 1$ for every natural number n. Since the zeros of $P_n^{(\lambda)}(x)$ coincide with the zeros of the Chebyshev polynomials of the first and of the second kind for $\lambda = 0$ and for $\lambda = 1$, respectively, then $x_{n1}(0) = \cos(\pi/2n)$ and $x_{n1}(1) = \cos(\pi/(n+1))$. If $x_{n1}(\lambda)$ is convex, then the expression

$$\mu x_{n1}(\lambda_1) + (1-\mu)x_{n1}(\lambda_2) - x_{n1}(\mu\lambda_1 + (1-\mu)\lambda_2)$$

must be positive for each $\mu \in [0,1]$ and for every pair of real parameters $\lambda_1, \lambda_2 \geqslant -\frac{1}{2}$. For $\mu = \frac{2}{3}$, $\lambda_1 = -\frac{1}{2}$ and $\lambda_2 = 1$ the above expression reduces to

$$\left(\tfrac{2}{3}\right) + \left(\tfrac{1}{3}\right)\cos(\pi/(n+1)) - \cos(\pi/2n),$$

which is positive only for $n = 1, \ldots, 6$, and negative for $n > 6$.

Various numerical experiments show that, when n is sufficiently large and fixed, the function $x_{n1}(\lambda)$ is concave in some interval $-\frac{1}{2} < \lambda < \lambda_0(n)$ and convex only for $\lambda > \lambda_0(n)$. Execute the simple MATHEMATICA 3.0 program

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tab1 = Table[N[FindRoot[GegenbauerC[10, -0.5 + k * Sqrt[2]/50, x]
== 0, {x, 1}], 16], {k, 1, 100}];
Table[N[tab1[[k - 1, 1, 2]] + tab1[[k + 1, 1, 2]] - 2 * tab1[[k, 1, 2]], 16],
{k, 2, 99}]
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The first command determines approximately the largest zeros of $P_{10}^{(\lambda)}(x)$ by Newton's method with an initial approximation $x_0 = 1$, when λ takes values at the points of the arithmetic mesh $-0.5 + k\varepsilon$, $k = 1, \ldots, 100$, with $\varepsilon = \sqrt{2}/50$. The second command calculates the second finite differences of $x_{10,1}(\lambda)$ at the mesh points. The first 62 numbers in the resulting table are negative and the remaining ones are positive. This shows that $x_{10,1}(\lambda)$ is concave for $-\frac{1}{2} < \lambda < \lambda_0(10)$ and convex for $\lambda > \lambda_0(10)$, where $\lambda_0(10) \approx 1.267766$.

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Kokologiannaki and Siafarikas' result [1] provides the upper bound $n/\sqrt{3}+\frac{1}{2}$ for $\lambda_0(n)$. However, the above arguments show that their theorem cannot be extended to the whole range of λ. Some additional examples as well as positive results on convexity and concavity properties of $x_{nk}(\lambda)$ will appear elsewhere.

References

[1] C.G. Kokologiannaki, P. Siafarikas, Convexity of the largest zero of the ultraspherical polynomials, Integral Transforms Special Funct. 4 (1996) 1–6.