

# Stabilization of bright solitons and vortex solitons in a trapless three-dimensional Bose-Einstein condensate by temporal modulation of the scattering length

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Using variational and numerical solutions of the mean-field Gross-Pitaevskii equation we show that a bright soliton can be stabilized in a trapless three-dimensional attractive Bose-Einstein condensate (BEC) by a rapid periodic temporal modulation of scattering length alone by using a Feshbach resonance. This scheme also stabilizes a rotating vortex soliton in two dimensions. Apart from possible experimental application in BEC, the present study suggests that the spatiotemporal solitons of nonlinear optics in three dimensions can also be stabilized in a layered Kerr medium with sign-changing nonlinearity along the propagation direction.

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## I. INTRODUCTION

Solitons are solutions of wave equation where localization is obtained due to a nonlinear attractive interaction. Solitons have been noted in optics [1], high-energy physics and water waves [1], and more recently Bose-Einstein condensates (BEC's) [3,4]. The bright solitons of attractive BEC's represent local maxima [4–6], whereas dark solitons of repulsive BEC's represent local minima [3].

A classic textbook example of solitons appears in the following one-dimensional (1D) nonlinear free Schrödinger equation in dimensionless units [2,7]:

$$\left[ -i \frac{\partial}{\partial t} - \frac{\partial^2}{\partial y^2} - |\Psi(y,t)|^2 \right] \Psi(y,t) = 0. \quad (1)$$

The solitons of this equation are localized solutions due to the attractive nonlinear interaction  $-|\Psi(y,t)|^2$  with wave function at time  $t$  and position  $y$ :  $\Psi(y,t) = \sqrt{2|\mathcal{E}|} \exp(-i\mathcal{E}t) \operatorname{sech}(y\sqrt{|\mathcal{E}|})$ , with  $\mathcal{E}$  the energy [7]. The Schrödinger equation with a nonlinear interaction  $-|\Psi|^2$  does not sustain a localized solitonic solution in 2D or 3D. However, a radially trapped and axially free version of this equation in 3D does sustain such a bright solitonic solution [5,6] which has been observed experimentally in BEC's [4].

To generate a soliton without a trap the repulsive kinetic pressure has to balance the attractive force. For a condensate of size  $L$ , the kinetic energy is proportional to  $L^{-2}$  whereas the attraction is proportional to  $L^{-D}$  in  $D$  dimensions. The effective potential, which is a sum of these two terms, has a confining minimum only for  $D=1$ , leading to a stable bound state [8,9]. Thus no stabilization can be obtained in 2D or 3D and any attempt to create a soliton leads to either collapse or an expansion to infinity.

A scheme of stabilization of a soliton in two dimensions has been suggested [8,10] recently by a rapid periodic temporal modulation of scattering length  $a$  of angular frequency  $\omega$  via  $a \rightarrow a_0[1 - c \sin(\omega t)]$  where  $a_0$  and  $c$  are constants and  $t$  is time. Such a modulation of the scattering length is possible by manipulating an external magnetic field near a Feshbach resonance [11] and has been employed in different studies of the BEC [12]. By such a modulation the atomic

interaction can be easily switched between attractive and repulsive, thus resulting in a rapid contraction and expansion of the condensate. If the constants of modulation are appropriately chosen, this leads to a stabilization of the condensate in 2D with breathing oscillations [8]. However, it has been “proved” by analytic and numerical calculations in Ref. [10] that such a stabilization does not take place in 3D.

As the problem of the stabilization of a soliton in a trapless condensate is of utmost interest in several areas—e.g., optics [2,13], nonlinear physics [2], and Bose-Einstein condensates—we reexamine this problem and find that a temporal modification of the scattering length can also lead to a stabilization of the trapless soliton in 3D. We use both variational as well as numerical solutions of the mean-field time-dependent Gross-Pitaevskii (GP) equation to establish our claim. To the best of our knowledge this is the first suggestion of the stabilization of a trapless soliton in 3D. We find numerically that an untrapped attractive condensate can maintain a reasonably constant spatial profile over a large interval of time through temporal modulation of the  $s$ -wave scattering length. We also point out a possible reason for the failure to stabilize a bright soliton in 3D in Ref. [10].

The present approach is also extended to stabilize vortex solitons in 2D [14] with angular momentum  $\hbar$  per atom in the axial (azimuthal) direction. The two-dimensional geometry can be achieved in an axially symmetric configuration by applying a strong trap in the axial direction [8]. This is also equivalent to applying a very weak trap in the radial direction. In both cases the radial dimension of the condensate is much larger than the axial dimension and a 2D treatment can be justified. Vortex solitons are rotating solitons of an attractive condensate and it is suggestive that a similar scheme can also be used to stabilize a vortex soliton of a trapless condensate in 3D. After the experimental observation [15] of a vortex in a rotating BEC and the theoretical prediction [6] of a radially trapped bright vortex soliton, the experimental stabilization of trapless vortex solitons seems viable. However, the stabilization of a vortex soliton in 3D calls for a full three-dimensional calculation and is beyond the scope of the present study. In the present study we stick to a two-dimensional circularly symmetric configuration. It is well known that in a real 3D problem, the vortices are

unstable against azimuthal perturbation which breaks the azimuthal symmetry and calls for a full 3D treatment. However, these degrees of freedom are expected to be partially suppressed in the limit of a very strong azimuthal trap or a very weak radial trap when the vortex dynamics becomes essentially two dimensional. In that limit a circularly symmetric 2D calculation for a bright vortex soliton should be sufficient. In this paper we find that such a 2D vortex is stable against radial perturbation. However, we have not established its stability under transverse perturbation. The stability under transverse perturbation can be tested by a calculation in Cartesian coordinates and would be an investigation of future interest.

In Sec. II we present the mean-field model which we use in our study. In Secs. III and IV, respectively, we present the variational and numerical results of our investigation and in Sec. V we present our conclusions.

## II. MEAN-FIELD MODEL

We use the mean-field GP equation for the present investigation [9]. In terms of an external reference angular frequency  $\Omega$  and length  $l \equiv \sqrt{\hbar/(m\Omega)}$  the GP equation for the time-dependent Bose-Einstein condensate wave function  $\Psi(\vec{r};t)$  at position  $\vec{r}$  and time  $t$  can be rewritten in dimensionless form as [14]

$$\left[ -i\frac{\partial}{\partial t} - \nabla_r^2 + \frac{1}{4}\left(\frac{\omega_\rho^2}{\Omega^2}\rho^2 + \frac{\omega_z^2}{\Omega^2}z^2\right) + 8\sqrt{2}\pi n|\Psi(\vec{r};t)|^2 \right] \Psi(\vec{r};t) = 0, \quad (2)$$

where length, time,  $\nabla^2$ , and wave function are expressed in units of  $l/\sqrt{2}$ ,  $\Omega^{-1}$ ,  $(l/\sqrt{2})^{-2}$ , and  $(l/\sqrt{2})^{-3/2}$ , respectively. Here nonlinearity  $n=Na/l$ ,  $m$  is the mass and  $N$  the number of atoms in the condensate, and  $a$  the atomic scattering length. The scattering length  $a$  and nonlinearity  $n$  are negative for an attractive condensate and positive for a repulsive condensate. In Eq. (2) there is an axially symmetric harmonic trap with angular frequency  $\omega_\rho$  in the radial direction  $\rho$  and  $\omega_z$  in the axial direction  $z$ . The normalization condition in Eq. (2) is  $\int d\vec{r}|\Psi(\vec{r};t)|^2=1$ .

The quasi-2D limit of Eq. (2) is achieved by considering  $\Omega=\omega_z \gg \omega_\rho$ . This condition is satisfied by taking the limit  $\omega_\rho \rightarrow 0$  for a fixed  $\Omega=\omega_z$ . This corresponds to a pancake-shaped condensate and we look for a solution of the form  $\Psi(\vec{r};t)=\mathcal{A}(z)\psi(\vec{\rho};t)$  with  $\mathcal{A}(z)$  satisfying the one-dimensional ground-state oscillator equation

$$-\frac{d^2}{dz^2}\mathcal{A}(z) + \frac{z^2}{4}\mathcal{A}(z) = \frac{1}{2}\mathcal{A}(z), \quad (3)$$

with  $|\mathcal{A}(z)|^2 = \sqrt{1/(2\pi)}\exp[-z^2/2]$  and  $\int_{-\infty}^{\infty}|\mathcal{A}(z)|^2 dz=1$ . Multiplying Eq. (2) by  $\mathcal{A}^*(z)$  and integrating over  $z$  we get the quasi-two-dimensional GP equation for  $\psi(\vec{\rho};t)$  [8]:

$$\left[ -i\frac{\partial}{\partial t} - \nabla_\rho^2 + \frac{1}{4}\frac{\omega_\rho^2}{\Omega^2}\rho^2 + 4n\sqrt{2}\pi|\psi(\vec{\rho};t)|^2 \right] \psi(\vec{\rho};t) = 0, \quad (4)$$

with normalization  $\int d\vec{\rho}|\psi(\vec{\rho};t)|^2=1$ . In a quantized vortex state [14], with each atom having angular momentum  $L\hbar$

along the  $z$  axis,  $\psi(\vec{\rho},t)=\varphi(\rho,t)\exp(iL\phi)$ , where  $\phi$  is the azimuthal angle. Then the radial part of the GP equation (4) becomes [14]

$$\left[ -i\frac{\partial}{\partial t} - \frac{\partial^2}{\partial \rho^2} - \frac{1}{\rho}\frac{\partial}{\partial \rho} + \frac{1}{4}\frac{\omega_\rho^2}{\Omega^2}\rho^2 d(t) + \frac{L^2}{\rho^2} + 4n\sqrt{2}\pi|\varphi(\rho;t)|^2 \right] \varphi(\rho;t) = 0, \quad (5)$$

with normalization  $2\pi\int_0^\infty d\rho\rho|\varphi(\rho;t)|^2=1$ . In Eq. (5) we have introduced a strength parameter  $d(t)$  with the radial trap. Normally, in the presence of the radial trap  $d(t)=1$ . When the radial trap is switched off  $d(t)$  will be reduced to 0.

The spherically symmetric limit of the three-dimensional GP equation (2) is obtained by taking  $\omega_z=\omega_\rho=\omega_r$ . In the spherically symmetric configuration,  $\Psi(\vec{r},t)=\varphi(r,t)$ . Then the radial part of the GP equation (2) becomes

$$\left[ -i\frac{\partial}{\partial t} - \frac{\partial^2}{\partial r^2} - \frac{2}{r}\frac{\partial}{\partial r} + \frac{\omega_r^2}{4\Omega^2}r^2 d(t) + 8\pi n\sqrt{2}|\varphi(r;t)|^2 \right] \varphi(r;t) = 0, \quad (6)$$

with normalization  $4\pi\int_0^\infty r^2 dr|\varphi(r;t)|^2=1$ . Here, as in Eq. (5),  $d(t)$  is a strength parameter which is to be reduced to 0 from 1 when the radial trap is switched off.

Equations (5) and (6) represent nonrotating ( $L=0$ ) and vortex solitons ( $L \neq 0$ ) rotating around  $z$  axis in the quasi-two-dimensional and spherically symmetric three-dimensional cases, respectively. To study the solitons we finally set  $d(t)=0$  in these equations. It should be noted that  $\Omega$  is supposed to be a constant reference frequency and not the trap frequencies  $\omega_\rho$  or  $\omega_z$ . However, we took in Eqs. (3)–(5) the condition  $\omega_z=\Omega$ . This does not correspond to any specialization but only simplifies the equations algebraically. Nevertheless, it is of advantage to take  $\Omega$  to have the same order of magnitude as an experimental trap frequency—for example,  $\Omega \equiv 2\pi \times 80$  Hz. With this value of  $\Omega$  the dimensionless time unit corresponds to  $\Omega^{-1}=1/(2\pi \times 80)$  s  $\approx 2$  ms. In Eq. (6),  $\Omega$  is a constant and  $\Omega \neq \omega_r$ .

We solve the GP equations (5) and (6) numerically using the split-step time-iteration method employing the Crank-Nicholson discretization scheme described recently [16]. The time iteration is started with the known oscillator solution of these equations with zero nonlinearity  $n$ . Then in the course of time iteration the attractive nonlinearity is switched on very slowly and in the initial stage the harmonic trap is also switched off slowly by changing  $d(t)$  from 1 to 0. If the nonlinearity is increased rapidly the system collapses. The tendency to collapse must be avoided to obtain a stabilized soliton. After switching off the harmonic trap in Eqs. (5) and (6) and after slowly introducing a final attractive nonlinearity  $n_0$ , if  $n$  is replaced by  $n_0[1-c \sin(\omega t)]$ , a stabilization of the final solution could be obtained for a suitably chosen  $c$  and a large  $\omega$ . The stabilization could be obtained for a range of values of  $c$  and  $\omega$  provided that  $n_0$  is negative corresponding to attraction. After some experimentation with Eqs. (5) and (6) we opted for the choice  $c=4$  and  $\omega=10\pi$  in all our cal-

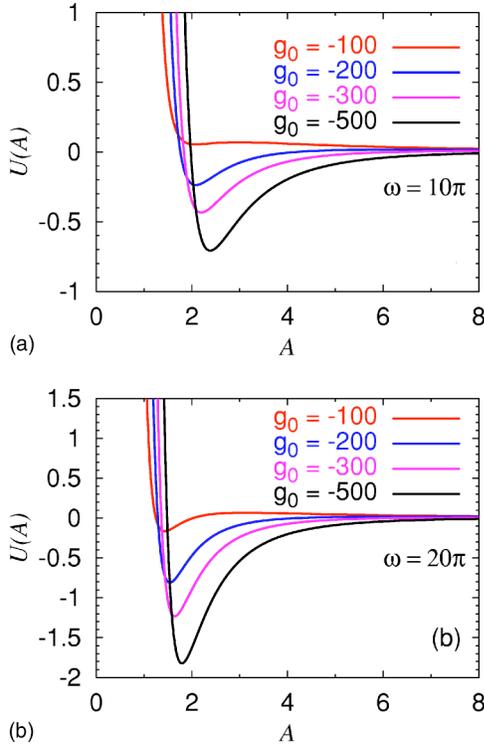


FIG. 1. The effective potential  $U(A)$  of Eq. (14) vs  $A$  in arbitrary units for  $g_0 = -100, -200, -300,$  and  $-500$  (from upper to lower curve) for (a)  $\omega = 10\pi$  and (b)  $\omega = 20\pi$ .

culations except in Fig. 1(b) in 2D and 3D—variational and numerical. In Fig. 1(b) we report some results for  $\omega = 20\pi$  for comparison.

### III. VARIATIONAL RESULTS

To understand how the stabilization can take place we employ a variational method with the following Gaussian wave function for the solution of Eqs. (5) and (6) [8,10]:

$$\varphi(r, t) = N(t)r^L \exp\left[-\frac{r^2}{2R^2(t)} + \frac{i}{2}\beta(t)r^2 + i\alpha(t)\right], \quad (7)$$

where  $N(t)$ ,  $R(t)$ ,  $\beta(t)$ , and  $\alpha(t)$  are the normalization, width, chirp, and phase of the soliton, respectively. In 3D,  $N(t) = [\pi^{3/4}R^{3/2}(t)]^{-1}$  and  $L=0$ , and in 2D,  $N(t) = [\pi^{1/2}R^{L+1}(t)\sqrt{L!}]^{-1}$ . The Lagrangian density for generating Eq. (6) with  $\omega_r=0$  is [10]

$$\mathcal{L}(\varphi) = \frac{i}{2}\left(\frac{\partial \varphi}{\partial t}\varphi^* - \frac{\partial \varphi^*}{\partial t}\varphi\right) - \left|\frac{\partial \varphi}{\partial r}\right|^2 - \frac{L^2|\varphi|^2}{r^2} - \frac{1}{2}g|\varphi|^4, \quad (8)$$

where  $g \equiv 8\pi n\sqrt{2}$  in 3D and  $g \equiv 4n\sqrt{2\pi}$  in 2D. The trial wave function (7) is substituted in the Lagrangian density and the effective Lagrangian is calculated by integrating the Lagrangian density:  $L_{\text{eff}} = \int \mathcal{L}(\varphi) d\vec{r}$ .

The Euler-Lagrange equations for  $R(t)$  and  $\beta(t)$  are then obtained from the effective Lagrangian in standard fashion in 3D:

$$\frac{dR(t)}{dt} = 2R(t)\beta(t), \quad (9)$$

$$\frac{d\beta(t)}{dt} = \frac{2}{R^4(t)} - 2\beta^2(t) + \frac{g}{2\sqrt{2\pi^3}R^5(t)}. \quad (10)$$

From Eqs. (9) and (10) we get the following second-order differential equation for the evolution of the width:

$$\frac{d^2R(t)}{dt^2} = \frac{4}{R^3(t)} + \frac{g_0 + g_1\sin(\omega t)}{\sqrt{2\pi^3}R^4(t)}, \quad (11)$$

with  $g = g_0 + g_1\sin(\omega t)$ , where  $g_0$  corresponds to the constant part of the scattering length and  $g_1$  to the oscillating part. We separate  $R(t)$  into a slowly varying part  $A(t)$  and a rapidly varying part  $B(t)$  by  $R(t) = A(t) + B(t)$ . Substituting this into Eq. (11) and retaining terms on the order of  $\omega^{-2}$  in  $B(t)$  we obtain the following equations of motion for  $B(t)$  and  $A(t)$ :

$$\frac{d^2B(t)}{dt^2} = \frac{g_1\sin(\omega t)}{\sqrt{2\pi^3}A^4(t)},$$

$$\frac{d^2A(t)}{dt^2} = \frac{4}{A^3(t)} + \frac{g_0}{\sqrt{2\pi^3}A^4(t)} - \frac{2\sqrt{2}g_1\langle B(t)\sin(\omega t) \rangle}{\pi^{3/2}A^5(t)},$$

where  $\langle \dots \rangle$  denotes the time average over rapid oscillations.

Using the solution  $B(t) = -g_1\sin(\omega t)/[\sqrt{2\pi^3}\omega^2A^4(t)]$ , the equation of motion for  $A(t)$  becomes

$$\frac{d^2A(t)}{dt^2} = \frac{4}{A^3} + \frac{g_0}{\sqrt{2\pi^3}A^4} + \frac{g_1^2}{\pi^3\omega^2A^9} \quad (12)$$

$$= -\frac{\partial}{\partial A} \left[ \frac{2}{A^2} + \frac{g_0}{3\sqrt{2\pi^3}A^3} + \frac{g_1^2}{8\pi^3\omega^2A^8} \right]. \quad (13)$$

The quantity in the square brackets in Eq. (13) is the effective potential  $U(A)$  of the equation of motion:

$$U(A) = \frac{2}{A^2} + \frac{g_0}{3\sqrt{2\pi^3}A^3} + \frac{g_1^2}{8\pi^3\omega^2A^8}. \quad (14)$$

Small oscillations around a stable configuration are possible when there is a minimum in this effective potential [8]. Unfortunately, this condition does not lead to a simple analytical solution. However, straightforward numerical study reveals that this effective potential has a minimum for a negative  $g_0$  corresponding to attraction with  $|g_0|$  above a critical value. For a numerical calculation the quantity  $g$  is taken to be of the form  $g = g_0 + g_1\sin(\omega t) = g_0[1 - 4\sin(\omega t)]$  so that  $g_1 = -4g_0$  with  $g_0$  negative (attractive).

In Figs. 1(a) and 1(b) we plot the effective potential  $U(A)$  vs  $A$  for different  $g_0$  for  $\omega = 10\pi$  and  $\omega = 20\pi$ , respectively. We find that, as the value of  $g_0$  is reduced, the effective potential develops a minimum which gradually becomes deeper and deeper. The depth of the minimum in the effective potential increases as  $\omega$  increases. For  $\omega = 10\pi$  and  $g_0 = -100$  there is no minimum in the effective potential  $U(A)$ , whereas a minimum has appeared for  $g_0 = -200$  which be-

comes deeper for  $g_0 = -300$  and  $-500$ . For  $\omega = 20\pi$  and  $g_0 = -100$ , a minimum has already appeared in Fig. 1(b). In the rest of this study we use the frequency  $\omega = 10\pi$ , although its actual value has no consequence on the calculation as long as it is large corresponding to rapid oscillations. A careful examination reveals that the threshold for the minimum in the present case is given by  $g_0 \approx -116$  for  $\omega = 10\pi$ . Hence, for  $\omega = 10\pi$  stabilization is not possible for  $g_0 = -100$ , and it is possible for  $g_0 < -116$ . Also there is no upper limit for  $|g_0|$  and stabilization is possible for an arbitrarily large  $|g_0|$ . As the first and third terms on the right-hand side (RHS) of Eq. (14) are positive, no stabilization is possible for a positive  $g_0$  corresponding to repulsion. We shall verify these findings by actual numerical calculation in the following.

Similarly, a two-dimensional soliton of Eq. (5) leads to the following equation of motion for the small oscillation of width  $A(t)$  for a general  $L$ :

$$\frac{d^2 A(t)}{dt^2} = - \frac{\partial}{\partial A} \left[ \frac{g_0 + 4\pi}{\pi A^2} + \frac{1}{4} \frac{g_1^2}{\pi^2 \omega^2 A^6} \right]. \quad (15)$$

A similar equation was obtained before for  $L=0$  [8]. The condition for small oscillations is given by  $g_0 < -4\pi$  or  $n < -\sqrt{\pi/2}$ , when the first term on the RHS of Eq. (15) becomes negative, allowing for the possibility of a minimum in the effective potential in the square brackets resulting in stable small oscillations.

#### IV. NUMERICAL RESULTS

With this preliminary variational study we turn to a full numerical investigation of Eqs. (5) and (6) in 2D and 3D. As a warm-up it is worthwhile to redo the numerical study for a  $L=0$  soliton in 2D and extend it to a  $L=1$  vortex soliton before considering a  $L=0$  soliton in 3D.

There could be many ways of numerical stabilization of the soliton. In the course of time evolution of the GP equation certain initial conditions are necessary for the stabilization of a soliton with a specific nonlinearity  $|n_0|$  above a critical value. As one requires a large (attractive) nonlinearity  $|n_0|$  for stabilization, one needs to reduce the harmonic trap frequency while increasing the nonlinearity  $|n|$ . Unless the trap frequency is reduced the system will collapse [14] due to attraction. In other words, one allows the system to expand and simultaneously increase the nonlinearity  $|n|$ . During this process the harmonic trap is removed, and after the final nonlinearity  $n_0$  is attained at time  $t_0$  the periodically oscillating nonlinearity  $n = n_0 [1 - 4 \sin\{10\pi(t - t_0)\}]$  is applied for  $t > t_0$ . If the size of the condensate is close to the desired size, a stabilization of the condensate for a large time is obtained. This procedure could also be followed in an experimental attempt to stabilize a soliton. Saito and Ueda [8] used a qualitatively similar, but quantitatively different, procedure for stabilization. The procedure of Saito and Ueda could also be applied successfully in the present context.

The correct implementation of the above calculational scheme is important for stabilization. If the (attractive) nonlinearity after switching off the harmonic trap is strong for the size of the condensate, the system becomes highly attrac-

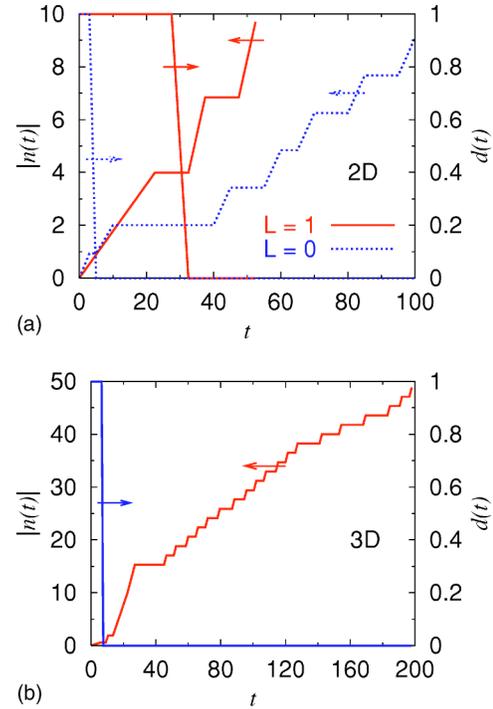


FIG. 2. Variation of nonlinearity parameter  $|n(t)|$  and the strength of radial trap  $d(t)$  in (a) Eq. (5) and (b) Eq. (6) in the initial stage of stabilization until the desired final nonlinearity  $n_0$  is attained at time  $t_0$ . For  $t > t_0$ , the oscillating nonlinearity  $n = n_0 [1 - 4 \sin\{10\pi(t - t_0)\}]$  is applied. In 2D,  $n_0 = -9.1$  for  $L=0$  and  $n_0 = -9.7$  for  $L=1$ ; in 3D,  $n_0 = -48.9$ .

tive in the final stage and it eventually collapses. If the nonlinearity after switching off the harmonic trap is weak for its size, the system becomes weakly attractive in the final stage and it expands to infinity. The final nonlinearity has to have an appropriate intermediate value, decided by trial, for eventual stabilization. In Figs. 2(a) and 2(b) we provide the actual time variation of nonlinearity  $|n(t)|$  as well as the strength parameter for the radial trap  $d(t)$  employed in Eqs. (5) and (6) for 2D and 3D, respectively. A fine-tuning of the final nonlinearity  $|n(t)|$  was needed for stabilization over a large interval of time, as reported in this paper.

The results of numerical calculations based on Eq. (5) are shown in Figs. 3(a) and 3(b) for the two-dimensional soliton ( $L=0$ ) and vortex soliton ( $L=1$ ), respectively, where we plot the radial part of the wave function at times  $t = 0, 50, 100, 150, 200, 250, 300$  after stabilization is obtained. With the present value of  $\Omega = 2\pi \times 80$  Hz,  $t = 300$  corresponds to 600 ms.

Next we turn to a numerical calculation in 3D. The relation between the constant  $g$  considered in the variational calculation and nonlinearity  $n$  in the GP equation is  $g \equiv 8\pi m \sqrt{2} \approx 35n$ . From the variational calculation presented in Fig. 1(a) for  $\omega = 10\pi$ , we find that the condition for stabilization of a soliton in 3D is  $g_0 < -116$  which corresponds to  $n_0 < -3.3$ , approximately. From a complete numerical solution of the GP equation (6) we also find that there is a threshold of nonlinearity for stabilization consistent with the variational calculation. In the numerical calculation it was

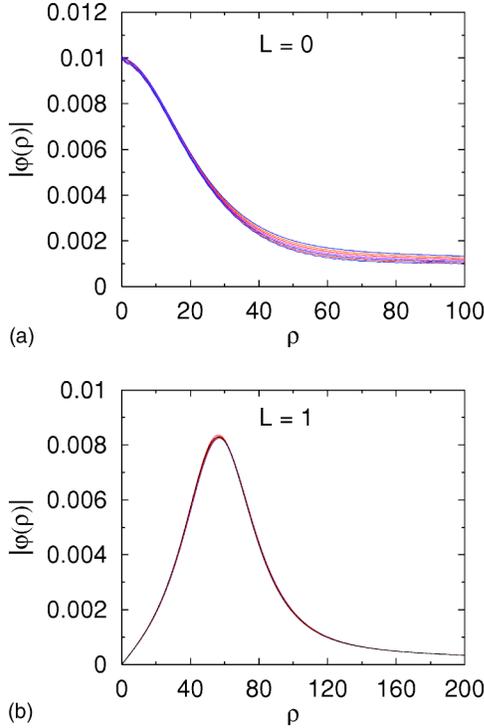


FIG. 3. (a) Wave function  $|\varphi(\rho)|$  of stabilized solitons in 2D with  $L=0$  and  $n=-9.1[1-4\sin(10\pi t)]$  in Eq. (5) at times  $t=0, 50, 100, 150, 200, 250, 300$ . (b) Same for vortex solitons with  $L=1$  and  $n=-9.7[1-4\sin(10\pi t)]$  in Eq. (5) at times  $t=0, 50, 100, 150, 200, 250, 300$ .

difficult to obtain accurately the threshold value of  $n_0$  for stabilization. However, we could not obtain stabilization of the soliton for  $n_0 > -10$ . The stabilization is possible for stronger  $n_0$  (larger  $|n_0|$ ). In Fig. 4 we plot the radial wave function of Eq. (6) in 3D at times  $t=0, 50, 100, 150, 200, 250, 300$  for  $n_0=-48.9$  after obtaining the stabilization. The narrow spread of the wave function over the large interval of time shows the quality of stabilization. In both Figs. 3 and 4 the results at intermediate times lie in the region covered by the plots. The plot of the full wave function, rather than that of mean radii or the wave function at a particular point versus time, clearly shows the degree of stabilization achieved.

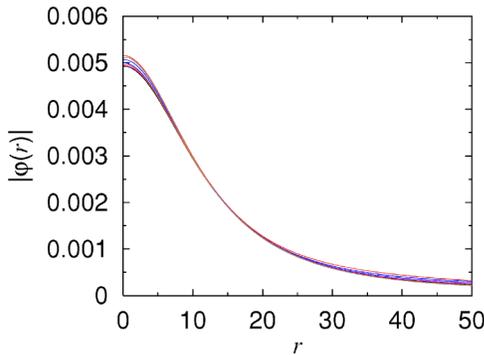


FIG. 4. Wave function  $|\varphi(r)|$  of stabilized soliton in 3D with  $n=-48.9[1-4\sin(10\pi t)]$  in Eq. (6) at times  $t=0, 50, 100, 150, 200, 250, 300$ .

Of the three cases presented in Figs. 3 and 4 the vortex soliton of Fig. 3(b) is the most stable with minimum oscillations. For the rotating (vortex) soliton, the outward *centrifugal* force approximately balances the attractive inward force (as seen in a rotating frame). The final exact balance is provided by the oscillating nonlinearity. However, for  $L=0$  there is no centrifugal force and stabilization is more difficult.

The stabilization in both 2D and 3D can be obtained for (attractive) BEC solitons with nonlinearity  $|n_0|$  larger than a critical value. In 2D the variational critical nonlinearity is  $n_{\text{crit}}=-\sqrt{\pi/2}$ , whereas the final average nonlinearities in Figs. 3(a) and 3(b) are  $n_0=-9.1$  and  $-9.7$ , respectively. In actual numerical calculations we found that the stronger the nonlinearity  $|n_0|$ , the more sustained was the stabilization of the soliton. The effective potential develops a deeper minimum for a larger nonlinearity  $|n_0|$ . The variational threshold for stabilization in 3D is  $|n_{\text{crit}}|\approx 3.3$ . In Fig. 4 the actual final average nonlinearity  $|n_0|=48.9$  is much larger than the variational threshold  $|n_{\text{crit}}|\approx 3.3$ .

Using a variational procedure alone, not quite identical with the present approach, Abdullaev *et al.* [10] also had found that stabilization of a soliton could be possible in 3D via a temporal modulation of the scattering length. However, they confirmed after further analytical and numerical studies that such a stabilization does not take place in 3D. Saito and Ueda [8], on the other hand, are silent about the possibility of the stabilization of a soliton in 3D. We point out one possible reason for the negative result obtained in 3D [10]. The nonlinearity parameter  $\Lambda=-1$  used in Ref. [10] for stabilizing a soliton in 3D corresponds in our notation to  $n_0=-\sqrt{2\pi^3}/[8\pi\sqrt{2}]=-\sqrt{\pi}/8\approx -0.22$ . [The relation between  $n_0$  and  $\Lambda$  of Ref. [10] follows from the present Eq. (11) and their Eq. (37).] The nonlinearity  $\Lambda=-1$  is much too weak for obtaining a stabilized soliton in 3D. It should be noted that the value of  $n_0$  used for the stabilization of a soliton in 3D in the present calculation is  $-48.9$ , whereas the variational threshold for stabilization is  $n_{\text{crit}}\approx -3.3$ . These values of nonlinearities are much stronger than the value  $n_0=-0.22$  used in Ref. [10].

## V. DISCUSSION AND CONCLUSION

In this paper we have discussed the stabilization of a bright vortex soliton in 2D and a bright soliton in 3D by a periodic temporal modulation of the scattering length. Now we compare the dimensionless parameters used in the simulations to typical numbers for an experimental system  $^{85}\text{Rb}$ . This system has a Feshbach resonance which can be used to vary the effective interaction between atoms by varying the atomic scattering length [17]. With the reference trap  $\Omega=2\pi\times 80\text{ Hz}$ , for  $^{85}\text{Rb}$  the dimensionless length parameter  $l=\sqrt{\hbar}/(m\Omega)\approx 1.2\ \mu\text{m}$ . In a 3D condensate of 10 000 atoms, the variational critical nonlinearity  $|n_{\text{crit}}|=3.3$  for this system corresponds to a scattering length  $a=n_{\text{crit}}l/N\approx -0.4\text{ nm}$ . The applied nonlinearity  $n_0=-48.9$  of the present 3D simulation corresponds to a scattering length  $a\approx -5.9\text{ nm}$ . The oscillating nonlinearity corresponds in this case ( $a\approx -5.9\text{ nm}$ ) to a variation of scattering length between  $-30\text{ nm}$  and  $18\text{ nm}$ . For 100 000 atoms the above values for scattering length will

be reduced by a factor of 10. Similar variations of the scattering length of  $^{85}\text{Rb}$  have already been realized in the laboratory via a Feshbach resonance in actual experiments [17]. Hence it might be possible to stabilize a  $^{85}\text{Rb}$  condensate using a Feshbach resonance. With  $\Omega=2\pi\times 80$  Hz, the interval of stabilization of 300 units of time in Figs. 3 and 4 corresponds to 600 ms, which is a reasonably large interval of time.

The present study has important consequences in the generation of a stable spatiotemporal soliton of nonlinear optics [2,18] which is an optical wave packet confined in all three directions and often referred to as a light bullet. The existence and stability of self-trapped beams in a nonlinear medium has been a subject of active research since its suggestion [18]. Such a spatiotemporal optical soliton satisfies an equation in an anomalously dispersive medium quite similar to Eq. (6) with  $\omega_r=0$  [2]. Hence a stable solution for a light bullet in actual 3D can be obtained through a modulation of the cubic Kerr nonlinearity. This modulation can be achieved in a layered nonlinear medium with sign-altering Kerr nonlinearity [10,13]. The possibility of such a stabilization in two space dimensions was demonstrated in [13], whereas its impossibility in 3D has been emphasized in [10].

In conclusion, from a numerical solution of the GP equation we find that it is possible to stabilize a matter-wave bright soliton in 3D and a vortex soliton in 2D by employing a rapid periodic modulation of the scattering length  $a$  via a Feshbach resonance with an attractive (negative) mean value  $a_0$  via  $a\rightarrow a_0[1-c\sin(\omega t)]$  with a large  $c$  and  $\omega$ . From a variational calculation we show that this oscillation produces a minimum in the effective potential, thus producing a potential well in which the soliton can be trapped and execute small oscillations. The sinusoidal variation of  $a$  is actually not needed for stabilization; any periodic fluctuation between positive and negative values stabilizes the soliton. This is of interest to investigate if such BEC “bullets” could be created experimentally in 3D. As the mathematical equation satisfied by a light bullet [2] in 3D is quite similar to the nonlinear three-dimensional equation (6) we suggest the possibility of creating light bullets in a layered Kerr medium with sign-altering nonlinearity.

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