



Instituto de Física Teórica
Universidade Estadual Paulista

Dissertação de Mestrado

IFT-D.005/15

Inflation in supergravity and the volume modulus

Heliudson de Oliveira Bernardo

Orientador

Horatiu Stefan Nastase



Julho de 2015

Aos meus pais e à minha eterna mestra

"I do not know what may appear to the world, but to myself I seem to have been only like a boy playing on the seashore, and diverting myself in now and then finding a smoother pebble or a prettier shell than ordinary, whilst the great ocean of truth lay all undiscovered before me."

Sir Isaac Newton

"The limits of my language means the limits of my world"

Ludwig Wittgenstein

Agradecimentos

Durante minha inacabada jornada pela busca ao conhecimento das leis fundamentais da natureza, tive o prazer de conhecer pessoas das quais lembrarei até o fim. É para todas essas pessoas que meus agradecimentos são dirigidos e, como são tantas, não conseguirei terminar de especificar todas. Mas, permitam-me começar:

Agradeço aos meus familiares, aos meus irmãos, irmãs e principalmente aos meus pais. Pode-se achar que eles foram as primeiras pessoas que eu conheci, mas as dificuldades vencidas até aqui só me permitiram que eu os entendessem (e conhecessem) ainda mais, principalmente minha mãe. Fica aqui a minha terna gratidão à minha mãe Sandra, meu pai Hélio e meus irmãos Adson e Kalliny. Agradeço também aos meus irmãos Helina e Breno que, mesmo longe, acreditam e confiam em mim.

Agradeço aos amigos que propiciaram e continuam gerando vários momentos de diversão e descontração, desde o fim da minha vida escolar até hoje : Rodrigo Rocha e o pessoal do RPG; Guilherme e os(as) colegas da física/matemática da UnB e ao Simão e sua família, que sempre me apoiaram e ajudaram em tudo que podiam.

Posso dizer que minha vida acadêmica foi marcada por profissionais da educação que foram além de suas funções e obrigações para me possibilitar a continuar seguindo em frente. Elas o faziam porque acreditavam no meu potencial mesmo quando eu não me importava com o que viria a acontecer: houve momentos em que só o desejo pelo próximo livro me faziam continuar. Agradeço ao professor Jailson Fagundes, por me ajudar a sair do mar sem ventos do senso comum, às professoras Márcia e Solange, pelas horas maravilhosas de álgebra e geometria e à Liliana Cardoso, por tentar me decifrar e ver em mim algo mais que um estudante comum: suas escolhas com relação à mim influenciaram muito o curso da já citada jornada. Agradeço a todas as professoras da sala de recursos de Taguatinga-DF especialmente à professora Karla, por me receber tão bem e se tornar uma grande amiga.

Agradeço exponencialmente à minha *eterna mestra* Valéria Cristina. Ela foi a primeira física que conheci e se tornou uma espécie de "mãe científica". Me ajudou a entender o lado "humano" da Física sem diminuir a beleza da ciência. Mestra, lembro de nossas conversas até hoje e às vezes me pego de surpresa querendo voltar

àquela época em que refletir sobre o conhecimento humano com você era como parar no tempo e rir juntos justificava tudo. Não esquecerei de você, jamais.

Agradeço ao meu orientador Horatiu Nastase pela paciência, simpatia e por ter me guiado à cosmologia. Agradeço também aos professores da época da graduação Arsen Melikyan, Olavo Leopoldino, Simone Bruschi, Vanessa de Andrade e Clóvis Maia.

Agradeço ao CNPq pelo apoio financeiro.

Por fim, agradeço minha companheira, amiga, "irmã" e querida Jéssica, por tudo que passamos juntos e por compartilhar sua jornada com a minha. Acredito que ela não saiba o que representa pra mim, e que talvez nem entenderia. Jéssica, você é muito especial para mim, obrigado por ter me permitido entrar em sua vida!

Resumo

Nesta dissertação de mestrado, nós implementamos inflação na supergravidade com o volume de dimensões extras grandes atuando como o inflaton. Trabalhamos no contexto de $\mathcal{N} = 1$, $d = 4$ supergravidade e com inflação gerada por um único campo escalar que assume valores subplanckianos. A Lagrangeana de matéria chiral acoplada com supergravidade é derivada desde os primeiros princípios e nós revisamos a cosmologia padrão e as consequências de um universo inflacionário. Então procedemos para a análise da inflação na supergravidade e explicamos o que são dimensões extras. Por fim, nós construímos um modelo de nova inflação e procuramos pela sua compatibilidade com os resultados observacionais obtidos pelo satélite Planck.

Palavras-Chave: Superespaço; Supergravidade; Inflação na supergravidade; Inflação com dimensões extras.

Áreas do conhecimento: Cosmologia; Teoria de Campos.

Abstract

In this master thesis we attempt to implement inflation in supergravity with the volume modulus of large extra dimensions acting as the inflaton. We work in the context $\mathcal{N} = 1$, $d = 4$ supergravity and with small single-field inflation. The Lagrangian of chiral matter coupled with supergravity is derived from first principles and we review the standard cosmology and the consequences of an inflationary universe. Then we proceed to the analysis of inflation in supergravity and explain what are extra dimensions. Lastly, we construct a model of new inflation and look for its compatibility with the observational results obtained from the Planck satellite.

Key Words: Superspace; Supergravity; Inflation in supergravity; Inflation with extra dimensions.

Areas: Cosmology; Field theory.

Contents

1	Introduction	1
2	Supersymmetry and Superspace	3
2.1	The Poincaré Algebra	3
2.2	The Supersymmetry algebra	5
2.3	Superspace	6
2.4	Differential forms in superspace	9
2.5	Superfields	12
3	Chiral matter and the supergravity multiplet	15
3.1	Chiral superfields and the Wess-Zumino model	15
3.2	Supersymmetric flat space	19
3.3	Local supersymmetry: Vielbein and Torsion	20
3.4	Bianchi identities and its solution	24
3.5	The supergravity transformations of the multiplet	28
4	Coupling supergravity with chiral matter	37
4.1	Chiral fields in curved superspace and the chiral densities	37
4.2	Action for pure supergravity	39
4.3	Action for chiral supergravity model	41
5	Standard Cosmology	45
5.1	Robertson-Walker metric and Friedmann models	46
5.2	The Λ CDM model	53
6	Inflation	55
6.1	The flatness and horizon problems	55

6.2	Inflation from scalar fields: The slow roll inflation	58
6.3	Primordial density power spectrum from inflation	61
7	Inflation in supergravity	69
7.1	Inflaton potential and the eta problem	69
7.2	New inflation in supergravity	71
7.3	Large extra dimensions	73
8	Inflation with large extra dimensions	77
8.1	Perturbative expansion of the superpotential	77
8.1.1	Isolated singular points of analytic functions	79
8.2	Models with essential singularity	80
8.3	Attempting to fit the parameters with Planck data	84
9	Conclusions	87
A	Notation and conventions	89
	Bibliography	94

List of Figures

7.1	Inflaton potential for Izawa and Yanagida model, in Planckian units, with $v \simeq 6.9 \times 10^{-7}$, $n = 4$, $g = 0.3$ and $N = 50$	74
8.1	Inflaton potential, in Planck units, for the parameter values (8.36). . .	85

Chapter 1

Introduction

In 1981 Allan Guth introduced the idea of inflation, an era of the very early universe in which the expansion of the space was accelerated [1]. Such an epoch is the most popular way to solve some shortcomings of the Big Bang cosmology, why the universe is so flat and so large, and it turns out that it can be used to explain the initial inhomogeneity of the cosmological fluid [2, 3]. The most common implementation of inflation is using scalar fields. There are many models using single or multi fields, but if they intend to be phenomenologically viable then it needs to satisfy some observational constraints, that comes from observations of the Cosmic Microwave Background (CMB).

The energy scale involved during inflation is much larger than the scale tested in the Large Hadron Collider. Therefore, it is natural to describe inflation in an extended version of the Standard Model. Supersymmetry is present in some extensions of the Standard Model as well as its gauged version, supergravity. Supergravity can be treated as a supersymmetric extension of General Relativity or a theory with local supersymmetry and is also an effective theory of the low energy limit of string theory. Supergravity is self consistent at the classical level, though at quantum level we should really consider the full string theory. Hence is possible to look for viable models of inflation in supergravity, a task that is not easily achieved. Once we have a model of inflation in supergravity, one can go further and try to find its origin in string theory. In doing that we have to learn how to deal with the extra dimensions that take part of the string inventory.

In the 1920s Theodor Kaluza and Oskar Klein introduced and study the possibility of extra spatial dimensions [4, 5]. They were interesting in the unification

of eletromagnetism with General Relativity in a single theory, that today is called Kaluza-Klein theory. More recently, it was proposed that large extra dimensions, dimensions with scales as large as a millimeter, can be used to solve the hierarchy problem of the Standard Model [6, 7]. According to what we know about the quantum world it seems to exist a desert between the electroweak and Planck scales, this fact being a manifestation of the hierarchy problem. Such desert is not predicted and would be nice if we had an explanation for it. The idea is that only gravity can propagate in the extra dimensions and it has the effect of bringing the Planck scale down to the TeV region.

In this work we explore the possibility of the volume modulus of the large extra dimensions driving the inflation in the supergravity context. We study single chiral field models in $\mathcal{N} = 1$, $d = 4$ supergravity. The thesis is divided as follows: chapters 2 to 3 describes what is supersymmetry and supergravity and how the Lagrangian for pure supergravity is constructed from geometrical methods in the superspace; chapter 4 shows the Lagrangian for the general chiral model coupled to supergravity; chapter 5 is a review of Friedmann-Lemaître-Robertson-Walker cosmology and the Λ CDM model; in chapter 6 we motivate and construct the slow roll inflation and in chapters 7 and 8 we study inflation in supergravity and its implementation with large extra dimensions, trying to construct a model of new inflation in supergravity.

Chapter 2

Supersymmetry and Superspace

2.1 The Poincaré Algebra

The Poincaré group $\mathcal{P}$ is the group of symmetries of Minkowski spacetime. It is the set of linear non-homogeneous transformations of a spacetime coordinate vector x^m

$$x'^m = \Lambda^m_n x^n + a^m \quad (2.1)$$

that leaves the Minkowski metric invariant. This means that the spacetime interval

$$ds^2 = \eta_{mn} dx^m dx^n \quad (2.2)$$

is the same in the two inertial reference systems with coordinates x^m and x'^m . We will denote (Λ, a) an element of the Poincaré group that transform x^m to x'^m according to (2.1). The need for invariance of (2.2) is equivalent to the equation

$$\eta_{mn} \Lambda^m_p \Lambda^n_q = \eta_{pq} \quad (2.3)$$

This group contains transformations that reverse time direction and change spatial orientation and we are not interested in such transformations. So we require the conditions

$$\Lambda^0_0 \geq 1, \quad \det \Lambda = 1 \quad (2.4)$$

to restrict our group elements to those that preserve parity and the direction of time. The homogeneous part of the Poincaré group is called Lorentz group $\mathcal{L}$ and consist of rotations in space and boosts in arbitrary directions.

Consider infinitesimal group elements, that is, elements that are close to the group identity $(\mathbb{I}, 0)$

$$(\Lambda, a) = (\mathbb{I} + \lambda, \epsilon) \quad (2.5)$$

with $\lambda^m_n \ll 1$ and $\epsilon \ll 1$. Using it in equation (1.3) we get

$$\begin{aligned} \eta_{mn}(\delta^m_p + \lambda^m_p)(\delta^n_q + \lambda^n_q) &= \eta_{pq} \\ \eta_{mn}(\delta^m_p \delta^n_q + \delta^m_p \lambda^n_q + \delta^n_q \lambda^m_p) &= \eta_{pq} \\ \eta_{pq} + \lambda_{pq} + \lambda_{qp} &= \eta_{pq} \end{aligned}$$

where we have neglected terms of order two in λ .

Therefore we have $\lambda_{pq} = -\lambda_{qp}$. As λ is a 4×4 matrix, this implies that λ has only $\frac{4(4-1)}{2} = 6$ independent parameters. This is the dimension of the Lorentz group and these parameters specifies the three angles of a general rotation and the three components of a velocity that define a boost. If we add the four parameters ϵ^m for translations, we get the dimension of the Poincaré group. Indeed, the Poincaré group is a 10-dimensional group.

The Lie algebra of the generators of the Poincaré group is [8]

$$\begin{aligned} [P_m, P_n] &= 0 \\ [M_{mn}, P_q] &= i\eta_{mq}P_n - i\eta_{nq}P_m \\ [M_{mn}, M_{pq}] &= i\eta_{mp}M_{nq} - i\eta_{mq}M_{np} + i\eta_{nq}M_{mp} - i\eta_{np}M_{mq} \end{aligned} \quad (2.6)$$

where $M_{mn} = -M_{nm}$ are the generators of Lorentz transformations and P_m are the generators of translations. Such commutator relations define a representation of the Poincaré algebra in the sense that any element $R(\Lambda, a)$ of a representation of the group can be infinitesimally written as

$$R(\Lambda, a) = \mathbb{I}_V + \frac{i}{2}\lambda^{mn}M_{mn} - ia^mP_m$$

where $\mathbb{I}_V$ is the identity of the vector space V where the linear map R acts.

2.2 The Supersymmetry algebra

In 1967 S. Coleman and J. Mandula showed that it is impossible to combine space-time and internal symmetries in a non-trivial manner [9]. More specifically, using very general assumptions, they showed that any Lie Group containing the Poincaré $\mathcal{P}$ group and an internal group $\mathcal{G}$, with generators X_r satisfying

$$[X_r, X_s] = f_{rs}^t X_t$$

must be a direct product of $\mathcal{P}$ and $\mathcal{G}$. In terms of the Lie Algebra, that means

$$[X_r, M_{mn}] = 0 = [X_r, P_m]$$

The Coleman-Mandula theorem applies to symmetries acting on elements of the S-matrix of an arbitrary process. Essentially, this no-go theorem says that if we use a Lie group that combine the internal and Poincaré groups in a non-trivial way, then the S-matrix is zero for all processes.

In 1971 Gelfand and Likhtman showed that if we consider a generalization of Lie groups, it is possible to achieve the desired combination between spacetime and internal symmetries [10]. As the structure of any Lie group is determined by the corresponding Lie algebra, such generalization can be viewed as an extension of the notion of a Lie algebra. This enlargement is the inclusion of the generators Q_α^i called *odd* generators, that satisfy anti-commutator relations, and a $\mathbb{Z}_2$ graded structure for the algebra. In other words, calling M_{mn} , P_m and X_r *even* generators, we have the following form for commutator and anti-commutator relations

$$\begin{aligned} [even, even] &= even \\ \{odd, odd\} &= even \\ [even, odd] &= odd \end{aligned}$$

These type of algebra is called graded Lie algebra or superalgebra. The i index in Q_α^i refer to a rotation in the odd generator space and take values $i = 1, \dots, N$. The meaning of the α index will be reveal shortly. If $N = 1$ the algebra is called supersymmetry algebra and those with $N > 1$ are called extended supersymmetry algebras. We will work only with $N = 1$ supersymmetry.

Using the graded structure and generalised Jacobi identities, we can construct the supersymmetry algebra or super-Poincaré algebra [11, 12, 13]

$$\begin{aligned}\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} &= 2\sigma_{\alpha\dot{\alpha}}{}^m P_m \\ \{Q_\alpha, Q_\beta\} &= 0 = \{\bar{Q}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}\}\end{aligned}\tag{2.7}$$

$$\begin{aligned}[P_m, Q_\alpha] &= 0 = [P_m, \bar{Q}_{\dot{\alpha}}] \\ [M_{nm}, Q_\alpha] &= i\sigma_{mn\alpha}{}^\beta Q_\beta\end{aligned}\tag{2.8}$$

plus the commutator relations (2.6). From this algebra we can see that the odd generator Q_α is a two component left-handed Weyl spinor and similarly, $\bar{Q}_{\dot{\alpha}}$ is a right-handed Weyl spinor. We can apply the exponential map to go from the algebra (2.6) and (2.7) to the super-Poincaré group $\mathcal{SP}$.

It was Haag, Sohnius and Lopuszanski, in 1975, who proved that the supersymmetry algebra is the only graded Lie algebra of the S-matrix consistent with relativistic quantum field theory. This is one of the reasons that make supersymmetry an interesting subject to physicists.

2.3 Superspace

The Minkowski spacetime can be identified with the coset space $\mathcal{P}/\mathcal{L}$. Roughly speaking, this can be seen from the fact that any element of the $\mathcal{P}$ group can be written as

$$p(a, \lambda) = e^{i(-a^m P_m + \frac{1}{2}\lambda^{mn} M_{mn})}\tag{2.9}$$

Therefore, as an element of $\mathcal{P}/\mathcal{L}$ is parametrized by a^m , we obtain an one-to-one correspondence between the coset manifold and Minkowski spacetime. We will now consider the coset space *super-Poincaré group/Lorentz group*, $\mathcal{SP}/\mathcal{L}$.

Introducing anticommuting parameters ξ^α and $\xi_{\dot{\alpha}}$,

$$\{\xi^\alpha, \xi^\beta\} = \{\xi^\alpha, Q_\beta\} = \cdots = \{\xi^\alpha, \xi_{\dot{\beta}}\} = 0\tag{2.10}$$

we can express (2.7) using commutators rather than anti-commutators:

$$\begin{aligned}
[\xi Q, \bar{\xi} \bar{Q}] &= 2\xi \sigma^m \bar{\xi} P_m \\
[\xi Q, \xi Q] &= 0 = [\bar{\xi} \bar{Q}, \bar{\xi} \bar{Q}] \\
[P^m, \xi Q] &= 0 = [P^m, \bar{\xi} \bar{Q}]
\end{aligned} \tag{2.11}$$

This suggests that a general element of the $\mathcal{SP}$ group is of the form

$$h(a, \theta, \bar{\theta}, \lambda) = e^{i(-x^m P_m + \theta Q + \bar{\theta} \bar{Q} + \frac{1}{2} \lambda^{mn} M_{mn})} \tag{2.12}$$

So, the coset space $\mathcal{SP}/\mathcal{L}$ is parametrized by eight parameters $(x^m, \theta, \bar{\theta})$ and this manifold is identified with the *superspace* in the same sense as $\mathcal{P}/\mathcal{L}$ is identified with Minkowski spacetime.

In order to understand what happens with the parameters by a supersymmetry transformation, we will use the Baker-Campbell-Hausdorff (BCH) formula

$$e^A e^B = e^{A+B + \frac{1}{2!}[A,B] + \frac{1}{3!}(\frac{1}{2}[[A,B],B] + \frac{1}{2}[A,[A,B]]) + \dots} \tag{2.13}$$

to multiply the group elements $h(0, \xi, \bar{\xi}), h(x^m, \theta, \bar{\theta}) \in \mathcal{SP}/\mathcal{L}$:

$$\begin{aligned}
h(0, \xi, \bar{\xi}) h(x^m, \theta, \bar{\theta}) &= e^{i(\xi Q + \bar{\xi} \bar{Q})} e^{i(-x^m P_m + \theta Q + \bar{\theta} \bar{Q})} \\
&= e^{-ix^m P_m + i(\theta + \xi)Q + i(\bar{\theta} + \bar{\xi})\bar{Q} + \frac{1}{2}[i(\xi Q + \bar{\xi} \bar{Q}), i(\theta Q + \bar{\theta} \bar{Q})]} \\
&= e^{-ix^m P_m + i(\theta + \xi)Q + i(\bar{\theta} + \bar{\xi})\bar{Q} - \frac{1}{2}([\xi Q, \bar{\theta} \bar{Q}] + [\bar{\xi} \bar{Q}, \theta Q])} \\
&= e^{-ix^m P_m + i(\theta + \xi)Q + i(\bar{\theta} + \bar{\xi})\bar{Q} - \frac{1}{2}(2\xi \sigma^m \bar{\theta} P_m - 2\theta \sigma^m \bar{\xi} P_m)} \\
&= e^{i[-(x^m + i\theta \sigma^m \bar{\xi} - i\bar{\xi} \sigma^m \bar{\theta})P_m + (\theta + \xi)Q + (\bar{\theta} + \bar{\xi})\bar{Q}]} \\
&= h(x^m + i\theta \sigma^m \bar{\xi} - i\bar{\xi} \sigma^m \bar{\theta}, \theta + \xi, \bar{\theta} + \bar{\xi})
\end{aligned}$$

It is clear from this calculation that group multiplication induces a motion in the parameter space

$$(x^m, \theta, \bar{\theta}) \rightarrow (x^m + i\theta \sigma^m \bar{\xi} - i\bar{\xi} \sigma^m \bar{\theta}, \theta + \xi, \bar{\theta} + \bar{\xi}) \tag{2.14}$$

such motion is generated by differential operators that represents the supersymmetry generators in this space. Hoping no confusion we will call these operators Q and $\bar{Q}$, the same name of the group generators.

$$\xi Q + \bar{\xi} \bar{Q} = \xi^\alpha \left(\frac{\partial}{\partial \theta^\alpha} - i \sigma_{\alpha\dot{\alpha}}^m \bar{\theta}^{\dot{\alpha}} \partial_m \right) + \bar{\xi}_{\dot{\alpha}} \left(\frac{\partial}{\partial \bar{\theta}_{\dot{\alpha}}} - i \theta^\alpha \sigma_{\alpha\dot{\beta}}^m \epsilon^{\dot{\beta}\dot{\alpha}} \partial_m \right) \quad (2.15)$$

It is straightforward to show the following relations

$$\begin{aligned} \{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} &= 2i \sigma_{\alpha\dot{\alpha}}^m \partial_m \\ \{Q_\alpha, Q_\beta\} &= 0 = \{\bar{Q}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}\} \end{aligned} \quad (2.16)$$

If we make the same analysis using left multiplication, that is, starting with $h(x^m, \theta, \bar{\theta})h(0, \xi, \bar{\xi})$ we obtain a motion generated by the differential operators

$$\begin{aligned} D_\alpha &= \frac{\partial}{\partial \theta^\alpha} + i \sigma_{\alpha\dot{\alpha}}^m \bar{\theta}^{\dot{\alpha}} \partial_m \\ \bar{D}_{\dot{\alpha}} &= -\frac{\partial}{\partial \bar{\theta}_{\dot{\alpha}}} - i \theta^\alpha \sigma_{\alpha\dot{\alpha}}^m \partial_m \end{aligned} \quad (2.17)$$

that satisfy

$$\begin{aligned} \{D_\alpha, \bar{D}_{\dot{\alpha}}\} &= -2i \sigma_{\alpha\dot{\alpha}}^m \partial_m \\ \{D_\alpha, D_\beta\} &= 0 = \{\bar{D}_{\dot{\alpha}}, \bar{D}_{\dot{\beta}}\} \end{aligned} \quad (2.18)$$

The Q and D operators are anticommuting

$$\{D_\alpha, Q_\beta\} = \{\bar{D}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}\} = \{D_\alpha, \bar{Q}_{\dot{\beta}}\} = \{\bar{D}_{\dot{\alpha}}, Q_\beta\} = 0 \quad (2.19)$$

These last anticommutation relations are of crucial importance to construct field theory in superspace in a manner that supersymmetry is kept manifest during calculations. Indeed, this is one of the motivations to define the *superspace*.

The motion in parameter space, equation (2.14), is in fact a coordinate transformation of superspace. We may write it in a more concise way, denoting elements of the superspace by

$$z^M \sim (x^m, \theta^\mu, \bar{\theta}_{\dot{\mu}}) \quad (2.20)$$

where M take values of vector and spinor indices. It is worth remarking that if M is an upper index then $\dot{\mu}$ will be a lower index. Because of the anticommuting nature of θ and $\bar{\theta}$, elements of the superspace satisfy

$$z^M z^N = (-1)^{n(N)m(M)} z^N z^M \quad (2.21)$$

where the functions $n(N)$ and $m(M)$ are zero when N and M are vector indices and one when N and M are spinor indices, respectively. We will drop the arguments and write just n and m for these functions.

With (2.20) we can write (2.14) as a supercoordinate transformation

$$z^M \rightarrow z'^M = z^M - \xi^M \quad (2.22)$$

where $\xi^M = - (i (\theta \sigma^m \bar{\xi} - \xi \sigma^m \bar{\theta}), \xi^\mu, \bar{\xi}_{\dot{\mu}})$. We want to write theories that are covariant under general coordinate transformation of superspace. This can be obtained by using differential forms in superspace and we will use this formalism to construct supergravity.

2.4 Differential forms in superspace

We will denote $\mathbf{dz}^M = (d\mathbf{x}^m, d\boldsymbol{\theta}^\mu, d\bar{\boldsymbol{\theta}}_{\dot{\mu}})$ the (super)coordinate basis for an one-form in superspace. Due to equation (2.21) exterior products in superspace satisfy

$$\begin{aligned} \mathbf{dz}^M \wedge \mathbf{dz}^N &= -(-1)^{nm} \mathbf{dz}^N \wedge \mathbf{dz}^M \\ \mathbf{dz}^M z^N &= (-1)^{nm} z^N \mathbf{dz}^M \end{aligned} \quad (2.23)$$

A zero-form is a function in the superspace $F(z)$ and a one-form $\boldsymbol{\Phi}$ is written as

$$\boldsymbol{\Phi}(z) = \mathbf{dz}^M \Phi_M(z) = d\mathbf{x}^m \Phi_m(z) + d\boldsymbol{\theta}^\mu \Phi_\mu(z) + d\bar{\boldsymbol{\theta}}_{\dot{\mu}} \bar{\Phi}^{\dot{\mu}}(z) \quad (2.24)$$

Similarly, a p -form is

$$\boldsymbol{\Omega}(z) = \mathbf{dz}^{M_1} \wedge \cdots \wedge \mathbf{dz}^{M_p} \Omega_{M_p \cdots M_1}(z) \quad (2.25)$$

Given a p -form $\boldsymbol{\Lambda}$ and a q -form $\boldsymbol{\Omega}$, the exterior product obey the following relations:

$$\begin{aligned}
\Lambda \wedge \Omega &= (-1)^{pq} \Omega \wedge \Lambda \\
\Lambda \wedge (\Omega \wedge \Psi) &= (\Lambda \wedge \Omega) \wedge \Psi \\
(a_1 \Lambda_1 + a_2 \Lambda_2) \wedge \Omega &= a_1 \Lambda_1 \wedge \Omega + a_2 \Lambda_2 \wedge \Omega
\end{aligned} \tag{2.26}$$

$$\tag{2.27}$$

where a_1, a_2 are numbers and Ψ is an arbitrary ranked form. Exterior derivatives map p -forms into $(p+1)$ -forms, according to

$$d\Omega(z) = dz^{M_1} \wedge \cdots \wedge dz^{M_p} \wedge dz^N \frac{\partial}{\partial z^N} \Omega_{M_p \dots M_1}(z) \tag{2.28}$$

and have the properties

$$\begin{aligned}
d(\Lambda + \Omega) &= d\Lambda + d\Omega \\
d(\Lambda \Omega) &= \Lambda d\Omega + (-1)^q (d\Lambda) \Omega \\
dd &= 0
\end{aligned} \tag{2.29}$$

With these operations we are able to construct theories covariantly under coordinate transformations. To elaborate *gauge* theories we need to consider the local structure of the gauge group and this is done by considering forms that span a representation of this group. Let X be an element of a group $\mathcal{G}$ in some representation, then a p -form Ω transforms as

$$\begin{aligned}
\Omega' &= \Omega X \\
\Omega'^a &= \Omega^b X_b^a
\end{aligned} \tag{2.30}$$

where a and b runs from 1 to the dimension of the representation. For Yang-Mills theory the group $\mathcal{G}$ is some compact Lie group, and for gravity theories this group is the Lorentz group $\mathcal{L}$.

Forms which transform as (2.30) are called tensors. The exterior derivative of a tensor is not a tensor. In general, we introduce a Lie algebra valued one-form

$$\phi = dz^M \phi_M^r(z) iT^r \tag{2.31}$$

that enable us to define the *covariant derivative*,

$$\begin{aligned}
\mathcal{D}\Omega &= d\Omega + \Omega\phi \\
&= dz^N \mathcal{D}_N \Omega \\
&= dz^{M_1} \wedge \cdots \wedge dz^{M_p} \wedge dz^N \frac{\partial}{\partial z^N} \Omega_{M_p \dots M_1}(z) + \\
&\quad + dz^{M_1} \wedge \cdots \wedge dz^{M_p} \wedge dz^N \phi_N{}^r \Omega_{M_p \dots M_1}(z) iT^r
\end{aligned} \tag{2.32}$$

of a p -form Ω . The one-form (2.31) is called *connection* and the matrices T are the generators of the structure group. The connection has the transformation law

$$\phi' = X^{-1} \phi X - X^{-1} dX \tag{2.33}$$

Using this transformation, one may show that the covariant derivative map tensors into tensors:

$$\mathcal{D}\Omega' = (\mathcal{D}\Omega)X \tag{2.34}$$

The *curvature* tensor $\mathbf{F}$ is defined as the covariant derivative of the connection

$$\mathbf{F} = d\phi + \phi \wedge \phi \tag{2.35}$$

it is a Lie algebra valued 2-form:

$$\begin{aligned}
\mathbf{F} &= \frac{1}{2} dz^M \wedge dz^N F_{NM}(z) \\
F_{NM} &= F_{NM}{}^r(z) iT^r
\end{aligned} \tag{2.36}$$

and its transformation law is $\mathbf{F}' = X^{-1} \mathbf{F} X$. There is no further tensor quantity that can be defined taking derivatives of (2.32) and (2.35). Derivatives of this equations, lead us to the *Bianchi identities*:

$$\mathcal{D}\mathcal{D}\Omega = \Omega \mathbf{F} \tag{2.37}$$

$$\mathcal{D}\mathbf{F} = 0 \tag{2.38}$$

These identities are geometric and has come from the fact that $d\mathbf{d} = 0$. They may be treated as the starting point of the construction of gauge field theory in superspace and supergravity.

2.5 Superfields

A *superfield* is a function of the superspace $F(z) = F(x, \theta, \bar{\theta})$. Since θ_α and $\bar{\theta}_{\dot{\alpha}}$ are anticommuting variables and α or $\dot{\alpha}$ take only two values, we have

$$\theta_\alpha \theta_\beta \theta_\delta = 0 = \bar{\theta}_{\dot{\alpha}} \bar{\theta}_{\dot{\beta}} \bar{\theta}_{\dot{\delta}} \quad (2.39)$$

Therefore, expanding a superfield $V(x, \theta, \bar{\theta})$ in terms of θ and $\bar{\theta}$ we get

$$\begin{aligned} V(x, \theta, \bar{\theta}) = & v(x) + \theta^\alpha \phi_\alpha(x) + \bar{\theta}_{\dot{\alpha}} \bar{\psi}^{\dot{\alpha}}(x) + \theta\theta f(x) + \bar{\theta}\bar{\theta} g(x) + \\ & + \theta\sigma^m \bar{\theta} a_m(x) + \bar{\theta}\bar{\theta} \theta^\alpha \lambda_\alpha(x) + \theta\theta \bar{\theta}_{\dot{\alpha}} \bar{\chi}^{\dot{\alpha}}(x) + \theta\theta \bar{\theta}\bar{\theta} d(x) \end{aligned} \quad (2.40)$$

where the coefficients of this expansion are called *component fields* of the superfield. Generalizing the notion of a tensor field to superspace allow us to find representations of the super-Poincaré group on superfields. A tensor superfield of Lorentz type $(\frac{n}{2}, \frac{m}{2})$ have components that transforms according to [13]

$$V'_{\alpha_1 \alpha_2 \dots \alpha_n \dot{\alpha}_1 \dot{\alpha}_2 \dots \dot{\alpha}_m}(z') = e^{\frac{i}{2} \lambda^{mn} M_{mn}} V_{\alpha_1 \alpha_2 \dots \alpha_n \dot{\alpha}_1 \dot{\alpha}_2 \dots \dot{\alpha}_m}(z) \quad (2.41)$$

under a general super-Poincaré transformation, parametrized by $(a, \xi, \bar{\xi}, \lambda)$. Note that the Lorentz generators M_{mn} acts on the external fields only and that z' is related to z by (2.22) plus an ordinary translation $x^m + a^m$. With this assertions, it is clear that the superfield $V(z)$ in (2.40) is a *scalar superfield*, that is, transforms as

$$V'(z') = V(z) \quad (2.42)$$

with respect to the super-Poincaré group. For a Poincaré transformation (λ, a) equation (2.42) yields

$$\begin{aligned}
V'(z') &= v'(x') + \theta'^\alpha \phi'_\alpha(x') + \bar{\theta}'_{\dot{\alpha}} \bar{\psi}'^{\dot{\alpha}}(x') + \theta' \theta' f'(x') + \bar{\theta}' \bar{\theta}' g'(x') + \\
&\quad + \theta' \sigma^m \bar{\theta}' a'_m(x') + \bar{\theta}' \bar{\theta}' \theta'^\alpha \lambda'_\alpha(x') + \theta' \theta' \bar{\theta}'_{\dot{\alpha}} \bar{\chi}'^{\dot{\alpha}}(x') + \theta' \theta' \bar{\theta}' \bar{\theta}' d'(x') \\
&= v'(x') + (\theta N)^\alpha \phi'_\alpha(x') + (\bar{\theta} N^\dagger)_{\dot{\alpha}} \bar{\psi}'^{\dot{\alpha}}(x') + \theta \theta f'(x') + \bar{\theta} \bar{\theta} g'(x') + \\
&\quad + (\theta N) \sigma^m (\bar{\theta} N^\dagger) a'_m(x') + \bar{\theta} \bar{\theta} (\theta N)^\alpha \lambda'_\alpha(x') + \theta \theta (\bar{\theta} N^\dagger)_{\dot{\alpha}} \bar{\chi}'^{\dot{\alpha}}(x') + \theta \theta \bar{\theta} \bar{\theta} d'(x') \\
&= v(x) + \theta^\alpha \phi_\alpha(x) + \bar{\theta}_{\dot{\alpha}} \bar{\psi}^{\dot{\alpha}}(x) + \theta \theta f(x) + \bar{\theta} \bar{\theta} g(x) + \theta \sigma^m \bar{\theta} a_m(x) + \\
&\quad + \bar{\theta} \bar{\theta} \theta^\alpha \lambda_\alpha(x) + \theta \theta \bar{\theta}_{\dot{\alpha}} \bar{\chi}^{\dot{\alpha}}(x) + \theta \theta \bar{\theta} \bar{\theta} d(x) \\
&= V(z)
\end{aligned}$$

where $N = e^{-\frac{i}{2} \lambda^{mn} \sigma_{mn}}$ is a matrix of $SL(2, \mathbb{C})$. From this calculation, we see that the component fields $v(x)$, $f(x)$, $g(x)$ and $d(x)$ are scalar fields, $a_m(x)$ is a vector field, $\phi_\alpha(x)$ and $\lambda_\alpha(x)$ are undotted spinor fields and $\bar{\chi}^{\dot{\alpha}}$ and $\bar{\psi}^{\dot{\alpha}}$ are dotted spinor fields. Supersymmetry transformations are generated by Q_α and $\bar{Q}^{\dot{\alpha}}$ defined in (2.15):

$$\delta V(z) = (\xi Q + \bar{\xi} \bar{Q}) V(z) \quad (2.43)$$

Matching the powers of θ and $\bar{\theta}$ in both sides of this equality, one may find the transformation laws for the component fields.

Chapter 3

Chiral matter and the supergravity multiplet

3.1 Chiral superfields and the Wess-Zumino model

In order to obtain irreducible representations of supersymmetry on superfields, we have to constrain the superfield without breaking supersymmetry. The *chiral constraint*,

$$\bar{D}_{\dot{\alpha}}\Phi = 0 \tag{3.1}$$

is invariant under supersymmetry transformations, because of (2.19). In this case, $\Phi(x, \theta, \bar{\theta})$ is called *chiral superfield* and is in fact an irreducible representation of supersymmetry. The chiral constraint is easily solved noting that $\bar{D}_{\dot{\alpha}}\theta = -\frac{\partial\theta}{\partial\bar{\theta}_{\dot{\alpha}}} - i\theta^{\alpha}\sigma_{\alpha\dot{\alpha}}{}^m\partial_m\theta = 0$ and

$$\begin{aligned} \bar{D}_{\dot{\alpha}}x^n &= -\frac{\partial x^n}{\partial\bar{\theta}_{\dot{\alpha}}} - i\theta^{\alpha}\sigma_{\alpha\dot{\alpha}}{}^m\partial_mx^n \\ &= -i\theta^{\alpha}\sigma_{\alpha\dot{\alpha}}{}^n = \bar{D}_{\dot{\alpha}}(-i\theta^{\alpha}\sigma_{\alpha\dot{\beta}}{}^n\bar{\theta}^{\dot{\beta}}) \Rightarrow \\ &\Rightarrow \bar{D}_{\dot{\alpha}}(x^n + i\theta\sigma^n\bar{\theta}) = 0 \end{aligned}$$

So, if we require Φ to be function of θ and $y^m \equiv x^m + i\theta\sigma^m\bar{\theta}$, then (3.1) is automatically satisfied. Therefore, a chiral superfield has the following expansion:

$$\begin{aligned}
\Phi(y, \theta) &= \phi(y) + \sqrt{2}\theta\psi(y) + \theta\theta F(y) \\
&= \phi(x) + i\theta\sigma^m\bar{\theta}\partial_m\phi(x) + \frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\partial^m\partial_m\phi(x) + \\
&\quad + \sqrt{2}\theta\psi(x) - \frac{i}{\sqrt{2}}\theta\theta\partial_m\psi(x)\sigma^m\bar{\theta} + \theta\theta F(x)
\end{aligned} \tag{3.2}$$

The conjugated superfield $\bar{\Phi}$ satisfies the *antichiral constraint* $D_\alpha\bar{\Phi} = 0$, and is called *antichiral superfield*. It is a function of $\bar{\theta}$ and $\bar{y}^m = x^m - i\theta\sigma^m\bar{\theta}$ and its expansion may be obtained conjugating the above one:

$$\begin{aligned}
\bar{\Phi}(y, \theta) &= \bar{\phi}(\bar{y}) + \sqrt{2}\bar{\theta}\bar{\psi}(\bar{y}) + \bar{\theta}\bar{\theta}\bar{F}(\bar{y}) \\
&= \bar{\phi}(x) - i\theta\sigma^m\bar{\theta}\partial_m\bar{\phi}(x) + \frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\partial^m\partial_m\bar{\phi}(x) + \\
&\quad + \sqrt{2}\bar{\theta}\bar{\psi}(x) + \frac{i}{\sqrt{2}}\bar{\theta}\bar{\theta}\theta\sigma^m\partial_m\bar{\psi}(x) + \bar{\theta}\bar{\theta}\bar{F}(x)
\end{aligned} \tag{3.3}$$

The most general action in superspace involving chiral and antichiral superfields is [11, 13]

$$S = \int d^4x d^4\theta K(\Phi, \bar{\Phi}) + \int d^4x d^2\theta W(\Phi) + \int d^4x d^2\bar{\theta}\bar{W}(\bar{\Phi}) \tag{3.4}$$

where K is called *Kähler potential* and W is the *superpotential*. They give kinetic terms and interactions for the components fields, respectively.

In 1974, Wess and Zumino studied a model of interacting chiral-antichiral scalar superfields, with the following action in superspace [14]

$$S = \int d^4x d^4\theta \bar{\Phi}\Phi + \left\{ \int d^4x d^2\theta (\lambda\Phi + \frac{1}{2}m\Phi^2 + \frac{1}{3}g\Phi^3) + c.c. \right\}. \tag{3.5}$$

This is the most general supersymmetric renormalizable action involving a single chiral field. This theory is called *the interacting Wess-Zumino model*. The action in terms of the components fields is [11]

$$\begin{aligned}
S &= \int d^4x \left\{ i\partial_m\bar{\psi}\bar{\sigma}^m\psi + \bar{\phi}\square\phi + \bar{F}F + \right. \\
&\quad \left. + \left[\left(\lambda F + m\phi F - \frac{1}{2}m\psi\psi + g\phi\phi F - g\psi\psi\phi \right) + c.c. \right] \right\}
\end{aligned} \tag{3.6}$$

Note that the component field F has zero propagator and so is no dynamical. Generally such kind of fields are called *auxiliary fields* and can be eliminated from the action using their Euler-Lagrange equations. In our case,

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial F} &= \bar{F} + \lambda + m\phi + g\phi^2 = 0 \\ \frac{\partial \mathcal{L}}{\partial \bar{F}} &= F + \bar{\lambda} + \bar{m}\bar{\phi} + \bar{g}\bar{\phi}^2 = 0\end{aligned}\tag{3.7}$$

Isolating F and its conjugate in this equations, we can write (3.6) in terms of dynamical fields only:

$$S = \int d^4x \left\{ \bar{\phi} \square \phi + i \partial_m \bar{\psi} \bar{\sigma}^m \psi - \frac{1}{2} m \psi \psi - \frac{1}{2} \bar{m} \bar{\psi} \bar{\psi} - g \psi \psi \phi - \bar{g} \bar{\psi} \bar{\psi} \bar{\phi} - \mathcal{V}(\phi, \bar{\phi}) \right\}\tag{3.8}$$

where $\mathcal{V}(\phi, \bar{\phi}) = (\lambda + m\phi + g\phi^2)(\bar{\lambda} + \bar{m}\bar{\phi} + \bar{g}\bar{\phi}^2)$ is the scalar potential. Note that, as expected of a supersymmetric theory, this potential is positive semidefinite.

In order to find the supersymmetry transformations for the component fields using (2.43), it is worthwhile to note that

$$\begin{aligned}\phi(x) &= \Phi|_{\theta=0=\bar{\theta}} \\ \psi_\alpha(x) &= \frac{1}{\sqrt{2}} D_\alpha \Phi|_{\theta=0=\bar{\theta}} \\ F(x) &= -\frac{1}{4} D^\alpha D_\alpha \Phi|_{\theta=0=\bar{\theta}}\end{aligned}\tag{3.9}$$

and

$$(\xi Q + \bar{\xi} \bar{Q})|_{\theta=0=\bar{\theta}} = (\xi D + \bar{\xi} \bar{D})|_{\theta=0=\bar{\theta}}\tag{3.10}$$

when acting in some superfield. With these remarks and the anticommutation relations (2.18) and (2.19), we have

$$\begin{aligned}\delta_\xi \phi &= (\xi Q + \bar{\xi} \bar{Q}) \Phi|_{\theta=0=\bar{\theta}} = (\xi D + \bar{\xi} \bar{D}) \Phi|_{\theta=0=\bar{\theta}} = \xi D \Phi|_{\theta=0=\bar{\theta}} \\ &= \sqrt{2} \xi \psi\end{aligned}\tag{3.11}$$

$$\begin{aligned}\delta_\xi \psi_\alpha &= \frac{1}{\sqrt{2}} D_\alpha \{ (\xi Q + \bar{\xi} \bar{Q}) \Phi \}|_{\theta=0=\bar{\theta}} \\ &= \frac{1}{\sqrt{2}} (\xi Q + \bar{\xi} \bar{Q}) D_\alpha \Phi|_{\theta=0=\bar{\theta}} \\ &= \frac{1}{\sqrt{2}} (\xi D + \bar{\xi} \bar{D}) D_\alpha \Phi|_{\theta=0=\bar{\theta}} \\ &= \frac{1}{\sqrt{2}} \left(\xi^\beta D_\beta D_\alpha + \bar{\xi}_{\dot{\beta}} \bar{D}^{\dot{\beta}} D_\alpha \right) \Phi|_{\theta=0=\bar{\theta}} \\ &= \frac{1}{\sqrt{2}} \left(\xi^\beta D_\beta D_\alpha - \bar{\xi}^{\dot{\beta}} \{D_\alpha, \bar{D}_{\dot{\beta}}\} + \bar{\xi}^{\dot{\beta}} D_\alpha \bar{D}_{\dot{\beta}} \right) \Phi|_{\theta=0=\bar{\theta}} \\ &= \frac{1}{\sqrt{2}} \left(2\xi_\alpha F + 2i\sigma_{\alpha\dot{\beta}}^m \bar{\xi}^{\dot{\beta}} \partial_m \phi \right)\end{aligned}\tag{3.12}$$

$$\begin{aligned}\delta_\xi F &= -\frac{1}{4} (\xi D + \bar{\xi} \bar{D}) D^2 \Phi|_{\theta=0=\bar{\theta}} \\ &= -\frac{1}{4} \bar{\xi} \bar{D} D^2 \Phi|_{\theta=0=\bar{\theta}} \\ &= \sqrt{2} i \bar{\xi} \bar{\sigma}^m \partial_m \psi\end{aligned}\tag{3.13}$$

Similarly,

$$\delta_\xi \phi = \sqrt{2} \bar{\xi} \bar{\psi}\tag{3.14}$$

$$\delta_\xi \bar{\psi}_\alpha = \sqrt{2} \bar{\xi}_{\dot{\alpha}} \bar{F} - \sqrt{2} i \xi^\beta \sigma_{\beta\dot{\alpha}}^m \partial_m \phi\tag{3.15}$$

$$\delta_\xi \bar{F} = \sqrt{2} i \xi \sigma^m \partial_m \bar{\psi}\tag{3.16}$$

It is easy to show that (3.6) is invariant under these transformations. If we eliminate the auxiliary field, we see that spinor transformation laws become nonlinear and even the closure of the supersymmetry algebra is lost. It can be restored only using the equations of motion for the spinor fields, that is, supersymmetry is closed only *on-shell*.

This feature is clarified if we consider the number of bosonic and fermionic degrees of freedom that we have in our model. When we eliminate F , there are 4 fermionic (corresponding to ψ_α and $\bar{\psi}^{\dot{\alpha}}$) and 2 bosonic (ϕ and $\bar{\phi}$) degrees of freedom. That is why we need to use dynamical equations to close the supersymmetry

algebra: it reduces the fermionic degrees to 2 and we have the matching between fermionic and bosonic degrees of freedom. So, auxiliary fields are required to obtain *off-shell* supersymmetry.

3.2 Supersymmetric flat space

In section 1.4, we have worked in the coordinate basis $\mathbf{dz}^M = (d\mathbf{x}^m, d\theta^\mu, d\bar{\theta}_{\dot{\mu}})$. In such basis, the exterior derivative $\mathbf{dz}^M \frac{\partial}{\partial z^M}$ does not map superfields into superfields. It happens because the operators $\frac{\partial}{\partial z^M}$ does not commute with the generators defined in (2.15). Due to (2.19), this can be solved writing the exterior derivative in terms of

$$\begin{aligned} D_a &= \frac{\partial}{\partial x^a} \\ D_\alpha &= \frac{\partial}{\partial \theta^\alpha} + i\sigma_{\alpha\dot{\alpha}}^m \bar{\theta}^{\dot{\alpha}} \frac{\partial}{\partial x^m} \\ \bar{D}^{\dot{\alpha}} &= \frac{\partial}{\partial \bar{\theta}_{\dot{\alpha}}} + i\theta^\alpha \sigma_{\alpha\dot{\beta}}^m \varepsilon^{\dot{\beta}\dot{\alpha}} \frac{\partial}{\partial x^m} \end{aligned} \quad (3.17)$$

So, we are looking for a new basis $\mathbf{e}^A(z) = \mathbf{dz}^M e_M^A(z)$ such that

$$\mathbf{dz}^M \frac{\partial}{\partial z^M} = \mathbf{e}^A D_A \quad (3.18)$$

The functions $e_M^A(z)$ has inverse $e_A^N(z)$, that is,

$$\begin{aligned} e_M^A(z) e_A^N(z) &= \delta_M^N \\ e_A^N(z) e_N^B(z) &= \delta_A^B \end{aligned} \quad (3.19)$$

where

$$\delta_M^N = \begin{pmatrix} \delta_m^{n} & 0 & 0 \\ 0 & \delta_\mu^{\nu} & 0 \\ 0 & 0 & \delta^{\dot{\mu}}_{\dot{\nu}} \end{pmatrix} \quad (3.20)$$

We can find these functions writing $D_A = e_A^N \frac{\partial}{\partial z^N}$ and using (3.17). Doing that, we obtain

$$e_A{}^M = \begin{pmatrix} \delta_a{}^m & e_a{}^\mu = 0 & e_{a\dot{\mu}} = 0 \\ i\sigma_{\alpha\dot{\alpha}}{}^m \bar{\theta}^{\dot{\alpha}} & \delta_\alpha{}^\mu & e_{\alpha\dot{\mu}} = 0 \\ i\theta^\alpha \sigma_{\alpha\dot{\beta}}{}^m \varepsilon^{\dot{\beta}\dot{\alpha}} & e^{\dot{\alpha}\mu} = 0 & \delta^{\dot{\alpha}}{}_{\dot{\mu}} \end{pmatrix} \quad (3.21)$$

and its inverse is

$$e_M{}^A = \begin{pmatrix} \delta_m{}^a & e_m{}^\alpha = 0 & e_{m\dot{\alpha}} = 0 \\ -i\sigma_{\mu\dot{\mu}}{}^a \bar{\theta}^{\dot{\mu}} & \delta_\mu{}^\alpha & e_{\mu\dot{\alpha}} = 0 \\ -i\theta^\rho \sigma_{\rho\dot{\nu}}{}^a \varepsilon^{\dot{\nu}\dot{\mu}} & e^{\dot{\mu}\alpha} = 0 & \delta^{\dot{\mu}}{}_{\dot{\alpha}} \end{pmatrix} \quad (3.22)$$

The last two equations define supersymmetric flat space. It is worthwhile to note that the exterior derivatives of the (new) basis forms are:

$$\begin{aligned} d\mathbf{e}^A &= d\mathbf{z}^M \wedge d\mathbf{z}^N \frac{\partial}{\partial z^N} e_M{}^A(z) \\ \Rightarrow \quad d\mathbf{e}^a &= \mathbf{e}^\alpha \wedge \mathbf{e}^{\dot{\alpha}} (-2i\sigma_{\alpha\dot{\alpha}}{}^a), \quad d\mathbf{e}^\alpha = 0 = d\mathbf{e}_{\dot{\alpha}} \end{aligned} \quad (3.23)$$

This result will be used to show that even the flat superspace has a non-zero torsion.

3.3 Local supersymmetry: Vielbein and Torsion

The Wess-Zumino model is an example of a theory that is invariant under *rigid* supersymmetry transformations (2.22). Now we wish to *gauge* such transformations, in other words, work with local supersymmetry. So, our transformations are:

$$(x^m, \theta, \bar{\theta}) \rightarrow (x^m + i\theta\sigma^m\bar{\xi}(x) - i\xi(x)\sigma^m\bar{\theta}, \theta + \xi(x), \bar{\theta} + \bar{\xi}(x)) \quad (3.24)$$

In fact, supergravity can be viewed as a supersymmetric theory of gravity or a theory of local supersymmetry.

In ordinary gravity, there is an issue about how to treat spinor fields in a curved space-time, in particular how to take their covariant derivatives. This comes from the fact that the group of general coordinate transformations has no spinor representations. This problem is solved using the tetrad formalism: at every point in space-time one defines a set of orthonormal vectors that are called vierbein (or

tetrad) field¹. The covariant derivative for spinors is constructed using the Lorentz group as structure group to define the connection (that is usually called *spin connection*) and the theory is further gauged by making the coefficients of Lorentz transformation space-time dependent (see [17, 18, 19]). The vierbein field theory is the correct way to describe relativistic field theory in a curved space-time. For a review of vierbein field theory, see [16].

As supersymmetry relates fermions and bosons, it is natural to use the vielbein formalism for supergravity. The vielbein forms

$$\mathbf{E}^A = dz^M E_M^A \quad (3.25)$$

are manifestly coordinate independent and it defines a local reference frame. The coefficients functions E_M^A are called vielbein fields and they change under coordinate transformations:

$$E_M^A(z') = \frac{\partial z^N}{\partial z'^M} E_N^A(z) \quad (3.26)$$

From this law, one can see that only the M index take part on the coordinate transformation. This kind of index will be called *world* or Einstein index. The A is a *local* or *tangent space* index and it is relate to the structure group, that we will take as the Lorentz group. We will use letters from the middle (beginning) of the alphabet to denote Einstein (Lorentz) index. So, under a local Lorentz transformations, we have

$$\begin{aligned} \delta \mathbf{E}^A &= \mathbf{E}^B M_B^A(z) \\ \delta E_N^A &= E_N^B M_B^A(z) \end{aligned} \quad (3.27)$$

The generators of the Lorentz group M_B^A may always be decomposed in its irreducible components M_b^a , M_β^α and $M^{\dot{\beta}}_{\dot{\alpha}}$ and such components are related by a simple identity. Using the σ matrices to write

$$\sigma_{\alpha\dot{\alpha}}^a \sigma_{\beta\dot{\beta}}^b M_{ab} = M_{\alpha\beta\dot{\alpha}\dot{\beta}} \quad (3.28)$$

and the antisymmetry of M_{ab} , one can show that

¹Einstein himself proposed a vierbein field theory approach for general relativity in 1928 [15]. See [16] for an English translation of this work.

$$\sigma_{\alpha\dot{\alpha}}{}^a\sigma_{\beta\dot{\beta}}{}^b M_{ab} = -2\varepsilon_{\alpha\beta}M_{\dot{\alpha}\dot{\beta}} + 2\varepsilon_{\dot{\alpha}\dot{\beta}}M_{\alpha\beta} \quad (3.29)$$

where $M^{\dot{\alpha}\dot{\beta}}$ and $M_{\alpha\beta}$ are symmetric in their indices. This type of decomposition is very useful because we can use the vielbein fields and its inverse to change between Lorentz and Einstein indices:

$$\begin{aligned} U_M &= E_M{}^A U_A \\ U_A &= E_A{}^M U_M \end{aligned} \quad (3.30)$$

and, therefore, work only with Lorentz indices.

The connection form is Lie algebra valued in its indices

$$\begin{aligned} \phi_A{}^B &= d\mathbf{z}^M \phi_{MA}{}^B \\ \phi_{MAB} &= -(-1)^{ab} \phi_{MBA} \end{aligned} \quad (3.31)$$

and, from (2.33), it transforms as

$$\delta\phi = \phi L - L\phi - d\mathbf{L} \quad (3.32)$$

Similarly to the connection coefficients, the curvature coefficients will have symmetry relations too. This is so because the curvature tensor is Lie algebra valued:

$$\begin{aligned} \mathbf{R}_A{}^B &= \frac{1}{2} d\mathbf{z}^M \wedge d\mathbf{z}^N R_{NMA}{}^B \\ &= \frac{1}{2} \mathbf{E}^C \wedge \mathbf{E}^D R_{DCA}{}^B \end{aligned} \quad (3.33)$$

Therefore

$$\begin{aligned} R_{NMAB} &= -(-1)^{ab} R_{NMBA} \\ \sigma_{\alpha\dot{\alpha}}{}^a\sigma_{\beta\dot{\beta}}{}^b R_{NMab} &= -2\varepsilon_{\alpha\beta}R_{NM\dot{\alpha}\dot{\beta}} + 2\varepsilon_{\dot{\alpha}\dot{\beta}}R_{NM\alpha\beta} \\ R_{NMA}{}^B &= -(-)^{mn} R_{MNA}{}^B \end{aligned} \quad (3.34)$$

where the last relation is due to the fact that $\mathbf{R}_A{}^B$ is a two-form. All other components of $\mathbf{R}$ vanish.

Together with the vielbein, the connection will be one of the dynamical variables in our theory. They carry a large number of component fields that will be reduced by coordinate transformations and covariant constraints. These constraints will be applied over the torsion tensor $\mathbf{T}^A$ that is defined as the covariant derivative of the vielbein:

$$\begin{aligned}\mathbf{T}^A &\equiv \mathcal{D}\mathbf{E}^A \\ &= d\mathbf{E}^A + \mathbf{E}^B \wedge \phi_B{}^A \\ &= \frac{1}{2}\mathbf{E}^C \wedge \mathbf{E}^B T_{BC}{}^A\end{aligned}\tag{3.35}$$

In flat superpace, we can set $\mathbf{E}^A = \mathbf{e}^A$ and $\phi = 0$. Then, from (3.23), the torsion is

$$\mathbf{T}^A = d\mathbf{e}^A\tag{3.36}$$

$$= \mathbf{e}^\alpha \wedge \mathbf{e}^{\dot{\alpha}} (-2i\sigma_{\alpha\dot{\alpha}}{}^a)\tag{3.37}$$

and the only non-zero components of the torsion are

$$T_{\alpha\dot{\beta}}{}^a = T_{\dot{\beta}\alpha}{}^a = 2i\sigma_{\alpha\dot{\beta}}{}^a\tag{3.38}$$

Further derivatives on the torsion and curvature leads to the Bianchi identities and, therefore, these are the only covariant tensors that can be obtained from our dynamical fields: the vielbein and the connection. The fields of the supergravity multiplet are "hidden" in its component fields. Solving the Bianchi identities restricts the number of independent components, but it does not give the minimal number of fields. We need additional constraints to get the minimum number of independent fields, these are $e_m{}^a(x)$ (graviton), $\psi_m{}^\alpha(x)$, $\bar{\psi}_{m\dot{\alpha}}(x)$ (gravitino) and $M(x)$, $b_a(x) = \bar{b}_a(x)$ (auxiliary fields).

In general relativity, we use the torsion-free constraint to express the spin connection entirely in terms of the vierbein. In supergravity we cannot require that all torsion components are zero. In that case we were excluding flat superpace from the solution².

²There is other reason for do not use the torsion-free condition: in supergravity, the curvature can be expressed completely in terms of the torsion. This is know as the Dragon theorem [20].

The desired covariant constraints are motivated by purely superspace arguments and turns out to be [11]

$$\begin{aligned}
T_{\alpha\beta}{}^c &= 0 = T_{\dot{\alpha}\dot{\beta}}{}^c \\
T_{\underline{\alpha}b}{}^c &= 0 = T_{a\underline{\beta}}{}^c \\
T_{\alpha\dot{\beta}}{}^a &= T_{\dot{\beta}\alpha}{}^a = 2i\sigma_{\alpha\dot{\beta}}{}^a \\
T_{\underline{\alpha}\underline{\beta}}{}^\gamma &= 0 \\
T_{ab}{}^c &= 0
\end{aligned} \tag{3.39}$$

where the underline in $\underline{\gamma}$ denotes γ or $\dot{\gamma}$. The Bianchi identities should be solved subject to these constraints.

3.4 Bianchi identities and its solution

Let us apply equation (2.37), in the context of supergravity, for the vielbein $\mathbf{E}^A$:

$$\begin{aligned}
\mathcal{D}\mathcal{D}\mathbf{E}^A &= \mathbf{E}^B \mathbf{R}_B{}^A \\
\mathcal{D}\mathbf{T}^A &= \mathbf{E}^B \mathbf{R}_B{}^A
\end{aligned} \tag{3.40}$$

where we used the definition of torsion, equation (3.35). Now we compute this equation in the vielbein basis:

$$\begin{aligned}
\mathcal{D}\mathbf{T}^A &= \frac{1}{2} \mathcal{D}(\mathbf{E}^B \wedge \mathbf{E}^C T_{CB}{}^A) \\
&= \frac{1}{2} \mathbf{E}^B \wedge \mathbf{E}^C \mathcal{D}T_{CB}{}^A + \frac{1}{2} \mathcal{D}(\mathbf{E}^B \wedge \mathbf{E}^C) T_{CB}{}^A \\
&= \frac{1}{2} \mathbf{E}^B \wedge \mathbf{E}^C \mathcal{D}T_{CB}{}^A + \frac{1}{2} \mathbf{E}^B \wedge \mathbf{T}^C T_{CB}{}^A - \frac{1}{2} \mathbf{T}^B \wedge \mathbf{E}^C T_{CB}{}^A
\end{aligned} \tag{3.41}$$

Putting it in (3.40) we obtain

$$\mathbf{E}^B \wedge \mathbf{E}^C \wedge \mathbf{E}^D (\mathcal{D}_D T_{CB}{}^A - R_{DCB}{}^A + T_{DC}{}^F T_{FB}{}^A) = 0 \tag{3.42}$$

Since each index in $BCDA$ stands for $(m, \mu, \dot{\mu})$, the last identity contains $3^3 + 3 = 30$ Lorentz-covariant components. Using complex conjugation and the constraints (3.39) we can reduce it to thirteen independent components [21]. These are:

1. $R_{\beta\delta\gamma\alpha} + R_{\delta\gamma\beta\alpha} + R_{\gamma\beta\delta\alpha} = 0$
2. $R_{\delta\dot{\gamma}\beta\alpha} + R_{\dot{\gamma}\beta\delta\alpha} = -2i\sigma_{\delta\dot{\gamma}}^f T_{\beta f\alpha} - 2i\sigma_{\beta\dot{\gamma}}^f T_{\delta f\alpha}$
3. $R_{\beta\delta\dot{\gamma}\dot{\alpha}} = -2i\sigma_{\delta\dot{\gamma}}^f T_{\beta f\dot{\alpha}} - 2i\sigma_{\beta\dot{\gamma}}^f T_{\delta f\dot{\alpha}}$
4. $R_{\delta b\gamma\alpha} + R_{\gamma b\delta\alpha} = -\mathcal{D}_\gamma T_{\delta b\alpha} - \mathcal{D}_\delta T_{\gamma b\alpha}$
5. $R_{b\delta\dot{\gamma}\dot{\alpha}} = \mathcal{D}_\delta T_{\dot{\gamma} b\dot{\alpha}} + \bar{\mathcal{D}}_{\dot{\gamma}} T_{\delta b\dot{\alpha}} + 2i\sigma_{\delta\dot{\gamma}}^f T_{fb\dot{\alpha}}$
6. $\bar{\mathcal{D}}_{\dot{\gamma}} T_{\delta b\alpha} + \bar{\mathcal{D}}_{\dot{\delta}} T_{\dot{\gamma} b\alpha} = 0$
7. $R_{\beta\dot{\delta}c\alpha} = -2i\sigma_{a\phi\dot{\delta}} T_{\dot{\beta}c}^\phi - 2i\sigma_{a\phi\dot{\beta}} T_{\dot{\delta}c}^\phi$
8. $R_{\beta\dot{\delta}c\alpha} = -2i\sigma_{a\phi\dot{\delta}} T_{\beta c}^\phi + 2i\sigma_{a\beta\dot{\phi}} T_{\dot{\delta}c}^\phi$
9. $R_{bd\dot{\gamma}\dot{\alpha}} = \mathcal{D}_b T_{d\dot{\gamma}\dot{\alpha}} + \mathcal{D}_d T_{\dot{\gamma} b\dot{\alpha}} + \bar{\mathcal{D}}_{\dot{\gamma}} T_{bd\dot{\alpha}} + T_{d\dot{\gamma}}^\phi T_{\phi b\dot{\alpha}} + T_{\dot{\gamma} b}^\phi T_{\phi d\dot{\alpha}}$
10. $\mathcal{D}_b T_{d\dot{\gamma}\alpha} + \mathcal{D}_d T_{\dot{\gamma} b\alpha} + \bar{\mathcal{D}}_{\dot{\gamma}} T_{bda} + T_{d\dot{\gamma}}^\phi T_{\phi b\alpha} + T_{\dot{\gamma} b}^\phi T_{\phi d\alpha} = 0$
11. $R_{\beta dca} + R_{c\beta da} + 2i\sigma_{a\beta\dot{\phi}} T_{dc}^\phi = 0$
12. $\mathcal{D}_b T_{dca} + \mathcal{D}_d T_{cba} + \mathcal{D}_c T_{bd\alpha} + T_{bd}^\phi T_{\phi c\alpha} + T_{dc}^\phi T_{\phi b\alpha} + T_{cb}^\phi T_{\phi d\alpha} = 0$
13. $R_{bdca} + R_{dcba} + R_{cbda} = 0$

where as in (3.39) the underlined indices represents ϕ and $\dot{\phi}$. The solutions to these identities are found by decomposing the torsion and curvature in terms of their irreducible components. This is more easily done by working with spinor notation, that can be obtained by using identities like (3.28) and (3.29). At the end, the result is that all the components of the curvature and the torsion can be written in terms of a chiral superfield R , a hermitian vector superfield $G_{\alpha\dot{\alpha}}$ and other chiral superfield $W_{\alpha\beta\gamma}$ that is totally symmetric in its indices. These superfields satisfy the following properties:

1. $\bar{\mathcal{D}}_{\dot{\alpha}} W_{\beta\gamma\delta} = 0 = \mathcal{D}_\alpha \bar{W}_{\dot{\beta}\dot{\gamma}\dot{\delta}}$
2. $\mathcal{D}^\alpha G_{\alpha\dot{\beta}} = \bar{\mathcal{D}}_{\dot{\beta}} R^\dagger \quad \bar{\mathcal{D}}^{\dot{\beta}} G_{\alpha\dot{\beta}} = \mathcal{D}_\alpha R$
3. $\bar{\mathcal{D}}_{\dot{\alpha}} R = 0$

4. $\bar{\mathcal{D}}^{\dot{\alpha}} \bar{W}_{\dot{\alpha}\dot{\beta}\dot{\delta}} + \frac{1}{2}i (\mathcal{D}_{\beta\dot{\beta}} G^{\beta}_{\dot{\delta}} + \mathcal{D}_{\beta\dot{\delta}} G^{\beta}_{\dot{\beta}}) = 0 = \mathcal{D}^{\alpha} W_{\alpha\beta\delta} + \frac{1}{2}i (\mathcal{D}_{\beta\dot{\beta}} G^{\dot{\beta}}_{\delta} + \mathcal{D}_{\delta\dot{\beta}} G^{\dot{\beta}}_{\beta})$
5. $(G_{\alpha\dot{\alpha}})^{\dagger} = G_{\alpha\dot{\alpha}}$
6. $(W_{\alpha\beta\gamma})^{\dagger} = \bar{W}_{\dot{\alpha}\dot{\beta}\dot{\gamma}}$

The proof of the solutions to (1) – (13) above, together with conditions (1) – (6) can be found in [21]. For now, we just list the results for the non-vanishing components of the curvature and the torsion. For the former, we have

$$\begin{aligned}
R_{\delta\gamma\phi\alpha} &= 4(\varepsilon_{\delta\phi}\varepsilon_{\gamma\alpha} + \varepsilon_{\gamma\phi}\varepsilon_{\delta\alpha}) R^{\dagger}, & R_{\dot{\delta}\dot{\gamma}\dot{\phi}\dot{\alpha}} &= 4(\varepsilon_{\dot{\delta}\dot{\phi}}\varepsilon_{\dot{\gamma}\dot{\alpha}} + \varepsilon_{\dot{\gamma}\dot{\phi}}\varepsilon_{\dot{\delta}\dot{\alpha}}) R \\
R_{\delta\dot{\gamma}\phi\alpha} &= -(\varepsilon_{\delta\phi}G_{\alpha\dot{\gamma}} + \varepsilon_{\delta\alpha}G_{\phi\dot{\gamma}}), & R_{\delta\dot{\gamma}\dot{\phi}\dot{\alpha}} &= -(\varepsilon_{\dot{\gamma}\dot{\phi}}G_{\delta\dot{\alpha}} + \varepsilon_{\dot{\gamma}\dot{\alpha}}G_{\delta\dot{\phi}}) \\
R_{\phi c\delta\alpha} &= -\frac{1}{2}\bar{\sigma}_c^{\dot{\gamma}\gamma} R_{\phi\gamma\dot{\delta}\alpha}, & R_{\phi c\dot{\delta}\dot{\alpha}} &= -\frac{1}{2}\bar{\sigma}_c^{\dot{\gamma}\gamma} R_{\phi\gamma\dot{\delta}\dot{\alpha}} \\
R_{\dot{\phi} c\delta\alpha} &= -\frac{1}{2}\bar{\sigma}_c^{\dot{\gamma}\gamma} R_{\dot{\phi}\gamma\dot{\delta}\alpha}, & R_{\dot{\phi} c\dot{\delta}\dot{\alpha}} &= -\frac{1}{2}\bar{\sigma}_c^{\dot{\gamma}\gamma} R_{\dot{\phi}\gamma\dot{\delta}\dot{\alpha}} \\
R_{cd\gamma\alpha} &= \frac{1}{4}\bar{\sigma}_c^{\dot{\phi}\phi}\bar{\sigma}_d^{\dot{\delta}\delta} R_{\phi\dot{\phi}\delta\dot{\delta}\gamma\alpha}, & R_{cd\dot{\gamma}\dot{\alpha}} &= \frac{1}{4}\bar{\sigma}_c^{\dot{\phi}\phi}\bar{\sigma}_d^{\dot{\delta}\delta} R_{\phi\dot{\phi}\delta\dot{\delta}\dot{\gamma}\dot{\alpha}}
\end{aligned}$$

$$\begin{aligned}
R_{\phi\gamma\dot{\gamma}\dot{\delta}\dot{\alpha}} &= 4i\varepsilon_{\phi\gamma}\bar{W}_{\dot{\gamma}\dot{\delta}\dot{\alpha}} + \frac{i}{2}(\varepsilon_{\dot{\gamma}\dot{\delta}}\mathcal{D}_{\phi}G_{\gamma\dot{\alpha}} + \varepsilon_{\dot{\gamma}\dot{\alpha}}\mathcal{D}_{\phi}G_{\gamma\dot{\delta}}) \\
R_{\dot{\phi}\gamma\dot{\gamma}\delta\alpha} &= 4i\varepsilon_{\dot{\phi}\gamma}W_{\delta\gamma\alpha} + \frac{i}{2}(\varepsilon_{\gamma\delta}\bar{\mathcal{D}}_{\dot{\phi}}G_{\alpha\dot{\gamma}} + \varepsilon_{\gamma\alpha}\bar{\mathcal{D}}_{\dot{\phi}}G_{\delta\dot{\gamma}}) \\
R_{\phi\gamma\dot{\gamma}\delta\alpha} &= i(\varepsilon_{\phi\delta}\varepsilon_{\gamma\alpha} + \varepsilon_{\phi\alpha}\varepsilon_{\gamma\delta})\mathcal{D}_{\beta}G^{\beta}_{\dot{\gamma}} + \frac{i}{2}(\varepsilon_{\phi\gamma}\mathcal{D}_{\delta} + \varepsilon_{\phi\delta}\mathcal{D}_{\gamma})G_{\alpha\dot{\gamma}} + \\
&\quad + \frac{i}{2}(\varepsilon_{\phi\gamma}\mathcal{D}_{\alpha} + \varepsilon_{\phi\alpha}\mathcal{D}_{\gamma})G_{\delta\dot{\gamma}} \\
R_{\dot{\phi}\gamma\dot{\gamma}\dot{\delta}\dot{\alpha}} &= i(\varepsilon_{\dot{\phi}\dot{\delta}}\varepsilon_{\dot{\gamma}\dot{\alpha}} + \varepsilon_{\dot{\phi}\dot{\alpha}}\varepsilon_{\dot{\gamma}\dot{\delta}})\bar{\mathcal{D}}_{\dot{\phi}}G_{\gamma\dot{\delta}} + \frac{i}{2}(\varepsilon_{\dot{\phi}\dot{\gamma}}\bar{\mathcal{D}}_{\dot{\delta}} + \varepsilon_{\dot{\phi}\dot{\delta}}\bar{\mathcal{D}}_{\dot{\gamma}})G_{\gamma\dot{\alpha}} + \\
&\quad + \frac{i}{2}(\varepsilon_{\dot{\phi}\dot{\gamma}}\bar{\mathcal{D}}_{\dot{\alpha}} + \varepsilon_{\dot{\phi}\dot{\alpha}}\bar{\mathcal{D}}_{\dot{\gamma}})G_{\gamma\dot{\delta}} \\
R_{\phi\dot{\phi}\delta\dot{\delta}\dot{\gamma}\dot{\alpha}} &= -2\varepsilon_{\phi\delta}\bar{U}_{\dot{\phi}\dot{\delta}\dot{\gamma}\dot{\alpha}} + 2\varepsilon_{\dot{\phi}\dot{\delta}}V_{\phi\delta\dot{\gamma}\dot{\alpha}} \\
R_{\phi\dot{\phi}\delta\dot{\delta}\gamma\alpha} &= 2\varepsilon_{\dot{\phi}\dot{\delta}}U_{\phi\delta\gamma\alpha} - 2\varepsilon_{\phi\delta}\bar{V}_{\dot{\phi}\dot{\delta}\gamma\alpha},
\end{aligned} \tag{3.43}$$

where

$$\begin{aligned}
U_{\gamma\delta\phi\alpha} = & -\frac{1}{4}(\mathcal{D}_\gamma W_{\delta\phi\alpha} + \mathcal{D}_\delta W_{\phi\alpha\gamma} + \mathcal{D}_\phi W_{\alpha\gamma\delta} + \mathcal{D}_\alpha W_{\gamma\delta\phi}) + \\
& + (\varepsilon_{\gamma\alpha}\varepsilon_{\phi\delta} + \varepsilon_{\delta\alpha}\varepsilon_{\phi\gamma}) \left[-2RR^\dagger + \frac{1}{8}G_{\beta\dot{\beta}}G^{\beta\dot{\beta}} + \right. \\
& \left. + \frac{1}{16}(\bar{\mathcal{D}}_{\dot{\beta}}\bar{\mathcal{D}}^{\dot{\beta}}R^\dagger + \mathcal{D}^\beta\mathcal{D}_\beta R) \right]
\end{aligned} \tag{3.44}$$

$$\begin{aligned}
V_{\phi\alpha\dot{\gamma}\dot{\delta}} = \bar{V}_{\dot{\gamma}\dot{\delta}\phi\alpha} = & \frac{1}{4}(G_{\phi\dot{\delta}}G_{\alpha\dot{\gamma}} + G_{\alpha\dot{\delta}}G_{\phi\dot{\gamma}}) + \frac{i}{8}(\mathcal{D}_{\alpha\dot{\gamma}}G_{\phi\dot{\delta}} + \mathcal{D}_{\phi\dot{\gamma}}G_{\alpha\dot{\delta}} + \\
& + \mathcal{D}_{\alpha\dot{\delta}}G_{\phi\dot{\gamma}} + \mathcal{D}_{\phi\dot{\delta}}G_{\alpha\dot{\gamma}}) + \frac{1}{8}(\bar{\mathcal{D}}_{\dot{\gamma}}\mathcal{D}_\alpha G_{\phi\dot{\delta}} + \bar{\mathcal{D}}_{\dot{\gamma}}\mathcal{D}_\phi G_{\alpha\dot{\delta}} + \\
& + \bar{\mathcal{D}}_{\dot{\delta}}\mathcal{D}_\alpha G_{\phi\dot{\gamma}} + \bar{\mathcal{D}}_{\dot{\delta}}\mathcal{D}_\phi G_{\alpha\dot{\gamma}})
\end{aligned} \tag{3.45}$$

For the torsion:

$$\begin{aligned}
T_{\dot{\delta}c}{}^\alpha &= -\frac{1}{2}\bar{\sigma}_c^{\dot{\phi}\phi}T_{\dot{\delta}\phi\dot{\phi}}{}^\alpha, & T_{\delta c}{}^{\dot{\alpha}} &= -\frac{1}{2}\bar{\sigma}_c^{\dot{\phi}\phi}T_{\delta\phi\dot{\phi}}{}^{\dot{\alpha}} \\
T_{\delta c}{}^\alpha &= -\frac{1}{2}\bar{\sigma}_c^{\dot{\phi}\phi}T_{\delta\phi\dot{\phi}}{}^\alpha, & T_{\dot{\delta}c}{}^{\dot{\alpha}} &= -\frac{1}{2}\bar{\sigma}_c^{\dot{\phi}\phi}T_{\dot{\delta}\phi\dot{\phi}}{}^{\dot{\alpha}} \\
T_{dc}{}^\alpha &= \frac{1}{4}\bar{\sigma}_d^{\dot{\delta}\delta}\bar{\sigma}_c^{\dot{\gamma}\gamma}T_{\delta\dot{\delta}\gamma\dot{\gamma}}{}^\alpha, & T_{dc}{}^{\dot{\alpha}} &= \frac{1}{4}\bar{\sigma}_d^{\dot{\delta}\delta}\bar{\sigma}_c^{\dot{\gamma}\gamma}T_{\delta\dot{\delta}\gamma\dot{\gamma}}{}^{\dot{\alpha}} \\
T_{\dot{\delta}\phi\dot{\phi}\alpha} &= -2i\varepsilon_{\dot{\delta}\dot{\phi}}\varepsilon_{\phi\alpha}R, & T_{\delta\phi\dot{\phi}\dot{\alpha}} &= -2i\varepsilon_{\delta\phi}\varepsilon_{\dot{\phi}\dot{\alpha}}R^\dagger \\
T_{\delta\phi\dot{\phi}\alpha} &= \frac{i}{4}(\varepsilon_{\phi\alpha}G_{\delta\dot{\phi}} - 3\varepsilon_{\delta\alpha}G_{\phi\dot{\phi}} - 3\varepsilon_{\delta\phi}G_{\alpha\dot{\phi}}) \\
T_{\dot{\delta}\phi\dot{\phi}\dot{\alpha}} &= \frac{i}{4}(\varepsilon_{\dot{\phi}\dot{\alpha}}G_{\phi\dot{\delta}} - 3\varepsilon_{\dot{\delta}\dot{\alpha}}G_{\phi\dot{\phi}} - 3\varepsilon_{\dot{\delta}\phi}G_{\phi\dot{\alpha}}) \\
T_{\delta\dot{\delta}\gamma\dot{\gamma}\alpha} &= -2\varepsilon_{\dot{\delta}\dot{\gamma}}W_{\delta\gamma\alpha} - \frac{1}{2}\varepsilon_{\delta\dot{\gamma}}(\varepsilon_{\delta\alpha}\bar{\mathcal{D}}_{\dot{\beta}}G_{\gamma}{}^{\dot{\beta}} + \varepsilon_{\gamma\alpha}\bar{\mathcal{D}}_{\dot{\beta}}G_{\delta}{}^{\dot{\beta}}) + \\
& + \frac{1}{2}\varepsilon_{\delta\gamma}(\bar{\mathcal{D}}_{\dot{\delta}}G_{\alpha\dot{\gamma}} + \bar{\mathcal{D}}_{\dot{\gamma}}G_{\alpha\dot{\delta}}) \\
T_{\delta\dot{\delta}\gamma\dot{\gamma}\dot{\alpha}} &= -2\varepsilon_{\delta\gamma}\bar{W}_{\dot{\delta}\dot{\gamma}\dot{\alpha}} - \frac{1}{2}\varepsilon_{\delta\gamma}(\varepsilon_{\dot{\delta}\dot{\alpha}}\mathcal{D}_{\dot{\beta}}G_{\gamma}{}^{\dot{\beta}} + \varepsilon_{\dot{\gamma}\dot{\alpha}}\mathcal{D}_{\dot{\beta}}G_{\delta}{}^{\dot{\beta}}) + \\
& + \frac{1}{2}\varepsilon_{\dot{\delta}\dot{\gamma}}(\mathcal{D}_{\delta}G_{\gamma\dot{\alpha}} + \mathcal{D}_{\gamma}G_{\delta\dot{\alpha}}),
\end{aligned} \tag{3.46}$$

and, from (3.39),

$$T_{\gamma\dot{\phi}}{}^c = T_{\dot{\phi}\gamma}{}^c = 2i\sigma_{\gamma\dot{\phi}}{}^c.$$

3.5 The supergravity transformations of the multiplet

In this section, we use the freedom in the choice of the transformation parameters ξ^A and L_{AB} to transform away some lowest components ($\theta = 0 = \bar{\theta}$) of the vielbein and the connection. At the end we find the supergravity multiplet and how it transform under a supersymmetry transformations. We begin by calculating the transformation of the vielbein fields under an infinitesimal superspace dependent transformation $z'^M = z^M - \xi^M(z)$ and a Lorentz transformation. Using (3.26) we have

$$\begin{aligned}\delta E_M'^A &= -\xi^L \partial_L E_M^A - (\partial_M \xi^L) E_L^A + E_M^B L_B^A \\ &= -\partial_M (\xi^L E_L^A) - \xi^L \partial_L E_M^A + (-1)^{lm} \xi^L \partial_M E_L^A + E_M^B L_B^A\end{aligned}\quad (3.47)$$

From (3.35) one can calculate the components of the torsion:

$$\begin{aligned}T_{LM}^A &= \partial_L E_M^A - (-1)^{lm} \partial_M E_L^A + (-1)^{l(b+m)} E_M^B \phi_{LB}^A - \\ &\quad - (-1)^{mb} E_N^B \phi_{MB}^A\end{aligned}\quad (3.48)$$

Using it we obtain

$$\begin{aligned}\delta E_M^A &= -\xi^L (T_{LM}^A - \phi_{LM}^A + (-1)^{ml} \phi_{ML}^A) - \partial_M \xi^A + E_M^B L_B^A \\ &= -\mathcal{D}_M \xi^A - \xi^L T_{LM}^A + \xi^L \phi_{LM}^A + E_M^B L_B^A\end{aligned}\quad (3.49)$$

If we choose

$$L_B^A = -\xi^L \phi_{LB}^A, \quad (3.50)$$

then the two last terms in (3.49) will cancel each other. This choice for L_B^A define a *supergauge transformation*. Under such kind of transformation, we have

$$\delta E_M^A = -\mathcal{D}_M \xi^A - \xi^L T_{LM}^A \quad (3.51)$$

It is possible to show that using this result and a further Lorentz transformation we can gauge away some components of $E_M^A(z)|_{\theta=0=\bar{\theta}}$, bringing it to the following form [11, 13]:

$$E_M^A(z)|_{\theta=0=\bar{\theta}} = \begin{pmatrix} e_m^a(x) & \frac{1}{2}\psi_m^\alpha(x) & \frac{1}{2}\bar{\psi}_{m\dot{\alpha}}(x) \\ 0 & \delta_\mu^\alpha & 0 \\ 0 & 0 & \delta^{\dot{\mu}}_{\dot{\alpha}} \end{pmatrix} \quad (3.52)$$

And the inverse vielbein will be

$$E_A^M(z)|_{\theta=0=\bar{\theta}} = \begin{pmatrix} e_a^m(x) & -\frac{1}{2}\psi_a^\mu(x) & -\frac{1}{2}\bar{\psi}_{a\dot{\mu}}(x) \\ 0 & \delta_\alpha^\mu & 0 \\ 0 & 0 & \delta^{\dot{\alpha}}_{\dot{\mu}} \end{pmatrix} \quad (3.53)$$

where $\psi_a^\mu = e_a^m \psi_m^\mu$ and $\bar{\psi}_{a\dot{\mu}} = e_a^m \bar{\psi}_{m\dot{\mu}}$.

Now we will show that the connection can be written in terms of the remaining fields e_m^a , ψ_m^α and $\bar{\psi}_{m\dot{\alpha}}$. Equation (3.32) gives

$$\begin{aligned} \delta\phi_{MA}^B &= -\xi^L \partial_L \phi_{MA}^B - (\partial_M \xi^L) \phi_{LA}^B + \phi_{MA}^C L_C^B - \\ &\quad - (-1)^{m(a+c)} L_A^C \phi_{MC}^B - \partial_M L_A^B \end{aligned} \quad (3.54)$$

and therefore, for a supergauge transformation (3.50),

$$\begin{aligned} \delta\phi_{MA}^B &= -\xi^L (\partial_L \phi_{MA}^B - (-1)^{lm} \partial_M \phi_{LA}^B + \\ &\quad + (-1)^{l(m+a+c)} \phi_{MA}^C \phi_{LC}^B - (-1)^{m(a+c)} \phi_{LA}^C \phi_{MC}^B) \end{aligned}$$

The expression in brackets of this equality is exactly the components of the curvature in the coordinate basis $\mathbf{d}z^M$. Thus

$$\delta\phi_{MA}^B = -\xi^L R_{LMA}^B. \quad (3.55)$$

Similarly to the vielbein's case we can use the result above, together with another Lorentz transformation, to get rid of $\phi_{\mu A}^B$ and $\phi_A^{\dot{\mu} B}$. The only non-zero lowest component of ϕ_{MA}^B will be

$$\phi_{mA}^B(z)|_{\theta=0=\bar{\theta}} = \omega_{mA}^B(x). \quad (3.56)$$

So, from (3.48) we have

$$\begin{aligned}
T_{nm}{}^a|_{\theta=0=\bar{\theta}} &= \partial_n e_m{}^a(x) - \partial_m e_n{}^a(x) + \omega_{nm}{}^a(x) - \omega_{mn}{}^a(x) \\
T_{nm}{}^a|_{\theta=0=\bar{\theta}} &= \frac{1}{2} (\partial_n \psi_m{}^\alpha - \partial_m \psi_n{}^\alpha) + \frac{1}{2} (\psi_m{}^\beta \omega_{n\beta}{}^\alpha - \psi_n{}^\beta \omega_{m\beta}{}^\alpha) \\
&= \frac{1}{2} (\tilde{\mathcal{D}}_n \psi_m{}^a - \tilde{\mathcal{D}}_m \psi_n{}^a) = \frac{1}{2} \psi_{nm}{}^\alpha(x) \\
T_{nm\dot{\alpha}}|_{\theta=0=\bar{\theta}} &= \frac{1}{2} \bar{\psi}_{nm\dot{\alpha}}(x),
\end{aligned} \tag{3.57}$$

where

$$\begin{aligned}
\omega_{nm}{}^a &= e_m{}^b \omega_{nb}{}^a \\
\tilde{\mathcal{D}}_n \psi_m{}^a &= \partial_n \psi_m{}^a + \psi_m{}^\beta \omega_{n\beta}{}^\alpha \\
\psi_{nm}{}^\alpha &= \tilde{\mathcal{D}}_n \psi_m{}^a - \tilde{\mathcal{D}}_m \psi_n{}^a
\end{aligned} \tag{3.58}$$

But, using the vielbein fields to write

$$T_{LM}{}^A = (-1)^{b(l+m)} E_M{}^B E_L{}^C T_{CB}{}^A \tag{3.59}$$

and applying the constraints on the torsion (3.39), we obtain

$$\begin{aligned}
T_{nm}{}^\alpha|_{\theta=0=\bar{\theta}} &= E_m{}^b E_n{}^c T_{cb}{}^\alpha + E_m{}^b E_n{}^\delta T_{\gamma b}{}^\alpha + E_m{}^\beta E_n{}^c T_{c\beta}{}^\alpha + \\
&+ E_m{}^b E_{n\dot{\delta}} T^{\dot{\delta}}{}_b{}^\alpha + E_{m\beta} E_n{}^c T_c{}^{\beta\alpha}
\end{aligned} \tag{3.60}$$

$$\begin{aligned}
T_{nm}{}^a|_{\theta=0=\bar{\theta}} &= E_m{}^\alpha E_{n\dot{\delta}} T^{\dot{\delta}}{}_\alpha{}^a + E_{m\dot{\beta}} E_n{}^\gamma T_\gamma{}^{\dot{\beta}}{}^a \\
&= \frac{i}{2} (\psi_n(x) \sigma^a \bar{\psi}_m(x) - \psi_m(x) \sigma^a \bar{\psi}_n(x))
\end{aligned} \tag{3.61}$$

Thus we can use this last equation and (3.57) to find the connection in terms of e , ψ and $\bar{\psi}$. This is easily done if we note that $\omega_{mnl} = e_{la} e_n{}^b \omega_{mb}{}^a = -\omega_{mln}$ and add the same equation with permuted indices. The result is

$$\begin{aligned}
\omega_{nml} &= \frac{1}{2} \left\{ e_{na} (\partial_m e_l{}^a - \partial_l e_m{}^a) - e_{ma} (\partial_l e_n{}^a - \partial_n e_l{}^a) - \right. \\
&- e_{la} (\partial_n e_m{}^a - \partial_m e_n{}^a) + \frac{i}{2} e_{na} (\psi_l \sigma^a \bar{\psi}_m - \psi_m \sigma^a \bar{\psi}_l) - \\
&- \frac{i}{2} e_{la} (\psi_m \sigma^a \bar{\psi}_n - \psi_n \sigma^a \bar{\psi}_m) - \frac{i}{2} e_{ma} (\psi_n \sigma^a \bar{\psi}_l - \psi_l \sigma^a \bar{\psi}_n) \left. \right\}
\end{aligned} \tag{3.62}$$

This is similar to ordinary relativity, in which the torsion is constrained to zero and we can find the connection in terms of the vierbein. With equations (3.60) and (3.57) it is possible to find

$$T_{\delta\dot{\gamma}\gamma\alpha}|_{\theta=0=\bar{\theta}} = \frac{1}{2} [\psi_{\delta\dot{\gamma}\gamma\alpha} + (\bar{\psi}_{\delta\dot{\delta}}{}^{\dot{\rho}} T_{\dot{\rho}\gamma\gamma\alpha} - \bar{\psi}_{\gamma\dot{\gamma}}{}^{\dot{\rho}} T_{\dot{\rho}\delta\dot{\delta}\alpha}) - (\psi_{\delta\dot{\delta}}{}^{\rho} T_{\rho\gamma\gamma\alpha} - \psi_{\gamma\dot{\gamma}}{}^{\rho} T_{\rho\delta\dot{\delta}\alpha})] \quad (3.63)$$

where we have defined

$$\begin{aligned} \psi_{\delta\dot{\gamma}\gamma} &= \sigma_{\delta\dot{\gamma}}^a e_a^m \psi_{m\gamma} \\ \psi_{\delta\dot{\delta}\gamma\alpha} &= \sigma_{\delta\dot{\delta}}^d \sigma_{\gamma\dot{\gamma}}^c e_c^n e_n^m \psi_{nma} \end{aligned} \quad (3.64)$$

Using this result and the solution to the Bianchi identities collected in the last section, we may calculate $W_{\alpha\beta\gamma}|$ and $\mathcal{D}_\alpha G_{\beta\dot{\gamma}}|$ in terms of e , ψ and $\bar{\psi}$:

$$W_{\delta\gamma\alpha}|_{\theta=0=\bar{\theta}} = \frac{1}{16} \sum_{(\delta\gamma\alpha)} \left(\frac{1}{3} \varepsilon^{\delta\dot{\gamma}} \psi_{\delta\dot{\delta}\gamma\alpha} - i \varepsilon^{\dot{\gamma}\dot{\delta}} \psi_{\delta\dot{\gamma}\gamma} G_{\alpha\dot{\delta}} \right) \quad (3.65)$$

$$\begin{aligned} \mathcal{D}_\delta G_{\alpha\dot{\alpha}}|_{\theta=0=\bar{\theta}} &= \frac{1}{4} \bar{\psi}_\alpha{}^{\dot{\gamma}}{}_{\delta\dot{\gamma}\dot{\alpha}} + \frac{1}{12} \varepsilon_{\delta\alpha} \bar{\psi}^{\gamma\dot{\gamma}}{}_{\gamma\dot{\alpha}\dot{\gamma}} + i \psi_{\alpha\dot{\alpha}\delta} R^\dagger - \\ &\quad - \frac{i}{4} (\bar{\psi}_{\alpha\dot{\rho}}{}^{\dot{\rho}} G_{\delta\dot{\alpha}} + \bar{\psi}_{\delta\dot{\rho}}{}^{\dot{\rho}} G_{\alpha\dot{\alpha}} - \bar{\psi}_{\delta\dot{\rho}}{}^{\dot{\rho}} G_{\alpha\dot{\rho}}) \end{aligned} \quad (3.66)$$

where the sum in the first equation is over all cyclic permutations of the indicated indices. From the supergauge transformations of scalar and vector superfields, we have

$$\begin{aligned} \delta R &= -\xi^C \mathcal{D}_C R \\ \delta G_a &= -\xi^C \mathcal{D}_C G_a + G_b L_a^b \end{aligned} \quad (3.67)$$

From which we can see that the lowest components of R and $G_{\alpha\dot{\beta}}$ cannot be gauged away. So, since it is not possible to write it in terms of the graviton and gravitino fields, we define

$$\begin{aligned}
G_a(z)|_{\theta=0=\bar{\theta}} &= -\frac{1}{3}b_a(x) \\
R(z)|_{\theta=0=\bar{\theta}} &= -\frac{1}{6}M(x).
\end{aligned} \tag{3.68}$$

These fields are auxiliary bosonic fields. They do indeed equalize the number of degrees of freedom between our fermionic and bosonic fields: $e_m^a(x)$ has sixteen components but we can use a Lorentz transformation ($L_{ab}| = -L_{ab}|$ with six independent components) and a coordinate tranformation ($\xi^a|$ with 4 components) to deal with six independent components only; Each $\psi_m^\alpha(x)$ and $\bar{\psi}_m^{\dot{\alpha}}(x)$ has eight components, but since we can make a supersymmetry transformation ($\xi^\alpha|, \bar{\xi}^{\dot{\alpha}}|$ each with 2 components), we have twelve fermionic degrees of freedom. The six missing bosonic degrees of freedom come from $M(x)$, $\bar{M}(x)$ and $b_a(x)$. The set of fields e , ψ , $\bar{\psi}$, b and M is the supergravity multiplet.

We have expressed some components of the torsion in terms of the vielbein component fields e , ψ and $\bar{\psi}$. Now we will apply the same procedure for the curvature. For the components of $\mathbf{R}$ we find

$$\begin{aligned}
R_{nma}{}^b(z)|_{\theta=0=\bar{\theta}} &\equiv \mathcal{R}_{nma}{}^b(x) \\
&= \partial_n \omega_{ma}{}^b - \partial_m \omega_{na}{}^b + \omega_{ma}{}^c \omega_{nc}{}^b - \omega_{na}{}^c \omega_{mc}{}^b
\end{aligned} \tag{3.69}$$

but, as we did for the connection, we can use

$$R_{nma}{}^b = (-1)^{cd} E_n{}^C E_m{}^D R_{CDa}{}^b \tag{3.70}$$

and the solution to the Bianchi identities to find a relation between the second derivatives of R and G and the first derivative of W in terms of e , ψ , $\bar{\psi}$, M and b . For further reference, we just list the result for the second derivative of R :

$$\begin{aligned}
\mathcal{D}\mathcal{D}R(z)|_{\theta=0=\bar{\theta}} &= -\frac{1}{3}e_a{}^m e_b{}^n \mathcal{R}_{mn}{}^{ab}(x) + \frac{4}{9}M(x)\bar{M}(x) + \frac{2}{9}b^a(x)b_a(x) + \\
&+ \frac{1}{12}\varepsilon^{klmn} (\bar{\psi}_k(x)\bar{\sigma}_l\psi_{mn}(x) + \psi_k(x)\sigma_l\bar{\psi}_{mn}(x)) + \\
&+ \frac{2}{3}i\bar{\psi}^m(x)\bar{\sigma}^n\psi_{mn}(x) - \frac{2}{3}ie_a{}^m \mathcal{D}_m b^a(x) + \\
&+ \frac{1}{3}\bar{\psi}(x)\bar{\psi}(x)M(x) - \frac{1}{3}\psi_m(x)\sigma^m\bar{\psi}_n b^n(x)
\end{aligned} \tag{3.71}$$

Although we are working in a gauge in which the vielbein fields and connection take the form (3.52) and (3.55), we can act any supersymmetry and Lorentz transformation on it, including those that change these equations. So, it is interesting to find transformations that preserve our gauge choice. We shall call these the *supergravity transformations*, that turns out to be defined by the following parameters:

$$\begin{aligned}
\xi^\alpha(z) &= \zeta^\alpha(x) & \bar{\xi}_{\dot{\alpha}}(z) &= \bar{\zeta}_{\dot{\alpha}}(x) \\
\xi^a(z) &= 2i(\theta\sigma^a\bar{\zeta}(x) - \zeta(x)\sigma^a\bar{\theta}) \\
L_{\alpha\beta}(z) &= \frac{1}{3}[\theta_\alpha(2\zeta_\beta(x)\bar{M}(x) - b_{\beta\dot{\gamma}}(x)\bar{\zeta}^{\dot{\gamma}}(x)) + \\
&\quad + \theta_\beta(2\zeta_\alpha(x)\bar{M}(x) - b_{\alpha\dot{\gamma}}(x)\bar{\zeta}^{\dot{\gamma}}(x))] \\
L_{\dot{\alpha}\dot{\beta}}(z) &= \frac{1}{3}[\bar{\theta}_{\dot{\alpha}}(2\bar{\zeta}_{\dot{\beta}}(x)M(x) - \zeta^\gamma(x)b_{\gamma\dot{\beta}}(x)) + \\
&\quad + \bar{\theta}_{\dot{\beta}}(2\bar{\zeta}_{\dot{\alpha}}(x)M(x) - \zeta^\gamma(x)b_{\gamma\dot{\alpha}}(x))] \\
L_{ab} &= \frac{1}{4}\bar{\sigma}_a^{\alpha\dot{\alpha}}\bar{\sigma}_b^{\beta\dot{\beta}}L_{\alpha\dot{\alpha}\beta\dot{\beta}} = \frac{1}{4}\bar{\sigma}_a^{\alpha\dot{\alpha}}\bar{\sigma}_b^{\beta\dot{\beta}}(-2\varepsilon_{\alpha\beta}L_{\dot{\alpha}\dot{\beta}} + 2\varepsilon_{\dot{\alpha}\dot{\beta}}L_{\alpha\beta}) \\
&= \frac{1}{2}(\bar{\sigma}_a\sigma_b\varepsilon)^{\dot{\alpha}\dot{\beta}}L_{\dot{\alpha}\dot{\beta}} - \frac{1}{2}(\varepsilon\sigma_a\bar{\sigma})^{\alpha\beta}L_{\alpha\beta}
\end{aligned} \tag{3.72}$$

We may now calculate the transformations of the supergravity multiplet under supergravity transformations. For the vierbein, we find

$$\begin{aligned}
\delta e_m^a &= \delta E_m^a| = -\mathcal{D}_m\xi^a| - \xi^BT_{Bm}^a| \\
&= -\xi^\beta T_{\beta m}^a| - \bar{\xi}_{\dot{\beta}}T^{\dot{\beta}}_m{}^a|
\end{aligned}$$

From the constraints on the torsion, we have $T_{\beta m}^a = 2iE_m^{\dot{\gamma}}\sigma_{\beta\dot{\gamma}}^a$ and $T_{\dot{\beta}m}^a = -2iE_m^\gamma\sigma_{\gamma\dot{\beta}}^a$, hence we get

$$\delta e_m^a(x) = i(\psi_m\sigma^a\bar{\zeta} - \zeta\sigma^a\bar{\psi}_m). \tag{3.73}$$

For the gravitino we have

$$\frac{1}{2}\delta\psi_m^a = \delta E_m^a| = -\mathcal{D}_m\xi^a| - \xi^BT_{Bm}^a|$$

From the solutions (3.46) of the Bianchi identities, we find

$$T^{\dot{\beta}}_c{}^\alpha = -i\bar{\sigma}^{\dot{\beta}\alpha}_c R$$

$$T_{\beta c}{}^\alpha = \frac{i}{8}\bar{\sigma}^{\dot{\gamma}\gamma}_c (3\varepsilon_{\beta\gamma} G^\alpha_{\dot{\gamma}} + \delta_\gamma{}^\alpha G_{\beta\dot{\gamma}} - 3\delta_\beta{}^\alpha G_{\gamma\dot{\gamma}})$$

and therefore

$$\delta\psi_m{}^\alpha = -2\mathcal{D}_m\xi^\alpha + ie_m{}^c \left[b_c\zeta^\alpha + \frac{1}{3}M(\varepsilon\sigma_c\bar{\zeta})^\alpha + \frac{1}{3}b^d(\zeta\sigma_d\bar{\sigma}_c)^\alpha \right] \quad (3.74)$$

The transformation of $\delta\bar{\psi}_{m\dot{\alpha}}$ can be easily computed by conjugating the last result:

$$\delta\bar{\psi}_m{}^{\dot{\alpha}} = -2\mathcal{D}_m\bar{\xi}_{\dot{\alpha}} - ie_m{}^c \left[b_c\bar{\zeta}_{\dot{\alpha}} + \frac{1}{3}\bar{M}(\zeta\sigma_c)_{\dot{\alpha}} - \frac{1}{3}b^d(\bar{\sigma}_c\sigma_d\bar{\zeta})_{\dot{\alpha}} \right] \quad (3.75)$$

It remains to calculate δM and δb_a . Remembering that R is chiral, this is given by (3.68):

$$-\frac{1}{6}\delta M = \delta R| = -\xi^\alpha \mathcal{D}_\alpha R|$$

$$-\frac{1}{3}\delta b_a = \delta G_a| = -(\xi^\alpha \mathcal{D}_\alpha + \bar{\xi}_{\dot{\alpha}} \bar{\mathcal{D}}^{\dot{\alpha}}) G_a| \quad (3.76)$$

Equation (3.66) together with property (2) from last section, allows to compute $\mathcal{D}_\alpha R$ in terms of the supergravity multiplet:

$$\mathcal{D}_\alpha R| = -\frac{1}{3}(\sigma^{ab})_\alpha{}^\beta \psi_{ab\beta} + \frac{i}{6}(\sigma^a\bar{\psi}_a)_\alpha M - \frac{i}{6}\psi_{a\alpha}b^a \quad (3.77)$$

where $\sigma^{ab} = \frac{1}{4}(\sigma^a\bar{\sigma}^b - \sigma^b\bar{\sigma}^a)$ and $\psi_{ab\beta}$ is defined in (3.58). Thus from (3.76), we obtain

$$\delta M = -\zeta(\sigma^a\bar{\sigma}^b\psi_{ab} + ib^a\psi_a - i\sigma^a\bar{\psi}_a M) \quad (3.78)$$

and

$$\delta b_{\alpha\dot{\alpha}} = \zeta^\delta \left[\frac{1}{4}\varepsilon_{\delta\alpha}\bar{\psi}^{\gamma\dot{\gamma}}_{\dot{\alpha}}\gamma_{\dot{\gamma}\dot{\alpha}} + \frac{3}{4}\bar{\psi}_\alpha{}^{\dot{\gamma}}\delta_{\dot{\gamma}\dot{\alpha}} - \frac{i}{2}\bar{M}\psi_{\alpha\dot{\alpha}} + \right.$$

$$\left. + \frac{i}{4}(\bar{\psi}_{\alpha\dot{\rho}}{}^{\dot{\rho}}b_{\delta\dot{\alpha}} + \psi_{\delta\dot{\rho}}{}^{\dot{\rho}}b_{\alpha\dot{\alpha}} - \bar{\psi}_{\delta\dot{\rho}}{}^{\dot{\rho}}b_{\alpha\dot{\rho}}) \right] - \bar{\zeta}^{\dot{\delta}} \left[\frac{1}{4}\varepsilon_{\dot{\delta}\dot{\alpha}}\psi_{\alpha}{}^{\gamma\dot{\gamma}}\gamma_{\dot{\gamma}\dot{\alpha}} + \frac{3}{4}\psi_{\dot{\delta}\dot{\alpha}}{}^{\gamma\dot{\gamma}} + \right.$$

$$\left. + \frac{i}{2}M\bar{\psi}_{\alpha\dot{\alpha}} - \frac{i}{4}(\psi_{\rho\dot{\alpha}}{}^{\rho}b_{\alpha\dot{\delta}} + \psi_{\rho\dot{\delta}}{}^{\rho}b_{\alpha\dot{\alpha}} - \psi_{\rho\dot{\delta}}{}^{\rho}b_{\rho\dot{\alpha}}) \right] \quad (3.79)$$

Equations (3.73), (3.74), (3.75), (3.78) and (3.79) are the desired supergravity transformation laws of the supergravity multiplet. In the next chapter, we shall find an action that is invariant under such transformations.

Chapter 4

Coupling supergravity with chiral matter

4.1 Chiral fields in curved superspace and the chiral densities

In flat superspace, chiral fields satisfy (3.1). The covariant constraint

$$\bar{\mathcal{D}}_{\dot{\alpha}}\Phi = 0 \tag{4.1}$$

is the condition that reduces to the previous in the flat limit and will be satisfied by any chiral field in curved superspace. This was already used to derive the transformations of $M(x)$ in (3.76). The components of Φ will be defined by analogy with (3.9):

$$\begin{aligned} \phi(x) &= \Phi| \\ \chi_{\alpha}(x) &= \frac{1}{\sqrt{2}} \mathcal{D}_{\alpha}\Phi| \\ F(x) &= -\frac{1}{4} \mathcal{D}^{\alpha}\mathcal{D}_{\alpha}\Phi|. \end{aligned} \tag{4.2}$$

The supersymmetry transformation of these components may be found from

$$\delta\Phi = -\xi^A \mathcal{D}_A\Phi, \tag{4.3}$$

with the parameters ξ^A given by (3.72). This yields

$$\begin{aligned}
\delta\phi &= -\sqrt{2}\zeta^\alpha\chi_\alpha \\
\delta\chi_\alpha &= -\sqrt{2}\zeta_\alpha F - i\sqrt{2}\sigma_{\alpha\dot{\beta}}{}^a\bar{\zeta}^{\dot{\beta}}\hat{D}_a\phi \\
\delta F &= -\frac{1}{3}\sqrt{2}\bar{M}\zeta^\alpha\chi_\alpha + \bar{\zeta}^{\dot{\alpha}}\left(\frac{1}{6}\sqrt{2}b_{\alpha\dot{\alpha}}\chi^\alpha - i\sqrt{2}\hat{D}_{\alpha\dot{\alpha}}\chi^\alpha\right),
\end{aligned} \tag{4.4}$$

here we have defined a supercovariant derivative:

$$\hat{D}_a\phi \equiv e_a{}^m\left(\partial_m\phi - \frac{1}{\sqrt{2}}\psi_m{}^\mu\chi_\mu\right). \tag{4.5}$$

In principle one could define the components in (4.2) as the coefficients of a θ^μ expansion of the chiral field. But θ^μ carries a world index and therefore such expansion is coordinate dependent. This motivate us to introduce a new set of variables Θ^α , with Lorentz index, in such a way that

$$\Phi = \phi(x) + \sqrt{2}\Theta^\alpha\chi_\alpha(x) + \Theta^\alpha\Theta_\alpha F(x). \tag{4.6}$$

In this case it is possible to define a new set of parameters $\lambda^M(x, \Theta)$ for which the transformation law of the chiral superfield is

$$\delta\Phi = -\lambda^M(x, \Theta)\partial_M\Phi, \tag{4.7}$$

where the partial derivatives $\partial_\alpha, \partial^{\dot{\alpha}}$ are taking with respect to the new variables Θ^α and $\bar{\Theta}_{\dot{\alpha}}$. Using (4.4) we can relate λ^M with ζ and the supergravity multiplet:

$$\begin{aligned}
\lambda^m &= 2i\Theta\sigma^m\bar{\zeta} + \Theta\Theta\bar{\psi}_n\bar{\sigma}^m\sigma^n\bar{\zeta} \\
\lambda^\alpha &= \zeta^\alpha - i\Theta\sigma^m\bar{\zeta}\psi_m{}^\alpha + \Theta\Theta\left[\frac{1}{3}\bar{M}\zeta^\alpha + \frac{1}{6}b_a(\varepsilon\sigma^a\bar{\zeta})^\alpha - \right. \\
&\quad \left. -i\omega_m{}^{\alpha\beta}(\sigma^m\bar{\zeta})_\beta - \frac{1}{2}\psi_n{}^\alpha(\bar{\psi}_m\bar{\sigma}^n\sigma^m\bar{\zeta})\right].
\end{aligned} \tag{4.8}$$

From these parameters we can define *chiral densities*. These are superfields that transforms as

$$\begin{aligned}
\delta\Omega &= -\partial_M[(-1)^m\lambda^M\Omega] \\
&= -\lambda^M\partial_M\Omega - (-1)^m(\delta_M\lambda^M)\Omega
\end{aligned} \tag{4.9}$$

So, the product of a chiral density and a chiral field transforms as a chiral density:

$$\begin{aligned}\delta(\Omega\Phi) &= (\delta\Omega)\Phi + \Omega(\delta\Phi) = -\partial_M [(-1)^m \lambda^M \Omega] \Phi - \Omega \lambda^M \partial_M \Phi \\ &= -\partial_M [(-1)^m \lambda^M \Omega\Phi]\end{aligned}\quad (4.10)$$

We can decompose Ω in terms of the component fields

$$\Omega = \varphi + \sqrt{2}\Theta\rho + \Theta\Theta f, \quad (4.11)$$

that has transformation laws given from (4.9):

$$\begin{aligned}\delta\varphi &= -\sqrt{2}\zeta\rho + i\varphi\psi\sigma\bar{\zeta} \\ \delta\rho_\alpha &= -\sqrt{2}\zeta_\alpha f - i\sqrt{2}\mathcal{D}_m(\sigma^m\bar{\zeta}\varphi)_\alpha + i\psi\sigma\bar{\zeta}\rho_\alpha + i(\sigma^m\bar{\zeta})_\alpha\psi_m\rho - \\ &\quad - \frac{1}{3}\sqrt{2}\zeta_\alpha\bar{M}\varphi - \frac{1}{6}\sqrt{2}\varphi(\sigma^a\bar{\zeta})_\alpha b_a + \frac{1}{2}\sqrt{2}\psi_{n\alpha}\bar{\psi}_m\bar{\sigma}^n\sigma^m\bar{\zeta}\varphi \\ \delta f &= \partial_m\left(-\varphi\bar{\psi}_n\bar{\sigma}^m\sigma^n\bar{\zeta} + i\sqrt{2}\rho\sigma^m\zeta\right).\end{aligned}\quad (4.12)$$

The Θ variable and a special chiral density will be used to construct invariant actions.

4.2 Action for pure supergravity

Consider the following action constructed from a chiral density Ω and a function $h(\Phi)$ of a chiral field,

$$S = \int d^4x d^2\Theta \Omega h(\Phi). \quad (4.13)$$

This action is invariant under supergravity transformations, because

$$\begin{aligned}\delta S &= \int d^4x d^2\Theta \delta(\Omega h(\Phi)) \\ &= - \int d^4x d^2\Theta \partial_M [(-1)^m \lambda^M \Omega h(\Phi)] \\ &= 0\end{aligned}\quad (4.14)$$

where we have used (4.10). In order to write the action for pure supergravity, we need to define a chiral density $\mathcal{E}$ which has components that depends on the vielbein [11, 12]. Using the notation of the last section for an arbitrary chiral density, its lowest component is defined as

$$a = \frac{1}{2} \det e_m{}^a \equiv \frac{1}{2} e, \quad (4.15)$$

and the other components may be computed from the transformations (4.10). As an example, from the transformations of the supergravity multiplet we have

$$\delta e = e e_a{}^m \delta e_m{}^a = i e e_a{}^m (\psi_m \sigma^a \bar{\zeta} - \zeta \sigma^a \bar{\psi}_m) \quad (4.16)$$

and comparing it with (4.12) we find that

$$\rho = \frac{i}{4} \sqrt{2} e \sigma^m \bar{\psi}_m \quad (4.17)$$

The highest component can be calculated in a similar way and is given by

$$f = -\frac{1}{2} e \bar{M} - \frac{1}{2} e \bar{\psi}_m \bar{\sigma}^{mn} \bar{\psi}_n. \quad (4.18)$$

Hence

$$\mathcal{E} = \frac{e}{2} [1 + i \Theta \sigma^a \bar{\psi}_a - \Theta \Theta (\bar{M} + \bar{\psi}_a \bar{\sigma}^{ab} \bar{\psi}_b)] \quad (4.19)$$

A natural chiral superfield that depends on the supergravity multiplet is R , defined in the last chapter. From equations (3.68), (3.71) and (3.77), we have, with $\mathcal{R} = e_a{}^m e_b{}^n R_{mn}{}^{ab}$,

$$\begin{aligned} R = & -\frac{1}{6} \left\{ M + \Theta (\sigma^a \bar{\sigma}^b \psi_{ab} - i \sigma^a \bar{\psi}_a M + i \psi_a b^a) + \right. \\ & + \Theta \Theta \left[-\frac{1}{2} \mathcal{R} + i \bar{\psi}^a \bar{\sigma}^b \psi_{ab} + \frac{2}{3} M \bar{M} + \frac{1}{3} b^a b_a - i e_a{}^m \mathcal{D}_m b^a + \right. \\ & \left. \left. + \frac{1}{2} \bar{\psi} \bar{\psi} M - \frac{1}{2} \psi_a \sigma^a \bar{\psi}_c b^c + \frac{1}{8} \varepsilon^{abcd} (\bar{\psi}_a \bar{\sigma}_b \psi_{cd} + \psi_a \sigma_b \bar{\psi}_{cd}) \right] \right\} \end{aligned} \quad (4.20)$$

Thus, from the same reasoning after (4.13), the invariant action for pure supergravity is

$$S_{sugra} = -\frac{6}{8\pi G} \int d^4x d^2\Theta \mathcal{E} R + c.c., \quad (4.21)$$

where G is the Newton's constant. It is straightforward to write this action in component fields:

$$S_{sugra} = \frac{1}{8\pi G} \int d^4x \left[-\frac{1}{2}e\mathcal{R} - \frac{1}{3}e\bar{M}M + \frac{1}{3}eb^ab_a + \frac{1}{2}e\varepsilon^{klmn} \left(\bar{\psi}_k\bar{\sigma}_l\tilde{\mathcal{D}}_m\psi_n - \psi_k\sigma_l\tilde{\mathcal{D}}_m\bar{\psi}_n \right) \right] \quad (4.22)$$

As expected this action contains the Einstein-Hilbert action (in vielbein formulation) and the kinetic terms for the gravitino and the auxiliary fields, M and b^a . Note that the connection is included in the derivative $\tilde{\mathcal{D}}_m$ that was defined in (3.58). We can now proceed to the construction of the coupling of chiral matter to supergravity.

4.3 Action for chiral supergravity model

We begin by writing the most general action for chiral matter in flat superspace (3.4) in the following form:¹

$$S = \int d^4x d^2\theta \left[\left(-\frac{1}{8}\bar{D}\bar{D} \right) K(\Phi, \bar{\Phi}) + W(\Phi) \right] + c.c. \quad (4.23)$$

and then go to the action in curved superspace by making the following replacements:

- $\theta \rightarrow \Theta$. As explained, we have defined Θ to get a coordinate independent expansion of chiral fields *in* curved superspaces. So, it is a natural prescription;
- $d^2\theta \rightarrow d^2\Theta 2\mathcal{E}$. This is the invariant measurement, as one can see from the transformation for the highest component of $\mathcal{E}$ in (4.12);
- $\bar{D}\bar{D} \rightarrow (\bar{\mathcal{D}}\bar{\mathcal{D}} - 8R)$. This is the generalization of the chiral projectors for curved superspace and comes from the fact that $\bar{\mathcal{D}}_{\dot{\alpha}}(\bar{\mathcal{D}}_{\dot{\beta}}\bar{\mathcal{D}}^{\dot{\beta}} - 8R)\bar{\Phi} = 0$.

Then, for the minimal chiral supergravity model, we have, setting $8\pi G = 1$:

¹In this section we follow [11] closely.

$$\begin{aligned}
S &= S_{sugra} + \left\{ \int d^4x d^2\Theta d^2\bar{\Theta} \left[-\frac{1}{8} (\bar{\mathcal{D}}\bar{\mathcal{D}} - 8R) K(\Phi, \bar{\Phi}) + W(\Phi) \right] + c.c. \right\} \\
&= \int d^4x d^2\Theta d^2\bar{\Theta} \left[-\frac{1}{8} (\bar{\mathcal{D}}\bar{\mathcal{D}} - 8R) \Omega(\Phi, \bar{\Phi}) + W(\Phi) \right] + c.c.
\end{aligned} \tag{4.24}$$

where $\Omega(\Phi, \bar{\Phi}) = K(\Phi, \bar{\Phi}) - 3$. The constant -3 in Ω gives the pure supergravity action. If we derive the action in component fields for $K(\Phi, \bar{\Phi}) = \Phi\bar{\Phi}$, i.e. the Kähler potential for flat superspace or *canonical* Kähler potential, we get a Brans-Dicke normalization for the gravitational action:

$$\mathcal{L}_{EH} \propto e\Omega\mathcal{R}$$

We need a Weyl rescaling of the vierbein and spinor fields in order to fix this normalization, that is, to work in the "Einstein frame" where we recover the standard Einstein-Hilbert action. When this is done, it turns out that the Lagrangian depends on Ω on through the combination $-3\ln(-\frac{\Omega}{3})$ and its derivatives. This motivates us to begin with the following Lagrangian,

$$\mathcal{L} = \frac{1}{8\pi G} \int d^2\Theta d^2\bar{\Theta} \left[\frac{3}{8} (\bar{\mathcal{D}}\bar{\mathcal{D}} - 8R) \exp \left\{ -\frac{8\pi G}{3} K(\Phi, \bar{\Phi}) \right\} + (8\pi G)W(\Phi) \right] + c.c. \tag{4.25}$$

This is the general chiral supergravity model and if we expand the exponential, we recover the minimal chiral coupling (4.24) plus extra terms. For an arbitrary number of chiral fields, the component form of (4.25) is

$$\begin{aligned}
\mathcal{L} &= -\frac{1}{2}e\mathcal{R} - eK_{i\bar{j}}\partial_m\phi^i\partial^m\bar{\phi}^{\bar{j}} - ieK_{i\bar{j}}\bar{\chi}^{\bar{j}}\bar{\sigma}^m\mathcal{D}_m\chi^i + e\varepsilon^{klmn}\bar{\psi}_k\bar{\sigma}_l\tilde{\mathcal{D}}_m\psi_n - \\
&\quad - \frac{1}{2}\sqrt{2}eK_{i\bar{j}}\partial_n\bar{\phi}^{\bar{j}}\chi^i\sigma^m\bar{\sigma}^n\psi_m - \frac{1}{2}\sqrt{2}eK_{i\bar{j}}\partial_n\phi^i\bar{\chi}^{\bar{j}}\bar{\sigma}^m\sigma^n\bar{\psi}_m - \\
&\quad - \frac{1}{8}e(K_{i\bar{j}}K_{k\bar{l}} - 2R_{i\bar{j}k\bar{l}})\chi^i\chi^k\bar{\chi}^{\bar{j}}\bar{\chi}^{\bar{l}} + \frac{1}{4}eK_{i\bar{j}}(i\varepsilon^{klmn}\psi_k\sigma_l\bar{\psi}_m + \\
&\quad + \psi_m\sigma^n\bar{\psi}^m)\chi^i\sigma_n\bar{\chi}^{\bar{j}} - e\exp\left(\frac{K}{2}\right)\left(\bar{W}\psi_a\sigma^{ab}\psi_b + W\bar{\psi}_a\bar{\sigma}^{ab}\bar{\psi}_b + \right. \\
&\quad + \frac{i}{2}\sqrt{2}D_iP\chi^i\sigma^a\bar{\psi}_a + \frac{i}{2}\sqrt{2}D_{\bar{i}}\bar{W}\bar{\chi}^{\bar{i}}\bar{\sigma}^a\psi_a + \frac{1}{2}\mathcal{D}_iD_jW\chi^i\chi^j + \frac{1}{2}\mathcal{D}_{\bar{i}}D_{\bar{j}}\bar{W}\bar{\chi}^{\bar{i}}\bar{\chi}^{\bar{j}} \Big) - \\
&\quad - e\exp(K)\left[K^{i\bar{j}}(D_iW)(D_{\bar{j}}\bar{W}) - 3\bar{W}W\right]
\end{aligned} \tag{4.26}$$

where we made the follow definitions:

$$\begin{aligned}
K_{ij} &= \frac{\partial^2 K}{\partial \phi_i \partial \phi_j}, & K_{i\bar{j}} &= \frac{\partial^2 K}{\partial \phi_i \partial \bar{\phi}_j}, & K^{i\bar{j}} &= K^{-1}{}_{i\bar{j}}, & \Gamma_{ij}^k &= K^{k\bar{l}} \frac{\partial K_{j\bar{l}}}{\partial \phi_i}; \\
D_i W &= \frac{\partial W}{\partial \phi_i} + \frac{\partial K}{\partial \phi_i} W; \\
\mathcal{D}_m \chi^i &= \partial_m \chi^i + \chi^i \omega_m + \Gamma_{jk}^i \partial_m \phi^j \chi^k - \frac{1}{4} \left(\frac{\partial K}{\partial \phi_j} \partial_m \phi^j - \frac{\partial K}{\partial \bar{\phi}_j} \partial_m \bar{\phi}^j \right) \chi^i; \\
\tilde{\mathcal{D}}_m \psi_n &= \partial_m \psi_n + \psi_n \omega_m + \frac{1}{4} \left(\frac{\partial K}{\partial \phi_j} \partial_m \phi^j - \frac{\partial K}{\partial \bar{\phi}_j} \partial_m \bar{\phi}^j \right) \psi_n; \\
\mathcal{D}_i D_j W &= \frac{\partial^2 W}{\partial \phi_i \partial \phi_j} + K_{ij} W + \frac{\partial K}{\partial \phi_i} D_j W + \frac{\partial K}{\partial \phi_j} D_i W - \frac{\partial K}{\partial \phi_i} \frac{\partial K}{\partial \phi_j} W - \\
&\quad - \Gamma_{ij}^k D_k W.
\end{aligned}$$

In the Lagrangian (4.26), the superpotential and Kähler potential depends only on the complex scalar fields and then the last line in this equation gives the supergravity scalar potential.

Chapter 5

Standard Cosmology

One of the most remarkable features of the observable Universe is the fact that in scales larger than 300 million light years it has a distribution of matter that is *homogeneous and isotropic*. This is a strong statement based on sky surveys observations. In such scale we can treat the constituents of the universe as clusters of galaxies, each one containing from a few dozens to hundreds galaxies. The extension of the assumption of homogeneity and isotropy to the entire universe is called *Cosmological Principle*. The main justification for this principle is the highly degree of isotropy of the cosmic microwave background.

Another extraordinary observation was the discovery of the expansion of the universe, discovered in the 1920s by Edwin Hubble. He measured that light from far galaxies were red-shifted and inferred that the galaxies were moving away from our galaxy and from each other. Moreover, the velocity v of the galaxy depended of its radial r distance by

$$v = H_0 r \tag{5.1}$$

where H_0 , which is constant, is the actual value of the *Hubble's constant* H . It is equal to

$$H_0 = h \cdot 100 \text{ kms}^{-1} \text{Mpc}^{-1}, \tag{5.2}$$

and h has a present value in the range 0.55 to 0.85. So, a galaxy 1 Mpc away has a velocity of about 70 kms^{-1} . Equation (5.1) is called *Hubble's law* and is a consequence of the expansion of the space itself. It is worth to keep in mind that

in cosmology we work with large spatial and temporal scales, such that when we talk about the *present* we are referring to lapse of time very small if compared with billion of years. To see this, we can invert the Hubble's constant at the present value:

$$H_0^{-1} \simeq \frac{3}{h} \times 10^{17} \text{s} \quad (5.3)$$

that is approximately $10h^{-1}$ billion of years.

In this chapter we shall see how Hubble's law is obtained only from the Cosmological Principle and what we get applying Einstein's general relativity for all the universe.

5.1 Robertson-Walker metric and Friedmann models

By translating the Cosmological Principle into a geometrical language we ended up with a specific form for the space-time metric, that is known as *Robertson-Walker* metric. In coordinates associated to observers that are at rest with respect to the expansion, i.e. *comoving* coordinates, this metric assumes the form [22]

$$ds^2 = -c^2 dt^2 + a^2(t) \left(\frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right) \quad (5.4)$$

where k codifies the geometry of the spatial section and take the values $k = 1$, $k = 0$ and $k = -1$ corresponding to a closed (S^3), flat (E^3) and open (hyperbolic) universe, respectively. The factor $a^2(t)$, called *scale factor*, describes an expanding or contracting universe and we know from observation that in our case the universe is expanding. The behavior of $a(t)$ and the value of k are related *dynamically* with the matter content of our universe via Einstein equations, that are not necessary to derive (5.4). It is noteworthy that this metric arises naturally from symmetry requirements of the Cosmological Principle, see ([23, 19]) for a deduction of this metric using only the homogeneity and isotropy of the space.

The Hubble expansion is a *kinematic* feature that can be obtained from (5.4), independent of the specific form of the scale factor. Suppose our galaxy is located at $r = 0$, and there is another galaxy at $r = r_g$. Using (5.4), one can compute its proper distance $L(t)$ from us at the cosmic time t :

$$L(t) = a(t) \int_0^{r_g} \frac{dr}{\sqrt{1 - kr^2}} = a(t)f(r), \quad (5.5)$$

where

$$f(r) = \begin{cases} \sin^{-1}(r) & (k = 1); \\ r & (k = 0); \\ \sinh^{-1}(r) & (k = -1). \end{cases} \quad (5.6)$$

Thus, the velocity of recession is

$$v = \dot{L}(t) = \dot{a}(t)f(r) = \frac{\dot{a}(t)}{a(t)}L, \quad (5.7)$$

this is exactly the Hubble law (5.1) with

$$H(t) = \frac{\dot{a}(t)}{a(t)}. \quad (5.8)$$

Note that "Hubble's constant" is not a constant in general. Another observational fact can be derived from the Robertson-Walker metric: the *cosmological red-shift*. Consider successive crests of light emitted from the above galaxy in $r = r_g$ at times t_g and $t + \Delta t_g$ and received by us at times t_0 and $t_0 + \Delta t_0$. Since the light travels radially and $ds^2 = 0$, we have for the first crest

$$\int_{t_g}^{t_0} \frac{dt}{a(t)} = \frac{1}{c} \int_{r_g}^0 \frac{dr}{\sqrt{1 - kr^2}}, \quad (5.9)$$

and for the second crest:

$$\int_{t_g + \Delta t_g}^{t_0 + \Delta t_0} \frac{dt}{a(t)} = \frac{1}{c} \int_{r_g}^0 \frac{dr}{\sqrt{1 - kr^2}}. \quad (5.10)$$

This implies

$$\int_{t_g + \Delta t_g}^{t_0 + \Delta t_0} \frac{dt}{a(t)} = \int_{t_g}^{t_0} \frac{dt}{a(t)} \quad (5.11)$$

Assuming that $a(t)$ is constant over the intervals Δt_g and Δt_0 , we find

$$\frac{\Delta t_g}{\Delta t_0} = \frac{a(t_g)}{a(t_0)} \quad (5.12)$$

and there is a shift z between the emitted and observed wavelengths of the light. Denoting the former $\lambda_g = c\Delta t_g$ and the last $\lambda_0 = c\Delta t_0$, we find:

$$z = \frac{\lambda_0 - \lambda_g}{\lambda_g} = \frac{a(t_0)}{a(t_g)} - 1 \quad (5.13)$$

Therefore, there is a red-shift if the universe is expanding, for $a(t_0) > a(t_g)$. Such cosmological red-shift was measured by Hubble and it is distinct from the gravitational red-shift and the Doppler shift in which the source of the light and the observer are in relative motion in the *same* inertial frame. The cosmological shift is due to the expansion of the space itself.

We now turn to the problem of what sort of universe are allowed by Einstein's General Relativity. This is possible provide we have an expression for the energy-momentum tensor that describes the degrees of freedom of the total matter in the universe. The model usually adopted is of a perfect fluid which has an energy-momentum tensor of form

$$T_{mn} = \frac{p}{c^2}g_{mn} + (\rho + \frac{p}{c^2})u_mu_v, \quad (5.14)$$

where u_m is the velocity vector field of the fluid, p is its pressure and ρ is the energy density. In comoving coordinates, this tensor is diagonal:

$$T_{00} = \rho, \quad T_{ii} = \frac{p}{c^2}, \quad T_{ij} = 0(i \neq j), \quad T_{0i} = 0 \quad (5.15)$$

Thus, we can use the Einstein field equations

$$R_{mn} - \frac{1}{2}g_{mn}R = \frac{8\pi G}{c^2}T_{mn}, \quad (5.16)$$

to relate $a(t)$, k , p and ρ ¹. The time-time component gives

$$\dot{a}(t) + kc^2 = \frac{8\pi G}{3}\rho a^2(t) \quad (5.17)$$

and the space-space components give

$$2a(t)\ddot{a}(t) + \dot{a}^2(t) + kc^2 = -\frac{8\pi G}{c^2}pa^2(t) \quad (5.18)$$

The second derivative of the scale factor in the last equation can be written in terms of the first derivative by differentiating (5.17). Substituting $\ddot{a}$ into equation (5.18) yields

$$\dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right) = 0 \quad (5.19)$$

¹For a good geometrical and concise treatment of the dynamics of the cosmology see [19].

It is possible to check that this equation expresses the conservation of the energy-momentum tensor (5.14). Equation (5.17) is called the *Friedmann equation* and we shall treat it and (5.19) as the fundamental equations.

Intuitively, we would expect that the energy content of the universe brakes its expansion. This is exactly what we get from Einstein's equations: if we subtract the Friedmann equation (5.17) from (5.18) we have

$$\ddot{a} = -\frac{4\pi G}{3} \left(\rho + 3\frac{p}{c^2} \right) a, \quad (5.20)$$

and hence if $\rho > 3\frac{p}{c^2}$ then $\ddot{a}$ is negative, so $\dot{a}$ is always decreasing in time, i.e. the universe was expanding faster in the past than it is now. Therefore, as a consequence of General Relativity, there must have been a time in which a was "zero". This is the initial singularity and relativity "breaks down" at such scales that we expect to be ruled by a quantum gravity theory.

From the Friedmann equation we can justify the connection of k and the geometry of the space. If $k = 0$ or $k = -1$, $\dot{a}^2$ cannot be zero and the universe expands forever. These are the flat and open universes. For $k = 1$ the expansion will stop when $\rho a^2 = \frac{3kc^2}{8\pi G}$ and after that the universe will start to contract, that is, for $k = 1$ we have a closed universe. The sign of the spatial curvature is determined by the value of the energy density ρ . To see this, let us define the *critical density*

$$\rho_c \equiv \frac{3H^2}{8\pi G} \quad (5.21)$$

where the subscript 0 indicates evaluation at the present (cosmological) time. Writing (5.17) for this moment

$$k = \frac{8\pi G}{3c^2} a_0^2 (\rho_0 - \rho_c) \quad (5.22)$$

one can see that the open, flat and closed cases correspond to universes that has $\rho > \rho_c$, $\rho = \rho_c$ and $\rho < \rho_c$, respectively.

The discussion about k above leave us with three unknowns ρ , p and a for the dynamics of the universe. But we have only two equations for these quantities, (5.17) and (5.19). Hence, a complete solution requires at least one more relation between these variables. In *Friedmann models* it is used an equation of state, relating ρ and p , to solve the dynamical equations. There are three interesting models: for dust, radiation and the vacuum.

The dust case is the one with zero pressure, $p = 0$, describing ordinary matter. In this case, equation (5.19) gives

$$\frac{\dot{\rho}}{\rho} = -3\frac{\dot{a}}{a} \Rightarrow \rho a^3 = \text{const.} = \rho_0 a_0^3 \quad (5.23)$$

This is just the conservation of mass for non-relativistic matter in an expanding space. We can use the Friedmann equation to find the scale factor in terms of time. For example, with $k = 0$, we obtain

$$\dot{a}^2 \propto a^{-1} \Rightarrow a(t) \propto t^{\frac{2}{3}} \quad (5.24)$$

For radiation we have $\rho = \frac{3p}{c^2}$. Then, using the energy-momentum conservation (5.19),

$$\frac{\dot{\rho}}{\rho} = -4\frac{\dot{a}}{a} \Rightarrow \rho a^4 = \text{const.} = \rho_0 a_0^4. \quad (5.25)$$

We see that there is an extra power of a in relation to the dust case. This can be understood by remembering the cosmological red-shift: since λ increases with the expansion of the space, the energy of a photon $E = ch/\lambda$ decreases by a factor a^{-1} and hence the energy density of a certain volume with fotons decreases by a factor a^{-4} . From (5.17) we can find $a(t)$. Again for flat geometry, we have

$$\dot{a} \propto a^{-2} \Rightarrow a(t) \propto t^{\frac{1}{2}} \quad (5.26)$$

The vacuum case is related with static solutions of our dynamical equations. In a static universe we have $\dot{a} = 0 = \ddot{a}$ and from (5.17) and (5.18) this implies

$$\rho = -\frac{3p}{c^2} \quad (5.27)$$

which, assuming positive energy density, means that the universe is occupied by a fluid with negative pressure and so corresponds to an unphysical solution. In order to get static solutions Einstein introduced an extra term into the field equations, the *cosmological constant* Λ [24]:

$$R_{mn} - \frac{1}{2}g_{mn}R - g_{mn}\Lambda = \frac{8\pi G}{c^2}T_{mn}. \quad (5.28)$$

Equations (5.17) and (5.18) become

$$\dot{a}(t) + kc^2 = \frac{8\pi G}{3}\rho a^2(t) + \frac{c^2}{3}\Lambda a^2(t) \quad (5.29)$$

$$2a(t)\ddot{a}(t) + \dot{a}^2(t) + kc^2 = -\frac{8\pi G}{c^2}pa^2(t) + c^2\Lambda a^2(t) \quad (5.30)$$

and equation (5.19) stay the same. Now a static universe is possible since the requirement $\dot{a} = 0 = \ddot{a}$ leads to

$$\Lambda = \frac{4\pi G}{c^2}(\rho + p) \quad (5.31)$$

As an example, for the dust model a static universe is achieved if $\Lambda = \frac{4\pi G}{c^2}\rho$ which gives $k = \Lambda a^2$. Therefore, this is a static and closed universe. From this example we see that the cosmological constant represents a force that balances the matter attraction in such way that the space remains static. The vacuum Friedmann model describes an empty flat space with cosmological constant, i.e. $\rho = 0 = k$. Inserting this in (5.29) gives

$$\dot{a}^2(t) = \frac{c^2}{3}\Lambda a^2(t) \quad (5.32)$$

which has the following solution,

$$a(t) = a(0) \exp \{Ht\} \quad (5.33)$$

and in this case, the Hubble constant is constant in time:

$$H = \sqrt{\frac{c^2\Lambda}{3}}. \quad (5.34)$$

This is the *de Sitter space* and it is expanding, as expected from the "repulsive force" described by Λ . In fact it is an accelerating expansion as well as observed in our universe. This fact leads to the conclusion that the cosmological constant is actually not zero. The origin of a non-vanishing cosmological constant is treated as the contribution of the repulsive energy to the total energy density of the universe. To see this, let us write (5.29) in the form

$$kc^2 = \frac{8\pi G}{3}a^2 \left(\rho + \frac{c^2\Lambda}{8\pi G} - \frac{3H^2}{8\pi G} \right) \quad (5.35)$$

As we observe decoupled matter and radiation in the universe, ρ is the sum of the energy density for matter ρ_m and for radiation ρ_r . Then we have, from (5.21)

$$kc^2 = \frac{8\pi G}{3}a^2 \left(\rho_r + \rho_m + \frac{c^2\Lambda}{8\pi G} - \rho_c \right), \quad (5.36)$$

from which we can see that

$$\rho_\Lambda \equiv \frac{c^2 \Lambda}{8\pi G} \quad (5.37)$$

is the contribution of the cosmological constant for the energy density. It is important to note that the requirement for a cosmological constant to explain the accelerated expansion observed is implemented as an extra content in the universe. This can be seen from the fact that the conservation of the energy momentum tensor is unchanged if we add a term proportional to the metric to it. In this case, if one compare the "extra" energy momentum tensor

$$T_{mn} = \Lambda g_{mn} \quad (5.38)$$

to the perfect fluid one, equation (5.14), then this correspond to an *extra* energy density component ρ_{extra} of ρ in (5.17) with the equation of state $\rho_{ex} = -p_{ex}$. Then we can use the conservation equation (5.19) to find that $\rho_{ex} = const.$ and hence solve the Friedmann equation for an universe dominated by a fluid described by (5.38). For a flat geometry, we have

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3} \rho_{ex}, \quad (5.39)$$

which implies that

$$a(t) = a(0) \exp \left\{ \left(\sqrt{\frac{8\pi G}{3} \rho_{ex}} \right) t \right\}. \quad (5.40)$$

Comparing it with (5.33) we find

$$\rho_{ex} = \frac{c^2 \Lambda}{8\pi G}, \quad (5.41)$$

which is exactly the energy density defined in (5.37). Therefore, the cosmological constant is a prescription to describe some ingredient of the content of our universe that has the equation of state, $\rho_\Lambda = -p_\Lambda$. The fundamental physical nature of this ingredient is still unknown and it is called *dark energy*. As the cosmological constant leads to the expanding empty de Sitter universe, dark energy seems to be related with the *vacuum energy*, that is, it would parametrize an energy density associated to empty space.

5.2 The Λ CDM model

The existence of a cosmic microwave background radiation (CMB), first discovered in 1965 by Arno Penzias and Robert Wilson ([25]) corroborate the idea of Ralph Alpher, Hans Bethe and George Gamow² that in 1948 published a paper in which they proposed a hot beginning of the universe with matter at high density and high temperature standing in thermal equilibrium through reactions like $e^+ + e^- \leftrightarrow \gamma + \gamma$ and $p + n \leftrightarrow d + \gamma$, [26, 27]. This equilibrium state give rise to a thermal radiation with a blackbody spectrum and as the universe expanded and cooled this radiation has remained with such character. The time in which the matter and radiation *decoupled* is known as *recombination* and happened when the universe was at an age of 300,000 years and a temperature of 3000 K. It was the first time that the photons were with energy small enough to form neutral atoms and was when the CMB was emitted. Moreover, the Alpher, Bethe and Gamow work provides an explanation to the measured abundances of elements. This is called *big bang nucleosynthesis*.

Observations of this cosmic radiation revealed that it is almost perfectly isotropic and that corresponds to the spectrum of a blackbody with a temperature of $2.72K$. The small degree of anisotropy in this background comes from the inhomogeneities in the cosmic dust that further result in the formation of the large structures now observed. These observations lead to constraints on the values of k and to a scenario consistent with a flat universe. This justifies the results of the previous section and allows to estimate the energy density of dark energy. For this purpose, let us introduce the following parameter

$$\Omega = \frac{\rho}{\rho_c} \quad (5.42)$$

Using it, we can write (5.36), with $k = 0$, as:

$$\Omega_r + \Omega_m + \Omega_\Lambda = 1 \quad (5.43)$$

where the subscripts corresponds to which component of the total energy density we used to write Ω . Recent estimates for the actual matter contribution gives $\Omega_m \approx 0.3$. Note that Ω_m includes the contribution of the baryonic matter and of the *dark matter*. Dark matter is made of particles that does not interact with photons

²This work is sometimes referred as the $\alpha\beta\gamma$ paper. Bethe was invited to allows the *alphabetical* allusion.

and then it is non-luminous. This kind of matter is needed to explain the rotation curve of gas well outside the visible part of galaxies and turns out to be crucial to the theory of structure formation. Up to date, the quantum constituents of dark matter are unknown. What exactly is dark matter and dark energy are the most important questions in modern cosmology.

For the radiation we can use the Stefan-Boltzmann law with $T = 2.72$ K:

$$\rho_r = \frac{4\sigma}{c}T^4 = 4.6 \times 10^{-31} \text{kgm}^{-3}, \quad (5.44)$$

where σ is the Stefan-Boltzmann constant. From the definition of the critical density with $H_0 = 72 \text{ kms}^{-1} \text{Mpc}^{-1}$, we have

$$\rho_c = 0.97 \times 10^{-26} \text{kgm}^{-3}, \quad (5.45)$$

and then $\Omega_r \approx 5 \times 10^{-5}$.

From the above discussion we see that the dark energy dominates the dynamics of the universe, with $\Omega \approx 0.7$. But this is the *actual* "hierarchy" of the contributions to the total energy density. In an universe containing matter and radiation as our does, their respective energy densities will change as expansion occurs. From equations (5.23) and (5.25) we have

$$\frac{\rho_r}{\rho_m} \propto \frac{1}{a(t)} \quad (5.46)$$

and therefore, if there is radiation at the present time, it must have dominated over the matter at early times. We now have an overall picture of the cosmic history of the universe: a flat space with an initial singularity at finite past from which hot matter and radiation comes at equilibrium (radiation dominated era) and cools conforms the space expands, followed by a matter dominated expansion during which galaxies form. If there is vacuum energy, its constant energy density dominates and the space begins an accelerated expansion. This is basically the content of what is called Λ Cold Dark Matter (ΛCDM) model, where Λ refers to the cosmological constant and the "cold" means that we can treat dark matter as non-relativistic matter.

Chapter 6

Inflation

6.1 The flatness and horizon problems

Despite from its success the Big Bang cosmology does not explain why the universe is so close to the flat geometry and what is the origin of the primordial inhomogeneity of the cosmic fluid, that stayed imprinted in the cosmic microwave background (CMB) as observed by its anisotropy. There is still another problem related to causality due to the existence of horizons. Let us focus on the flatness problem first.

We can write the equation of state in a general form as

$$p = w\rho \tag{6.1}$$

so that the dust and radiation cases corresponds to $w = 0$ and $w = \frac{1}{3}$ respectively. Using equations (5.17), (5.18) and (5.36) to find how the density parameter Ω evolves with respect to the scale factor, we have:

$$\frac{d\Omega}{d\ln a} = (1 + 3w)\Omega(\Omega - 1) \tag{6.2}$$

Although $\Omega = 1$ for flat geometry, in general Ω is not constant. For matter or radiation we have $(1 + 3w) > 0$ which leads to the conclusion that flat space is an unstable fixed point: if Ω begins small than 1 then it evolves for values even small than 1; if Ω begins greater than 1 then it tends to increase even more. That is, we have

$$\frac{d|\Omega - 1|}{d\ln a} > 0, \quad (1 + 3w) > 0 \tag{6.3}$$

From this we see that an universe closed to the flat geometry today is a highly fine-tuned state. To have a quantitative idea, from the CMB measurements we know that at present time, $|\Omega_0 - 1| < 0.02$. So it is very reasonable to take $\Omega_0 = 1 \pm 0.05$, that implies that at the recombination we have $\Omega_{rec} = 1 \pm 0.0004$, and at the time of primordial nucleosynthesis, $\Omega_{nuc} = 1 \pm 10^{-12}$. This fine-tuned situation is called *flatness problem*.

Due to the finite age of the universe, photons can only have traveled a finite distance since the Big Bang. Thus, the universe has a horizon: any observer sees himself at the center of spherical horizon. This *defines* his observable universe. The size of the horizon at time t after the Big Bang can be calculated from the Robertson-Walker metric (5.4):

$$r_h(t) = \int_0^t \frac{dt'}{a(t')}, \quad (6.4)$$

where we have putted $t = 0$ at the Big Bang. This is the coordinate (comoving) size, in order to obtain the proper size of the horizon we need to multiply by the scale factor

$$r_h^{prop}(t) = a(t) \int_0^t \frac{dt'}{a(t')} \quad (6.5)$$

As an example, for a 14-billion year-old universe with flat geometry and matter domination, we have, from the solution (5.24) and normalizing a as $a_0 = 1$, $r_h^{prop} = 3t_0 \simeq 13\text{Gpc}$.

To examine the causal structure of an expanding universe, let us work with the *conformal time*, τ . This is defined in such way that the flat Robertson-Walker metric is written as

$$ds^2 = a^2(\tau) \left[-d\tau^2 + dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right]. \quad (6.6)$$

Then, this metric is conformally related to Minkowsky metric and since the conformal factor does not affect null geodesics, we can analyze causal structure as in Minkowsky spacetime. From the last equation above, we have

$$d\tau = \frac{dt}{a(t)} \quad (6.7)$$

and then,

$$r_h(t) = \int_0^t \frac{dt'}{a(t')} = \int_0^\tau d\tau' = \tau. \quad (6.8)$$

So, the comoving horizon size is the age of the universe in conformal time. The expanding universe has finite age, and therefore unlike the Minkowski spacetime we cannot prolong the light cone of an observer to infinite past: the light cone "stops" at $\tau = 0$. In this way, two points on the CMB at 180° apart in the sky will have past light cones that do not overlap: they "stop" at the spatially infinite surface at $\tau = 0$. Thus, these points are causally disconnected and there is no way for such points reach the thermal equilibrium observed. A more quantitative explanation of this is that the comoving horizon size at the time of the recombination t_{rec} is much less than the comoving distance that the radiation travels after decoupling [28, 29]:

$$\int_0^{t_{rec}} \frac{dt}{a(t)} \ll \int_{t_{rec}}^{t_0} \frac{dt}{a(t)}. \quad (6.9)$$

Therefore, the Big Bang cosmology does not describe why the temperature of the CMB is so isotropic. This is the *horizon problem*.

Let us again use the Friedmann equation, in the form

$$|\Omega - 1| = \frac{|k|}{a^2 H^2} \quad (6.10)$$

In the Big Bang model, aH is always decreasing, and then Ω evolves away from flatness. But if we require

$$\frac{d}{dt} \left(\frac{H^{-1}}{a} \right) < 0 \quad (6.11)$$

which means that the Hubble length H^{-1} in comoving coordinates decreases, then Ω evolves *towards* flatness. The last condition is equivalent to $\ddot{a} > 0$, this is the definition of *inflation* an epoch in which the cosmological expansion is accelerated. From equation (6.2) inflation is achieved if $(1 + 3w) < 0$:

$$\frac{d|\Omega - 1|}{d \ln a} < 0, \quad (1 + 3w) < 0. \quad (6.12)$$

and we can see that this result implies inflation using (5.18). Inflation solves the flatness problem automatically, but any model has to describe enough inflation in order to do not lose the almost flat character after the end of inflation. From (6.11)

it is possible to conclude that inflation solves the horizon problem: since Hubble length is shrinking during inflation, distances that we can see before inflation are much larger than distance that we can see after inflation. This is true for scales compared to the horizon too, that is, distances that are smaller than the horizon before inflation are "red-shifted" to scales larger than the horizon after inflation.

Inflation complements the standard Big Bang model and need to end to give place to the conventional behavior. It is an epoch of the very early universe and solves the horizon and flatness problems, shortcomings of the Big Bang cosmology.

As an example let us consider the de Sitter case, for which we have $p = -\rho$ and

$$\frac{d \ln \Omega}{d \ln a} = 2(1 - \Omega). \quad (6.13)$$

We can see the solution to the horizon problem using the conformal time:

$$d\tau = \frac{dt}{a(t)} = \exp \{-Ht\} dt \quad (6.14)$$

and so,

$$\tau = -\frac{1}{aH} \quad (6.15)$$

then, during inflation, the conformal time is negative and in this case the value $\tau = 0$ represents the transition of inflationary expansion to the radiation dominated era. The two opposite points in the sky cited before can now share a casual past in the negative domain of the conformal time.

6.2 Inflation from scalar fields: The slow roll inflation

The de Sitter example of last section is an explicit picture of inflation. But the physics responsible for accelerated expansion at early times cannot be Einstein's cosmological constant, because it dominates over matter and radiation contributions at later times. We need a transition between the inflation era and the radiation dominated phase, the "vacuum" energy that drives inflation must be time-dependent. For this purpose, inflation is implemented with scalar fields. We are interesting in models with single real scalar field, which in this case is called *the inflaton*.

Thus, it is natural to begin with the action for the inflaton minimally coupled to gravity:

$$S = \int d^4x \sqrt{-g} \left[\frac{1}{16\pi G} R - \frac{1}{2} g^{mn} \partial_m \phi \partial_n \phi - V(\phi) \right] \quad (6.16)$$

where $V(\phi)$ is the potential energy of the scalar field. Assuming flat geometry, the equation of motion for ϕ is

$$\ddot{\phi} + 3H\dot{\phi} - \nabla^2 \phi + \frac{\delta V}{\delta \phi} = 0 \quad (6.17)$$

We are interesting in an homogeneous field for which $\nabla \phi = 0$. The equation of motion simplifies to

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0, \quad (6.18)$$

where the prime represents derivative with respect to the field. The energy-momentum tensor of the scalar field is

$$T_{mn} = \partial_m \phi \partial_n \phi - g_{mn} \left(\frac{1}{2} g^{pq} \partial_p \phi \partial_q \phi + V(\phi) \right) \quad (6.19)$$

and for a homogeneous scalar field, it has the same form of the energy-momentum tensor for a perfect fluid, with energy density and pressure as follows:

$$\begin{aligned} \rho &= \frac{1}{2} \dot{\phi}^2 + V(\phi) \\ p &= \frac{1}{2} \dot{\phi}^2 - V(\phi) \end{aligned} \quad (6.20)$$

Then we see that the de Sitter limit $p \simeq -\rho$ is the regime in which the potential energy dominates the kinetic energy, $\dot{\phi}^2 \ll V(\phi)$. In this case the universe expands almost exponentially. The implementation with scalar fields and the last approximation frequently leads to a quasi-de Sitter universe. For this reason, it is convenient to define the number of e-folds N as

$$dN \equiv -H dt, \quad (6.21)$$

note that N decreases as time (and the scale factor) increases. Inserting (6.20) into the Friedmann equation (5.17) and (5.18) gives

$$H^2 = \frac{8\pi G}{3} \left(\frac{1}{2} \dot{\phi}^2 + V(\phi) \right); \quad (6.22)$$

$$\left(\frac{\ddot{a}}{a} \right) = -\frac{4\pi G}{3} (\rho + 3p) \equiv H^2(1 - \epsilon), \quad (6.23)$$

where we have defined the dimensionless parameter

$$\epsilon = \frac{3}{2} \left(\frac{p}{\rho} + 1 \right) = 4\pi G \left(\frac{\dot{\phi}}{H} \right)^2 \quad (6.24)$$

because the de Sitter limit is equivalent to $\epsilon \rightarrow 0$, that is the case of in which the potential energy dominates the kinetic energy, and

$$H^2 \simeq \frac{8\pi G}{3} V(\phi) \quad (6.25)$$

We will assume that the frictional term in the equation (6.18) dominates the "acceleration" term $\ddot{\phi}$:

$$\ddot{\phi} \ll 3H\dot{\phi} \quad (6.26)$$

and then the equation of motion for the scalar field is approximately

$$3H\dot{\phi} + V'(\phi) \simeq 0. \quad (6.27)$$

The assumption (6.26) can be written in terms of another dimensionless parameter $|\eta| \ll 1$, where we defined

$$\eta \equiv -\frac{\ddot{\phi}}{H\dot{\phi}} \quad (6.28)$$

Using equations (6.25) and (6.26) we can write the parameters ϵ and η approximately as

$$\begin{aligned} \epsilon &= 4\pi G \left(\frac{\dot{\phi}}{H} \right)^2 \simeq \frac{1}{16\pi G} \left(\frac{V'(\phi)}{V(\phi)} \right)^2; \\ \eta &= -\frac{\ddot{\phi}}{H\dot{\phi}} \simeq \frac{1}{8\pi G} \left(\frac{V''(\phi)}{V(\phi)} \right). \end{aligned} \quad (6.29)$$

The approximations (6.25) and (6.26) are called *slow roll* approximation and it is valid as long as the parameters, called *slow roll parameters*, satisfy

$$\epsilon \ll 1, \quad |\eta| \ll 1. \quad (6.30)$$

The first of these conditions tell us that the field has a very flat potential, $V'(\phi) \ll V(\phi)$ and the second slow roll condition informs that the curvature $V''(\phi)$ of the potential must also be small.

Using the definition of ϵ we can write the number of e-folds as an integral in the field space:

$$N = - \int H dt = - \int \frac{H}{\dot{\phi}} d\phi = 2\sqrt{\pi G} \int \frac{d\phi}{\sqrt{\epsilon}} \simeq 8\pi G \int_{\phi_e}^{\phi_N} \frac{V(\phi)}{V'(\phi)} d\phi \quad (6.31)$$

where ϕ_e is the field value at the end of the inflation, that will can be obtained by breaking the slow conditions $\epsilon(\phi) = 1$, and ϕ_N is just an adjusted value that leads to a certain number N of e-folds. In order to get enough inflation we need to have $N > 60$.

We now can put all together and construct a qualitative picture of slow roll inflation. At very early times the energy density of the universe is dominated by the potential energy of a scalar field, the inflaton. The potential is nearly constant and inflation stands as the inflaton *slowly rolls* this region, the space expanding almost exponentially during it. Inflation ends when the field enters a steeper and more curved region and the slow roll conditions are not satisfied anymore. To obtain the standard Big Bang cosmology, the energy of the inflaton must decay into the Standard Model particles in a process that is called *reheating*. The theory of *reheating* is far for complete, as it depends on the extension of the Standard Model to high energies. This process is model dependent, the inflaton would or not be a fundamental field. This single viewed of the dynamics of inflation driven by scalar fields is an effective representation of some underlying theory that is not yet established. It is remarkable that only the potential energy of the inflaton is important for the physics of slow roll inflation. Thus the potential specifies the model.

6.3 Primordial density power spectrum from inflation

In addition to its success solving the flatness and horizon problems, it is remarkable that the primordial density perturbation of the cosmic fluid can be obtained from inflation. This is done by considering the quantum nature of the inflaton: in essence,

the primordial inhomogeneity comes from the *quantum fluctuations* of the inflaton. It is worthwhile to remember that is this primordial inhomogeneity that leads to the formation of the large structures of our universe. It was this density perturbation that leads to the anisotropies in the cosmic microwave background. Indeed, by observation of the *CMB* one can construct the *power spectrum* of this perturbation on the cosmic fluid.

We begin by solving the Klein-Gordon equation for a free scalar field φ , that is not the inflaton and does not affect the scale factor: is just an arbitrary scalar field in a fixed background. For an flat expanding space, the equation of motion is like (6.18) but with $V = 0$ and in conformal coordinates, which gives

$$\varphi'' + 2 \left(\frac{a'}{a} \right) \varphi' - \nabla^2 \varphi = 0, \quad (6.32)$$

where now the prime indicates derivative with respect to the conformal time, $' = d/d\tau$. Consider its Fourier expansion in comoving wavevectors $\mathbf{k}$:

$$\varphi(\tau, \mathbf{x}) = \int \frac{d^3 k}{(2\pi)^{3/2}} (\varphi_{\mathbf{k}}(\tau) b_{\mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{x}} + \bar{\varphi}_{\mathbf{k}}(\tau) \bar{b}_{\mathbf{k}} e^{-i\mathbf{k} \cdot \mathbf{x}}). \quad (6.33)$$

Then, the equation of motion for the mode $\varphi_{\mathbf{k}}$ is

$$\varphi_{\mathbf{k}}'' + 2 \left(\frac{a'}{a} \right) \varphi_{\mathbf{k}}' + k^2 \varphi_{\mathbf{k}} = 0. \quad (6.34)$$

Using the field redefinition $u_k \equiv a(\tau) \phi_{\mathbf{k}}(\tau)$ gives

$$u_k'' + \left(k^2 - \frac{a''}{a} \right) u_k = 0 \quad (6.35)$$

We shall analyze this equation in two regimes:

- *Long wavelength limit*, $k \ll a''/a$. This limit, we have

$$a'' u_k = a u_k'' \quad (6.36)$$

with solution $u_k \propto a$ which implies that $\varphi_k = \text{const.}$. Then, in this limit, the field modes φ_k are not dynamical, but has a non-zero amplitude. This is a example of *mode freezing*: the mode asymptote to a constant;

- *Short wavelength limit, $k \gg a''/a$.* In this case the equation reduces to Klein-Gordon equation in Minkowski space, but with conformal coordinates:

$$u_k'' + k^2 u_k = 0 \quad (6.37)$$

that has the solution

$$u_k(\tau) = \frac{1}{\sqrt{2k}} (A_k e^{-ik\tau} + B_k e^{ik\tau}) \quad (6.38)$$

This can be identified with the exact Minkowskian solution in the ultraviolet limit.

Hence, all we need to do is set the boundary conditions on field perturbations in the ultraviolet limit (short wavelength limit). But in such limit we have a Minkowskian description of the fields, equation (6.37), and we know how to quantize fields in Minkowski spacetime. In fact, one of the conditions comes from quantization and the other comes from the vacuum selection. Together they specify A_k and B_k .

Under canonical quantization, the coefficients $b_{\mathbf{k}}$ and $\bar{b}_{\mathbf{k}}$ of the expansion (6.33) are promoted to annihilation and creation operators with the commutation relation

$$[\hat{b}_{\mathbf{k}}, \hat{b}_{\mathbf{k}'}^\dagger] = \delta^3(\mathbf{k} - \mathbf{k}') \quad (6.39)$$

The quantum field $\hat{\varphi}(\tau, \mathbf{x})$ is written in terms of these operators and has a canonical conjugate momentum

$$\hat{\Pi}(\tau, \mathbf{x}) = a^2(t) \frac{\partial \varphi}{\partial \tau}. \quad (6.40)$$

The commutation relation

$$[\hat{\varphi}(\tau, \mathbf{x}), \hat{\Pi}(\tau, \mathbf{x}')] = i\delta^3(\mathbf{x} - \mathbf{x}'), \quad (6.41)$$

leads to

$$u_k \frac{\partial \bar{u}_k}{\partial \tau} - \bar{u}_k \frac{\partial u_k}{\partial \tau} = i \quad (6.42)$$

which gives, for the ultraviolet mode solution (6.38),

$$|A_k|^2 - |B_k|^2 = 1. \quad (6.43)$$

The second condition on these constants, comes from the vacuum selection. We choose the vacuum which gives the usual Minkowski vacuum in the ultraviolet limit: $A_k = 1, B_k = 0 \rightarrow u_k(\tau) \propto e^{-ik\tau}$.

Now we can apply this analysis to the inflaton field. In this case we need to take into account the evolution of the background spacetime that is encoded in $a(\tau)$. A good approximation is to consider the slow parameter ϵ as constant, but in general this is not the case. To an arbitrary equation of state

$$\tau = - \left(\frac{1}{aH} \right) \left(\frac{1}{1 - \epsilon} \right) \quad (6.44)$$

and from (5.17) and (5.18) with flat geometry we have

$$\frac{a''}{a} = a^2 H^2 (2 - \epsilon). \quad (6.45)$$

Then equation (6.35) is

$$u_k'' + [k^2 - a^2 H^2 (2 - \epsilon)] u_k = 0 \quad (6.46)$$

Using (6.44) to write aH in terms of the conformal time, we have

$$\tau^2 (1 - \epsilon)^2 u_k'' + [(k\tau)^2 (1 - \epsilon)^2 - (2 - \epsilon)] u_k = 0 \quad (6.47)$$

This is a Bessel equation and its solutions depends on the first and second Bessel functions, J_ν and Y_ν [30]:

$$u_k \propto \sqrt{-k\tau} (J_\nu(-k\tau) \pm iY_\nu(-k\tau)), \quad (6.48)$$

where

$$\nu = \frac{3 - \epsilon}{2(1 - \epsilon)} \quad (6.49)$$

For the de Sitter case, $\epsilon = 0$ and the Bessel index is $\nu = 3/2$ and the solution is

$$u_k \propto \left(\frac{k\tau - i}{k\tau} \right) e^{\pm i k \tau} \quad (6.50)$$

The short wavelength limit is $-k\tau \rightarrow -\infty$. This can be seen from the equation (6.44):

$$(-k\tau)(1 - \epsilon) = \frac{k}{aH}, \quad (6.51)$$

since the quantity (k/aH) represents the wavenumber k in units of the comoving horizon size $r_H = (aH)^{-1}$, then the limit $-k\tau \rightarrow 0$ corresponds to $(k/aH) \ll 1$, the long wavelength limit. Taking the short wavelength limit to the de Sitter solution

and selecting the vacuum, $u_k \propto e^{ik\tau}$, with the normalization fixed by the canonical quantization, we have:

$$u_k = \frac{1}{\sqrt{2k}} e^{-ik\tau}. \quad (6.52)$$

Thus, the exact solution to the mode function of inflaton in the de Sitter case is

$$u_k = \frac{1}{\sqrt{2k}} \left(\frac{k\tau - i}{k\tau} \right) e^{-ik\tau}. \quad (6.53)$$

This solution is valid for all wavelengths: we use the short limit just to find the proper normalization. Let us confirm the long wavelength limit, $-k\tau \rightarrow 0$:

$$u_k \rightarrow \frac{1}{\sqrt{2k}} \left(\frac{i}{(-ik\tau)} \right) = \frac{i}{\sqrt{2k}} \left(\frac{aH}{k} \right) \propto a. \quad (6.54)$$

Comparing with the solution of (6.36), we see that these are consistent. Therefore, the field amplitude φ_k is, in the long wavelength limit

$$|\varphi_k| = \left| \frac{u_k}{a} \right| = \frac{H}{\sqrt{2k^3}} = \text{const.} \quad (6.55)$$

The amplitude of the quantum fluctuations can be obtained from the two-point correlation function of the quantum field $\hat{\varphi}$:

$$\langle 0 | \hat{\varphi}(\tau, \mathbf{x}) \hat{\varphi}(\tau, \mathbf{x}') | 0 \rangle = \int \frac{d^3k}{(2\pi)^3} |\varphi|^2 e^{i\mathbf{k} \cdot (\mathbf{x} - \mathbf{x}')} \equiv \int \frac{dk}{k} P(k) e^{i\mathbf{k} \cdot (\mathbf{x} - \mathbf{x}')}, \quad (6.56)$$

where we have defined the *power spectrum* $P(k)$ as

$$P(k) \equiv \left(\frac{k^3}{2\pi^2} \right) \left| \frac{u_k}{a} \right|^2 = \left(\frac{H}{2\pi} \right)^2, \quad (6.57)$$

note that this power spectrum is scale independent. For a more general model, the space is *not exactly* de Sitter and the power spectrum of the field fluctuations will only be approximately scale invariant. It is conventional to evaluate the power spectrum at $aH = k$, the *horizon crossing*.

The primordial density perturbation, that is responsible for structure formation, is generated by the fluctuations of the inflaton. Using the general relativistic perturbation theory, it is possible to show that the power spectrum of the primordial

density perturbations $\mathcal{R}$ is related to the power spectrum of the inflaton fluctuations $P(k)$, by [28, 31]:

$$\mathcal{R} = \left(\frac{H}{\dot{\phi}}\right)^2 \left(\frac{H}{2\pi}\right)^2 \quad (6.58)$$

where the right hand side is evaluated at the horizon exit, $k = aH$. Using the slow roll approximation (6.25) and (6.26), we have

$$\mathcal{R} \simeq \frac{(8\pi G)^2 V}{24\pi^2 \epsilon} \quad (6.59)$$

The spectral index n_s is a parameter that measures the departure from scale invariance:

$$n_s - 1 \equiv \frac{d \ln \mathcal{R}}{d \ln k} \quad (6.60)$$

It is possible to use the slow roll conditions to write it as

$$n_s - 1 = -6\epsilon + 2\eta \quad (6.61)$$

The tensor perturbations are related to the production of gravitational waves. Since the transverse and longitudinal polarization states of the gravitational waves evolves as independent scalars (see [19] for instance), we can use the power spectrum of the inflaton fluctuation to calculate the power spectrum of the gravity waves as the sum of the 2 correlated function for the separate polarizations:

$$P_T = 2 \times (8\pi G) \left(\frac{H}{2\pi}\right)^2 \quad (6.62)$$

where the factor $(8\pi G)$ comes from the Einstein-Hilbert Lagrangian. This power spectrum has a spectral index $n_T = -2\epsilon$. Then we have a consistent condition between the power spectra P_T and $\mathcal{R}$

$$r = \frac{P_T}{\mathcal{R}} \simeq 16\epsilon \quad (6.63)$$

Therefore, for each model specified by the inflaton potential, we have three independent parameters to calculate: the amplitudes of the tensor and scalar power spectra P_t and $\mathcal{R}$ respectively, the scalar spectral index n_s and the tensor to scalar ratio

r . The observational constraints on these parameters are, from the Planck results [32, 33]

$$\begin{aligned} n_s &= 0.9603 \pm 0.0073 \\ r &< 0.11 \\ \mathcal{R} &\simeq 2.2 \times 10^{-9} \end{aligned} \tag{6.64}$$

Chapter 7

Inflation in supergravity

7.1 Inflaton potential and the eta problem

Up to now, there is no fundamental theory for Planckian scales and it is at such scales that inflaton dynamics take place. So, the origin of inflation is not known and would be explained by a theory that should reduce to the Standard Model (SM) at low energies. Many extended versions of the SM includes supersymmetry and this motivates the investigation of inflation in supergravity.

Although supergravity is the low energy *effective field theory* of superstring, it is possible to implement inflation in supergravity without requiring string theory. Therefore, we can construct viable models for inflation dealing with chiral matter and its coupling to supergravity only, at least in our approach.

The model building is specified giving a Kähler potential $K(\Phi, \bar{\Phi})$ and the superpotential $W(\Phi)$ for the superfield, the inflaton being one component of the complex scalar field in the multiplet [34]. We shall focus in single chiral superfield models [35, 36]. From equation (4.26) we see that the potential for the scalar fields is

$$V = e^{\frac{K}{M_{Pl}^2}} \left[\left(\frac{\partial W}{\partial \phi_i} + M_{Pl}^{-2} \frac{\partial K}{\partial \phi_i} W \right) K_{i\bar{j}}^{-1} \left(\frac{\partial \bar{W}}{\partial \bar{\phi}_j} + M_{Pl}^{-2} \frac{\partial K}{\partial \bar{\phi}_j} \bar{W} \right) - 3M_{Pl}^{-2} W \bar{W} \right] \quad (7.1)$$

where K and W appearing in this formula depends only on the complex scalar field in the multiplet (the other component fields does not take part on it, see section (4.3)) and

$$K_{i\bar{j}} = \frac{\partial^2 K}{\partial \phi_i \partial \bar{\phi}_j} \quad (7.2)$$

It is worthwhile to note that the limit $M_{Pl} \rightarrow \infty$ in this potential leads to the scalar potential in (3.8). This is the case of local supersymmetry.

Note that the exponential factor in (7.1) prevents the inflaton from taking large values, greater than the Planck scale, at least for almost canonical Kähler potentials. So, it is natural work with *small field* inflation, in which the inflaton take subplanckian values. In order to realize large field inflation, we usually need a fine tuned Kähler potential (or even an inflaton non-depended one¹) .

In general, it is not so simple to choose a superpotential and Kähler potential that gives a scalar potential suitable for inflation, that is, a potential that yields inflation parameters satisfying slow-roll conditions (6.30) and the observational constraints (6.64). This can be seen taking almost canonical Kähler potential, at least next to the origin of the field space,

$$K(\Phi_i, \bar{\Phi}_i) = \sum_i \Phi_i \bar{\Phi}_i + \dots \quad (7.3)$$

and, from string theory, the coefficients of higher order terms in that approximation are of order 1 in planckian units. Then (7.1) can be approximated as

$$\begin{aligned} V &= e^{M_{Pl}^{-2} \phi_i \bar{\phi}_i + \dots} \left\{ \left[\frac{\partial W}{\partial \phi_i} + M_{Pl}^{-2} (\bar{\phi}_i + \dots) W \right] (\delta_{ij} + \dots) \left[\frac{\partial \bar{W}}{\partial \bar{\phi}_j} + M_{Pl}^{-2} (\phi_j + \dots) \bar{W} \right] - \right. \\ &\quad \left. - 3 M_{Pl}^{-2} W \bar{W} \right\} \\ &= \frac{\partial W}{\partial \phi_i} \frac{\partial \bar{W}}{\partial \bar{\phi}_i} + \frac{\partial W}{\partial \phi_i} \frac{\partial \bar{W}}{\partial \bar{\phi}_i} \frac{\phi_j \bar{\phi}_j}{M_{Pl}^2} + \dots \end{aligned} \quad (7.4)$$

where the derivatives are evaluated at the origin. From this we can see that any scalar field, including the inflaton, get a effective mass squared $\frac{\partial W}{\partial \phi_i} \frac{\partial \bar{W}}{\partial \bar{\phi}_i}$. This gives a contribution to the slow-roll parameter η that breaks the slow-roll conditions (6.30)

$$\eta = M_{Pl}^2 \frac{V''}{V} \sim M_{Pl}^2 \frac{|\frac{\partial W}{\partial \phi_i}|^2}{M_{Pl}^2 |\frac{\partial W}{\partial \phi_i}|^2} \sim 1 \quad (7.5)$$

¹For instance, one can use a shift symmetry that is broken by a small breaking term. Topological and chaotic inflations are of this class.

Therefore, any model of inflation in supergravity have to evade this problem, that is called the *eta problem*. This is commonly done by appealing to (an unwelcome) amount of fine-tuning in the parameters of the specific model.

7.2 New inflation in supergravity

New inflation was introduced in 1982 by Linde [37] and Albrecht and Steinhardt [38] and has the feature that the inflaton slowly rolls towards a minimum of the potential, that is, a plateau following by a drop to the minimum. So, the potential is very flat and the inflaton end up at a value at most of order M_{Pl} , this make it a good class of models to realize small field inflation. It is easy to see that the plateau can be achieved by

$$K(\Phi, \bar{\Phi}) = \Phi \bar{\Phi}, \quad W(\Phi) = \Phi \quad (7.6)$$

which gives

$$V = e^{\phi \bar{\phi}} \left\{ |1 + \phi \bar{\phi}|^2 - 3\phi \bar{\phi} \right\} \simeq 1 + O(\phi^4) \quad (7.7)$$

So, this would be a good starting point, to which we need to add other features, including a minimum.

As a example of new inflation in supergravity, we will present a single-field model of inflation that was introduced by Izawa and Yanagida [39], in 1996. In their work, the superpotential for the chiral superfield is given by

$$W(\Phi) = v^2 \Phi - \frac{g}{n+1} \Phi^{n+1} \quad (7.8)$$

where $v \ll 1$, g is of order unity, $n \geq 3$ and we assume that both v and g are real and positive. We will analyze a simplest version of the model, in which the Kähler potential is the canonical one

$$K(\Phi, \bar{\Phi}) = \Phi \bar{\Phi} \quad (7.9)$$

For these K and W , we have

$$K_{i\bar{j}} = \delta_{ij} = 1$$

$$\frac{\partial W}{\partial \phi} + \frac{\partial K}{\partial \phi} W = v^2 - g\phi^n + \phi\bar{\phi} \left(v^2 - \frac{g}{n+1}\phi \right)$$

Therefore, using (7.1) we get

$$V = e^{\phi\bar{\phi}} \left[\left| v^2 (1 + \phi\bar{\phi}) - g\phi^n \left(1 + \frac{\phi\bar{\phi}}{n+1} \right) \right|^2 - 3\phi\bar{\phi} \left| v^2 - \frac{g}{n+1}\phi \right|^2 \right] \quad (7.10)$$

The inflaton ρ is identified as the real part of the complex field ϕ , $\rho \equiv \sqrt{2} \text{Re}\phi$. As $v \ll 1$ and ρ is subplanckian, the inflation dynamics is ruled by the following potential:

$$V(\rho) \simeq v^4 - \frac{2g}{2^{n/2}} v^2 \rho^n + \frac{g^2}{2^n} \rho^{2n} \quad (7.11)$$

This potential has a minimum at

$$\rho_{min} = \sqrt{2} \left(\frac{v^2}{g} \right)^{\frac{1}{n}}$$

Using the fact that, during inflation, the potential is dominated by the first constant term v^4 , the slow-roll parameters are given by

$$\epsilon = \frac{1}{2} \left(\frac{V'}{V} \right)^2 \simeq \frac{1}{2} \left(\frac{-2^{1-\frac{n}{2}} g n v^2 \rho^{n-1}}{v^4} \right)^2 \simeq \frac{g^2 n^2}{2^{n-1} v^4} \rho^{2(n-1)} \quad (7.12)$$

$$\eta = \frac{V''}{V} \simeq \left(\frac{-\frac{2g v^2 n(n-1)}{2^{\frac{n}{2}}} \rho^{n-2}}{v^4} \right) \simeq -\frac{n(n-1)g}{2^{\frac{n-2}{2}} v^2} \rho^{n-2}$$

Inflation ends at a value ρ_e in which $|\eta|$ becomes as larger as unity, that is,

$$\rho_e = \sqrt{2} \left(\frac{v^2}{g n(n-1)} \right)^{\frac{1}{n-2}} \quad (7.13)$$

and we can use this result to calculate N , the number of e-folds, that is given by equation (6.31):

$$\begin{aligned}
N &\simeq \int_{\rho_e}^{\rho_N} \frac{V}{V'} d\rho \\
&= \left(-\frac{2^{\frac{n-2}{2}} v^2}{gn} \right) \int_{\rho_e}^{\rho_N} \rho^{1-n} d\rho \\
&= \left(-\frac{2^{\frac{n-2}{2}} v^2}{gn(2-n)} \right) \left[\rho_N^{2-n} - 2^{\frac{2-n}{2}} \left(\frac{v^2}{gn(n-1)} \right)^{\frac{2-n}{n-2}} \right] \\
&= \left(-\frac{2^{\frac{n-2}{2}} v^2}{gn(2-n)} \right) \left[\rho_N^{2-n} - 2^{\frac{2-n}{2}} \frac{gn(n-1)}{v^2} \right] \\
&= \frac{2^{\frac{n-2}{2}} v^2}{gn(n-2)} \rho_N^{2-n} - \frac{n-1}{n-2}
\end{aligned} \tag{7.14}$$

where ϕ_N is the inflaton initial value that corresponds to an e-folding number equal to N . Inverting this equation we can express the spectral index and amplitude of the primordial density perturbations in terms of N :

$$\begin{aligned}
\mathcal{R}_s &= \frac{V}{24\pi^2 \epsilon} \simeq \frac{2^{n-4} v^8}{3\pi^2 n^2 g^2} \frac{1}{\rho_N^{2(n-1)}} \simeq \frac{1}{24\pi^2} (N(n-2))^{\frac{2(n-1)}{n-2}} (gn)^{\frac{2}{n-2}} v^{\frac{4(n-3)}{n-2}} \\
n_s - 1 &= -6\epsilon + 2\eta \simeq 2\eta \simeq -\frac{gn(n-1)}{2^{\frac{n-4}{2}} v^2} \rho_N^{n-2} \simeq -\frac{2}{N} \frac{n-1}{n-2}
\end{aligned} \tag{7.15}$$

Inserting the observational constraint for the primordial amplitude, we can obtain v for specific values of n , g and N . For example, putting $R_s^2 = 2.442 \times 10^{-9}$ we get $v \simeq 6.9 \times 10^{-7}$ for $n = 4$, $g = 0.3$ and $N = 50$. The potential is showed in figure 7.1 for these parameters.

In this case, the spectral index is $n_s \simeq 0.94$ and the tensor fluctuations are not significant. This is in tension with the observations, but it can be improved by take account higher order terms in the Kähler potential, as it was done in the original paper, or by allowing a small constant term in the superpotential during inflation as did in [40].

7.3 Large extra dimensions

We will focus on supergravity inflation models with large extra dimension, so we will explain what these are.

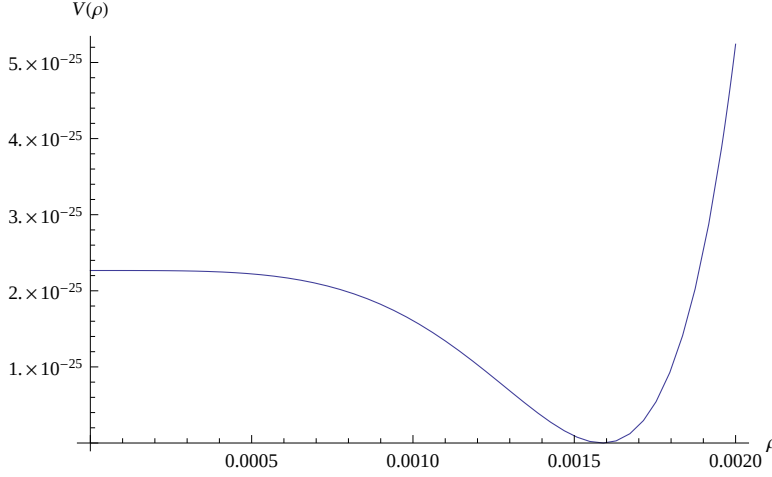


Figure 7.1: Inflaton potential for Izawa and Yanagida model, in Planckian units, with $v \simeq 6.9 \times 10^{-7}$, $n = 4$, $g = 0.3$ and $N = 50$.

The solution to the hierarchy problem of Standard Model is one of the greatest motivations to the construction of theories beyond SM. Let us compare the electroweak and Planck scales, $m_{EW} \sim 1$ TeV and $M_{Pl} \sim 10^{18}$ GeV,

$$\frac{m_{EW}}{M_{Pl}} \sim 10^{-17} \quad (7.16)$$

The gravitational interaction becomes as strong as the other gauge interactions at scales next to M_{Pl} and there is an unexplained desert between m_{EW} and M_{Pl} , probably ruled by effective field theories that describe the physics beyond SM.

These energy scales have distance scales associated, 10^{-16} cm for electroweak and 10^{-33} cm for Planck scale. The former distance scale is of the order of the distances that our present-day particle accelerators are exploring, while gravitational forces have not been tested at distances $\sim 10^{-33}$ cm. Torsion pendulum based experiments have probed Newton's law only until $\sim 10^{-5}$ cm. Then, treat M_{Pl} as a fundamental energy scale is based on the assumption that gravity does not change over almost thirty orders of magnitude.

A simple solution to this problem was proposed by Nima Arkani-Hamed, Savas Dimopoulos and Gia Dvali [6], in the end of the last century. The idea is to low the Planck scale to the TeV region, supposing n extra compact spatial dimensions of radius $\sim R$. The ratio (7.16) may be explained using the following argument. Consider the (Newtonian) gravitational potential $V(r)$ at a distance $r \ll R$ from a

body with mass m at the origin. It can be calculated from the Poisson equation

$$\nabla^2 V = 4\pi G_{(4+n)} \rho_{(4+n)} = \frac{4\pi}{M_{Pl(4+n)}^{2+n}} \rho_{(4+n)} \quad (7.17)$$

where, in this case, $\rho_{(4+n)} = m\delta^{3+n}(0)$. All the subscripts shown in parenthesis denote the dimensionality of spacetime. This equation can be solved and gives

$$V(r) = \frac{m}{M_{Pl(4+n)}^{2+n}} \frac{1}{r^{n+1}} \quad (7.18)$$

If we now calculate the potential at a distance $r \gg R$, we recover the usual formula

$$V(r) = \frac{m}{M_{Pl}^2} \frac{1}{r} = \frac{m}{M_{Pl(4+n)}^{n+2}} \frac{1}{R^n r} \quad (7.19)$$

where the second equality comes from the fact that $\rho_{(4)} = R^n \rho_{(4+n)}$. Therefore our usual 4 dimensional Planck scale satisfy

$$M_{Pl}^2 = M_{Pl(4+n)}^{n+2} R^n \quad (7.20)$$

and requiring $m_{EW} \sim M_{Pl(4+n)}$ we have

$$R \sim \frac{M_{Pl}^{\frac{2}{n}}}{m_{EW}^{\frac{n}{2+n}}} \quad (7.21)$$

Putting $n = 1$, we get $R \sim 10^{13}\text{cm}$ and this case is obviously excluded. But even with $n = 2$, $R \sim 10^{-6}\text{cm}$! This show that cases $n \geq 2$ are allowed by present experimental bounds on gravitational forces.

We should note that the most common way to deal with extra dimensions is the usual Kaluza-Klein one, that the extra dimensions have Planck size. This would be natural in a string theoretic setting.

Nevertheless, we want to consider the possibility of large extra dimensions in supergravity inflation. A natural light scalar that can play the role of inflation is the volume modulus of the extra dimensions. Indeed, a big problem with large extra dimensions is the stability of compactification of internal dimensions, more specifically the stabilization of the volume modulus of the compactified space [41, 42]. This volume appears to us (4 dimensional space) as a light scalar field and its kinetic term, via dimensional reduction, appears from the Kähler potential

$$K(\Phi, \bar{\Phi}) = -\delta \ln(-i(\Phi - \bar{\Phi})) \quad (7.22)$$

where δ depends on the number n of extra dimensions. As an example, $\alpha = 2$ and 3 for $n = 2$ and 6, respectively [43]. In general, the relation between δ and n can be obtained directly from the dimensional reduction procedure.

Chapter 8

Inflation with large extra dimensions

8.1 Perturbative expansion of the superpotential

We want to construct a model of small-field new inflation in supergravity that comes from a single chiral field, with a superpotential that leads to an inflaton potential favored by Planck results [44]. As the inflaton takes subplanckian values, let us make the following expansion for the superpotential

$$W(\Phi) = \frac{1}{\Phi^n} (1 + a\Phi + b\Phi^2 + c\Phi^3) \quad (8.1)$$

and take the canonical Kähler potential $K = \Phi\bar{\Phi}$. We have

$$\begin{aligned} \frac{\partial W}{\partial \Phi} + \frac{\partial K}{\partial \Phi} W &= -n\Phi^{-n-1} (1 + a\Phi + b\Phi^2 + c\Phi^3) + \Phi^{-n} (a + 2b\Phi + 3c\Phi^2) + \\ &\quad + \bar{\Phi}\Phi^{-n} (1 + a\Phi + b\Phi^2 + c\Phi^3) \\ \frac{\partial \bar{W}}{\partial \bar{\Phi}} + \frac{\partial K}{\partial \bar{\Phi}} \bar{W} &= -n\bar{\Phi}^{-n-1} (1 + a\bar{\Phi} + b\bar{\Phi}^2 + c\bar{\Phi}^3) + \bar{\Phi}^{-n} (a + 2b\bar{\Phi} + 3c\bar{\Phi}^2) + \\ &\quad + \Phi\bar{\Phi}^{-n} (1 + a\bar{\Phi} + b\bar{\Phi}^2 + c\bar{\Phi}^3) \end{aligned}$$

and therefore, the scalar potential is

$$\begin{aligned} V(\phi) = e^{|\phi|^2} \Big\{ &|-n\phi^{-n-1} (1 + a\phi + b\phi^2 + c\phi^3) + \phi^{-n} (a + 2b\phi + 3c\phi^2) + \\ &+ \bar{\phi}\phi^{-n} (1 + a\phi + b\phi^2 + c\phi^3)|^2 - 3|\phi|^{-2n} |1 + a\phi + b\phi^2 + c\phi^3|^2 \Big\} \quad (8.2) \end{aligned}$$

We will identify the inflaton ρ as the imaginary part of ϕ , $\rho \equiv \sqrt{2} \text{Im}\phi$. So, the inflaton potential comes from

$$V(\phi = i\sigma) = e^{\sigma^2} \left\{ [n^2 \sigma^{-2(n+1)} - 5n \sigma^{-2n} + \sigma^{-(n-1)}] [1 + (a^2 - 2b)\sigma^2 + (b^2 - 2ac)\sigma^4 + c^2 \sigma^6] + 2 [n \sigma^{-2n-1} + \sigma^{-2n+1}] [(a^2 - 2b)\sigma + (2b^2 - 4ac)\sigma^3 + 3c\sigma^5] + \sigma^{-2n} [a^2 + (4b^2 - 6ac)\sigma^2 + 9c^2 \sigma^4] \right\}$$

that may be further approximated as

$$\begin{aligned} V(\sigma) &\simeq e^{\sigma^2} \left\{ n^2 \sigma^{-2n-2} + \sigma^{-2n} [-5n + n^2(a^2 - 2b) + 2n(a^2 - 2b) + a^2] + \right. \\ &\quad \left. + \sigma^{-2n+2} [1 - 5n(a^2 - 2b) + n^2(b^2 - 2ac) + 2(a^2 - 2b) + \right. \\ &\quad \left. + 2n(2b^2 - 4ac) + 4b^2 - 6ac] \right\} \\ &\equiv e^{\sigma^2} [n^2 \sigma^{-2n-2} + A\sigma^{-2n} + B\sigma^{-2n+2}] \end{aligned} \quad (8.3)$$

where, in the last line, we defined

$$\begin{aligned} A &\equiv -5n + n^2(a^2 - 2b) + 2n(a^2 - 2b) + a^2 \\ B &\equiv 1 - 5n(a^2 - 2b) + n^2(b^2 - 2ac) + 2(a^2 - 2b) + \\ &\quad + 2n(2b^2 - 4ac) + 4b^2 - 6ac \end{aligned} \quad (8.4)$$

Now we can calculate the slow-roll parameters and analyze its dependence in n . As $\rho = \sqrt{2}\sigma$, we have

$$\begin{aligned} \epsilon &= \frac{1}{2} \left(\frac{1}{V} \frac{d\sigma}{d\rho} \frac{dV}{d\sigma} \right)^2 = \frac{1}{4} \left\{ [2\sigma (n^2 \sigma^{-2n-2} + A\sigma^{-2n} + B\sigma^{-2n+2}) - \right. \\ &\quad \left. - (2n+2)n^2 \sigma^{-2n-3} - 2nA\sigma^{-2n-1} - (2n-2)B\sigma^{-2n+1}] \times \right. \\ &\quad \left. \times [n^2 \sigma^{-2n-2} + A\sigma^{-2n} + B\sigma^{-2n+2}]^{-1} \right\}^2 \\ &= \frac{1}{4} \left\{ 2\sigma + \frac{1}{\sigma} \frac{-n^2(2n+2) - 2nA\sigma^2 - (2n-2)B\sigma^4}{n^2 + A\sigma^2 + B\sigma^4} \right\}^2 \end{aligned}$$

If $n \neq 0$ and $n \neq -1$, then we have

$$\epsilon \simeq \frac{1}{4\sigma^2} (2n+2)^2 \gg 1 \quad (8.5)$$

and so this possibility is excluded. For $n = 0$, we get

$$\epsilon \simeq \frac{1}{4} \left(2\sigma + \frac{2B\sigma^4}{A\sigma^3} \right)^2 = \sigma^2 \left(1 + \frac{B}{A} \right)^2 \quad (8.6)$$

with, in that case, $A = a^2$ and $B = 1 + 2(a^2 - 2b) + 4b^2 - 6ac$ and we can, in principle, adjust the parameters to get $\epsilon \ll 1$. Assuming that we can set $\frac{B}{A} \sim 1$, we have the following number of e-folds

$$N = \int_{\rho_e}^{\rho_N} \frac{1}{\sqrt{2\epsilon}} d\rho = \int_{\rho_e}^{\rho_N} \frac{1}{\sqrt{2\sigma} \left(1 + \frac{B}{A} \right)} d\sigma \sim \int_{\rho_e}^{\rho_N} \frac{1}{2\sqrt{2}\sigma} d\sigma \quad (8.7)$$

So, $N > 60$ implies $\frac{\sigma_N}{\sigma_e} > e^{120\sqrt{2}}$. It is a fine tuning issue and would be nice if we do not need to deal with it.

For $n = -1$, we have

$$\epsilon \simeq \frac{1}{4} \left(2\sigma + \frac{1}{\sigma} 2A\sigma^2 \right)^2 = \sigma^2 (1 + A)^2 \quad (8.8)$$

and it leads to a similar fine tuning problem as before.

The cases with $n \leq 0$ correspond to polynomial superpotentials, being a Taylor expansion of the superpotential around zero. If $n > 0$ the superpotential has a *pole* of order n at $\Phi = 0$. These cases are excluded if we do not wish the fine tune in the value of the inflaton at the beginning of inflation. So, the remained possibility is a superpotential with an *essential singularity*. We now turn to a (very) brief review of isolated singularities of analytic functions.

8.1.1 Isolated singular points of analytic functions

Let f be an analytic function defined in an open set $U \subset \mathbb{C}$. A point $z_0 \in \mathbb{C}$ is called *isolated singularity* if f is defined and analytic around z_0 , but not at z_0 . In other words, z_0 is a point at which $f'(z_0)$ fails to exist and does exist around this point. It is possible to construct the Laurent series of f around its singular point z_0 ,

$$f(z) = \sum_{i=-\infty}^{\infty} a_i (z - z_0)^i \quad (8.9)$$

and we can classify the singularity in three cases:

1. *Pole*. There exist $j > 0$ such that $a_{-j} \neq 0$ and $a_i = 0$ for all $i < -j$. In this case we say that z_0 is a pole of order j of f , and the Laurent series is

$$f(z) = \frac{a_{-j}}{(z - z_0)^j} + \frac{a_{-j+1}}{(z - z_0)^{j-1}} + \cdots + \frac{a_{-1}}{(z - z_0)} + \sum_{i \geq 0} (z - z_0)^i \quad (8.10)$$

2. *Removable singularity*. In this case, $a_i = 0$ for all $i < 0$ and the Laurent series reduces to Taylor expansion. Moreover, f can be extended for z_0 putting $f(z_0) = a_0$.
3. *Essential singularity*. There is infinitely many terms with negative powers of $(z - z_0)$ in the Laurent series.

These possibilities are exclusives, if one happens then the others cannot be. This allow us to define an essential singularity as a singular point that is not a pole or a removable singularity.

Now one can see that we have proved the need for an essential singularity in the superpotential, at least if we want to get rid of the fine-tuning mentioned before.

8.2 Models with essential singularity

In the previous analysis we considered the canonical Kähler potential. But we wish to build a model that takes into account the existence of large extra dimensions and we want eventually to consider that Kähler potential and superpotential can be obtained from string theory. Therefore we need to modify the standard compactification result of (7.22) in order to get a canonical Kähler potential at small field values. This can be achieved by considering nonperturbative corrections of the form:

$$K(\Phi, \bar{\Phi}) = -\delta \ln \left(-i (\Phi - \bar{\Phi}) + \lambda e^{-\alpha \Phi \bar{\Phi}} \right) \quad (8.11)$$

It reduces to $K(\Phi, \bar{\Phi}) \simeq \delta \alpha \Phi \bar{\Phi}$ for small Φ , and this is the case since we are working with small field inflation and the inflaton is effectively Φ . So, we can choose $\alpha = \frac{1}{\delta}$ and apply the results of the last section.

Our task is to choose a superpotential that gives an inflaton potential suitable for new inflation. If we fix the Kähler potential to be the canonical one, then our analysis in last section applies and so we require the superpotential to have an essential singularity at the origin. A simple function with an essential singularity at origin is $e^{\frac{1}{z}}$. So, let us see what happens if we add this factor to the superpotential in (7.6),

$$W(\Phi) = e^{\frac{c}{\Phi}} \Phi \quad (8.12)$$

which, with canonical Kähler potential, gives

$$V(\phi) = e^{\bar{\phi}\phi} \left\{ \left| \left(1 - \frac{c}{\phi} \right) + \bar{\phi}\phi \right|^2 - 3\bar{\phi}\phi \right\} \quad (8.13)$$

and the potential for the inflaton $\rho = \sqrt{2} \text{Im}\phi$ is

$$V(\rho) = e^{\frac{\rho^2}{2}} \left(1 - \frac{\rho^2}{2} + \frac{\rho^4}{4} + \frac{2c^2}{\rho^2} \right) \quad (8.14)$$

The slow-roll parameter ϵ is, during inflation,

$$\epsilon = \frac{1}{2} \left(\frac{V'}{V} \right)^2 \simeq \frac{1}{2} \left[2 \left(\frac{2}{\rho^2} + \frac{2\sqrt{2}}{\rho^3} \right) \right]^2 \simeq \frac{8}{\rho^4} \gg 1 \quad (8.15)$$

and it is not viable for inflation. To give a flavor of what terms should be used to give a nice potential, let us consider other superpotential:

$$W(\Phi) = B e^{-\beta e^{\frac{c}{i\Phi}}} \Phi \quad (8.16)$$

then,

$$\frac{\partial W}{\partial \Phi} + \frac{\partial K}{\partial \Phi} W = B e^{-\beta e^{\frac{c}{i\Phi}}} \left(\frac{\beta c}{i\Phi} e^{\frac{c}{i\Phi}} + 1 + \Phi \bar{\Phi} \right) \quad (8.17)$$

$$\frac{\partial W}{\partial \Phi} + \frac{\partial K}{\partial \Phi} W = B e^{-\beta e^{-\frac{c}{i\Phi}}} \left(\frac{-\beta c}{i\Phi} e^{-\frac{c}{i\Phi}} + 1 + \Phi \bar{\Phi} \right) \quad (8.18)$$

and

$$V = B^2 e^{\phi\bar{\phi}\left(e^{\frac{c}{i\phi}} + e^{-\frac{c}{i\bar{\phi}}}\right)} \left[\frac{\beta^2 c^2}{\phi\bar{\phi}} e^{\frac{c}{i\phi} - \frac{c}{i\bar{\phi}}} + \frac{\beta c}{i\phi} e^{\frac{c}{i\phi}} + \frac{\beta c}{i} \bar{\phi} e^{\frac{c}{i\bar{\phi}}} - \frac{\beta c}{i\bar{\phi}} e^{-\frac{c}{i\phi}} + 1 + \phi\bar{\phi} - \frac{\beta c}{i} \phi e^{-\frac{c}{i\phi}} + \phi\bar{\phi} + (\phi\bar{\phi})^2 - 3\phi\bar{\phi} \right] \quad (8.19)$$

as before, we take $\phi = i\sigma$ to be pure imaginary to find the potential for the inflaton:

$$V(\sigma) = \frac{B^2}{\sigma^2} e^{-2\beta e^{-\frac{c}{\sigma}} - \frac{2c}{\sigma} + \sigma^2} \left[\beta^2 c^2 - 2c\beta\sigma e^{\frac{c}{\sigma}} (1 + \sigma^2) + e^{\frac{2c}{\sigma}} \sigma^2 (1 - \sigma^2 + \sigma^4) \right]$$

and we can approximate it as

$$V(\sigma) \simeq B^2 e^{-2\beta e^{-\frac{c}{\sigma}} + \sigma^2} \left(1 - \frac{2\beta c}{\sigma} e^{-\frac{c}{\sigma}} \right) \quad (8.20)$$

To calculate ϵ , we need the ratio

$$\frac{V'}{V} = \frac{2\sigma - 2\beta c e^{-\frac{c}{\sigma}} \left(2 + \frac{c}{\sigma^3} \right) + \frac{4\beta^2 c^2}{\sigma^3} e^{-\frac{2c}{\sigma}}}{1 - \frac{2\beta c}{\sigma} e^{-\frac{c}{\sigma}}} \simeq 2\sigma \quad (8.21)$$

in which we have assumed that $\frac{c}{\sigma} > 1$ and $\frac{\beta c e^{-\frac{c}{\sigma}}}{\sigma} \ll 1$, that are reasonable assumptions because $\sigma \ll 1$, during inflation. This gives $\epsilon \simeq \sigma^2$, which lead to the discussed fine-tuning problem. Therefore, (8.16) is not a good superpotential.

The factor 2σ in the numerator of (8.21) comes from the exponential term with the Kähler potential that gives e^{σ^2} , this justify why we have maintained such term in our approximation. If there is a term $e^{a\sigma}$ in the potential then we are going to have a factor a in the ϵ , that we could adjust to evade the tuning. Moreover, we want to take the inflaton as the volume modulus of large extra dimensions and so we expect that the potential do not prohibit this volume to take small values long after inflation, that is, our potential should not grows up to infinity for large field values. For this purpose we introduce a factor of $e^{-\beta_1 \Phi^4}$ in the superpotential to win over the e^{σ^2} in the inflaton potential. Hence, in face of these considerations, let us take our superpotential as

$$W(\Phi) = B e^{ib\Phi} e^{-\beta_1 \Phi^4} e^{-\beta_2 e^{\frac{c}{i\Phi}}} \Phi \quad (8.22)$$

from which we obtain, using the canonical Kähler potential,

$$\begin{aligned}\frac{\partial W}{\partial \Phi} + \frac{\partial K}{\partial \Phi} W &= B e^{ib\Phi - \beta_1 \Phi^4 - \beta_2 e^{\frac{c}{i\Phi}}} \left\{ \left(ib - 4\beta_1 \Phi^3 + \frac{\beta_2 c}{i\Phi^2} e^{\frac{c}{i\Phi}} \right) \Phi + 1 + \Phi \bar{\Phi} \right\} \\ \frac{\partial \bar{W}}{\partial \bar{\Phi}} + \frac{\partial K}{\partial \bar{\Phi}} \bar{W} &= B e^{-ib\bar{\Phi} - \beta_1 \bar{\Phi}^4 - \beta_2 e^{-\frac{c}{i\bar{\Phi}}}} \left\{ \left(-ib - 4\beta_1 \bar{\Phi}^3 - \frac{\beta_2 c}{i\bar{\Phi}^2} e^{-\frac{c}{i\bar{\Phi}}} \right) \bar{\Phi} + \right. \\ &\quad \left. + 1 + \Phi \bar{\Phi} \right\}\end{aligned}\quad (8.23)$$

then,

$$\begin{aligned}V(\phi) &= B^2 e^{\phi\bar{\phi} + ib(\phi - \bar{\phi}) - \beta_1(\phi^4 + \bar{\phi}^4) - \beta_2 \left(e^{\frac{c}{i\phi}} + e^{-\frac{c}{i\bar{\phi}}} \right)} \left\{ \left| \left(ib - 4\beta_1 \phi^3 + \right. \right. \right. \\ &\quad \left. \left. + \frac{\beta_2 c}{i\phi^2} e^{\frac{c}{i\phi}} \right) \phi + 1 + \phi \bar{\phi} \right|^2 - 3\phi \bar{\phi} \right\}\end{aligned}\quad (8.24)$$

With $\phi = i\sigma$ we have:

$$\begin{aligned}V(i\sigma) &= B^2 e^{\sigma^2 - 2b\sigma - 2\beta_1 \sigma^4 - 2\beta_2 e^{-\frac{c}{\sigma}}} \left\{ \left[\left(-b - 4\beta_1 \sigma^3 - \frac{\beta_2 c}{\sigma^2} e^{-\frac{c}{\sigma}} \right) \sigma + \right. \right. \\ &\quad \left. \left. + 1 + \sigma^2 \right]^2 - 3\sigma^2 \right\} \\ &= B^2 e^{\sigma^2 - 2b\sigma - 2\beta_1 \sigma^4 - 2\beta_2 e^{-\frac{c}{\sigma}}} \left\{ \left[b^2 + 8\beta_1 b\sigma^3 + 16\beta_1^2 \sigma^6 + \right. \right. \\ &\quad \left. \left. + \frac{2\beta_2 c}{\sigma} e^{-\frac{c}{\sigma}} (b + 4\beta_1 \sigma^3) + \frac{\beta_2^2 c^2}{\sigma^4} e^{-\frac{2c}{\sigma}} \right] \sigma^2 - 2 \left(b + 4\beta_1 \sigma^3 + \right. \right. \\ &\quad \left. \left. + \frac{\beta_2 c}{\sigma^2} e^{-\frac{c}{\sigma}} \right) (\sigma + \sigma^3) + 1 + \sigma^4 - \sigma^2 \right\}\end{aligned}\quad (8.25)$$

Assuming that $\frac{\beta_2 c e^{-\frac{c}{\sigma}}}{\sigma} \ll 1$, as before, and that b is such that $b\sigma \ll 1$, we can write

$$\begin{aligned}V(i\sigma) &\simeq e^{\sigma^2 - 2b\sigma} (b^2 \sigma^2 - 2b\sigma - \sigma^2) \\ &\simeq B^2 (1 - 4b\sigma + 7b^2 \sigma^2)\end{aligned}\quad (8.26)$$

As $\rho = \sqrt{2}\sigma$, the inflaton potential is

$$V_{inf}(\rho) = B^2 \left(1 - 2\sqrt{2}b\rho + \frac{7}{2}b^2\rho^2 \right) \quad (8.27)$$

Now we can calculate the slow-roll parameters:

$$\epsilon = \frac{1}{2} \left(\frac{V'_{inf}}{V_{inf}} \right)^2 \simeq 4b^2 \quad (8.28)$$

$$\eta = \left(\frac{V''_{inf}}{V_{inf}} \right) \simeq 7b^2 \quad (8.29)$$

Note that this time we do not have the fine-tuning issue and so we can proceed to calculate the spectral index of the scalar power spectrum n_s , the tensor-to-scalar ratio r and the amplitude of the primordial perturbations $\mathcal{R}^2(k_0)$. Furthermore the number of e-folds is estimated as

$$N \simeq \int_{\rho_e}^{\rho_i} \frac{V_{inf}}{V'_{inf}} d\rho \simeq \int_{\rho_e}^{\rho_i} \frac{-1}{2\sqrt{2}b} d\rho \simeq \frac{\rho_e}{2\sqrt{2}b} \quad (8.30)$$

where ρ_e is the field value at the end of inflation and we considered the initial value of the inflaton ρ_i as zero. Requiring $N > 60$ implies $\rho_e > 120\sqrt{2}b$, and from the condition $b\sigma \ll 1$ there is a specific range for ρ_e

$$120\sqrt{2}b < \rho_e < \frac{\sqrt{2}}{b} \quad (8.31)$$

and our choice of parameters must respect this range.

The potential (8.22) first appeared in [43] in the context of chameleonic inflation.

8.3 Attempting to fit the parameters with Planck data

As discussed in section (6.3), Planck mission gives [33]

$$\begin{aligned} n_s &= 0.9603 \pm 0.0073 \\ r &< 0.11 \\ \mathcal{R} &\simeq 2.2 \times 10^{-9} \end{aligned} \quad (8.32)$$

From equations (6.59), (6.61) and (6.63) we have

$$n_s - 1 = -6\epsilon + 2\eta \simeq -10b^2 \quad (8.33)$$

$$r = 16\epsilon \simeq 64b^2 \quad (8.34)$$

$$\mathcal{R} \simeq \frac{V_{inf}}{24\pi^2\epsilon} \simeq \frac{B^2}{96\pi^2b^2} \quad (8.35)$$

The last equation allows us to find the scale of the inflaton potential in terms of b : $B \simeq b10^{-3}$. After this constraint, our model has 4 independent parameters that we will take as

$$b = 10^{-3}; \quad \beta_1 = 1; \quad \beta_2 = 10; \quad c = 1 \quad (8.36)$$

With these parameter values, conditions $\frac{\beta_2 c e^{-\frac{c}{\sigma}}}{\sigma} \ll 1$, $\frac{b\rho}{\sqrt{2}} \ll 1$ and (8.31) are clearly achieved. The full inflaton potential is plotted in figure 8.1 for these parameters.

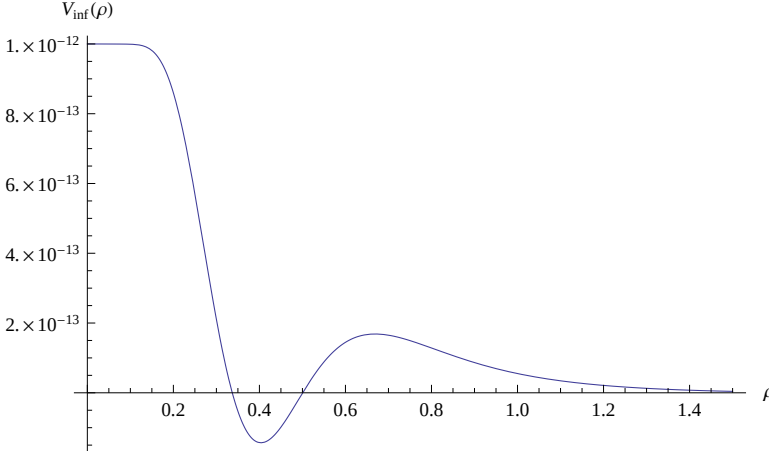


Figure 8.1: Inflaton potential, in Planck units, for the parameter values (8.36).

The tensor to scalar ratio is $r \simeq 64 \times 10^{-6}$ and it is in agreement with (6.64). Although these set of parameters seems to work, it provides only a tiny deviation from scale invariance: $n_s - 1 \simeq 10^{-5}$. This is incompatible with the Planck data and we could not find another parameter values that gives an acceptable spectral index. Therefore, our simple model of inflation in supergravity driven by the volume modulus of large extra dimensions is not phenomenologically viable. That is a

negative result but it is worth to remember that the conclusions about essential singularities in small field models of inflation in supergravity has some degree of generality and hence are very important.

Chapter 9

Conclusions

We saw that the inflation in supergravity is not easily obtained. When we focus on the class of potentials favored by the Planck results, those of new inflation with a plateau, there are not many models that can achieved it. In fact, we have made a more general analysis, trying to find the behavior of the potential in the inflationary region by assuming small field inflation, that is a very reasonable assumption in the context of new inflation. Our treatment of the extra dimension was facilitated by the form of the Kähler potential that comes from dimension reduction procedure. The fine-tuning of the parameters shows almost inevitable but we have find a nice potential by using the result that the superpotential needs to have an essential singularity.

An important point to note is that the Izawa and Yanagida model for $n = 4$, described in section (7.2), does not have an essential singularity. This model was not found by our perturbative analysis of the superpotential because we did not consider the case when all the coefficients in the expansion were zero until the fifth power of the superfield in (8.1). This is a rather special case, and we considered here general cases. More importantly, the model is incompatible with large extra dimension, since it has $K = \Phi\bar{\Phi}$ and the potential blows up after the inflationary and reheating (minimum) regions. Thus ϕ cannot escape the small field region.

Unfortunately, our model leads to a spectral tilt that is in tension with the observational data. Note that the same problem happens to the Izawa and Yanagida model. In their model, this is solved by the introduction of a small constant term in the superpotential during inflation as in [40]. Then, stays here the perspective of the possibility to make the same procedure to fix our model with an essential

singularity.

Let us remember that inflation in supergravity is specified by the Kähler potential and the superpotential. In principle we do not need to motivate the choice of these two functions: the model justifies itself if it is consistent with the observations. Therefore, a priori, we do not need to relate the model in (8.22) to large extra dimensions. But the possibility to correct the Kähler potential as in (8.11), that leads to requirement of an essential singularity, and the using of the long field behavior expected from a volume modulus, leads to a restricted form for the potential. This motivates the idea that we have found the unique superpotential with these features.

Appendix A

Notation and conventions

In general we tried to follow the conventions of [11]. We used the mostly plus convention for the metric: $\eta_{mn} = \text{diag}(-1, 1, 1, 1)$ and worked in Planck units $c = \hbar = k_B = 1$ otherwise stated the contrary. The Levi-Civita symbol is defined by $\varepsilon_{0123} = -1$ and the spin metric is

$$\varepsilon_{21} = \varepsilon^{12} = 1, \quad \varepsilon_{12} = \varepsilon^{21} = -1, \quad \varepsilon_{11} = \varepsilon_{22} = 0 \quad (\text{A.1})$$

Since $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ are related by complex conjugation, we have $(\bar{\psi}_\alpha) = (\bar{\psi})_{\dot{\alpha}}$, where as in the text, the bar in the left hand side indicates complex conjugation. It was used the following sum convention for Weyl spinors:

$$\psi^\alpha = \varepsilon^{\alpha\beta} \psi_\beta, \quad \psi_\alpha = \varepsilon_{\alpha\beta} \psi^\beta \quad (\text{A.2})$$

$$\psi\chi = \psi^\alpha \chi_\alpha = -\psi_\alpha \chi^\alpha = \chi^\alpha \psi_\alpha = \chi\psi \quad (\text{A.3})$$

$$\bar{\psi}\bar{\chi} = \bar{\psi}_{\dot{\alpha}} \bar{\chi}^{\dot{\alpha}} = -\bar{\psi}^{\dot{\alpha}} \bar{\chi}_{\dot{\alpha}} = \bar{\chi}_{\dot{\alpha}} \bar{\psi}^{\dot{\alpha}} = \bar{\chi}\bar{\psi} \quad (\text{A.4})$$

The hermitian sigma matrices are

$$\sigma^0 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (\text{A.5})$$

$$\sigma^2 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (\text{A.6})$$

and

$$\bar{\sigma}^{m\alpha\dot{\alpha}} = \varepsilon^{\dot{\alpha}\dot{\beta}} \varepsilon^{\alpha\beta} \sigma_{\beta\dot{\beta}}^m \quad (\text{A.7})$$

$$\bar{\sigma}^0 = \sigma^0 \quad (\text{A.8})$$

$$\bar{\sigma}^{1,2,3} = -\sigma^{1,2,3} \quad (\text{A.9})$$

These matrices satisfy many identities, see [11, 13]. From the γ matrices,

$$\gamma^m = \begin{pmatrix} 0 & \sigma^m \\ \bar{\sigma}^m & 0 \end{pmatrix} \quad (\text{A.10})$$

we can construct the four component formalism of Dirac spinors Ψ_D ,

$$\Psi_D = \begin{pmatrix} \chi_\alpha \\ \bar{\psi}^\alpha \end{pmatrix} \quad (\text{A.11})$$

and for Majorana spinors Ψ_M ,

$$\Psi_M \begin{pmatrix} \chi_\alpha \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix} \quad (\text{A.12})$$

We end this addendum listing some identities that are commonly used in some calculations of chapter 2, 3 and 4:

$$A_{\alpha\beta} = -A_{\beta\alpha} \rightarrow A_{\alpha\beta} = -\frac{1}{2} \varepsilon_{\alpha\beta} (\varepsilon^{\gamma\delta} A_{\gamma\delta}) \quad (\text{A.13})$$

$$\theta \sigma^m \bar{\theta} \sigma^n \bar{\theta} = -\frac{1}{2} \theta \theta \bar{\theta} \bar{\theta} \eta^{mn} \quad (\text{A.14})$$

$$(\theta\phi)(\theta\psi) = -\frac{1}{2}(\phi\psi)(\theta\theta) \quad (\text{A.15})$$

$$\varepsilon^{\alpha\beta} \frac{\partial}{\partial \theta^\alpha} \frac{\partial}{\partial \theta^\beta} \theta\theta = 4 = \varepsilon_{\dot{\alpha}\dot{\beta}} \frac{\partial}{\partial \bar{\theta}_{\dot{\alpha}}} \frac{\partial}{\partial \bar{\theta}_{\dot{\beta}}} \bar{\theta}\bar{\theta} \quad (\text{A.16})$$

$$\chi \sigma^m \bar{\psi} = -\bar{\psi} \bar{\sigma}^m \chi, \quad \chi \sigma^m \bar{\sigma}^n \psi = \psi \sigma^n \bar{\sigma}^m \chi. \quad (\text{A.17})$$

Bibliography

- [1] A. H. Guth, “Inflationary universe: A possible solution to the horizon and flatness problems,” *Physical Review D* **23** (1981) no. 2, 347.
- [2] A. A. Starobinsky, “Relict gravitation radiation spectrum and initial state of the universe,” *JETP lett* **30** (1979) 682.
- [3] V. F. Mukhanov and G. Chibisov, “Quantum fluctuations and a nonsingular universe,” *JETP Letters* **33** (1981) no. 10, 532–535.
- [4] T. Kaluza, “On the problem of unity in physics,” in *Unified Field Theories of > 4 Dimensions*, vol. 1, p. 427. 1983.
- [5] O. Klein, “Quantum theory and five-dimensional relativity theory,” *Z. phys* **37** (1926) no. 12, 895–906.
- [6] N. Arkani-Hamed, S. Dimopoulos, and G. Dvali, “The hierarchy problem and new dimensions at a millimeter,” *Physics Letters B* **429** (1998) no. 3, 263–272.
- [7] I. Antoniadis and B. Pioline, “Large dimensions and string physics at a TeV,” in *Beyond the desert: Accelerator, non-accelerator and space approaches into the next millennium. Proceedings, 2nd International Conference on particle physics beyond the standard model, Ringberg Castle, Tegernsee, Germany, June 6-12, 1999*, pp. 373–384. 1999. [arXiv:hep-ph/9906480](#) [hep-ph].
- [8] S. Weinberg, *The quantum theory of fields*, vol. 1. Cambridge university press, 1995.
- [9] S. Coleman and J. Mandula, “All possible symmetries of the S matrix,” *Physical Review* **159** (1967) no. 5, 1251.

- [10] Yu. A. Golfand and E. P. Likhtman, “Extension of the algebra of Poincaré group generators and violation of P invariance,” *JETP Lett.* **13** (1971) 323–326.
- [11] J. Wess and J. Bagger, *Supersymmetry and supergravity*. Princeton University Press, 1992.
- [12] P. C. West, *Introduction to supersymmetry and supergravity*. World Scientific, 1990.
- [13] I. L. Buchbinder and S. M. Kuzenko, *Ideas and methods of supersymmetry and supergravity*. IOP, 1998.
- [14] J. Wess and B. Zumino, “Supergauge transformations in four-dimensions,” *Nucl. Phys. B* **70** (1974) no. 3, 9.
- [15] A. Einstein, “Riemann-geometrie mit aufrechterhaltung des begriffes des fernparallelismus,”.
- [16] J. Yopez, “Einstein’s vierbein field theory of curved space,” *arXiv preprint arXiv:1106.2037* (2011) .
- [17] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation*. Macmillan, 1973.
- [18] R. M. Wald, *General relativity*. University of Chicago press, 2010.
- [19] L. Ryder, *Introduction to general relativity*. Cambridge University Press, 2009.
- [20] N. Dragon, “Torsion and curvature in extended supergravity,” *Zeitschrift für Physik C Particles and Fields* **2** (1979) no. 1, 29–32.
- [21] R. Grimm, J. Wess, and B. Zumino, “A complete solution of the Bianchi identities in superspace with supergravity constraints,” *Nuclear Physics B* **152** (1979) no. 2, 255–265.
- [22] H. P. Robertson, “Kinematics and world-structure,” *The Astrophysical Journal* **82** (1935) 284.
- [23] S. Weinberg, *Gravitation and cosmology: Principle and applications of general theory of relativity*. John Wiley and Sons, Inc., New York, 1972.

- [24] A. Einstein, “Cosmological considerations on the general theory of relativity,” *The Principle of Relativity. Dover Books on Physics. June 1, 1952. 240 pages.* 0486600815, p. 175-188 **1** (1952) 175–188.
- [25] A. A. Penzias and R. W. Wilson, “A measurement of excess antenna temperature at 4080 mc/s.,” *The Astrophysical Journal* **142** (1965) 419–421.
- [26] R. A. Alpher, H. Bethe, and G. Gamow, “The origin of chemical elements,” *Physical Review* **73** (1948) no. 7, 803.
- [27] G. Gamow, “Expanding universe and the origin of elements,” *Physical Review* **70** (1946) no. 7-8, 572.
- [28] A. R. Liddle and D. H. Lyth, *Cosmological inflation and large-scale structure*. Cambridge University Press, 2000.
- [29] A. R. Liddle, “An introduction to cosmological inflation,” *arXiv preprint astro-ph/9901124* (1999) .
- [30] W. H. Kinney, “TASI lectures on inflation,” *arXiv preprint 0902.1529* (2009) .
- [31] D. H. Lyth and A. R. Liddle, *The primordial density perturbation: Cosmology, inflation and the origin of structure*. Cambridge University Press, 2009.
- [32] P. Ade, N. Aghanim, C. Armitage-Caplan, M. Arnaud, M. Ashdown, F. Atrio-Barandela, J. Aumont, C. Baccigalupi, A. J. Banday, R. Barreiro, *et al.*, “Planck 2013 results. xvi. cosmological parameters,” *Astronomy & Astrophysics* **571** (2014) A16.
- [33] P. Ade, N. Aghanim, C. Armitage-Caplan, M. Arnaud, M. Ashdown, F. Atrio-Barandela, J. Aumont, C. Baccigalupi, A. Banday, R. Barreiro, *et al.*, “Planck 2013 results. xxii. constraints on inflation,” *Astronomy & Astrophysics* **571** (2014) A22.
- [34] M. Yamaguchi, “Supergravity-based inflation models: a review,” *Classical and Quantum Gravity* **28** (2011) no. 10, 103001.
- [35] S. V. Ketov and T. Terada, “Inflation in supergravity with a single chiral superfield,” *Physics Letters B* **736** (2014) 272–277.

- [36] D. Roest, M. Scalisi, and I. Zavala, “Kähler potentials for planck inflation,” *Journal of Cosmology and Astroparticle Physics* **2013** (2013) no. 11, 007.
- [37] A. D. Linde, “A new inflationary universe scenario: A possible solution of the horizon, flatness, homogeneity, isotropy and primordial monopole problems,” *Physics Letters B* **108** (1982) no. 6, 389–393.
- [38] A. Albrecht and P. J. Steinhardt, “Cosmology for grand unified theories with radiatively induced symmetry breaking,” *Physical Review Letters* **48** (1982) no. 17, 1220.
- [39] K.-I. Izawa and T. Yanagida, “Natural new inflation in broken supergravity,” *Physics Letters B* **393** (1997) no. 3, 331–336.
- [40] F. Takahashi, “New inflation in supergravity after planck and LHC,” *Physics Letters B* **727** (2013) no. 1, 21–26.
- [41] S. Kachru, A. D. Linde, R. E. Kallosh, and S. Trivedi, “de Sitter vacua in string theory,” *Phys. Rev. D* **68** (2003) no. hep-th/0301240, 046005.
- [42] A. Linde, “Inflationary cosmology after Planck 2013,” *arXiv preprint arXiv:1402.0526* (2014) .
- [43] K. Hinterbichler, J. Khoury, H. Nastase, and R. Rosenfeld, “Chameleonic inflation,” *Journal of High Energy Physics* **2013** (2013) no. 8, 1–27.
- [44] J. Martin, C. Ringeval, and V. Vennin, “Encyclopaedia inflationaris,” *arXiv preprint arXiv:1303.3787* (2013) .