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Câmpus de São José do Rio Preto

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Piecewise smooth vector fields: Closing lemma, Topological entropy and Shifts

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Tese apresentada como parte dos requisitos para obtenção do título de Doutor em Matemática, junto ao Programa de Pós-Graduação em Matemática, do Instituto de Biociências, Letras e Ciências Exatas da Universidade Estadual Paulista "Júlio de Mesquita Filho", Câmpus de São José do Rio Preto.

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A minha família,
dedico

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"A preocupação com o próprio homem e seu destino deve ser sempre o principal objetivo de todos os empreendimentos técnológicos(...) para que criações de nossa mente sejam uma benção, e não uma maldição, para a humanidade. Nunca se esqueçam disso quando estiverem às voltas com os seus diagramas e equações."

(Einstein, 1931, citado em [14], tradução nossa)

RESUMO

Recentemente, a teoria sobre campos vetoriais suaves por partes (CVSPs) tem passado por importantes desenvolvimentos. Uma linha de investigação desses campos vetoriais é procurar estabelecer resultados análogos aos já clássicos sobre campos vetoriais suaves, como os teoremas de Poincaré-Bendixson e Índice de Poincaré. Nessa linha de trabalho, nós abordamos o clássico problema do Closing Lemma para CVSPs e apresentamos uma resposta positiva para o caso C^0.

Outra possível linha de investigação é estudar as diferenças entre CVSPs e suaves. A maior parte delas surge do fato de que não há unicidade de trajetória por um ponto para CVSPs. Isto implica resultados como a existência de um CVSP planar caótico. Nesta linha de trabalho, propomos um novo modo de abordar CVSPs, através da construção de um espaço métrico de todas as possíveis trajetórias, usamos isso para definir entropia topológica de um CVSP; provamos a existências de CVSPs com entropia positiva (finita e infinita) e damos condições suficientes para que um CVSP tenha entropia infinita. Além disso, a partir desse espaço métrico, propomos um modo de conjugar a dinâmica de um CVSP com a aplicação shift em espaços de sequências.

Palavras-chave: Sistemas dinâmicos, Campos de vetores suaves por partes, Closing Lemma, Entropia topológica, Shifts.

ABSTRACT

Recently, the theory concerning piecewise smooth vector fields (PSVFs for short) has been undergoing important improvements. One line of investigation of these vector fields is to seek to establish results analogous to those already well known for the smooth case, such as Poincaré Bendixson and Poincaré Index Theorems. On this line of work, we tackle the classical problem of Closing Lemma in the setting of PSVFs and provide a positive answer for the case C^0.
Another possible line of investigation is studying the differences between PSVFs and smooth vector fields. Most of them arises from the fact that there is no uniqueness of trajectory passing through a point for a PSVF. It implies results like the existence of a planar PSVF that is chaotic. On this line, we propose a new way of looking at PSVFs, by the construction of a metric space of all possible trajectories, we use it to define topological entropy of a PSVF; prove the existence of PSVFs of positive (finite and infinite) entropy and give a sufficient condition for a PSVF to have infinite topological entropy. Moreover, from this metric space, we propose a way of conjugating the dynamics of PSVFs and shift spaces.

Keywords: Dynamical Systems, Piecewise smooth vector fields, Closing Lemma, Topological entropy, Shifts.

Contents

CHAPTER 1

Introduction

Ordinary Differential Equations (ODEs) has been extensively studied since mid 17th century from pure and applied points of view and had contributions from many important mathematicians like Newton, Euler, the Bernoulli family, Laplace, Cauchy and others. But it was Poincaré, at the end of 19th century ([30]), that started what today is called Qualitative Theory of Ordinary Differential Equations. Since then, many tools were developed to study the behaviour of solutions of an ODE without explicitly solving it, with aid from all three major areas: Topology, Analysis and Algebra.

From the beginning, it was clear the vast applications of these equations in modelling natural phenomena, like Newton's second law of motion and Maxwell's equations of electromagnetism. More recently, ODEs and their associated vector fields were used to model phenomena that are subject to sudden change (e.g. caused by a on-off switch), so they have two or more distinct ODEs that rule this phenomena. These systems with two or more laws of interaction were called Piecewise Smooth Vector Fields (PSVFs for short). Some examples of these are: a system that model the dynamics of a cancer patient being submitted to a severe drug treatment that is abruptly interrupted in order to provide the immune system recovery (see [37, 38]). The same holds for recent protocols of HIV treatment (see [9]). There is also examples in ecology, modelling predator preys populations when predator suddenly change its food preference (see [11]; and electric engineering: [12, 43].

The general mathematical theory about PSVFs is being constructed in recent years. The textbook [13] is a good reference for general theory and some applications, and also [19] and [24] are very comprehensive on theory of PSVFs. There are some similarities with the smooth vector fields, and some versions of classical theorems have been published (e.g. the *Poincaré-Bendixson Theorem*, *Poincaré Index Theorem*, *Poincaré recurrence Theorem*, and *Peixoto Theorem* see [5, 7, 15, 40]), but there are many behaviours which are typical to PSVFs that are not possible in the smooth case. Describing this has been a main source of research for the mathematicians dedicated to this area. For example: it is possible to construct a planar PSVF that is chaotic ([6]), whereas this feature is impossible for the smooth case. The main aspect that leads to these differences is the non uniqueness of trajectories passing through a point of the domain of the PSVF. In Chapter 2, we provide the main ideas and general definitions concerning PSVFs, topological entropy and shift spaces that we use along this text.

Chapter 3 is dedicated to a version of the historical problem of Closing Lemma for PSVFs. The contents of this chapter is compiled at preprint [3]. The question here is the following:

"Given a nonperiodic point x_0 such that the trajectory through x_0 return to an arbitrarily small neighborhood of x_0 infinitely many times, does there exist an arbitrarily small C^r perturbation of this system which has a closed trajectory through x_0?"

Probably who first stated this problem was Poincaré ([31]) and the fact that this problem is the *10th Problem* of the 21st century list compiled by Smale ([39]) certifies the relevance of this question.

Nowadays, the term Closing Lemma actually refers to a family of different results, that involves a point p with some recurrence property and some additional hypothesis on the dimension of space, on the vector field (or diffeomorphism) and others in order to achieve the existence of the closed orbit through p.

Because of this, many versions of Closing Lemmas were published through the years. In some of them a closed trajectory is achieved (see [28, 29, 33, 34, 35]) but in other cases it is not (see [20, 22, 36]). The surveys [1, 32] give a good vision of this problem's history and advances achieved in this subject.

As far as the author know, the paper [8] is the only one studying a version of the Closing Lemma for PSVFs. In fact, it proves that the Closing Lemma is false for a specific example of PSVF.

In this work, we use the same recurrence property of [8] that is the following: "p is *non-trivially recurrent* of Z if it belongs to α or $\omega-$ limit sets of itself and it is not periodic" (see Definition 2.1.6), and require an additional hypothesis on the transversality between the trajectory through p and the switching manifold Σ. With this we provide a *positive answer* for the Closing Lemma C^0 for PSVFs.

Let γ_p denote a trajectory through $p \in \Sigma$, and p_{-1}, p_1 denote the first hits of this trajectory in the discontinuity Σ for negative and positive time, respectively. The main result of Chapter 3 is the following:

Theorem 3.1.1 . *Let $Z = (X, Y)$ be a PSVF defined in $\mathbb{R}^n$, with switching manifold given by $\Sigma = f^{-1}(0)$ and $p \in \Sigma$ a nontrivially recurrent point for Z. Suppose one of the following holds:*

(a) $p \in \omega(p)$ and γ_p is transversal to Σ at p and p_{-1}.

(b) $p \in \alpha(p)$ and γ_p is transversal to Σ at p and p_1.

Then for every C^0-neighbourhood $\mathcal{V}$ of Z, there exists $\widetilde{Z} \in \mathcal{V}$ and a closed trajectory $\gamma_p^{\widetilde{Z}}$ of $\widetilde{Z}$ passing through p.

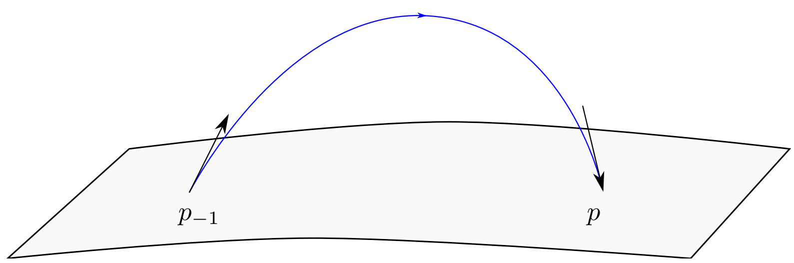

Figure 1.1: Case (a) of Theorem 3.1.1. Source: Prepared by the author.

Chapters 4 and 5 are devoted to present a new way of looking at PSVFs and how that made it possible to study *topological entropy* of these systems and conjugate its dynamics with *symbolic dynamics*. The content of these chapters are compiled at preprints [2, 4].

Given that one of the most relevant characteristic about PSVFs is the non-uniqueness of trajectory through a point it is hard to define a time-one map and define the topological entropy from that, as it is done in the smooth case. So, in Chapter 4, a space Ω of all possible trajectories of a given PSVF Z is constructed with an appropriate metric, and a flow $\{F_t\}_{t\in\mathbb{R}}$ that summarizes all the dynamics of the PSVF. With this flow, topological entropy of a PSVF Z ($h(Z)$) is defined as $h(Z) := h(F_1)$. Details are given in Sections 2.2 and 4.1. The main results of Chapter 4 are:

Theorem 4.2.1 . *Given $\alpha \in \mathbb{Z}, \alpha \geqslant 2$ there exists a PSVF Z such that $h(Z) = \log \alpha$*

Theorem 4.3.1 . *The PSVF Z defined by:*

$$Z(x,y) = \begin{cases} X(x,y) = (1,-2x), & \textit{for} \quad y \geq 0 \\ Y(x,y) = (-2,-4x^3+2x), & \textit{for} \quad y \leq 0 \end{cases}. \tag{1.1}$$

has infinite topological entropy.

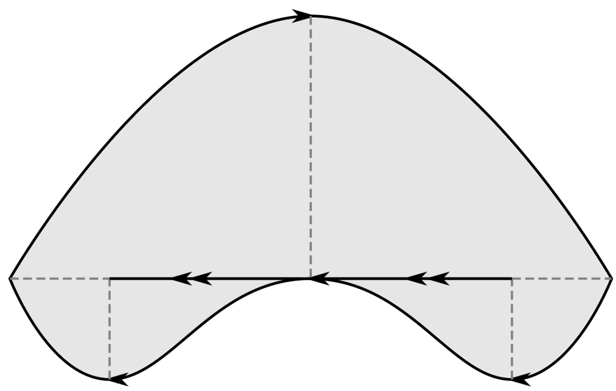

Figure 1.2: Vector field from Theorem 4.3.1. Source: [4].

Based on Theorem 4.3.1, sufficient conditions for a PSVF Z to have infinite topological entropy are given. Moreover, some examples of planar PSVFs with null, positive and infinite entropy are presented. Achieving this in $\mathbb{R}^2$ is one more example of how complex the dynamics of PSVFs can be. To the best of author's knowledge, there is no work related to the topological entropy of PSVFs in the literature.

Chapter 5 also features a metric space Ω of all possible trajectories of a PSVF Z, but in this one, the metric is defined in a slightly different manner, but there is still a flow $\{T_t\}$ on Ω inherited from the PSVF and conjugations between T_1 and *shift spaces* are constructed.

The first results concern the full shift of two symbols, and subshifts of finite type with $2k$ symbols, for any $k \in \mathbb{N}$. In the other ones we use shift spaces with infinite symbols: a subshift of $\mathbb{Z}^{\mathbb{Z}}$ and $(0,1]^{\mathbb{Z}}$.

Moreover, we define a geometric structure (*homoclinic loops*) that enables us to construct these conjugations on a larger class of PSVFs:

Definition 5.2.1 . *A k-*homoclinic loop *of a planar PSVF $Z = (X, Y)$ is a global trajectory presenting k distinct visible-visible two-fold singularities $p_1, \dots, p_k$ in such a way that, after it passes through p_i, the trajectory reaches Σ either in p_{i-1} or p_{i+1} when $i = 2, \dots, k-1$. When $i = 1$ (resp., $i = k$), after passes through p_i, the trajectory reaches Σ either in a crossing point or p_{i+1} (resp., p_{i-1}) (see Figure 1.3).*

*Moreover, an ∞-*homoclinic loop *of a planar PSVF is a global trajectory of Z presenting ∞ visible-visible two-fold singularities $p_1, p_2, \dots$ in such a way that, after it passes through p_i, the trajectory reaches Σ either in p_{i-1} or p_{i+1}.*

The main results of Chapter 5 are the following:

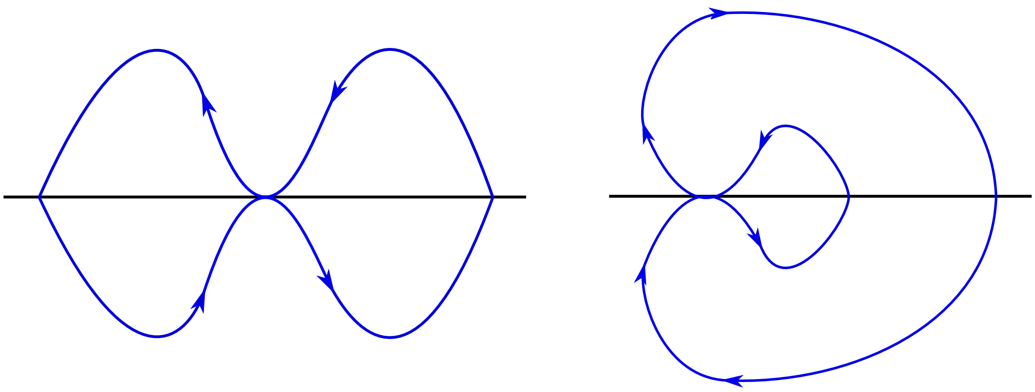

Figure 1.3: 2-homoclinic loops. Source: [2].

Theorem 5.2.1 . *(i) There exists a conjugacy between the time-one map of the PSVF*

$$Z_2(x,y) = \begin{cases} X_2(x,y) = \left(1, \frac{x}{2} - 4x^3\right), & \textit{for} \quad y \geq 0 \\ Y_2(x,y) = \left(-1, \frac{x}{2} - 4x^3\right), & \textit{for} \quad y \leq 0 \end{cases}, \tag{1.2}$$

restricted to an invariant compact set $\Lambda_2 \subset \mathbb{R}^2$ and the full shift of two symbols $\sigma : \{0,1\}^{\mathbb{Z}} \to \{0,1\}^{\mathbb{Z}}$

(ii) For each $k \geqslant 3$, there exists a planar PSVF

$$Z_k(x,y) = \begin{cases} X_k(x,y) = (1, P_k'(x)), & \textit{for} \quad y \geq 0 \\ Y_k(x,y) = (-1, P_k'(x)), & \textit{for} \quad y \leq 0 \end{cases}, \tag{1.3}$$

such that, restricted to an invariant compact set Λ_k, its time-one map is conjugate to a subshift of $2(k-1)$ symbols.

(iii) There exists a conjugacy between the time-one map of the PSVF

$$Z_\infty(x,y) = \begin{cases} X_\infty(x,y) = (1, 2\operatorname{sen}(2\pi x)), & \textit{for} \quad y \geq 0 \\ Y_\infty(x,y) = (-1, 2\operatorname{sen}(2\pi x)), & \textit{for} \quad y \leq 0 \end{cases}, \tag{1.4}$$

restricted to an invariant set $\Lambda_\infty \subset \mathbb{R}^2$ and a subshift over a countable alphabet $\sigma : \Theta_\infty \to \Theta_\infty$, where $\Theta_\infty \subset \mathbb{Z}^{\mathbb{Z}}$.

(iv) The return map of the PSVF

$$Z(x,y) = \begin{cases} X(x,y) = (1, -2x), & \textit{for} \quad y \geq 0 \\ Y(x,y) = (-2, -4x^3 + 2x), & \textit{for} \quad y \leq 0 \end{cases}, \tag{1.5}$$

restricted to an invariant compact set Λ, is conjugated to $\sigma : (0,1]^{\mathbb{Z}} \to (0,1]^{\mathbb{Z}}$.

In the next theorem, we use the $\Sigma-$equivalence between two PSVFs, that, in vague words, is stated as the existence of a homeomorphism that takes the switching manifold and the orbits of the first PSVF to the switching manifold and orbits of the other one, respectively (see details in Definition 5.2.2).

Theorem 5.2.2 . *(i) The PSVF Z_2 of Theorem A, restrict to Λ_2, is Σ-equivalent to any PSVF presenting a 1-homoclinic loop.*

(ii) The PSVF Z_k of Theorem A, restrict to Λ_k, is Σ-equivalent to any PSVF presenting a $(k-1)$-homoclinic loop.

(iii) The PSVF Z_∞ of Theorem A, restrict to Λ_∞, is Σ-equivalent to any PSVF presenting a ∞-homoclinic loop.

(iv) The PSVF Z of Theorem A, restrict to Λ, is Σ-equivalent to any PSVF $\widetilde{Z}$ presenting a compact region $\widetilde{\Lambda}$ bounded by a trajectory of $\widetilde{Z}$ passing through a invisible-visible two-fold $\widetilde{p}$. Moreover, except for $\widetilde{p}$, the PSVF $\widetilde{Z}$ has just more two invisible tangential singularities.

In the final chapter, some concluding remarks and future perspectives of this work are presented.

CHAPTER 2

Preliminaries

2.1 Basic Notions on PSVFs

Let V be an open region of $\mathbb{R}^n$. Consider a codimension one manifold Σ of $\mathbb{R}^n$ given by $\Sigma = f^{-1}(0)$, where $f : V \to \mathbb{R}$ is a smooth function having $0 \in \mathbb{R}$ as a regular value (i.e. $\nabla f(p) \neq 0$, for any $p \in f^{-1}(0)$). Assume that Σ is an embedded submanifold of V. We call Σ the *switching manifold* that is the separating boundary of the regions $\Sigma^+ = \{q \in V \mid f(q) \geq 0\}$ and $\Sigma^- = \{q \in V \mid f(q) \leq 0\}$.

Designate by $\mathfrak{X}^r$ the space of C^r-vector fields on $V \subset \mathbb{R}^n$ endowed with the C^r-topology, with $r \geq 1$ large enough for our purposes. Call $\mathscr{Z}^r$ the space of piecewise smooth vector fields (PSVFs for short) $Z : V \to \mathbb{R}^n$ such that

$$Z(q) = \begin{cases} X(q), & \text{for} \quad q \in \Sigma^+, \\ Y(q), & \text{for} \quad q \in \Sigma^-, \end{cases} \tag{2.1}$$

where $X = (X_1, \dots, X_n), Y = (Y_1, \dots, Y_n) \in \mathfrak{X}^r$. We denote (2.1) simply as $Z = (X, Y)$ when the switching manifold is implicit. Note that Z is multi-valued on Σ. We endow $\mathscr{Z}^r$ with the product topology, i.e., in this case,

$$\|Z\|_{C^r} = \max\{|X|_{C^r}, |Y|_{C^r}\},$$

where $|\cdot|_{C^r}$ denotes the classical C^r-norm for the smooth vector fields X and Y restricted to Σ^+ and Σ^-, respectively.

In order to define rigorously the flow of Z we distinguish whether this point is at $\Sigma^\pm \setminus \Sigma$ or Σ. For the first two regions, the local trajectory is defined by X and Y respectively, as usual, but for Σ we divide this into three regions based on the contact between the vector fields X, Y and Σ characterized by the Lie derivative $Xf(p) = \langle \nabla f(p), X(p)\rangle$, where $\langle \cdot , \cdot \rangle$ is the usual inner product. We also use higher order derivatives given by $X^k f = X(X^{k-1}f) = \langle \nabla X^{k-1}f, X\rangle$.

These three regions are:

• Crossing Region: $\Sigma^c = \{p \in \Sigma \,|\, Xf(p) \cdot Yf(p) > 0\}$. Moreover, we denote $\Sigma^{c+} = \{p \in \Sigma \mid Xf(p) > 0, Yf(p) > 0\}$ and $\Sigma^{c-} = \{p \in \Sigma \mid Xf(p) < 0, Yf(p) < 0\}$.

• Sliding Region: $\Sigma^s = \{p \in \Sigma \mid Xf(p) < 0, Yf(p) > 0\}$.

• Escaping Region: $\Sigma^e = \{p \in \Sigma \mid Xf(p) > 0, Yf(p) < 0\}$.

These regions are relatively open in Σ and their definitions exclude the points where $Xf(p) \cdot Yf(p) = 0$. These points are on the boundary of those regions.

Any $q \in \Sigma$ such that $Xf(q) \cdot Yf(q) = 0$ is called a *tangential singularity* (or also *tangency point*) and we denote the set of these points by Σ^t. If there exists an orbit of the vector field $X|_{\Sigma^+}$ (respec. $Y|_{\Sigma^-}$) reaching $q \in \Sigma^t$ in a finite time, then such tangency is called a *visible tangency* for X (respec. Y), otherwise we call q an *invisible tangency* for X (respec. Y).

We may also distinguish a particular tangential singularity called *two-fold*, which is a common tangency q of both X and Y (that is, $Xf(q) = Yf(q) = 0$) satisfying $X^2f(q)), Y^2f(q)) \neq 0$. A two-fold is called

1. *visible*, if it is a visible tangency for X and Y;

2. *invisible*, if it is an invisible tangency for X and Y;

3. *visible-invisible*, if it is a visible tangency for X and an invisibe tangency of Y, or vice-versa.

If $V \subset \mathbb{R}^2$, we say that a tangency point p is *singular* if p is an invisible tangency for both X and Y. On the other hand, a tangency point p is *regular* if it is not singular.

For higher dimensional spaces, there are some aspects other than that. In particular, if q is an invisible two-fold such that the sets $S_X = \{p \in \Sigma;\ Xf(p) = 0\}$ and $S_Y = \{p \in \Sigma;\ Yf(p) = 0\}$ are transverse at q, then q is said to be a *T-singularity* of Z.

In $\Sigma^{s,e}$, the definition of local orbit is given by Filippov convention [16]. Consider the following:

Definition 2.1.1. *Given a point $p \in \Sigma^s \cup \Sigma^e \subset \Sigma$, we define the* sliding vector field *at p as the linear convex combination of $X(p)$ and $Y(p)$. That is, it is given by the expression*

$$Z^s(p) = \frac{Yf(p)X(p) - Xf(p)Y(p)}{Yf(p) - Xf(p)}. \tag{2.2}$$

It is worth mentioning that Z^s can be extended to $\partial\Sigma^s$ using the normalized sliding vector field

$$Z^s_N(q) = Yf(q)X(q) - Xf(q)Y(q), q \in \Sigma^s.$$

It is clear that if $q \in \Sigma^s$ then $q \in \Sigma^e$ for $(-Z)$ and we can define the *escaping vector field* Z^e on Σ^e associated to Z by $Z^e = -(-Z)^s$. We will use the notation Z^T to both, Z^s and Z^e.

Now we establish the classical convention on the trajectories of a PSVF:

Definition 2.1.2. *The* local trajectory *$\phi_Z(t, p)$ of a PSVF $Z = (X, Y)$ through $p \in U$ is defined as follows:*

(i) For $p \in \Sigma^+$ and $p \in \Sigma^-$ the trajectory is given by $\phi_Z(t, p) = \phi_X(t, p)$ and $\phi_Z(t, p) = \phi_Y(t, p)$ respectively, where $t \in I$.

(ii) For $p \in \Sigma^{c+}$ and taking the origin of time at p, the trajectory is defined as $\phi_Z(t, p) = \phi_Y(t, p)$ for $t \in I \cap \{t \leq 0\}$ and $\phi_Z(t, p) = \phi_X(t, p)$ for $t \in I \cap \{t \geq 0\}$. If $p \in \Sigma^{c-}$ the definition is the same reversing time.

(iii) For $p \in \Sigma^e$ and taking the origin of time at p, the trajectory is defined as $\phi_Z(t, p) = \phi_{Z^s}(t, p)$ for $t \in I \cap \{t \leq 0\}$ and $\phi_Z(t, p)$ is either $\phi_X(t, p)$ or $\phi_Y(t, p)$ or $\phi_{Z^s}(t, p)$ for $t \in I \cap \{t \geq 0\}$. For $p \in \Sigma^s$ the definition is the same reversing time.

(iv) For p a regular tangency point and taking the origin of time at p, the trajectory is defined as $\phi_Z(t, p) = \phi_1(t, p)$ for $t \in I \cap \{t \leq 0\}$ and $\phi_Z(t, p) = \phi_2(t, p)$ for $t \in I \cap \{t \geq 0\}$, where each ϕ_1, ϕ_2 is either ϕ_X or ϕ_Y or ϕ_{Z^T}.

(v) For $p \in V \subset \mathbb{R}^2$ a singular tangency point, $\phi_Z(t, p) = p$ for all $t \in \mathbb{R}$.

Definition 2.1.3. *The* global trajectory (orbit) $\Gamma_Z(t, p_0)$ *of* Z *passing through* p_0 *is a union*

$$\Gamma_Z(t, p_0) = \bigcup_{i \in \mathbb{Z}} \{\sigma_i(t, p_i) : t_i \leq t \leq t_{i+1}\}$$

of preserving-orientation local trajectories $\sigma_i(t, p_i)$ *satisfying* $\sigma_i(t_{i+1}, p_i) = \sigma_{i+1}(t_{i+1}, p_{i+1}) = p_{i+1}$ *and* $t_i \to \pm\infty$ *as* $i \to \pm\infty$.

Definition 2.1.4. *A set* A *is* $Z-$invariant *if for each* $p \in A$ *and any global trajectory* $\Gamma_Z(t, p)$ *passing through* p *it holds* $\Gamma_Z(t, p) \subset A$.

Definition 2.1.5. *Let* $Z \in \Omega^r$. *We say*

(i) p *is a* single periodic point *for* Z *if there exists only one closed trajectory of* Z *by it;*

(ii) p *is a* multi-periodic point *for* Z *if there exist more than one closed trajectories of* Z *by it.*

Definition 2.1.6. *Let* $Z \in \Omega^r$. *We say*

(i) p *is a* nontrivially recurrent point *for* Z *if it is not a single periodic point or a fixed point of* Z *and belongs to the* ω *or* α*-limit set of itself;*

(ii) p *is a* nonwandering point *for* Z *if each one of its neighbourhoods meets arbitrarily large iterations of itself.*

2.2 Basic Notions on Entropy

The definitions and results of this section can be found and further studied at [21]. Let $f : K \to K$ a continuous map of a metric space (K, d), $f^i = f \circ f \circ \ldots \circ f$ (i-times) and $f^0(x) = x$. Define a sequence of metrics d_n^f, $n = 1, 2, \ldots$, by:

$$d_n^f(x, y) = \max_{0 \leqslant i \leqslant n-1} d(f^i(x), f^i(y)).$$

Denote $B_f(x, \varepsilon, n) = \{y \in K \mid d_n^f(x, y) < \varepsilon\}$ the open ball centered at x, with radius ε with respect to metric d_n^f.

Definition 2.2.1. *A set* $E \subset K$ *is* $\varepsilon-$dense with respect to d_n^f *(or* $(n, \varepsilon)-$*dense), if* $K \subset \bigcup_{x \in E} B_f(x, \varepsilon, n)$. *Then the* $\varepsilon-$capacity *of* d_n^f*, denoted* $S_d(f, \varepsilon, n)$*, is the minimal cardinality of a* $(n, \varepsilon)-$*dense set, equivalently, the cardinality of a minimal* $(n, \varepsilon)-$*dense set:*

$$S_d(f, \varepsilon, n) = \min\{\#E \mid E \subset K \text{ is } (n, \varepsilon)\text{-denso}\}.$$

Definition 2.2.2. *Consider the exponential growth rate*

$$h_d(f, \varepsilon) := \limsup_{n \to \infty} \frac{1}{n} \log S_d(f, \varepsilon, n)$$

and define the topological entropy *of* f*, denoted* $h(f)$ *by:*

$$h(f) := \lim_{\varepsilon \to 0} h_d(f, \varepsilon).$$

There exists analogous definitions for flows, but in this work we use the equivalent one that follows:

Definition 2.2.3. *Let X is a vector field, and φ is the flow defined by this field, we define the* time-one map *of this field by $X^1(x) = \varphi(x, 1)$. And the* topological entropy of the flow *is defined by $h(\varphi) = h(X^1)$.*

Definition 2.2.4. *Let $f : M \to M$ and $g : N \to N$ be two maps. A map $H : M \to N$ is called a* topological semi-conjugacy from f to g *provided that (i) H is continuous, (ii) H is onto, and (iii) $H \circ f = g \circ H$. We may also refer to f being* topologically semi-conjugate to g by H.

The map H is called topological conjugacy *if it is a semi-conjugacy and (iv) H is one to one and onto and has a continuous inverse (so H is a homeomorphism). We also say that f and g are* topologically conjugate *by H, or simply that f and g are* conjugate.

The following Propositions give some useful properties of topological entropy. Their proofs can be found at [21].

Proposition 2.2.1. *Let $f : M \to M$ and $g : N \to N$ be two maps, and $H : M \to N$ a semi-conjugacy from f to g. Then $h(f) \geqslant h(g)$. Moreover, if H is a topological conjugacy, then $h(f) = h(g)$.*

Proposition 2.2.2. *If Λ is a closed $f-$invariant set, then $h(f|_\Lambda) \leqslant h(f)$*

Proposition 2.2.3. *Let $f : M \to M$ a map. Then $h(f^m) = |m|h(f)$.*

2.3 Basic Notions on Symbolic Dynamics

Consider a set $\mathcal{A}_k$ with k elements (say $\mathcal{A}_k = \{0, 1, \ldots, k-1\}$) with the discrete topology. Now, consider $\mathcal{A}_k^{\mathbb{Z}}$, i.e., all the sequences $x = (x_j)_{j\in\mathbb{Z}}$, with $x_j \in \mathcal{A}_k$, for all j and the product topology of all discrete topologies.

Definition 2.3.1. *Let $x = (x_j)_{j\in\mathbb{Z}}$ and $y = (y_j)_{j\in\mathbb{Z}}$ two elements of $\mathcal{A}_k^{\mathbb{Z}}$. Define $d : \mathcal{A}_k^{\mathbb{Z}} \times \mathcal{A}_k^{\mathbb{Z}} \to \mathbb{R}$ by:*

$$d(x, y) = \sum_{j\in\mathbb{Z}} \frac{|x_j - y_j|}{2^{|j|}}$$

Definition 2.3.2. *Define $\sigma_k : \mathcal{A}_k^{\mathbb{Z}} \to \mathcal{A}_k^{\mathbb{Z}}$ given by $\sigma((a_j)) = (b_j)$, where $b_j = a_{j+1}$. The map σ is called* two-sided full shift *and the discrete flow $(\mathcal{A}_k^{\mathbb{Z}}, \sigma)$ is called* symbolic flow *or* shift system.

Proposition 2.3.1. *The following holds:*

(i) The function d defined above is a metric on $\mathcal{A}_k^{\mathbb{Z}}$ and induces the product topology;

(ii) The space $\mathcal{A}_k^{\mathbb{Z}}$ is a compact Hausdorff space;

(iii) σ_k is a homeomorphism.

Proof. In [23], p. 48. the authors assert that the results in items (i) and (ii) of this proposition are true, but without a proof. So, for completeness, we proceed the proof below.

(i) The convergence of d follows from comparison test with the geometric series and the properties for it to be a metric follows easily from absolute value in the numerator. To see that the topology is the same, consider $\mathcal{U}$ an open basic set and $x \in \mathcal{U}$. Without loss of generality, one can assume that $\mathcal{U} = \prod_{j<-\alpha} \mathcal{A} \times \{x_{-\alpha}\} \times \cdots \times \{x_\alpha\} \times \prod_{j>\alpha} \mathcal{A}$, for some $\alpha \in \mathbb{N}$. That is, any $y \in \mathcal{U}$ coincides with x in all entries between $-\alpha$ and α.

Let $0 < r < \frac{1}{2^\alpha}$ and consider the open ball $B = B(x,r)$. Let $y \in B$. Suppose there exists $\beta \in \mathbb{Z}$, $-\alpha \leqslant \beta \leqslant \alpha$ such that $y_\beta \neq x_\beta$. Then

$$d(x,y) = \sum_{j\in\mathbb{Z}} \frac{|x_j - y_j|}{2^{|j|}} \geqslant \frac{1}{2^{|\beta|}} \geqslant \frac{1}{2^\alpha} > r$$

This is a contradiction, since $y \in B$. Then $y \in \mathcal{U}$.

On the other hand, let $x \in \mathcal{A}_k^{\mathbb{Z}}$ and $B = B(x,r)$ an open ball. There exists $\alpha \in \mathbb{N}$, such that $\frac{k-1}{2^{\alpha-1}} < r$. Let $\mathcal{U}$ be an basic open set as before. We have $x \in \mathcal{U}$ and let $y \in \mathcal{U}$. Then:

$$d(x,y) = \sum_{j\in\mathbb{Z}} \frac{|x_j - y_j|}{2^{|j|}} = \sum_{\substack{j\in\mathbb{Z}\\ |j|>\alpha}} \frac{|x_j - y_j|}{2^{|j|}} \leqslant 2(k-1) \sum_{j=\alpha+1}^{\infty} \frac{1}{2^j} = \frac{k-1}{2^{\alpha-1}} < r$$

(ii) Each copy of $\mathcal{A}$ is compact, then, by Tychonoff's Theorem, $\mathcal{A}_k^{\mathbb{Z}}$ is compact. Moreover, it is a metric space, hence it is Hausdorff

(iii) It is enough to note that σ_k takes basic open set to basic open set.

□

When it is clear by the context, the subscript k may be suppressed in the sequel.

Definition 2.3.3. *Let $K \subset \mathcal{A}^{\mathbb{Z}}$. We say (K,σ) is a* subshift *if K is closed and invariant for σ.*

There are some works (see [18] and [26]) considering one and two sided shifts over enumerable alphabets. To the best of our knowledge, there is not a classic way to work with them, and when considering the full shift, it is difficult to define a topology over it in order to make it metrizable.

In this work, we consider a shift over $\mathbb{Z}$ presenting some restrictions over the sequences which are allowed to occur. These features allow us to define a metric over this space.

Definition 2.3.4. *Let $\Theta_\infty \subset \mathbb{Z}^{\mathbb{Z}}$ be the set of bi-infinite sequences with integer entries $(x_j)_{j\in\mathbb{Z}}$, such that the difference between two consecutive entries is at most 2 . That is:*

$$\Theta_\infty = \{(x_j)_{j\in\mathbb{Z}} \mid x_j \in \mathbb{Z} \textit{ and } |x_{j+1} - x_j| \leqslant 2, \forall j \in \mathbb{Z}\}$$

We can define over Θ_∞ the same metric given before.

Definition 2.3.5. *Consider $x = (x_j)_{j\in\mathbb{Z}}$ and $y = (y_j)_{j\in\mathbb{Z}}$ two elements of Θ_∞. Define $d : \Theta_\infty \times \Theta_\infty \to \mathbb{R}$ by:*

$$d(x,y) = \sum_{j\in\mathbb{Z}} \frac{|x_j - y_j|}{2^{|j|}}$$

Proposition 2.3.2. *The function d defined above is a metric over Θ_∞.*

Proof. Once proven the convergence of d, the properties for it to be a metric follows straightforward from the absolute value on numerator of the series.

Let $x = (x_j)_{j\in\mathbb{Z}}, y = (y_j)_{j\in\mathbb{Z}} \in \Theta_\infty$, then for all $j \in \mathbb{Z}$:

$$|x_j - y_j| \leqslant |x_j - x_{j-1}| + |x_{j-1} - y_{j-1}| + |y_{j-1} - y_j| \leqslant |x_{j-1} - y_{j-1}| + 4$$

Analogously, one can show: $|x_j - y_j| \leqslant |x_{j+1} - y_{j+1}| + 4$. And, recursively, we obtain:

$$\forall j \in \mathbb{Z} : |x_j - y_j| \leqslant |x_0 - y_0| + 4|j|.$$

Now we are in conditions to prove the convergence of d. Given $N \in \mathbb{N}$:

$$\begin{aligned}
\sum_{j=-N}^{N} \frac{|x_j - y_j|}{2^{|j|}} &= |x_0 - y_0| + \sum_{j=1}^{N} \frac{|x_j - y_j|}{2^j} + \sum_{j=1}^{N} \frac{|x_{-j} - y_{-j}|}{2^j} \leqslant \\
&\leqslant |x_0 - y_0| + \sum_{j=1}^{N} \frac{|x_0 - y_0| + 4j}{2^j} + \sum_{j=1}^{N} \frac{|x_0 - y_0| + 4j}{2^j} \\
&\leqslant |x_0 - y_0| + 2|x_0 - y_0| \sum_{j=1}^{N} \frac{1}{2^j} + 8 \sum_{j=1}^{N} \frac{j}{2^j} \leqslant \\
&\leqslant 3|x_0 - y_0| + 16 < \infty
\end{aligned}$$

Making $N \to \infty$, we have that d converges.

□

Two important properties of a dynamical system (X, f) are defined below:

Definition 2.3.6. *Let (X, f) be a dynamical system. We say that f is* topologically transitive *if for every open sets $\mathcal{U}, \mathcal{V}$, there exists n, such that $f^n(\mathcal{U}) \cap \mathcal{V} \neq \emptyset$. And f is* topologically mixing *if for every open sets $\mathcal{U}, \mathcal{V}$, there exists n_0, such that $f^n(\mathcal{U}) \cap \mathcal{V} \neq \emptyset$ for all $n \geqslant n_0$.*

The following propositions can be found at [21].

Proposition 2.3.3. *If f is topologically mixing, then f is topologically transitive.*

Proposition 2.3.4. *The two-sided shift σ_k is topologically mixing.*

CHAPTER 3

Closing Lemma for Piecewise Smooth Vector Fields

Many problems have haunted mathematicians minds for centuries, some of them were solved, others that were believed to be a simple problem, with a difficult solution turned out to be dismembered into many others smaller problems, still with the same difficult. One of these problems is the *Closing Lemma*. A classic question of Dynamical Systems, with version about diffeomorphisms and vector fields that can be enunciated as follows:

"Given a nonperiodic point x_0 of $X \in \mathfrak{X}^r$ such that the trajectory through x_0 returns to an arbitrarily small neighborhood of x_0 infinitely many times, does there exist an arbitrarily small C^r-perturbation of X which has a closed trajectory through x_0."

Intuitively it is a simple statement, that at first glance, many think of it as a direct statement, but it has been theme of discussions extensively, until 1964, when Pugh proved the C^1 case [33] for diffeomorphisms, flows, and vector fields.

The only paper about this question on the subject of PSVFs is [8], as far as the author know. In this chapter, we present another result regarding this question.

3.1 Main results

Now we establish some notation that will be used in the sequel: given $Z = (X, Y)$ and $p \in \mathbb{R}^n$ and $q = \gamma_p(t)$ for some global trajectory γ_p of Z, the arc of the trajectory γ_p from p to q is denoted by $\widehat{pq}$. Also, if $p \in \Sigma$, we denote by p_1 and p_{-1} the first intersections of γ_p and Σ for positive and negative time, that is, $\widehat{pp_1} \cap \Sigma = \{p, p_1\}$ and $\widehat{p_{-1}p} \cap \Sigma = \{p_{-1}, p\}$.

Theorem 3.1.1. *Let $Z = (X, Y)$ be a PSVF defined in $\mathbb{R}^n$, with switching manifold given by $\Sigma = f^{-1}(0)$ and $p \in \Sigma$ a nontrivially recurrent point for Z (see Definition 2.1.6). Suppose one of the following holds:*

(a) $p \in \omega(p)$ and γ_p is transversal to Σ at p and p_{-1}.

(b) $p \in \alpha(p)$ and γ_p is transversal to Σ at p and p_1.

Then for every C^0-neighbourhood $\mathcal{V}$ of Z, there exists $\widetilde{Z} \in \mathcal{V}$ and a closed orbit $\gamma_p^{\widetilde{Z}}$ of $\widetilde{Z}$ passing through p.

Using similar arguments as in the proof of Theorem 3.1.1, we obtain the following result for nonwandering points.

Proposition 3.1.1. *Let $Z = (X, Y)$ be a PSVF defined in $\mathbb{R}^n$, with switching manifold given by $\Sigma = f^{-1}(0)$ and $p \in \Sigma$ a nonwandering point for Z (see Definition 2.1.6), such that γ_p is transversal to Σ at p_{-1} and p. Then for every C^0-neighbourhood $\mathcal{V}$ of Z there exists $\widetilde{Z} \in \mathcal{V}$, a point p^* in a neighbourhood of p and a closed orbit $\gamma_{p^*}^{\widetilde{Z}}$ through p^*.*

3.2 Proof of the main results

3.2.1 Proof of Theorem 3.1.1

Suppose (a) holds (if (b) holds, the proof is analogous). In this case, $p \in \omega(p)$ and γ_p is transversal to Σ at p_{-1}. From this hypothesis, together with $\widehat{p_{-1}p} \cap \Sigma = \{p_{-1}, p\}$ we have that f does not change sign on $\widehat{p_{-1}p}$, that is, this trajectory is entirely determined by either X or Y. Suppose it is X and let $\varphi(t, p)$ (or $\varphi_t(p)$) denote the flow of X. Then there exists $t_0 > 0$ such that $p = \varphi(t_0, p_{-1})$. Moreover, $f(\varphi(t_0, p_{-1})) = 0$. Now,

$$\frac{\partial}{\partial t}(f \circ \varphi)|_{(t_0,p_{-1})} = df|_p \frac{\partial}{\partial t}\varphi|_{(t_0,p_{-1})} = df|_p X_{\varphi(t_0,p_{-1})} = Xf(p) \neq 0$$

By the Implicit Function Theorem, there exists an open neighbourhood $V_{p_{-1}}$ of p_{-1}, and a function τ, such that $\tau(p_{-1}) = t_0$ and $f(\varphi(\tau(x), x)) = 0$, for all $x \in V_{p_{-1}}$.

Let $\mathcal{U}$ be the intersection between this neighbourhood and Σ and define $F : \mathcal{U} \to \Sigma$, by $F(x) = \varphi(\tau(x), x)$. We have that $F : \mathcal{U} \to \widetilde{\mathcal{U}}$ is a diffeomorphism, where $\widetilde{\mathcal{U}}$ is the image of $\mathcal{U}$ by F.

Note that, even though the vector field X is defined on Σ^+ its trajectories through $\mathcal{U}$ can be extended to an open set beyond Σ^+, that is, for every $x \in \mathcal{U}$, there exists an open interval $(a(x), b(x)) \supset [0, \tau(x)]$, for which $\varphi_t(x)$ is defined. Consider the open set

$$C = \{\varphi(t, x) \mid x \in \mathcal{U}, a(x) < t < b(x)\},$$

that is, the disjoint union of arcs of orbits through points of $\mathcal{U}$. Observe that it can occur $f(\varphi(t, x)) = 0$ for some point $x \in \mathcal{U}$ and $0 < t < \tau(x)$, but in this case $\mathcal{U}$ can be chosen smaller so it doesn't happen.

We have that $\mathcal{U}$ is an $(n-1)$-dimensional open neighbourhood of $p_{-1} \in \Sigma$, then there exists a bijection $\widetilde{H} : B(0, \delta) \subset \mathbb{R}^{n-1} \to \mathcal{U}$, of class C^∞ with $\widetilde{H}(0) = p_{-1}$ and such that $d\widetilde{H}|_y$ is injective for every $y \in B(0, \delta)$.

Now, consider

$$D = \{(t, y) \in \mathbb{R}^n \mid y \in B_{\mathbb{R}^{n-1}}(0, \delta) \text{ and } a(\widetilde{H}(y)) < t + t_0 < b(\widetilde{H}(y))\}$$

and define the map $H : D \to C$ by $H(t, y) = \varphi(t + t_0, \widetilde{H}(y))$.

Clearly H is bijective. Moreover, if $(u, v) \in \mathbb{R} \oplus \mathbb{R}^{n-1}$, then:

$$\begin{aligned} dH|_{(t,y)}.(u, v) &= \left.\frac{d\varphi}{dt}\right|_{(t+t_0,\widetilde{H}(y))} u + d\varphi_{t+t_0}|_{\widetilde{H}(y)} d\widetilde{H}|_y.v \\ &= X(\varphi_{t+t_0}(\widetilde{H}(y)))u + d\varphi_{t+t_0}|_{\widetilde{H}(y)} d\widetilde{H}|_y.v \\ &= d\varphi_{t+t_0}|_{\widetilde{H}(y)} \left(uX(\widetilde{H}(y)) + d\widetilde{H}|_y.v\right) \end{aligned}$$

Now, $d\varphi_t|_{\widetilde{H}(y)}$ is an isomorphism for all t and $\mathbb{R}X(\widetilde{H}(y)) \oplus d\widetilde{H}|_y\mathbb{R}^{n-1} = \mathbb{R}^n$, for all $y \in B(0, \delta)$, since $Xf(\widetilde{H}(y)) \neq 0$. Thus, by the Inverse Function Theorem, H is a diffeomorphism and $dH|_{(t,y)}e_1 = X(\varphi_t(\widetilde{H}(y)))$. Moreover $H(0, 0) = p$ and $H(-t_0, 0) = p_{-1}$.

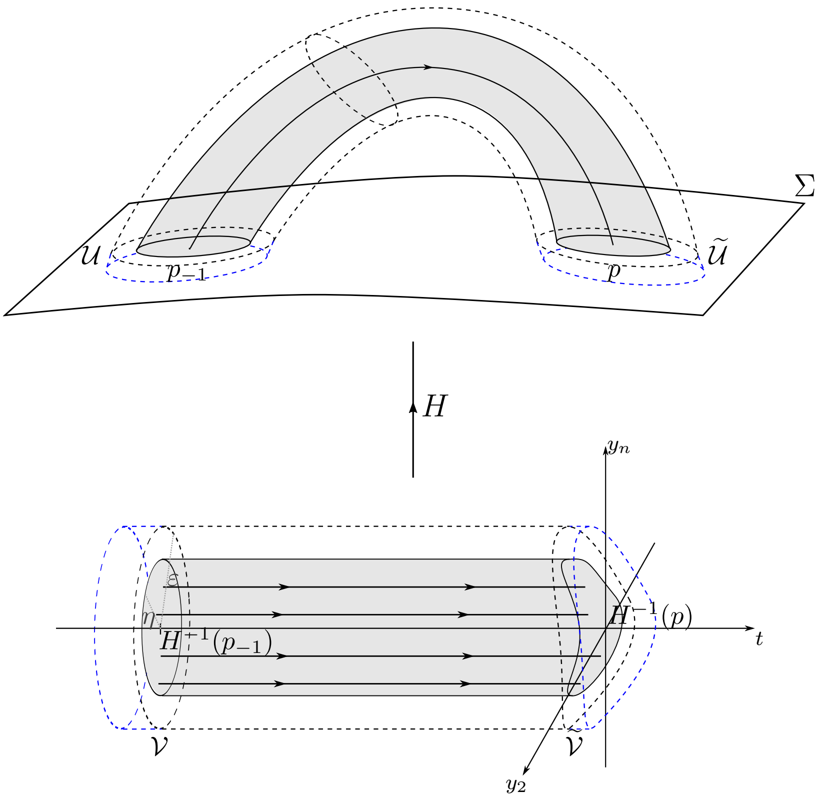

Figure 3.1: Diffeomorphism H in $\mathbb{R}^3$. Source: [3].

The construction above is often mentioned as a *flow box*, but in our case we require that it intersects Σ in two distinct regions, and the neighbourhood $H^{-1}(\mathcal{U})$ is contained entirely in the plane $t = -t_0$. To simplify notation, let $\mathcal{V} = H^{-1}(\mathcal{U})$ and $\widetilde{\mathcal{V}} = H^{-1}(\widetilde{\mathcal{U}})$.

Let $\varepsilon > 0$ small enough. Since $p \in \omega(p)$, there exists $q \in \gamma_p^+$, such that $H^{-1}(q) \in H^{-1}(B(p,\varepsilon)) \cap \widetilde{\mathcal{V}}$. We take q to be the first point of γ_p^+ for which this occurs.

Let $q_{-1} = F^{-1}(q)$, then $H^{-1}(q_{-1}) = (-t_0, y^0) \in \mathcal{V}$ and $H^{-1}(q) = (t^*, y^0)$, where $y^0 = (y_2^0, \ldots, y_n^0)$. Consider the vector field $\widetilde{X} = \left(1, -\frac{y_2^0}{t_0}, \ldots, -\frac{y_n^0}{t_0}\right)$ defined over D.

Note that there is an integral curve of $\widetilde{X}$ that connects $H^{-1}(q_{-1})$, and $H^{-1}(p)$.

Now $\|H^{-1}(q) - H^{-1}(p)\| = \|(t^*, y^0) - (0,0)\| < \varepsilon$, then y^0 is in the $(n-1)$-dimensional ball $B(0,\varepsilon)$, and there exists $\eta > 0$ such that $\|y^0\| < \eta < \varepsilon$. This implies that $y^0 \in \overline{B(0,\eta)} \subset B(0,\varepsilon)$.

Consider the following standard bump function $b : D \to \mathbb{R}$ by

$$b(t,y) = \frac{g(\varepsilon - \|y\|)}{g(\varepsilon - \|y\|) + g(\|y\| - \eta)},$$

where $g : \mathbb{R} \to \mathbb{R}$ is the C^∞ function:

$$g(t) = \begin{cases} e^{-\frac{1}{t^2}}, & \text{for} \quad t > 0, \\ 0, & \text{for} \quad t \leqslant 0. \end{cases}$$

The function b vanishes outside the open cylinder of radius ε and is 1 inside the closed cylinder of radius η. Now, define $X_\varepsilon = X + b(\widetilde{X} - X)$ for all $(t,y) \in D$. Then, we have that:

$$|X_\varepsilon - X|_{C^0} = \left| \left(0, -\frac{y_2^0 b}{t_0}, \ldots, -\frac{y_n^0 b}{t_0}\right) \right|_{C^0} < \|b\|_\infty \varepsilon = \varepsilon$$

So the vector fields X and X_ε are ε-C^0-close, which implies that the PSVF $Z = (X, Y)$ and $Z_\varepsilon = (X_\varepsilon, Y)$ are also ε-C^0-close.

Besides that, the orbit $\widetilde{\gamma}_p$ from Z_ε is closed. In fact, the trajectory from p to q_{-1} lies almost completely outside of supp(b), except for two segments immediately after leaving p, and immediately before reaching q_{-1}. But note that, although we use this set to perturb the vector field X, once we define Z_ε, these two segments are not part of the trajectory because they are confined in Σ^-, where the dynamics is given by Y and remains the same. Hence, $\widetilde{\gamma}_p$ coincides with γ_p from p to q_{-1} and once it reaches q_{-1} the flow is given by X_ε, so it returns to p, closing the trajectory. □

Scholium 3.2.1. *The proof of Theorem 3.1.1 above cannot be extended to prove the result in C^r-neighborhoods with $1 \leqslant r \leqslant \infty$.*

Proof. Let X and X_ε be given as above, then

$$|X_\varepsilon - X|_{C^r} = \left|\left(0, -\frac{y_2^0 b}{t_0}, \dots, -\frac{y_n^0 b}{t_0}\right)\right|_{C^r} < \|b\|_{C^r}\varepsilon.$$

Now, observe that $b(y) = 1$, for all $\|y\| = \eta$ and $b(y) = 0$, for all $\|y\| = \varepsilon$, then we obtain, from Mean Value Theorem:

$$1 \leqslant \|b'\|_\infty(\varepsilon - \eta) \Longrightarrow \|b'\|_\infty \varepsilon > 1.$$

Since $\|b\|_{C^r} \geqslant \|b'\|_\infty$, it follows that $\|b\|_{C^r}\varepsilon > 1$, which does not imply that X_ε and X are ε-close in $\mathfrak{X}^r$. □

Remark 3.2.1. *The scholium 3.2.1 is still true if we choose another partition of unity function $\widetilde{b}$, that is, $\widetilde{b} = 1$, for $\|y\| \leqslant \eta$ and $\widetilde{b} = 0$, for $\|y\| \geqslant \varepsilon$.*

Remark 3.2.2. *In a larger open set, containing the ε-cylinder chosen to construct the partition of unity, we may control the closeness, but we cannot guarantee that the orbit is not modified by the new vector field. It may be the case where the orbit goes from q_{-1} to p, but it does not return to q_{-1}.*

3.2.2 Proof of Proposition 3.1.1

Consider the same neighbourhoods $\mathcal{U}$ and $\widetilde{\mathcal{U}}$ given in the beginning of proof of Theorem 3.1.1. Since p is nonwandering, each one of its neighbourhoods intersects itself in arbitrarily large iterations. That is, given $\varepsilon > 0$, there exists $p^* \in B(p, \varepsilon) \cap \Sigma$ such that, $q = \gamma_{p^*}(t) \in B(p, \varepsilon) \cap \Sigma \subset \widetilde{\mathcal{U}}$ for some $t > 0$. Let $q_{-1} = F^{-1}(q) \in \mathcal{U}$.

Using the same coordinates as before (diffeomorphism H), we construct X_ε analogously, but in this case, X_ε has a trajectory from q_{-1} to p^*. Then

$$|X_\varepsilon - X|_{C^0} < \|b\|_{C^0} 2\varepsilon,$$

and the PSVF $Z_\varepsilon = (X_\varepsilon, Y)$ has a closed orbit through p^*. □

3.3 A class of $3D$ Filippov systems presenting a nontrivially recurrent point

In what follows, we introduce a class of Filippov systems having a nontrivially recurrent point in such a way that the orbit through this recurrent point is a crossing orbit, and therefore this class will satisfy the hypotheses of Theorem 3.1.1.

Consider that Σ is the plane $\{z = 0\}$ in $\mathbb{R}^3$, and let $Z_{\alpha,\beta} = (X_\alpha, Y_\beta)$ be a Filippov system, where X_α and Y_α are given by

$$X_\alpha(x, y, z) = \begin{pmatrix} \alpha \\ 1 \\ -y \end{pmatrix} \text{ and } Y_\beta(x, y, z) = \begin{pmatrix} 1 \\ \beta \\ x \end{pmatrix}, \tag{3.1}$$

where $\alpha\beta > 1$ and $\alpha, \beta < 0$.

For each α, β satisfying the conditions above, the Filippov system $Z_{\alpha,\beta}$ has a T-singularity at the origin and it follows from [17, 42] that $Z_{\alpha,\beta}$ is locally structurally stable at the origin. Also in [42] is proved that $Z_{\alpha,\beta}$ has a pair of cones, C^{uns} and C^{st}, with vertexes at the origin, which are foliated by crossing orbits of $Z_{\alpha,\beta}$, in such a way that the orbit passing through a point of $C^{\text{st}}/C^{\text{uns}}$ spiral toward/away from the origin (see Figure 3.2). These cones are the stable and unstable (crossing) invariant manifolds of $Z_{\alpha,\beta}$ at the origin, and they are usually called as the *nonsmooth diabolo* associated to the T-singularity.

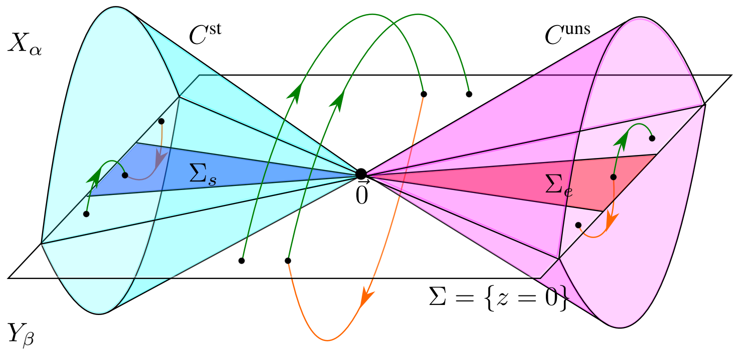

Figure 3.2: Phase space of $Z_{\alpha,\beta}$. Source: [3].

Our construction relies on the following ideas. First we use these known systems $Z_{\alpha,\beta}$ to produce a communication between these two cones in an open region M (which will be a solid cylinder) of $\mathbb{R}^3$, via gluing techniques. Such process will give rise to a Filippov system $\widetilde{Z}_{\alpha,\beta}$ having a new switching manifold given by Σ and a plane Σ_1 transverse to Σ. Finally, we will use $\widetilde{Z}_{\alpha,\beta}$ and the ideas of the Sotomayor-Teixeira regularization method (see [25, 27]) to produce a new Filippov system $\widehat{Z}_{\alpha,\beta}$ defined on M with a regular switching manifold Σ which has a recurrent point satisfying the conditions of Theorem 3.1.1. Figure 3.3 illustrates the construction of $\widehat{Z}_{\alpha,\beta}$.

For $\star = $ uns, st, consider a plane $\Pi_\star$ transversal to the cone $C^\star$ in such a way that X and Y are transverse to $\Pi_\star$ at $C^\star \cap \Pi_\star \cap \{z \geq 0\}$ and $C^\star \cap \Pi_\star \cap \{z \leq 0\}$, respectively. Denote the circle $C^\star \cap \Pi_\star$ by $\mathcal{C}^\star$.

Consider the planes

- $\Pi_j = \{(x, y, z); z = a_j\},\ j = 1, 2$, with $a_2 < 0 < a_1$;
- $\Pi_j = \{(x, y, z); y = a_j\},\ j = 3, 4$, with $a_4 < 0 < a_3$,

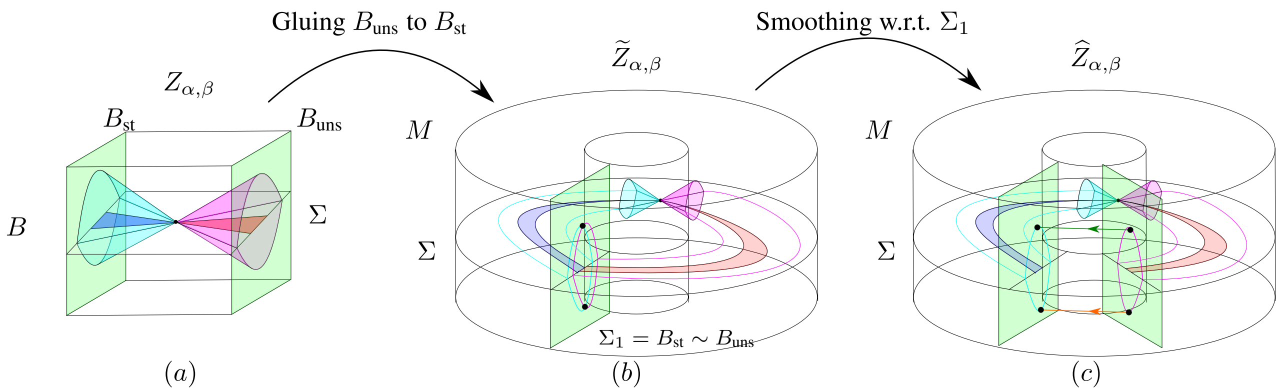

Figure 3.3: Process of construction of the Filippov system $\widehat{Z}_{\alpha,\beta}$. In (a) we have the phase space of $Z_{\alpha,\beta}$ restricted to a box B. In (b), we have the Filippov system $\widetilde{Z}_{\alpha,\beta} = (\widetilde{X}_{\alpha}, \widetilde{Y}_{\beta})$ (with switching manifold $\Sigma \cup \Sigma_1$) induced in the manifold M obtained by gluing the sides B_{st} and B_{uns} of B. In (c) we have the Filippov system $\widehat{Z}_{\alpha,\beta}$ (with switching manifold Σ) obtained by regularizing the vector fields $\widetilde{X}_{\alpha}$ and $\widetilde{Y}_{\beta}$ restricted to Σ^+ and Σ^-, respectively, with respect to Σ_1. Source: [3].

and choose the constants $a_1, \cdots, a_4$ in such a way that the box B delimited by Π_j, $j = 1, \cdots, 4$, and $\Pi_{\text{uns}}, \Pi_{\text{st}}$, satisfies that $\mathcal{C}^\star$ is contained in the interior of the side of the box B contained in $\Pi_\star$. Denote the side of B contained in $\Pi_\star$ by $B_\star$, $\star =$ uns,st.

Now, let $\sigma : B_{\text{uns}} \to B_{\text{st}}$ be a diffeomorphism such that:

1. σ is Σ-invariant, i.e. $\pi_3 \circ \sigma(p_1, p_2, 0) = 0$ for every $(p_1, p_2, 0) \in B_{\text{uns}}$:
2. σ sends $B_{\text{uns}} \cap \{\pm z > 0\}$ onto $B_{\text{st}} \cap \{\pm z > 0\}$;
3. σ sends ∂B_{uns} onto ∂B_{st}.
4. The topological circles $\sigma(\mathcal{C}^{\text{uns}})$ and $\mathcal{C}^{\text{st}}$ are in general position, i.e., they intersect transversally at two points contained in $B_{\text{st}} \cap \{z \neq 0\}$. See Figure 3.4.

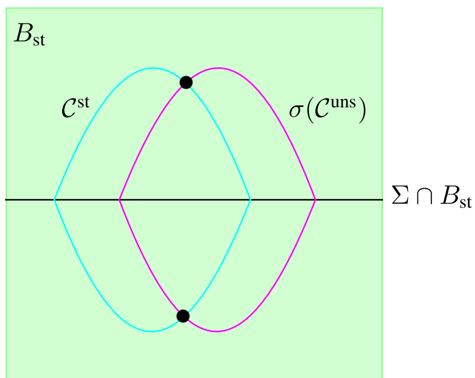

Figure 3.4: Topological circles $\sigma(\mathcal{C}^{\text{uns}})$ and $\mathcal{C}^{\text{st}}$ in the plane B_{st}. Source: [3].

Now, consider the equivalence relation on B induced by σ which consists on

$$p \sim q,\ p \in B_{\text{uns}},\ q \in B_{\text{st}} \Longleftrightarrow q = \sigma(p),$$

and let $M = B/\sim$ be the quotient manifold associated to $\sim$. Notice that M can be seen as a region of $\mathbb{R}^3$ delimited by two cylinders. See Figure 3.3-(b).

Also, the vector field X_α (resp. Y_β) is defined in B, but it does not have to induce a smooth vector field $\tilde{X}_\alpha$ (resp. $\tilde{Y}_\beta$) in M, since $X_\alpha(p)$ and $X_\alpha(\sigma(p))$ do not coincide in general. Nevertheless, the vector field X_α (resp. Y_β) induces a vector field $\tilde{X}_\alpha$ (resp. $\tilde{Y}_\beta$) in M which is bi-valued on B_{st}. Therefore $Z_{\alpha,\beta}$ induces a Filippov system $\tilde{Z}_{\alpha,\beta} = (\tilde{X}_\alpha, \tilde{Y}_\beta)$ having two switching manifolds $\Sigma = \{(x, y, z);\ z = 0\}$ and $\Sigma_1 = B_{\text{st}}$. Notice that $\tilde{Z}_{\alpha,\beta}$ presents two crossing self-connections at the origin (T-singularity) which crosses the both switching manifolds. See Figure 3.3-(b).

Now, in order to eliminate the discontinuity in Σ_1, we consider the following regularizations. Let B_{st} be given as the zero of a function $g : M \to \mathbb{R}$ and consider:

$$X_{\alpha,\delta}(x,y,z) = \frac{X_\alpha(x,y,z) + X_\alpha(\sigma(x,y,z))}{2} + \phi\left(\frac{g(x,y,z)}{\delta}\right)\frac{X_\alpha(x,y,z) - X_\alpha(\sigma(x,y,z))}{2},$$

$$Y_{\beta,\delta}(x,y,z) = \frac{Y_\beta(x,y,z) + Y_\beta(\sigma(x,y,z))}{2} + \phi\left(\frac{g(x,y,z)}{\delta}\right)\frac{Y_\beta(x,y,z) - Y_\beta(\sigma(x,y,z))}{2},$$

where $\phi : \mathbb{R} \to \mathbb{R}$ is a C^r function such that

$$\begin{cases} \phi(x) = -1, \text{ if } x \leq -1, \\ \phi|_{[-1,1]} \text{ is increasing}, \\ \phi(x) = 1, \text{ if } x \geq 1. \end{cases}$$

Remark 3.3.1. *Notice that the regularization process is applied separately to the vector fields X_α and Y_β, and thus we obtain smooth vector fields $X_{\alpha,\delta}$ and in $Y_{\beta,\delta}$ defined in $M \cap \Sigma^+$ and $M \cap \Sigma^-$, respectively.*

For $\delta^* > 0$ sufficiently small fixed, consider the Filippov system $\widehat{Z}_{\alpha,\beta} = (X_{\alpha,\delta^*}, Y_{\beta,\delta^*})$ in M with switching manifold given by the plane Σ.

Since $\tilde{Z}_{\alpha,\beta}$ has two crossing self-connections at the origin, it follows from this process that $\widehat{Z}_{\alpha,\beta}$ also has two crossing self-connections at the origin (see [17]). Moreover, the invariant cones of the T-singularity intersect themselves in general position at the plane B_{st}. See Figure 3.3-(c).

In [17], it is also proved that the presence of these self-connections at the origin of the Filippov system $\widehat{Z}_{\alpha,\beta}$ give rise to a nontrivially recurrent point $p \in \Sigma^c$ (see Theorem B in the referred paper), in such a way that the orbit passing through p is of crossing type and crosses Σ infinitely many times. Therefore, Theorem 3.1.1 applies to the class of Filippov systems $\widehat{Z}_{\alpha,\beta}$ in M with $\alpha\beta > 1$ and $\alpha, \beta < 0$.

CHAPTER 4

Topological Entropy of PSVFs

4.1 General Theory on Entropy for PSVFs

In this section, we adapt Definition 2.2.3 for the non-smooth case and prove some preliminary results of this theory. As far as the author knows, this is the first work to cover these aspects of non-smooth vector fields.

Let $Z = (X, Y)$ a PSVF defined over a compact 2-dimensional surface M and $\Omega = \{\gamma :$ global trajectory of $Z\}$.

Definition 4.1.1. *Let Ω be the set of all global trajectories, as before. Define $\rho : \Omega \times \Omega \to \mathbb{R}$ by:*

$$\rho(\gamma_1, \gamma_2) = \sum_{i \in \mathbb{Z}} \frac{1}{2^{|i|}} \int_i^{i+1} |\gamma_1(t) - \gamma_2(t)| dt,$$

where $|\gamma_1(t) - \gamma_2(t)|$ denotes the distance between the points $\gamma_1(t)$ and $\gamma_2(t)$.

Proposition 4.1.1. *The space (Ω, ρ) is a metric space.*

Proof. Let $\gamma_1, \gamma_2 \in \Omega$. Observe that M being compact, implies $|\gamma_1(t) - \gamma_2(t)|$ is bounded for all $t \in \mathbb{R}$, thus the series above converges for any γ_1, γ_2.

If $\rho(\gamma_1, \gamma_2) = 0$ then $\int_i^{i+1} |\gamma_1(t) - \gamma_2(t)| dt = 0$ for all $i \in \mathbb{Z}$ which implies $\gamma_1(t) = \gamma_2(t)$ for all $t \in \mathbb{R}$ and therefore $\gamma_1 = \gamma_2$.

The fact $\rho(\gamma_1, \gamma_2) = \rho(\gamma_2, \gamma_1)$ follows immediately from $|\gamma_1(t) - \gamma_2(t)| = |\gamma_2(t) - \gamma_1(t)|$.

And, finally, for the triangle inequality part it is enough to notice that $|\gamma_1(t) - \gamma_3(t)| \leqslant |\gamma_1(t) - \gamma_2(t)| + |\gamma_2(t) - \gamma_3(t)|$ for all $t \in \mathbb{R}$ gives the inequality $\rho(\gamma_1, \gamma_2) \leqslant \rho(\gamma_1, \gamma_3) + \rho(\gamma_3, \gamma_2)$. $\square$

Propositions 4.1.2, 4.1.3 give some important insights about how this metric behaves.

Proposition 4.1.2. *Given t_0, ε_0, then there exists $\delta > 0$ such that, for all $\gamma_1, \gamma_2 \in \Omega$ if $\rho(\gamma_1, \gamma_2) < \delta$, then for every $|t| < t_0$, it holds $|\gamma_1(t) - \gamma_2(t)| \leqslant \varepsilon_0$.*

Proof. Suppose, by contradiction, that for every δ, there is $-t_0 < t_\delta < t_0$, such that $|\gamma_1(t_\delta) - \gamma_2(t_\delta)| > \varepsilon_0$.

By continuity, there is open interval $J = (\alpha_\delta, \beta_\delta) \ni t_\delta$, such that for all $t \in J$, $|\gamma_1(t) - \gamma_2(t)| > \varepsilon_0$. Moreover, we know that γ_1, γ_2 are differentiable by parts. Hence it makes sense to talk about their derivatives (which could differ if taken by left or right), and we know that because the vector fields that generate the Filippov system are all bounded, these derivatives are all uniformly bounded.

Now,

$$\rho(\gamma_1,\gamma_2) > \frac{1}{2^{t_0}}\int_J |\gamma_1(t)-\gamma_2(t)|dt > \frac{\varepsilon_0}{2^{t_0}}(\beta_\delta-\alpha_\delta) \Longrightarrow$$

$$\Longrightarrow \frac{\varepsilon_0}{2^{t_0}}(\beta_\delta-\alpha_\delta) < \rho(\gamma_1,\gamma_2) < \delta.$$

Which implies that $\beta_\delta-\alpha_\delta$ goes to zero as $\delta\to 0$, and then the derivative of this function would explode at some point of J, but it can not happen, since the derivatives are bounded. □

Proposition 4.1.3. *Given $t_0>0$, and $(\gamma_n)\subset\Omega$ such that $\gamma_n\to\gamma\in\Omega$, then for all $|t|<t_0$, $\gamma_n(t)\to\gamma(t)$.*

Proof. Let $\varepsilon>0$ given. There exists $\delta>0$, such that

$$\rho(\gamma_n,\gamma)<\delta \Rightarrow |\gamma_n(t)-\gamma(t)|<\varepsilon,\ \forall |t|<t_0.$$

And there is $N\in\mathbb{N}$ such that, if $n>N$ then $\rho(\gamma_n,\gamma)<\delta$. Hence, if $n>N$, then $|\gamma_n(t)-\gamma(t)|<\varepsilon$, for every $|t|<t_0$. □

In this space of all orbits, we can define a natural flow determined by the PSVF Z as $F:\mathbb{R}\times\Omega\to\Omega$, $F(t,\gamma)(.)=\gamma(.+t)$. And then we have the *time-one map*, $F_1(\gamma)=\gamma(.+1)$.

Proposition 4.1.4. *The map $F_1:\Omega\to\Omega$ defined above is a homeomorphism.*

Proof. The function F_1 is invertible, with inverse $(F_1)^{-1}(\gamma)(.)=\gamma(.-1)$.

Now,

$$\int_i^{i+1}|F_1(\gamma_1)(t)-F_1(\gamma_2)(t)|dt = \int_i^{i+1}|\gamma_1(t+1)-\gamma_2(t+1)|dt =$$
$$= \int_{i+1}^{i+2}|\gamma_1(t)-\gamma_2(t)|dt.$$

Using the relation above, we obtain:

$$\begin{aligned}
\rho(F_1(\gamma_1),F_1(\gamma_2)) &= \sum_{i\in\mathbb{Z}}\frac{1}{2^{|i|}}\int_i^{i+1}|F_1(\gamma_1)(t)-F_1(\gamma_2)(t)|dt = \\
&= \sum_{i\in\mathbb{Z}}\frac{1}{2^{|i|}}\int_{i+1}^{i+2}|\gamma_1(t)-\gamma_2(t)|dt = \\
&= \lim_{k\to\infty}\left(\sum_{i=0}^{k}\frac{1}{2^i}\int_{i+1}^{i+2}|\gamma_1(t)-\gamma_2(t)|dt + \sum_{i=-k}^{-1}\frac{1}{2^{-i}}\int_{i+1}^{i+2}|\gamma_1(t)-\gamma_2(t)|dt\right) = \\
&= \lim_{k\to\infty}\left(2\sum_{i=0}^{k}\frac{1}{2^{i+1}}\int_{i+1}^{i+2}|\gamma_1(t)-\gamma_2(t)|dt + \frac{1}{2}\sum_{i=-k}^{-1}\frac{1}{2^{-i-1}}\int_{i+1}^{i+2}|\gamma_1(t)-\gamma_2(t)|dt\right) = \\
&= \lim_{k\to\infty}\left(2\sum_{j=1}^{k+1}\frac{1}{2^j}\int_{j}^{j+1}|\gamma_1(t)-\gamma_2(t)|dt + \frac{1}{2}\sum_{i=-k+1}^{0}\frac{1}{2^{-j}}\int_{j}^{j+1}|\gamma_1(t)-\gamma_2(t)|dt\right) \leqslant \\
&\leqslant 2\lim_{k\to\infty}\left(\sum_{j=1}^{k+1}\frac{1}{2^j}\int_{j}^{j+1}|\gamma_1(t)-\gamma_2(t)|dt + \sum_{j=-k+1}^{0}\frac{1}{2^{-j}}\int_{j}^{j+1}|\gamma_1(t)-\gamma_2(t)|dt\right) = \\
&\leqslant 2\sum_{j=-\infty}^{\infty}\frac{1}{2^{|j|}}\int_{j}^{j+1}|\gamma_1(t)-\gamma_2(t)|dt = 2\rho(\gamma_1,\gamma_2).
\end{aligned}$$

Hence F_1 is continuous. The proof of continuity of the inverse is analogous. □

Definition 4.1.2. *In the conditions above, we define the topological entropy h of $Z = (X, Y)$ on M, as the topological entropy of the map F_1 in Ω, that is, $h(Z) := h(F_1)$.*

Note that, if we consider $Z = (X, X)$ then Z is a smooth vector field. In this case, the value given by Definition 4.1.2 should coincide with the value obtained in Definition 2.2.3. The next proposition assures it.

Proposition 4.1.5. *Let $Z = (X, X)$ a PSVF defined on M, then the function $f : M \to \Omega$ defined by $f(x) = \gamma_x$ is a homeomorphism and the following diagram commutes:*

$$\begin{tikzcd}
(M,d) \arrow[r, "X^1"] \arrow[d, "f"] & (M,d) \arrow[d, "f"] \\
(\Omega,\rho) \arrow[r, "F_1"] & (\Omega,\rho)
\end{tikzcd}$$

Figure 4.1:

Proof. Let φ denote the flow of X and define $f : M \to \Omega$ as $f(x) = \gamma_x$, where $\gamma_x(t) = \varphi(t, x)$ for all t. From the Existence and Uniqueness Theorem, the function f is well-defined and it has inverse $f^{-1}(\gamma) = \gamma(0)$.

Let $x_1, x_2 \in M$, $\gamma_{x_1} = f(x_1), \gamma_{x_2} = f(x_2)$. Since φ is smooth on M, there exists a constant k_1 such that $|\gamma_{x_1}(t) - \gamma_{x_2}(t)| \leqslant k_1|x_1 - x_2|$, for all t. This implies that if $|x_1 - x_2| < \varepsilon$, then $\int_i^{i+1} |\gamma_{x_1}(t) - \gamma_{x_2}(t)|dt < k_1\varepsilon$.

Hence

$$\rho(\gamma_{x_1}, \gamma_{x_2}) = \sum_{i\in\mathbb{Z}} \frac{1}{2^{|i|}} \int_i^{i+1} |\gamma_{x_1}(t) - \gamma_{x_2}(t)|dt < 3k_1\varepsilon.$$

Thus f is continuous.

Now, by Proposition 4.1.2, given $\varepsilon > 0$, there is $\delta > 0$, such that if $\rho(\gamma_1, \gamma_2) < \delta$ then $\gamma_1(0) = x_1$ and $\gamma_2(0) = x_2$ are $\varepsilon-$close, which proves that f^{-1} is continuous.

Moreover, for all $x \in M$ and all $t \in \mathbb{R}$, $f(X^1(x))(t) = f(\varphi(1, x))(t) = \gamma_{\varphi(1,x)}(t) = \gamma_x(t + 1) = F_1(\gamma_x)(t) = F_1(f(x))(t)$. Hence

$$f \circ X^1 = F_1 \circ f.$$

□

Corollary 4.1.1. *The entropy of $Z = (X, X)$ (Definition 4.1.2) and entropy of X (Definition 2.2.3) are the same.*

Proof. Propositions 2.2.1 and 4.1.5. □

Corollary 4.1.2. *The space (Ω, ρ) is compact in this case.*

Proof. We have that M is compact and f is a homeomorphism. □

Observe that it is not always the case that Ω is compact, as it can be seen in Remark **??**.

Theorem 4.1.1. *Let Z be a PSVF over M, and $K \subset M$ a closed invariant set for Z. Then $h(Z) \geqslant h(Z|_K)$.*

Proof. Let Ω represent the set of all trajectories of Z over M. Since $K \subset M$ is closed and $Z-$invariant, the set $\Omega_K \subset \Omega$, of all trajectories through points of K is invariant for F_1. Moreover, given any sequence $(\gamma_n) \subset \Omega_K$, such that $\gamma_n \to \gamma$, we have that, $\gamma \in \Omega_K$, otherwise $\gamma \subset M \setminus K$, which is open, then for every point x of γ, there is ε_x such that $B(x, \varepsilon_x) \subset M \setminus K$, which implies γ cannot be limit of trajectories of Ω_K. Thus $\Omega_K \subset \Omega$ is closed.

Now, $\Omega_K \subset \Omega$ is closed and F_1-invariant, then $h(Z_K) = h(F_1|_{\Omega_K}) \leqslant h(F_1) = h(Z)$, by Proposition 2.2.2. □

4.2 Finite Positive Topological Entropy

In this section we prove results about the existence of PSVFs of finite positive topological entropy.

Theorem 4.2.1. *Given $\alpha \in \mathbb{Z}, \alpha \geqslant 2$ there exists PSVF Z such that $h(Z) = \log \alpha$.*

Proof. Consider the polynomial PSVF $Z = (X, Y)$ with $\Sigma = \{y = 0\}$, where $X = (1, 1 - x)$ and $Y = (-1, 1 - x)$. They are both symmetric and the point $(1, 0)$ is an invisible-invisible two-fold, and there is a closed trajectory that goes through the origin $(0, 0)$.

Let $\alpha = 3$, and consider six rays from the origin (one at each multiple of $\frac{\pi}{3}$). In that way, the plane is divided into six different regions. Number each region from 1 to 6, counter clockwise. We define X in region 1, and Y in region 6. Now, define a vector field in each one of these regions, such that in regions 3 and 5 we have a phase portrait that is the one of X rotated, and in regions 2 and 4 the phase portrait is symmetric to the one of Y (see Figure 4.2). Now there are three closed arcs that goes through the origin. That defines a PSVF with six different regions such that, apart from three invisible two-folds, and the origin, every other point is either regular or crossing, and every trajectory that does not goes through the origin is closed.

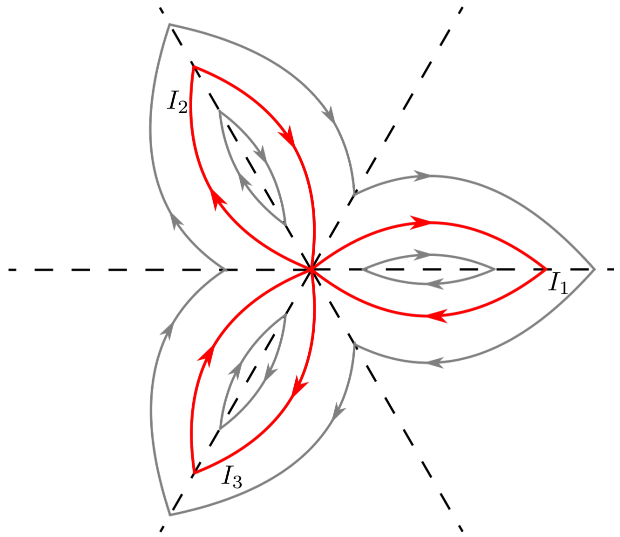

Figure 4.2: Set highlighted in red is invariant for Z. Source: [4].

The union of the curves I_1, I_2, I_3 highlighted in Figure 4.2 is invariant to the PSVF and a trajectory in this is the amalgamation of these three different arcs in every possible combination.

Let $\Omega^* = \{\gamma \in \Omega \mid \gamma(0) = 0\}$. Observe that the function $F_1 : \Omega^* \to \Omega^*$ is well defined since the arcs I_j can be adjusted so it has time duration 1.

For any $\gamma, \psi \in \Omega^*$ and for all $i \in \mathbb{Z}$, we have that $\gamma(i) = \psi(i) = 0$, and the integral

$$\int_i^{i+1} |\gamma(t) - \psi(t)| dt$$

vanishes when the same arc is used at the same time in the construction of γ and ψ or it equals a constant μ, if different arcs occur (because of the symmetry of phase portrait this value μ is the same for any pair I_j, I_k with $j \neq k$).

Let γ_j represents a trajectory starting with I_j, for $j = 1, 2, 3$. Consider $\varepsilon = 2\mu$. The open ball $B(\gamma_j, \varepsilon, 1)$ contains all trajectories that begins with one arc I_j. Then $\Omega^* \subset B(\gamma_1, 2\mu, 1) \cup B(\gamma_2, 2\mu, 1) \cup B(\gamma_3, 2\mu, 1)$, which implies that $S_\rho(F_1, 2\mu, 1) \leqslant 3$.

Since $\max\{\rho(\gamma, \psi), \gamma, \psi \in \Omega^*\} = 3\mu$, it does not occur $S_\rho(F_1, 2\mu, 1) = 1$, then suppose by contradiction that $S_\rho(F_1, 2\mu, 1) = 2$ and let $\psi_1, \psi_2 \in \Omega^*$ such that

$$\Omega^* \subset B(\psi_1, 2\mu, 1) \cup B(\psi_2, 2\mu, 1).$$

Since there are three options I_j, it is possible to construct ψ^* that does not coincide with either ψ_1 or ψ_2 at any given time. So, $\rho(\psi^*, \psi_1) = \rho(\psi^*, \psi_2) = 3\mu$ and, therefore,

$$\Omega^* \not\subset B(\psi_1, 2\mu, 1) \cup B(\psi_2, 2\mu, 1).$$

Thus $S_\rho(F_1, 2\mu, 1) = 3$.

Now fix γ_{11} a trajectory that begins with the concatenation of I_1 twice in a row; γ_{12} for one starting with I_1 and followed by I_2 and so on. In general, fix γ_{jk} for a trajectory beginning with the concatenation of I_j followed by I_k, for $j, k = 1, 2, 3$.

Given any $\psi \in \Omega^*$, it coincides with some γ_{jk} in the interval $[0, 2]$. From which, we get that $\rho(\gamma_{jk}, \psi) < 2\mu$ and $\rho(F_1(\gamma_{jk}), F_1(\psi)) < 2\mu$, then $\rho_2^{F_1}(\gamma_{jk}, \psi) < 2\mu$. Thus,

$$\Omega^* \subset \bigcup_{j,k=1,2,3} B(\gamma_{jk}, 2\mu, 2),$$

hence $S_\rho(F_1, 2\mu, 2) \leqslant 9$.

Once again, suppose for contradiction, that $S_\rho(F_1, 2\mu, 2) < 9$. Then there exist at most 8 trajectories ψ_l, such that

$$\Omega^* \subset \bigcup_l B(\psi_l, 2\mu, 2). \tag{4.1}$$

There exists a trajectory ψ^* that does not coincide with any ψ_l at the interval $[-1, 2]$. And, for any l,

$$\rho(\psi^*, \psi_l) \geqslant \frac{1}{2}\int_{-1}^{0} |\psi^*(t) - \psi_l(t)| dt + \int_0^1 |\psi^*(t) - \psi_l(t)| dt + \frac{1}{2}\int_1^2 |\psi^*(t) - \psi_l(t)| dt = 2\mu.$$

Then $\rho_2^{F_1}(\psi^*, \psi_l) \geqslant 2\mu$, which contradicts (4.1). Hence $S_\rho(F_1, 2\mu, 2) = 9$.

In general, for n fixed, there are 3^n fixed trajectories $\gamma_{j_0 \cdots j_{n-1}}$, $j_i \in \{1, 2, 3\}$, for which the open balls $B_{F_1}(\gamma_{j_0 \cdots j_{n-1}}, 2\mu, n)$ cover Ω^*. Thus $S_\rho(F_1, 2\mu, n) = 3^n$.

Now consider $\varepsilon = \dfrac{\mu}{2^m}$. For $n = 1$ (metric $\rho_1^{F_1}$), we choose trajectories $\gamma_{j_{-m} \cdots j_m}$ to be the centers of the open balls. And for a fixed n, we choose trajectories $\gamma_{j_{-m} \cdots j_{m+n-1}}$. Then $S_\rho(F_1, \dfrac{\mu}{2^m}, n) = 3^{2m+n}$.

So,

$$h(F_1) = \lim_{m\to\infty} \lim_{n\to\infty} \frac{1}{n} \log S_\rho(F_1, \frac{\mu}{2^m}, n) = \lim_{m\to\infty} \lim_{n\to\infty} \frac{1}{n} \log 3^{2m+n} = \log 3.$$

Now, for any $\alpha \in \mathbb{Z}, \alpha \geqslant 2$, we can construct a PSVF analogous to the one above (with 2α regions) and, by the same calculations, conclude $h(F_1) = \log \alpha$. □

Consider the figure with three arcs (Figure 4.2), in demonstration of Theorem 4.2.1. If we consider $\widetilde{Z}$ the restriction of Z to $I_1 \cup I_2$ (the union of only two arcs), then $h(\widetilde{Z}) = \log 2$.

Corollary 4.2.1. *More generally, given Z with 2α regions, as in proof of Theorem 4.2.1 (with α closed arcs), we can consider $\widetilde{Z}$ as the restriction of Z to any number $\beta < \alpha$ of closed arcs, and then $h(\widetilde{Z}) = \log \beta$.*

4.3 Infinite Topological Entropy

In this section, we prove the existence of a PSVF of infinite topological entropy with an example, and next we give sufficient conditions for a PSVF Z to have $h(Z) = \infty$.

Example 4.3.1. *Consider the PSVF:*

$$Z(x,y) = \begin{cases} X(x,y) = (1, -2x), & \textit{for} \quad y \geq 0 \\ Y(x,y) = (-2, -4x^3 + 2x), & \textit{for} \quad y \leq 0 \end{cases}. \tag{4.2}$$

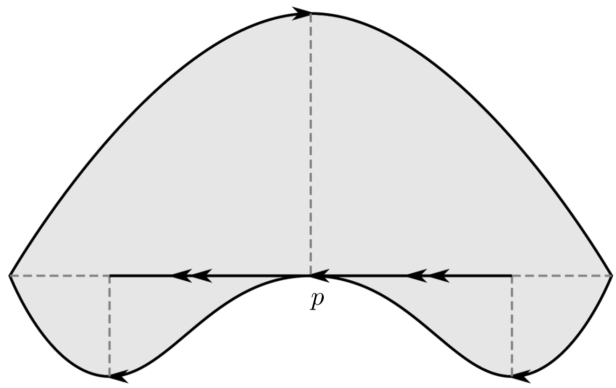

Figure 4.3: Set K is invariant for Z. Source: [4].

It has a compact invariant set $K = \{(x,y) \in \mathbb{R}^2 : -1 \leqslant x \leqslant 1$ and $x^4/2 - x^2/2 \leqslant y \leqslant 1 - x^2\}$. Furthermore, K is a chaotic non-trivial minimal set for Z (see [5]). As before, we will consider $\Omega = \{\gamma$ global trajectory of $Z|_K\}$ the set of all possible trajectories contained in the set K (see Figure 4.3).

Remark 4.3.1. *The space Ω is not compact in this case. In fact, consider $(\gamma_n) \subset \Omega$ a sequence of closed simple trajectories through p, such that the period of each γ_n is half of the previous period. Since they are closing in to p, we have that $\int_i^{i+1} |\gamma_n(t) - p| dt \to 0$. Then (γ_n) converges to a fixed trajectory $\gamma^*(t) \equiv p$, but $\gamma^* \notin \Omega$.*

We will show that $h(Z) = \infty$.

Definition 4.3.1. *We say that a trajectory γ escapes $\overline{\Sigma^e}$ at a time s_0 if $\gamma(s_0) \in \overline{\Sigma^e}$ and there is $s_1 > s_0$, such that $\gamma(s) \notin \overline{\Sigma^e}$, for all $s_0 < s < s_1$. We may also add that γ escapes to Σ^+ or Σ^-, depending on which of these regions the points $\gamma(s)$ are located in. Alternatively, we may say that $\gamma(s_0)$ is a* escape point of γ.

Since every trajectory of Ω intersects $\overline{\Sigma^e}$ in positive time, define the function $\tau : \Omega \to \mathbb{R}_+, \tau(\gamma) = \tau_\gamma$, as the minimum $t > 0$, for which γ escapes $\overline{\Sigma^e}$, in other words, the time τ_γ is the "next escape time" for γ (see Figure 4.4).

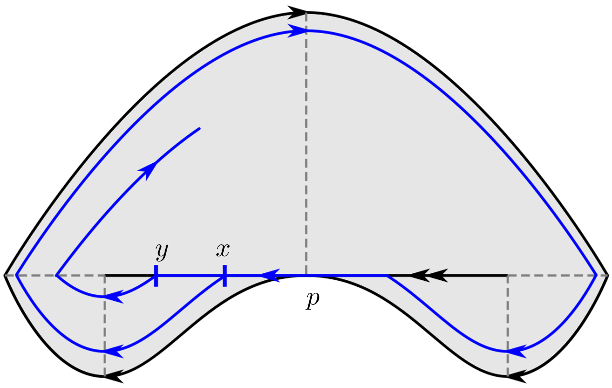

Figure 4.4: Arc of trajectory γ, such that, $\gamma(0) = x$, and $\gamma(\tau_\gamma) = y$. Both x and y are escape points of γ. Source: [4].

Now, consider a subset $R \subset K$, bounded by two distinct simple closed trajectories, and let

$$S = \{\gamma \in \Omega \mid \forall t, \gamma(t) \in R \textit{ and } \gamma \textit{ escapes to } \Sigma^- \textit{ at } t = 0\}$$

that is, S is the set of trajectories entirely contained in R that escapes at $s_0 = 0$ (see Figure 4.5). We may restrict R in order to obtain some positive integer $k > 1$ for which we have $k - 1 < \tau(\gamma) < k$ for every $\gamma \in S$. And define the map:

$$\begin{aligned} P : S &\to S \\ \gamma &\mapsto P(\gamma), \end{aligned}$$

where $P(\gamma)(\,\cdot\,) = \gamma(\,\cdot\, + \tau_\gamma)$.

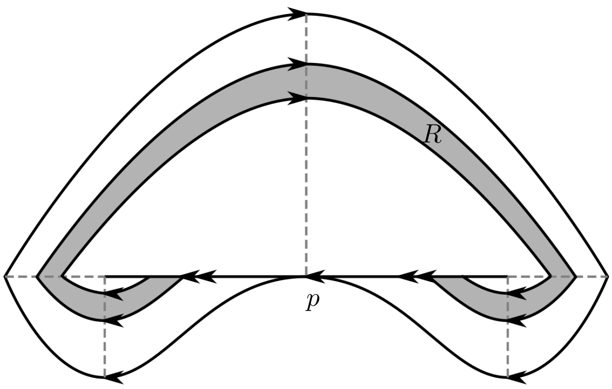

Figure 4.5: Region R. Source: [4].

In the following, we prove some properties of τ and P, that will be useful next.

Proposition 4.3.1. *The function $\tau : D \to \mathbb{R}_+$ is continuous.*

Proof. Let $\gamma_1, \gamma_2 \in S, \varepsilon > 0$ and consider $t_0 > k$ large enough. For simplicity, let $\tau_1 = \tau_{\gamma_1}$ and $\tau_2 = \tau_{\gamma_2}$ and assume $\tau_1 < \tau_2$.

By Proposition 4.1.2, there is $\delta > 0$ such that, for any $\gamma_1, \gamma_2 \in S$, if $\rho(\gamma_1, \gamma_2) < \delta$, then for every $|t| < t_0$, $|\gamma_1(t) - \gamma_2(t)| \leqslant \varepsilon$. This implies that $\gamma_1(\tau_1)$ and $\gamma_2(\tau_1)$ are ε close to each other and also $\gamma_1(\tau_2)$ and $\gamma_2(\tau_2)$.

Now, there exists $\bar{t} > 0$, that is the minimum time for which $B(\gamma_1(\tau_1 + \bar{t}), \varepsilon) \cap \overline{\Sigma^e} = \emptyset$. That is, $\bar{t}$ is the time required to γ_1 distance ε from Σ^e. It clearly depends continuously of ε.

Since $|\gamma_1(\tau_2) - \gamma_2(\tau_2)| < \varepsilon$, and $\gamma_2(\tau_2) \in \Sigma^e$, we must have $\tau_1 < \tau_2 < \tau_1 + \bar{t}$.

Intuitively, from the moment γ_1 escapes Σ^e, there is little time to γ_2 escape it too, for if it does not, they would distance themselves.

□

Proposition 4.3.2. *The map $P : S \to S$ as defined above is continuous.*

Proof. Let $\gamma \in S$ and $\gamma_n \to \gamma$. We will show that $P(\gamma_n) \to P(\gamma)$.

Let $\varepsilon > 0$ then there is $t_0 > 0$ such that

$$\sum_{|i| \geqslant t_0} \frac{1}{2^{|i|}} \int_i^{i+1} |P(\gamma_n)(t) - P(\gamma)(t)| dt < \frac{\varepsilon}{2}.$$

Let $\varepsilon_0 > 0$ such that

$$\forall |t| < t_0, |P(\gamma_n)(t) - P(\gamma)(t)| < \varepsilon_0 \Rightarrow \sum_{|i| < t_0} \frac{1}{2^{|i|}} \int_i^{i+1} |P(\gamma_n)(t) - P(\gamma)(t)| dt < \frac{\varepsilon}{2}.$$

Now, by Propositions 4.1.3 and 4.3.1, and considering the fact that γ is piecewise differentiable with bounded derivatives, we have that, there is N large enough such that if $n > N$:

$$\begin{aligned} |P(\gamma_n)(t) - P(\gamma)(t)| &= |\gamma_n(t + \tau_{\gamma_n}) - \gamma(t + \tau_\gamma)| \leqslant \\ &\leqslant |\gamma_n(t + \tau_{\gamma_n}) - \gamma(t + \tau_{\gamma_n})| + |\gamma(t + \tau_{\gamma_n}) - \gamma(t + \tau_\gamma)| < \\ &< \frac{\varepsilon_0}{2} + M|\tau_{\gamma_n} - \tau_\gamma| < \varepsilon_0. \end{aligned}$$

Hence, if $n > N$, $\rho(P(\gamma_n), P(\gamma)) < \varepsilon$.

□

Proposition 4.3.3. *The map P has infinite topological entropy.*

Proof. Consider a linear parametrization $\theta : \overline{\Sigma^e} \cap R \to [0, 1]$. Then construct a function itinerary $s : S \to [0, 1]^{\mathbb{Z}}$ given by: $s(\gamma) = (s_j(\gamma))_{j \in \mathbb{Z}}$, such that $s_0(\gamma) = \theta(\gamma(0))$, and $s_j(\gamma) = \theta(P^j(\gamma)(0))$.

Now, let $(b_j)_{j \in \mathbb{Z}} = \sigma \circ s(\gamma)$, and $(a_j)_{j \in \mathbb{Z}} = s \circ P(\gamma)$, where $\sigma : [0, 1]^{\mathbb{Z}} \to [0, 1]^{\mathbb{Z}}$ is the shift map. Then:

$$b_j = s_{j+1}(\gamma) = \theta(P^{j+1}(\gamma)(0)) = \theta(P^j(P(\gamma))(0)) = s_j(P(\gamma)) = a_j.$$

Hence $\sigma \circ s = s \circ P$.

$$\begin{array}{ccc} S & \xrightarrow{P} & S \\ \Big\downarrow s & & \Big\downarrow s \\ [0,1]^{\mathbb{Z}} & \xrightarrow{\sigma} & [0,1]^{\mathbb{Z}} \end{array}$$

The map s is continuous. In fact,

$$|s_j(\gamma_1) - s_j(\gamma_2)| = |\theta(P^j(\gamma_1)(0)) - \theta(P^j(\gamma_2)(0))| = \theta'|P^j(\gamma_1)(0) - P^j(\gamma_2)(0)|.$$

Given $\varepsilon > 0$, there exists $\eta > 0$ and $N_0 \in \mathbb{N}$, such that, if $|s_j(\gamma_1) - s_j(\gamma_2)| < \eta$ for all $-N_0 < j < N_0$, then $d(s(\gamma_1), s(\gamma_2)) < \varepsilon$.

But, since P is continuous, there exists $\delta > 0$, such that $\theta'|P^j(\gamma_1)(0) - P^j(\gamma_2)(0)| < \eta$, for all $-N_0 < j < N_0$. Hence s is continuous.

By Proposition 2.2.1, $h(P) = \infty$. □

Theorem 4.3.1. *The PSVF Z presented in (4.2) has infinite topological entropy.*

Proof. In order to prove this, we show that the following diagram commutes:

$$\begin{array}{ccc} S & \xrightarrow{F_1^k} & F_1^k(S) \\ \Big\downarrow P & & \Big\downarrow P \\ S & \xrightarrow{P} & S \end{array}$$

where $k > 1$ is the integer given above, such that $k - 1 < \tau_\gamma < k$, for all $\gamma \in S$.

Given $\gamma \in S$, and t, on one side we have

$$P \circ P(\gamma)(t) = P(\gamma)(t + \tau_{P(\gamma)}) = \gamma(t + \tau_{P(\gamma)} + \tau_\gamma)$$

and on the other side

$$P \circ F_1^k(\gamma)(t) = F_1^k(\gamma)(t + \tau_{F_1^k(\gamma)}) = \gamma(t + \tau_{F_1^k(\gamma)} + k).$$

Then it is enough to show that, for any $\gamma \in S$:

$$\tau_{P(\gamma)} + \tau_\gamma = k + \tau_{F_1^k(\gamma)}. \tag{4.3}$$

In fact, on the one hand, $\tau_{P(\gamma)} + \tau_\gamma$ is exactly the time at which γ escapes $\overline{\Sigma^e}$ for the second time. And since $k - 1 < \tau_\gamma < k$, we have $k \leqslant 2k - 2 < \tau_{P(\gamma)} + \tau_\gamma < 2k$.

That is,

$$\tau_{P(\gamma)} + \tau_\gamma = \min\{s > k \mid \gamma(s) \text{ is a escape point of } \gamma\}. \tag{4.4}$$

On the other hand:

$$\begin{aligned} k + \tau_{F_1^k(\gamma)} &= k + \min\{t > 0 \mid F_1^k(\gamma)(t) \text{ is a escape point of } F_1^k(\gamma)\} = \\ &= k + \min\{t > 0 \mid \gamma(t+k) \text{ is a escape point of } \gamma\} = \\ &= k + \min\{s - k > 0 \mid \gamma(s) \text{ is a escape point of } \gamma\} = \\ &= \min\{s > k \mid \gamma(s) \text{ is a escape point of } \gamma\}. \end{aligned}$$

From the above equation and (4.4), we get (4.3).

Since P is onto and continuous it works as a semi-conjugacy between F_1^k and P, then by Proposition 2.2.1, $h(F_1) \geqslant h(P) = \infty$. Thus $h(Z) = \infty$.

□

In the following theorem, we give sufficient conditions for a PSVF Z to have infinity topological entropy, but first we state a lemma that will be useful to complete the proof of the theorem.

Lemma 4.3.1. *Let Z and $\widetilde{Z}$ two PSVF defined over a compact invariant set $K \subset \mathbb{R}^2$, such that $Z = c \cdot \widetilde{Z}$, where $c > 0$ is an integer. If $h(\widetilde{Z}) = \infty$, then $h(Z) = \infty$.*

Proof. Let γ be a trajectory of Z. It is not difficult to see that $\widetilde{\gamma}(t) = \gamma(\frac{1}{c}t)$ is trajectory of $\widetilde{Z}$. If Ω and $\widetilde{\Omega}$ represent the spaces of trajectories of Z and $\widetilde{Z}$, respectively, we can define a continuous bijection between them $H : \Omega \to \widetilde{\Omega}$, simply by $H(\gamma)(t) = \widetilde{\gamma}(t) = \gamma(\frac{1}{c}t)$.

Consider the following diagram:

$$\begin{tikzcd} \Omega \arrow[r, "F_1"] \arrow[d, "H"] & \Omega \arrow[d, "H"] \\ \widetilde{\Omega} \arrow[r, "\widetilde{F_1}^c"] & \widetilde{\Omega} \end{tikzcd}$$

The diagram above commutes. In fact, for any $t \in \mathbb{R}$, we have:

$$H \circ F_1(\gamma)(t) = F_1(\gamma)(\frac{1}{c}t) = \gamma(\frac{1}{c}t + 1).$$

On the other hand:

$$\widetilde{F_1}^c \circ H(\gamma)(t) = H(\gamma)(t + c) = \gamma(\frac{1}{c}t + 1).$$

Therefore $h(Z) = h(F_1) = h(\widetilde{F_1}^c) = \infty$. □

Theorem 4.3.2. *Let $Z = (X, Y)$ be a PSVF defined over a compact invariant set $K \subset \mathbb{R}^2$, with $\Sigma^e \neq \emptyset$. Assume there exists a closed interval $J \subset \Sigma^e$, such that for all $x \in J$, there is a trajectory γ_x that escapes through x to Σ^+ (or Σ^-) and γ_x intersects J in finite time (see Figure 4.6). Then $h(Z) = \infty$.*

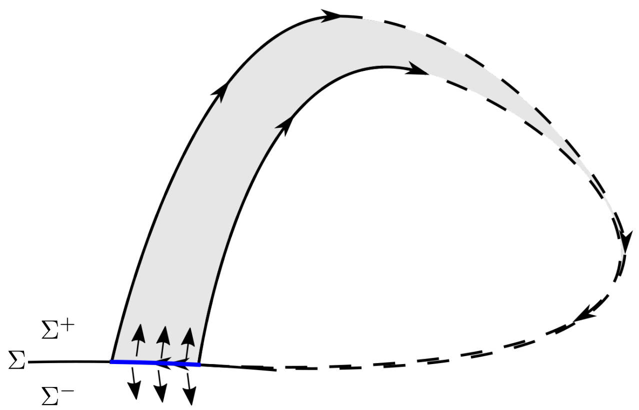

Figure 4.6: Every point of J (in blue) has a trajectory that escapes to Σ^+ and returns to J. Source: [4].

Proof. From hypothesis, we get that for every $x, y \in J$ there is an arc of trajectory escaping at x and going to y. Consider S to be the set of every possible trajectory γ that is formed by a concatenation of these arcs and $\gamma(0) \in J$ is a escape point. On this set S, define the functions τ and P in the same way as defined before.

Note that Propositions 4.3.1 to 4.3.3 can be repeated word by word, since they do not depend on the vector field.

In the example, we required an integer $k > 1$ such that $k - 1 < \tau_\gamma < k$ for all $\gamma \in S$. If we can restrict to $J_0 \subset J$ in which this inequality holds, then the Theorem 4.3.1 can also be repeated which concludes the proof.

If it is not the case, we must do some considerations first: since J is closed, τ has a minimum value $M > 0$, in this case less than 1, so let $c > 0$ be the smallest integer such that $1 < c \cdot M$ and define $\widetilde{Z} = \frac{1}{c} Z$. Each one of the hypothesis are still true for $\widetilde{Z}$, and now we can find an integer k satisfying that inequality. Hence $h(\widetilde{Z}) = \infty$ and the Lemma 4.3.1 concludes the proof.

□

4.4 Examples

Example 4.4.1. *Consider the PSVF $Z = (X, Y)$ where $X = \left(1, \frac{x}{2} - 4x^3\right)$ and $Y = \left(-1, \frac{x}{2} - 4x^3\right)$ and $\Sigma = \{y = 0\}$, we have that $h(Z) = \log 2$. In fact, there is a set Λ that is $Z-$invariant, that is the union of two symmetric integral curves of X and Y (see Figure 4.7) in which we can distinguish two arcs I_1, I_2 and apply the same calculations from the proof of Theorem 4.2.1. Note that this example presents a polynomial PSVF that satisfies Theorem 4.2.1 for $\alpha = 2$ with only two regions, using a visible two-fold.*

Example 4.4.2. *In [10], the authors present in details a PSVF Z tangent to $\mathbb{S}^2$, without equilibrium points that has a non-trivial minimal set diffeomorphic to that presented in Example 4.3.1 (see Figure 4.8). Thus, by Theorem 4.1.1, $h(Z) = \infty$.*

Example 4.4.3. *Let Z be a PSVF defined over $\mathbb{S}^2$, with discontinuity on $\mathbb{S}^1$, and Σ^+, Σ^- are the northern and southern hemispheres. Consider X^+ being a vector field without singularities, and X^- a vector field with a visible attracting node. Then it has a escaping region in some segment on $\mathbb{S}^1$ and all global orbits tend to p in positive time. And in negative time it reaches the escaping region (see Figures 4.9 and 4.10).*

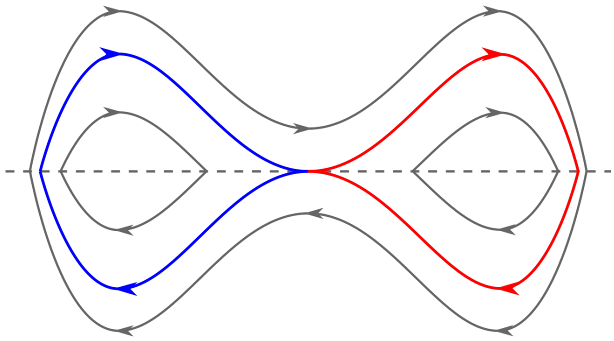

Figure 4.7: Polynomial PSVF with just two regions and topological entropy equal $\log 2$. Source: [4].

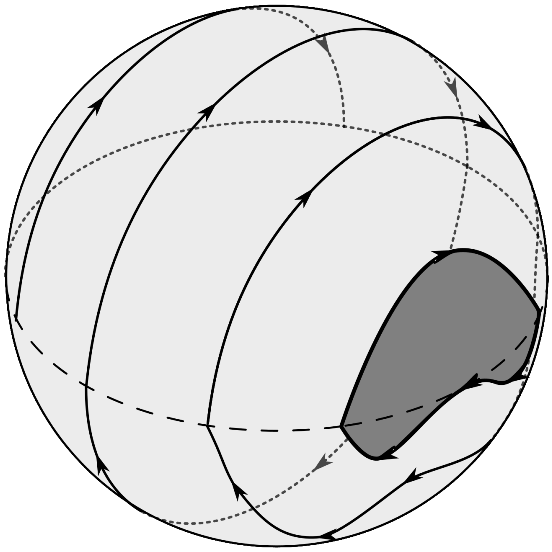

Figure 4.8: PSVF tangent to $\mathbb{S}^2$ with infinite topological entropy. Source: [4].

Now, for all $\varepsilon > 0$, there exists $m \in \mathbb{N}$ such that given any $\gamma \in \Omega$, for any $n > m$, $|F_1^n(\gamma)(0) - p| < \varepsilon$, because every trajectory tends to p. This implies that $\rho(F_1^n(\gamma), \gamma_p) < \varepsilon$.

Then the ε-capacity, $S_\rho(F_1, \varepsilon, n)$ does not grow with n, then $\limsup \frac{1}{n} \log S_\rho(F_1, \varepsilon, n) = 0$. Hence $h(Z) = 0$.

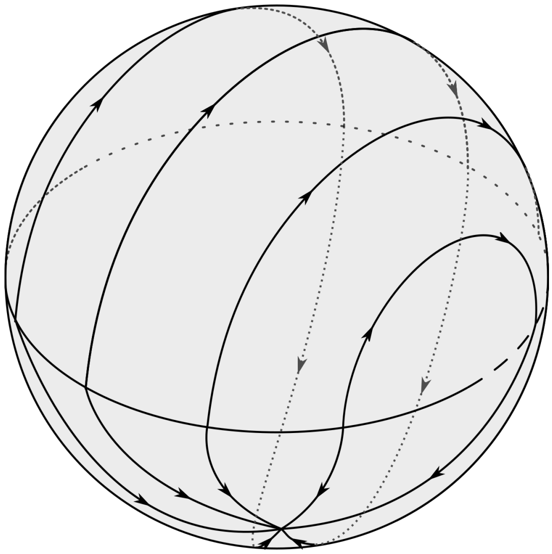

Figure 4.9: PSVF tangent to $\mathbb{S}^2$ with null topological entropy. Source: [4].

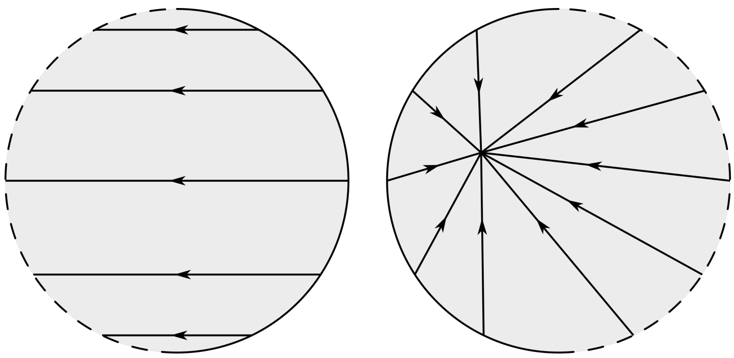

Figure 4.10: Northern and southern hemispheres in Example 4.4.3. Source: [4].

CHAPTER 5

PSVFs and Shifts

5.1 Introduction

In this chapter we propose yet another way of constructing the metric space Ω of all possible trajectories, using the definition of *Hausdorff distance* between two closed subsets of a metric space (see Definition 5.3.4).

Because it is another metric, some of the proofs here are similar to those from the previous chapter, but for sake of completeness we kept them.

In the previous chapter, when using the metric $|a-b|$ between the points, we assume some extra structure on the space where the PSVF is defined and this structure was actually useful in the proofs of Propositions 4.1.2 and 4.1.3. Now in this chapter we do not involve integrals on defining the metric, but we give up this extra structure.

With this space Ω, we define again the flow on this set of orbits and conjugate the time one map of this flow with shift spaces.

5.2 Main results

Given a flow $\varphi(t,x)$ of a vector field W defined in an open set $\mathcal{U}$, the time-one map is the function $T_1 : \mathcal{U} \to \mathcal{U}$, $T_1(x) = \varphi(1,x)$. The next result relates time-one maps of PSVFs (see Definition 5.3.1) and (sub)shifts.

Theorem 5.2.1. *(i) There exists a conjugacy between the time-one map of the PSVF*

$$Z_2(x,y) = \begin{cases} X_2(x,y) = \left(1, \frac{x}{2} - 4x^3\right), & \textit{for} \quad y \geq 0 \\ Y_2(x,y) = \left(-1, \frac{x}{2} - 4x^3\right), & \textit{for} \quad y \leq 0 \end{cases}, \tag{5.1}$$

restricted to an invariant compact set $\Lambda_2 \subset \mathbb{R}^2$ and the full shift of two symbols $\sigma : \{0,1\}^{\mathbb{Z}} \to \{0,1\}^{\mathbb{Z}}$.

(ii) For each $k \geqslant 3$, there exists a planar PSVF

$$Z_k(x,y) = \begin{cases} X_k(x,y) = (1, P_k'(x)), & \textit{for} \quad y \geq 0 \\ Y_k(x,y) = (-1, P_k'(x)), & \textit{for} \quad y \leq 0 \end{cases}, \tag{5.2}$$

where

$$P_k(x) = -\left(x + \frac{k-1}{2}\right)\left(x - \frac{k-1}{2}\right)\prod_{i=1}^{k-1}\left(x - \left(i - \frac{k}{2}\right)\right)^2 \tag{5.3}$$

such that, restricted to an invariant compact set Λ_k, the time-one map is conjugated to a subshift of $2(k-1)$ symbols.

(iii) There exists a conjugacy between the time-one map of the PSVF

$$Z_\infty(x,y) = \begin{cases} X_\infty(x,y) = (1, 2\,\mathrm{sen}(2\pi x)), & \text{for} \quad y \geq 0 \\ Y_\infty(x,y) = (-1, 2\,\mathrm{sen}(2\pi x)), & \text{for} \quad y \leq 0 \end{cases}, \tag{5.4}$$

restricted to an invariant set $\Lambda_\infty \subset \mathbb{R}^2$ and a subshift over an infinite alphabet $\sigma : \Theta_\infty \to \Theta_\infty$.

(iv) The return map of the PSVF

$$Z(x,y) = \begin{cases} X(x,y) = (1, -2x), & \text{for} \quad y \geq 0 \\ Y(x,y) = (-2, -4x^3 + 2x), & \text{for} \quad y \leq 0 \end{cases}, \tag{5.5}$$

restricted to an invariant compact set Λ, is conjugated to $\sigma : (0,1]^{\mathbb{Z}} \to (0,1]^{\mathbb{Z}}$.

In Section 5.5 is exhibited an example of PSVF whose time one map is conjugated to subshift of 4-symbols.

In the next result we state that the canonical forms presented in items (i) to (iv) of Theorem 5.2.1 represent a bigger class of PSVFs conjugated to the (sub)shifts mentioned. Before announcing it, we define a geometric structure that is present in these PSVFs:

Definition 5.2.1. *A* k-homoclinic loop *of a planar PSVF is a global trajectory of $Z = (X, Y)$ presenting k distinct visible-visible two-fold singularities $p_1, \ldots, p_k$ in such a way that, after passes through p_i, the trajectory reaches Σ either in p_{i-1} or p_{i+1} when $i = 2, \ldots, k-1$. When $i = 1$ (resp., $i = k$), after passes through p_i, the trajectory reaches Σ either in a sewing point or p_{i+1} (resp., p_{i-1}).*

Moreover, an ∞-homoclinic loop *of a planar PSVF is a global trajectory of Z presenting ∞ visible-visible two-fold singularities $p_1, p_2, \ldots$ in such a way that, after passes through p_i, the trajectory reaches Σ either in p_{i-1} or p_{i+1}.*

The homoclinic loops Γ defined above are regular *if $X(p)$ and $Y(p)$ point to opposite direction, for all two-fold singularity $p \in \Gamma$ (remember that $X(p)$ and $Y(p)$ are parallel). Otherwise, the homoclinic loop is* singular. *Figures 5.1 and 5.6 illustrate some regular k-homoclinic loops.*

Another important definition is the concept of equivalence between two PSVFs.

Definition 5.2.2. *Two PSVFs $Z = (X, Y)$, $\widetilde{Z} = (\widetilde{X}, \widetilde{Y}) \in \mathcal{Z}^r$, defined in U, $\widetilde{U}$ respectively and with switching manifold Σ and $\widetilde{\Sigma}$ are* **Σ**-equivalent *if there exists an orientation preserving homeomorphism $h : U \to \widetilde{U}$ that sends $U \cap \Sigma$ to $\widetilde{U} \cap \widetilde{\Sigma}$, the orbits of X restricted to $U \cap \Sigma^+$ to the orbits of $\widetilde{X}$ restricted to $\widetilde{U} \cap \widetilde{\Sigma}^+$, the orbits of Y restricted to $U \cap \Sigma^-$ to the orbits of $\widetilde{Y}$ restricted to $\widetilde{U} \cap \widetilde{\Sigma}^-$ and the orbits of Z^T restricted to Σ to the orbits of $\widetilde{Z}^T$ restricted to $\widetilde{\Sigma}$.*

Now, for the Theorem:

Theorem 5.2.2. *(i) The PSVF Z_2 of Theorem A, restrict to Λ_2, is Σ-equivalent to any PSVF presenting a 1-homoclinic loop.*

(ii) The PSVF Z_k of Theorem A, restrict to Λ_k, is Σ-equivalent to any PSVF presenting a $(k-1)$-homoclinic loop.

(iii) The PSVF Z_∞ of Theorem A, restrict to Λ_∞, is Σ-equivalent to any PSVF presenting a ∞-homoclinic loop.

(iv) *The PSVF Z of Theorem A, restrict to Λ, is Σ-equivalent to any PSVF $\widetilde{Z}$ presenting a compact region $\widetilde{\Lambda}$ bounded by a trajectory of $\widetilde{Z}$ passing through a invisible-visible two-fold $\widetilde{p}$. Moreover, except for $\widetilde{p}$, the PSVF $\widetilde{Z}$ has just more two invisible tangential singularities.*

We stress that the restriction to the pictured sets of both PSVFs in Figure 5.1, according to Theorem B item (i), are Σ-equivalents to Z_2 restricted to Λ_2. That means that our result is not restrict to general PSVFs with invariant sets with the shape of a "figure eight lying". The same holds for Z_k, with $k \in \{3, 4, \ldots\} \cup \{\infty\}$.

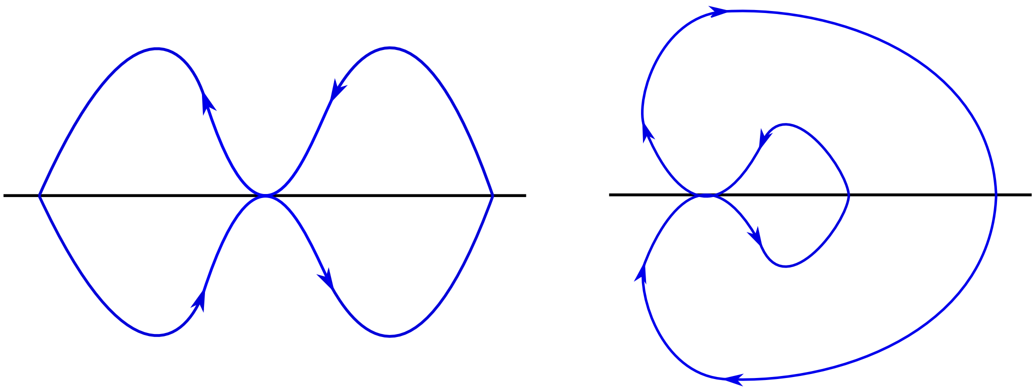

Figure 5.1: Disctint kinds of PSVFs whose restriction to an invariant set is Σ-equivalent to a restriction of Z_2 to Λ_2. Also, it is important to highlight that these PSVFs could have non-polynomial expressions. An analogous remark could be done for the other cases in Theorem 5.2.1. Source: [2].

As an immediate consequence of the previous results we have that:

Corollary 5.2.1. *The time-one maps of PSVFs presented in Theorems 5.2.1 and 5.2.2 have periodic points of any period and are topologically mixing (see Definition 2.3.6).*

Corollary 5.2.2. *All PSVFs presented in items (i) to (iv) of Theorems 5.2.1 and 5.2.2 are chaotic.*

5.3 Proof of Theorem 5.2.1

For each natural number $k \geqslant 2$, let:

$$P_k(x) = -\left(x + \frac{k-1}{2}\right)\left(x - \frac{k-1}{2}\right)\prod_{i=1}^{k-1}\left(x - \left(i - \frac{k}{2}\right)\right)^2 \tag{5.6}$$

and

$$P_\infty(x) = 1 - \cos(2\pi x). \tag{5.7}$$

Lemma 5.3.1. *Let P_k and P_∞ as defined above. Then:*

(i) *P_k has $2k$ roots, being 2 simple roots at $r_0 = \frac{1-k}{2}$ and $r_1 = \frac{k-1}{2}$ and $k-1$ roots of multiplicity two at $p_j = j - \frac{k}{2}$ for $j = 1, \ldots, k-1$. Moreover, $P_k'(r_0) > 0$, $P_k'(r_1) < 0$, $P_k'(p_j) = 0$ and $P_k''(p_j) > 0$ for every $j = 1, \ldots, k-1$.*

(ii) *P_∞ has infinitely many roots placed at $p_j = j$ for $j \in \mathbb{Z}$. Moreover $P'_\infty(p_j) = 0$ and $P''_\infty(p_j) > 0$. As consequence, each p_j is a root of multiplicity two.*

Proof. The placement of those roots follows easily from the factored form of P_k and the expression for P_∞ presented in (5.3) and (5.7).

For the derivatives part, if $k < \infty$, put $g(x) = \left(x + \frac{k-1}{2}\right)\left(x - \frac{k-1}{2}\right)$ and $h(x) = \prod_{i=1}^{k-1}\left(x - \left(i - \frac{k}{2}\right)\right)^2$, then $P_k(x) = -g(x)h(x)$ and

$$P'_k(x) = -g(x)h'(x) - g'(x)h(x) = -g(x)h'(x) - 2xh(x) \Rightarrow$$

$$\Rightarrow P'_k(r_0) = -2r_0 h(r_0) > 0 \text{ and } P'_k(r_1) = -2r_1 h(r_1) < 0.$$

Since p_j are roots of multiplicity two of $h(x)$, then $h(x) = h'(x) = 0$, which implies $P'_k(p_j) = 0$ and:

$$P''_k(p_j) = -g(p_j)h''(p_j).$$

Now, we have

$$h''(p_j) = 2\prod_{\substack{i=1\\ i\neq j}}^{k-1}\left(p_j - \left(i - \frac{k}{2}\right)\right)^2 > 0.$$

And, if $|x| < \frac{k-1}{2}$:

$$g(x) = \left(x - \frac{1-k}{2}\right)\left(x - \frac{k-1}{2}\right) < 0.$$

Thus, $P''_k(p_j) > 0$.

For P_∞, we have:

$$P'_\infty(x) = 2\,\mathrm{sen}(2\pi x) \Rightarrow P'_\infty(j) = 0, \forall j \in \mathbb{Z}$$

and

$$P''_\infty(x) = 4\cos(2\pi x) \Rightarrow P''_\infty(j) = 1 > 0, \forall j \in \mathbb{Z}.$$

□

Now, for every $k \geqslant 2$ or $k = \infty$, define:

$$Z_k(x,y) = \begin{cases} X_k(x,y) = (1, P'_k(x)), & \text{for } \quad y \geq 0 \\ Y_k(x,y) = (-1, P'_k(x)), & \text{for } \quad y \leq 0. \end{cases} \tag{5.8}$$

Lemma 5.3.2 (Tangencies of Z_k)**.** *Let Z_k be as above, then the following holds:*

- *If $k < \infty$ then $(r_0, 0)$ and $(r_1, 0)$ are crossing points of Z_k;*
- *The points $(p_j, 0)$ are visible-visible two folds of Z_k, $j = 1, \ldots, k-1$ (or $j \in \mathbb{Z}$, if $k = \infty$);*
- *$\Sigma^s \cup \Sigma^e = \emptyset$.*

Proof. The switching manifold is given by $\Sigma = f^{-1}(0)$, where $f(x,y) = y$, then $X_k f(r_0, 0).Y_k f(r_0, 0) = (P'_k(r_0))^2 > 0$. Analogous for $(r_1, 0)$.

From Lemma 5.3.1, $X_k f(p_j, 0) = Y_k f(p_j, 0) = P'_k(p_j) = 0$ and $X_k^2 f(p_j, 0) = Y_k^2 f(p_j, 0) = P''_k(p_j) > 0$. □

For each $k < \infty$, let

$$\gamma_k^X = \{(x, P_k(x)) \mid x \in [r_0, r_1]\} \text{ and } \gamma_k^Y = \{(x, -P_k(x)) \mid x \in [r_0, r_1]\}$$

and, for $k = \infty$,

$$\gamma_\infty^X = \{(x, P_\infty(x)) \mid x \in \mathbb{R}\} \text{ and } \gamma_\infty^Y = \{(x, -P_\infty(x)) \mid x \in \mathbb{R}\}.$$

Define $\Lambda_k = \gamma_k^X \cup \gamma_k^Y$ for both $k < \infty$ or $k = \infty$.

Proposition 5.3.1. *Λ_k is an invariant set for Z_k.*

Proof. Note that γ_k^X and γ_k^Y are integral curves of X_k and Y_k, respectively. Moreover, γ_k^X and γ_k^Y coincide at $(p_j, 0)$, for all j and at $(r_0, 0), (r_1, 0)$ (in the case $k < \infty$). By Definitions 2.1.2, 2.1.3 and the description of these points given by Lemma 5.3.2 we obtain that Λ_k is invariant. □

Our main goal is to prove the conjugacy between the time-one map of the fields Z_k restricted to Λ_k and a two-sided shift space. But, since a PSVF does not give uniqueness of trajectory through a point, a map $T_1 : \Lambda_k \to \Lambda_k, T_1(x) = \varphi(1, x)$, where φ is the flow, is not well-defined, for it may have more than one image (depending on the flow chosen). One way of avoiding this is to work with the space of all possible trajectories, so T_1 will be well-defined.

In that spirit, we will consider the set $\Omega_k = \{\gamma \text{ global trajectory of } Z_k \mid \gamma(0) \in \Lambda_k\}$ throughout this chapter and we re-define the *time-one map* as follows:

Definition 5.3.1. *We will call* time-one map *the function $T_1 : \Omega_k \to \Omega_k$, given by $T_1(\gamma)(.) = \gamma(. + 1)$.*

Proposition 5.3.2. *For any $k < \infty$ (or $k = \infty$), let $\gamma \in \Omega_k$ be a global trajectory, then for all $t \in \mathbb{R}$, there exists an unique $t^* \in [t, t+1)$ such that $\gamma(t^*) \in \{(p_j, 0)\}_{j=1}^{k-1}$ (respectively, $\gamma(t^*) \in \{(p_j, 0)\}_{j\in\mathbb{Z}}$).*

Proof. Follows immediately from the expression of Z_k and Lemmas 5.3.1 e 5.3.2. □

The region Λ_k can be partitioned into arcs that goes from p_j to the adjacent ones (p_{j+1} and p_{j-1}) or to itself (only when $k < \infty$). So, consider $k < \infty$ to be fixed and let $I_0 = \{(x, P_k(x)), x \in [r_0, p_1)\} \cup \{(x, -P_k(x)), x \in [r_0, p_1)\}$, i.e., the arc from p_1 to itself passing through r_0. For any $j = 1, \dots, k-2$, let $I_{2j-1} = \{(x, P_k(x)), x \in (p_j, p_{j+1})\}$ and $I_{2j} = \{(x, -P_k(x)), x \in (p_j, p_{j+1})\}$, i.e, the arcs from p_j to p_{j+1} and from p_{j+1} to p_j, respectively. And, $I_{2k-3} = \{(x, P_k(x)), x \in (p_{k-1}, r_1]\} \cup \{(x, -P_k(x)), x \in (p_{k-1}, r_1]\}$. In short, we enumerate these arcs top to bottom, left to right (see Figure 5.6).

For the case $k = \infty$ we will separate Λ_∞ into arcs, in the same way as we done for the finite case: let $I_{2j} = \{(x, P_\infty(x)) \mid j < x < j+1\}$ and $I_{2j+1} = \{(x, -P_\infty(x)) \mid j < x < j+1\}$.

Definition 5.3.2. *Let $s : \Omega_k \to \mathcal{A}_{2k-2}^{\mathbb{Z}}$, given by $s(\gamma) = (s_j(\gamma))_{j\in\mathbb{Z}}$ where:*

$$s_j(\gamma) = \begin{cases} n, & \textit{if} \quad \gamma(j) \in I_n \\ m, & \textit{if} \quad \gamma(j) \in \{(p_l, 0)\} \textit{ and } \gamma(j + 1/2) \in I_m \end{cases}$$

The sequence $s(\gamma)$ is called the itinerary *of γ.*

For $k < \infty$, it is clear that s is well-defined. For $k = \infty$ we have to analyse if $s(\gamma) \in \Theta_\infty$. From Definitions 2.1.2 and 2.1.3, of local and global trajectories, we see that if γ goes through some compartment I_{2l}, then the next one must be the one below it (I_{2l+1}) or the one to its right (I_{2l+2}) and if it goes through I_{2l+1} then the following one must be above it (I_{2l}) or the one to its left (I_{2l-1}). In any case, $0 < |s_j(\gamma) - s_{j+1}(\gamma)| \leqslant 2$, so $s(\gamma) \in \Theta_\infty$. From Proposition 5.3.2, $(s_j(\gamma))_{j\in\mathbb{Z}}$ encodes every compartment I_j that the trajectory γ visits in positive and negative time.

Note that, given $\gamma \in \Omega_k$, there exists an infinity amount of different trajectories with the same itinerary of γ, simply by changing the initial condition of it, without modifying the compartment it is located. To avoid such situation, we will consider two trajectories with the same itinerary to be equivalent. That is, we consider the equivalence relation:

Definition 5.3.3. *Let $\gamma_1, \gamma_2 \in \Omega_k$. We say $\gamma_1 \sim \gamma_2$ if and only if $s(\gamma_1) = s(\gamma_2)$. Denote $\overline{\Omega}_k = \Omega_k/\sim$.*

Remark 5.3.1. *Observe that, given $\overline{\gamma} \in \overline{\Omega}_k$, there exists a representative $\gamma^* \in \overline{\gamma}$ such that $\gamma^*(0) \in \{(p_j, 0), j = 1, \dots, k-1\}$, because if $\gamma(0) \in \{(p_j, 0), j = 1, \dots, k-1\}$, simply take $\gamma^* = \gamma$, if not, by Proposition 5.3.2 there exists unique $-1 < t^* < 0$ such that $\gamma(t^*) \in \{(p_j, 0), l = 1, \dots, k-1\}$. Moreover, if $s(\gamma) = (s_j)_{j\in\mathbb{Z}}$, then $\gamma((t^* + j, t^* + j + 1)) = I_{s_j}$. Which implies that $\gamma^*((j, j+1)) = I_{s_j}$ and, consequently, $\gamma^*(j + \frac{1}{2}) \in I_{s_j}$. Then $s(\gamma^*) = s(\gamma)$.*

In the following we construct a metric on the space $\overline{\Omega}_k$, but for this, we use the Hausdorff distance between two closed sets that is defined below:

Definition 5.3.4. *The Hausdorff distance between the sets A and B is given by*

$$d_H(A, B) = max\{\sup_{x\in A}\inf_{y\in B} d(x, y), \sup_{y\in B}\inf_{x\in A} d(x, y)\}$$

Definition 5.3.5. *Now, define $\rho_k : \overline{\Omega}_k \times \overline{\Omega}_k \to \mathbb{R}$, by*

$$\rho_k(\overline{\gamma}_1, \overline{\gamma}_2) = \sum_{i=-\infty}^{\infty} \frac{d_i(\overline{\gamma_1}, \overline{\gamma_2})}{2^{|i|}}$$

where $d_i(\overline{\gamma_1}, \overline{\gamma_2}) = d_H(\gamma_1^*([i, i+1]), \gamma_2^*([i, i+1]))$, d_H is the Hausdorff distance and γ_1^*, γ_2^* are those representatives given in the previous remark.

In order to simplify notation, in the sequel we only refer to $\gamma \in \overline{\Omega}_k$, meaning the equivalence class $\overline{\gamma}$ with the representative γ^*.

Proposition 5.3.3. *$(\overline{\Omega}_k, \rho_k)$ is a metric space.*

Proof. Let us show that ρ_k is well-defined, i.e., the series in the previous definition converges. For $k < \infty$, there exists $M > 0$ such that $d_i(\gamma_1, \gamma_2) \leqslant M$, for any $i \in \mathbb{Z}$, since $\gamma_1([i, i+1]), \gamma_2([i, i+1])$ are closed subsets of Λ_k which is compact. So $\rho_k(\gamma_1, \gamma_2) \leqslant M\left(1 + 2\sum\limits_{i=1}^{\infty}\frac{1}{2^i}\right) = 3M$, i.e. it converges for any $\gamma_1, \gamma_2 \in \overline{\Omega}_k$.

For the case $k = \infty$, note that, Λ_∞ is contained in the horizontal strip $\mathbb{R} \times [-2, 2]$. Then, for any $\gamma \in \overline{\Omega}_\infty$, the arcs $\gamma([i-1, i]), \gamma([i, i+1]) \subset [\lambda, \lambda + 2] \times [-2, 2]$ with diameter $2\sqrt{5}$. Then, for every $i \in \mathbb{Z}$:

$$d_H(\gamma([i-1, i]), \gamma([i, i+1])) \leqslant 2\sqrt{5}$$

Now, given $\gamma_1, \gamma_2 \in \overline{\Omega}_\infty$ for every $i \in \mathbb{Z}$:

$$\begin{aligned}
d_i(\gamma_1,\gamma_2) &= d_H(\gamma_1([i,i+1]),\gamma_2([i,i+1])) \\
&\leqslant d_H(\gamma_1([i,i+1]),\gamma_1([i-1,i])) + d_H(\gamma_1([i-1,i]),\gamma_2([i-1,i])) \\
&\qquad\qquad\qquad\qquad\qquad + d_H(\gamma_2([i-1,i]),\gamma_2([i,i+1])) \\
&\leqslant d_H(\gamma_1([i-1,i]),\gamma_2([i-1,i])) + 4\sqrt{5} \\
&\leqslant d_{i-1}(\gamma_1,\gamma_2) + 4\sqrt{5}
\end{aligned}$$

In an analogous way, can be shown that

$$d_{i-1}(\gamma_1,\gamma_2) \leqslant d_i(\gamma_1,\gamma_2) + 4\sqrt{5}$$

Recursively, follows that for every $i \in \mathbb{Z}$:

$$d_i(\gamma_1,\gamma_2) \leqslant d_0(\gamma_1,\gamma_2) + 4\sqrt{5}\,|i|$$

So we have

$$\begin{aligned}
\rho_\infty(\gamma_1,\gamma_2) = \sum_{i=-\infty}^{\infty} \frac{d_i(\gamma_1,\gamma_2)}{2^{|i|}} &\leqslant \sum_{i=-\infty}^{\infty} \frac{d_0(\gamma_1,\gamma_2) + 4\sqrt{5}\,|i|}{2^{|i|}} \\
&\leqslant 3d_0(\gamma_1,\gamma_2) + 8\sqrt{5}\sum_{i=1}^{\infty}\frac{i}{2^i} < \infty.
\end{aligned}$$

So ρ_k is well defined for every $k \in \mathbb{N} \cup \{\infty\}$. Let us show it is a metric: every summand in the definition of ρ_k is non-negative, then $\rho_k(\gamma_1,\gamma_2) = 0$ if and only if $d_i(\gamma_1,\gamma_2) = 0$ for every $i \in \mathbb{Z}$. Hence $\gamma_1([i,i+1]) = \gamma_2([i,i+1])$, for all $i \in \mathbb{Z}$. Then $\gamma_1 = \gamma_2$.

From $d_i(\gamma_1,\gamma_2) = d_i(\gamma_2,\gamma_1)$, we obtain $\rho_k(\gamma_1,\gamma_2) = \rho_k(\gamma_2,\gamma_1)$.

Now, for every $i \in \mathbb{Z}$, we get:

$$d_i(\gamma_1,\gamma_2) \leqslant d_i(\gamma_1,\gamma_3) + d_i(\gamma_3,\gamma_2) \Rightarrow$$

$$\sum_{i=-N}^{N} \frac{d_i(\gamma_1,\gamma_2)}{2^{|i|}} \leqslant \sum_{i=-N}^{N} \frac{d_i(\gamma_1,\gamma_3)}{2^{|i|}} + \sum_{i=-N}^{N} \frac{d_i(\gamma_3,\gamma_2)}{2^{|i|}}, \forall N \Rightarrow$$

$$\rho_k(\gamma_1,\gamma_2) \leqslant \rho_k(\gamma_1,\gamma_3) + \rho_k(\gamma_3,\gamma_2)$$

Therefore, $(\overline{\Omega}_k, \rho_k)$ is a metric space. □

Let $\overline{T_1} : \overline{\Omega}_k \to \overline{\Omega}_k$ the function induced by T_1, that is, $\overline{T_1}(\overline{\gamma}) = \overline{T_1(\gamma)}$. It does not depend on the representative, for if $s(\gamma_1) = s(\gamma_2) = (s_j)_{j\in\mathbb{Z}}$, then, for all $j \in \mathbb{Z}$:

$$\gamma_1(j), \gamma_2(j) \in I_{s_j} \Rightarrow \gamma_1(j+1), \gamma_2(j+1) \in I_{s_{j+1}}$$

$$\Rightarrow T_1(\gamma_1)(j), T_1(\gamma_2)(j) \in I_{s_{j+1}} \Rightarrow s(T_1(\gamma_1)) = s(T_1(\gamma_2)).$$

Proposition 5.3.4. *The function $\overline{T_1}$ given above is a homeomorphism.*

Proof. The function T_1 clearly is invertible, with inverse $(T_1)^{-1}(\gamma)(.) = \gamma(. - 1)$, and it is straightforward to see that $\overline{T_1}^{-1} = \overline{T_1^{-1}}$.

Now,

$$d_i(\overline{T_1}(\gamma_1), \overline{T_1}(\gamma_2)) = d_H(T_1(\gamma_1)([i,i+1]), T_1(\gamma_2)([i,i+1])) =$$

$$= d_H(\gamma_1([i+1,i+2]),\gamma_2([i+1,i+2])) = d_{i+1}(\gamma_1,\gamma_2).$$

Then

$$\begin{aligned}
\rho(\overline{T_1}(\gamma_1),\overline{T_1}(\gamma_2)) &= \sum_{i=-\infty}^{\infty} \frac{d_i(T_1(\gamma_1),T_1(\gamma_2))}{2^{|i|}} = \sum_{i=-\infty}^{\infty} \frac{d_{i+1}(\gamma_1,\gamma_2)}{2^{|i|}} = \\
&= \lim_{k\to\infty}\left(\sum_{i=0}^{k} \frac{d_{i+1}(\gamma_1,\gamma_2)}{2^{i}} + \sum_{i=-k}^{-1} \frac{d_{i+1}(\gamma_1,\gamma_2)}{2^{-i}}\right) = \\
&= \lim_{k\to\infty}\left(2\sum_{i=0}^{k} \frac{d_{i+1}(\gamma_1,\gamma_2)}{2^{i+1}} + \frac{1}{2}\sum_{i=-k}^{-1} \frac{d_{i+1}(\gamma_1,\gamma_2)}{2^{-i-1}}\right) \leqslant \\
&\leqslant 2\lim_{k\to\infty}\left(\sum_{j=1}^{k+1} \frac{d_{j}(\gamma_1,\gamma_2)}{2^{j}} + \sum_{j=-k+1}^{0} \frac{d_{j}(\gamma_1,\gamma_2)}{2^{-j}}\right) = \\
&\leqslant 2\sum_{j=-\infty}^{\infty} \frac{d_j(\gamma_1,\gamma_2)}{2^{|j|}} = 2\rho(\gamma_1,\gamma_2).
\end{aligned}$$

Hence T_1 is continuous. The proof of continuity of the inverse is analogous. □

Now let $\overline{s} : \overline{\Omega}_k \to \{0,1,\ldots,2k-3\}^{\mathbb{Z}}$ the function induced by s, that is, $\overline{s}(\gamma) = s(\gamma)$. It does not depend on the representative chosen, because of the equivalence relation and it is one-to-one.

Proposition 5.3.5. *The map $\overline{s}$ is a homeomorphism onto its image.*

Proof. **Case** $k < \infty$ Put $\mu = \min\{d_H(I_l,I_j) \mid l \neq j, \text{and } l,j = 1,\ldots,2k-3\}$ and $\gamma_1,\gamma_2 \in \overline{\Omega}_k$. Suppose $\rho_k(\gamma_1,\gamma_2) < \frac{\mu}{2^N}$. Then, for any $-N \leqslant i \leqslant N$:

$$d_i(\gamma_1,\gamma_2) < \mu$$

because, on the contrary, we have $\rho(\gamma_1,\gamma_2) = \sum \frac{d_i(\gamma_1,\gamma_2)}{2^{|i|}} \geqslant \frac{\mu}{2^N}$.

Now

$$d_i(\gamma_1,\gamma_2) < \mu \Rightarrow d_i(\gamma_1,\gamma_2) = 0$$

and therefore, $\gamma_1((i,i+1)) = \gamma_2((i,i+1))$ implying that $s_i(\gamma_1) = s_i(\gamma_2)$, for all $-N \leqslant i \leqslant N$. So

$$\begin{aligned}
d(s(\gamma_1),s(\gamma_2)) &= \sum_{i\in\mathbb{Z}} \frac{|s_i(\gamma_1)-s_i(\gamma_2)|}{2^{|i|}} \\
&= \sum_{\substack{i\in\mathbb{Z}\\ |i|>N}} \frac{|s_i(\gamma_1)-s_i(\gamma_2)|}{2^{|i|}} \leqslant \frac{2k-3}{2^{N-1}}.
\end{aligned}$$

This proves that $\overline{s}$ is continuous.

The same argument reverses itself in order to show $\overline{s}$ is open: let $\gamma_1,\gamma_2 \in \overline{\Omega}_k$, and $N \in \mathbb{N}$

$$d(\overline{s}(\gamma_1),\overline{s}(\gamma_2)) < \frac{1}{2^N} \Rightarrow s_i(\gamma_1) = s_i(\gamma_2), \forall -N \leqslant i \leqslant N.$$

Then $\gamma_1([i,i+1]) = \gamma_2([i,i+1])$, for all $-N \leqslant i \leqslant N$. Hence

$$\rho(\gamma_1,\gamma_2) = \sum_{\substack{i\in\mathbb{Z}\\ |i|>N}} \frac{d_i(\gamma_1,\gamma_2)}{2^{|i|}} \leqslant \frac{M}{2^{N-1}},$$

where $M > 0$, such that $\operatorname{diam}(\Lambda_k) < M$.

Case $k = \infty$ Again, let $\mu = \min\{d_H(I_l, I_j) \mid l \neq j, \text{ and } l, j \in \mathbb{Z}\}$ and $\gamma_1, \gamma_2 \in \overline{\Omega}_\infty$, such that $\rho(\gamma_1, \gamma_2) < \frac{\mu}{2^N}$. Then, for any $-N \leqslant i \leqslant N$ we get:

$$d_i(\gamma_1, \gamma_2) < \mu \Rightarrow d_i(\gamma_1, \gamma_2) = 0,$$

which implies $s_i(\gamma_1) = s_i(\gamma_2)$, for all $-N \leqslant i \leqslant N$. So

$$\begin{aligned} d(s(\gamma_1), s(\gamma_2)) &= \sum_{\substack{i \in \mathbb{Z} \\ |i| > N}} \frac{|s_i(\gamma_1) - s_i(\gamma_2)|}{2^{|i|}} \\ &\leqslant \sum_{\substack{i \in \mathbb{Z} \\ |i| > N}} \frac{|s_0(\gamma_1) - s_0(\gamma_2)| + 4|i|}{2^{|i|}} \\ &\leqslant 2 \sum_{i=N+1}^{\infty} \frac{4i}{2^i} = \frac{1}{2^{N-4}}. \end{aligned}$$

If $\gamma_1, \gamma_2 \in \overline{\Omega}_\infty$ and $N \in \mathbb{N}$ are such that $d(\overline{s}(\gamma_1), \overline{s}(\gamma_2)) < \frac{1}{2^N}$, then for all $-N \leqslant i \leqslant N$:

$$s_i(\gamma_1) = s_i(\gamma_2) \Rightarrow \gamma_1([i, i+1]) = \gamma_2([i, i+1]).$$

Hence

$$\begin{aligned} \rho(\gamma_1, \gamma_2) &= \sum_{\substack{i \in \mathbb{Z} \\ |i| > N}} \frac{d_i(\gamma_1, \gamma_2)}{2^{|i|}} \leqslant \sum_{\substack{i \in \mathbb{Z} \\ |i| > N}} \frac{d_0(\gamma_1, \gamma_2) + 4\sqrt{5}|i|}{2^{|i|}} \\ &\leqslant \frac{8\sqrt{5}}{2^N} \sum_{i=1}^{\infty} \frac{i}{2^i} = \frac{\sqrt{5}}{2^{N-4}}. \end{aligned}$$

Therefore $\overline{s}$ is a homeomorphism over its image. $\square$

Now we are in conditions to prove items (i) to (iii) of Theorem 5.2.1.

5.3.1 Proof of item *(i)* of Theorem 5.2.1

Proposition 5.3.6. *The function $\overline{s} : \overline{\Omega}_2 \to \{0,1\}^{\mathbb{Z}}$ is a conjugation between $\overline{T_1}$ and σ, i.e., $\overline{s} \circ \overline{T_1} = \sigma \circ \overline{s}$ (see Figure 5.2).*

$$\begin{tikzcd} \overline{\Omega}_2 \arrow[r, "\overline{T_1}"] \arrow[d, "\overline{s}"] & \overline{\Omega}_2 \arrow[d, "\overline{s}"] \\ \{0,1\}^{\mathbb{Z}} \arrow[r, "\sigma"] & \{0,1\}^{\mathbb{Z}} \end{tikzcd}$$

Figure 5.2:

Proof. Let us show that $\overline{s}(\overline{\Omega}_2) = \{0,1\}^{\mathbb{Z}}$. Given $(s_j) \in \{0,1\}^{\mathbb{Z}}$, construct γ by concatenating the arcs I_0 and I_1 according to (s_j), so γ is a trajectory with $\gamma(0) = p_1$, and $\gamma((0,1)) = I_{s_0}$. Then $\gamma(1) = p_1$ and $\gamma((1,2)) = I_{s_1}$. In general, for all $j \in \mathbb{Z}$, $\gamma(j) = p_1$ and $\gamma((j,j+1)) = I_{s_j}$. By Definition 2.1.3, γ is a global trajectory of Z_2 and therefore $\gamma \in \Omega_2$. Moreover, $s(\gamma) = \overline{s}(\gamma) = (s_j)_{j\in\mathbb{Z}}$.

Now, for the commutative part, let $\gamma \in \overline{\Omega}_2$, $(a_j)_{j\in\mathbb{Z}} = \overline{s}(\gamma)$, and $(b_j)_{j\in\mathbb{Z}} = \overline{s}(\overline{T_1}(\gamma))$, then:

$$b_j = \begin{cases} 0, & \text{if} \quad \overline{T_1}(\gamma)(j) \in I_0 \\ 1, & \text{if} \quad \overline{T_1}(\gamma)(j) \in I_1 \end{cases} = \begin{cases} 0, & \text{if} \quad \gamma(j+1) \in I_0 \\ 1, & \text{if} \quad \gamma(j+1) \in I_1 \end{cases} = a_{j+1} \Rightarrow$$

$\Rightarrow (b_j)_{j\in\mathbb{Z}} = \sigma((a_j)_{j\in\mathbb{Z}})$, i.e., $\overline{s} \circ \overline{T_1} = \sigma \circ \overline{s}$. □

5.3.2 Proof of item *(ii)* of Theorem 5.2.1

Proposition 5.3.7. *The function $\overline{s} : \overline{\Omega}_k \to \overline{s}(\overline{\Omega}_k)$ is a conjugation between $\overline{T_1}$ and σ, i.e., $\overline{s} \circ \overline{T_1} = \sigma \circ \overline{s}$ (see Figure 5.3).*

$$\begin{tikzcd} \overline{\Omega}_k \arrow[r, "\overline{T_1}"] \arrow[d, "\overline{s}"] & \overline{\Omega}_k \arrow[d, "\overline{s}"] \\ \overline{s}(\overline{\Omega}_k) \arrow[r, "\sigma"] & \overline{s}(\overline{\Omega}_k) \end{tikzcd}$$

Figure 5.3:

Proof. Let $\gamma \in \overline{\Omega}_k$, $(a_j)_{j\in\mathbb{Z}} = \overline{s}(\gamma)$, and $(b_j)_{j\in\mathbb{Z}} = \overline{s}(\overline{T_1}(\gamma))$. Then:

$$\begin{aligned} b_j &= m, \quad \text{if} \quad \overline{T_1}(\gamma)(j) \in I_m \\ &= m, \quad \text{if} \quad \gamma(j+1) \in I_m \\ &= a_{j+1} \Rightarrow \end{aligned}$$

$\Rightarrow (b_j)_{j\in\mathbb{Z}} = \sigma((a_j)_{j\in\mathbb{Z}})$.

It remains to show that $\overline{s}(\overline{\Omega}_k)$ is a subshift. We have that $\sigma = \overline{s} \circ \overline{T_1} \circ \overline{s}^{-1}$, implies $\overline{s}(\overline{\Omega}_k)$ is invariant for σ.

Now, we show $\overline{s}(\overline{\Omega}_k)$ is closed in $\mathcal{A}_{2k-2}^{\mathbb{Z}}$. Let $(a_j) \in \left(\mathcal{A}_{2k-2}^{\mathbb{Z}} \setminus \overline{s}(\overline{\Omega}_k)\right)$, this means that there exists a finite word $a_\alpha a_{\alpha+1} \ldots a_{\alpha+l}$ that corresponds to a sequence of arcs $I_{a_\alpha} \ldots I_{a_{\alpha+l}}$ that does not form a trajectory of Z_k. Since every sequence in the cylinder $[a_\alpha \ldots a_{\alpha+l}]$ contains this finite word, we have that $[a_\alpha \ldots a_{\alpha+l}] \subset \left(\mathcal{A}_{2k-2}^{\mathbb{Z}} \setminus \overline{s}(\overline{\Omega}_k)\right)$. Thus $\overline{s}(\overline{\Omega}_k)$ is closed.

Hence $\overline{s}$ is a conjugation between both systems. □

5.3.3 Proof of item *(iii)* of Theorem 5.2.1

Proposition 5.3.8. *The function $\overline{s} : \overline{\Omega}_\infty \to \overline{s}(\overline{\Omega}_\infty) \subset \Theta_\infty$ is a conjugation between $\overline{T_1}$ and σ, i.e., $\overline{s} \circ \overline{T_1} = \sigma \circ \overline{s}$ (see Figure 5.4).*

$$\begin{tikzcd}
\overline{\Omega}_\infty \arrow[r, "\overline{T_1}"] \arrow[d, "\overline{s}"] & \overline{\Omega}_\infty \arrow[d, "\overline{s}"] \\
\Theta_\infty \arrow[r, "\sigma"] & \Theta_\infty
\end{tikzcd}$$

Figure 5.4:

Proof. Let $\gamma \in \overline{\Omega}_\infty$, $(a_j)_{j\in\mathbb{Z}} = \overline{s}(\gamma)$, and $(b_j)_{j\in\mathbb{Z}} = \overline{s}(\overline{T_1}(\gamma))$. Then:

$$\begin{aligned} b_j &= m, \quad \text{if} \quad \overline{T_1}(\gamma)(j) \in I_m \\ &= m, \quad \text{if} \quad \gamma(j+1) \in I_m \\ &= a_{j+1} \Rightarrow \end{aligned}$$

$\Rightarrow (b_j)_{j\in\mathbb{Z}} = \sigma((a_j)_{j\in\mathbb{Z}})$.

By Proposition 5.3.5, there exists $\overline{s}^{-1} : \overline{s}(\overline{\Omega}_\infty) \to \overline{\Omega}_\infty$ and it is continuous and then $\sigma = \overline{s} \circ \overline{T_1} \circ \overline{s}^{-1}$, which proves that $\sigma(\overline{s}(\overline{\Omega}_\infty)) = \overline{s}(\overline{\Omega}_\infty)$. $\square$

5.3.4 Proof of item *(iv)* of Theorem 5.2.1

Consider the PSVF

$$Z(x,y) = \begin{cases} X(x,y) = (1, -2x), & \text{for} \quad y \geq 0 \\ Y(x,y) = (-2, -4x^3 + 2x), & \text{for} \quad y \leq 0. \end{cases} \tag{5.9}$$

This PSVF has a compact invariant set $\Lambda = \{(x,y) \in \mathbb{R}^2 : -1 \leqslant x \leqslant 1 \text{ and } x^4/2 + x^2/2 \leqslant y \leqslant 1 - x^2\}$. Furthermore, Λ is a chaotic non-trivial minimal set for Z (see [5]). As before, we will consider $\Omega = \{\gamma \text{ global trajectory of } Z|_\Lambda\}$ the set of all possible trajectories contained in the set Λ.

In cases *(i)-(iii)* of Theorem 5.2.1, one could consider the set $\{p_j\}$, as a "recurrent" set, in the sense that every trajectory visits it and returns to it infinitely many times. Moreover, the time between visits is exactly 1 (see Proposition 5.3.2), and that was the basic setting that allowed the previous constructions. In this case we consider the set $K = \{0\} \times (0,1]$, since every local trajectory intersects it transversally, and every point of K reaches $p = (0,0)$ in finite time (positive and negative), thus every trajectory visits it infinitely many times. Thus, the set K will play the role of $\{p_j\}$ in previous cases. This time, we cannot adjust the expression of Z in order to make the time between visits equal some constant. So we define the function: $\eta : \Omega \to \mathbb{R}$, as $\eta(\gamma) = \min\{t > 0 \mid \gamma(t) \in K\}$ to be the least positive value of t for which $\gamma(t) \in K$. For simplicity we may denote $\eta_\gamma = \eta(\gamma)$.

Now define the map $\mathcal{T} : \Omega \to \Omega$, as $\mathcal{T}(\gamma)(.) = \gamma(. + \eta_\gamma)$. This map takes a trajectory to another one that starts one loop ahead.

Lemma 5.3.3. *Given a global trajectory $\gamma \in \Omega$, there exists a bi-infinite increasing sequence $(t_j^\gamma)_{j\in\mathbb{Z}}$, such that $\gamma(t_j^\gamma) \in K$.*

Proof. We define $t_0 = \max\{t \leqslant 0 \mid \gamma(t) \in K\}$, and for $j > 0$, let $t_j^\gamma = \sum_{k=0}^{j-1} \eta_{\mathcal{T}^k(\gamma)}$. For $j < 0$, consider $\widetilde{\eta}(\gamma) = \max\{t < 0 \mid \gamma(t) \in K\}$, and the analogous for $\mathcal{T}$, that is, $\widetilde{\mathcal{T}} : \Omega \to \Omega$, $\widetilde{\mathcal{T}}(\gamma)(.) = \gamma(. + \widetilde{\eta}_\gamma)$. Then and $t_j = t_0 + \sum_{k=0}^{-j-1} \widetilde{\eta}_{\widetilde{\mathcal{T}}^k(\gamma)}$.

Given the sequence constructed above, $\gamma(t_0^\gamma) \in K$,

$$\gamma(t_1^\gamma) = \gamma(\eta_\gamma) = \mathcal{T}(\gamma)(0) \in K$$

When $j > 1$, we get

$$\begin{aligned}\gamma(t_j^\gamma) &= \gamma(\eta_\gamma + \eta_{\mathcal{T}(\gamma)} + \ldots + \eta_{\mathcal{T}^{j-1}(\gamma)}) = \mathcal{T}(\gamma)(\eta_{\mathcal{T}(\gamma)} + \ldots + \eta_{\mathcal{T}^{j-1}(\gamma)}) = \\ &= \mathcal{T}^2(\gamma)(\eta_{\mathcal{T}^2(\gamma)} + \ldots + \eta_{\mathcal{T}^{j-1}(\gamma)}) = \ldots = \mathcal{T}^{j-1}(\gamma)(\eta_{\mathcal{T}^{j-1}(\gamma)}) = \\ &= \mathcal{T}^j(\gamma)(0) \in K.\end{aligned}$$

For $j < 0$ it is analogous. □

Lemma 5.3.4. *Given a global trajectory $\gamma \in \Omega$ and (t_j^γ) given above, consider the sequence $(t_j^{\mathcal{T}(\gamma)})$. It holds $t_j^{\mathcal{T}(\gamma)} = t_{j+1}^\gamma - \eta_\gamma$.*

Proof. In fact,

$$t_0^{\mathcal{T}(\gamma)} = 0 = \eta_\gamma - \eta_\gamma = t_1^\gamma - \eta_\gamma.$$

For $j > 0$:

$$t_j^{\mathcal{T}(\gamma)} = \sum_{k=0}^{j-1} \eta_{\mathcal{T}^{k+1}(\gamma)} = \sum_{k=0}^{j} \eta_{\mathcal{T}^k(\gamma)} - \eta_\gamma = t_{j+1}^\gamma - \eta_\gamma.$$

And if $j < 0$:

$$t_j^{\mathcal{T}(\gamma)} = t_0 + \sum_{k=0}^{-j-1} \widetilde{\eta}_{\widetilde{\mathcal{T}}^k(\mathcal{T}(\gamma))} = \sum_{k=0}^{j} \eta_{\mathcal{T}^k(\gamma)} - \eta_\gamma = t_{j+1}^\gamma - \eta_\gamma.$$

□

Definition 5.3.6. *Consider $y : K \to (0, 1]$ the projection on y-coordinate (that is $y(0, \beta) = \beta$), and define the* itinerary map $s : \Omega \to (0, 1]^{\mathbb{Z}}$, *as $s(\gamma) = (s_j(\gamma))_{j\in\mathbb{Z}}$, where $s_j(\gamma) = y(\gamma(t_j^\gamma))$. In this way, every single trajectory of Ω can be encoded by the y-coordinates of its beats on K.*

Clearly s is well-defined, by construction and it is onto, because given a sequence $(x_j)_{j\in\mathbb{Z}} \in (0, 1]^{\mathbb{Z}}$ it is possible to construct a trajectory $\gamma \in \Omega$ by concatenating the correct arcs in order to get $s(\gamma) = (x_j)_{j\in\mathbb{Z}}$.

As in previous cases, there are infinitely many trajectories that describe the same curve, simply by changing the initial condition (a shift in time), and the function s takes all these trajectories to the same sequence. So, we consider the set $\overline{\Omega} = {}^{\Omega}\!/_{s}$, that is, two trajectories γ_1, γ_2 are equivalent if $s(\gamma_1) = s(\gamma_2)$.

Remark 5.3.2. *Every trajectory $\gamma \in \Omega$ is equivalent to another one γ^* such that $\gamma^*(0) \in K$.*

Remark 5.3.3. *Another way of avoiding the infinitely many trajectories that looks like the same would be to restrict our function to a subset $\Omega_* \subset \Omega$, where $\Omega_* = \{\gamma$ global trajectory of $Z|_\Lambda \mid \gamma(0) \in K\}$, but we chose to work in a more general setting, without restricting the dynamics as it was done in cases* (i)-(iii) *of Theorem 5.2.1.*

Definition 5.3.7. *Let $\overline{\gamma_1}, \overline{\gamma_2} \in \overline{\Omega}$, define $\rho : \overline{\Omega} \times \overline{\Omega} \to \mathbb{R}$, by*

$$\rho(\overline{\gamma_1}, \overline{\gamma_2}) = \sum_{i \in \mathbb{Z}} \frac{d_i(\gamma_1^*, \gamma_2^*)}{2^{|i|}}$$

where $d_i(\gamma_1^*, \gamma_2^*) = d_H(\gamma_1^*([t_i^{\gamma_1^*}, t_{i+1}^{\gamma_1^*}]), \gamma_2^*([t_i^{\gamma_2^*}, t_{i+1}^{\gamma_2^*}]))$. That is, at each step i, we take the Hausdorff distance between the i-th loops of both trajectories.

Proposition 5.3.9. *$(\overline{\Omega}, \rho)$ is a metric space.*

Proof. The function ρ defined above is well-defined, since Λ is a bounded set, which implies there exists $C > 0$, such that $d_i(\gamma_1^*, \gamma_2^*) < C$, for all $i \in \mathbb{Z}$.

The proof of those properties for it to be a metric is analogous to the one given in Proposition 5.3.3. □

Let $\overline{\mathcal{T}} : \overline{\Omega} \to \overline{\Omega}$ the function induced by $\mathcal{T}$, that is, $\overline{\mathcal{T}}(\overline{\gamma}) = \overline{\mathcal{T}(\gamma)}$. It does not depend on the representative, for if $s(\gamma_1) = s(\gamma_2) = (s_j)_{j \in \mathbb{Z}}$, then, for all $j \in \mathbb{Z}, y(\gamma_1(t_j^{\gamma_1})) = y(\gamma_2(t_j^{\gamma_2})) = s_j$.

$$\begin{aligned} y(\mathcal{T}(\gamma_1)(t_j^{\mathcal{T}(\gamma_1)})) &= y(\gamma_1(t_j^{\mathcal{T}(\gamma_1)} + \eta_{\gamma_1})) = y(\gamma_1(t_{j+1}^{\gamma_1} - \eta_{\gamma_1} + \eta_{\gamma_1})) \\ &= y(\gamma_1(t_{j+1}^{\gamma_1})) = y(\gamma_2(t_{j+1}^{\gamma_2})) = y(\mathcal{T}(\gamma_2)(t_j^{\mathcal{T}(\gamma_2)})) \\ &\Rightarrow s(\mathcal{T}(\gamma_1)) = s(\mathcal{T}(\gamma_2)). \end{aligned}$$

Proposition 5.3.10. *The function $\overline{\mathcal{T}} : \overline{\Omega} \to \overline{\Omega}$ is a homeomorphism.*

Proof. The function $\mathcal{T}$ clearly is invertible, with inverse $(\mathcal{T})^{-1}(\gamma)(.) = \gamma(. - \eta_\gamma)$, and it is straightforward to see that $(\overline{\mathcal{T}})^{-1} = \overline{\mathcal{T}^{-1}}$.

Now,

$$\begin{aligned} d_i(\mathcal{T}(\gamma_1), \mathcal{T}(\gamma_2)) &= d_H(\mathcal{T}(\gamma_1)([t_i^{\mathcal{T}(\gamma_1)}, t_{i+1}^{\mathcal{T}(\gamma_1)}]), \mathcal{T}(\gamma_2)([t_i^{\mathcal{T}(\gamma_2)}, t_{i+1}^{\mathcal{T}(\gamma_2)}])) \\ &= d_H(\gamma_1([t_{i+1}^{\gamma_1}, t_{i+2}^{\gamma_1}]), \gamma_2([t_{i+1}^{\gamma_2}, t_{i+2}^{\gamma_2}])) = d_{i+1}(\gamma_1, \gamma_2). \end{aligned}$$

Then:

$$\begin{aligned} \rho(\overline{\mathcal{T}}(\gamma_1), \overline{\mathcal{T}}(\gamma_2)) &= \sum_{i=-\infty}^{\infty} \frac{d_i(\mathcal{T}(\gamma_1), \mathcal{T}(\gamma_2))}{2^{|i|}} = \sum_{i=-\infty}^{\infty} \frac{d_{i+1}(\gamma_1, \gamma_2)}{2^{|i|}} = \\ &= \lim_{k \to \infty} \left(\sum_{i=0}^{k} \frac{d_{i+1}(\gamma_1, \gamma_2)}{2^i} + \sum_{i=-k}^{-1} \frac{d_{i+1}(\gamma_1, \gamma_2)}{2^{-i}} \right) = \\ &= \lim_{k \to \infty} \left(2 \sum_{i=0}^{k} \frac{d_{i+1}(\gamma_1, \gamma_2)}{2^{i+1}} + \frac{1}{2} \sum_{i=-k}^{-1} \frac{d_{i+1}(\gamma_1, \gamma_2)}{2^{-i-1}} \right) \leqslant \\ &\leqslant 2 \lim_{k \to \infty} \left(\sum_{j=1}^{k+1} \frac{d_j(\gamma_1, \gamma_2)}{2^j} + \sum_{j=-k+1}^{0} \frac{d_j(\gamma_1, \gamma_2)}{2^{-j}} \right) = \\ &\leqslant 2 \sum_{j=-\infty}^{\infty} \frac{d_j(\gamma_1, \gamma_2)}{2^{|j|}} = 2\rho(\gamma_1, \gamma_2). \end{aligned}$$

Hence $\mathcal{T}$ is continuous. The proof of continuity of the inverse is analogous. □

On the space $(0, 1]^{\mathbb{Z}}$ we consider the same metric from those spaces before, that is, given $(x_j)_{j \in \mathbb{Z}}, (y_j)_{j \in \mathbb{Z}} \in (0, 1]^{\mathbb{Z}}$ the distance between them is the real number $d(x, y) = \sum_{j \in \mathbb{Z}} \frac{|x_j - y_j|}{2^{|j|}}$.

Proposition 5.3.11. *The function* $\overline{s} : \overline{\Omega} \to (0,1]^{\mathbb{Z}}$ *is a homeomorphism.*

Proof. The function s is onto, which implies that $\overline{s}$ is onto and clearly it is injective because it is defined on the quotient ${}^{\Omega}\!/_{S}$. So it remains to show the continuity part.

Note that $|s_j(\gamma_1) - s_j(\gamma_2)| = ||\gamma_1(t_j^{\gamma_1}) - \gamma_2(t_j^{\gamma_2})|| \leqslant d_j(\gamma_1, \gamma_2)$ for all $j \in \mathbb{Z}$. Then:

$$d(s(\gamma_1), s(\gamma_2)) = \sum_{j\in\mathbb{Z}} \frac{|s_j(\gamma_1) - s_j(\gamma_2)|}{2^{|j|}} \leqslant \sum_{j\in\mathbb{Z}} \frac{d_j(\gamma_1,\gamma_2)}{2^{|j|}} = \rho(\gamma_1, \gamma_2).$$

Thus $\overline{s}$ is continuous.

The same argument reverses itself in order to show $\overline{s}$ is open: let $\gamma_1, \gamma_2 \in \overline{\Omega}$, and $\varepsilon > 0$ such that

$$d(\overline{s}(\gamma_1), \overline{s}(\gamma_2)) < \varepsilon.$$

Then $d_i(\gamma_1, \gamma_2) < M\varepsilon$, where $M > 0$ is such that $\operatorname{diam}(\Lambda) < M$.

Hence

$$\rho(\gamma_1, \gamma_2) = \sum_{i\in\mathbb{Z}} \frac{d_i(\gamma_1,\gamma_2)}{2^{|i|}} \leqslant 3M\varepsilon.$$

□

Proposition 5.3.12. *Let* $\sigma : (0,1]^{\mathbb{Z}} \to (0,1]^{\mathbb{Z}}$ *be the shift map. Then* $s \circ \mathcal{T} = \sigma \circ s$*, i.e., the following diagram commutes (Figure 5.5):*

$$\begin{tikzcd}
\Omega \arrow[r, "\mathcal{T}"] \arrow[d, "s"] & \Omega \arrow[d, "s"] \\
(0,1]^{\mathbb{Z}} \arrow[r, "\sigma"] & (0,1]^{\mathbb{Z}}
\end{tikzcd}$$

Figure 5.5:

Proof. Let $\gamma \in \Omega$, $(a_j)_{j\in\mathbb{Z}} = s(\gamma)$, and $(b_j)_{j\in\mathbb{Z}} = s(\mathcal{T}(\gamma))$. Then:

$$\begin{aligned} b_j &= s_j(\mathcal{T}(\gamma)) = y(\mathcal{T}(\gamma)(t_j^{\mathcal{T}(\gamma)})) = y(\gamma(t_j^{\mathcal{T}(\gamma)} + \eta_\gamma)) \\ &= y(\gamma(t_{j+1}^{\gamma} - \eta_\gamma + \eta_\gamma)) = s_{j+1}(\gamma) = a_{j+1} \end{aligned}$$

□

5.4 Proof of Theorem 5.2.2

Definition 5.4.1. *Two arcs* $\overline{A\,B}$ *and* $\overline{C\,D}$ *are parameterized by an* arc length parametrization *if there exists a homeomorphism* $h : \overline{A\,B} \to \overline{C\,D}$ *such that, for all* $x \in \overline{A\,B}$*, then* $h(x)$ *is the point on* $\overline{C\,D}$ *satisfying*

$$\frac{\textit{length}\ (\overline{A\,x})}{\textit{length}\ (\overline{A\,B})} = \frac{\textit{length}\ (\overline{C\,h(x)})}{\textit{length}\ (\overline{C\,D})}.$$

Proposition 5.4.1. *(i) Let $\widetilde{Z_k}$ (respectively $\widetilde{Z_\infty}$) be a PSVF presenting a $(k-1)$-homoclinic loop in the set $\widetilde{\Lambda_k}$ (respectively ∞-homoclinic loop in the set $\widetilde{\Lambda_\infty}$). Then Z_k and $\widetilde{Z_k}$ restricted to Λ_k and $\widetilde{\Lambda_k}$, respectively, are Σ-equivalent (respectively Z_∞ and $\widetilde{Z_\infty}$).*

(ii) The PSVF Z of Theorem A, restrict to Λ, is Σ-equivalent to any PSVF $\widetilde{Z}$ presenting a compact region $\widetilde{\Lambda}$ bounded by a trajectory of $\widetilde{Z}$ passing through a invisible-visible two-fold $\widetilde{p}$. Moreover, except for $\widetilde{p}$, the PSVF $\widetilde{Z}$ has just more two invisible tangential singularities.

Proof. (i) • Consider Z_2 and $\widetilde{Z_2}$ restricted to Λ_2 and $\widetilde{\Lambda}_2$. Take $h : \Lambda_2 \to \widetilde{\Lambda}_2$ such that: $h(p_1) = \widetilde{p}_1$. Now, consider the positive arc of trajectory γ_X^{+,p_1} of X starting at p_1 and finishing at r_1. Analogously, consider the positive arc of trajectory $\widetilde{\gamma}_{\widetilde{X}}^{+,\widetilde{p}_1}$ of $\widetilde{X}$ starting at $\widetilde{p}_1$and finishing at $\widetilde{r}_1$. Using the arc length parameterization we get $h(\gamma_X^{+,p_1}) = \widetilde{\gamma}_{\widetilde{X}}^{+,\widetilde{p}_1}$. Now, it is enough repeat this procedure and extend h to γ_X^{-,p_1}, γ_Y^{+,p_1} and γ_Y^{-,p_1}.

• For Z_k and Z_∞ it is enough repeat the previous construction changing p_1 by p_j and r_0 by p_{j+1} when it is necessary.

(ii) • Consider Z and $\widetilde{Z}$ restricted to Λ and $\widetilde{\Lambda}$. Take $h : \Lambda \to \widetilde{\Lambda}$ such that $h(p) = \widetilde{p}$. Since $\Sigma^e = (q_1, p) \times \{0\}$ and $\widetilde{\Sigma}^e \subset \widetilde{\Sigma}$ is a continuous curve connecting $\widetilde{q}_1$ and $\widetilde{p}$, by arc length parameterization we get $h(\Sigma^e) = \widetilde{\Sigma}^e$. Also, we can extend h in such a way that $h(q_1) = \widetilde{q}_1$.

For each $u \in \Sigma^e \cup \{q_1\}$, consider the positive arc of trajectory $\gamma_X^{+,u}$ of X starting at u and finishing at $v \in \Sigma^s$. Analogously, consider the positive arc of trajectory $\widetilde{\gamma}_{\widetilde{X}}^{+,\widetilde{u}}$ of $\widetilde{X}$ starting at $\widetilde{u} = h(u)$ and finishing at $\widetilde{v} \in \widetilde{\Sigma}^s$. Using the arc length parameterization we get $h(\gamma_X^{+,u}) = \widetilde{\gamma}_{\widetilde{X}}^{+,\widetilde{u}}$. Moreover, $h(\Sigma^s) = \widetilde{\Sigma}^s$ and $h(q_2) = \widetilde{q}_2$.

For each $u \in \Sigma^e \cup \{p\}$, consider the positive arc of trajectory $\gamma_Y^{+,u}$ of Y starting at u and finishing at $v \in \Sigma_+^c$. Analogously, consider the positive arc of trajectory $\widetilde{\gamma}_{\widetilde{Y}}^{+,\widetilde{u}}$ of $\widetilde{Y}$ starting at $\widetilde{u} = h(u)$ and finishing at $\widetilde{v} \in \widetilde{\Sigma}_+^c$. Using the arc length parameterization we get $h(\gamma_Y^{+,u}) = \widetilde{\gamma}_{\widetilde{Y}}^{+,\widetilde{u}}$. Moreover, $h(\Sigma_+^c) = \widetilde{\Sigma}_+^c$ and $h(s_1) = \widetilde{s}_1$.

For each $u \in \Sigma_+^c$, it is possible to repeat the previous argument and conclude that $h(\gamma_X^{+,u}) = \widetilde{\gamma}_{\widetilde{X}}^{+,\widetilde{u}}$, $h(\Sigma_-^c) = \widetilde{\Sigma}_-^c$ (and $h(s_2) = \widetilde{s}_2$).

For each $u \in \Sigma_-^c$, consider the positive arc of trajectory $\gamma_Y^{+,u}$ of Y starting at u and finishing at $v \in \Sigma^s \cup \{p\}$. Analogously, consider the positive arc of trajectory $\widetilde{\gamma}_{\widetilde{Y}}^{+,\widetilde{u}}$ of $\widetilde{Y}$ starting at $\widetilde{u} = h(u)$ and finishing at $\widetilde{v} \in \widetilde{\Sigma}^s \cup \{\widetilde{p}\}$. Using the arc length parameterization we get $h(\gamma_Y^{+,u}) = \widetilde{\gamma}_{\widetilde{Y}}^{+,\widetilde{u}}$.

□

5.5 Examples

Example 5.5.1. *Consider the case $k = 3$. Then $P_3 = -x^6 + \frac{3x^4}{2} - \frac{9x^2}{16} + \frac{1}{16}$ with roots ± 1 and $\pm\frac{1}{2}$ where the first two are simple and the latter are roots of multiplicity 2. Moreover the PSVF Z_3 is:*

$$Z_3(x,y) = \begin{cases} X_3(x,y) = \left(1, -6x^5 + 6x^3 - \frac{9x}{8}\right), & \textit{for} \quad y \geq 0 \\ Y_3(x,y) = \left(-1, -6x^5 + 6x^3 - \frac{9x}{8}\right), & \textit{for} \quad y \leq 0, \end{cases} \tag{5.10}$$

and the points $p_1 = (-\frac{1}{2}, 0)$ *and* $p_2 = (\frac{1}{2}, 0)$ *are visible-visible two fold of* Z_3*.*

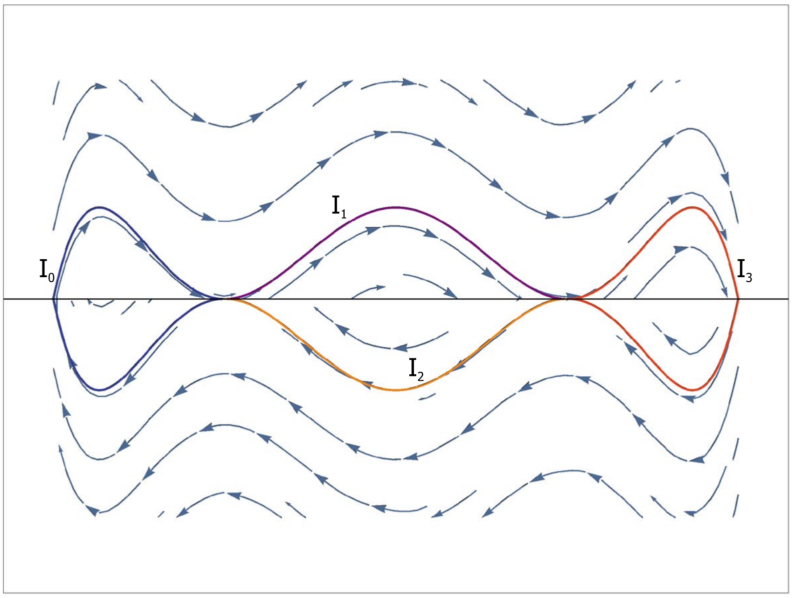

Figure 5.6: Case $k = 3$ of Theorem 5.2.1. Source: [2].

The invariant region is the set:

$$\Lambda_3 = \{(x, P(x)) \mid -1 \leqslant x \leqslant 1\} \cup \{(x, -P(x)) \mid -1 \leqslant x \leqslant 1\},$$

that is partitioned into the arcs

$$\begin{aligned} I_0 &= \left\{(x, P(x)) \mid -1 \leqslant x < -\frac{1}{2}\right\} \cup \left\{(x, -P(x)) \mid -1 \leqslant x < -\frac{1}{2}\right\} \\ I_1 &= \left\{(x, P(x)) \mid -\frac{1}{2} < x < \frac{1}{2}\right\} \\ I_2 &= \left\{(x, -P(x)) \mid -\frac{1}{2} < x < \frac{1}{2}\right\} \\ I_3 &= \left\{(x, P(x)) \mid \frac{1}{2} < x \leqslant 1\right\} \cup \left\{(x, -P(x)) \mid \frac{1}{2} \leqslant x \leqslant 1\right\} \end{aligned}$$

as shows Figure 5.6 (the arcs are blue, purple, orange and red, respectively).

Now, Ω_3 *is the set of all trajectories contained in* Λ_3 *and* $s : \Omega_3 \to \{0, 1, 2, 3\}^{\mathbb{Z}}$*. If we take* $\overline{\Omega}_3 = {}^{\Omega_3}\!/_{S}$ *and the functions* $\overline{s}$ *and* $\overline{T_1}$ *as before, we have that* $\overline{s}(\overline{\Omega}_3)$ *is a subshift of* $\{0, 1, 2, 3\}^{\mathbb{Z}}$ *and* $\overline{s}$ *is a conjugation between* $\overline{T_1}$ *and the shift* σ*. In fact, it is easy to see that such subshift is associated to the transition matrix:*

$$M = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

by analysing which arc can be reached from the other ones given the orientation of the flow.

CHAPTER 6

Concluding Remarks and Future Work

While writing this thesis, despite the results obtained and compiled at preprints [2, 3, 4] we had a lot of questions that remains open. Below we present some of these questions.

In Chapter 3, we shed light to a very difficult and poorly exploited theme in the context of PSVFs. We use a bump-function argument in order to obtain a C^0-close PSVF presenting a closed orbit passing through a non-trivially recurrent point. Moreover, while trying to improve this approach to obtain a C^k-version, we actually proved that this argument cannot work (see Scholium 3.2.1).

So as a future work, we hope that a C^1-Closing Lemma for PSVFs can be proved using some adaptation of the techniques used in the classical case.

Another aspect that should be analyzed is the comparison between the C^0- Closing Lemmas for PSVFs and smooth vector fields. We think that a regularizing process (e.g. Sotomayor-Teixeira regularization) can establish a good relation between these results.

The techniques used and results proved in Chapters 4 and 5 of this thesis offer a new theoretical framework in PSVFs and the author believes that many more results about Ω and properties of the PSVF can be studied from this.

Since we provided two distinct ways of defining the metric of Ω, some work can be done in discussing whether they are equivalent or not (under some assumptions), and if they are not, which one may be more useful and in what aspects?

Also, one could try to answer questions that better describe the space itself: under which hypothesis on the PSVF the space Ω is compact (or complete, or some other topological property)?

On the topic of topological entropy: it is not always true that two equivalent smooth flows have the same entropy ([41]), but since there are a few distinct ways of defining equivalence between two PSVFs, it may be true for some of these definitions. Also, how the properties of topological transitivity and chaoticity of PSVFs relate to topological entropy?

Moreover, we constructed examples of PSVFs with topological entropy equal to $\log \alpha$, where $\alpha \in \mathbb{N}$. Is it possible to construct a PSVF whose entropy equals any constant $r \in \mathbb{R}$?

Furthermore, in this work, we considered only two dimensional surfaces, so one possible problem to be solved is how to make it work on 3-dimensional spaces? Even though there is a natural way to extend these definitions and techniques to higher dimensional spaces, there are also big differences that need to be dealt with, for example the fact that $\Sigma^{s,e}$ are no longer 1-dimensional.

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