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Bayesian analysis of wind potential of Presidente Prudente under the Weibull distribution adjusted to wind speed

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Program: Matemática Aplicada e Computacional

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Aos 06 dias do mês de outubro do ano de 2025, às 17h, por meio de Videoconferência, realizou-se a defesa de DISSERTAÇÃO DE MESTRADO de LETICIA SILVA JUSTINO DE OLIVEIRA, intitulada **"Bayesian analysis of wind potential of Presidente Prudente under the Weibull distribution adjusted to wind speed".** A Comissão Examinadora foi constituída pelos seguintes membros: Prof. Dr. FERNANDO ANTONIO MOALA (Orientador(a) - Participação Presencial) do(a) Departamento de Estatística / Faculdade de Ciencias e Tecnologia de Presidente Prudente, Prof. Dr. GUILHERME APARECIDO SANTOS AGUILAR (Participação Presencial) do(a) Departamento de Estatística / Faculdade de Ciencias e Tecnologia de Presidente Prudente, Profa. Dra. ELIANE REGINA RODRIGUES (Participação Presencial) do(a) Departamento de Matematicas / Universidad Nacional Autónoma de México. Após a exposição pela mestranda e arguição pelos membros da Comissão Examinadora que participaram do ato, de forma presencial e/ou virtual, a discente recebeu o conceito final: APROVADA. Nada mais havendo, foi lavrada a presente ata, que após lida e aprovada, foi assinada pelo(a) Presidente(a) da Comissão Examinadora.

Prof. Dr. FERNANDO ANTONIO MOALA

To my parents,
Viviane and Valmir

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"In God we trust. All others must bring data."
W. Edwards Deming

Resumo

A demanda por novas fontes de energia tem crescido nos últimos anos. Muitos países têm feito uso de seus recursos de energia renovável, como vento, solar e ondas, não apenas para suprir a crescente demanda por energia, mas também por razões ambientais. O principal objetivo do presente estudo é estimar a potência eólica utilizando a distribuição de Weibull de dois parâmetros, ajustada para a velocidade do vento e potencial eólico. Assumindo as priori de Jeffreys, a priori de Tibshirani, a priori de referência e a priori vaga Gamma para os parâmetros do modelo, introduzimos uma análise Bayesiana utilizando métodos MCMC (Markov Chain Monte Carlo). Para testar a abordagem, comparamos os métodos de máxima verossimilhança e Bayesianos para estimar os parâmetros da distribuição de Weibull com dados de velocidade do vento do Instituto Nacional de Meteorologia do Brasil.

Palavra chave:: Análise Bayesiana, distribuição de Weibull, função de máxima verossimilhança, priori de Jeffreys, priori de Tibshirani, priori Referência, priori vaga Gamma, Markov Chain Monte Carlo, velocidade do vento, potencial eólico.

Abstract

The demand for new sources of energy has been grown in the recent years. Many countries have done use of its renewable energy resources, such as wind, solar and waves, not only to find the increasing energy demand, but also for environmental reasons. The main objective of the present study is to estimate wind power using the two-parameters Weibull distribution fitted for wind speed and adjusted to wind potential. Assuming the Jeffreys prior, Tibshirani's prior, the reference prior, and the Gamma vague prior for the parameters of the model, we introduce a Bayesian analysis using MCMC (Markov Chain Monte Carlo) methods. To test the approach, we compare both maximum likelihood and Bayesian methods to estimate the Weibull parameters for wind speed data from Brazilian National Meteorological Institute.

Keywords: Bayesian analysis, Weibull distribution, likelihood function, Jeffreys prior, Tibshirani's prior, reference prior, vague gamma prior, Markov Chain Monte Carlo, wind speed, wind power.

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GWEC: Global Wind Energy Council.

GW: Gigawatt.

ABEEólica: Brazilian Wind Energy Association.

INMET: National Meteorological Institute.

MCMC: Markov Chain Monte Carlo.

MLE: Maximum Likelihood Estimators.

EB: Empirical Bayes.

Sumário

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1

Introduction

Energy sources can be categorized into two types: renewable and non-renewable. Non-renewable energies are those generated by exhaustible sources, whose reserves are not naturally renewed in a short space of time. Examples of this type of energy include oil, coal, natural gas, and nuclear energy.

Burning coal, oil and natural gas produce pollutants and greenhouse gases, significantly contributing to global warming and climate change. Oil extraction and transportation can lead to spills, while coal mining needs precautions to prevent rain from washing waste into rivers. Nuclear energy poses risks of radiation leaks. Additionally, the extraction processes for oil, natural gas, and coal require measures to protect workers' health and prevent accidents.

As a result, renewable energy sources are gaining more and more ground, because not only they are considerably inexhaustible, but their environmental impact is also lower than that of non-renewable energy sources. Examples include wind, hydroelectric, and solar energy, among others.

Considered the "cleanest" source on the planet, wind energy is a good alternative for increasing and optimizing the use of renewable energy. Like most terrestrial energy sources, wind comes from the sun, which radiates an immense amount of energy to Earth in one hour, of which approximately 1 to 2% is converted into wind, generated by the uneven heating of air masses in the Earth's atmosphere.

According to Justus et al. [1978], solar heating and the Earth's rotation establish a seasonal pattern of air circulation in the atmosphere. This statement is still valid. Solar heating causes temperature differences between different regions of the Earth, creating areas of high and low pressure. Air tends to move from areas of high pressure to areas of low pressure, generating wind. Additionally, the Earth's rotation influences wind direction through the Coriolis effect, which deflects air movement to the right in the northern hemisphere and to the left in the southern hemisphere. This atmospheric circulation pattern is responsible for seasonal and regional variations in wind speed and direction.

According to Global Wind Energy Council (GWEC) data, by the end of 2020, the global installed capacity of onshore and offshore wind energy reached 743 Gigawatt (GW), an increase of 53 GW over the previous year. That represents, approximately, 5% of global electricity generation capacity.

According to Brazilian Wind Energy Association (ABEEólica), Brazil has an industry with 32 GW of installed capacity in operation, distributed over 1.063 wind farms and more than 11.358 wind turbines in 12 states. With an estimated potential of more than 1.500 GW in onshore and offshore wind and ranking sixth in the Global Ranking of Installed Onshore Capacity, the Brazilian wind industry has a critical role to play in helping to overcome the climate emergency.

For the city of Presidente Prudente, in the northwest of the state of São Paulo, very few studies have been carried out to date. Using the maximum likelihood method and wind speed data from August 2002, Rinaldi [2002] concluded that the region's wind potential was low.

The Weibull distribution has been widely used in studies of this type, relating to wind speed frequency analysis, as well as in identifying wind potential in various parts of the world, highlighting the pioneering studies by Henesey [1978] and Justus et al. [1978]. In Brazil, the first studies were carried out by Bastos et al. [1986], in which the wind zoning of the Brazilian Northeast was carried out, but no type of probability distribution adjustment was made. It was only in 1990 that Bandeira [1990], using the data from the above-mentioned study, adjusted the Weibull distribution to the hourly velocity averages.

In this paper, the National Meteorological Institute (INMET) provided the data used in this paper. This data refers to the average hourly wind speed during the month of August for each year from 2003 to 2024 in Presidente Prudente. The parameters of the Weibull distribution are estimated using the maximum likelihood method and the Bayesian method, in which Jeffreys non-informative prior was used for both parameters of the distribution. The Markov Chain Monte Carlo (MCMC) method was used for Bayesian estimation.

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2

General aspects of wind power

Wind originates from the sun, and for wind energy to be considered technically useful, its density must be greater than or equal to 500 W/m^2 at a height of 50 m, which requires a minimum wind speed of 7 to 8 m/s Grubb and Meyer [1993].

In this study, as in many others, the average values of wind speed and direction for each hour and each observation point were used. The variability of wind power is another significant characteristic, a general idea of the potential's regularity is given by the standard deviation of the wind speed. The higher the standard deviation, the less regular the wind power.

It is also worth noting the height at which the anemometer is placed, as wind speed varies with height. Free air above 100 meters flows at nearly twice the speed recorded at sea level. The anemometer was at an altitude of 431.92 meters when the information was recorded.

A wind turbine captures a part of the kinetic energy from the wind that goes through the blade sweep area by the rotor and transforms it in electric power. The available power, P, for a wind speed V is calculated by the following equation, Burton et al. [2002]:

$$P = \frac{1}{2}\rho A V^3, \tag{2.1}$$

where ρ is the average air density (1.225 kg/m^3 at average atmospheric pressure at sea level and at 15°C), which depends on altitude, air pressure and temperature.

The output of power wind from the wind turbine is obtained multiplying P, given in (2.1), by the power coefficient of the machine, C_p, whose maximum value is the limit of Betz $C_p = 0.593$, Burton et al. [2002], that is,

$$P = 0.59\rho A V^3. \tag{2.2}$$

It is easily verified that, if X follows a $W(\alpha, \beta)$ distribution then $Y = X^k$ also follows a Weibull distribution but with parameters α^k and β/k for $k > 0$.

Once the Weibull distribution has been fitted for the wind speed data, we are able to proceed with calculating the energy and power gains, according to Theorem below.

Theorem 1: If the wind speed V follows a Weibull distribution with scale and shape parameters α and β, respectively, then the distribution of P given in (2.2) also follows a Weibull distribution but with parameters $\alpha^* = 0.59\rho A\alpha^3$ and $\beta^* = \frac{\beta}{3}$. Thus, the power wind density function is given by

$$g(p|\alpha,\beta) = \frac{\beta}{3}(0.59\rho A\alpha^3)^{\frac{\beta}{3}} p^{\frac{\beta}{3}-1} \exp\left\{-\left(\frac{p}{0.59\rho A\alpha^3}\right)^{\frac{\beta}{3}}\right\}, \tag{2.3}$$

denoted by $P \sim Weibull(0.59\rho A\alpha^3, \frac{\beta}{3})$ with ρ and A known.

Proof: Let V be the wind speed and the wind potential (2.1) and (2.2). By using the transformation of variables we have,

$$g(p) = f(v)\left|\frac{dv}{dp}\right| \text{ with } v = \left(\frac{p}{0.59\rho A}\right)^{\frac{1}{3}}.$$

Therefore,

$$g(p) = \frac{\beta}{\alpha}\left(\frac{p}{0.59\rho A\alpha^3}\right)^{\frac{\beta}{3}} \exp\left\{-\left(\frac{p}{0.59\rho A\alpha^3}\right)^{\frac{\beta}{3}}\right\}\left(\frac{1}{0.59\rho A\alpha^3}\right)^{\frac{1}{3}} \frac{p^{\frac{-2}{3}}}{3},$$

That is,

$$g(p|\alpha,\beta) = \frac{\beta}{3}(0.59\rho A\alpha^3)^{-\frac{\beta}{3}} p^{\frac{\beta}{3}-1} \exp\left\{-\left(\frac{p}{0.59\rho A\alpha^3}\right)^{\frac{\beta}{3}}\right\}. \quad \blacksquare$$

It is important to note that wind power is proportional to the cube of its speed, such that small increases in speed produce substantial increases in power.

This property of the Weibull distribution and a reasonable fitting for the wind speed data have made this distribution more frequently used in applications related to the wind power. However, the supposition that Weibull distribution is one that better is fitted to the wind power (and also to the wind speed), as a result of theoretical relationship, should be checked by fitting distributions to the cubic of wind speed Conradsen et al. [1984]. On the other hand, if the Weibull distribution is not well fitted to the wind speed, this distribution will not represent the best fitting model for the power wind.

It is also important to observe that the probability distribution of the wind power has parameters dependent on the parameters from the speed wind probability distribution, usually derived by maximum likelihood estimation.

The expected monthly or annual wind power per unit area of a region based on the Weibull distribution (5.1) for the wind speed is expressed as

$$P_E = E(P) = 0.59\rho A\alpha^3\Gamma\left(1 + \frac{3}{\beta}\right). \tag{2.4}$$

Details of this calculation can be found in Celik [2003].

The expectancy (2.4) provides the energy earnings at any one specific wind turbine location.

There are several methods presented in the literature to estimate the parameters α and β. The main frequentist method used to estimate both parameters α and β is the Maximum Likelihood Estimation (MLE). In general, this estimator is preferred to the alternative frequentist methods (moments and least squares estimators), since usually it is considered to be more robust and produces more accurate results. In this paper, we applied both maximum likelihood and Bayesian methods to the data. We also construct intervals to reflect the uncertainty about the unknown parameters α and β, for both the Bayesian and MLE estimation method.

In this work, a wind turbine with a swept area $A = 1\,\mathrm{m}^2$ and a standard average air density $\rho = 1.225\,\mathrm{kg/m}^3$, corresponding to the density value at sea level and 15ºC, was used as a reference. This standard value is commonly adopted in wind energy studies to allow consistent comparisons between different regions and turbine models.

However, it is important to note that the air density in Presidente Prudente, a region characterized by higher average temperatures and moderate altitude, tends to be slightly lower. Thus, while the use of the standard value may lead to a minor overestimation of the theoretical wind power potential, it remains suitable for comparative and methodological purposes, as the proportional relationships between the Weibull parameters and the estimated power are preserved. Based on these reference conditions, the potential in watts generated by a wind turbine with the above dimensions was obtained.

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3

Bayesian Inference

3.1 Bayesian Paradigm

Bayesian paradigm states that all inference must be on the basis of the a posteriori distribution of the unknown quantities of the model, conditional on the data.

The difference between classical inference and Bayesian inference is that in the latter, the parameter of interest, θ, is treated as a random variable and therefore has a prior probability distribution. This distribution must probabilistically represent the knowledge regarding θ before the experiment is conducted. If there is no prior information about the parameter, a non-informative prior is used, which aims to avoid influencing the results. Examples of non-informative priors include the Jeffreys [1998] prior and the uniform prior.

After collecting the data, the knowledge about the parameter is updated using Bayes' Theorem. The new distribution, called the a posteriori distribution $p(\theta|x)$, combines the a priori with the information in the data. Bayes' Theorem allows us to calculate the a posteriori distribution according to the following expression:

$$p(\theta|x) = \frac{p(\theta, x)}{p(x)} = \frac{p(x|\theta)p(\theta)}{p(x)} = \frac{L(\theta|x)\pi(\theta)}{\int L(\theta|x)\pi(\theta)d\theta} \propto L(\theta|x)\pi(\theta), \tag{3.1}$$

where $L(\theta|x)$ is the likelihood function of the data given the parameter and $\pi(\theta)$ is the a priori distribution.

3.2 Non-Informative Priors

An Non-Informative prior is a priori distribution chosen on the basis of formal criteria and aims to provide inferences that are not dependent on the specific analyst. During this section, we will present the prior distributions of Jeffreys, Tibshirani, Reference and vague Gamma.

3.2.1 Jeffreys Prior

Jeffreys [1998] proposed a prior distribution that is known to represent a situation with little prior information available about the parameters and has since played a role in Bayesian inference. This distribution is derived from the Fisher Information matrix $I(\phi)$ as

$$\pi(\phi) \propto \sqrt{\det I(\phi)}, \tag{3.2}$$

where ϕ can be real or a vector and

$$I_{ij}(\boldsymbol{\phi}) = -E\left(\frac{\partial^2 \, log \, f(x \mid \boldsymbol{\phi})}{\partial \boldsymbol{\phi}_i \partial \boldsymbol{\phi}_j}\right). \tag{3.3}$$

Box and Tiao [1973] explain the derivation of non-informative Jeffreys priors in terms of a likelihood *data translated* by the data.

The Jeffreys prior is widely used because of its invariance property under one-to-one transformations of ϕ. Despite many desirable properties of the Jeffreys prior, there is an ongoing discussion about whether the multivariate Jeffreys prior is appropriate. The multiparametric version of the Jeffreys prior can lead to a transformation in which the likelihood is not *data translated* by the data (see Box and Tiao [1973] for details).

3.2.2 Tibshirani Prior

Tibshirani [1989] proposes a procedure for obtaining an objective prior for a parametric vector $\boldsymbol{\phi}$ such that the marginal posterior interval for a specific parameter of interest, such as $\delta = t(\boldsymbol{\phi})$, has a frequentist coverage probability close to a given confidence level α.

The procedure requires an orthogonal reparametrization (see Cox and Reid [1987]) of the parameter vector $\boldsymbol{\phi}$, where $\delta = t(\boldsymbol{\phi})$ is the parameter of interest, and $\boldsymbol{\lambda} = (\lambda_1, ..., \lambda_{k-1})$ is the nuisance parameter.

Suppose the parameter vector $\boldsymbol{\phi}$ is composed of $\boldsymbol{\phi} =(\phi_1, \boldsymbol{\phi_2})$, where ϕ_1 is the parameter of interest and $\boldsymbol{\phi_2} = (\phi_{21}, \phi_{22}, \ldots, \phi_{2q})$ is the vector of nuisance parameters, with $q = k-1$.

The original likelihood function L is expressed in terms of $(\phi_1, \phi_{21}, \phi_{22}, \ldots, \phi_{2q})$, and the goal is to derive a new parametrization $(\delta, \lambda_1, \lambda_2, \ldots, \lambda_q)$, where $\delta = \phi_1$ is orthogonal to each λ_t for $t = 1, 2, \ldots, q$.

The proposed criterion to achieve this reparametrization involves solving a system of q equations of the following form:

$$\sum_{i=1}^{q} i^*_{\phi_{2r}\phi_{2s}} \frac{\partial \phi_{2r}}{\partial \phi_1} = -i^*_{\phi_1\phi_{2s}} \ , \ s = 1, ..., q \tag{3.4}$$

where i^* are the elements of the Fisher information matrix obtained in the parametrization $(\phi_1, \boldsymbol{\phi}_2)$, assuming that the transformation $(\phi_1, \boldsymbol{\phi}_2) \rightarrow (\delta, \boldsymbol{\lambda})$ has a non-zero Jacobian.

Thus, the Fisher information matrix $k \times k$ for the new orthogonal parameters δ and $\boldsymbol{\lambda}$ is given by:

$$I(\delta, \boldsymbol{\lambda}) = \begin{bmatrix} I_{\delta\delta}(\delta, \boldsymbol{\lambda}) & 0 \\ 0 & I_{\boldsymbol{\lambda}\boldsymbol{\lambda}}(\delta, \boldsymbol{\lambda}) \end{bmatrix}, \tag{3.5}$$

where $I_{\boldsymbol{\lambda}\boldsymbol{\lambda}}(\delta, \boldsymbol{\lambda})$ denotes the submatrix of $I(\delta, \boldsymbol{\lambda})$.

Accordingly, the weak prior:

$$\pi(\delta, \boldsymbol{\lambda}) \propto g(\boldsymbol{\lambda})\sqrt{I_{\delta\delta}(\delta, \boldsymbol{\lambda})}, \tag{3.6}$$

leads, asymptotically in sample size, to an appropriate frequentist one-tailed posterior interval for δ, where $g(\boldsymbol{\lambda}) > 0$ is an arbitrary function.

The corresponding prior for $\boldsymbol{\phi}$ is given by:

$$\pi(\boldsymbol{\phi}) \propto g(\boldsymbol{\lambda}) I_{\delta\delta}(\delta, \boldsymbol{\lambda})^{1/2} \left| J(\delta, \boldsymbol{\lambda}) \right|, \tag{3.7}$$

where $|J(\delta, \boldsymbol{\lambda})|$ is the Jacobian of the transformation from $(\delta, \boldsymbol{\lambda})$ to $\boldsymbol{\phi}$.

Due to the non-uniqueness in the choice of orthogonal reparametrization, the class of orthogonal parameters takes the form $g(\lambda)$, where $g(\cdot)$ is any reparametrization function. This non-uniqueness is reflected in the function $g(\cdot)$ corresponding to equation (3.6). One possibility, in the case of a single nuisance parameter, is to require that (δ, λ) also satisfies Stein's condition (see Tibshirani [1989]) for λ, with δ taken as the nuisance parameter.

3.2.3 Reference Prior

The reference prior, initially described by Bernardo [1979] and later expanded by Berger and Bernardo [1992], is a well-known class of non-informative priors. The concept behind the reference prior is to derive a prior $\pi(\phi)$ that maximizes the expected posterior information of the parameters when independent replications of an experiment are considered. This expected information is measured relative to the information in the prior.

A natural way to measure the expected information about ϕ, given the data $\boldsymbol{x}$, is through the following expression:

$$I(\phi) = E_{\boldsymbol{x}}[K(p(\phi|\boldsymbol{x}), \pi(\phi))], \tag{3.8}$$

where the Kullback-Leibler distance, defined as

$$K(p(\phi|\boldsymbol{x}), \pi(\phi)) = \int_{\Phi} p(\phi|\boldsymbol{x}) \ln \frac{p(\phi|\boldsymbol{x})}{\pi(\phi)} d\phi, \tag{3.9}$$

quantifies the difference between the posterior distribution and the prior distribution. The reference prior is, therefore, the prior $\pi(\phi)$ that maximizes this expected Kullback-Leibler distance, given the experimental data.

The prior density $\pi(\phi)$ that maximizes the functional in Equation (3.8) is found using the calculus of variations, but the solution is not explicit. However, when the posterior

$p(\phi|\boldsymbol{x})$ is asymptotically normal, this approach leads to Jeffreys' prior in the case of a single parameter. If we are interested in a particular parameter while treating others as nuisance parameters, the situation changes, and the appropriate reference prior is not simply a multivariate Jeffreys prior. Bernardo suggests that when nuisance parameters are present, the reference prior should reflect which parameters are of primary interest.

The reference prior in this case is derived as follows, with a focus on two parameters. For more parameters, see Berger and Bernardo [1992].

Let $\theta = (\theta_1, \theta_2)$ be the full parameter vector, where $\theta_1 \in \Theta_1$ is the parameter of interest, and $\theta_2 \in \Theta_2$ is the nuisance parameter. The procedure is as follows:

First step: Determine the conditional reference prior for θ_2 given θ_1, denoted by $\pi_2(\theta_2|\theta_1)$:

$$\pi_2(\theta_2|\theta_1) = \sqrt{\mathrm{I}_{22}(\theta_1, \theta_2)}, \tag{3.10}$$

where $\mathrm{I}_{22}(\theta_1, \theta_2)$ is the (2,2) entry of the Fisher Information matrix.

Second step: Normalize $\pi_2(\theta_2|\theta_1)$.

If $\pi_2(\theta_2|\theta_1)$ is improper, select a sequence of subsets $\Omega_1 \subseteq \Omega_2 \subseteq \cdots \to \Omega$ where $\pi_2(\theta_2|\theta_1)$ is proper. Define

$$c_m(\theta_1) = \frac{1}{\int_{\Omega_m} \pi_2(\theta_2|\theta_1) d\theta_2}, \tag{3.11}$$

$$p_m(\theta_2|\theta_1) = c_m(\theta_1)\pi_2(\theta_2|\theta_1) 1_{\Omega_m}(\theta_2). \tag{3.12}$$

Third step: Find the marginal reference prior for θ_1, i.e., the reference prior for the experiment after marginalizing over $p_m(\theta_2|\theta_1)$. The marginal reference prior is given by:

$$\pi_m(\theta_1) \propto \exp\left\{ \frac{1}{2} \int_{\Omega_m} p_m(\theta_2|\theta_1) \ln \left| \frac{\det I(\theta_1, \theta_2)}{I_{22}(\theta_1, \theta_2)} \right| d\theta_2 \right\}. \tag{3.13}$$

Fourth step: Calculate the reference prior for (θ_1, θ_2) when θ_2 is a nuisance parameter:

$$\pi(\theta_1, \theta_2) = \lim_{m \to \infty} \left(\frac{c_m(\theta_1)\pi_m(\theta_1)}{c_m(\theta_1^*)\pi_m(\theta_1^*)} \right) \pi(\theta_2|\theta_1), \tag{3.14}$$

where θ_1^* is any fixed point in Θ_1 with positive density for all π_m.

3.3 Vague Priors

In addition to using objective priors to represent the lack of information for a prior, especially in situations where we do not have expert opinion to build our prior, the specifications of these prior distributions can be intentionally vague so that the choice of prior does not influence the posterior distribution too much.

We can assume prior independence of the parameters, and therefore the joint prior is given by,

$$\pi(\alpha, \beta) = \pi(\alpha)\pi(\beta). \tag{3.15}$$

3.3.1 Gamma Prior

It is typically difficult to specify informative beliefs about the shape and scale parameters α and β. Since these parameters have positive support, we can assign a weakly informative prior, such as a gamma prior with small values of the shape and rate hyperparameters. Thus, we consider the following prior distribution:

$$\alpha \sim \Gamma(a_1, b_1) \quad \text{e} \quad \beta \sim \Gamma(a_2, b_2), \tag{3.16}$$

where the hyperparameters "a_i" and "b_i", $i = 1, 2$, are known.

From the prior distribution for each parameter, the joint prior density for the parameters (α, β) is given by

$$\pi(\alpha, \beta) \propto \alpha^{a_1 - 1} \beta^{a_2 - 1} \exp\left(-\left(\frac{\alpha}{b_1} + \frac{\beta}{b_2}\right)\right). \tag{3.17}$$

We could choose hyperparameters $a_i = b_i = 0.01$ or 0.001, $i = 1, 2$, which suggests little prior information about the parameters. Values of $a \to 0$ and $b \to 0$ suggest a vague prior distribution (lack of information).

3.4 MCMC Algorithm

Markov Chain Monte Carlo (MCMC) is a powerful class of algorithms used in Bayesian inference to sample from complex probability distributions.

As Bayesian inference relies on the posterior distribution, $p(\theta|x)$, which is often analytically intractable due to the complexity of the likelihood $L(\theta|x)$ and prior $\pi(\theta)$, MCMC methods provide a way to approximate this posterior distribution through simulation. The most popular algorithms are Gibbs Samplings and Metropolis-Hastings.

3.4.1 Metropolis-Hastings

According to Paulino et al. [2003], the Metropolis-Hastings algorithm is a powerful Markov Chain Monte Carlo (MCMC) method that enables sampling from an unknown posterior distribution $p(\theta \mid x)$, even when its density is not explicitly known.

This algorithm operates iteratively to sample from the target distribution by approximating the posterior density through a rejection mechanism. It uses a proposal distribution $q(\theta' \mid \theta^{(t)})$, which depends only on the current state $\theta^{(t)}$, and generates a random value from a uniform distribution to decide whether to accept the proposed sample.

The Metropolis-Hastings algorithm can be summarized as follows:

1. **Initialization**: Start with an initial guess $\theta^{(0)}$ for the Markov chain and set $t = 1$.

2. **Proposal Generation**: Generate a candidate θ^* from the proposal distribution $q(\theta^* \mid \theta^{(t)})$.

3. **Acceptance Ratio Calculation**: Compute the acceptance ratio α as:

$$\alpha = \min\left(1, \frac{p(x \mid \theta^*)q(\theta^{(t)} \mid \theta^*)}{p(x \mid \theta^{(t)})q(\theta^* \mid \theta^{(t)})}\right).$$

4. **Uniform Random Value Generation**: Generate a value u from the uniform distribution $\mathrm{Uniform}(0, 1)$.

5. **Acceptance or Rejection**: Accept the candidate θ^* if $u \leq \alpha$; otherwise, retain the previous state $\theta^{(t)}$.

6. **Update and Repeat**: Update t to $t + 1$ and repeat the process until the chain reaches the desired stationary distribution.

Capítulo

4

Materials and data

The dataset used in this study was obtained from the Brazilian government agency INMET. The original records consist of wind speed measurements, expressed in meters per second, collected at hourly intervals during the month of August for each year from 2003 to 2024. For the purposes of this work, these hourly observations were aggregated to compute the monthly mean for August of each year, resulting in a total of 21 average wind speed values. The mean was chosen as a representative measure since the data showed low dispersion and the median values were very close to the mean, indicating a symmetric and consistent pattern. The measurements were taken by an anemograph located in the town of Presidente Prudente, São Paulo State, at latitude -22.11999999, longitude -51.40861111, and altitude 431.92 meters.

Identifying an appropriate probability distribution to describe wind speed variability is of great importance, as it provides the basis for estimating the potential amount of wind-generated energy. In this context, the two-parameter Weibull distribution has been widely used to fit wind speed data and to assess wind power potential in several countries. See, for instance, Justus and Mikhail [1976], Petersen et al. [1981], Conradsen et al. [1984], Genc et al. [2005], Akpinara and Akpinarb [2009], Bhattacharya and Bhattacharjee [2010].

Traditionally, the maximum likelihood estimation (MLE) method has been the primary approach for fitting the two-parameter Weibull distribution to wind speed data. Conradsen et al. [1984] and Genc et al. [2005] discussed three classical methods for parameter estimation: (i) maximum likelihood estimation (MLE), (ii) minimum chi-square, and (iii) least squares. Their results showed that MLE provides the best overall properties among these approaches. However, the effectiveness of MLE largely depends on the availability of large datasets.

As an alternative, Bayesian statistical analysis has also been employed to estimate the parameters of the Weibull distribution fitted to wind speed data. The Bayesian approach offers several advantages, making it a valuable tool for inference in this context. For example, Smith and Naylor [1987] presented a Bayesian analysis of the three-parameter

Weibull distribution and compared the results with MLE in a wind speed study. Similarly, Panga et al. (2001) applied Bayesian estimation with Markov Chain Monte Carlo (MCMC) techniques using uniform priors, and Erto et al. [2010] also proposed a Bayesian inference approach via MCMC.

Despite the increasing use of Bayesian methods for modeling wind speed, relatively few studies have applied this methodology to wind power estimation.

Therefore, this work focuses on Bayesian inference under the Weibull distribution to model both wind speed data and wind power potential. The Weibull distribution is particularly suitable for this purpose due to its flexibility in characterizing wind speed distributions. In addition, a simulation study by Sun [1997] showed that, for small sample sizes, the reference prior outperforms the Jeffreys prior in terms of frequentist coverage probabilities.

In this study, we consider a more comprehensive set of important noninformative priors when both parameters α and β are of interest. Specifically, we focus on the Jeffreys prior, Tibshirani's prior, the reference prior, and the vague Gamma prior.

Furthermore, the study was extended to an application which, in the absence of expert input, employed elicitation-based and MLE-based Bayesian methodologies to real wind data from Presidente Prudente, and compared the Bayesian results with those obtained by classical MLE estimation.

Capítulo

5

Bayesian analysis of wind potential under the W(α,β) adjusted to wind speed

5.1 Weibull Distribution Fitted to Wind Speed

For the prediction of wind energy, the first step of a data analysis is the wind speed probability distribution.

Several probability distributions are used, such as Weibull, Rayleigh and Lognormal, to fit the wind speed distribution over a time period. The two-parameters Weibull distribution is accepted as the best among the models given because one can adjust the parameters to suit a period of time, usually one month or one year. Furthermore, since the Weibull distribution leads to an another Weibull for the distribution of the cube of the wind speed, it has also extensively been used in wind power analysis for many decades.

The two-parameter Weibull distribution, denoted by $W(\alpha, \beta)$, has the following density:

$$f(v \mid \alpha, \beta) = \frac{\beta}{\alpha}\left(\frac{v}{\alpha}\right)^{\beta-1} \exp\left\{-\left(\frac{v}{\alpha}\right)^{\beta}\right\}, \quad \text{for all } v > 0, \tag{5.1}$$

depending on the parameters $\alpha > 0$ and $\beta > 0$. Parameter α is called scale parameter and β as shape parameter. Thus, the probability that wind speed is greater than or equal to a speed is provided by:

$$P\{V \geq v_0\} = 1 - P\{V \leq v_0\} = \exp\left\{-\left(\frac{v_0}{\alpha}\right)^{\beta}\right\}. \tag{5.2}$$

The mean wind speed v_m and the variance are defined by:

$$v_m = E(V) = \alpha\Gamma\left(1 + \frac{1}{\beta}\right), \tag{5.3}$$

$$Var(V) = \alpha^2 \left(\Gamma \left(1 + \frac{2}{\beta} \right) - \Gamma^2 \left(1 + \frac{1}{\beta} \right) \right), \tag{5.4}$$

where,

$$\Gamma(k) = \int_0^{inf} u^{k-1} \exp(-u) \, du. \tag{5.5}$$

5.2 Maximum Likelihood Estimation for (α,β)

Suppose we have independent identically distributed wind speed $\boldsymbol{v} = (v_1, v_2, \ldots, v_n)^t$ from $W(\alpha, \beta)$. The likelihood function in the parameters α and β, based on $\boldsymbol{v}$, is therefore:

$$L(\alpha, \beta \mid \boldsymbol{v}) = \left(\frac{\beta}{\alpha}\right)^n \left[\prod_{i=1}^{n} \left(\frac{v_i}{\alpha}\right)^{\beta-1}\right] \exp\left\{-\sum_{i=1}^{n} \left(\frac{v_i}{\alpha}\right)^{\beta}\right\}. \tag{5.6}$$

Applying the logarithm to the equation 5.6

$$l(\alpha, \beta) = lnL(\alpha, \beta \mid \boldsymbol{v}) = nln\beta - n\beta ln\alpha + (\beta - 1) \sum_{i=1}^{n} lnv_i - \sum_{i=1}^{n} \left(\frac{v_i}{\alpha}\right)^{\beta}. \tag{5.7}$$

Calculating the partial derivatives regarding α and β, we have:

$$\frac{\partial \ell}{\partial \alpha} = -\frac{n\beta}{\alpha} + \frac{\beta}{\alpha} \sum_{i=1}^{n} \left(\frac{v_i}{\alpha}\right)^{\beta} \tag{5.8}$$

and

$$\frac{\partial \ell}{\partial \beta} = \frac{n}{\beta} - n \log \alpha + \sum_{i=1}^{n} \log v_i - \sum_{i=1}^{n} \left(\frac{v_i}{\alpha}\right)^{\beta} \log \left(\frac{v_i}{\alpha}\right). \tag{5.9}$$

The maximum likelihood estimators $\hat{\alpha}$ and $\hat{\beta}$ can be obtained solving $\frac{\partial \ell}{\partial \alpha} = 0$ and $\frac{\partial \ell}{\partial \beta} = 0$ simultaneously. A convenient way to do this is to notice that making (5.8) equal to zero, where

$$-\frac{n\beta}{\alpha} + \frac{\beta}{\alpha} \sum_{i=1}^{n} \left(\frac{v_i}{\alpha}\right)^{\beta} = \frac{\beta}{\alpha} \left[-n + \sum_{i=1}^{n} \left(\frac{v_i}{\alpha}\right)^{\beta}\right] = 0 \tag{5.10}$$

$$-n + \sum_{i=1}^{n} \left(\frac{v_i}{\alpha}\right)^{\beta} = 0 \tag{5.11}$$

$$\sum_{i=1}^{n} v_i^{\beta} = n\alpha^{\beta}, \tag{5.12}$$

obtaning,

$$\hat{\alpha} = \left(\frac{\sum_{i=1}^{n} v_i^{\beta}}{n} \right)^{1/\beta}. \tag{5.13}$$

By substituting (5.13) in the equation (5.9), we obtain

$$\frac{n}{\beta} - n \log \alpha + \sum_{i=1}^{n} \log v_i - \sum_{i=1}^{n} \left(\frac{v_i}{\alpha} \right)^{\beta} \log \left(\frac{v_i}{\alpha} \right) = 0 \tag{5.14}$$

$$\frac{n}{\beta} - n \cdot \frac{1}{\beta} \log \left(\frac{\sum v_i^{\beta}}{n} \right) + \sum \log v_i - \sum \frac{n v_i^{\beta}}{\sum v_j^{\beta}} \left[\log v_i - \frac{1}{\beta} \log \left(\frac{\sum v_j^{\beta}}{n} \right) \right] = 0 \tag{5.15}$$

$$\frac{n}{\beta} - n \cdot \frac{1}{\beta} \log \left(\frac{\sum v_i^{\beta}}{n} \right) + \sum \log v_i - n \frac{\sum v_i^{\beta} \log v_i}{\sum v_j^{\beta}} + n \cdot \frac{1}{\beta} \log \left(\frac{\sum v_j^{\beta}}{n} \right) = 0 \tag{5.16}$$

$$\frac{n}{\beta} + \sum \log v_i - n \frac{\sum v_i^{\beta} \log v_i}{\sum v_j^{\beta}} = 0 \tag{5.17}$$

$$\frac{\sum_{i=1}^{n} v_i^{\hat{\beta}} \log v_i}{\sum_{i=1}^{n} v_i^{\hat{\beta}}} - \frac{1}{n} \sum_{i=1}^{n} \log v_i - \frac{1}{\hat{\beta}} = 0. \tag{5.18}$$

To find $\hat{\alpha}$ and $\hat{\beta}$ we can therefore determine $\hat{\beta}$ as the solution of (5.18) and then obtain $\hat{\alpha}$ in 5.13. Since (5.18) cannot be solved analytically for $\hat{\beta}$, we should use a numerical method, for instance the Newton-Raphson Method.

The estimation with confidence intervals for the parameters can be obtained by the asymptotic normal approximation of the maximum likelihood estimators (MLE). Bain [1978] states the following asymptotic properties held for the MLE Weibull parameters:

$$\text{i. } \left(\sqrt{n}(\hat{\alpha} - \alpha), \sqrt{n}(\hat{\beta} - \beta) \right) \xrightarrow{d} N_2(\mathbf{0}, \boldsymbol{\Sigma}), \quad \text{as } \mathbf{n} \to \infty, \text{where} \tag{5.19}$$

$$\Sigma = \begin{bmatrix} \frac{1.109\alpha^2}{\beta} & 0.257\alpha \\ 0.257\alpha & 0.608\beta^2 \end{bmatrix}, \tag{5.20}$$

$$\text{ii. } \frac{\sqrt{n}(\hat{\beta} - \beta)}{\beta} \xrightarrow{d} N(0, 0.608), \quad \text{as } n \to \infty, \tag{5.21}$$

$$\text{iii. } \sqrt{n} \log(\hat{\alpha}/\alpha) \xrightarrow{d} N(0, 1.109), \quad \text{as } n \to \infty. \tag{5.22}$$

Interval estimation or tests of hypotheses concerning α and β may be based on the above asymptotic distributions.

It is known that the wind power varies as the cube of the wind speed, therefore, we can estimate the parameters of the Weibull distribution for wind speed variations, and subsequently use the MLE to estimate the wind power earnings, due to the invariance property of MLE under reparameterizations.

The expected wind speed and expected wind power are estimated by substituting the estimators of α and β in the equations (5.1) and (5.2), obtained from MLE. From (5.2) and (5.3), the MLE of expected wind energy is given by

$$\hat{v}_m = \hat{\alpha}\Gamma\left(1+\frac{1}{\hat{\beta}}\right) \tag{5.23}$$

and

$$\hat{P}_E = 0.59pA\alpha^3\Gamma\left(1+\frac{3}{\hat{\beta}}\right), \tag{5.24}$$

respectively.

5.3 Bayesian Estimation for (α,β)

The Bayesian inference is an attractive framework for estimation of parameters from Weibull distribution and has grown in popularity in recent years. Besides offering easier interpretability of parameter estimation it gives more reliable results without any need for asymptotic approximations. Furthermore, inference about any functions of the parameters, or predictions, may be obtained easily. To progress with the Bayesian analysis it is necessary to specify a joint prior distribution over the parameter space.

5.3.1 Jeffreys Prior for (α,β)

The Jeffreys prior is derived from Fisher Information matrix $I(\alpha,\beta)$ as

$$\pi(\alpha,\beta) \propto \sqrt{\det I(\alpha,\beta)}. \tag{5.25}$$

where the Fisher Information matrix $I(\alpha,\beta)$ has elements given by

$$I_{11} = -E\left(\frac{\partial^2}{\partial\theta^2}ln\,L\right), \quad I_{12} = -E\left(\frac{\partial^2}{\partial\theta\partial\lambda}ln\,L\right), \quad I_{22} = -E\left(\frac{\partial^2}{\partial\lambda^2}ln\,L\right). \tag{5.26}$$

Box and Tiao [1973] give an explanation of the derivation of non-informative Jeffreys priors in terms of "data translated" likelihood.

Jeffreys prior is widely used due to its invariance property under one-to-one transformations of (α,β).

By considering the Weibull distribution (5.1), the Fisher Information matrix $I(\alpha,\beta)$ is given by

$$I(\alpha,\beta) = \begin{bmatrix} \left(\frac{\beta}{\alpha}\right)^2 & -\frac{\Gamma'(2)}{\alpha} \\ -\frac{\Gamma'(2)}{\alpha} & \frac{1}{\beta^2}\left(1+\Gamma''(2)\right) \end{bmatrix}. \tag{5.27}$$

where $\Gamma'(2) = 1-\gamma$ and $\gamma = 0.577215\ldots$ Moala [2010].

From (5.25) and (5.27), the Jeffreys prior for (α, β) parameters is given by:

$$\pi(\alpha, \beta) \propto \frac{1}{\alpha}. \tag{5.28}$$

Therefore, by the combination of the joint prior distribution and the likelihood function, we have that the joint posteriori for the parameters of the distribution is

$$p(\alpha, \beta|\boldsymbol{v}) \propto k_1 \frac{1}{\alpha}\left(\frac{\beta}{\alpha}\right)^n \prod_{i=1}^{n}\left(\frac{v_i}{\alpha}\right)^{\beta-1} \exp\left(-\sum_{i=1}^{n}\left(\frac{v_i}{\alpha}\right)^{\beta}\right). \tag{5.29}$$

since $\int_0^\infty v^{-(x+1)} \exp\left(-\frac{a}{v^b}\right) dv = \frac{1}{b} a^{-\frac{x}{b}} \Gamma\left(\frac{x}{b}\right)$, the normalizing constant is

$$k_1^{-1} = \Gamma(n) \int_0^\infty \beta^{n-1} \prod_{i=1}^{n} v_i^{\beta-1} \left(\sum_{i=1}^{n} v_i^\beta\right)^{-n} d\beta. \tag{5.30}$$

which can be calculated via numerical integration. To estimate the parameters of interest, the marginal posterior distribution was used, integrating with respect to the parameter *nuisance*[1] of the joint density. Thus,

$$p(\alpha|\boldsymbol{v}) \propto \frac{\frac{1}{\alpha^{n+1}} \int_0^\infty \beta^n \prod_{i=1}^{n}\left(\frac{v_i}{\alpha}\right)^{\beta-1} \exp\left(-\sum_{i=1}^{n}\left(\frac{v_i}{\alpha}\right)^{\beta}\right) d\beta}{\Gamma(n) \int_0^\infty \beta^{n-1} \prod_{i=1}^{n} v_i^{\beta-1} \left(\sum_{i=1}^{n} v_i^\beta\right)^{-n} d\beta}. \tag{5.31}$$

Now, by direct integration of the parameter α of the joint density, the marginal a posteriori distribution of the unknown parameter β is calculated, i.e,

$$p(\beta|\boldsymbol{v}) \propto \frac{\beta^{n-1} \prod_{i=1}^{n} v_i^{\beta-1} \left(\sum_{i=1}^{n} v_i^\beta\right)^{-n}}{\int_0^\infty \beta^{n-1} \prod_{i=1}^{n} v_i^{\beta-1} \left(\sum_{i=1}^{n} v_i^\beta\right)^{-n} d\beta}. \tag{5.32}$$

Although the marginal posterior of β was obtained without resorting to integration, it is not possible to identify a known distribution for the density $p(\beta|\boldsymbol{v})$, because of this obstacle, the Metropolis-Hasting algorithm was used to obtain the marginal densities of the parameters α and β of $p(\alpha, \beta|\boldsymbol{v})$ and therefore estimate the wind potential.

5.3.2 Tibshirani Prior for (α, β)

The Tibshirani prior specifies a weak prior of the form:

$$\pi(\delta, \lambda) \propto g(\lambda)\sqrt{I_{11}(\delta, \lambda)}, \tag{5.33}$$

where $g(\lambda) > 0$ is an arbitrary positive function that can be chosen based on the context of the analysis.

[1]Parameters that are not of interest, i.e., it is not the main objective to make inferences about them.

For the Weibull model (5.1), we suggest an orthogonal reparameterization in terms of (δ, λ), where $\delta = \beta$ represents the key parameter and λ is the nuisance parameter. The orthogonal parameter λ is obtained by solving the following differential equation:

$$i^*_{\alpha\alpha} \frac{\partial \alpha}{\partial \beta} = -i^*_{\alpha\beta}. \tag{5.34}$$

Using the Weibull parameters from (5.1) and the matrix in (5.27), we derive the following differential equation:

$$\left(\frac{\beta}{\alpha}\right)^2 \frac{\partial \alpha}{\partial \beta} = \frac{\Gamma'(2)}{\alpha}. \tag{5.35}$$

The solution to this is:

$$\log \alpha + \log \left(\exp \left\{ \frac{\Gamma'(\alpha)}{\beta} \right\} \right) = a(\lambda), \tag{5.36}$$

where $a(\lambda)$ is an arbitrary function of λ. Choosing $a(\lambda) = \log \lambda$ results in:

$$\lambda = \alpha \exp \left\{ \frac{\Gamma'(2)}{\beta} \right\}. \tag{5.37}$$

The Fisher Information matrix for the orthogonal parameters, derived from (5.27) and (5.37), is given by:

$$I(\delta, \lambda) = \begin{pmatrix} \frac{1}{\delta^2}(1 + \Gamma''(2)\,[\Gamma'(2)]^2) & 0 \\ 0 & \frac{\delta^2}{\lambda^2} \end{pmatrix}. \tag{5.38}$$

From the combination of (5.33) and (5.38), the prior distribution for (δ, λ) is defined as:

$$\pi(\delta, \lambda) \propto g(\lambda) \frac{1}{\delta}, \tag{5.39}$$

where $g(\lambda)$ is an arbitrary function.

The function $g(\lambda)$ can be any positive function reflecting the inherent flexibility in the orthogonal reparameterization approach. In cases where the choice of reparameterization satisfies Stein's condition (see Tibshirani, 1989), the prior becomes:

$$\pi(\delta, \lambda) \propto \frac{1}{\delta\lambda}. \tag{5.40}$$

Thus, the prior expressed in terms of (α, β) is given by:

$$\pi(\alpha, \beta) \propto \frac{1}{\alpha\beta}. \tag{5.41}$$

Finally, combining the prior distribution and the likelihood function allows us to evaluate the posterior distribution for the parameter of interest.

By combining the prior distribution from (5.41) with the likelihood function from equation (5.6), the posterior distribution for the quantity of interest can be expressed as:

$$p(\alpha, \beta \mid v) = k_3 \, \frac{\beta^{n-1}}{\alpha^{n+1}} \prod_{i=1}^{n} \left(\frac{v_i}{\alpha}\right)^{\beta-1} \exp\left\{-\sum_{i=1}^{n} \left(\frac{v_i}{\alpha}\right)^{\beta}\right\}, \tag{5.42}$$

where

$$k_3^{-1} = \Gamma(n) \int_0^{\infty} \beta^{n-2} \prod_{i=1}^{n} {v_i}^{\beta-1} \left[\sum_{i=1}^{n} {v_i}^{\beta}\right]^{-n} d\beta \, d\alpha. \tag{5.43}$$

To extract the marginal distributions of interest, we need to integrate out specific parameters from the joint posterior distribution in Equation (5.42), as follows:

$$p(\alpha \mid v) = k_3 \, \frac{1}{\alpha^{n+1}} \int_0^{\infty} \beta^{n-1} \prod_{i=1}^{n} \left(\frac{v_i}{\alpha}\right)^{\beta-1} \exp\left\{-\sum_{i=1}^{n} \left(\frac{v_i}{\alpha}\right)^{\beta}\right\} d\beta \tag{5.44}$$

$$p(\beta \mid v) = \frac{\beta^{n-2} \prod_{i=1}^{n} {v_i}^{\beta-1} \left[\sum_{i=1}^{n} {v_i}^{\beta}\right]^{-n}}{\int_0^{\infty} \beta^{n-2} {v_i}^{\beta-1} \prod_{i=1}^{n} \left[\sum_{i=1}^{n} {v_i}^{\beta}\right]^{-n} d\beta}. \tag{5.45}$$

Given the complexity of these integrals, it is generally challenging to derive analytical solutions for the marginal posterior distributions of the parameters. Therefore, numerical methods, such as Markov Chain Monte Carlo (MCMC) or importance sampling, are often employed to compute the posterior distributions efficiently.

5.3.3 Reference Prior for (α,β)

With α as the parameter of interest and β as the nuisance parameter, we derive the reference prior for the parameters of the Weibull model given in (5.1).

Using the approach proposed by Berger and Bernardo [1992]-Bernardo [1979], we find that the reference prior for the nuisance parameter beta, conditional on the parameter of interest alpha, is given by following improper prior:

$$\pi(\beta|\alpha) \propto \sqrt{I(\alpha, \beta)} = \frac{1}{\beta}. \tag{5.46}$$

As in Sun [1997], a natural sequence of compact sets for (α, β) is $(l_1 n, l_2 n) \times (q_1 n, q_2 n)$, to ensure that $l_1 i, q_1 i \to 0$ and $l_2 i, q_2 i \to \infty$ when $i \to \infty$. Thus, the normalizing constant is,

$$c_i(\alpha) = \frac{1}{\int_{q_1 i}^{q_2 i} \frac{1}{\beta} d\beta} = \frac{1}{\log q_2 i - \log q_1 i}. \tag{5.47}$$

From (3.13), the marginal reference prior for α will be

$$\pi_i(\alpha) \propto exp\left\{\frac{1}{2} \int_{q_1 i}^{q_2 i} c_i(\alpha) \frac{1}{\beta} log\left[\frac{\frac{1}{\alpha^2}[1 + \Gamma''(2) - \Gamma'(2)^2]}{\frac{1}{\beta^2}[1 + \Gamma''(2)]}\right] d\beta\right\}, \tag{5.48}$$

therefore,

$$\pi_i(\alpha) \propto \exp\left\{\frac{1}{2}\int_{q_{1i}}^{q_{2i}} c_i(\alpha)\frac{1}{\beta}\left[2\log\beta - 2\log\alpha + \log\frac{[1+\Gamma''(2)-\Gamma'(2)^2]}{[1+\Gamma''(2)]}\right]d\beta\right\}. \quad (5.49)$$

$$\pi_i(\alpha) \propto \exp\left\{\int_{q_{1i}}^{q_{2i}} c_i(\alpha)\frac{\log\beta}{\beta}\,d\beta - \int_{q_{1i}}^{q_{2i}} c_i(\alpha)\frac{\log\alpha}{\beta}\,d\beta + \frac{1}{2}c_i(\alpha)\log\frac{[1+\Gamma''(2)-\Gamma'(2)^2]}{[1+\Gamma''(2)]}\int_{q_{1i}}^{q_{2i}}\frac{1}{\beta}\,d\beta\right\}. \quad (5.50)$$

$$\pi_i(\alpha) \propto \exp\left\{\frac{1}{2}c_i(\alpha)\left[(\log q_{2i})^2 - (\log q_{1i})^2\right] - c_i(\alpha)\log\alpha\left[\log q_{2i} - \log q_{1i}\right]\right.$$
$$\left. + \frac{1}{2}c_i(\alpha)\log\frac{[1+\Gamma''(2)-\Gamma'(2)^2]}{[1+\Gamma''(2)]}\left[\log q_{2i} - \log q_{1i}\right]\right\}. \quad (5.51)$$

Assuming that the integration limits satisfy the relation,

$$c_i(\alpha)\left(\log q_{2i} - \log q_{1i}\right) = 1,$$

we obtain,

$$\pi_i(\alpha) \propto \frac{1}{\alpha}\sqrt{\frac{1+\Gamma''(2)-\Gamma'(2)^2}{1+\Gamma''(2)}}exp\left\{\frac{1}{2}c_i(\alpha)\int_{q_1 i}^{q_2 i}\frac{log\beta}{\beta}d\beta\right\}. \quad (5.52)$$

Thus, the resulting marginal referential prior for α is as follows:

$$\pi_i(\alpha) \propto \frac{1}{\alpha}\sqrt{\frac{1+\Gamma''(2)-\Gamma'(2)^2}{1+\Gamma''(2)}}exp\left\{\frac{1}{4}(\log q_1 i + \log q_2 i)\right\}. \quad (5.53)$$

The global referential prior for (α, β), with α representing the parameter of interest, considering $\alpha^* = 1$, is given by,

$$\pi_\alpha(\alpha,\beta) \propto \lim_{i\to\infty}\left(\frac{c_i(\alpha)\pi_i(\alpha)}{c_i(\alpha^*)\pi_i(\alpha^*)}\right)\pi(\beta|\alpha) \propto \frac{1}{\alpha\beta}. \quad (5.54)$$

A parallel approach is employed to obtain the reference prior, this time with the beta parameter of interest and the alpha nuisance. As a result, the prior is also,

$$\pi_\beta(\alpha,\beta) \propto \frac{1}{\alpha\beta}. \quad (5.55)$$

It should be noted that priors (5.54) and (5.55) are identical to Tibishirani's prior. Consequently, the Bayesian results presented in the previous section can be directly applied here. In addiction, when β is known, the reference prior is identical to the Jeffrey´s prior, but when β is unknown, these two priors differ by a factor $\frac{1}{\beta}$.

5.3.4 Gamma Prior for (α,β)

In this section, we will derive the prior and posterior distributions for the parameters of the Weibull distribution, considering a Gamma prior for both the shape α and scale β parameters.

We assume that both parameters α and β follow independent Gamma distributions:
- $\alpha \sim \text{Gamma}(a_1, b_1)$, with pdf:

$$p(\alpha) = \frac{b_1^{a_1}}{\Gamma(a_1)}\alpha^{a_1-1}e^{-b_1\alpha} \tag{5.56}$$

- $\beta \sim \text{Gamma}(a_2, b_2)$, with pdf:

$$p(\beta) = \frac{b_2^{a_2}}{\Gamma(a_2)}\beta^{a_2-1}e^{-b_2\beta} \tag{5.57}$$

The posterior distribution is proportional to the product of the likelihood function and the prior distributions:

$$p(\alpha, \beta|\boldsymbol{v}) \propto L(\alpha, \beta|\boldsymbol{v})p(\alpha)p(\beta). \tag{5.58}$$

Substituting the expressions for the likelihood and the priors, we get:

$$p(\alpha, \beta|\boldsymbol{v}) \propto \left(\frac{\beta^n}{\alpha^n}\prod_{i=1}^{n} v_i^{\beta-1}\right) e^{-\sum_{i=1}^{n}(v_i/\alpha)^\beta}\frac{b_1^{a_1}}{\Gamma(a_1)}\alpha^{a_1-1}e^{-b_1\alpha}\frac{b_2^{a_2}}{\Gamma(a_2)}\beta^{a_2-1}e^{-b_2\beta}. \tag{5.59}$$

Simplifying further:

$$p(\alpha, \beta \mid v) \propto \alpha^{a_1-1-n}\,\beta^{n+a_2-1}\left(\prod_{i=1}^{n} v_i^{\beta-1}\right)\exp\left\{-b_1\alpha - b_2\beta - \sum_{i=1}^{n}\left(\frac{v_i}{\alpha}\right)^\beta\right\}. \tag{5.60}$$

This is the joint posterior distribution for α and β.

5.4 Data Analysis

In this section, our aim is estimate the wind power of Presidente Prudente city (Brazil), using frequentist and Bayesian approaches for estimation of the parameters of Weibull distribution fitted to the wind speed data and hence estimation of wind power of the area. The data were obtained from the government agency INMET and consists of average wind speeds in meters per second, taken at hourly intervals during August from 2003 to 2024.

5.4.1 Goodness-of-Fit

Although it is generally accepted that the Weibull distribution provides a suitable model for wind speed variation, goodness-of-fit tests (such as Chi-square and Kolmogorov-Smirnov tests) were applied to validate this model for our data.

For Presidente Prudente, the Weibull distribution showed the best fit to the observed data for representing wind speed. The Kolmogorov-Smirnov test confirmed that the fitted Weibull distribution achieved a p-value greater than 0.8 across all time periods considered, indicating a good fit to the data.

Furthermore, we provide the empirical histogram with the computed Weibull fit superimposed on the same graph for comparison purposes. This is accompanied by theoretical and empirical cumulative probability plots, which provide additional insight into the fit quality.

Another graphical method used to verify the adequacy of a given distribution to observed data is the Quantile-Quantile Plot (Q-Q plot). Figure 5.1 presents graphs for the fitted Weibull distribution. These plots show that the Weibull distribution provides a good fit to the observed data.

Additionally, theoretical and empirical cumulative probability plots and percentile plots can be utilized for further assessment.

Thus, the Weibull probability distribution was assumed to model the wind speed data based on nonparametric statistical tests and graphical analyses performed for the study location.

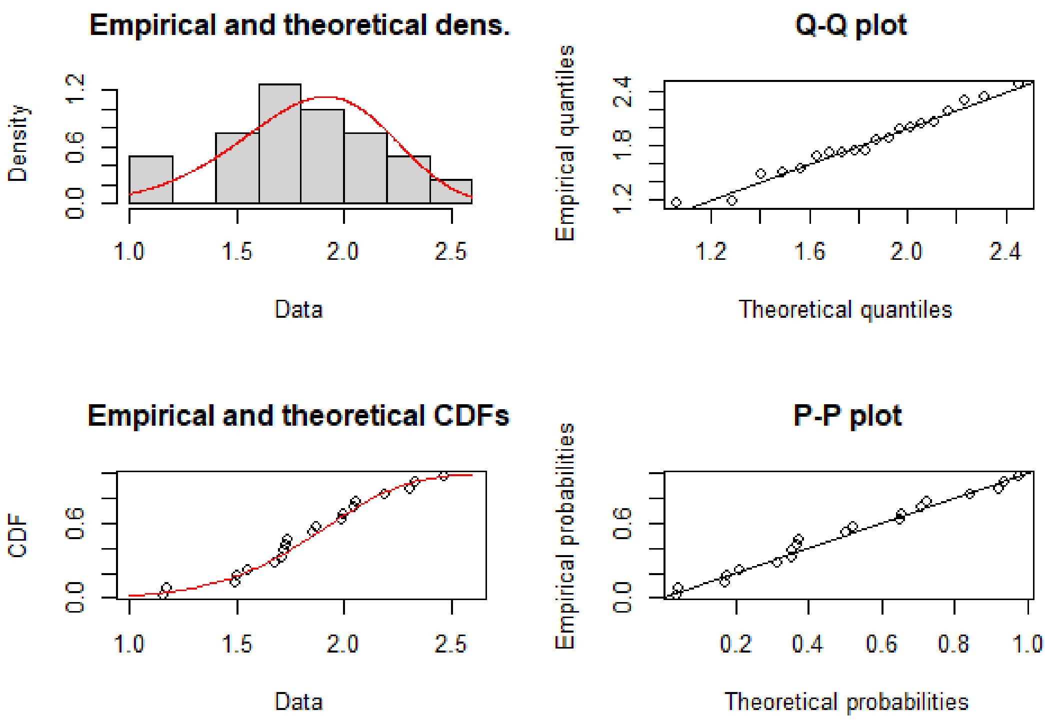

Figura 5.1: Weibull distribution fit to wind speed data.

5.4.2 Maximum likelihood and Bayesian estimation

Based on the identification of the probability distributions of the speed wind and wind power for the researched place, the Weibull parameters α and β were estimated according to the Maximum likelihood and Bayesian approaches.

The following figure illustrates the distribution of wind speed and potential power.

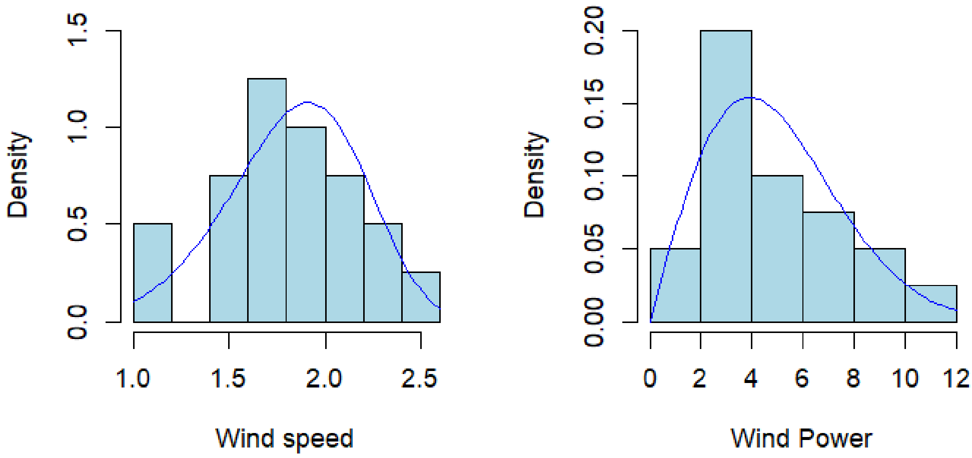

Figura 5.2: Distribution of wind speed and potential power.

First of all, we need to appeal to MCMC algorithm to obtain a sample of values of α and β from the joint posterior. This way, we can extract characteristics of marginal posterior distributions such as Bayes estimator, mode and confiance intervals. The chain is run for 50000 iterations. The MCMC output and autocorrelation plot for posterior distribution are showed in Figure 5.3 and graphical representations of the marginal posterior densities for the parameters α and β are plotted in Figure (5.2).

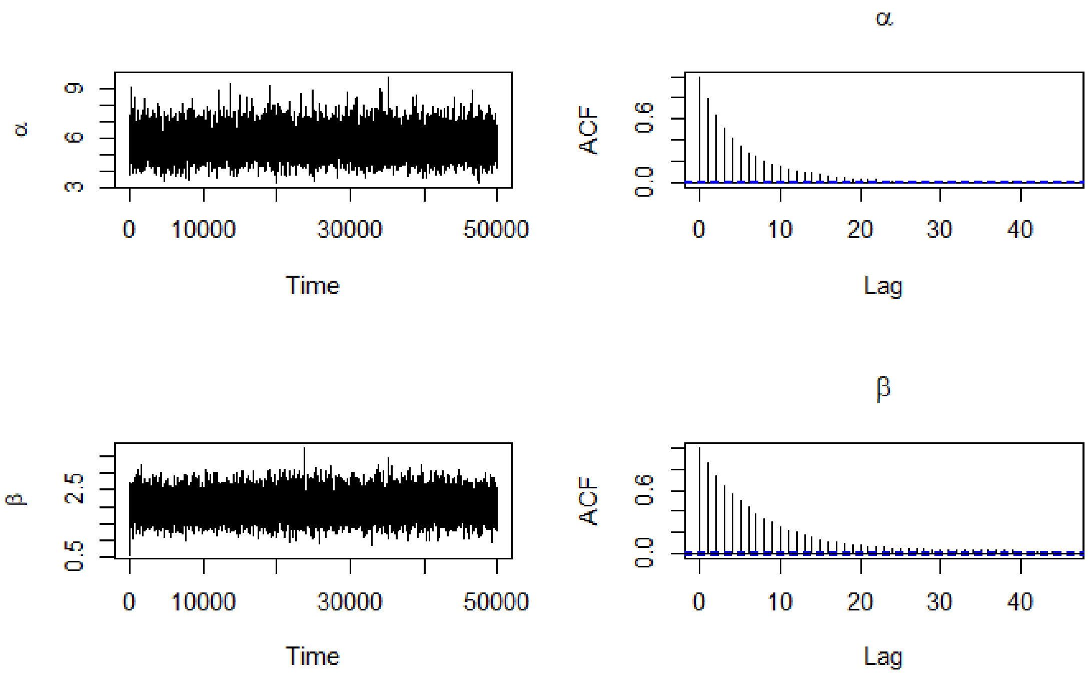

Figura 5.3: MCMC output and autocorrelation plot for the parameters of Jeffreys posterior distributions.

The MCMC diagnostic plots in Figure 5.3 provide insights into the convergence and mixing of the chains. The trace plots on the left show the sampled values of α and β over 50,000 iterations. Both plots exhibit stationary behavior, with the chains fluctuating around stable mean values, indicating that they have likely reached convergence and are sampling from their respective posterior distributions. These results correspond to the analysis under the Jeffreys prior, however, the behaviour of the chains was similar for the other prior specifications, exhibiting comparable mixing and stability patterns.

There is no evident trend or drift in either trace plot, which further supports the conclusion that the chains have reached equilibrium. The autocorrelation function (ACF) plots on the right reveal the degree of correlation between samples at different lags.

Both parameters show high autocorrelation at short lags, which gradually diminishes as the lag increases, approaching zero around lag 30–40. This pattern suggests adequate mixing of the chains, although the presence of autocorrelation at short lags implies that adjacent samples are not entirely independent.

Overall, the diagnostics in Figure 5.3, indicate that the MCMC process for both α and β has likely converged, with the chains exploring the posterior distributions effectively despite some short-term autocorrelation.

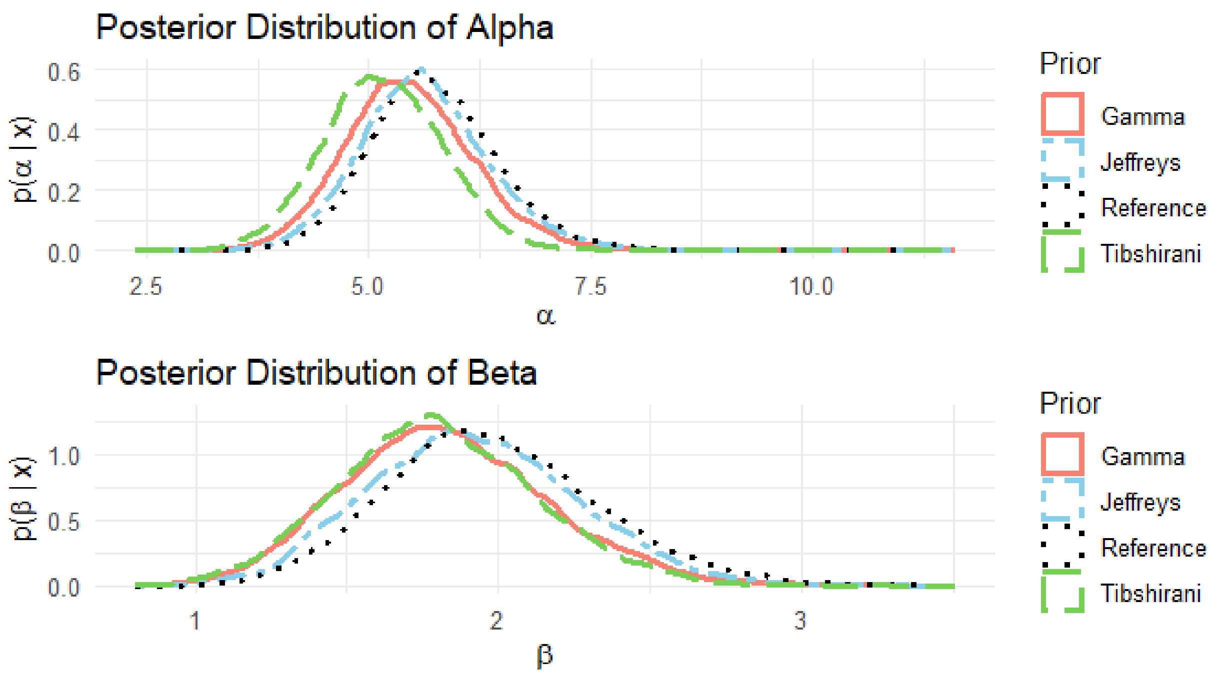

Figura 5.4: Marginal posterior densities of parameters α and β for the wind speed data.

The plots in Figure 5.4 illustrate the marginal posterior distributions for the parameters α and β in a model of wind speed. These distributions represent the updated beliefs about the values of α and β after incorporating observed data, providing insights into the uncertainty associated with each parameter.

In the upper plot, depicting the posterior distribution for α, we observe a density peak around $\alpha = 6$. This peak suggests that the most probable values for α are centered near this region. The distribution has a moderately spread shape, with a rightward tail extending towards higher values of α, indicating that while values around 6 are most likely, there is some probability mass extending towards higher values. This spread suggests a degree of uncertainty surrounding the estimation of α.

The lower plot shows the posterior distribution for β, which peaks more sharply at $\beta = 2$. The sharper peak implies a more concentrated probability mass around this estimate, suggesting that β is more precisely estimated relative to α. The distribution's tail, extending to the right, indicates a lower but non-negligible probability of larger values for β, albeit with decreasing likelihood.

These posterior distributions indicate that β has a narrower credible interval compared to α, suggesting less uncertainty in the estimate of β. The wider spread for α implies greater variability, which could reflect a higher sensitivity of this parameter to the observed data on wind speed. Overall, the posterior distributions provide a probabilistic view of the likely parameter values, where α appears less precisely estimated than β.

The data used were the averages of the original records, which consisted of wind speed measurements expressed in meters per second. These were collected at hourly intervals throughout August each year from 2003 to 2024, resulting in 21 observations.

In view of one of the main objectives of the work, the confidence interval was calculated for the maximum likelihood estimate (MLE) and the credibility interval for the Bayesian methods.

Tabela 5.1: Point estimators for α and β using the wind speed data.

	MLE	Jeffreys	Tibshirani	Reference	Vague Gamma
α	1.972955	1.976191	1.965312	1.983492	1.970901
β	5.963418	5.738381	5.430381	5.926586	5.529280

Tabela 5.2: Point estimators for α and β using the wind power data.

	MLE	Jeffreys	Tibshirani	Reference	Vague Gamma
α	5.5506	5.641221	5.181392	5.715020	5.479578
β	1.987806	1.919848	1.802412	1.983823	1.834641

Examining the Table 5.1 and 5.2 we conclude that both the maximum likelihood estimation (MLE) and Bayesian approaches produce similar point estimates for both parameters, $\alpha$and β, and for the different prior distributions. The 95% confidence intervals from the maximum likelihood estimate (MLE) and the 95% credible intervals from the Bayesian approach for each parameter are displayed in the table. 5.3 and 5.4.

Tabela 5.3: (a) Wind speed

	MLE	Jeffreys	Tibshirani	Reference	Vague Gamma
α (CI 95%)	(1.820; 2.126)	(1.814; 2.143)	(1.795; 2.139)	(1.825; 2.151)	(1.800; 2.144)
Range	0.306	0.329	0.344	0.327	0.344
Acceptance rate	–	0.200	0.209	0.237	0.252
β (CI 95%)	(3.933; 7.994)	(3.840; 7.819)	(3.632; 7.468)	(4.076; 8.051)	(3.670; 7.601)
Range	4.061	3.979	3.836	3.975	3.931
Acceptance rate	–	0.202	0.218	0.210	0.221

Tabela 5.4: (b) Wind power

	MLE	Jeffreys	Tibshirani	Reference	Vague Gamma
α (CI 95%)	(4.356; 6.945)	(4.297; 7.242)	(3.884; 6.561)	(4.428; 7.221)	(4.147; 7.020)
Range	2.589	2.946	2.676	2.792	2.873
Acceptance rate	–	0.487	0.475	0.546	0.565
β (CI 95%)	(1.311; 2.665)	(1.302; 2.616)	(1.194; 2.490)	(1.353; 2.686)	(1.209; 2.539)
Range	1.354	1.314	1.296	1.333	1.330
Acceptance rate	–	0.203	0.206	0.209	0.221

Despite minor differences, the Bayesian and MLE estimates are consistent, with intervals of similar widths for all prior distributions, indicating robustness of the results across different inferential approaches. The estimations of α and β for each method and the expected wind power (P_E) are shown in Table 5.5.

Tabela 5.5: Expected wind power potential P_E obtained from the estimated parameters α and β.

Method	$\hat{\alpha}$	$\hat{\beta}$	$\hat{P_E}$
MLE	5.5506	1.987806	165.3724
Jeffreys	5.6412	1.919848	180.4164
Tibshirani	5.1814	1.802412	151.0034
Reference	5.7150	1.983823	180.8940
Vague Gamma	5.4796	1.834641	174.6073

The analyses presented in this chapter highlighted the use of the Weibull distribution to estimate wind speed and wind power through both classical and Bayesian approaches. The results confirmed the consistency of the estimates under different inferential methods and priors. The methodological framework developed here establishes the basis for the subsequent chapter, where elicitation strategies are explored as an alternative for incorporating prior information into the Bayesian analysis of wind potential. The findings for the wind speed and wind power show that there is no feasibility of wind turbines in the region os Presidente Prudente.

Capítulo

6

Elicitation Applied to the Weibull Distribution

6.1 Introduction

One of the greatest advantages of Bayesian inference is its ability to combine expert knowledge about an unknown parameter of interest with empirical data. This process of formal integration is known as elicitation.

The incorporation of expert insights leads to a more precise and up-to-date understanding of the parameter, further highlighting the value of Bayesian methods.

Despite its importance in statistical analysis, prior knowledge is often overlooked in Bayesian studies. This may stem from difficulties in subjectively quantifying expert opinions or from the assumption that with large datasets, prior information has minimal influence on results. Moreover, the availability of techniques for specifying non-informative priors leads many practitioners to disregard expert input in favor of neutral assumptions. However, when analyzing novel or rare phenomena—such as modeling earthquake impacts, studying rare diseases, or other data-sparce scenarios—expert judgment becomes a critical source of information. In addition, elicitation is widely applicable across various scientific fields, particularly in cases where experimental data are limited due to collection challenges or high costs.

The Weibull distribution parameters, typically interpreted as shape and scale, directly influence energy output estimates and, consequently, the economic feasibility of wind projects. Given the sensitivity of energy production forecasts to these parameters, incorporating expert knowledge through elicitation strengthens the modeling process and enables more robust decision making.

Furthermore, the elicitation process fosters structured collaboration between analysts and domain experts. By translating qualitative insights into quantitative priors, it enables transparent and rigorous Bayesian modeling, particularly in the early stages of assessments, where measurement data may be limited or incomplete.

This chapter explores the practical application of elicitation in the context of the Weibull distribution for wind energy modeling. It presents methodologies for integrating expert knowledge into prior distributions and discusses how this approach can enhance the probability of wind resource assessments in uncertain conditions.

6.2 Laplace Method for Integral Approximation

Often, we are unable to solve integrals analytically, that is to say, we cannot determine a closed-form expression for the integral. One possible solution is to use numerical approximation methods. However, in Bayesian inference, we need an expression for these integrals, rather than just their numerical value.

One of the most popular methods of approximating integrals is given by the **Laplace Method** proposed by Tierney and Kadane [1989].

The Laplace Method for Integral Approximation is a technique widely used before MCMC in Bayesian Inference to calculate marginal posteriors, posterior moments, and predictive densities.

It is based on a second-order Taylor expansion applied to the exponential term of the integrand. In this section, we provide a brief and informal description of the method and its properties. For more details, see Kass et al. [1990].

Let $h : \mathbb{R} \to \mathbb{R}$ be a differentiable function of θ where $-h$ has a unique maximum point $\hat{\theta}$; the Laplace Method approximates an integral of the form,

$$I = \int_0^\infty f(\theta) \exp\{-nh(\theta)\}\, d\theta \tag{6.1}$$

by expanding h and f around $\hat{\theta}$.

The function $f(\theta)$ is chosen so that $\hat{\theta}$ can be obtained explicitly.

For the unidimensional parameter case, we obtain:

$$\int_0^\infty f(\theta) \exp\{-nh(\theta)\}\, d\theta \cong \left(\frac{2\pi}{nh''(\hat{\theta})}\right)^{1/2} f(\hat{\theta}) \exp\{-nh(\hat{\theta})\} \tag{6.2}$$

where $h''(\hat{\theta})$ is the second derivative of h evaluated at $\hat{\theta}$.

Kass, Tierney, and Kadane argue that:

$$I = \hat{I}\left\{1 + \mathrm{O}(n^{-1})\right\} \tag{6.3}$$

where

$$\hat{I} = \left(\frac{2\pi}{nh''(\hat{\theta})} \right)^{1/2} f(\hat{\theta}) \exp\{-nh(\hat{\theta})\}$$

For the m-dimensional case:

$$\int_0^\infty f(\theta) \exp\{-nh(\theta)\}\, d\theta \cong \left(\frac{2\pi}{n} \right)^{m/2} \left[\det\left(H(\hat{\theta}) \right) \right]^{-1/2} f(\hat{\theta}) \exp\{-nh(\hat{\theta})\} \tag{6.4}$$

where $H(\hat{\theta})$ is the Hessian matrix of h evaluated at $\hat{\theta}$.

In fact, the Laplace method converts an integration problem into an optimization solution.

The approximation will be very accurate when the posterior distribution is approximately normal. However, if this is not the case, achieving good approximations requires careful choice of parameterization, particularly for small sample sizes.

Achcar (1990) derived a formula to assess the error of the Laplace approximation, as well as a reparametrization technique for achieving accurate approximations.

6.3 Weibull Distribution through Laplace application

The probability density function of Weibull distribution for a wind speed variable V is given in (5.1) and the probability that wind speed is greater than or equal to a speed is provided by (5.2).

The cumulative Weibull distribution function is:

$$F(v) = 1 - \exp\left\{ -\left(\frac{v}{\alpha} \right)^\beta \right\}. \tag{6.5}$$

In a Bayesian framework, a prior distribution must be specified for the parameters. One may assume independent Gamma priors for α and β, namely Gamma(a, b) and Gamma(c, d), resulting in the joint prior distribution:

$$\pi(\alpha, \beta) = \frac{bd}{\Gamma(a)\Gamma(c)} (b\alpha)^{a-1} (d\beta)^{c-1} \exp\{-(b\alpha + d\beta)\}. \tag{6.6}$$

Eliciting prior distributions for α and β directly is often difficult in wind energy studies. Experts are unlikely to provide prior density values, but they may be more comfortable providing quantiles or probabilities for wind speeds.

Thus, the elicitation process can be based on quantiles of the wind speed distribution. The expert may be asked to provide the wind speed values corresponding to certain cumulative probabilities. For example, if the expert believes that the wind speed is greater than v with 25% probability, then this implies

$$P(V \geq v_i) = 1 - F(v_i) = \exp\left\{-\left(\frac{v_i}{\alpha}\right)^{\beta}\right\} = 0.25, \quad i = 1, \ldots, k. \tag{6.7}$$

Suppose four percentiles are elicited, the 10^{th}, 25^{th}, 50^{th}, and 75^{th}, this percentiles were chosen, as these are commonly used in the literature.

Note that, in fact, the expert is being asked about $P(V \geq v_i)$, given by:

$$\begin{aligned} P(V \geq v_i) &= \int_0^\infty \int_0^\infty P(V > v_i|\alpha, \beta)\pi(\alpha, \beta)\, d\alpha d\beta \\ &= \frac{bd}{\Gamma(a)\Gamma(b)} \int_0^\infty \int_0^\infty (b\alpha)^{a-1}(d\beta)^{c-1} \exp\{-(b\alpha + d\beta)\} \exp\left\{-\left(\frac{v_i}{\alpha}\right)^{\beta}\right\} d\alpha d\beta, \end{aligned} \tag{6.8}$$

with $i = 1, ..., 4$, which will be a function of the hyperparameters a, b, c, and d.

Note that there will be four expressions (6.8), one for each elicited percentile.

This integral has no closed-form solution, but can be approximated using Laplace's method of Tierney and Kadane [1989], yielding:

$$P(V \geq v_i) = \frac{2\pi\ bd}{\Gamma(a)\Gamma(b)} \exp\left\{-\left(\frac{v_i}{\hat{\alpha}}\right)^{\hat{\beta}}\right\} (\hat{b}\hat{\alpha})^{a-1}(\hat{d}\hat{\beta})^{c-1} \exp\{-(\hat{b}\hat{\alpha}+\hat{d}\hat{\beta})\} \cdot \frac{\hat{\alpha}\hat{\beta}}{\sqrt{(a-1)(c-1)}}, \tag{6.9}$$

where $\hat{\alpha} = \frac{a-1}{b}$ and $\hat{\beta} = \frac{c-1}{d}$ with $a > 1$, $c > 1$, $i = 1, \ldots, 4$.

Thus, a nonlinear system of four equations with four unknowns a, b, c, and d is obtained.

Hence, by solving the nonlinear system of equations derived from (6.9) for each elicited quantile, one can determine the hyperparameters a, b, c, and d of the prior distribution defined in (6.6).

6.4 Numerical Illustration

Although the Laplace method is traditionally employed for integral approximation in Bayesian analysis, this study adopted Monte Carlo simulation through the Metropolis-Hastings algorithm. This choice is justified by the greater flexibility of MCMC in representing the posterior distribution without relying on quadratic approximations around the maximum likelihood estimate.

Samples of sizes $n = 5$, $n = 8$, and $n = 12$ were generated from a Weibull distribution with parameters $\alpha = 6$ and $\beta = 1.5$, which are typical for modeling wind power. The selected samples are shown in Table 6.1.

To illustrate the case in which prior expert information is available, the 10^{th}, 25^{th}, 50^{th}, and 75^{th} percentiles were elicited using the quantiles of the Weibull distribution

Tabela 6.1: Simulated samples

Size	Sample											
$n = 5$	8.74	4.15	0.89	7.13	1.01							
$n = 8$	4.03	3.98	5.13	7.67	0.51	0.79	10	5.29				
$n = 12$	6.86	1.08	9.25	3.85	6.41	9.12	3.73	1.86	0.65	9.26	6.75	2.85

with parameters $\alpha = 6$ and $\beta = 1.5$. Thus, the obtained probabilities are exact, and the hyperparameters of the prior distribution defined in (6.6) were: $a = 56.25$, $b = 9.19$, $c = 42.74$, and $d = 27.60$.

However, it is very difficult for the expert, even with considerable knowledge of the subject, to provide exact probabilities. Therefore, in order to represent the possibility of imprecision in the elicitation process, a convex combination of the exact elicitation and a weakly informative prior was considered. Specifically, the hyperparameters with error were defined as

$$\text{hip}_{err} = 0.9 \cdot \text{hip}_{exact} + 0.1 \cdot \text{hip}_{ni}, \tag{6.10}$$

where $\text{hip}_{\text{exact}}$ denotes the hyperparameters obtained from the exact elicitation process, hip_{ni} represents the hyperparameters of a weakly informative prior, and hip_{err} corresponds to the resulting hyperparameters incorporating elicitation error through a convex combination. In this case, the hyperparameters were: $a = 50.63$, $b = 8.27$, $c = 38.47$, and $d = 24.84$.

A Markov chain with 20,000 iterations was generated using the Metropolis-Hastings algorithm in R, with the objective of simulating samples of the posterior values of α, β, and the probability $P(V \geq 7)$. Convergence diagnostics were verified using formal tests such as Geweke and Gelman-Rubin diagnostics.

Table 6.2 shows the estimates of the unknown parameters and their variances for different sample sizes. Values in parentheses indicate deviation from defined parameters, $\alpha = 6$ and $\beta = 1.5$.

Tabela 6.2: Posterior estimates of the parameters $\alpha = 6$, $\beta = 1.5$, and $\hat{P}(V \geq 7) = 0.284$ for different sample sizes and priors.

Sample size	Prior	$\hat{\alpha}$	$\hat{\beta}$	$P(V \geq 7)$
$n = 5$	Elicited + error	5.82 (0.58)	1.50 (0.18)	0.27 (0.05)
	Elicited exact	5.83 (0.56)	1.51 (0.17)	0.27 (0.05)
	Non-informative	4.76 (1.69)	1.09 (0.37)	0.22 (0.10)
$n = 8$	Elicited + error	5.86 (0.58)	1.53 (0.17)	0.27 (0.05)
	Elicited exact	5.91 (0.54)	1.53 (0.17)	0.27 (0.05)
	Non-informative	5.12 (1.29)	1.27 (0.32)	0.22 (0.09)
$n = 12$	Elicited + error	6.02 (0.53)	1.57 (0.17)	0.28 (0.05)
	Elicited exact	5.95 (0.51)	1.57 (0.16)	0.27 (0.05)
	Non-informative	6.11 (1.23)	1.51 (0.31)	0.29 (0.09)

For all sample sizes, the estimates obtained using elicitation and elicitation with error were closer to the true value and, exhibited lower average bias compared to those using non-informative priors. As the sample size increases, the prior influence diminishes and results become more similar. Figures 6.1, 6.2, and 6.3 illustrate this.

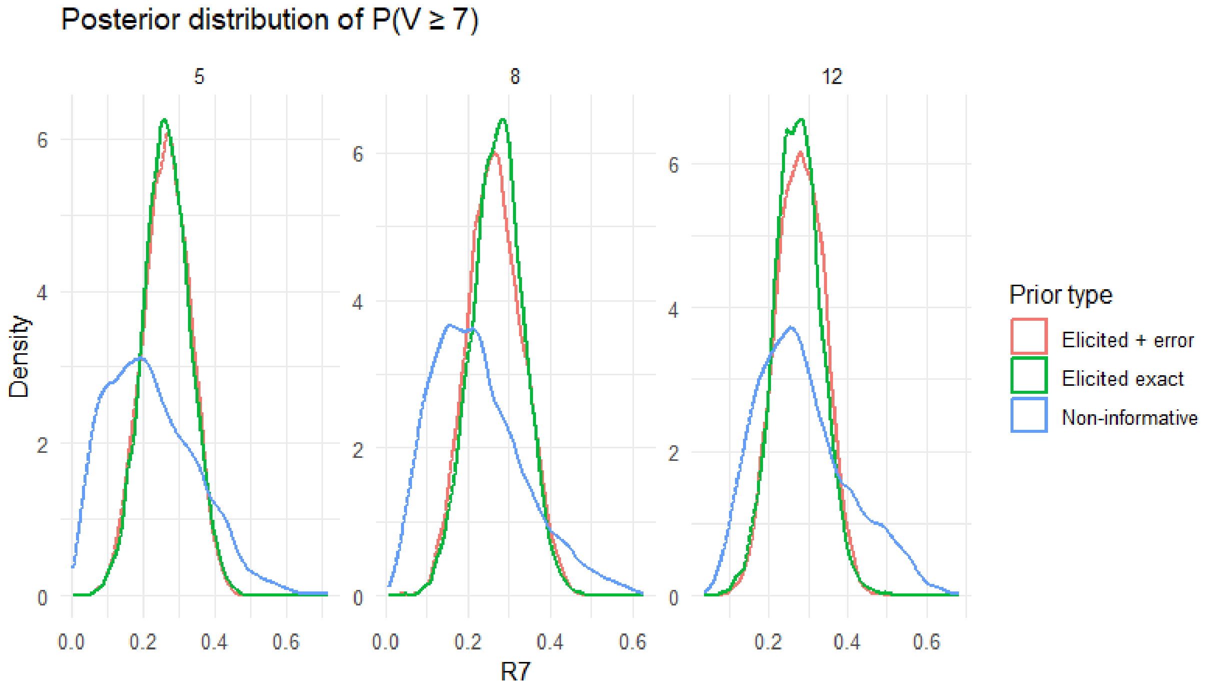

Figura 6.1: Posterior density of $P(V \geq 5)$ for $n = 5$, $n = 8$ and $n = 12$.

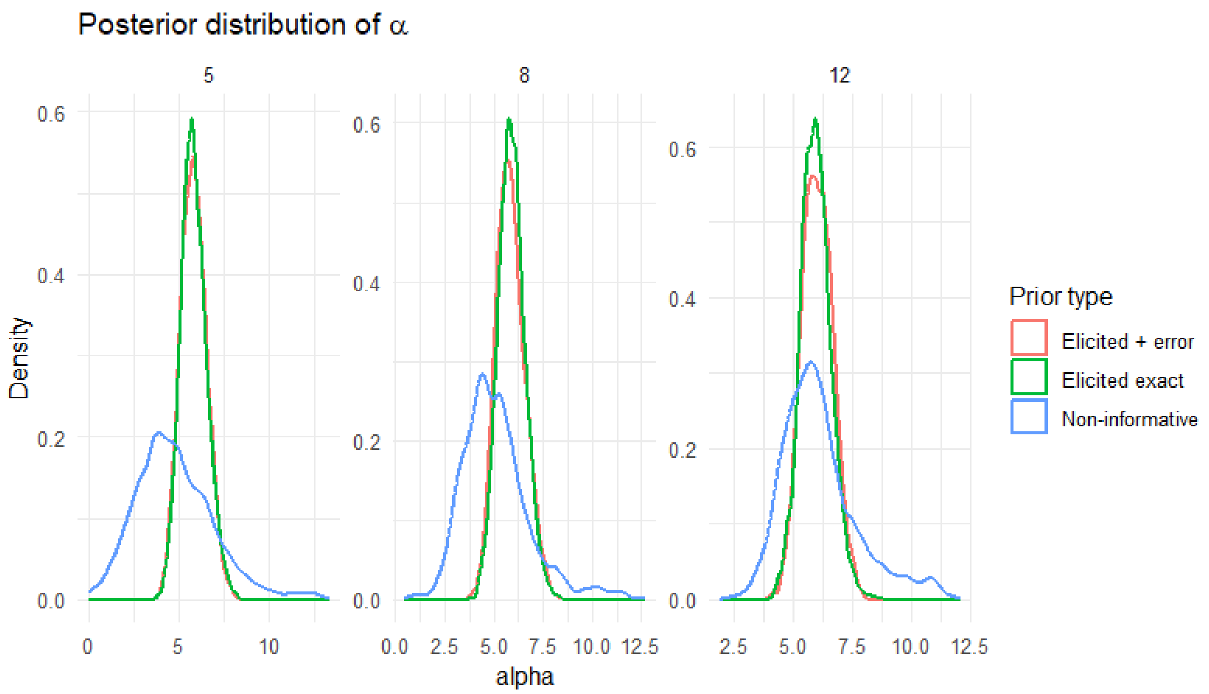

Figura 6.2: Posterior density of α for $n = 5$, $n = 8$ and $n = 12$.

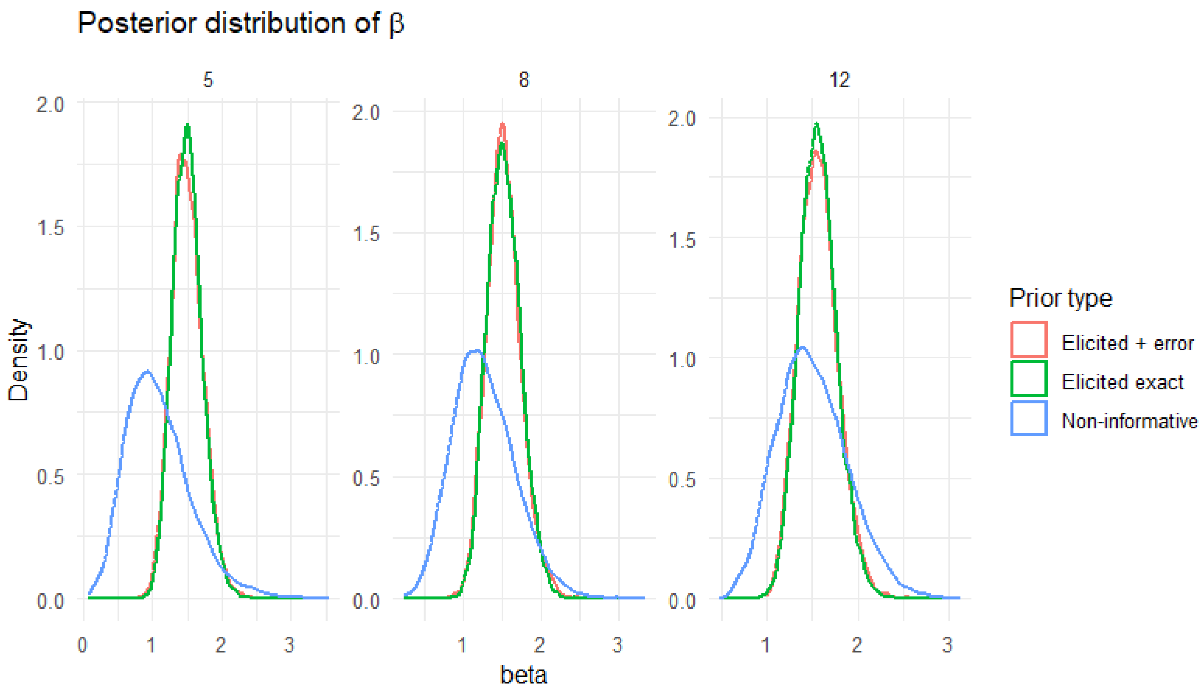

Figura 6.3: Posterior density of β for $n = 5$, $n = 8$ and $n = 12$.

The results obtained proved relevant to situations where there is a shortage of data and prior expert knowledge. Therefore, elicitation is of paramount importance in statistical studies, particularly when the lack of information from the sample makes conventional statistical analysis impractical.

Capítulo

7

Elicitation Applied to Wind Speed Data

7.1 Introduction

In the absence of direct expert knowledge, an alternative elicitation strategy was adopted by exploiting empirical information derived from the observed wind speed data. This methodology is based on the estimation of Weibull parameters through Maximum Likelihood Estimation (MLE), followed by the construction and comparison of different prior distributions. Such an approach is justified, since in practice, expert judgment is often unavailable or highly subjective, and empirical elicitation provides a data-driven surrogate.

The Weibull distribution, parameterized by the scale (α) and shape (β), is widely used in wind speed modeling due to its flexibility in capturing the variability of the data. Using the MLE method, we obtained the initial point estimates $\hat{\alpha} = 1.97$ and $\hat{\beta} = 5.96$, which served as reference values.

Three distinct prior formulations were constructed and compared:

1. **Non-informative prior:** A vague specification using Gamma$(0.01, 0.01)$ for both parameters, which ensures propriety while remaining weakly informative. This formulation approximates a flat prior, allowing the posterior distribution to be primarily driven by the observed data.

2. **Elicitation based on empirical quantiles (Specialist surrogate – MLE):** In this case, the Laplace elicitation system of equations was solved by matching the empirical quantiles of the wind speed distribution with the exceedance probabilities implied by the Weibull distribution. The resulting hyperparameters were:

$$a = 2.9605, \quad b = 0.9841, \quad c = 5.2015, \quad d = 0.6175$$

3. **Elicitation based on Bayesian Inference (MLE-informed):** In this approach, the system of equations for prior elicitation was also solved, but instead of relying on empirical quantiles, the theoretical quantiles of the Weibull distribution fitted via MLE were employed. This ensures that the elicited prior remains consistent with the parametric structure suggested by the likelihood function, while still being derived from a formal elicitation procedure. The resulting hyperparameters were:

$$a = 2.1505, \quad b = 0.5954, \quad c = 2.4367, \quad d = 0.2057.$$

The wind speed data were obtained from the meteorological station at Presidente Prudente (INMET), with a sample size of $n = 21$.

Posterior inference was performed via Markov Chain Monte Carlo (MCMC), using six parallel chains and 900,000 retained samples after burn-in, ensuring stability of the estimates. The 10^{th}, 25^{th}, 50^{th}, and 75^{th} percentiles were extracted.

This MLE-based Bayesian approach provides a balance between fully subjective elicitation and data-driven methods. It is particularly useful when historical data are available but expert judgment is not, allowing the integration of prior structure into the analysis and enhancing both robustness and interpretability.

7.2 Application and Results

The convergence analysis of the MCMC chains, using the Gelman-Rubin potential scale reduction factor ($\hat{R}$), indicated values equal to or very close to 1 for both parameters under all approaches, confirming good convergence and the stability of the posterior estimates.

Table 7.1 presents the posterior mean estimates. The values are very close across the three methods, reinforcing the robustness of the results. Notably, the estimates obtained using elicited priors were slightly closer to the reference MLE values and exhibited lower variability compared to those obtained under the non-informative prior.

Tabela 7.1: Posterior mean estimates of α, β, and $P(V \geq 7)$.

Prior type	$\hat{\alpha}$	$\hat{\beta}$	$P(V \geq 7)$
Elicited - Bayesian	1.99 (0.0645)	5.97 (0.814)	5.20×10^{-12} (1.04×10^{-11})
Elicited - MLE	1.99 (0.0637)	6.03 (0.786)	4.94×10^{-13} (9.88×10^{-13})
Non-informative	1.98 (0.0670)	5.73 (0.816)	8.41×10^{-11} (1.68×10^{-10})

Tabela 7.2: 95% confidence intervals for α and β.

	Elicited - Bayesian	Elicited - MLE	Non-informative
$\hat{\alpha}$	(1.83; 2.15)	(1.83; 2.15)	(1.81; 2.15)
Range	0.32	0.32	0.34
$\hat{\beta}$	(4.11; 8.10)	(4.22; 8.08)	(3.86; 7.87)
Range	3.99	3.86	4.01

The posterior distributions of $P(V \geq 7)$ exhibited very similar behavior across all methods. According to the fitted model, the probability of wind speeds reaching or exceeding 7 m/s at the studied site is essentially zero. This result highlights a limited potential for high wind events, which directly constrains the suitability of the location for wind power generation.

Figures 7.1 and 7.2 show the posterior densities of α and β under the three priors. The results demonstrate that the data are highly informative: the posterior distributions overlap substantially, with only minimal differences in the tails and density heights.

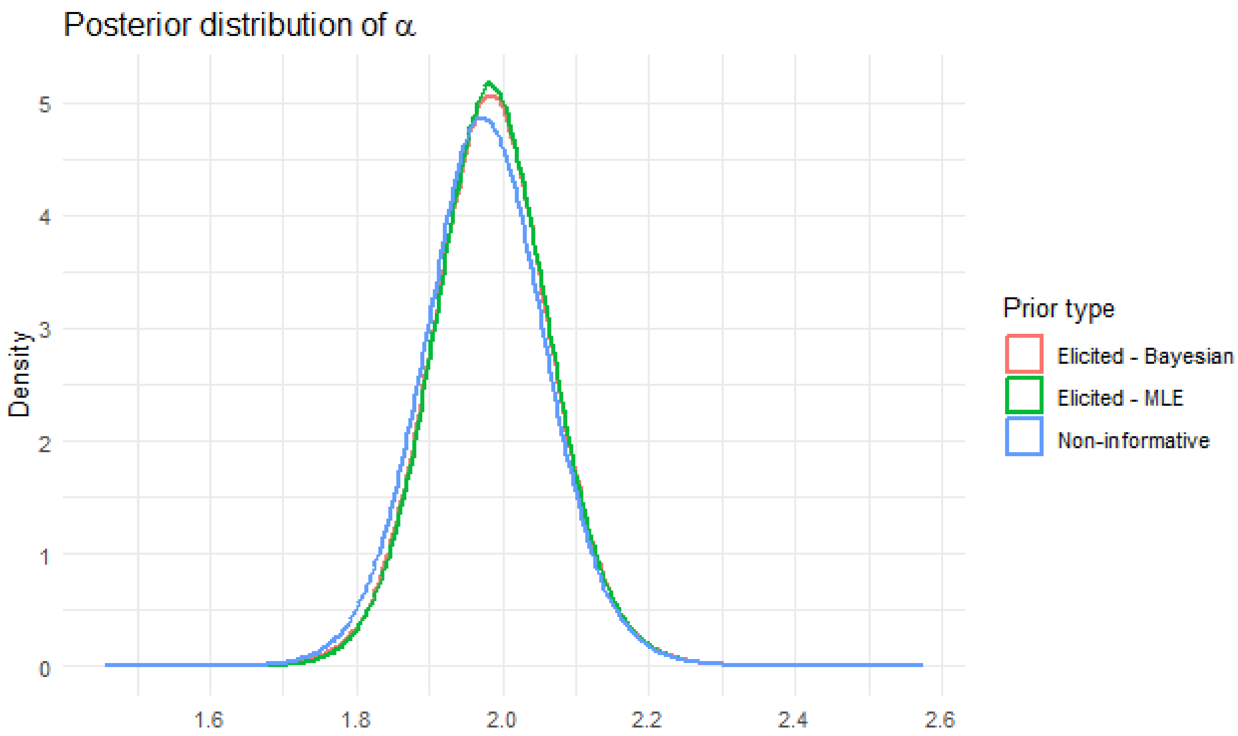

Figura 7.1: Posterior density of α under different priors.

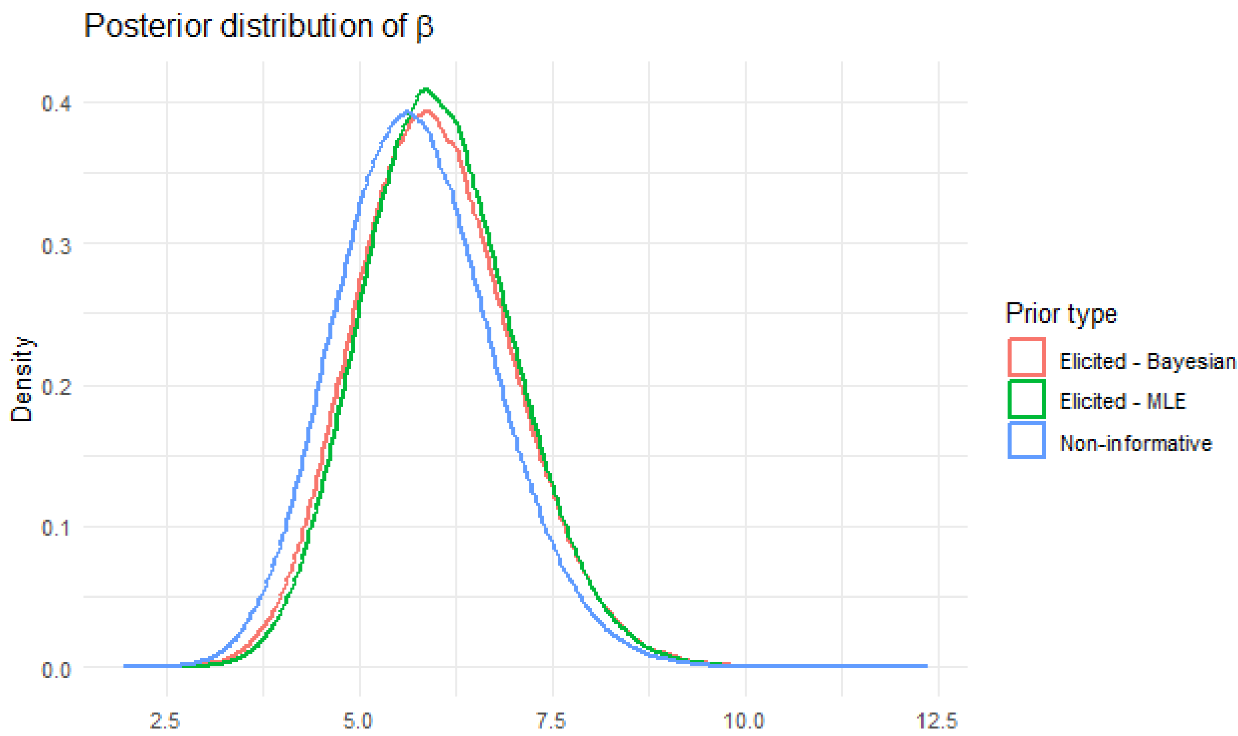

Figura 7.2: Posterior density of β under different priors.

Afterwards, the parameters α and β for wind power data were obtained in order to calculate the potential power using the equation 2.4.

Regarding the wind power potential, the posterior means obtained under the three methods were also very similar, further reinforcing the reliability of the results. The elicited priors, whether based on MLE or Bayesian inference, provided posterior distributions that were slightly more precise and consistent with the empirical evidence than those obtained under the non-informative prior.

Tabela 7.3: Posterior mean estimates of α, β, and wind power potential P_E.

Prior type	$\hat{\alpha}$	$\hat{\beta}$	$\hat{P}_E$
Elicited - Bayesian	5.69 (0.554)	1.99 (0.271)	177.716
Elicited - MLE	5.70 (0.548)	2.01 (0.262)	176.816
Non-informative	5.616 (0.570)	1.91 (0.272)	179.035

Tabela 7.4: 95% confidence intervals for α and β (wind power).

	Elicited - Bayesian	Elicited - MLE	Non-informative
$\hat{\alpha}$	(4.40; 7.19)	(4.43; 7.19)	(4.29; 7.16)
Range	2.79	2.76	2.87
$\hat{\beta}$	(1.37; 2.70)	(1.41; 2.69)	(1.29; 2.62)
Range	1.33	1.28	1.33

In summary, the results indicate that the elicitation based on MLE and Bayesian Inference was valid and consistent, and representing a valuable alternative in scenarios where expert knowledge is unavailable. Although the specific application to Presidente Prudente suggests a modest wind potential, the methodology itself is general and can be extended to other sites and datasets, thus contributing to more reliable assessments of renewable energy resources.

Capítulo

8

Conclusions

This work presented a comprehensive study on the application of the Weibull distribution to estimate wind potential, exploring both classical and Bayesian approaches under different inferential strategies. The analysis demonstrated that the two-parameter Weibull model provides a suitable framework for describing wind speed distributions and, consequently, for estimating wind power potential with precision.

In the Bayesian framework, several prior structures were evaluated, including Jeffreys, Tibshirani, reference, and gamma priors. The results showed consistency across all priors, with credible intervals of similar ranges, which highlights the reliability and robustness of Bayesian inference for wind energy modeling. Furthermore, the Bayesian approach proved to be extremely flexible, allowing inference for any quantity of interest without relying on asymptotic approximations.

An alternative elicitation strategy was proposed, based on the estimation of Weibull parameters through Maximum Likelihood Estimation (MLE) and the subsequent construction of informative priors using the Laplace system of equations. This approach was implemented in two ways: (i) by matching empirical quantiles of the observed wind speed distribution to their theoretical exceedance probabilities, and (ii) by adopting the theoretical quantiles of the Weibull distribution fitted via MLE. Both procedures ensured that the elicited priors remained consistent with the data, while the non-informative Gamma$(0.01, 0.01)$ prior served as a benchmark. Posterior inference was carried out via MCMC, with convergence diagnostics confirming the stability of the estimates. The predominance of the likelihood over the priors was evident, reinforcing the robustness of the methodology.

From an applied perspective, the analyses, whether using non-informative priors, empirical elicitation, or MLE-informed elicitation, converged to the same conclusion, the region of Presidente Prudente does not present favorable wind conditions for large-scale wind energy generation. The posterior probability of wind speeds exceeding 7 m/s was essentially zero, and the estimates of wind speed and wind power consistently indicated the infeasibility of installing wind turbines in the area.

Despite these regional limitations, the methodologies developed and applied in this study remain of great relevance. They can be adapted to other locations where wind conditions are more favorable, provided that reliable data on wind speed distributions are available. The Brazilian National Meteorological Institute (INMET) provides such datasets for most cities in Brazil, making it possible to extend these methods to evaluate the wind potential nationwide.

Finally, future research could benefit from incorporating additional variables, such as wind direction, seasonal variability, and topographical effects, to further improve the accuracy of wind potential assessments. Moreover, with the increasing availability of high-resolution and real-time data, Bayesian methods combined with elicitation strategies have the potential to significantly enhance decision-making processes in renewable energy planning.

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