

## Dressing approach to the nonvanishing boundary value problem for the AKNS hierarchy

This article has been downloaded from IOPscience. Please scroll down to see the full text article.

2012 J. Phys.: Conf. Ser. 343 012039

(<http://iopscience.iop.org/1742-6596/343/1/012039>)

View [the table of contents for this issue](#), or go to the [journal homepage](#) for more

Download details:

IP Address: 200.145.3.40

The article was downloaded on 18/07/2013 at 21:19

Please note that [terms and conditions apply](#).

# Dressing approach to the nonvanishing boundary value problem for the AKNS hierarchy

J F Gomes, Guilherme S França and A H Zimerman

Instituto de Física Teórica - IFT/UNESP  
Rua Dr. Bento Teobaldo Ferraz, 271, Bloco II  
01140-070, São Paulo - SP, Brazil

E-mail: [jfg@ift.unesp.br](mailto:jfg@ift.unesp.br), [guisf@ift.unesp.br](mailto:guisf@ift.unesp.br), [zimerman@ift.unesp.br](mailto:zimerman@ift.unesp.br)

**Abstract.** We propose an approach to the nonvanishing boundary value problem for integrable hierarchies based on the dressing method. Then we apply the method to the AKNS hierarchy. The solutions are found by introducing appropriate vertex operators that takes into account the boundary conditions.

## 1. Introduction

The structure of affine Kac-Moody algebras has provided considerable insight underlining the construction of integrable hierarchies in 1+1 spacetime dimensions. An integrable hierarchy consists of a series of equations of motion with a common algebraic structure. They are constructed out of a zero curvature condition which involves a pair of gauge potentials, lying within the affine Lie algebra. A crucial ingredient is the decomposition of the affine Lie algebra  $\hat{\mathcal{G}}$  into graded subspaces, determined by a grading operator  $Q$ . The integrable hierarchy is further specified by a constant, grade one semi-simple element  $E^{(1)}$ . The choice of  $\hat{\mathcal{G}}$ ,  $Q$  and  $E^{(1)}$  are the basic ingredients that fully determine and classify the entire hierarchy (see for instance ref. [1] for a review).

Representation theory of Kac-Moody algebras play also an important role in constructing systematically soliton solutions of integrable hierarchies. The main idea of the dressing method consists in, by gauge transformation, to map a simple vacuum configuration into a non trivial one- or multi-soliton solution. The pure gauge solution of the zero curvature representation leads to explicit spacetime dependence for the vacuum configuration which in turn, generates by gauge transformation, the non-trivial soliton solutions of the hierarchy [2, 3]. It becomes clear that the solutions are classified into conjugacy classes according to the choice of vacuum configuration. In general, the vacuum is taken as the zero field configuration and the soliton solutions are constructed and classified in terms of vertex operators [4].

A dressing method approach to nontrivial constant vacuum configuration was proposed in [5] when considering the negative even flows of the modified Korteweg-de Vries (mKdV) hierarchy, which do not admit trivial vacuum configuration, i.e. vanishing boundary condition. A deformed vertex operator was introduced and the method was further applied to the whole

mKdV hierarchy with nonvanishing boundary condition and to a hierarchy containing the Gardner equation [6]. Here in these notes, we shall provide further clarifications discussing nonvanishing boundary conditions for integrable hierarchies in general, constructed with the structure proposed in [7] (and references therein). We point out that not all the models within the hierarchy admit solutions with nonvanishing (constant) boundary condition. We then consider as an example the Ablowitz-Kaup-Newell-Segur (AKNS) hierarchy with nontrivial constant vacuum configuration. Results for the focusing nonlinear Schroedinger (NLS) equation were recently obtained in ref. [8]. A hybrid of the dressing and Hirota method for a multi-component generalization of the AKNS hierarchy was also considered recently in [9].

We first describe the general aspects of the dressing method in section 3, together with a discussion on the possible different boundary conditions. We point out the requirements for the individual models of the hierarchy to admit a nonvanishing constant boundary value solutions. Later in section 4, we discuss the construction of the AKNS hierarchy, followed by the construction of its solutions, section 5. In section 6 we present the deformed vertex operators used to construct solitonic solutions with nonvanishing boundary condition and illustrate with explicit examples.

## 2. Algebraic concepts and integrable hierarchies

In this section we introduce the main algebraic concepts used for the construction of integrable models and to obtain its solutions. A more detailed exposition can be found in [10, 11].

Let  $\mathcal{G}$  be a semi-simple finite-dimensional Lie algebra. The infinite-dimensional loop algebra  $L(\mathcal{G})$  is defined as the tensor product of  $\mathcal{G}$  with integer powers of the so called complex spectral parameter  $\lambda$ ,  $L(\mathcal{G}) \equiv \mathcal{G} \otimes \mathbb{C}[\lambda, \lambda^{-1}]$ . If  $T_a \in \mathcal{G}$  then  $T_a^n \equiv T_a \otimes \lambda^n \in L(\mathcal{G})$  for  $n \in \mathbb{Z}$ . The commutator of the loop algebra is given by

$$[T_a^n, T_b^m] \equiv [T_a, T_b] \otimes \lambda^{n+m}, \quad (1)$$

where  $[T_a, T_b]$  is the  $\mathcal{G}$  commutator.

The central extension is performed by the introduction of the operator  $\hat{c}$ , which commutes with all the others  $[\hat{c}, T_a^n] = 0$ . Furthermore, consider the spectral derivative operator  $\hat{d} \equiv \lambda \frac{d}{d\lambda}$  such that  $[\hat{d}, T_a^n] = nT_a^n$ , so it measures the *power* of the spectral parameter. The affine Kac-Moody Lie algebra is defined by  $\hat{\mathcal{G}} \equiv L(\mathcal{G}) \oplus \mathbb{C}\hat{c} \oplus \mathbb{C}\hat{d}$  and for  $T_a^n, T_b^m \in \hat{\mathcal{G}}$  the commutator is provided by

$$[T_a^n, T_b^m] \equiv [T_a, T_b] \otimes \lambda^{n+m} + n\delta_{n+m,0} \langle T_a | T_b \rangle \hat{c}, \quad (2)$$

where  $\langle T_a | T_b \rangle$  is the Killing form on  $\mathcal{G}$ .

The algebra  $\hat{\mathcal{G}}$  can be decomposed into  $\mathbb{Z}$ -graded subspaces by the introduction of a grading operator  $Q$  such that

$$\hat{\mathcal{G}} = \bigoplus_{j \in \mathbb{Z}} \hat{\mathcal{G}}^{(j)}, \quad (3)$$

where  $\hat{\mathcal{G}}^{(j)} \equiv \{T_a^n \in \hat{\mathcal{G}} \mid [Q, T_a^n] = jT_a^n\}$ . The integer  $j$  is the *grade* of the operators defined with respect to  $Q$ . It follows from the Jacobi identity that if  $T_a^n \in \hat{\mathcal{G}}^{(i)}$  and  $T_b^m \in \hat{\mathcal{G}}^{(j)}$  then  $[T_a^n, T_b^m] \in \hat{\mathcal{G}}^{(i+j)}$ .

Let  $E$  be a semi-simple element of  $\hat{\mathcal{G}}$ . Its kernel is defined by  $\mathcal{K} \equiv \{T_a^n \in \hat{\mathcal{G}} \mid [E, T_a^n] = 0\}$ . Its complement is called the image subspace  $\mathcal{M}$  and  $\hat{\mathcal{G}} = \mathcal{K} \oplus \mathcal{M}$ . It can be readily verified from this definition and the Jacobi identity that  $[\mathcal{K}, \mathcal{K}] \subset \mathcal{K}$  and  $[\mathcal{K}, \mathcal{M}] \subset \mathcal{M}$ . We also assume the symmetric space structure  $[\mathcal{M}, \mathcal{M}] \subset \mathcal{K}$ .

Consider the linear system in 1+1 spacetime dimension

$$(\partial_x + U)\Psi = 0, \quad (\partial_t + V)\Psi = 0, \quad (4)$$

where  $U, V \in \widehat{\mathcal{G}}$  and  $\Psi$  is an element of the Lie group of  $\widehat{\mathcal{G}}$ . The compatibility of this system ensures integrability, yielding the zero curvature equation

$$[\partial_x + U, \partial_t + V] = 0. \quad (5)$$

The general structure to construct nontrivial integrable models through an algebraic approach was discussed in [7, 10] (see also references therein). Let  $Q$  be a grading operator and choose a constant, independent of  $x$  and  $t$ , grade one semi-simple element  $E = E^{(1)}$ . Let  $A^{(0)}(x, t)$  be a linear combination of zero grade operators lying entirely in  $\mathcal{M}$ . The coefficients of this linear combination are spacetime functions, i.e. the fields of the model. Therefore, if  $U$  and  $V$  have the following algebraic structure [7]

$$U \equiv E^{(1)} + A^{(0)}(x, t), \quad (6a)$$

$$V \equiv \sum_{i=-p}^q D^{(i)}(x, t) = D^{(q)} + D^{(q-1)} + \dots + D^{(-p)}, \quad (6b)$$

where  $p, q \geq 0$  are nonnegative integers, the zero curvature equation (5) can be solved grade by grade, determining each  $D^{(i)}$  in terms of the fields in  $A^{(0)}(x, t)$ ; the zero grade projection will provide the nonlinear integrable field equations. The explicit examples considered in the following sections will illustrate this assertion. The particular case where  $V = D^{(-1)}$  is equivalent to a relativistic model obtained from a Hamiltonian reduction of the Wess-Zumino-Novikov-Witten model. The infinite number differential equations obtained from this algebraic structure, one for each choice of  $p$  and  $q$ , constitute an integrable hierarchy immediately classified by  $\widehat{\mathcal{G}}$ ,  $Q$  and  $E^{(1)}$ .

Consider a unitary highest weight representation of  $\widehat{\mathcal{G}}$ , where the hermitian relations are  $(T_a^n)^\dagger = T_a^{-n}$ . There exists a finite set of states  $\{|\mu_k\rangle\}$ ,  $k = 0, 1, \dots, \text{rank } \widehat{\mathcal{G}}$ , that are annihilated by every operator with a positive power on the spectral parameter,

$$T_a^n |\mu_k\rangle = 0, \quad n > 0. \quad (7)$$

The fundamental state  $|\mu_0\rangle$  is annihilated by every operator, including  $\{T_a^0\}$ , except the central term. Under the action of  $\hat{c}$  all states obey

$$\hat{c}|\mu_k\rangle = |\mu_k\rangle. \quad (8)$$

### 3. Dressing method

Let us briefly review the dressing method [2, 3]. Assume a known solution of the zero curvature equation (5), in pure gauge form, called vacuum

$$U_0 = -\partial_x \Psi_0 \Psi_0^{-1}, \quad V_0 = -\partial_t \Psi_0 \Psi_0^{-1}. \quad (9)$$

The dressing method arises as a factorization problem

$$\Theta_-^{-1} \Theta_+ = \Psi_0 g \Psi_0^{-1}, \quad (10)$$

where  $g$  is an arbitrary constant group element, independent of  $x$  and  $t$ , and  $\Theta_\pm$  are factorized into positive and negative grade operators, respectively, according to a grading operator (3),

$$\Theta_- = \exp(m^{(-1)}) \exp(m^{(-2)}) \dots, \quad \Theta_+ = \exp(m^{(0)}) \exp(m^{(1)}) \dots \quad (11)$$

where  $m^{(j)} \in \widehat{\mathcal{G}}^{(j)}$ . The factorization problem (10) ensures that, the dressing operators  $\Theta_\pm$  map a vacuum solution into a new solution by the gauge transformations

$$U = \Theta_\pm U_0 \Theta_\pm^{-1} - \partial_x \Theta_\pm \Theta_\pm^{-1}, \quad (12a)$$

$$V = \Theta_\pm V_0 \Theta_\pm^{-1} - \partial_t \Theta_\pm \Theta_\pm^{-1}. \quad (12b)$$

### 3.1. Vanishing boundary condition

Taking a zero field configuration  $A^{(0)} \rightarrow 0$  in (6) we have the constant vacuum operators

$$U_0 = E^{(1)} \equiv \Omega^{(1)}, \quad (13a)$$

$$V_0 = D_0^{(q)} + D_0^{(-p)} \equiv \Omega^{(q)} + \Omega^{(-p)}. \quad (13b)$$

where  $\Omega^{(j)} \in \mathcal{K}$ . This conclusion comes from the projection of the zero curvature equation into each graded subspace and the resulting equations for the highest and lowest grades [7]. Using (13a) in the gauge transformation (12a) with  $\Theta_+$ , taking into account the central extension<sup>1</sup> of the loop algebra denoted by  $-\nu_x \hat{c}$ , and projecting the resulting equation into a zero graded subspace we get

$$A^{(0)} - \nu_x \hat{c} = -\partial_x \exp(m^{(0)}) \exp(-m^{(0)}). \quad (14)$$

Assume a pure gauge connection, involving zero grade generators only, for the field operator

$$A^{(0)} = -B_x B^{-1}. \quad (15)$$

Substituting (15) into (14) gives

$$\exp(m^{(0)}) = B \exp(\nu \hat{c}). \quad (16)$$

Consider the fundamental state  $|\mu_0\rangle$  and generic states  $|\mu_k\rangle, |\mu_l\rangle$  that are not annihilated by some operators  $T_a^0$ . In fact, the generic states are chosen as the ones that survive under the action of the zero grade operators associated with the fields in  $A^{(0)}$ . This will be clear when we consider a concrete example. Therefore, projecting (10) between appropriate highest weight states, we get the implicit solution of the model

$$\langle \mu_k | B | \mu_l \rangle = \frac{\tau_{kl}}{\tau_{00}}, \quad (17)$$

where we have defined the tau functions

$$\tau_{00} \equiv \langle \mu_0 | \Psi_0 g \Psi_0^{-1} | \mu_0 \rangle, \quad \tau_{kl} \equiv \langle \mu_k | \Psi_0 g \Psi_0^{-1} | \mu_l \rangle. \quad (18)$$

The vacuum operators (13) will commute, since for zero field configuration they lie in  $\mathcal{K}$ , so integrating (9) yields

$$\Psi_0 = \exp\{-\Omega^{(1)}x - (\Omega^{(q)} + \Omega^{(-p)})t\}. \quad (19)$$

Soliton solutions arise by choosing  $g$  in terms of vertex operators in the form [4]

$$g = \prod_{i=1}^N \exp(X_i). \quad (20)$$

Suppose the vertex is an eigenstate of the vacuum, i.e.

$$[\Omega^{(1)}x + (\Omega^{(q)} + \Omega^{(-p)})t, X_i] = -\eta_i(x, t)X_i. \quad (21)$$

<sup>1</sup> In the construction of integrable models only the loop algebra plays a significant role, but to employ the highest weight representation the central term needs to be carefully considered. The equations of motion are invariant under  $U \rightarrow U + f\hat{c}$ . We are choosing  $f = -\nu_x$  for a convenient notation.

The function  $\eta_i$  contains the dispersion relation in an explicit spacetime dependence. Then, the tau functions (18) can be computed in an explicit way,

$$\Psi_0 g \Psi_0^{-1} = \Psi_0 \left( \prod_{i=1}^N e^{X_i} \right) \Psi_0^{-1} = \prod_{i=1}^N \Psi_0 e^{X_i} \Psi_0^{-1} = \prod_{i=1}^N e^{\Psi_0 X_i \Psi_0^{-1}}. \quad (22)$$

Using the expansion  $e^A B e^{-A} = B + [A, B] + \frac{1}{2!} [A, [A, B]] + \dots$  with the eigenvalue equation (21) we obtain the  $N$ -soliton solution

$$\begin{aligned} \tau_{kl} &= \langle \mu_k | \prod_{i=1}^N \exp(e^{\eta_i} X_i) | \mu_l \rangle = \langle \mu_k | \prod_{i=1}^N (1 + e^{\eta_i} X_i) | \mu_l \rangle \\ &= \delta_{kl} + a_1 e^{\eta_1} + a_2 e^{\eta_2} + a_3 e^{\eta_3} \\ &\quad + a_{12} e^{\eta_1 + \eta_2} + a_{13} e^{\eta_1 + \eta_3} + a_{23} e^{\eta_2 + \eta_3} + a_{123} e^{\eta_1 + \eta_2 + \eta_3} + \dots, \end{aligned} \quad (23)$$

where we have assumed the nilpotency property of the vertex,  $\langle \mu_k | (X_i)^n | \mu_l \rangle = 0$  for  $n \geq 2$ , and defined the following matrix elements

$$a_i = \langle \mu_k | X_i | \mu_l \rangle, \quad a_{ij} = \langle \mu_k | X_i X_j | \mu_l \rangle, \quad a_{ijk} = \langle \mu_k | X_i X_j X_k | \mu_l \rangle, \quad \text{etc.} \quad (24)$$

### 3.2. Nonvanishing boundary condition

Let us now assume a constant, nonzero, field configuration  $A^{(0)} \rightarrow A_0^{(0)}$ . Then (6) yields the constant vacuum operators

$$U_0 = E^{(1)} + A_0^{(0)}, \quad (25a)$$

$$V_0 = D_0^{(n)} + D_0^{(n-1)} + \dots. \quad (25b)$$

The analogous of eq. (14) becomes

$$A^{(0)} - \nu_x \hat{c} = \exp(m^{(0)}) A_0^{(0)} \exp(-m^{(0)}) - \partial_x \exp(m^{(0)}) \exp(-m^{(0)}). \quad (26)$$

Assuming the form (16), then (26) yields a relation between the physical fields in  $A^{(0)}$ , functionals associated with zero grade generators in  $B$  and the nonzero vacuum, i.e.

$$A^{(0)} = B A_0^{(0)} B^{-1} - B_x B^{-1}. \quad (27)$$

When  $A_0^{(0)} \rightarrow 0$  we recover the relation (15). The general solution of the model is still given in the form

$$\langle \mu_k | B | \mu_l \rangle = \frac{\tau_{kl}}{\tau_{00}} = \frac{\langle \mu_k | \Psi_0 g \Psi_0^{-1} | \mu_l \rangle}{\langle \mu_0 | \Psi_0 g \Psi_0^{-1} | \mu_0 \rangle} \quad (28)$$

but now with  $\Psi_0$  satisfying the linear system (9) with the more general vacuum operators (25) and henceforth, depending upon the constant vacuum solution. We emphasize that the implicit solution (28) and (27) is valid for an arbitrary vacuum solution, even if it is a general spacetime function. For the general case, however, the difficulty arises in integrating the linear system (9), while for a constant vacuum solution this is trivial.

Assuming a constant vacuum, the operators (25) does not lie purely in  $\mathcal{K}$  anymore, but they are supposed to be solution of the zero curvature equation, i.e.  $[U_0, V_0] = 0$ , so integrating (9) gives

$$\Psi_0 = \exp(-U_0 x - V_0 t). \quad (29)$$

Assuming again that  $g$  has the form (20) and that there exists vertex operators satisfying eigenvalue equations like (21), but now with operators (25), it will be possible to compute the tau functions in (28). Because (25) depend on the constant vacuum fields it suggests that for (21) to be valid,  $X_i$  will also depend on these constants fields, so the dispersion relation and multi-solitonic interaction terms will depend on the boundary conditions.

### 3.3. Models admitting nonvanishing boundary value solutions

Not all the models in the hierarchy defined by (6) have solutions with nonvanishing boundary condition. The vacuum must be a solution of the zero curvature equation, but for this to be true, in the case of constant boundary condition, a restriction on the structure of the Lax pair is required.

The operator (6b) depends on the fields. When taking the fields to vanish, only the operator with highest (and/or lowest) grade that is a constant operator purely in  $\mathcal{K}$  remains, see (13b). Clearly, in this situation there is no restriction besides the own algebraic construction (6).

For a nonzero constant field configuration, other operators in (25b) that previously vanished now remain, and the whole combination should commute with (25a). In (25a) we have the structure  $\Omega^{(1)} \equiv \mathcal{K}^{(1)} + \mathcal{M}^{(0)}$ . If (25b) keeps the same algebraic structure, i.e. if it is formed by a linear combination of operators of the kind

$$\Omega^{(j)} \equiv \mathcal{K}^{(j)} + \mathcal{M}^{(j-1)} = \lambda^{j-1} \Omega^{(1)} \quad (30)$$

where  $j \leq n$  and  $\lambda$  is the loop algebra spectral parameter, then we will have  $[U_0, V_0] = 0$ . Note that in (30) it is required at least two operators differing by one grade. This will not be possible for the first model of the negative hierarchy,  $V = \sum_{i=-p}^{-1} D^{(i)}$ , that contains only one operator  $V = D^{(-1)}$ . These are the relativistic models of the hierarchy, like sinh-Gordon and Lund-Regge for instance. In sum, at least the relativistic models of any hierarchy constructed with the structure (6) will not have solutions with nonvanishing (constant) boundary condition.

## 4. The AKNS hierarchy

The previously discussed approach for a nontrivial vacuum solution was considered for the mKdV hierarchy in [5, 6]. Now we present another example by considering the non-abelian AKNS hierarchy. Let  $\hat{\mathcal{G}} = \hat{A}_1 \sim \hat{s}\ell_2$  with the following commutation relations

$$[H^n, H^m] = 2n\delta_{n+m,0}\hat{c}, \quad [H^n, E_{\pm\alpha}^m] = \pm 2E_{\pm\alpha}^{n+m}, \quad [E_{\alpha}^n, E_{-\alpha}^m] = H^{n+m} + n\delta_{n+m,0}\hat{c} \quad (31)$$

Consider the homogeneous grading operator  $Q = \hat{d}$ , which decomposes the algebra into the graded subspaces

$$\hat{\mathcal{G}}^{(j)} = \{H^j, E_{\alpha}^j, E_{-\alpha}^j\}. \quad (32)$$

Choosing the constant semi-simple element in (6a) as  $E^{(1)} = H^1$ , we have  $A^{(0)} = qE_{\alpha}^0 + rE_{-\alpha}^0$  where  $q = q(x, t)$  and  $r = r(x, t)$  are both fields of the models within the hierarchy. Let

$$D^{(j)} = a_j H^j + b_j E_{\alpha}^j + c_j E_{-\alpha}^j \in \hat{\mathcal{G}}^{(j)} \quad (33)$$

where  $a_j, b_j, c_j$  are functions of  $x$  and  $t$ . Neglecting for the moment the central terms (i.e.  $\hat{c} = 0$ ) and considering only positive grade operators in (6b) we have the zero curvature equation for the positive AKNS hierarchy

$$[\partial_x + H^1 + qE_{\alpha}^0 + rE_{-\alpha}^0, \partial_t + D^{(n)} + D^{(n-1)} + \dots + D^{(0)}] = 0, \quad (34)$$

where  $n = 1, 2, 3, \dots$  is an arbitrary positive integer. Solving (34) grade by grade, starting from the highest one, we can determine all coefficients in (33) in terms of the fields  $q$  and  $r$ . The zero grade projection provides the field equations.

For the case where  $n = 2$ , fixing the two arbitrary constant coefficients as  $a_2 = 1$  and  $a_1 = 0$ , and ignoring possible integration constants as usual, we get the famous AKNS system

$$q_t = -\frac{1}{2}q_{xx} + q^2 r, \quad r_t = \frac{1}{2}r_{xx} - qr^2. \quad (35)$$

Choosing  $q = \varphi$ ,  $r = \sigma\varphi^*$ ,  $t \rightarrow -it$  and  $x \rightarrow ix$  we obtain the well known NLS equation

$$i\varphi_t - \frac{1}{2}\varphi_{xx} - \sigma|\varphi|^2\varphi = 0. \quad (36)$$

The cases  $\sigma = \pm 1$  correspond to the *focusing* and *defocusing* NLS equations, respectively. Note that the nonzero constant fields  $q \rightarrow q_0$ ,  $r \rightarrow r_0$  are not solution of (35). If we do not ignore an integration constant arising when solving for the  $a_0$  coefficient in (33), but denote it conveniently as  $\frac{1}{2}q_0r_0$ , we get the system

$$q_t = -\frac{1}{2}q_{xx} + (qr - q_0r_0)q, \quad r_t = \frac{1}{2}r_{xx} - (qr - q_0r_0)r, \quad (37)$$

whose Lax pair is given by

$$U = H^1 + qE_\alpha^0 + rE_{-\alpha}^0, \quad (38a)$$

$$V = H^2 + qE_\alpha^1 + rE_{-\alpha}^1 - \frac{1}{2}(qr - q_0r_0)H^0 - \frac{1}{2}q_xE_\alpha^0 + \frac{1}{2}r_xE_{-\alpha}^0. \quad (38b)$$

Note that now  $q \rightarrow q_0$ ,  $r \rightarrow r_0$  is a solution of (37). The choice  $q = \varphi$ ,  $r = \sigma\varphi^*$ ,  $t \rightarrow -it$  and  $x \rightarrow ix$  yields the *focusing* and *defocusing* NLS equations admitting nonvanishing (constant) boundary value solution

$$i\varphi_t - \frac{1}{2}\varphi_{xx} - \sigma(|\varphi|^2 - A^2) = 0, \quad (39)$$

where  $A^2 \equiv |\varphi_0|^2$  is a real constant. Solutions of this equation with  $\sigma = 1$  (focusing) was the object under study in the recent paper of Zakharov and Gelash [8].

Solving (34) for the  $n = 3$  case we get

$$q_t = \frac{1}{4}q_{xxx} - \frac{3}{2}qrq_x, \quad r_t = \frac{1}{4}r_{xxx} - \frac{3}{2}qrr_x, \quad (40)$$

whose Lax pair given by (38a) and

$$V = H^3 + qE_\alpha^2 + rE_{-\alpha}^2 - \frac{1}{2}qrH^1 - \frac{1}{2}q_xE_\alpha^1 + \frac{1}{2}r_xE_{-\alpha}^1 \\ + \frac{1}{4}(rq_x - qr_x)H^0 + \frac{1}{4}(q_{xx} - 2q^2r)E_\alpha^0 + \frac{1}{4}(r_{xx} - 2q^2r)E_{-\alpha}^0. \quad (41)$$

The constant fields  $q \rightarrow q_0$ ,  $r \rightarrow r_0$  are solution of (40). Note that taking  $r = \pm q$  in (40) yields the mKdV equation,  $r = \pm 1$  the KdV equation and  $r = \alpha + \beta q$  the Gardner equation with arbitrary coefficients

$$4q_t = q_{xxx} - 6\alpha qq_x - 6\beta q^2q_x. \quad (42)$$

Solutions with nonvanishing boundary condition for the mKdV hierarchy and solutions of a deformed hierarchy containing the Gardner equation were considered in [6].

The first negative flow of the AKNS hierarchy is the Lund-Regge model, that as discussed in section 3.3, does not have nonvanishing (constant) boundary value solutions.

## 5. AKNS solutions

Let

$$\Omega^{(n)} \equiv H^n + q_0E_\alpha^{n-1} + r_0E_{-\alpha}^{n-1}. \quad (43)$$

Taking  $q \rightarrow q_0$ ,  $r \rightarrow r_0$  in (38) and (41) yield the respective vacuum Lax operators

$$U_0 = \Omega^{(1)} \quad \text{for (38a),} \quad (44a)$$

$$V_0 = \Omega^{(2)} \quad \text{for (38b),} \quad (44b)$$

$$V_0 = \Omega^{(3)} - \frac{1}{2}q_0r_0\Omega^{(1)} \quad \text{for (41).} \quad (44c)$$



Let

$$B = \exp(\chi E_{-\alpha}^0) \exp(\phi H^0) \exp(\psi E_{\alpha}^0). \quad (45)$$

Solving (27) will imply in the following relations

$$\begin{aligned} \phi_x &= r_0 \psi (1 + \chi \psi e^{2\phi}) - q_0 \chi e^{2\phi} + \chi \psi_x e^{2\phi}, \\ q &= e^{2\phi} (q_0 - r_0 \psi^2 - \psi_x), \\ r &= r_0 e^{-2\phi} + e^{2\phi} (q_0 \chi^2 - r_0 \chi^2 \psi^2) - \chi_x - \chi^2 \psi_x e^{2\phi}. \end{aligned} \quad (46)$$

Introducing the auxiliary functions

$$\tilde{\chi} \equiv \chi e^{\phi}, \quad \tilde{\psi} \equiv \psi e^{\phi}, \quad \Delta \equiv 1 + \tilde{\chi} \tilde{\psi}, \quad (47)$$

the expressions (46) can be further simplified to

$$q = \frac{q_0}{\Delta} e^{2\phi} - \frac{\tilde{\psi}_x}{\Delta} e^{\phi}, \quad r = r_0 \Delta e^{-2\phi} - \tilde{\chi}_x e^{-\phi}. \quad (48)$$

Taking into account the central terms (i.e.  $\hat{c} \neq 0$ ), consider the highest weight states of  $\hat{sl}_2$ ;  $\{|\mu_0\rangle, |\mu_1\rangle\}$ . These states obey the following actions,

$$H^n |\mu_k\rangle = 0, \quad E_{\pm\alpha}^n |\mu_k\rangle = 0, \quad (\text{for } n > 0) \quad (49)$$

$$E_{\alpha}^0 |\mu_k\rangle = 0, \quad H^0 |\mu_0\rangle = 0, \quad H^0 |\mu_1\rangle = |\mu_1\rangle, \quad \hat{c} |\mu_k\rangle = |\mu_k\rangle. \quad (50)$$

The adjoint relations are

$$(H^n)^{\dagger} = H^{-n}, \quad (E_{\pm\alpha}^n)^{\dagger} = E_{\mp\alpha}^{-n}, \quad \hat{c}^{\dagger} = \hat{c}, \quad \hat{d}^{\dagger} = \hat{d}. \quad (51)$$

Besides the highest weight states we use the same notation to denote the particular state

$$|\mu_2\rangle = E_{-\alpha}^0 |\mu_1\rangle \neq 0. \quad (52)$$

Using these states in (28) with (45), we have the following tau functions

$$\begin{aligned} \tau_{00} &= e^{\nu} = \langle \mu_0 | \Psi_0 g \Psi_0^{-1} | \mu_0 \rangle, \\ \tau_{11} &= e^{\nu+\phi} = \langle \mu_1 | \Psi_0 g \Psi_0^{-1} | \mu_1 \rangle, \\ \tau_{12} &= \psi e^{\nu+\phi} = \langle \mu_1 | \Psi_0 g \Psi_0^{-1} | \mu_2 \rangle, \\ \tau_{21} &= \chi e^{\nu+\phi} = \langle \mu_2 | \Psi_0 g \Psi_0^{-1} | \mu_1 \rangle, \end{aligned} \quad (53)$$

from which we have

$$e^{\phi} = \frac{\tau_{11}}{\tau_{00}}, \quad \chi = \frac{\tau_{21}}{\tau_{11}}, \quad \psi = \frac{\tau_{12}}{\tau_{11}}, \quad \tilde{\chi} = \frac{\tau_{21}}{\tau_{00}}, \quad \tilde{\psi} = \frac{\tau_{12}}{\tau_{00}}. \quad (54)$$

Substituting the tau functions in (48) we get the explicit dependence for the nonvanishing boundary value solutions of the AKNS hierarchy

$$q = q_0 \frac{\tau_{11}^2}{\tau_{00}^2 + \tau_{12}\tau_{21}} + \frac{\tau_{11}}{\tau_{00}} \frac{(\tau_{00})_x \tau_{12} - \tau_{00}(\tau_{12})_x}{\tau_{00}^2 + \tau_{12}\tau_{21}}, \quad (55a)$$

$$r = r_0 \frac{\tau_{00}^2 + \tau_{12}\tau_{21}}{\tau_{11}^2} + \frac{(\tau_{00})_x \tau_{21} - \tau_{00}(\tau_{21})_x}{\tau_{00}\tau_{11}}. \quad (55b)$$

## 6. Vertex operators

We now introduce vertex operators that are eigenstates of the nontrivial vacuum Lax operators (44). Consider  $g$  in the usual solitonic form (20), with vertex operators defined by

$$X_i \equiv \sum_{n=-\infty}^{\infty} \left[ \frac{1}{2} \left( \kappa_i - \frac{q_0 r_0}{\kappa_i} \right) \right]^{-n} X_i^{(n)} \quad (56a)$$

$$\text{where } X_i^{(n)} \equiv -\frac{q_0}{\kappa_i} H^n + \frac{q_0(\kappa_i^2 - q_0 r_0)}{\kappa_i(\kappa_i^2 + q_0 r_0)} \delta_{n0} \hat{c} - \frac{q_0^2}{\kappa_i^2} E_\alpha^n + E_{-\alpha}^n, \quad (56b)$$

$$Y_i \equiv \sum_{n=-\infty}^{\infty} \left[ \frac{1}{2} \left( \kappa_i - \frac{q_0 r_0}{\kappa_i} \right) \right]^{-n} Y_i^{(n)} \quad (56c)$$

$$\text{where } Y_i^{(n)} \equiv -\frac{r_0}{\kappa_i} H^n - \frac{r_0(\kappa_i^2 - q_0 r_0)}{\kappa_i(\kappa_i^2 + q_0 r_0)} \delta_{n0} \hat{c} + E_\alpha^n - \frac{r_0^2}{\kappa_i^2} E_{-\alpha}^n, \quad (56d)$$

where  $\kappa_i$  is a complex parameter. In the limit  $q_0, r_0 \rightarrow 0$  these vertex operators recover the well known vertices used for zero boundary condition [12]. A direct calculation will give the following eigenvalue equations, for  $\Omega^{(n)}$  defined by (43),

$$[\Omega^{(n)}, X_i] = - \left[ \frac{1}{2} \left( \kappa_i - \frac{q_0 r_0}{\kappa_i} \right) \right]^{n-1} \left[ \kappa_i + \frac{q_0 r_0}{\kappa_i} \right] X_i, \quad (57a)$$

$$[\Omega^{(n)}, Y_i] = + \left[ \frac{1}{2} \left( \kappa_i - \frac{q_0 r_0}{\kappa_i} \right) \right]^{n-1} \left[ \kappa_i + \frac{q_0 r_0}{\kappa_i} \right] Y_i. \quad (57b)$$

Therefore, the spacetime dependence of the solutions of system (37) will be given by

$$\begin{aligned} \eta_i &= + \left( \kappa_i + \frac{q_0 r_0}{\kappa_i} \right) x + \frac{1}{2} \left( \kappa_i^2 - \frac{(q_0 r_0)^2}{\kappa_i^2} \right) t, \\ \xi_i &= - \left( \kappa_i + \frac{q_0 r_0}{\kappa_i} \right) x - \frac{1}{2} \left( \kappa_i^2 - \frac{(q_0 r_0)^2}{\kappa_i^2} \right) t, \end{aligned} \quad (58)$$

while for the system (40) we have

$$\begin{aligned} \eta_i &= + \left( \kappa_i + \frac{q_0 r_0}{\kappa_i} \right) x + \frac{1}{4} \left( \kappa_i^3 - 3\kappa_i q_0 r_0 - 3 \frac{(q_0 r_0)^2}{\kappa_i} + \frac{(q_0 r_0)^3}{\kappa_i^3} \right) t, \\ \xi_i &= - \left( \kappa_i + \frac{q_0 r_0}{\kappa_i} \right) x - \frac{1}{4} \left( \kappa_i^3 - 3\kappa_i q_0 r_0 - 3 \frac{(q_0 r_0)^2}{\kappa_i} + \frac{(q_0 r_0)^3}{\kappa_i^3} \right) t. \end{aligned} \quad (59)$$

The matrix elements involving one vertex operator only are

$$\langle \mu_0 | Y_i | \mu_0 \rangle = -\frac{r_0(\kappa_i^2 - q_0 r_0)}{\kappa_i(\kappa_i^2 + q_0 r_0)}, \quad \langle \mu_0 | X_i | \mu_0 \rangle = \frac{q_0(\kappa_i^2 - q_0 r_0)}{\kappa_i(\kappa_i^2 + q_0 r_0)}, \quad (60a)$$

$$\langle \mu_1 | Y_i | \mu_1 \rangle = \frac{-2\kappa_i r_0}{\kappa_i^2 + q_0 r_0}, \quad \langle \mu_1 | X_i | \mu_1 \rangle = \frac{-2q_0^2 r_0}{\kappa_i(\kappa_i^2 + q_0 r_0)}, \quad (60b)$$

$$\langle \mu_1 | Y_i | \mu_2 \rangle = 1, \quad \langle \mu_1 | X_i | \mu_2 \rangle = -\frac{q_0^2}{\kappa_i^2}, \quad (60c)$$

$$\langle \mu_2 | Y_i | \mu_1 \rangle = -\frac{r_0^2}{\kappa_i^2}, \quad \langle \mu_2 | X_i | \mu_1 \rangle = 1. \quad (60d)$$

Unlike the case of vanishing boundary condition, the solution involving only one vertex is not trivial. Choosing  $g = e^{X_1}$  yields

$$\tau_{00} = 1 + \langle \mu_0 | X_1 | \mu_0 \rangle e^{\eta_1}, \quad (61a)$$

$$\tau_{11} = 1 + \langle \mu_1 | X_1 | \mu_1 \rangle e^{\eta_1}, \quad (61b)$$

$$\tau_{12} = \langle \mu_1 | X_1 | \mu_2 \rangle e^{\eta_1}, \quad (61c)$$

$$\tau_{21} = \langle \mu_2 | X_1 | \mu_1 \rangle e^{\eta_1}. \quad (61d)$$

Analogous expressions are also valid for  $Y_1$ , replacing  $\eta_1 \rightarrow \xi_1$ . Substituting the tau functions (61) in (55) we get the explicit 1-vertex solution of the positive AKNS hierarchy.

For a solution involving two vertices like  $g = e^{X_1} e^{Y_2}$ , we have more complicated matrix elements, but its general form is given by

$$\tau_{00} = 1 + \langle \mu_0 | X_1 | \mu_0 \rangle e^{\eta_1} + \langle \mu_0 | Y_2 | \mu_0 \rangle e^{\xi_2} + \langle \mu_0 | X_1 Y_2 | \mu_0 \rangle e^{\eta_1 + \xi_2}, \quad (62a)$$

$$\tau_{11} = 1 + \langle \mu_1 | X_1 | \mu_1 \rangle e^{\eta_1} + \langle \mu_1 | Y_2 | \mu_1 \rangle e^{\xi_2} + \langle \mu_1 | X_1 Y_2 | \mu_1 \rangle e^{\eta_1 + \xi_2}, \quad (62b)$$

$$\tau_{12} = \langle \mu_1 | X_1 | \mu_2 \rangle e^{\eta_1} + \langle \mu_1 | Y_2 | \mu_2 \rangle e^{\xi_2} + \langle \mu_1 | X_1 Y_2 | \mu_2 \rangle e^{\eta_1 + \xi_2}, \quad (62c)$$

$$\tau_{21} = \langle \mu_2 | X_1 | \mu_1 \rangle e^{\eta_1} + \langle \mu_2 | Y_2 | \mu_1 \rangle e^{\xi_2} + \langle \mu_2 | X_1 Y_2 | \mu_1 \rangle e^{\eta_1 + \xi_2}. \quad (62d)$$

Besides the matrix elements involving one vertex operator (60), the nilpotency property was explicitly verified and the following two vertices matrix elements calculated,

$$\langle \mu_0 | X_i Y_j | \mu_0 \rangle = \frac{(\kappa_i^2 - q_0 r_0)(\kappa_j^2 - q_0 r_0)(\kappa_i \kappa_j + q_0 r_0)^2}{(\kappa_i^2 + q_0 r_0)(\kappa_j^2 + q_0 r_0)\kappa_i \kappa_j (\kappa_i - \kappa_j)^2}, \quad (63a)$$

$$\langle \mu_1 | X_i Y_j | \mu_1 \rangle = \frac{\{q_0 r_0 (\kappa_i - \kappa_j)^2 + (\kappa_i \kappa_j - q_0 r_0)^2\}(\kappa_i \kappa_j + q_0 r_0)^2}{(\kappa_i^2 + q_0 r_0)(\kappa_j^2 + q_0 r_0)\kappa_i^2 (\kappa_i - \kappa_j)^2}, \quad (63b)$$

$$\langle \mu_1 | X_i Y_j | \mu_2 \rangle = -2q_0 \frac{(\kappa_i \kappa_j - q_0 r_0)(\kappa_i \kappa_j + q_0 r_0)^2}{(\kappa_i^2 + q_0 r_0)(\kappa_j^2 + q_0 r_0)(\kappa_i - \kappa_j)\kappa_i^2}, \quad (63c)$$

$$\langle \mu_2 | X_i Y_j | \mu_1 \rangle = -2r_0 \frac{(\kappa_i \kappa_j - q_0 r_0)(\kappa_i \kappa_j + q_0 r_0)^2}{(\kappa_i^2 + q_0 r_0)(\kappa_j^2 + q_0 r_0)(\kappa_i - \kappa_j)\kappa_j^2}. \quad (63d)$$

The matrix elements (63) and (60) recover known results when  $q_0, r_0 \rightarrow 0$ , corresponding to the vanishing boundary condition [12].

## 7. Conclusions

We have shown how to employ the algebraic dressing method, using vertex operators, to construct nonvanishing (constant) boundary value solutions for integrable hierarchies. This method was applied previously for the mKdV hierarchy [5, 6] and in this letter for the AKNS hierarchy, completing the possible classical models obtained using the algebra  $\hat{s}\ell_2$  with the construction of integrable hierarchies [7]. We have algebraically explained how the dispersion relation and multi-soliton interaction depend on the boundary conditions, here arising as a consequence of the vertex operators (56).

Our solutions are quite general, solving at once various individual models contained in the AKNS hierarchy like the focusing and defocusing NLS, mKdV, KdV and Gardner equations. We also have obtained an explicit 2-soliton solution, involving two vertex operators. Of course higher multi-soliton solutions can be constructed following the same approach, we only face a technical difficulty in computing the matrix elements but the procedure is well established.

## Acknowledgments

We thank CNPQ and Fapesp for financial support. We would like to thank L. Feher for suggestions.

## References

- [1] Aratyn H, Gomes J F, Nissimov E, Pacheva S and Zimerman A H 2000 *Proc. NATO Advanced Research Workshop on Integrable Hierarchies and Modern Physical Theories (NATO ARW - UIC 2000)(Chicago)* arXiv:nlin/0012042v1
- [2] Babelon O and Bernard D 1993 *Int. J. Mod. Phys. A* **8** 507 hep-th/9206002
- [3] Ferreira L A, Miramontes J L and Guillén J S 1997 *J. Math. Phys.* **38** 882 arXiv:hep-th/9606066v1
- [4] Olive D I, Turok N and Underwood J W R 1993 *Nucl. Phys. B* **409** 509 arXiv:hep-th/9305160v1
- [5] Gomes J F, França G S, de Melo G R and Zimerman A H 2009 *J. Physics A* **42** 445204 arXiv:0906.5579 [hep-th]
- [6] Gomes J F, França G S and Zimerman A H 2011 Nonvanishing boundary condition for the mKdV hierarchy and the Gardner equation *accepted for publication J. Phys. A* arXiv:1110.3247v1 [nlin.SI]
- [7] Aratyn H, Gomes J F and Zimerman A H 2004 Algebraic construction of integrable and super integrable hierarchies *Proc. XI International Conference on Symmetry Methods in Physics (SYMPHYS-11)(Prague, Czech Republic)* arXiv:hep-th/0408231v1
- [8] Zakharov V and Gelash A 2011 Soliton on unstable condensate arXiv:1109.0620v2 [nlin.SI]
- [9] Assunção A O, Blas H and da Silva M J B F 2011 Generalized AKNS system, non-vanishing boundary conditions and N-dark-dark solitons arXiv:1110.3108v1 [nlin.SI]
- [10] Leznov A N and Saveliev M V 1983 *Commun. Math. Phys.* **89** 59
- [11] Goddard P and Olive D 1988 *Kac-Moody and Virasoro algebras: a reprint volume for physicists* (World Scientific)
- [12] Cabrera-Carnero I, Gomes J F, Gueuvoghlianian E P, Sotkov G M and Zimerman A H 2001 Non abelian Toda models and constrained KP hierarchies *Talk given at the 7th International Wigner Symposium (Wigsym 7), College Park, Maryland, 2001* arXiv:hep-th/0109117