JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **221,** 538–543 (1998)
ARTICLE NO. AY985912

Characterization of Generalized Bessel Polynomials in Terms of Polynomial Inequalities*

Eliana X. L. de Andrade, Dimitar K. Dimitrov,[†] and A. Sri Ranga

Departamento de Ciências de Computação e Estatística, IBILCE, Universidade Estadual Paulista, São José do Rio Preto, 15054-000 SP, Brazil

Submitted by Robert A. Gustafson

Received July 12, 1996

Generalized Bessel polynomials (GBPs) are characterized as the extremal polynomials in certain inequalities in L^2 norm of Markov type.

1. INTRODUCTION AND STATEMENT OF RESULTS

Generalized Bessel polynomials $y_n(x; \alpha, \beta)$ are defined by

$$y_n(x; \alpha, \beta) := \sum_{k=0}^{n} (-n)_k (n + \alpha - 1)_k \frac{(-x/\beta)^k}{k!}$$
$$= {}_2F_0(-n, n + \alpha - 1; -; -x/\beta), \tag{1}$$

where $(a)_k$ denotes the Pochhammer symbol, $(a)_0 := 1$, $(a)_k := a(a+1) \cdots (a + k - 1)$, $k \geq 1$. For $\alpha = \beta = 2$ they reduce to the simple Bessel polynomials $y_n(x) := y_n(x; 2, 2)$. A recent review of Srivastava [5] contains comprehensive historical information and new results about GBPs. Here we mention only the following orthogonality relation given in [5]:

$$\int_0^\infty x^{\alpha-2} e^{-\beta/x} y_j(x; \alpha, \beta) y_m(x; \alpha, \beta)\, dx$$
$$= \beta^{\alpha-1} \frac{j!}{1 - \alpha - 2j} \Gamma(2 - \alpha - j)\, \delta_{jm}, \tag{2}$$
$$\mathrm{Re}(\alpha) < 1 - m - j, \qquad \mathrm{Re}(\beta) > 0, \qquad m, j \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}.$$

* Research supported by the Brazilian foundations FAPESP and CNPq and the Bulgarian Science Foundation under Grant MM-414.

[†] On leave from the University of Rousse, Bulgaria. E-mail: dimitrov@nimitz.dcce.ibilce.unesp.br.

0022-247X/98 $25.00

Even though the parameter β does not have essential influence in our investigation, we keep it in order to maintain the common notation $y_n(x; \alpha, \beta)$.

Observe that, for $\alpha, \beta \in \mathbb{R}$, $\beta > 0$, $n \in \mathbb{N}_0$, and $2n < 1 - \alpha$,

$$\|p\|_{\alpha, \mathscr{P}_n} = \left(\int_0^\infty x^{\alpha - 2} e^{-\beta/x} p^2(x)\, dx \right)^{1/2} \tag{3}$$

is a well-defined norm in the space $\mathscr{P}_n$ of real algebraic polynomials of degree not exceeding n. Then under the same restrictions on α, β, and n, $\|p'\|_{\alpha+2, \mathscr{P}_{n-1}}$ is well defined because the inequalities $2n < 1 - \alpha$ and $2(n-1) < 1 - (\alpha + 2)$ are equivalent.

First we prove an inequality of Markov type for the norm (3) for which GBPs are the extremal polynomials:

THEOREM 1. *Let $\alpha, \beta \in \mathbb{R}$, $\beta > 0$, $n \in \mathbb{N}_0$, and $2n < 1 - \alpha$. For every $p \in \mathscr{P}_n$ and any positive integer k, $k \leq n$,*

$$\|p^{(k)}\|_{\alpha + 2k, \mathscr{P}_{n-k}} \leq \left(\frac{n!}{(n-k)!} \prod_{j=1}^{k} (4 - \alpha - n - 3j) \right)^{1/2} \|p\|_{\alpha, \mathscr{P}_n}. \tag{4}$$

Moreover, for each k, $k = 1, \ldots, n$, equality is attained if and only if p is a constant multiple of $y_n(x; \alpha, \beta)$.

It is clear that the expression $n!/(n-k)!\prod_{j=1}^{k}(4 - \alpha - n - 3j)$, which appears in the best constant

$$M_n(k, \alpha) = \left(\frac{n!}{(n-k)!} \prod_{j=1}^{k} (4 - \alpha - n - 3j) \right)^{1/2}$$

in (4), is positive because $2n < 1 - \alpha$.

Inequalities similar to (4) which characterize the classical Jacobi and general Laguerre polynomials are obtained by the second author in [1] and independently by Guessab and Milovanović [2] and Min [4]. Chapter 6 of Milovanović, Mitrinović, and Rassias's book [3] provides complete information about various inequalities for polynomials.

Next we shall prove an inequality of Markov type for a special type of "quasi-polynomials." Let $\alpha, \beta \in \mathbb{R}$, $\beta > 0$, $n \in \mathbb{N}_0$, and $2n < -1 - \alpha$. The set of "quasi-polynomials $\mathscr{Q}_n(\alpha)$ is defined by

$$\mathscr{Q}_n(\alpha) := \left\{ q(x) = x^\alpha e^{-\beta/x} p(x) : p \in \mathscr{P}_n \right\}.$$

Then $\mathscr{Q}_n(\alpha)$ can be equipped with the norm

$$\|q\|_{\alpha, \mathscr{Q}_n(\alpha)} = \left(\int_0^\infty x^{-\alpha} e^{\beta/x} q^2(x)\, dx \right)^{1/2}. \tag{5}$$

Since for any $q \in \mathscr{Q}_n(\alpha)$ its derivative belongs to $\mathscr{Q}_{n+1}(\alpha - 2)$ and the restriction $2n < -1 - \alpha$ is equivalent to $2(n+1) < -1 - (\alpha - 2)$, then $\|q'\|_{\alpha, \mathscr{Q}_{n+1}(\alpha-2)}$ is well defined.

THEOREM 2. *Let $\alpha, \beta \in \mathbb{R}$, $\beta > 0$, $n \in \mathbb{N}_0$, and $2n < -1 - \alpha$. Then the inequality*

$$\|q'\|_{\alpha-2, \mathscr{Q}_{n+1}(\alpha-2)} \leq \big((n+1)(-\alpha - n)\big)^{1/2} \|q\|_{\alpha, \mathscr{Q}_n(\alpha)} \tag{6}$$

holds for every $q \in \mathscr{Q}_n(\alpha)$. Moreover, equality is attained if and only if q is a constant multiple of $x^{\alpha} e^{-\beta/x} y'_{n+1}(x; \alpha, \beta)$.

Note that the expression $(n+1)(-\alpha - n)$ which appears on the right-hand side of (6) is positive because $2n < -1 - \alpha$.

2. PROOFS

It is well known that the generalized Bessel polynomial of degree n satisfies the second-order differential equation

$$x^2 y'' + (\alpha x + \beta) y' - n(n + \alpha - 1) y = 0. \tag{7}$$

Obviously, this can be rewritten in the self-adjoint form:

$$\left(x^{\alpha} e^{-\beta/x} y'\right)' = n(n + \alpha - 1) x^{\alpha-2} e^{-\beta/x} y. \tag{8}$$

It follows immediately from (1) that

$$y'_n(x; \alpha, \beta) = \frac{n(n + \alpha - 1)}{\beta} y_{n-1}(x; \alpha + 2, \beta). \tag{9}$$

Proof Theorem 1. First we prove (4) for $k = 1$. Let p be an arbitrary polynomial of degree n. Under the restrictions on α, β, and n, the polynomials $y_0(x; \alpha, \beta), \ldots, y_n(x; \alpha, \beta)$ are orthogonal polynomials. Hence they form a basis in $\mathscr{P}_n$ and p can be uniquely represented in the form $p(x) = \sum_{j=0}^{n} a_j \tilde{y}_j(x; \alpha, \beta)$, where

$$\tilde{y}_j(x; \alpha, \beta) := \left(\frac{1 - \alpha - 2j}{\beta^{\alpha-1} j! \Gamma(2 - \alpha - j)}\right)^{1/2} y_j(x; \alpha, \beta), \qquad j = 0, \ldots, n,$$

are the orthonormalized GBPs. Therefore

$$\|p\|^2_{\alpha, \mathscr{P}_n} = \sum_{j=0}^{n} a_j^2.$$

Since $p'(x) = \sum_{j=1}^{n} a_j \tilde{y}_j'(x;\alpha,\beta)$ then (9) yields

$$p'(x) = \sum_{j=1}^{n} a_j \left(\frac{1-\alpha-2j}{\beta^{\alpha-1} j!\Gamma(2-\alpha-j)} \right)^{1/2} \times \frac{j(j+\alpha-1)}{\beta} y_{j-1}(x;\alpha+2,\beta).$$

Note that the polynomials $y_0(x;\alpha+2,\beta),\dots,y_{n-1}(x;\alpha+2,\beta)$ are orthogonal on $(0,\infty)$ with respect to the weight function $x^{\alpha}e^{-\beta/x}$. The requirement $\alpha < 1-2n$ is equivalent to $\alpha+2 < 1-2(n-1)$. Therefore we can apply (2) for $\alpha = \alpha+2$ and $m, j \le n-1$. Thus

$$\begin{aligned} \|p'\|^2_{\alpha+2,\mathscr{P}_{n-1}} &= \sum_{j=1}^{n} \frac{1-\alpha-2j}{\beta^{\alpha-1} j!\Gamma(2-\alpha-j)} \frac{j^2(j+\alpha-1)^2}{\beta^2} \beta^{\alpha+1} \\ &\quad \times \frac{(j-1)!}{1-\alpha-2j} \Gamma(1-\alpha-j) a_j^2 \\ &= \sum_{j=1}^{n} j(1-\alpha-j) a_j^2. \end{aligned}$$

The statement of the theorem for $k=1$ will be proved if we show that the solution of the extremal problem

$$\max\left\{ \sum_{j=0}^{n} j(1-\alpha-j)a_j^2 \middle/ \sum_{j=0}^{n} a_j^2 : a_0,\dots,a_n \in \mathbb{R},\ \sum_{j=0}^{n} a_j^2 \neq 0 \right\} \quad (10)$$

is $M_n^2(1,\alpha)$ and it is attained for $a_0 = \cdots = a_{n-1} = 0$. It is well known and easy to see that (10) is equivalent to the problem of determining

$$\max_{1\le j\le n} j(1-\alpha-j).$$

Since $f(j) := j(1-\alpha-j)$ is a concave binomial with zeros at 0 and $1-\alpha$, then f is an increasing function of j for $j \in (0,(1-\alpha)/2)$. On the other hand, the requirement $\alpha < 1-2n$ is equivalent to $n < (1-\alpha)/2$. Hence $f(j) < f(n)$ for $j = 0,\dots,n-1$. Therefore the solution of the extremal problem (10) is $n(1-\alpha-n) = M_n^2(1,\alpha)$ and the maximal value is attained only for $a_0 = \cdots = a_{n-1} = 0$ and an arbitrary nonzero a_n. This completes the proof for $k=1$. The inequalities (4) for $k = 2,\dots,n$ follow by repeated application of that for $k=1$.

Proof of Theorem 2. Let $q \in \mathscr{Q}_n(\alpha)$ be arbitrary. Since the polynomials $y_1'(x;\alpha,\beta),\dots,y_{n+1}'(x;\alpha,\beta)$ form a basis in $\mathscr{Q}_n(\alpha)$, then q can be uniquely represented in the form $q(x) = x^{\alpha}e^{-\beta/x}\sum_{j=1}^{n+1} a_j y_j'(x;\alpha,\beta)$. Keeping in mind that under the restrictions on α, β, and n, the polynomials $y_j'(x;\alpha,\beta)$, $j=1,\ldots,n+1$, are orthogonal on $(0,\infty)$ with respect to the weight function $x^{\alpha}e^{-\beta/x}$, and using (9) and (2) for $\alpha=\alpha+2$, we get

$$\begin{aligned}\|q\|^2_{\alpha,\mathscr{Q}_n} &= \int_0^\infty x^{\alpha}e^{-\beta/x}\left(\sum_{j=1}^{n+1} a_j y_j'(x;\alpha,\beta)\right)^2 dx\\ &= \sum_{j=1}^{n+1} a_j^2 \| y_j'(x;\alpha,\beta)\|^2_{\alpha+2,\mathscr{P}_{n-1}}\\ &= \sum_{j=1}^{n+1} a_j^2\beta^{\alpha-1}j^2(j+\alpha-1)^2\frac{(j-1)!}{1-\alpha-2j}\Gamma(1-\alpha-j). \qquad (11)\end{aligned}$$

On the other hand, (8) yields

$$q'(x)=x^{\alpha-2}e^{-\beta/x}\sum_{j=1}^{n+1} a_j j(j+\alpha-1)y_j(x;\alpha,\beta).$$

Then

$$\begin{aligned}\|q'\|^2_{\alpha-2,\mathscr{Q}_{n+1}} &= \int_0^\infty x^{\alpha-2}e^{-\beta/x}\left(\sum_{j=1}^{n+1} a_j j(j+\alpha-1)y_j(x;\alpha,\beta)\right)^2 dx\\ &= \sum_{j=1}^{n+1} a_j^2\beta^{\alpha-1}j^2(j+\alpha-1)^2\frac{j!}{1-\alpha-2j}\Gamma(2-\alpha-j).\end{aligned} \qquad (12)$$

Introducing the notation

$$b_j^2 := a_j^2\beta^{\alpha-1}j^2(j+\alpha-1)^2\frac{(j-1)!}{1-\alpha-2j}\Gamma(1-\alpha-j),$$

we can rewrite (11) and (12) in the form

$$\|q\|^2_{\alpha,\mathscr{Q}_n} = \sum_{j=1}^{n+1} b_j^2$$

and

$$\|q'\|^2_{\alpha-2,\mathscr{Q}_{n+1}} = \sum_{j=1}^{n+1} j(1-\alpha-j)b_j^2,$$

respectively. Note that the requirements of the theorem are equivalent to $n + 1 < (1 - \alpha)/2$. Then the same reasoning as in the proof of Theorem 1 completes the proof.

REFERENCES

1. D. K. Dimitrov, Bernstein inequality in L_2 norm, *Math. Balkanica* (*N.S.*) **7** (1993), 131–136.
2. A. Guessab and G. V. Milovanović, Weighted L^2-analogues of Bernstein's inequality and classical orthogonal polynomials, *J. Math. Anal. Appl.* **182** (1994), 244–249.
3. G. V. Milovanović, D. S. Mitrinović, and Th. M. Rassias, "Topics in Polynomials: Extremal Problems, Inequalities, Zeros," World Scientific, Singapore, 1994.
4. G. H. Min, Bernstein–Markov inequalities in L^2 spaces and their optimal constants II, *J. Math. Res. Exposition* **14** (1994), 135–138.
5. H. M. Srivastava, Orthogonality relations and generating functions for the generalized Bessel polynomials, *Appl. Math. Comput.* **61** (1994), 99–134.