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The CHY Formalism of Scattering Amplitudes:

From Ambitwistor Strings to Tropical Grassmannian Soft Theorems

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THE CHY FORMALISM OF SCATTERING AMPLITUDES:

FROM AMBITWISTOR STRINGS TO TROPICAL

GRASSMANNIAN SOFT THEOREMS

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Resumo

Nesta tese, revisaremos o formalismo CHY de amplitude de espalhamento e sua tentadora origem em termos de cordas com ambitwistors. Em particular, trabalhamos em um modelo de cordas com twistor-ambitwistor que, assim como a teoria de corda com twistor original, utiliza variáveis twistor em sua construção. Encontramos o gerador BRST para este modelo, o qual foi um problema em aberto no trabalho que primeiro introduziu esse modelo. Na segunda metade desta tese, revisaremos uma generalização recém introduzida para o formalismo CHY para espaços projetivos complexos em dimensões mais altas. Estudamos o soft limit da generalização da amplitude escalar biadjunto $m_n^{(k)}$, a qual conjectura-se uma relação com estruturas geométricas conhecidas como Grassmanianas tropicais TrG(k,n). Utilizando o formalismo CHY junto com o teorema global de resíduos provaremos a soft factorization para amplitudes $m_n^{(k)}$ para k e n arbitrários. Encontraremos que os soft factors possuem a forma de amplitudes $m_n^{(2)}$, implicando que todas as amplitudes de espalhamento da teoria de escalares biadjuntos podem ser interpretada como uma família infinita de soft factors. De passagem, a dualidade Grassmaniana revela que as amplitudes generalizadas $m_n^{(k)}$ com $k > 2$ não só satisfazem um soft theorem, mas também um "hard" theorem não trivial.

Palavras-chaves: Amplitudes de espalhamento; Formalismo CHY; Corda (Twistor) Ambitwistorial; Geometria Tropical e Grassmanianas; Soft Theorems.

Áreas do conhecimento: Teoria Quântica de Campos, Teoria de Cordas.

Abstract

In this thesis we review the CHY formalism of scattering amplitudes and discuss their tantalizing origin in terms of ambitwistor strings. In particular, we work a twistor-ambitwistor string model that, as the original twistor string, uses twistor variables in its construction. We find the BRST generator for this model, which was an open problem from the first work that introduced such model. In the second half of this thesis we review a recently introduced generalization of the CHY formalism to complex projective spaces of higher dimension. We study the soft limit of the generalization of the biadjoint scalar amplitudes $m_n^{(k)}$, which have been conjectured to have a relation to a geometrical structure known as the tropical Grassmannian Tr G(k,n). Using the CHY formulation along with the Global Residue Theorem, we prove the soft factorization for $m_n^{(k)}$ amplitudes for arbitrary k and n. We find that the soft factors take the form of $m_n^{(2)}$ amplitudes, which entails that all scattering amplitudes of the ordinary biadjoint scalar theory can be interpreted as an infinite family of soft factors. In-passing, Grassmannian duality reveals that generalized amplitudes $m_n^{(k)}$ with $k > 2$ satisfy not only a soft theorem, but also a non-trivial "hard" theorem.

Key-words: Scattering Amplitudes; CHY Formalism; (Twistor) Ambitwistor Strings; Tropical Geometry and Grassmannians; Soft Theorems.

Field of knowledge: Quantum Field Theory, String Theory.

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Contents

A Note on Conventions: Throughout this thesis, Latin indices from the middle of the alphabet correspond to ten-dimensional vector indices $m, n, \ldots = 0, \ldots 9$, whereas Greek indices from the beginning of the alphabet correspond to Majorana-Weyl spinor indices $\alpha, \beta, \ldots = 1, \ldots 16$.

Chapter I

Introduction

In the standard textbook formulation of Quantum Field Theory (QFT), the concept of scattering amplitude plays a central role as the main observable that ones wishes to compute and understand. The main tool that there is at one's disposal are Feynman rules and Feynman diagrams arising from a Lagrangian, which makes manifest properties such as locality, though at the expense of introducing additional redundancies in the description of the theory in the form of gauge invariance. In hindsight, one of the very early hints of the fact that the standard approach of Feynman diagrams was not the best one to compute scattering amplitudes (and that therefore a better understanding of field theories in general was needed) was found by Parke and Taylor as early as in 1986 [1], where they conjectured an astonishingly simple formula for the scattering of n gluons in the maximally helicity violating sector. This, despite the fact that with standard computational techniques even the simplest cases of four or five point amplitudes required heavy calculations involving hundreds or thousands of terms.

The decisive step that unleashed all further discoveries on the subject (now dubbed just as "amplitudes") was taken in 2003 by Witten, whom successfully combined string theory with twistor theory so as to propose a new formulation for the tree level scattering matrix of gauge bosons in terms of a "connected formula" for gauge boson amplitudes [2]. Non-trivial evidence for the validity of the formula was given shortly after by Roiban, Spradlin and Volovich [3]. Furthermore, in [4], Berkovits proposed an alternative string theory, also based on twistor worldsheet variables, from which he found the same formulation. Since that initial spark, a plethora of new results arose along with the development of novel "on-shell methods" to compute amplitudes, such as the spinor-helicity formalism, Britto-Cachazo-Feng-Witten recursion relations [5,6], (super)momentum twistors [7], among others. Furthermore, striking results such as the dual conformal/infinite dimensional Yangian symmetry of (four dimensional) $\mathcal{N} = 4$ Super Yang-Mills [8] were consequently found, as an example of properties deeply hidden under the conventional Feynman diagram wisdom. A general review of these methods and results can be found in [9].

Although the original idea of the "twistor string" was successful in that provided a novel way to represent tree amplitudes, and certainly important from a historical point of view, it suffered from many limitations. First, it described only a very specific case: $\mathcal{N} = 4$ Super Yang-Mills theory, in a fixed number of dimensions: $d = 4$. Secondly, it was later found [10] that the original twistor string at loop level did not coincide with pure Super Yang-Mills theory as conformal supergravity states actually contributed at loop level (thus rendering the theory unphysical). Naturally, finding a similar formulation for further theories in such a way that these drawbacks were not present quickly became an open problem. Indeed, in the astounding series of works by Cachazo, He, and Yuan (CHY) [11–15] this is what effectively is done for tree level amplitudes, although giving up on supersymmetry and therefore on considering fermions. The result is now known as *the CHY formalism of scattering amplitudes*, which provides a unified description for tree level scattering amplitudes of a large number of theories of massless particles in an arbitrary number of dimensions. In the CHY formalism, a n-particle scattering amplitude is expressed as an integral over the moduli space of n-punctured Riemann spheres. The integral is localized on the support of the so-called *scattering equations*, which relate the space of n punctures on the Riemann sphere $\mathbb{CP}^1$ to the space of kinematic invariants for n massless particles.

The CHY formalism, through the scattering equations and the idea of integrating over an auxiliary space, allows us to bypass the use of Feynman diagrams (and by that matter, gauge invariance and manifest locality) to compute tree level amplitudes for the scattering of massless particles. In spite of this, the actual power of the formalism resides not in computation, but rather in the conceptual understanding that it has provided us into the structure of (perturbative) field theory. Indeed, new insights as well as new relations in the so-called Kawai-Lewellen-Tye (KLT) relations [16], a unified point of view for gravity, Yang-Mills theory, and a number of other seemingly unrelated theories such as the nonlinear-sigma-model [15], novel results in the study of soft limits and soft theorems [17–21], as well as new perspectives in the study of loop amplitudes [22,23], are several examples of what the CHY formalism has allowed us to understand in the previous few years.

Considering the pervasiveness of the CHY formalism at the moment of describing the scattering of massless particles, we are instinctively tempted to ask for the origin of the formalism, in a similar way as Feynman diagrams arise from standard QFT, and the twistor string comes from combining string theory and twistor theory in a suitable way. Perhaps unsurprisingly (considering the intuition provided by the integration of punctures over a Riemann sphere), the answer came from string theory, in a certain limit known as *ambitwistor strings*. This was originally shown in [24] by Mason and Skinner, with a pure spinor superstring version by Berkovits coming shortly after in [25]. Unlike string theory, ambitwistor strings are worldsheet models that are subject not only to the Virasoro constraint, but also to the massless constraint $P^2 = 0$, and may be seen as models obtained from complexifying a massless particle action. Effectively, this limit eliminates all massive modes originally present in standard string theories, leaving us only with the massless states as described by the CHY formulae. Scattering amplitudes are then computed in ambitwistor strings with a prescription similar to the one found in string theory (see chapter 5 in [26]), and that leads to the localization over the scattering equations as stated by CHY. Different formulae then correspond to different matter models in the ambitwistor string action, and although the original CHY formalism concerned only bosons, fermions can in principle be accommodated in the ambitwistor string construction via supersymmetry. Albeit not an issue at tree level (where CHY is valid), avoidance of anomalies in the ambitwistor string shows that $d = 26$ for the bosonic case and $d = 10$ for the supersymmetric case, just as in standard string theory.

Since ambitwistor string models are capable of handling fermions through supersymmetry, and at the same time provide the CHY formulae in the bosonic case, they are a natural candidate to start looking for connected formulae in further supersymmetric theories in different number of dimensions (in a similar spirit as in the original twistor string). Indeed, that has been the path taken in recent years. For instance, ambitwistor strings using twistor variables have been used in [27] to obtain amplitudes with arbitrary amounts of supersymmetry in four dimensions.

Similar formulae in six dimensions were found in [28, 29], and in [30] such formulae were provided in ten and eleven dimensions in terms of an extension of the scattering equations known as the polarized scattering equations (also present in 6D in [29]). Nevertheless, in this last case a BRST analysis of the constraints and anomalies lacked. In an alternative line of developments, Berkovits, Guillen, and Mason (BGM) have recently introduced in [31] a new ambitwistor string model, also constructed from worldsheet supertwistor variables

$$(\lambda^\alpha, w_\alpha, \psi^m),$$

and that could also in principle provide connected formulae in ten dimensions (here, λ and w are bosonic spinors of different chirality and ψ is a fermionic vector). This ambitwistor model is based on a covariant/twistorial description of the superparticle developed by Berkovits in [32], and is constructed by promoting the worldline to a Riemann sphere and time derivatives to anti-holomorphic derivatives $\bar{\partial}$, in accordance to the procedure described in the paper by Mason and Skinner [24]. In order to correctly describe the superparticle, the variables appearing in Berkovits' model are subject to the reducible constraints

$$\begin{aligned} &v = (\lambda\gamma^m\lambda)\psi_m = 0, \\ &G^\alpha = (\lambda\gamma^m\lambda)(\gamma_m w)^\alpha - 2\lambda^\alpha(\lambda w) + 2i\psi^m\psi^n(\gamma_m\gamma_n\lambda)^\alpha = 0. \end{aligned}$$

Since these constraints are reducible $H^m \equiv (\lambda\gamma^m G) - 4i\psi^m v = 0$, $(\lambda\gamma^m\lambda)H_m = 0$, a covariant BRST analysis -similar to the one found in the Ramond-Neveu-Schwarz (RNS) [33] or the Berkovits superstring [34]- requires, apart from the conformal ghosts (c, b) and the ghosts $(g_\alpha, f^\alpha), (\beta, \gamma)$ associated to the previous constraints, the introduction of two generations of ghosts-for-ghosts (s_m, t^m), (η, ρ). Because of these reducibility issues, the authors in [31] performed a light-cone analysis to show the equivalence of the model to the light-cone RNS ambitwistor string, thus leaving the construction of the BRST generator [1] and the corresponding covariant description as an open problem. In this thesis, we provide a first step in tackling this problem. In particular, we find the BRST generator which turns out to be given by the BRST current

$$\begin{aligned} q(z) = \Big[&cT + c\partial cb + g_\alpha G^\alpha + \gamma T_F + s_m(\lambda\gamma^m f) + s_m(2\psi^m\beta) + \eta(\lambda\gamma_m\lambda)t^m \\ &+2(\lambda^\alpha g_\alpha)(g_\beta f^\beta) - 2(\lambda^\alpha g_\alpha)(\gamma\beta) - 2\big[\lambda^\alpha(\gamma_n)_{\alpha\beta}(\gamma^m)^{\beta\gamma}g_\gamma\big]s_m t^n \\ &+4(\lambda^\alpha g_\alpha)\eta\rho + 2\eta^{nm}s_n s_m\rho - \eta\beta^2 - 16\lambda^\alpha\partial g_\alpha \Big](z), \end{aligned} \tag{1.1}$$

and that in principle allows us to perform a covariant BRST analysis of the BGM model.

In a different exploration frontier, Cachazo, Early, Guevara, and Mizera have recently intro-

[1] A first step was already taken in [35], though effectively only a bosonic particle analysis is carried out.

duced in [36] a direct connection between scattering amplitudes and a geometric structure known as *the tropical Grassmannian* $\mathrm{Tr}\,\mathrm{G}(k,n)$ [37]. Such relation was originally found by generalizing the scattering equations and the CHY formalism to configuration spaces of n points living in $\mathbb{CP}^{k-1}$, thus effectively replacing the integration over punctured Riemann spheres in the CHY formulae to integration over higher-dimensional punctured "world-volumes". The original CHY formalism is then recovered when $k = 2$. Although finding a physical origin for these formulae (if any) remains an open problem, one may still consider the generalization of what is arguably the simplest theory that is known to have a CHY form; namely, biadjoint scalar theory, in order to get further insight into the structure of these theories. Moreover, biadjoint scalar theory presents further interest, as it has been uncovered as the heart of many constructions recently studied in amplitudes, ranging from the KLT relations [13, 15, 16, 38, 39] (as we shall see in subsection 2.4.1), to the kinematic associahedron [40–42].

The connection stemming from [36] states that the amplitudes of the biadjoint theory for general k are directly related to a subset of facets of the tropical Grassmannian $\mathrm{Tr}\,\mathrm{G}(k,n)$. These objects are polyhedral fans whose rays correspond to kinematic invariants and whose facets can be mapped to Feynman diagrams, in the general k sense. The most familiar case is the $\mathrm{Tr}\,\mathrm{G}(2,n)$ family which facets compute the standard biadjoint amplitudes $m_n^{(2)}$. Further, it was observed by Arkani-Hamed, Bai, He and Yang (ABHY) [40] that such amplitudes correspond to the canonical form of the (kinematic) associahedron $\mathcal{A}^{\mathrm{kin}}_{n-3}$. [2] Such structure is indeed dual to the positive part of $\mathrm{Tr}\,\mathrm{G}(2,n)$ [44]. For $k = 3$ the generalized amplitudes $m_n^{(3)}$ were first obtained from $\mathrm{Tr}\,\mathrm{G}(3,6)$ in [36] and later from $\mathrm{Tr}\,\mathrm{G}(3,7)$ in [45]. On the other hand, a geometric interpretation was provided in [46], which linked these amplitudes to certain cluster polytopes [47–50] dual to the tropical Grassmannians, therefore extending the associahedron interpretation of ABHY. As it turns out, for $\mathrm{Tr}\,\mathrm{G}(3,n)$ the $n = 6, 7, 8$ amplitudes can be obtained from volumes of the D_4, E_6 and E_8 cluster polytopes respectively. In this way, the amplitudes given in [36, 45] were recovered, and a precise form of the biadjoint amplitude for $\mathrm{Tr}\,\mathrm{G}(3,8)$ was proposed. Since then, further developments in the subject have been worked out in [51–58]. Note that due to the conjectured relation, we refer to the amplitudes of the generalized biadjoint scalar theory as tropical Grassmannian amplitudes interchangeably from now on.

Although the relation with the CHY formalism makes the tropical Grassmannian an interesting geometric structure to study on its own right, it is still unclear how a suitable realization of unitarity or locality emerge for the generalized amplitudes. In this work we take a first step in this direction by studying their soft limit [59, 60]. Focusing on $k = 3$, we first argue that such operation can be defined from a purely geometrical perspective as an embedding of the ordered facets of $\mathrm{Tr}\,\mathrm{G}(3, 5)$ into $\mathrm{Tr}\,\mathrm{G}(3, 6)$. We then introduce the analytic definition of the limit in terms of kinematic invariants and study it from the CHY perspective. Indeed, one of the early successes of the $k = 2$ CHY formulation was to make the well-known soft theorem of Weinberg completely straightforward to derive. This was important evidence supporting the conjectural CHY expression for gauge theory amplitudes, later proven in [61].

We will show that tropical Grassmannian amplitudes indeed satisfy a soft theorem analogous to the $k = 2$ case. Furthermore, we will show that for $k > 2$ we find a non-trivial "hard" theorem induced by the Grassmannian duality $m_n^{(k)} = m_n^{(n-k)}$. For any value of k, we will show that the

[2] As the biadjoint theory arises as the low energy limit of disk integrals, the connection between biadjoint theory and the associahedron was introduced in [41] from a string theory perspective, see also [43].

soft factor is obtained from the positive facets of the smaller object $\mathrm{Tr}\,\mathrm{G}(2,k+2)$. In practice, this means that the soft factor is given explicitly by the ordinary cubic biadjoint scalar amplitudes $m_{k+2}^{(2)}$, up to appropriate relabelings of its variables.

In order to outline our strategy let us briefly rederive the soft theorem for $k=2$ biadjoint scalar amplitudes, corresponding to $\mathrm{Tr}\,\mathrm{G}(2,n)$. These amplitudes are given by the integration over $n-3$ punctures on $\mathbb{CP}^1$ localized by $n-3$ multidimensional contours $|E_i|=\epsilon$ [12,13], i.e.

$$m_n^{(2)}[\mathbb{I}|\mathbb{I}] = \int \prod_i^{n-3}{}' \frac{d\sigma_i}{E_i} \left(\frac{1}{\sigma_{12}\cdots\sigma_{n1}} \right)^2, \quad \sigma_{ij} = \sigma_i - \sigma_j\,. \tag{1.2}$$

Taking particle 1 to be soft, i.e. setting $s_{1i} = \tau \hat{s}_{1i}$ with $\tau \to 0$, the above formula behaves as

$$m_n^{(2)}[\mathbb{I}_n|\mathbb{I}_n] = \frac{1}{\tau} \int \prod_{i\neq 1}^{n-4}{}' \frac{d\sigma_i}{E_i} \left(\frac{1}{\sigma_{23}\cdots\sigma_{n2}} \right)^2 \times \int \frac{d\sigma_1}{\hat{E}_1} \left(\frac{\sigma_{n2}}{\sigma_{n1}\sigma_{12}} \right)^2 + \mathcal{O}(\tau^0), \tag{1.3}$$

where $\hat{E}_1 = \sum_{j\neq 1} \frac{\hat{s}_{1j}}{\sigma_{1j}}$. In the $\mathbb{CP}^1$ case, the integral over σ_1 corresponds to the soft factor and can be computed via a direct application of the residue theorem. Deforming the contour $\hat{E}_1 \to 0$ to enclose the other singularities in the integrand, where puncture σ_1 collides with either σ_n or σ_2, we find explicitly

$$\begin{aligned} \tau S^{(2)} := \int_{\hat{E}_1\to 0} \frac{d\sigma_1}{\hat{E}_1} \left(\frac{\sigma_{n2}}{\sigma_{n1}\sigma_{12}} \right)^2 &= - \int_{\sigma_{n1}\to 0} \frac{d\sigma_1}{\hat{E}_1} \left(\frac{\sigma_{n2}}{\sigma_{n1}\sigma_{12}} \right)^2 - \int_{(\sigma_{12})\to 0} \frac{d\sigma_1}{\hat{E}_1} \left(\frac{\sigma_{n2}}{\sigma_{n1}\sigma_{12}} \right)^2 \\ &= \frac{1}{\hat{s}_{n1}} + \frac{1}{\hat{s}_{12}}. \end{aligned} \tag{1.4}$$

Thus, we have easily found the soft factor which corresponds to the standard four-point amplitude $m_4^{(2)}(\mathbb{I}_4|\mathbb{I}_4) = 1/s_{41} + 1/s_{12}$, once we perform the replacements $s_{41} \to \hat{s}_{n1}$ and $s_{12} \to \hat{s}_{12}$. From (1.3) we then obtain the standard soft theorem

$$m_n^{(2)}[\mathbb{I}_n|\mathbb{I}_n] = S^{(2)} \times m_{n-1}^{(2)}[2\ldots n|2\ldots n] + \mathcal{O}(\tau^0). \tag{1.5}$$

In this work we will extend this result and show that for $\mathrm{Tr}\,\mathrm{G}(k,n)$ the soft factor $S^{(k)}$ is given by $m_{k+2}^{(2)}$, or alternatively by the facets of the positive $\mathrm{Tr}\,\mathrm{G}(2,k+2)$.

To prove the above statements for higher k is far more delicate and requires to implement multidimensional contour deformations using the Global Residue Theorem (GRT) [62]. A quick introduction to the necessary ideas to implement this theorem is summarized in an appendix. We first prove directly the case $k=3$ and use it to gain some intuition for higher k. The definitive proof of the soft theorem for general k is achieved in two steps. We first use the GRT to establish a recursion relation for the soft factor $S^{(k)}$ in terms of the lower soft factors $S^{(j)}$, with $j < k$. We then show that this recursion relation is satisfied by the biadjoint amplitude $m^{(2)}_{k+2}$ with appropriate relabelings.

This thesis is organized in the following way: In chapter 2 we provide a general review of the CHY formalism. We start describing the scattering equations and several properties satisfied by them, after which we list a series of amplitude formulae that can be constructed in CHY form. We close providing some applications of the CHY formalism. In chapter 3 we start with a review of the necessary ideas of ambitwistor strings that we would like to apply to the BGM twistor-ambitwistor string described earlier, and we proceed to describe such model in detail. We conclude this chapter by presenting original research results obtained by the author in collaboration with M. Guillen, which consists in solving for the BRST generator of the BGM twistor-ambitwistor string model. In chapter 4 we review the extension of the CHY formalism to higher dimensional complex projective spaces $\mathbb{CP}^{k-1}$ introduced in [36], and how the corresponding generalization of biadjoint amplitudes is surprisingly connected to the tropical Grassmannian. Chapter 5 is based on the work [63] of the author in collaboration with A. Guevara and it has considerable overlap with such work. In this chapter we study the soft limit of the generalization of biadjoint amplitudes to all (k, n), where we provide a direct proof of the soft theorem, and we shall find new insights into the connection between the extension of CHY and the geometry of the tropical Grassmannian. We close with a brief conclusion in chapter 6. Appendix A contains a quick introduction to the theory of reducible constrained systems with the goal of outlining the basic tools used to construct the BRST generator of the BGM model in chapter 3. Appendix B contains an introduction to the Global Residue Theorem used in chapter 5.

Chapter II

The CHY Formalism of Scattering Amplitudes

In this chapter we review the fundamentals of the CHY formalism of scattering amplitudes [11–15]. We begin reviewing the scattering equations, which are the main pillar over which the CHY formalism rests, with a short remark in their "projective origin" so as to pave the way for their generalization to $\mathbb{CP}^{k-1}$ introduced in chapter 4. We then describe the CHY formalism in general terms, and proceed to describe along the lines of CHY several theories with direct physical interest. We conclude by showing how CHY can be applied to KLT-relations and soft theorems.

2.1 The Scattering Equations

The scattering equations were first found in their more modern incarnation through the connection between the scattering data of n massless particles and maps from the n-punctured Riemann sphere into the null cone. Specifically, let $\{k_1^m, \ldots, k_n^m\}$ be a set of n null vectors $k_a^2 = 0$ constrained to satisfy the momentum conservation condition $\sum_a k_a = 0$. The corresponding map is given by

$$k_a^m = \frac{1}{2\pi i} \oint_{|\sigma - \sigma_a| = \epsilon} P^m, \qquad \forall a \in \{1, 2, \ldots, n\}, \tag{2.1}$$

where P^m is a differential form defined such that it has poles as defined by (2.1). In the Riemann sphere $\mathbb{CP}^1$ this has a unique solution given by

$$P^m(\sigma) = \sum_{a=1}^{n} \frac{k_a^m}{\sigma - \sigma_a}, \tag{2.2}$$

and where momentum conservation guarantees that there are no poles at infinity. Asking for P^μ to live in the null cone implies then

$$P^2 = \sum_{a,b} \frac{k_a \cdot k_b}{(\sigma - \sigma_a)(\sigma - \sigma_b)} = 0, \tag{2.3}$$

and taking residues at every possible pole $\sigma^* = \sigma_a$, one is left with a set of conditions known as the *scattering equations*:

$$E_a = \sum_{\substack{b=1\\ b\neq a}}^{n} \frac{k_a \cdot k_b}{(\sigma_a - \sigma_b)} = 0. \tag{2.4}$$

Alternatively, the scattering equations over $\mathbb{CP}^1$ can be seen as the conditions for finding the critical points of the following potential function (that has already appeared in other contexts such as [64, 65]):

$$\mathcal{S} = \sum_{1\leq a<b\leq n} s_{ab} \log{(ab)}, \tag{2.5}$$

where s_{ab} are so far abstract objects indexed by the particle labels. We don't assume any further defining conditions over them. (ab) is the $\mathrm{SL}(2,\mathbb{C})$ invariant determinant, as a function of the homogeneous coordinates σ_a and σ_b of the points a and b:

$$(ab) = \begin{vmatrix} \sigma_{a,1} & \sigma_{b,1} \\ \sigma_{a,2} & \sigma_{b,2} \end{vmatrix}. \tag{2.6}$$

In order for $\mathcal{S}$ to be properly defined over the Riemann sphere $\mathbb{CP}^1$ it must be invariant under the rescaling $(\sigma_{a,1}, \sigma_{a,2}) \sim t_a(\sigma_{a,1}, \sigma_{a,2})$, for arbitrary t_a (the defining condition of a projective space). As a consequence, the s_{ab}'s must satisfy

$$\sum_{\substack{b=1\\ b\neq a}}^{n} s_{ab} = 0, \qquad \forall a, \tag{2.7}$$

which we recognize as the momentum conservation condition for kinematic invariants in QFT. Due to the fact that the abstract objects s_{ab} satisfy this property, we identify them as kinematic invariants $s_{ab} = 2k_a \cdot k_b$ from now on. This observation provides us with a novel conceptual understanding of the principle of momentum conservation: it is a direct consequence of the projective invariance of $\mathcal{S}$.

Inhomogeneous coordinates over $\mathbb{CP}^1$ are defined as the coordinates obtained after rescaling the first homogeneous coordinate to 1:

$$(\sigma_{a,1}, \sigma_{a,2}) \sim (1, \sigma_a), \qquad \sigma_a = \sigma_{a,2}/\sigma_{a,1}. \tag{2.8}$$

We can use these inhomogeneous coordinates to find the conditions for the critical points of $\mathcal{S}$, which leads to the scattering equations connecting the position of the punctures on $\mathbb{CP}^1$ to the space of kinematic invariants:

$$E_a = \frac{\partial \mathcal{S}}{\partial x_a} = \sum_{\substack{b=1 \\ b\neq a}}^{n} \frac{s_{ab}}{\sigma_a - \sigma_b} = 0. \tag{2.9}$$

The scattering equations satisfy the following properties:

Property 2.1.1. *The scattering equations are covariant with weight 2 under a* $\mathrm{SL}(2,\mathbb{C})$ *transformation:.*

Proof. Recall how a $\mathrm{SL}(2,\mathbb{C})$ transformation acts over the σ's:

$$\sigma_a \longrightarrow \frac{\alpha\sigma_a + \beta}{\gamma\sigma_a + \delta}, \qquad \text{with} \ \ (\alpha\delta - \beta\gamma) = 1. \tag{2.10}$$

Then, we have that $(\sigma_a - \sigma_b) \longrightarrow \frac{(\sigma_a - \sigma_b)}{(\gamma\sigma_a+\delta)(\gamma\sigma_b+\delta)}$, and thus the scattering equations transform as

$$\begin{aligned} &\sum_{\substack{b=1 \\ b\neq a}}^{n} \frac{s_{ab}}{\sigma_a - \sigma_b} \longrightarrow \sum_{\substack{b=1 \\ b\neq a}}^{n} \frac{s_{ab}}{\sigma_a - \sigma_b}(\gamma\sigma_a + \delta)(\gamma\sigma_b + \delta) \\ &= (\gamma\sigma_a + \delta)^2 \sum_{\substack{b=1 \\ b\neq a}}^{n} \frac{s_{ab}}{\sigma_a - \sigma_b} - \gamma(\gamma\sigma_a + \delta) \sum_{\substack{b=1 \\ b\neq a}}^{n} s_{ab} = (\gamma\sigma_a + \delta)^2 \sum_{\substack{b=1 \\ b\neq a}}^{n} \frac{s_{ab}}{\sigma_a - \sigma_b}, \end{aligned} \tag{2.11}$$

where in the last equality we have used the momentum conservation condition (2.7). In addition, this shows that a solution to the scattering equations indeed gives a configuration of n points on $\mathbb{CP}^1$. ■

Property 2.1.2. *There are* $(n-3)$ *independent scattering equations.*

Proof. The statement follows from the relations:

$$\sum_{a=1}^{n} \sigma_a^m \sum_{\substack{b=1 \\ b\neq a}}^{n} \frac{s_{ab}}{\sigma_a - \sigma_b} = 0 \ , \qquad m = 0, 1, 2. \tag{2.12}$$

The case $m = 0$ is trivially checked using antisymmetry in the labels a and b. The $m = 1$ and $m = 2$ conditions hold by antisymmetry and momentum conservation:

$$\sum_{a=1}^{n} \sigma_a \sum_{\substack{b=1 \\ b\neq a}}^{n} \frac{s_{ab}}{\sigma_a - \sigma_b} = \frac{1}{2} \sum_{\substack{a,b \\ b\neq a}}^{n} s_{ab} = 0, \tag{2.13}$$

$$\sum_{a=1}^{n} \sigma_a^2 \sum_{\substack{b=1 \\ b\neq a}}^{n} \frac{s_{ab}}{\sigma_a - \sigma_b} = \sum_{a=1}^{n} \sigma_a \sum_{\substack{b=1 \\ b\neq a}}^{n} s_{ab} + \sum_{\substack{a,b \\ b\neq a}}^{n} \frac{\sigma_a \sigma_b}{\sigma_a - \sigma_b} s_{ab} = 0, \tag{2.14}$$

which finishes the proof. ■

Property 2.1.3. *The scattering equations have* $(n-3)!$ *independent solutions.*

Proof. Let us call the number of solutions $\mathcal{N}_n$ and assume that the number of solutions do not change under a smooth variation of the coefficients in the scattering equations. Thus, let us perform the soft limit variation $k_n = \tau \hat{k}_n$ with $\tau \to 0$ and $\hat{k}_n$ fixed. The scattering equations then split between the equation corresponding to the soft particle and the rest:

$$E_a \longrightarrow \sum_{b=1}^{n-1} \frac{s_{ab}}{\sigma_a - \sigma_b} + \tau \frac{k_a \cdot \hat{k}_n}{\sigma_a - \sigma_n} = 0, \quad a \in \{1, \ldots, n-1\}, \tag{2.15}$$

$$E_n \longrightarrow \tau \hat{E}_n = \sum_{b=1}^{n-1} \frac{\hat{k}_n \cdot k_b}{\sigma_n - \sigma_b} = 0. \tag{2.16}$$

As we take $\tau \to 0$ we see that the first set just becomes the set of scattering equations for $n-1$ particles with $\mathcal{N}_{n-1}$ solutions. Assume we have solved for this set, so the second equation can be seen as a polynomial equation of degree $(n-3)$ on σ_n (due to momentum conservation). This last equation has then $(n-3)$ solutions, so we arrive at $\mathcal{N}_n = (n-3)\mathcal{N}_{n-1}$. Since for $n = 4$ we just get a linear equation we know that $\mathcal{N}_4 = 1$. Solving the recursion with this initial condition we get that $\mathcal{N}_n = (n-3)!$, as desired.

An alternative way of showing this (more rigorously), consists on using the polynomial form of the scattering equations found by Dolan and Goddard [66]:

$$\sum_{\substack{S_i \subset \{2,\dots,n-1\} \\ |S_i|=i}} s_{\{1\}\cup S_i}\sigma_{S_i}, \quad i = 1,\dots,n-3, \tag{2.17}$$

where $s_{S_j} = \sum_{a,b\in S_j} k_a \cdot k_b$, and $\sigma_{S_j} = \prod_{a\in S_j}\sigma_a$. These polynomials are equivalent to the scattering equations after gauge fixing $\{\sigma_1, \sigma_2, \sigma_n\} \to \{\infty, \sigma^*, 0\}$. Bezout's theorem [62] then guarantees that the scattering equations have $(n-3)!$ solutions. ■

2.2 The CHY Formalism

The scattering equations are the main ingredient in the CHY formalism of scattering amplitudes, according to which an amplitude for the scattering of n massless particles can be written as

$$\mathcal{A}_n = \int \frac{d^n\sigma}{\mathrm{Vol}[\mathrm{SL}(2,\mathbb{C})]}\left[\prod_m{}' \delta\big(E_m\big)\right] \mathcal{I}\big(\{\sigma\},\{s,\epsilon\}\big) = \int d\mu_n\, \mathcal{I}. \tag{2.18}$$

We call $d\mu_n$ the CHY integration measure. The proposal is that the measure and the scattering equations are universal to all theories of massless particles, while the theory specific information (polarization and spin of particles, color ordering...) is contained in the integrand $\mathcal{I}$. We now explain the meaning of all the symbols defining $d\mu_n$.

First, we need to notice that due to $\mathrm{SL}(2,\mathbb{C})$ invariance we can always fix three σ's to arbitrary positions. We remove this redundancy in the integral by taking the quotient by $Vol[\mathrm{SL}(2,\mathbb{C})]$. In practical terms, this amounts to define an invariant measure given by the replacement

$$d^n\sigma \longrightarrow \begin{vmatrix} 1 & 1 & 1 \\ \sigma_a & \sigma_b & \sigma_c \\ \sigma_a^2 & \sigma_b^2 & \sigma_c^2 \end{vmatrix} \frac{d^n\sigma}{d\sigma_a\ d\sigma_b\ d\sigma_c} = \sigma_{ab}\sigma_{bc}\sigma_{ca}\frac{d^n\sigma}{d\sigma_a\ d\sigma_b\ d\sigma_c}, \tag{2.19}$$

where we have defined $\sigma_{ab} = \sigma_a - \sigma_b$.

In order to localize the integral over the correct number of independent equations, we have to consider just $n-3$ scattering equations at the cost of a jacobian factor that makes the choice of the scattering equations permutation invariant:

$$\prod_m{}' \delta\big(E_m\big) = \begin{vmatrix} 1 & 1 & 1 \\ \sigma_a & \sigma_b & \sigma_c \\ \sigma_a^2 & \sigma_b^2 & \sigma_c^2 \end{vmatrix} \prod_{m\neq a,b,c} \delta\big(E_m\big) = \sigma_{ab}\sigma_{bc}\sigma_{ca} \prod_{m\neq a,b,c} \delta\big(E_m\big). \tag{2.20}$$

Property 2.2.1 *the CHY measure $d\mu_n$ transforms covariantly with weight -4 under a* $\mathrm{SL}(2,\mathbb{C})$ *transformation.*

Proof. Apply (2.10) and (2.11) over (2.20), and factor out of the delta functions the factors coming from the $\mathrm{SL}(2,\mathbb{C})$ transformation. One obtains a -2 weight contribution. Similarly, applying (2.10) over (2.19), and using that

$$d\sigma_a \longrightarrow \frac{\alpha d\sigma_a}{\gamma\sigma_a+\delta} - \frac{(\alpha\sigma_a+\beta)\gamma d\sigma_a}{(\gamma\sigma_a+\delta)^2} = (\gamma\sigma_a+\delta)^{-2}d\sigma_a, \tag{2.21}$$

one obtains another -2 weight. Adding both contributions, one obtains a total -4 weight. ■

Clearly, the amplitude must be invariant under a $\mathrm{SL}(2,\mathbb{C})$ transformation over the auxiliary Riemann sphere. This imposes a consistency condition over a valid integrand $\mathcal{I}$; namely, that it must transform covariantly with weight 4 under $\mathrm{SL}(2,\mathbb{C})$. Despite this fact, it will be further useful to consider quantities that transform covariantly under $\mathrm{SL}(2,\mathbb{C})$ with weights different than 4. This way, we can multiply different such quantities in order to arrive to different CHY-integrands. In particular, we define a half-integrand to be a quantity that transforms covariantly with weight 2 under $\mathrm{SL}(2,\mathbb{C})$.

Localizing over the system of equations given by the scattering equations gives rise to a jacobian given by the matrix

$$\Phi_{ab} = \frac{\partial^2\mathcal{S}}{\partial\sigma_a\partial\sigma_b}. \tag{2.22}$$

This matrix has rank $n-3$, and therefore its determinant vanishes. In order to construct a permutation invariant jacobian, one must remember that not all differentials and scattering equations appear in the integral. This means that one has to remove the rows i,j,k associated with the deltas that one has removed, and one has to remove the columns p,q,r associated with the σ's that one has fixed. We will also absorb the Vandermonde determinants appearing in (2.19) and (2.20)

$$V_{abc} = (\sigma_a-\sigma_b)(\sigma_b-\sigma_c)(\sigma_c-\sigma_a), \tag{2.23}$$

into the definition of the jacobian, and define the permutation invariant quantity

$$\det{}'\Phi = \frac{\det\Phi^{ijk}_{pqr}}{V_{ijk}V_{pqr}}, \tag{2.24}$$

where Φ^{ijk}_{pqr} means the Φ matrix with the rows (ijk) removed and the columns (pqr) removed.

The scattering equations completely localize the integral, which means that, all in all, the formula (8) can be expressed in the following form as a sum over the solutions to the scattering equations

$$\mathcal{A}_n = \sum_{\{\sigma\}\in\text{solutions}} \frac{\mathcal{I}(\{\sigma\},\{s,\epsilon\})}{\det'\Phi}. \tag{2.25}$$

We remark two properties regarding the terms found in this expression. First, each term is not (necessarily) local, but rather the whole summed expression is. That is, there may be terms that do not map to any contribution coming from a Feynman diagram, but the whole expression is equivalent to a sum of Feynman diagrams contributions, each of which is local by construction.

Secondly, in the previous sum each contribution is gauge invariant on its own. This is reminiscent of the decomposition of an amplitude in terms of partial amplitudes, in which case the full amplitude also decomposes into a sum of terms, each of which is gauge invariant on their own. As we shall see, for Yang-Mills theory we can perform the decomposition (2.25) and the partial amplitudes decomposition simultaneously. For gravity, we would only have the gauge invariant decomposition (2.25).

2.3 Tree Level Amplitudes from CHY

In this section we summarize the main theories that can be written in the Cachazo-He-Yuan formalism.

2.3.1 The Building Blocks

The first building block that we introduce is the Parke-Taylor factor

$$\mathrm{PT}_n[\alpha] = \frac{1}{(\sigma_{\alpha(1)}-\sigma_{\alpha(2)})(\sigma_{\alpha(2)}-\sigma_{\alpha(3)})\cdots(\sigma_{\alpha(n)}-\sigma_{\alpha(1)})}\ , \tag{2.26}$$

where α is a permutation of n elements, and whenever no confusion arises we write $\mathrm{PT}_n[\alpha]$ just as $\mathrm{PT}[\alpha]$. The reason why we consider building blocks that depend on an ordering is because we are going to use them to construct color-ordered amplitudes rather than the full amplitudes themselves. [3]

One can build a second quantity that allows to introduce particles with integer spin in the CHY formalism from the following $2n \times 2n$ matrix, which decomposes in block form:

$$\Psi(k,\epsilon,\sigma) = \begin{pmatrix} A & -C^t \\ C & B \end{pmatrix}, \tag{2.27}$$

where

[3] A general tree amplitude associated to a group G can always be expanded as $A_n = \sum_\sigma A_n[1\sigma(2...n)]Tr[T^{a_1}T^{\sigma(a_2}...T^{a_n)}]$, where T^a are the generators of the group G. The (gauge invariant) quantities $A_n[1\sigma(2...n)]$ are called color-ordered (or partial) amplitudes. For more details see [9].

$$A_{ab} = \frac{k_a \cdot k_b}{\sigma_a - \sigma_b}, \qquad B_{ab} = \frac{\epsilon_a \cdot \epsilon_b}{\sigma_a - \sigma_b}, \qquad C_{ab} = \frac{\epsilon_a \cdot k_b}{\sigma_a - \sigma_b} \tag{2.28}$$

$$A_{aa} = 0, \qquad B_{aa} = 0, \qquad C_{aa} = -\sum_{\substack{c=1\\c\neq a}}^{n} \frac{\epsilon_a \cdot k_c}{\sigma_a - \sigma_c}. \tag{2.29}$$

Here ϵ_a corresponds to the polarization vector of the a-th particle, which satisfies the transversality condition $\epsilon_a \cdot k_a = 0$ (for each a). We stress the following property of the matrix Ψ:

Property 2.3.1. *The matrix Ψ has (at most) rank $2n-2$.*

Proof. Notice that Ψ has the following two eigenvectors with zero eigenvalue:

$$v_1 = (1, \ldots, 1, 0, \ldots, 0), \qquad v_\sigma = (\sigma_1, \ldots, \sigma_n, 0, \ldots, 0). \tag{2.30}$$

That v_1 has zero eigenvalue follows easily from the scattering equations and the definition of the entries in (2.27). That v_σ has zero eigenvalue can be seen as follows:

$$\sum_{\substack{b=1\\b\neq a}}^{n} \frac{k_a \cdot k_b}{\sigma_a - \sigma_b}\sigma_b = -\sum_{\substack{b=1\\b\neq a}}^{n} k_a \cdot k_b + \sigma_a \left(\sum_{\substack{b=1\\b\neq a}}^{n} \frac{k_a \cdot k_b}{\sigma_a - \sigma_b} \right) = 0,$$

for the entries in the A matrix, and

$$\sum_{\substack{b=1\\b\neq a}}^{n} \frac{\epsilon_a \cdot k_b}{\sigma_a - \sigma_b}\sigma_b - \sigma_a \sum_{\substack{b=1\\b\neq a}}^{n} \frac{\epsilon_a \cdot k_b}{\sigma_a - \sigma_b} = -\epsilon_a \cdot \sum_{\substack{b=1\\b\neq a}}^{n} k_b = \epsilon_a \cdot k_a = 0,$$

for the entries in the C matrix. Here, we have used momentum conservation, the null condition $k_a^2 = 0$ (for each a), the scattering equations, and that the polarization vector is transverse $\epsilon_a \cdot k_a = 0$. ■

A natural scalar quantity associated to a $2n \times 2n$ matrix T_n corresponds to its determinant. When one considers an anti-symmetric matrix such as the one in (2.27), it is well-known that the determinant is actually a perfect square, so we may instead consider the building block:

$$\mathrm{Pf}(T_n) = \frac{1}{2^n n!} \sum_{\sigma} \mathrm{sgn}(\sigma) \prod_{i=1}^{n} T_{\sigma(2i-1),\sigma(2i)}, \tag{2.31}$$

where σ is a permutation of $2n$ elements and $\mathrm{sgn}(\sigma) = +1$ when σ is an even permutation while $\mathrm{sgn}(\sigma) = -1$ for an odd permutation. This is a quantity known as the Pfaffian of T_n, and is such that $\det T_n = (\mathrm{Pf}\ T_n)^2$. When an anti-symmetric matrix T is odd-dimensional its determinant vanishes and we have that $\mathrm{Pf}\ T = 0$.

Since the matrix Ψ has (at most) rank $2n-2$, its Pfaffian vanishes trivially. We build a non-trivial quantity by removing two rows and columns i, j such that $1 \leq i < j \leq n$, after which we take the Pfaffian of the remaining matrix Ψ^{ij}_{ij}. We multiply by an additional factor to ensure permutation invariance in the rows and columns removed:

$$\mathrm{Pf}'\big[\Psi\big] = \frac{(-1)^{i+j}}{\sigma_j - \sigma_i}\mathrm{Pf}\big[\Psi^{ij}_{ij}\big]. \tag{2.32}$$

We call this construction the *reduced Pfaffian* of Ψ. We now list and prove several properties that the reduced Pfaffian satisfies which lead to its consideration as a building block for amplitude formulae.

Property 2.3.2 *The reduced Pfaffian* $\mathrm{Pf}'\big[\Psi\big]$ *is independent of the choice of the rows and columns* i, j *removed.*

Proof. First, note that changing two rows and two columns of a matrix changes the Pfaffian by a sign. If we exchange i and j, then the rows and columns $i+n$ and $j+n$ in Ψ^{ij}_{ij} get interchanged and the Pfaffian is modified by a sign. This sign is however cancelled by the prefactor in (2.32) and the reduced Pfaffian is invariant. When we exchange a and b different than i and j, then we have to exchange rows and columns a, b as well as rows and columns $a+n$, $b+n$. The two minus signs cancel and the reduced Pfaffian is invariant.

Let us show now that the Pfaffians of Ψ^{12}_{12} and Ψ^{13}_{13} are equal (by the previous paragraph, showing that this is the case is sufficient to complete the proof). Notice that $(\sigma_1 - \sigma_3)\mathrm{Pf}\Psi^{12}_{12}$ is the same as the Pfaffian of a matrix constructed from Ψ^{12}_{12} with the first row and column multiplied by $(\sigma_1 - \sigma_3)$. We further compute over this matrix by adding to its first row and column the sum of each $(p-2)$-th row and column multiplied by $(\sigma_1 - \sigma_p)$, for $p = 4 \ldots n$ (which is allowed since the Pfaffian is invariant under the simultaneous addition of multiples of rows and columns). We perform an analogous construction over $(\sigma_1 - \sigma_2)\mathrm{Pf}\Psi^{13}_{13}$. It is straightforward to check that both constructions agree in the support of the scattering equations, up to a sign in the first row/column. We can pull out this sign out of the Pfaffian and write $(\sigma_1 - \sigma_3)\mathrm{Pf}\Psi^{12}_{12} = -(\sigma_1 - \sigma_2)\mathrm{Pf}\Psi^{13}_{13}$. Thus, we conclude that the reduced Pfaffian is independent of the choice of i and j. ■

Property 2.3.3 *The reduced Pfaffian* $\mathrm{Pf}'\big[\Psi\big]$ *is covariant with weight 2 under* $\mathrm{SL}(2, \mathbb{C})$.

Proof. This property is clear when we use (2.10) over each entry of Ψ. For most entries is obvious that they transform as $\Psi_{ab} \to (\gamma\sigma_a + \delta)(\gamma\sigma_b + \delta)\Psi_{ab}$, although the C_{aa} entries are a bit less trivial:

$$C_{aa} = -\sum_{\substack{b=1 \\ c\neq b}}^{n} \frac{\epsilon_a \cdot k_b}{\sigma_a - \sigma_b} \longrightarrow -\sum_{\substack{b=1 \\ b\neq a}}^{n} \frac{\epsilon_a \cdot k_b}{\sigma_a - \sigma_b}(\gamma\sigma_a + \delta)(\gamma\sigma_b + \delta)$$

$$= (\gamma\sigma_a + \delta)^2 C_{aa} - \gamma(\gamma\sigma_a + \delta)\epsilon_a \cdot \sum_{\substack{b=1 \\ b\neq a}}^{n} k_b = (\gamma\sigma_a + \delta)^2 C_{aa}, \tag{2.33}$$

where in the last equality we used momentum conservation and that $\epsilon_a \cdot k_a = 0$. Pulling out of the Pfaffian the factors $(\gamma\sigma_a + \delta)$ appearing in each row and column a, $a+n$, we arrive at the desired property. ■

Property 2.3.4 *The reduced Pfaffian* $\mathrm{Pf}'\big[\Psi\big]$ *is gauge invariant.*

Proof. Let us perform the replacement $\epsilon_a \to k_a$. Since

$$C_{aa} = -\sum_{\substack{b=1 \\ c\neq b}}^{n} \frac{\epsilon_a \cdot k_b}{\sigma_a - \sigma_b} \longrightarrow \sum_{\substack{b=1 \\ c\neq b}}^{n} \frac{k_a \cdot k_b}{\sigma_a - \sigma_b} = 0, \tag{2.34}$$

by the scattering equations, the previous replacement transforms Ψ into a matrix where the columns a and $a+n$ are identical. Hence, the new Pfaffian vanishes, which implies the gauge invariance of the corresponding amplitude. ■

Property 2.3.5 *The reduced Pfaffian* $\mathrm{Pf}'\big[\Psi\big]$ *is multi-linear in the polarization vectors* ϵ_a*.*

Proof. This follows easily from the following Pfaffian expansion of a $2n \times 2n$ anti-symmetric matrix T (analogous to the determinant expansion in minors):

$$\mathrm{Pf}(T) = \sum_{\substack{b=1 \\ b\neq a}}^{2n} (-1)^{a+b+1+\theta(a-b)} t_{ab} \mathrm{Pf}(T_{\hat{a}\hat{b}}), \tag{2.35}$$

where the index a can be selected arbitrarily, θ is the step function, and $T_{\hat{a}\hat{b}}$ is the T matrix with rows and columns a and b removed. Choosing the index a in Ψ so that the respective row/column contains the corresponding polarization vector ϵ_a, we see that all polarization vector factorize and we get a multi-linear expression. ■

The previous observations immediately suggest that we use $\mathrm{Pf}'\big[\Psi\big]$ as a building block to construct amplitude formulae for theories with a gauge invariance. Of course, as we shall see this suggestion is indeed correct.

A third building block comes from considering the following $n \times n$ matrix, corresponding to the first block defining the matrix Ψ in (2.27):

$$\mathrm{A}_{ab} = \begin{cases} \frac{k_a \cdot k_b}{\sigma_a - \sigma_b} & \text{if} \quad a \neq b, \\ \\ 0 & \text{if} \quad a = b. \end{cases} \tag{2.36}$$

This matrix inherits properties analog to the ones that Ψ satisfies; namely, in the support of the scattering equations A has at least two eigenvectors with zero eigenvalue:

$$v_1' = (1, \ldots, 1), \qquad v_\sigma' = (\sigma_1, \ldots, \sigma_n), \tag{2.37}$$

and we build a non-trivial half-integrand by constructing the reduced Pfaffian of A:

$$\mathrm{Pf}'\mathrm{A} = \frac{(-1)^{i+j}}{\sigma_j - \sigma_i} \mathrm{Pf}\big[\mathrm{A}_{ij}^{ij}\big]. \tag{2.38}$$

In a way completely analogous to the case of Ψ, this expression is permutation invariant and covariant with weight 1 under a $\mathrm{SL}(2, \mathbb{C})$ transformation. Therefore, two powers of this object are needed to construct a half integrand.

Finally, we can introduce another building block that is independent of momentum or polarization from the matrix

$$\chi_{a,b} = \begin{cases} \frac{\delta^{I_a, I_b}}{\sigma_a - \sigma_b} & \text{if} \quad a \neq b, \\ 0 & \text{if} \quad a = b, \end{cases} \tag{2.39}$$

and taking the Pfaffian $\mathrm{Pf}\,\chi$. Here we have allowed the particles to be associated to a gauge group, so the delta is non-zero only when the particles carry the same flavor index I_a (in other words, we may think of this delta as the trace of two generators of the group $\mathrm{Tr}(T^a T^b) = \delta^{ab}$). It is straightforward to see that this building block has weight 1 under $\mathrm{SL}(2, \mathbb{C})$.

2.3.2 The Amplitudes

The first theory that one can write in CHY form arises from considering the reduced Pfaffian $\mathrm{Pf}'[\Psi]$ and take it to be a half-integrand. Then, a full integrand is given by two copies of $\mathrm{Pf}'[\Psi]$:

$$M_n = \int d\mu_n \, \mathrm{Pf}' \, \Psi_n(k, \epsilon, \sigma) \, \mathrm{Pf}' \, \Psi_n(k, \tilde{\epsilon}, \sigma). \tag{2.40}$$

Notice that the polarization vectors can be taken to be in principle different in each factor. The symmetric-traceless combination of the polarization vectors in this formula gives rise to a closed formula for the tree level n-graviton amplitude in Einstein gravity. One can see here one of the first great advantages of the CHY formalism: one is able to write a formula valid at all multiplicities and study directly general properties of the amplitudes. Notice that this happens in sharp contrast to the standard Feynman-diagrammatic procedure based on the perturbative expansion of the Einstein-Hilbert Lagrangian, and which yields an infinite number of interaction vertices! Meanwhile, the trace part and anti-symmetric combinations give rise to dilaton, and anti-symmetric field amplitudes respectively.

The tree level n-gluon amplitude in Yang-Mills theory can be written as a product of a Parke-Taylor factor and $\mathrm{Pf}'[\Psi]$:

$$A_n^{YM}[\alpha] = \int d\mu_n \, \mathrm{PT}[\alpha] \, \mathrm{Pf}' \, \Psi_n(k, \epsilon, \sigma). \tag{2.41}$$

A proof of the correctness of this formula was given in [61], based on BCFW recursion relations [5].

Next, we can combine two copies of the Parke-Taylor factor $\mathrm{PT}[\alpha]$ with two different orderings α and β, and we arrive at

$$m[\alpha|\beta] = \int d\mu_n \, \mathrm{PT}[\alpha] \, \mathrm{PT}[\beta]. \tag{2.42}$$

The double-color-ordered amplitudes computed by this formula correspond to the amplitudes of the biadjoint scalar theory, where the scalars interact according to an interaction vertex

$$- f_{abc} f_{a'b'c'} \phi^{aa'} \phi^{bb'} \phi^{cc'}, \tag{2.43}$$

with f being the structure constants of $U(N)$ and $U(\tilde{N})$ groups. This scalar theory is interesting on its own right, being directly related to the Kawai-Lewellen-Tye (KLT) kernel [15, 16] which gives rise to the double copy relations between Einstein gravity and Yang-Mills theories, as we shall see in 2.4.1. In addition, it also has has an elegant relation to a polytope structure known as the kinematic associahedron [40]. At this point, however, we are just thinking of this theory as just one more CHY-theory that we can construct from the previous building blocks.

We can use now the third building block (2.38) and combine it with any of the other possibilities to see if what we get describes a meaningful theory. First, let us combine one Parke-Taylor factor and two powers of this reduced Pfaffian:

$$A_n^{\text{NLSM}}[\alpha] = \int d\mu_n \; \text{PT}[\alpha] \, \left[\text{Pf}'\text{A}(s,\sigma)\right]^2. \tag{2.44}$$

It can be checked (see [15]) that the previous amplitudes match the ones coming from the following Lagrangian describing a theory of massless scalars, flavored under a $U(N)$ group:

$$\mathcal{L}_{\text{NLSM}} = \frac{1}{8\lambda^2}\text{Tr}(\partial_\mu U^\dagger \partial^\mu U), \qquad U = (\mathbb{I} + \lambda\Phi)(\mathbb{I} - \lambda\Phi)^{-1}. \tag{2.45}$$

This Lagrangian is known as the chiral Lagrangian (in Cayley parametrization), and it appears in the context of effective field theories describing the low energy behavior of nuclear interactions (e.g. the scattering of pions). The theory obtained is called the $U(N)$ nonlinear-sigma-model (NLSM). As it is straightforward to see from the fact that $\text{Pf}\, A_n = 0$ when n is odd, all amplitudes at odd n vanish (which signals a broken symmetry). An obvious implication of this fact is a phenomenon known as the Adler's zero [67], which consists in that if we take a single soft limit for n even, we obtain a vanishing result. As we shall see in subsection 2.4.3 there is more to this limit than this apparent trivial result, and the novel features are manifest when writing these amplitudes in CHY-form.

Multiplying the half-integrands (2.38) and (2.32) one gets

$$A_n^{BI} = \int d\mu_n \, \left[\text{Pf}'\text{A}(s,\sigma)\right]^2 \text{Pf}' \, \Psi(s,\epsilon,\sigma), \tag{2.46}$$

which is a closed formula for the scattering of photons in Born-Infeld theory (a non-linear generalization of Maxwell's electrodynamics in which photons self-interact), whose Lagrangian is

$$\mathcal{L}_{BI} = l^{-2}\sqrt{-\det(\eta_{\mu\nu} + lF_{\mu\nu})} - l^{-2}. \tag{2.47}$$

it is easy to see that this theory also presents the Adler's zero phenomenon.

Another CHY integrand can easily be found taking four powers of (2.38). The theory obtained corresponds to a so-called Galileon theory with a special $\mathbb{Z}_2$-symmetry, and so is dubbed a special Galileon theory:

$$A_n^{sGal} = \int d\mu_n \, \left[\text{Pf}'\text{A}(s,\sigma)\right]^4. \tag{2.48}$$

We can now consider the last of the building blocks considered in the last subsection. This last building block is special in that it will let us consider theories where more than one different kind of boson is interacting. For instance

$$A_n^{YMS}[\alpha] = \int d\mu_n \, \Big[\mathrm{PT}[\alpha]\mathrm{Pf}[\chi]_{:s}\mathrm{Pf}'\,[\Psi]_{:\hat{s}}\Big], \tag{2.49}$$

where $\mathrm{Pf}[\Psi]_{:\hat{s}} = \mathrm{Pf}[\Psi]_{g,s:g}$ is the minor of Ψ obtained by deleting rows and columns with scalar labels in the second block of Ψ, corresponds to a theory of gluons interacting with scalars according to the Lagrangian

$$\mathcal{L}^{YMS} = -\frac{1}{2}\mathrm{Tr}(D_\mu\Phi^I D^\mu\Phi^I) - \frac{1}{4}\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}) - \frac{g^2}{4}\mathrm{Tr}\Big([\Phi^I,\Phi^J]^2\Big). \tag{2.50}$$

Similarly

$$A_n^{DBI} = \int d\mu_n \, \Big[[\mathrm{Pf}'\mathrm{A_n}]^2\mathrm{Pf}[\chi]_{:s}\mathrm{Pf}'\,[\Psi]_{:\hat{s}}\Big] \tag{2.51}$$

computes amplitudes for the Dirac-Born-Infeld theory, whose Lagrangian is given by

$$\mathcal{L}^{DBI} = l^{-2}\sqrt{-\det\left(\eta_{\mu\nu} + l^2\partial_\mu\phi\partial_\nu\phi + lF_{\mu\nu}\right)} - l^{-2}. \tag{2.52}$$

Finally, Einstein-Maxwell amplitudes can be found by the CHY formula

$$A_n^{EM} = \int d\mu_n \, \Big[\mathrm{Pf}'\,\Psi(\epsilon)\mathrm{Pf}[\chi]_{:s}\mathrm{Pf}'\,[\Psi]_{:\hat{s}}(\tilde{\epsilon})\Big], \tag{2.53}$$

where in this case we must think of the gauge group as $U(1)^M$ since it is associated to the photons interacting in Einstein-Maxwell theory.

2.4 Applications

Apart from the obvious fact that we can compute directly amplitudes using the CHY formulae, there are several further applications of the formalism. In this section we provide several such applications that were either first seen through CHY, or that are particularly interesting to understand under this formalism. See [15] for a more thorough analysis of these applications as well as other ones.

2.4.1 KLT Relations

Something very special happens when an integrand can be split into two half-integrands, each with $\mathrm{SL}(2,\mathbb{C})$ weight 2 in each σ_a:

$$\mathcal{I}(k,\epsilon,\tilde{\epsilon},\sigma)=\mathcal{I}^{(L)}(k,\epsilon,\sigma)\mathcal{I}^{(R)}(k,\tilde{\epsilon},\sigma). \tag{2.54}$$

In order to see what happens in this case let us take the following vectors defined over the $(n-3)!$-dimensional space of solutions to the scattering equations:

$$e_i^{(I)}=\frac{\mathcal{I}_n^{(I)}(\sigma^{(i)})}{\det'\Phi(\sigma^{(i)})},\qquad i=1,\ldots,(n-3)!,\quad I=L,R. \tag{2.55}$$

In this case we can rewrite (2.25) as the sum

$$\mathcal{A}_n=\sum_{i,j=1}^{(n-3)!}\det{}'\Phi(\sigma^{(i)})\delta_{i,j}e_i^{(L)}e_j^{(R)}=e^{(L)t}\,\mathrm{D}\,e^{(R)}, \tag{2.56}$$

where we introduced the diagonal matrix $\mathrm{D}_{ii}=\det'\Phi(\sigma^{(i)})$.

Now, we introduce two vectors over an auxiliary $(n-3)!$-dimensional space: $L(\sigma)$ and $R(\sigma)$, with their entries being (arbitrary) rational functions of the σ_a's, and we ask for them to have weight 2 under $\mathrm{SL}(2,\mathbb{C})$. Further, we can construct two $(n-3)!\times(n-3)!$ matrices L,R by evaluating L or R on each solution to the scattering equations:

$$\mathrm{L}_\alpha^{(i)}=L(\sigma^{(i)})_\alpha,\qquad \mathrm{R}_\alpha^{(i)}=R(\sigma^{(i)})_\alpha. \tag{2.57}$$

With the matrices D, L, and R constructed, we can build the matrix

$$\mathrm{m}_{\alpha,\beta}=(\mathrm{R}\ \mathrm{D}^{-1}\ \mathrm{L})_{\alpha\beta}=\sum_{i=1}^{(n-3)!}\frac{R_\alpha(\sigma^{(i)})L_\beta(\sigma^{(i)})}{\det'\Phi(\sigma^{(i)})}=\int d\mu_n R_\alpha(\sigma)L_\beta(\sigma), \tag{2.58}$$

and by this definition, we can solve for D, $\mathrm{D}=\mathrm{L}\,(\mathrm{m}^{-1})\,\mathrm{R}$, and introduce back in (2.56)

$$\mathcal{A}_n=e^{(L)t}\,\mathrm{L}\,(\mathrm{m}^{-1})\,\mathrm{R}\,e^{(R)}=\sum_{\alpha,\beta}\mathcal{A}^{(L)}(\alpha)\,(\mathrm{m}^{-1})_{\alpha\beta}\,\mathcal{A}^{(R)}(\beta), \tag{2.59}$$

where we defined

$$\mathcal{A}^{(L)}(\alpha) = \int d\mu_n \, L_\alpha(\sigma) \, \mathcal{I}_n^{(L)}(k, \epsilon, \sigma), \tag{2.60}$$

and similarly for $\mathcal{A}^{(R)}(\alpha)$. We obtain a clear pattern: we have decomposed the amplitude $\mathcal{A}_n$ into a sum (2.59) of terms, each of which takes the form of a product of CHY-amplitudes (2.60), connected by the inverse of (2.58), which also takes the form of a CHY amplitude! Indeed, let us make this construction more precise building an explicit example. Take, for instance

$$L_\alpha = \frac{1}{(\sigma_1 - \sigma_{\alpha(2)})(\sigma_{\alpha(2)} - \sigma_{\alpha(3)}) \ldots (\sigma_{\alpha(n-2)} - \sigma_{n-1})(\sigma_{n-1} - \sigma_n)(\sigma_n - \sigma_1)}, \tag{2.61}$$

$$R_\alpha = \frac{1}{(\sigma_1 - \sigma_{\alpha(2)})(\sigma_{\alpha(2)} - \sigma_{\alpha(3)}) \ldots (\sigma_{\alpha(n-2)} - \sigma_n)(\sigma_n - \sigma_{n-1})(\sigma_{n-1} - \sigma_1)}, \tag{2.62}$$

where α is a permutation of $(n-3)!$ elements. Notice that we can then recognize the auxiliary space constructed before precisely as this space of permutations. We can also recognize that in this case the matrix (2.58)

$$\mathrm{m}_{\alpha\beta} = \int d\mu_n \left[\frac{1}{\sigma_{1,\alpha(2)} \cdots \sigma_{\alpha(n-2),n-1} \sigma_{n-1,n} \sigma_{n,1}} \right] \times \left[\frac{1}{\sigma_{1,\beta(2)} \cdots \sigma_{\beta(n-2),n} \sigma_{n,n-1} \sigma_{n-1,1}} \right] \tag{2.63}$$

is nothing else than a matrix whose entries are biadjoint scalar amplitudes (2.42) with different choices of orderings α,β. Now, to see the first manifestation of the procedure just constructed let us take both left and right half-integrands to be the reduced Pfaffians

$$\mathcal{I}_n^{(L,R)}(k, \epsilon, \sigma) = \mathrm{Pf}'\big[\Psi\big]. \tag{2.64}$$

As it is straightforward to see, this is the case where $\mathcal{A}_n$ in (2.56) is a n-point graviton amplitude. The decomposition (2.59) then gives

$$M_n = \sum_{\alpha,\beta} A_n^{YM}[\alpha] \, (\mathrm{m}^{-1})_{\alpha\beta} \, A_n^{YM}[\beta], \tag{2.65}$$

which allows us to see graviton amplitudes as a combination of Yang-Mills amplitudes. We use the symbol $\overset{\text{KLT}}{\otimes}$ whenever a splitting of this form takes place. For instance, in this particular case we write

$$\text{Gravity} = \text{YM} \overset{\text{KLT}}{\otimes} \text{YM}. \tag{2.66}$$

The KLT-relations are known from long ago in the context of string theory [16], where they provide a relation between open and closed string amplitudes, and in the field theory limit they reduce to the relation (2.65) between gravity and Yang-Mills amplitudes. Let us notice, however, that the CHY approach provides us with an additional piece of insight that is not obvious from this original point of view: the fact that the KLT-kernel is actually controlled by an underlying physical theory; namely, biadjoint scalar theory.

Clearly, a similar decomposition takes place if we identify the corresponding half-integrands. Using the different CHY-integrals described in subsection 2.3, we arrive at the following relations between vastly different theories:

- $$\text{BI} = \text{NLSM} \overset{\text{KLT}}{\otimes} \text{YM}, \tag{2.67}$$
- $$\text{sGal} = \text{NLSM} \overset{\text{KLT}}{\otimes} \text{NLSM}, \tag{2.68}$$
- $$\text{EM} = \text{YMS} \overset{\text{KLT}}{\otimes} \text{YM}, \tag{2.69}$$
- $$\text{DBI} = \text{NLSM} \overset{\text{KLT}}{\otimes} \text{YMS}, \tag{2.70}$$
- $$\text{EYM} = \text{gen.YMS} \overset{\text{KLT}}{\otimes} \text{YM}. \tag{2.71}$$

Except for the last line, all KLT-theories here listed were written down in subsection 2.3. The last line provide a KLT-decomposition for Einstein-Yang-Mills theory in terms of Yang-Mills amplitudes and a certain generalization of Yang-Mills-Scalar theory. The details for considering these theories in terms of CHY can be found in [15].

2.4.2 Soft Theorems

Another property that is straightforward to study with the CHY formalism is the behaviour of different amplitudes in the soft limit. To see how is this the case let us study the soft limit in the case of Yang-Mills theory. We take $k_n = \tau \hat{k}_n$ with $\tau \to 0$ and $\hat{k}_n$ fixed, and the scattering equations decouple as in (2.15). We compute the reduced Pfaffian of Ψ by choosing the rows and columns i and j to remove as different than n, and further, expand the Pfaffian according to (2.35) with $a = n$. We get

$$\text{Pf}\big[\Psi^{ij}_{ij}\big] \longrightarrow C_{nn}\, \text{Pf}\big[\Psi^{ijn(2n)}_{ijn(2n)}\big]. \tag{2.72}$$

The last matrix $\Psi^{ijn(2n)}_{ijn(2n)}$ is equal to the matrix obtained from $n-1$ particles removing rows and columns i and j. Now, take an ordering α with labels (a,n,b) neighboring n. The integral we have to evaluate to obtain the soft factor is then:

$$\oint_\Gamma d\sigma_n \frac{\sum_{a\neq n} \frac{\epsilon_n\cdot k_a}{\sigma_{na}}}{\sum_{a\neq n} \frac{\hat{k}_n\cdot k_a}{\sigma_{na}}} \frac{\sigma_{a,b}}{\sigma_{a,n}\sigma_{n,b}}, \tag{2.73}$$

where we wrote all delta functions as contour integrals, with the contour Γ enclosing the zeros defined by the denominator in the first factor (i.e. the n-th scattering equation in the soft limit). We evaluate this integral through the residue theorem, which picks the poles at $\sigma_n^* = \sigma_1, \sigma_{n-1}$ (there are no residues at infinity), and we get

$$\mathcal{S}^{\mathrm{YM}}(a,n,b) = \frac{\epsilon_n\cdot k_a}{\hat{k}_n\cdot k_a} - \frac{\epsilon_n\cdot k_b}{\hat{k}_n\cdot k_b}. \tag{2.74}$$

A similar computation can be done for graviton amplitudes. Nevertheless, let us perform this computation using the KLT-decomposition (2.65) as it will be useful for us later in the next subsection (and for the sake of diversity). We start from

$$A_n^{\mathrm{GR}} = \sum_{a,b=2}^{n-2} \sum_{\omega,\tilde{\omega}\in S_{n-4}} A_n^{\mathrm{YM}}[1,\omega,n-1,a,n] \left[\mathrm{m}^{-1}\right] A_n^{\mathrm{YM}}[1,\tilde{\omega},n-1,n,b], \tag{2.75}$$

where for compactness we have omitted the orderings in the kernel m^{-1}, and it is implicit that they are the same as the orderings in the respective amplitudes. As can be seen, we need the behavior of biadjoint amplitudes in the soft limit. Fortunately for us, this behaviour is quite simple for the orderings considered [21]:

$$m_n[1,\omega,n-1,a,n|1,\tilde{w},n-1,n,b] = \tau^{-1}\frac{\delta_{ab}}{\hat{s}_{ab}} m_{n-1}[1,\omega,n-1,a|1,\tilde{w},n-1,a]. \tag{2.76}$$

The graviton soft limit can then be recovered from (2.76) and (2.74) over (2.75), obtaining:

$$\begin{aligned} A_n^{\mathrm{GR}} \longrightarrow & \tau^{-1}\sum_{a=2}^{n-2} \hat{s}_{an}\left(\frac{\epsilon_n\cdot k_a}{\hat{k}_n\cdot k_a} - \frac{\epsilon_n\cdot k_1}{\hat{k}_n\cdot k_1}\right)\left(\frac{\tilde{\epsilon}_n\cdot k_{n-1}}{\hat{k}_n\cdot k_{n-1}} - \frac{\tilde{\epsilon}_n\cdot k_a}{\hat{k}_n\cdot k_a}\right)\left[A_{n-1}^{\mathrm{YM}} \overset{\mathrm{KLT}}{\otimes} A_{n-1}^{\mathrm{YM}}\right], \\ = & -\tau^{-1}\left(\sum_a^{n-1} \frac{\epsilon_{\mu\nu}k_a^\mu k_a^\nu}{\hat{k}_n\cdot k_a}\right) A_{n-1}^{\mathrm{GR}}. \end{aligned} \tag{2.77}$$

2.4.3 Extension of Theories from Soft Limits

We conclude this chapter by showing a simple but striking and insightful application first recognized through CHY [21], which consists in taking a soft limit in an apparently trivial case: a vanishing soft limit. For this purpose let us take the NLSM CHY-formula (2.44), and perform the soft limit over particle n as in the previous subsection. Choose rows and columns 1 and $n-1$ when computing the reduced Pfaffian, and use as in the previous subsection (2.35) with $a = n$:

$$\mathrm{Pf}'\mathrm{A} = \frac{\tau}{\sigma_{1,n-1}} \sum_{b=2}^{n-2} (-)^b \frac{\hat{s}_{bn}}{\sigma_{bn}} \left[\mathrm{Pf}\mathrm{A}^{1,n-1}_{1,n-1}\right]_{\hat{b}\hat{n}}. \tag{2.78}$$

As we know from (2.15), the scattering equation for the n-th particle contributes with an additional factor of τ in the denominator: $E_n \to \tau \hat{E}_n$, and is clear then that the soft limit vanishes. Write the Parke-Taylor factor in the following form:

$$\mathrm{PT}[\mathbb{I}_n] = \frac{\sigma_{n-1,1}}{\sigma_{n-1,n}\sigma_{n,1}} \mathrm{PT}[\mathbb{I}_{n-1}]. \tag{2.79}$$

Since in the soft limit the scattering equations for particles different than n decouple from the n-th equation, we can consistently separate the integral over the σ_n puncture and study the soft limit. Also, we write the delta function localizing over the scattering equations as a contour integral. We have then that in the soft limit

$$A_n^{\mathrm{NLSM}}[\mathbb{I}_n] \xrightarrow[\tau\to 0]{} -\tau \int d\mu_{n-1} \mathrm{PT}[\mathbb{I}_{n-1}] \oint_\Gamma \frac{d\sigma_n}{\hat{E}_n} \frac{1}{\sigma_{n-1,n}\sigma_{n,1}\sigma_{1,n-1}} \left\{ \sum_{b=2}^{n-2} (-)^b \frac{\hat{s}_{bn}}{\sigma_{bn}} \left[\mathrm{Pf}\mathrm{A}^{1,n-1}_{1,n-1}\right]_{\hat{b}\hat{n}} \right\}^2, \tag{2.80}$$

where Γ encloses the poles in σ_n defined by $\hat{E}_n$. Evaluating the contour integral by deforming Γ (notice there are no poles at infinity), we pick poles at $\sigma_n^* = \sigma_2, \ldots, \sigma_{n-2}$. Thus, one gets:

$$A_n^{\mathrm{NLSM}}[\mathbb{I}_n] = \tau \sum_{a=2}^{n-2} \hat{s}_{an} A_{n-1}^{\mathrm{NLSM}\oplus\phi^3}[\mathbb{I}_{n-1}|n-1, a, 1] + \mathcal{O}(\tau^2), \tag{2.81}$$

where

$$A_n^{\mathrm{NLSM}\oplus\phi^3}[\alpha|\beta] = \int d\mu_n \, \mathrm{PT}_n[\alpha] \left(\mathrm{PT}_{|\beta|}[\beta] \, \mathrm{Pf}\mathrm{A}_{\bar{\beta}} \right). \tag{2.82}$$

Thus, we see that despite the non-linear-sigma-model has a vanishing soft limit, there is a non-trivial quantity associated to it. Furthermore, this quantity takes the form of a CHY-amplitude! To be more precise, the coefficient controlling the soft limit expansion corresponds to a theory of

NLSM scalars enhanced by interactions with biadjoint scalars. To see this notice that the particles participating in the interaction have the ordering α in common. Biadjoint scalars (described by two Parke-Taylor factors) are described by the additional ordering β in the second Parke-Taylor factor, and $\mathrm{A}_{\bar{\beta}}$ is a $|\bar{\beta}| \times |\bar{\beta}|$ matrix of the form (2.36), whose labels corresponds to the particles in the complement set of the labels β, and that describes the NLSM scalars.

Of course, we can apply a similar procedure to any theory with a vanishing soft limit such as the special Galileon (2.48) or Born-Infeld theories (2.46). In order to not over-extend ourselves let us outline this procedure (which shows some differences with the case just presented) only with Born-Infeld theory, since it is better physically motivated. Other cases then follow in an analogous way. We start from the following KLT-decomposition of Born-Infeld:

$$A_n^{\mathrm{BI}} = \sum_{a,b=2}^{n-2} \sum_{\omega,\tilde{\omega}\in S_{n-4}} A_n^{\mathrm{NLSM}}[1,\omega,n-1,a,n] \; \left[\mathrm{m}^{-1}\right] \; A_n^{\mathrm{YM}}[1,\tilde{\omega},n-1,n,b], \tag{2.83}$$

where as in (2.75) we omitted the orderings in m^{-1}. We now proceed similarly as in the previous subsection, applying the soft limit over the known expressions for NLSM, Yang-Mills and biadjoint amplitudes. The result is

$$\begin{aligned} A_n^{\mathrm{BI}} &\xrightarrow[\tau\to 0]{} \tau \sum_{a=2}^{n-2} \sum_{\substack{c=2\\ c\neq a}}^{n-1} \hat{s}_{an}\hat{s}_{cn} \mathcal{S}^{\mathrm{YM}}[n-1,n,a] \left(A_{n-1}^{\mathrm{NLSM}\oplus\phi^3}[a,c,1|\cdot] \overset{\mathrm{KLT}}{\otimes} A_{n-1}^{\mathrm{YM}}[\cdot] \right) + \mathcal{O}(\tau^2) \\ &= -\tau \sum_{a=2}^{n-2} \sum_{\substack{c=2\\ c\neq a}}^{n-1} \hat{s}_{an}\hat{s}_{cn} \left(\frac{\epsilon_n \cdot k_a}{\hat{k}_n \cdot k_a} - \frac{\epsilon_n \cdot k_b}{\hat{k}_n \cdot k_b} \right) A_n^{\mathrm{BI}\oplus\mathrm{YM}}(a,c,1) + \mathcal{O}(\tau^2), \end{aligned} \tag{2.84}$$

where we have again defined an extended theory. In this case, the extension of Born-Infeld is given by:

$$A^{\mathrm{BI}\oplus\mathrm{YM}}[\alpha] = \int d\mu_n \Big[\mathrm{PT}[\alpha] \; (\mathrm{Pf}\; \mathrm{A}_{\bar{\alpha}})^2 \Big] \mathrm{Pf}' \big[\Psi_n \big], \tag{2.85}$$

which one may interpret as a theory of Born-Infeld photons interacting as well with gluons.

Chapter III

A Supertwistor Description of Ambitwistor Strings

Arguably, the most striking fact about the CHY formulae is the fact that such formulae exist at all, and yet, the CHY formalism as originally conceived makes no reference to their origin. As we shall see, ambitwistor strings provide such origin in terms of a certain infinite tension limit of string theory. Consequently, we begin this chapter by briefly reviewing the general framework of ambitwistor strings. Then, we present an ambitwistor string model recently developed by Berkovits, Guillen, and Mason in [31] based on a twistorial description for the ten-dimensional massless superparticle (introduced by Berkovits in [32]). To define this twistor-ambitwistor string model, we present the Brink-Schwarz action for the ten-dimensional superparticle [68] and then proceed to introduce the necessary supertwistor variables in order to eventually arrive to the desired model. Finally, we work out the BRST generator of the twistor-ambitwistor model, which was an open problem from the original work [31]. We conclude this chapter with some discussions and ideas for further research.

3.1 A Summary on Ambitwistor Strings

Let us consider the bosonic ambitwistor string action given that it is the simplest case that contains all the ingredients that allows us to show how ambitwistor strings give rise to the CHY formulae (although note that in this specific case the theory considered is ill-defined [69]). As a motivation, let us begin showing how to arrive to the specific action we want to consider by an appropriate $\alpha' \to 0$ limit of the standard bosonic string. Write the bosonic string action in the following way:

$$S_{bos} = \frac{1}{2\pi\alpha'}\int_{\Sigma} \frac{1}{\sqrt{1-e'\bar{e}'}}\big(\partial X \cdot \bar{\partial} X + e'\partial X \cdot \partial X + \bar{e}'\bar{\partial} X \cdot \bar{\partial} X\big). \tag{3.1}$$

We perform now the following two operations: First, we re-scale the Lagrange multipliers e' and $\bar{e}'$ according to $e = \alpha' e'$ and $\bar{e} = \alpha'^{-2}\bar{e}'$. Secondly, we introduce Lagrange multipliers P and $\bar{P}$ so that we can write the action as

$$S_{bos} = \frac{1}{2\pi}\int_\Sigma \frac{1}{\sqrt{1-\alpha' e\bar{e}}}\big(P\cdot\bar{\partial}X + \alpha'\bar{P}\cdot\partial X - \alpha'^2 P\cdot\bar{P} + eP^2 + \alpha'\bar{e}\bar{P}^2\big), \tag{3.2}$$

which up to the re-scalings over e and $\bar{e}$ we see is equivalent to (3.1) after solving for P and $\bar{P}$. The bosonic ambitwistor string action is then obtained by taking $\alpha' \to 0$, which gives

$$\bar{S}_{bos} = \int P_m\bar{\partial}X^m - \frac{e}{2}P_m P^m. \tag{3.3}$$

An alternative way to arrive to this action, more important for our purposes, consists in taking the massless free particle action [4]

$$S = \int P_m\dot{X}^m - \frac{e}{2}P_m P^m, \tag{3.4}$$

and replacing time derivatives by anti-holomorphic derivatives $\bar{\partial}$ and promoting the worldline of the particle to a Riemann sphere. Thus, we have in this procedure an almost-algorithmic way of constructing ambitwistor actions by complexifying a massless particle action.

Just as the bosonic string, the bosonic ambitwistor action (3.3) is invariant under reparametrizations:

$$\delta X^m = v\partial X^m, \quad \delta P_m = \partial(vP_m), \quad \delta e = v\partial e - e\partial v. \tag{3.5}$$

However, unlike the bosonic string, it is separately invariant under the transformations generated by the massless constraint $P^2 = 0$:

$$\delta X^m = \alpha P^m, \quad \delta P_m = 0, \quad \delta e = \bar{\partial}\alpha. \tag{3.6}$$

The BRST operator associated to the previous constraints is thus given by (see appendix A):

$$Q = \oint cT + \frac{\tilde{c}}{2}P^2, \tag{3.7}$$

[4] It is straightforward to see that $S = \int P_\mu\dot{X}^\mu - \frac{e}{2}(P_\mu P^\mu - m^2)$ reduces to the more standard $S = m\int\sqrt{\dot{X}^2}$ after solving for the non-dynamical variables e and P. The advantage of using the former is that such version admits the massless limit $m \to 0$, (3.4).

where c and $\tilde{c}$ are the corresponding ghosts, and

$$T = P_m \partial X^m + c\partial b + 2(\partial c)b + \tilde{b}\partial\tilde{c} \tag{3.8}$$

is the standard contribution to the BRST generator associated to the Virasoro constraint. Also in parallel to the bosonic string, the BRST generator for the bosonic ambitwistor string is nilpotent $Q^2 = 0$ only when $d = 26$. Nilpotency in the supersymmetric case is achieved when $d = 10$.

Physical states in the ambitwistor string belong to different cohomology classes of the BRST operator Q. In particular, vertex operators, describing external states through operators via the operator-state correspondence, satisfy that $Q, V = 0$ then implying the linearized target space equations of motion. In the bosonic ambitwistor string we have that such operators are given by

$$c\tilde{c}V = c\tilde{c}P_m P_n \epsilon^{mn} e^{ik\cdot X}, \tag{3.9}$$

and the condition that vertex operators lie in the cohomology of Q imply that

$$k^2 = 0, \qquad \epsilon^{mn} k_n = 0, \tag{3.10}$$

while gauge invariances correspond to BRST-trivial states which corresponds to

$$\epsilon^{mn} = k^{(m}\epsilon^{n)}, \tag{3.11}$$

which we can identify as linearized diffeomorphism invariance. Integrated vertex operators are given by

$$\int_\Sigma \mathcal{V} = \int_\Sigma \bar{\delta}(k\cdot P)V = \int_\Sigma \bar{\delta}(k\cdot P) P_m P_n \epsilon^{mn} e^{ik\cdot X} \tag{3.12}$$

To see how the delta function appears we use a quick argument from [23]. We start noticing that when gauge-fixing the Lagrange multiplier e we add to the action a term (ch. 11 in [73]) $\{Q, \tilde{b}F(e)\}$, with $F(e)$ being a gauge-fixing functional. We would like to set $e = 0$, but notice that the transformations (3.6) change e just by a Dolbeault-exact term $\bar{\partial}\alpha$. Thus, we can just vary e within a fixed Dolbeault cohomology class and we are at most allowed to choose the gauge-fixing functional to be

$$F(e) = e - \sum_{r=1}^{n-3} s_r \mu_r,$$

where μ_r is a basis of the different cohomology classes. Integrating out the coefficients s_r we then obtain a delta function, and the integral extracts the residue of P^2 at the location of the vertex effectively leaving us with a δ-function that forces this residue to vanish, which is essentially the content of the delta function in (3.12).

A tree level scattering amplitude is computed in the ambitwistor worldsheet model with a prescription similar to the one in bosonic string theory: computing a worldsheet correlation function involving three fixed unintegrated vertex operators and arbitrarily many integrated ones:

$$\mathcal{A}_n = \left\langle c_1 \tilde{c}_1 V_1 c_2 \tilde{c}_2 V_2 c_3 \tilde{c}_3 V_3 \int \mathcal{V}_4 \cdots \int \mathcal{V}_n \right\rangle. \tag{3.13}$$

The c and $\tilde{c}$ ghosts then give rise to the Vandermonde determinants in (2.24), similarly as in the standard bosonic string theory (see section 6.3 in [26]). In order to see how the scattering equations (and thus, the CHY formulae) arise from this prescription, notice that we can incorporate the exponentials $e^{ik_a \cdot X}$ into the action in the previous correlation function. Therefore, effectively we have

$$\frac{1}{2\pi} \int_{\Sigma} P_m \bar{\partial} X^m + i \sum_{a=1}^{n} (k_a \cdot X)\, \delta^2(\sigma - \sigma_a). \tag{3.14}$$

Since this expression now contains the whole dependence on X, we can integrate such variable out. The zero modes then lead to an overall momentum conservation delta function $\delta(\sum k_a)$, while the rest of the modes give the relation

$$\bar{\partial} P^m = 2\pi i \sum_a k_a^m \, \delta^2(\sigma - \sigma_a). \tag{3.15}$$

This expression has the following unique solution on the Riemann sphere:

$$P^m(\sigma) = \sum_{a=1}^{n} \frac{k_a^m}{(\sigma - \sigma_a)}, \tag{3.16}$$

which is nothing else than the vector (2.2) that we already saw in subsection 2.1 led to the scattering equations! Indeed, using the delta functions appearing in the integrated vertex operators we readily see that the scattering amplitudes are localized over the scattering equations

$$\sum_{b \neq a} \frac{k_a \cdot k_b}{(\sigma_a - \sigma_b)} = 0. \tag{3.17}$$

Thus, we have found a tantalizing construction which naturally leads to the CHY formulae. Of course, we have not provided the specific integrands for the different theories presented in subsection 2.3. In particular, the specific theory corresponding to the bosonic ambitwistor string here considered does not seem to correspond to a physically reasonable theory (see [69]). However, although we do not consider other ambitwistor strings here, there exist different matter models in the ambitwistor action that do lead to CHY-expressions corresponding to the better physically motivated theories presented in subsection 2.3. See, for example, [24, 25, 70, 71].

3.2 The Ten-Dimensional Massless Superparticle with Supertwistor Variables

3.2.1 The Brink-Schwarz Superparticle

The Brink-Schwarz superparticle [68] is described by the following action:

$$S_{BS} = \int d\tau \Big[P_m\Big(\dot{X}^m - i\dot{\theta}\gamma^m\theta\Big) + \frac{1}{2}eP^2\Big], \tag{3.18}$$

where X^m is a $SO(1,9)$ vector, and P_m is the corresponding conjugate momentum to X^m, θ^α is a fermionic $SO(1,9)$ Majorana-Weyl spinor, and e is a Lagrange multiplier which enforces the massless condition $P^2 = 0$. The ten-dimensional Pauli matrices $(\gamma^m)_{\alpha\beta}$, $(\gamma^m)^{\alpha\beta}$ are real, symmetric, 16×16 matrices that satisfy

$$(\gamma^m)^{\alpha\beta}(\gamma^n)_{\beta\delta} + (\gamma^n)^{\alpha\beta}(\gamma^m)_{\beta\delta} = 2\eta^{mn}\delta^\alpha_\delta. \tag{3.19}$$

The Brink-Schwarz action is invariant under the following symmetries:

Property 3.2.1. *The Brink-Schwarz action is invariant under global supersymmetry:*

$$\delta\theta^\alpha = \epsilon^\alpha, \qquad \delta X^m = -i(\delta\theta\gamma^m\theta), \qquad \delta P^m = 0, \qquad \delta e = 0. \tag{3.20}$$

Proof. Indeed, we have:

$$\delta_{susy}S_{BS} = \int d\tau \Big[P_m\Big(\delta\dot{X}^m - i\delta\dot{\theta}\gamma^m\theta - i\dot{\theta}\gamma^m\delta\theta\Big)\Big], \tag{3.21}$$

but because the supersymmetry is global $\delta\dot{\theta} = 0$:

$$\delta_{susy}S_{BS} = \int d\tau \Big[P_m\Big(-i(\epsilon\gamma^m\dot{\theta}) - i\dot{\theta}\gamma^m\epsilon\Big)\Big] = 0, \tag{3.22}$$

as desired. ∎

Property 3.2.2. *The Brink-Schwarz action is invariant under (local) κ-symmetry:*

$$\delta\theta^\alpha = P^m(\gamma_m\kappa), \quad \delta X^m = -i(\theta\gamma^m\delta\theta), \quad \delta e = 4i\dot{\theta}^\alpha\kappa_\alpha, \quad \delta P^m = 0, \tag{3.23}$$

Proof. In this case we have:

$$\delta_\kappa S_{BS} = \int d\tau \Big[P_m\Big(-2i\dot{\theta}\gamma^m\delta\theta\Big) + \frac{1}{2}\delta e P^2\Big] \quad = \int d\tau\Big[P_m\Big(-2i\dot{\theta}\gamma^m P^n(\gamma_n\kappa)\Big) + 2i\dot{\theta}^\alpha\kappa_\alpha P^2\Big],$$

and using the Dirac algebra we can rewrite the first term:

$$= \int d\tau\Big[-2i\dot{\theta}\kappa P^2 + 2i\dot{\theta}\kappa P^2\Big], \tag{3.24}$$

which is trivially zero. ■

Property 3.2.3. *The Brink-Schwarz action is invariant under the gauge symmetry*

$$\delta e = \dot{\epsilon}, \quad \delta X^m = -\epsilon P^m, \quad \delta(\theta^\alpha, P_m) = 0. \tag{3.25}$$

Proof. In this case we have:

$$\begin{aligned}\delta S_{BS} &= \int d\tau\Big[P_m\Big(-\dot{\epsilon}P^m - \epsilon\dot{P^m}\Big) + \frac{1}{2}\dot{\epsilon}P^2\Big]\\ &= \int d\tau\Big[-\epsilon P_m\dot{P^m} - \frac{1}{2}\dot{\epsilon}P^2\Big] = \int d\tau\Big[-\frac{1}{2}\frac{d}{d\tau}(\epsilon P^2)\Big] = 0,\end{aligned} \tag{3.26}$$

provided we ignore boundary terms. ■

3.2.2 The Berkovits Superparticle

The ambitwistor string that we are going to consider is constructed from a description of the ten-dimensional massless superparticle introduced by Berkovits in [32], and that makes use of supertwistor variables $\mathcal{Z} = (\lambda^\alpha, w_\alpha, \psi_m)$, where ψ^m is a fermionic real ten-dimensional vector, and λ^α, w_α are 16 component bosonic spinors of opposite chirality.

The model is constructed from the crucial observation that we can solve the massless constraint $P^2 = 0$ by taking advantage of the ten-dimensional γ-matrix special identity (also valid in three, four, and six dimensions):

$$(\gamma^m)_{\alpha\beta}(\gamma_m)_{\gamma\delta} + (\gamma^m)_{\alpha\delta}(\gamma_m)_{\beta\gamma} + (\gamma^m)_{\alpha\gamma}(\gamma_m)_{\delta\beta} = (\gamma^m)_{\alpha(\beta}(\gamma_m)_{\gamma\delta)} = 0. \tag{3.27}$$

It is straightforward to see that the previous relation allows us to solve the massless constraint by writing P^m in terms of the bosonic spinors λ^α in the following way:

$$P^m = \lambda^\alpha \gamma^m_{\alpha\beta} \lambda^\beta. \tag{3.28}$$

A quick counting reveals that the left hand side has 9 degrees of freedom. So far, we have not constrained the spinors λ^α and it may appear we have an inconsistency as they would have 16 degrees of freedom. As a point of fact, to describe the superparticle, $\mathcal{Z}$ must satisfy the fermionic and bosonic constraints:

$$v = (\lambda\gamma^m\lambda)\psi_m = 0, \tag{3.29}$$

$$G^\alpha = (\lambda\gamma^m\lambda)(\gamma_m w)^\alpha - 2\lambda^\alpha(\lambda w) + 2i\psi^m\psi^n(\gamma_m\gamma_n\lambda)^\alpha = 0. \tag{3.30}$$

These constraints are not all independent. Indeed, let us compute $(\lambda\gamma^m G)$:

$$\begin{aligned}\lambda^\alpha\gamma^m_{\alpha\beta}G^\beta =&(\lambda\gamma^n\lambda)(\lambda\gamma^m\gamma_n w) - 2(\lambda\gamma^m\lambda)(\lambda w) + 2i\psi^p\psi^q(\lambda\gamma^m\gamma_p\gamma_q\lambda)\\ =&2(\lambda\gamma^m\lambda)(\lambda w) - (\lambda\gamma^n\lambda)(\lambda\gamma_n\gamma^n w) - 2(\lambda\gamma^m\lambda)(\lambda w)\\ &+ 4i\psi^m\psi^q(\lambda\gamma_q\lambda) - 2i\psi^p\psi^q(\lambda\gamma_p\gamma^m\gamma_q\lambda) = 4i\psi^m v,\end{aligned} \tag{3.31}$$

where we have used (3.19) in the second equality, and in the third equality we have used the 10D special identity (3.27) and noticed that the last term is zero due to symmetry over the indices p and q. That is, all in all, we have the following constraints over constraints:

$$H^m \equiv (\lambda\gamma^m G) - 4i\psi^m v = 0. \tag{3.32}$$

Furthermore, these constraints over constraints are not all independent, since the 10D special identity implies that

$$(\lambda\gamma^m\lambda)H_m = 0. \tag{3.33}$$

One has then a total of $16 - 10 + 1 = 7$ independent constraints, therefore matching the number of degrees of freedom in each side of (3.28).

The constraints (3.29) and (3.30) are first class and generate the following gauge transformations:

$$\begin{aligned}
\delta_\eta w_\alpha &= 2(\gamma^m\lambda)_\alpha(\eta\gamma_m w) - 2\eta_\alpha(\lambda w) - 2(\lambda\eta)w_\alpha + 2i\psi^m\psi^n(\eta\gamma_m\gamma_n)_\alpha \\
\delta_\eta\lambda_\alpha &= -(\gamma^m\eta)^\alpha(\lambda\gamma_m\lambda) + 2(\lambda\eta)\lambda^\alpha \\
\delta_\eta\psi^m &= \psi^n(\eta\gamma_n\gamma^m\lambda) - \psi^n(\eta\gamma^m\gamma_n\lambda) \\
\delta_\zeta w_\alpha &= 2\zeta(\gamma^m\lambda)_\alpha\psi^m \\
\delta_\zeta\psi^m &= \zeta(\lambda\gamma^m\lambda),
\end{aligned} \tag{3.34}$$

where η_α and ζ are bosonic $SO(1,9)$ spinors and fermionic scalar parameters.

In order to describe the ten-dimensional superparticle we relate the standard ten-dimensional superspace variables (X^m, θ^α) and the just introduced supertwistor variables $(\lambda^\alpha, w_\alpha, \psi_m)$ using the "incidence relations":

$$w_\alpha = X_m(\gamma^m\lambda)_\alpha - i\psi_m(\gamma^m\theta)_\alpha, \qquad \psi^m = (\lambda\gamma^m\theta), \tag{3.35}$$

which are invariant under (3.23) and (3.25). These relations define a trajectory on a superparticle worldline [72].

We can relate the Brink-Schwarz action to these new variables using the incidence relations:

$$\begin{aligned}
\lambda^\alpha\dot{w}_\alpha &= \dot{X}_m(\lambda\gamma^m\lambda) + X_m(\lambda\gamma^m\dot{\lambda}) - i\dot{\psi}_m(\lambda\gamma^m\theta) - i\psi_m(\lambda\gamma^m\dot{\theta}) \\
&= \dot{X}_m P^m + \frac{1}{2}X_m\dot{P}^m - i\dot{\psi}_m\psi^m - i\psi_m(\lambda\gamma^m\dot{\theta}),
\end{aligned}$$

which implies

$$\dot{X}^m P_m = 2\lambda^\alpha\dot{w}_\alpha + 2i\dot{\psi}^m\psi_m + 2i\psi^m(\lambda\gamma_m\dot{\theta}) - \partial_\tau(X^m P_m). \tag{3.36}$$

Meanwhile, we can recover the 'θ-term' in (3.18) using the fermionic incidence relation and the 10-D special identity (3.27):

$$\begin{aligned}
\psi^m(\lambda\gamma_m\dot{\theta}) &= \lambda^\alpha\lambda^\gamma\theta^\beta\dot{\theta}^\delta(\gamma^m)_{\alpha\beta}(\gamma_m)_{\gamma\delta} \\
&= -\lambda^\alpha\lambda^\gamma\theta^\beta\dot{\theta}^\delta(\gamma^m)_{\alpha\gamma}(\gamma_m)_{\delta\beta} - \lambda^\alpha\lambda^\gamma\theta^\beta\dot{\theta}^\delta(\gamma^m)_{\alpha\delta}(\gamma_m)_{\beta\gamma} \\
&= -(\lambda\gamma^m\lambda)(\theta\gamma_m\dot{\theta}) - (\lambda\gamma^m\theta)(\lambda\gamma^m\dot{\theta}),
\end{aligned}$$

from which we conclude

$$-iP^m(\dot{\theta}\gamma_m\theta) = -2i\psi^m(\lambda\gamma_m\dot{\theta}). \tag{3.37}$$

We see that the Brink-Schwarz action (3.18) describing the ten-dimensional massless superparticle can be re-expressed in terms of supertwistor variables:

$$S_{ST} = \int d\tau \Big[2\lambda^\alpha \dot{w}_\alpha + 2i\dot{\psi}^m \psi_m + \mu_\alpha G^\alpha + \chi v \Big], \tag{3.38}$$

provided we ignore boundary terms. The Lagrange multipliers μ_α and χ enforce the constraints (3.29) and (3.30).

After all this construction one may wonder what is the advantage of Berkovits' superparticle model (3.38) over the Brink-Schwarz one (3.18). Let us provide a very brief answer to this issue. In a nutshell, the answer relies in the set of constraints. Although apparently simpler, the Brink-Schwarz action presents a set of second class constraints (see appendix A for definitions) that are not known how to be separated from the first class constraints in a manifestly covariant manner, thus rendering quantization of such model difficult (except for light-cone gauge). In contrast, in Berkovits' superparticle model all constraints are first-class, thus making manifestly covariant quantization of the superparticle feasible. The same comments then extend to the corresponding ambitwistor strings constructed from these superparticle models.

3.3 The Berkovits-Guillen-Mason Supertwistor-Ambitwistor String

The ambitwistor string theory that we shall consider corresponds to the heterotic version of the natural passage from Berkovits' superparticle action to an ambitwistor action by replacing time derivatives by anti-holomorphic derivatives and the worldline by a Riemann sphere, as described in subsection 3.1. That is:

$$S_{ST} = \int_\Sigma \Big[\frac{1}{2}\lambda\bar{\partial}w - \frac{1}{2}w\bar{\partial}\lambda - \frac{1}{2}\psi_m\bar{\partial}\psi^m + \mu_\alpha G^\alpha + \chi T_F \Big] + S_J, \tag{3.39}$$

where S_J stands for a current algebra as in the $SO(32)$ or $E_8 \times E_8$ heterotic superstring. The constraints are

$$T_F = (\lambda\gamma^m\lambda)\psi_m, \tag{3.40}$$

$$G^\alpha = (\lambda\gamma^m\lambda)(\gamma_m w)^\alpha - 2\lambda^\alpha(\lambda w) + \psi^m\psi^n(\gamma_m\gamma_n\lambda)^\alpha = 0. \tag{3.41}$$

The operator product expansions (OPEs) satisfied by the canonical variables are given by: [5]

$$\lambda^\alpha(z)w_\beta(w) \to \frac{\delta^\alpha_\beta}{z-w}, \tag{3.42}$$

$$\psi^m(z)\psi^n(w) \to \frac{\eta^{mn}}{z-w}. \tag{3.43}$$

Meanwhile, the stress-energy tensor is given by

$$T_B(z) = \frac{1}{2}\partial w_\alpha \lambda^\alpha - \frac{1}{2}w_\alpha \partial\lambda^\alpha - \frac{1}{2}\psi^m\partial\psi_m + T_J, \tag{3.44}$$

where T_J is the stress-energy tensor associated to the current algebra.

The constraints in the model that we are considering satisfy the following algebra of constraints:

$$G^\alpha(z)T_F(\omega) \longrightarrow -\frac{2}{(z-w)}\lambda^\alpha T_F(w), \tag{3.45}$$

$$T_B(z)G^\alpha(w) \longrightarrow \frac{3}{2(z-w)^2}G^\alpha(w) + \frac{1}{(z-w)}\partial G^\alpha(w), \tag{3.46}$$

$$T_B(z)T_F(w) \longrightarrow \frac{3}{2(z-w)^2}T_F(w) + \frac{1}{(z-w)}\partial T_F(w), \tag{3.47}$$

$$T_F(z)T_F(w) \longrightarrow 0, \tag{3.48}$$

$$T_B(z)T_B(w) \longrightarrow \frac{-\frac{1}{2}(2D-4) + \frac{D}{4} + \frac{c_J}{2}}{(z-w)^4} + \frac{2}{(z-w)^2}T_B(w) + \frac{1}{(z-w)}\partial T_B(w), \tag{3.49}$$

$$\begin{aligned} G^\alpha(z)G^\beta(w) \longrightarrow & -\frac{4}{(z-w)}\lambda^{[\alpha}G^{\beta]}(w) - \frac{56}{(z-w)^2}\lambda^\alpha\lambda^\beta(w) - \frac{36}{(z-w)}\partial\lambda^\beta\lambda^\alpha(w) \\ & - \frac{20}{(z-w)}\partial\lambda^\alpha\lambda^\beta(w) + \frac{16}{(z-w)^2}(\gamma^m)^{\alpha\beta}(\lambda\gamma_m\lambda)(w) + \frac{16}{(z-w)}(\gamma_m)^{\alpha\beta}(\partial\lambda\gamma^m\lambda)(w). \end{aligned} \tag{3.50}$$

These OPEs can be found as follows:

$T_F(z)T_F(w)$ **OPE:** It is straightforward to see that

$$T_F(z)T_F(w) \to \text{regular}, \tag{3.51}$$

[5]The conventions used to take OPEs are summarized in equation (3.69), where we have also written down the corresponding expression for the ghosts used later.

since the OPE contracts two ψ's, and $(\lambda\gamma^m\lambda)(\lambda\gamma_m\lambda) = 0$ by the ten-dimensional special gamma-matrix identity (we can evaluate at $z = w$ because the contribution is a simple pole). ■

$G^\alpha(z)T_F(w)$ **OPE:** By direct computation, we have:

$$G^\alpha(z)T_F(w) \to \Big[(\lambda\gamma^m\lambda)\gamma_m^{\alpha\beta} - 2\lambda^\alpha\lambda^\beta\Big]_{(z)}(2\gamma^n_{\beta\gamma}\lambda^\gamma\psi_n)_{(w)}(z-w)^{-1}$$
$$-\Big[\psi^n(\gamma_p\gamma_n\lambda)^\alpha - \psi^m(\gamma_m\gamma_p\lambda)^\alpha\Big]_{(z)}(\lambda\gamma^p\lambda)_{(w)}(z-w)^{-1}. \tag{3.52}$$

Notice that there are no contractions possible that could lead to a double pole, so we can evaluate at $z = w$ (further terms are regular). We use now the gamma-matrix algebra and the special 10-D identity. The first term vanishes:

$$2(\lambda\gamma^m\lambda)\gamma_m^{\alpha\beta}\gamma^n_{\beta\gamma}\lambda^\gamma\psi_n - 4\lambda^\alpha(\lambda\gamma^n\lambda)\psi_n = -2(\lambda\gamma^m\lambda)(\gamma^n)^{\alpha\beta}(\gamma_m)_{\beta\gamma}\lambda^\gamma\psi_n = 0,$$

while for the second term:

$$-\Big[\psi^n(\gamma_p\gamma_n\lambda)^\alpha - \psi^m(\gamma_m\gamma_p\lambda)^\alpha\Big](\lambda\gamma^p\lambda) = -2\psi_p\lambda^\alpha(\lambda\gamma^p\lambda) = -2\lambda^\alpha T_F(w).$$

We can thus conclude that the OPE is:

$$G^\alpha(z)T_F(w) \to -\frac{2}{(z-w)}\lambda^\alpha T_F(w), \tag{3.53}$$

as stated in equation (3.45). ■

$G^\alpha G^\beta$ **OPE:** This OPE is more complicated due the appearance of double poles (DP). Let us concentrate first in the simple pole (SP) terms (i.e. "the classical contribution"). In detail:

$$\begin{aligned}
&G^\alpha(z)G^\beta(w) \underset{\text{SP}}{\longrightarrow}\\
&\big[2(\gamma^m\lambda)_\gamma(\gamma_m w)^\alpha - 2\delta^\alpha_\gamma(\lambda w) - 2\lambda^\alpha w_\gamma + \psi^m\psi^n(\gamma_m^{\alpha\delta}\gamma_{n\delta\gamma})\big]_{(z)}\big[(\lambda\gamma^p\lambda)\gamma_p^{\beta\gamma} - 2\lambda^\beta\lambda^\gamma\big]_{(w)}\\
&-[(\lambda\gamma^p\lambda)\gamma_p^{\alpha\gamma} - 2\lambda^\alpha\lambda^\gamma]_{(z)}[2(\gamma^m\lambda)_\gamma(\gamma_m w)^\beta - 2\delta^\beta_\gamma(\lambda w) - 2\lambda^\beta w_\gamma + \psi^m\psi^n(\gamma_m^{\beta\delta}\gamma_{n\delta\gamma})]_{(w)}\\
&-[\psi^n(\gamma_m\gamma_n\lambda)^\alpha - \psi^n(\gamma_n\gamma_m\lambda)^\alpha]_{(z)}[\psi^p(\gamma^m\gamma_p\lambda)^\beta - \psi^p(\gamma_p\gamma^m\lambda)^\beta]_{(w)}/(z-w).
\end{aligned} \tag{3.54}$$

It is straightforward to see that the bosonic terms reduce to

$$\text{Bosonic Terms} = \frac{-2\lambda^\alpha[(\lambda\gamma^m\lambda)(\gamma_m w)^\beta - 2\lambda^\beta(\lambda w)] + 2\lambda^\beta[(\lambda\gamma^m\lambda)(\gamma_m w)^\alpha - 2\lambda^\alpha(\lambda w)]_{(w)}}{(z-w)}, \quad (3.55)$$

after use of the Dirac algebra and the 10-D special identity. We have introduced two terms that actually cancel to make obvious how the bosonic terms of the constraint algebra arise. Meanwhile, the terms involving the fermionic variables ψ have two type of contributions:

$$\begin{aligned}\psi^m\psi^n\gamma_m^{\alpha\delta}\gamma_{n\delta\gamma}(\lambda\gamma^p\lambda)\gamma_p^{\gamma\beta} - 2\lambda^\beta\psi^m\psi^n(\gamma_m\gamma_n\lambda)^\alpha - \psi^m\psi^n\gamma_m^{\beta\delta}\gamma_{n\delta\gamma}(\lambda\gamma^p\lambda)\gamma_p^{\alpha\gamma} + 2\lambda^\alpha\psi^m\psi^n(\gamma_m\gamma_n\lambda)^\beta \\ = -2\lambda^\beta\psi^m\psi^n(\gamma_m\gamma_n\lambda)^\alpha + 2\lambda^\alpha\psi^m\psi^n(\gamma_m\gamma_n\lambda)^\beta, \quad (3.56)\end{aligned}$$

and

$$\begin{aligned}&[\psi^n(\gamma_m\gamma_n\lambda)^\alpha - \psi^n(\gamma_n\gamma_m\lambda)^\alpha][\psi^p(\gamma^m\gamma_p\lambda)^\beta - \psi^p(\gamma_p\gamma^m\lambda)^\beta] \\ &= [2\psi^m\lambda^\alpha - 2\psi^n(\gamma_n\gamma_m\lambda)^\alpha][2\psi_m\lambda^\beta - 2\psi^p(\gamma_p\gamma^m\lambda)^\beta] \\ &= 4\lambda^\alpha\psi^m\psi^n(\gamma_m\gamma_n\lambda)^\beta - 4\lambda^\beta\psi^m\psi^n(\gamma_m\gamma_n\lambda)^\alpha, \quad (3.57)\end{aligned}$$

where in the last equality we used the 10D special identity. Thus, adding (3.55), (3.56), and (3.57) we see we obtain the first term in (3.50). The rest of the terms come from double pole (DP) contributions that we handle next. In the following, a $(z-w)^{-2}$ factor is assumed to be carried around in the calculation:

$$\begin{aligned}G^\alpha(z)G^\beta(w) \underset{\text{DP}}{\longrightarrow} \frac{1}{2}\Big\{ &- [2(\gamma^m\lambda)_\gamma\gamma_m^{\alpha\delta} - 2\delta_\gamma^\alpha\lambda^\delta - 2\delta_\gamma^\delta\lambda^\alpha]_{(z)}[2(\gamma^p\lambda)_\delta\gamma_p^{\beta\gamma} - 2\delta_\delta^\beta\lambda^\gamma - 2\delta_\delta^\gamma\lambda^\beta]_{(w)} \\ &- [2(\gamma^p\lambda)_\delta\gamma_p^{\alpha\gamma} - 2\delta_\delta^\alpha\lambda^\gamma - 2\delta_\gamma^\delta\lambda^\alpha]_{(z)}[2(\gamma^m\lambda)_\gamma\gamma_m^{\beta\delta} - 2\delta_\gamma^\beta\lambda^\delta - 2\delta_\gamma^\delta\lambda^\beta]_{(w)} \\ &- [(\gamma_m\gamma_n\lambda)^\beta - (\gamma_n\gamma_m\lambda)^\alpha]_{(z)}[(\gamma^m\gamma^n\lambda)^\beta - (\gamma^n\gamma^m\lambda)^\beta]_{(w)}\Big\} \\ = &- 4[(\gamma^m\lambda)_\gamma\gamma_m^{\alpha\delta} - \delta_\gamma^\alpha\lambda^\delta - \delta_\gamma^\delta\lambda^\alpha]_{(z)}[(\gamma^n\lambda)_\delta\gamma_n^{\beta\gamma} - \delta_\delta^\beta\lambda^\gamma - \delta_\delta^\gamma\lambda^\beta]_{(w)} \\ &- 2(\gamma^{mn})_\gamma^\alpha(\gamma_{mn})_\delta^\beta\lambda^\gamma(z)\lambda^\delta(w), \quad (3.58)\end{aligned}$$

where $\gamma^{mn} = \frac{1}{2}(\gamma^m\gamma^n - \gamma^n\gamma^m)$. We use the following identity to massage the second term:

$$(\gamma^{mn})_\beta^\alpha(\gamma_{mn})_\lambda^\gamma = 4\gamma_{\beta\gamma}^m\gamma_m^{\alpha\gamma} - 2\delta_\beta^\alpha\delta_\lambda^\gamma - 8\delta_\lambda^\alpha\delta_\beta^\gamma. \quad (3.59)$$

The second term in (3.58) then reads:

$$- 2(\gamma^{mn})^{\alpha}_{\gamma}(\gamma_{mn})^{\beta}_{\delta}\lambda^{\gamma}_{(z)}\lambda^{\delta}_{(w)} = -8\gamma^{\alpha\beta}_{m}(\lambda_{(z)}\gamma^{m}\lambda_{(w)}) + 4\lambda^{\alpha}_{(z)}\lambda^{\beta}_{(w)} + 16\lambda^{\alpha}_{(w)}\lambda^{\beta}_{(z)}. \tag{3.60}$$

Expanding the first term in (3.58), each separate term gives (recalling that $\delta^{\alpha}_{\alpha} = 16$ and $\delta^{m}_{m} = 10$):

$$\begin{aligned}
&- 4[(\gamma^{m}\lambda)_{\gamma}\gamma^{\alpha\delta}_{m} - \delta^{\alpha}_{\gamma}\lambda^{\delta} - \delta^{\delta}_{\gamma}\lambda^{\alpha}]_{(z)}[(\gamma^{n}\lambda)_{\delta}\gamma^{\beta\gamma}_{n} - \delta^{\beta}_{\delta}\lambda^{\gamma} - \delta^{\gamma}_{\delta}\lambda^{\beta}]_{(w)} \\
=&- 4(\gamma^{\alpha\delta}_{m}\gamma^{n}_{\delta\epsilon}\lambda^{\epsilon}_{(w)})(\gamma^{\beta\gamma}_{n}\gamma^{m}_{\gamma\phi}\lambda^{\phi}_{(z)}) + 4\gamma^{\alpha\beta}_{m}(\lambda_{(z)}\gamma^{m}\lambda_{(w)}) + 40\lambda^{\alpha}_{(z)}\lambda^{\beta}_{(w)} \\
&+ 4\gamma^{\alpha\beta}_{n}(\lambda_{(z)}\gamma^{n}\lambda_{(w)}) - 4\lambda^{\alpha}_{(w)}\lambda^{\beta}_{(z)} - 4\lambda^{\alpha}_{(z)}\lambda^{\beta}_{(w)} \\
&+ 40\lambda^{\alpha}_{(z)}\lambda^{\beta}_{(w)} - 4\lambda^{\alpha}_{(z)}\lambda^{\beta}_{(w)} - 64\lambda^{\alpha}_{(z)}\lambda^{\beta}_{(w)}.
\end{aligned} \tag{3.61}$$

The first term in the right hand side can be rewritten in the following way:

$$\begin{aligned}
&- 4(\gamma^{\alpha\delta}_{m}\gamma^{n}_{\delta\epsilon}\lambda^{\epsilon}_{(w)})(\gamma^{\beta\gamma}_{n}\gamma^{m}_{\gamma\phi}\lambda^{\phi}_{(z)}) \\
=&- 4\Big[\gamma^{\alpha\delta}_{(m}\gamma_{n)\delta\epsilon} + \gamma^{\alpha\delta}_{[m}\gamma_{n]\delta\epsilon}\Big]\Big[\gamma^{(n\beta\gamma}\gamma^{m)}_{\gamma\phi} + \gamma^{(n\beta\gamma}\gamma^{m)}_{\gamma\phi}\Big]\lambda^{\epsilon}_{(w)}\lambda^{\phi}_{(z)} \\
=&- 4\Big[\eta^{nm}\delta^{\alpha}_{\epsilon} + (\gamma^{mn})^{\alpha}_{\epsilon}\Big]\Big[\eta_{nm}\delta^{\beta}_{\phi} - (\gamma_{mn})^{\beta}_{\phi}\Big]\lambda^{\epsilon}_{(w)}\lambda^{\phi}_{(z)} \\
=&- 40\lambda^{\alpha}_{(w)}\lambda^{\beta}_{(z)} + 4\big(4\gamma^{m}_{\epsilon\phi}\gamma^{\alpha\beta}_{m} - 2\delta^{\alpha}_{\epsilon}\delta^{\beta}_{\phi} - 8\delta^{\alpha}_{\phi}\delta^{\beta}_{\epsilon}\big)\lambda^{\epsilon}_{(w)}\lambda^{\phi}_{(z)} \\
=&\ + 16\gamma^{\alpha\beta}_{m}(\gamma_{(z)}\gamma^{m}\lambda_{(w)}) - 48\lambda^{\alpha}_{(w)}\lambda^{\beta}_{(z)} - 32\lambda^{\alpha}_{(z)}\lambda^{\beta}_{(w)}.
\end{aligned} \tag{3.62}$$

Adding these terms to the rest of terms in (3.61) and (3.60) one obtains a total double pole contribution:

$$G^{\alpha}(z)G^{\beta}(w) \underset{\text{DP}}{\longrightarrow} \frac{16\gamma^{\alpha\beta}_{m}(\lambda_{(z)}\gamma^{m}\lambda_{(w)}) - 20\lambda^{\alpha}_{(z)}\lambda^{\beta}_{(w)} - 36\lambda^{\alpha}_{(w)}\lambda^{\beta}_{(z)}}{(z-w)^2}. \tag{3.63}$$

Thus, expanding z around w we see that we recover all the remaining terms in the $G^{\alpha}G^{\beta}$ OPE in (3.45). ∎

It is straightforward to derive the rest of the OPEs (the ones involving T_B) by noticing that since T_B is the stress energy tensor, its OPE with the rest of the variables is just determined by the conformal weight of the latter.

One can use this algebra to find the BRST generator and in principle the corresponding BRST-closed vertex operators. Nevertheless, because the constraints are reducible, the authors in [31] took an alternative approach of performing a light-cone analysis to show the equivalence between this model and the light-cone RNS ambitwistor string. In the next section we shall redress this situation, explicitly finding the BRST generator for this model.

3.4 Uncovering the BRST Generator

The strategy we use to obtain the BRST generator for the ambitwistor string considered is the following. We first consider the bosonic part of the model, and up to contributions that only involve single contractions in the OPE. We can think of this first case as finding the BRST generator for the bosonic massless particle described using ten-dimensional twistor variables. For this case, a systematic procedure to find the BRST generator is known, and the analysis can be further simplified using a general result that is valid when the constraints are linear in the momenta. We refer the reader to appendix A for a review of this method and the theory of reducible constrained systems in general, which we use in all the rest of this section. Afterwards, we generalize to the case with supersymmetry, although we still just consider contributions coming from single contractions in the OPE (that is, we are effectively considering the BRST generator for the Berkovits' superparticle model). In this instance, a systematic approach to build the BRST generator still exists, but no further simplifications are available to us since the constraints are not linear in the momenta. Notwithstanding this fact, we use the intuition developed from the bosonic case to find the BRST generator for the superparticle. We use the BRST generator of the twistor superparticle model as a starting point to find the BRST generator of the twistor ambitwistor string.

The BGM twistor-ambitwistor string shows two new obstacles to find the corresponding nilpotent BRST generator. First, one has to consider the Virasoro constraint. Secondly, there are terms in the constraint algebra that are not present in the particle case, which appear due to the presence of double poles in the OPE. In this case we have no systematic shortcut to find the BRST generator, but as we shall see, the double pole contributions can be handled by the introduction of just one new term. Apart from this term, and besides the standard Virasoro constraint contribution, no new terms are necessary to ensure nilpotency of the BRST generator.

Correspondingly to the strategy just described, we now consider the bosonic particle described using ten-dimensional twistor variables (or equivalently, just the bosonic piece of the full supersymmetric case). In this simplified scenario the variables are subject to the constraints

$$F^{\alpha} = (\lambda\gamma^{m}\lambda)(\gamma_{m}w)^{\alpha} - 2\lambda^{\alpha}(\lambda w) = 0. \tag{3.64}$$

These constraints are reducible in virtue of the special ten-dimensional gamma-matrix identity (3.27). Indeed, it is easy to check using this identity that

$$\lambda\gamma^{m}F = 0, \qquad (\lambda\gamma_{m}\lambda)\lambda\gamma^{m}F = 0. \tag{3.65}$$

Following the theory of reducible constrained systems we extend our phase space to include a fermionic ghost-antighost pair (g_{α}, f^{α}) associated to the bosonic constraint (3.64). Since the constraints are reducible we consider as well a second generation of bosonic ghosts (s_{m}, t^{m}), and a third generation of fermionic ghosts (η, ρ). The ghost numbers are :

$$\begin{aligned}
\mathrm{gh}(g_\alpha, f^\alpha) &= (1, -1) \\
\mathrm{gh}(s_m, t^m) &= (2, -2) \\
\mathrm{gh}(\eta, \rho) &= (3, -3)
\end{aligned} \tag{3.66}$$

We now look for the BRST generator. Following appendix A, the BRST generator must have the form

$$Q = g_\alpha F^\alpha + s_m(\lambda\gamma^m f) + \eta(\lambda\gamma_m\lambda)t^m + \bar{Q}, \tag{3.67}$$

where $\bar{Q}$ stands for terms with a product of at least two ghosts and one antighost, or two antighosts and one ghost. These terms are added in such a way that the BRST generator is nilpotent. In order to find the $\bar{Q}$ term, we make use now of the following proposition [74] (we sketch a proof of this proposition in appendix A):

Proposition 3.4.1. *In the case where the constraints are linear in the momenta, the BRST generator can be taken to be linear in the extended phase space momenta (that is, including antighosts).*

Since the constraints (3.64) are linear in the momenta w_α, the previous proposition highly constraints the possible terms and leaves us with just a few ones to consider. Indeed, considering the rules detailed in appendix A for the construction of the BRST generator, and using the previous proposition at each antighost number (constructing all possible combinations of ghosts at fixed antighost number that will lead to a fermionic and total ghost number one contribution):

- antighost number 1: $(\lambda g)(gf)$,
- antighost number 2: $\mathcal{J}_m^{\alpha\beta\gamma} g_\alpha g_\beta g_\gamma t^m$, $\mathcal{B}_n^{\alpha m} g_\alpha s_m t^n$,
- antighost number 3: $\mathcal{N}^{\alpha\beta\gamma\delta} g_\alpha g_\beta g_\gamma g_\delta \rho$, $\mathcal{P}_m^{\alpha\beta} g_\alpha g_\beta s_m \rho$, $\eta^{mn} s_m s_n \rho$, $(\lambda g)\eta\rho$.

From the index structure of these terms we can further see that it must be that $\mathcal{J}_m^{\alpha\beta\gamma} = \mathcal{N}^{\alpha\beta\gamma\delta} = \mathcal{P}_m^{\alpha\beta} = 0$. This is due to the fact that these factors must be constructed from γ-matrices, or λ's, both of which will lead to symmetric factors in the α indices that vanish when contracted with a product of (fermionic) g_α-ghosts. Furthermore, we have fixed the dependence in λ in the antighost number 1 and last antighost number 3 term by noticing that by conformal weight arguments one λ must be present.[6] We then contract in the only possible way that does not lead to a vanishing contribution. The form of the $ss\rho$ term at antighost number three is also fixed by similar considerations. [7]

[6] g, f, η, and ρ have conformal weight $-\frac{1}{2}$, $\frac{3}{2}$, -2 and 3 respectively. Since the BRST generator has conformal weight one, we need to multiply by a factor with conformal weight $\frac{1}{2}$. The only options are λ^α and w_α, but the latter is forbidden as it would give a contribution non-linear in the momenta.

[7] $s_m s_n \rho$ already has conformal weight one and the only symmetric object in m and n that does not involve further variables is the metric.

After taking these considerations into account, the BRST generator for the bosonic twistor particle must take the following form:

$$\begin{aligned} q =\, & g_\alpha F^\alpha + s_m(\lambda\gamma^m f) + \eta(\lambda\gamma_m\lambda)t^m \\ & + x(\lambda g)(gf) + \mathcal{B}_n^{\alpha m} g_\alpha s_m t^n + y(\lambda g)\eta\rho + z\eta^{mn}s_m s_n\rho, \end{aligned} \tag{3.68}$$

where x, y, z and $\mathcal{B}_n^{\alpha m}$ are coefficients to be determined by the nilpotency of the BRST generator.

Although currently we are going to consider only contributions coming from single contractions in the OPE, we will need the full OPE later. The convention for the OPE used throughout this thesis is summarized in the following exponential:

$$\begin{aligned} \exp\Bigg\{ \int \frac{d^2z_1 d^2z_2}{(z_1-z_2)} \Bigg[& \frac{\delta}{\delta\lambda^\alpha(z_1)}\frac{\delta}{\delta w_\alpha(z_2)} - \frac{\delta}{\delta w_\alpha(z_1)}\frac{\delta}{\delta\lambda^\alpha(z_2)} - \frac{\delta}{\delta\psi^m(z_1)}\frac{\delta}{\delta\psi_m(z_2)} \\ & - \frac{\delta}{\delta f^\alpha(z_1)}\frac{\delta}{\delta g_\alpha(z_2)} - \frac{\delta}{\delta g_\alpha(z_1)}\frac{\delta}{\delta f^\alpha(z_2)} + \frac{\delta}{\delta\gamma(z_1)}\frac{\delta}{\delta\beta(z_2)} - \frac{\delta}{\delta\beta(z_1)}\frac{\delta}{\delta\gamma(z_2)} \\ & + \frac{\delta}{\delta s_m(z_1)}\frac{\delta}{\delta t^m(z_2)} - \frac{\delta}{\delta t^m(z_1)}\frac{\delta}{\delta s_m(z_2)} - \frac{\delta}{\delta\eta(z_1)}\frac{\delta}{\delta\rho(z_2)} - \frac{\delta}{\delta\rho(z_1)}\frac{\delta}{\delta\eta(z_2)} \\ & - \frac{\delta}{\delta b(z_1)}\frac{\delta}{\delta c(z_2)} - \frac{\delta}{\delta c(z_1)}\frac{\delta}{\delta b(z_2)} \Bigg] \Bigg\}, \end{aligned} \tag{3.69}$$

where we have also considered conformal ghosts, fermionic variables and the $\beta\gamma$-system (associated to the fermionic constraint (3.40)) OPE that we will need later.

The x coefficient is determined by the antighost number 0 computation in (A.14), $\mathcal{B}_n^{\alpha m}$ is determined by the antighost number 1 computation, and y and z are then determined by the antighost number 2 computation. From proposition 3.4.1. we know that the antighost number 3 computation must vanish (as one may directly check), since we do not have freedom to add further terms. We proceed then to consider different antighost number computations:

Antighost number 0: The x coefficient is chosen to cancel the contribution coming from the FF algebra $g_\alpha(z)g_\beta(w)F^\alpha(z)F^\beta(w) \to -4g_\alpha g_\beta\lambda^\alpha F^\beta(w)(z-w)^{-1}$. Clearly, this can be done by choosing $x=2$, so that the contraction over the (f,g) ghosts cancels the previous OPE. Alternatively, that $x=2$ comes from the fact that at this antighost number we can see this calculation as the one for an irreducible non-abelian algebra

$$[G_a, G_b] = C_{ab}^c G_c, \tag{3.70}$$

in which case the formula for the BRST generator is known:

$$Q = c^a G_a - \frac{1}{2} c^b c^c C^a_{cb} b_a. \tag{3.71}$$

Effectively, we can see the antighost number 0 computation as a special case of this formula, since other antighost number terms do not contribute. ■

Antighost number 1: We have four OPEs to consider at this antighost number. First, we have the (λ, w) OPE between $(g_\alpha F^\alpha, 2(\lambda g)(gf))$ and the (λ, w) OPE between $(g_\alpha F^\alpha, s_m(\lambda\gamma^m f))$. Then, we have the (g, f) OPE between $(2(\lambda g)(gf), s_m(\lambda\gamma^m f))$, and finally the (s, t) OPE between $(s_m(\lambda\gamma^m f), \mathcal{B}^{\alpha m}_n g_\alpha s_m t^n)$. We see right away that the first OPE is zero because $\frac{\delta F^\alpha}{\delta w_\gamma}$ is symmetric in α and γ, while $g_\alpha g_\gamma$ is antisymmetric. The rest of the computation goes as follows:

$$g_\alpha F^\alpha_{(z)} s_m(\lambda\gamma^m f)_{(w)} \xrightarrow[(\lambda,\, w)]{} -g_\alpha\big[(\lambda\gamma^n\lambda)\gamma^{\alpha\phi}_n - 2\lambda^\alpha\lambda^\phi\big] s_m \gamma^m_{\phi\gamma} f^\gamma(w)/(z-w). \tag{3.72}$$

Let us massage this term a bit to compare better to the other OPEs later:

$$\begin{aligned}(\lambda\gamma^n\lambda)\gamma^{\alpha\phi}_n\gamma^m_{\phi\gamma} - 2\lambda^\alpha\lambda^\phi\gamma^m_{\phi\gamma} &= 2(\lambda\gamma^m\lambda)\delta^\alpha_\gamma - (\lambda\gamma^n\lambda)\gamma_{n\phi\gamma}\gamma^{m\alpha\phi} - 2\lambda^\alpha(\lambda^\phi\gamma^m_{\phi\gamma}) \\ &= 2(\lambda\gamma^m\lambda)\delta^\alpha_\gamma + \lambda^\beta\lambda^\delta\big(\gamma^n_{\beta\phi}\gamma_{n\gamma\delta} + \gamma^n_{\beta\gamma}\gamma_{n\phi\delta}\big)\big) - 2\lambda^\alpha(\lambda^\phi\gamma^m_{\phi\gamma}) \\ &= 2(\lambda\gamma^m\lambda)\delta^\alpha_\gamma + 2\big((\lambda\gamma^n\gamma^m)^\alpha(\lambda\gamma_n)_\gamma\big) - 2\lambda^\alpha(\lambda^\phi\gamma^m_{\phi\gamma}).\end{aligned} \tag{3.73}$$

Compute now the remaining OPEs:

$$2(\lambda g)(gf)_{(z)} s_m(\lambda\gamma^m f)_{(w)} \xrightarrow[(g,\, f)]{} [2s_m(\lambda\gamma^m\lambda) - 2(\lambda g)s_m(\lambda\gamma^m f)]_{(w)}/(z-w), \tag{3.74}$$

$$s_m(\lambda\gamma^m f)_{(z)}\mathcal{B}^{\alpha m}_n g_\alpha s_m t^n_{(w)} \xrightarrow[(s,\, t)]{} (\lambda\gamma^m f)\mathcal{B}^{\alpha m}_n g_\alpha s_m/(z-w). \tag{3.75}$$

We see then that the contributions in (3.74) cancel two of the terms in (3.72) under the massage performed in (3.73). Thus, to cancel the remaining term we choose $\mathcal{B}^{\alpha m}_n$ to be

$$\mathcal{B}^{\alpha m}_n = -2(\lambda\gamma_n\gamma^m)^\alpha. \tag{3.76}$$

This finishes the computation at antighost number 1. ■

Antighost number 2: One can separate this computation in three different cases noticing that $q_{(z)}q_{(w)}$ has ghost number 2. The different cases come from the three different ways of obtaining in the OPE, from the terms in (3.68), an antighost number 2 quantity with total ghost number 2. One can only obtain contributions proportional to $(ggst)$, $(ct\eta)$, and (sst), and since these contributions do not mix with each other they must vanish separately. Combinations $(gt\eta)$, and (sst) fix y and z respectively (and thus the $(ggst)$ contribution vanishes on its own). One can now proceed to find y and z:

$$\begin{aligned}
-2(\lambda\gamma_n\gamma^m g)s_m t^n_{(z)}\eta[(\lambda\gamma_p\lambda)t^p]_{(w)} &\xrightarrow[(s,\, t)]{} -2(\lambda\gamma_n\gamma^p g)t^n\eta(\lambda\gamma_p\lambda)_{(w)}/(z-w) \\
&= 4\eta t^n(\lambda\gamma_n\lambda)(\lambda g)_{(w)}/(z-w),
\end{aligned} \tag{3.77}$$

$$y(\lambda g)\eta\rho_{(z)}\eta[(\lambda\gamma_m\lambda)t^m]_{(w)} \xrightarrow[(\eta,\, \rho)]{} y(\lambda g)\eta[(\lambda\gamma_m\lambda)t^m]_{(w)}/(z-w). \tag{3.78}$$

Thus, in order to have a cancellation we must have that $y = 4$. Computing z:

$$\begin{aligned}
-2(\lambda\gamma_n\gamma^m g)s_m t^n_{(z)}s_p(\lambda\gamma^p f)_{(w)} &\xrightarrow[(f,\, g)]{} -2(\lambda\gamma_n\gamma^m)^\gamma s_m t^n s_p\lambda^\alpha\gamma^p_{\alpha\gamma}(w)/(z-w) \\
&= -2(\lambda\gamma_n\gamma^m\gamma^p\lambda)t^n s_m s_p(w)/(z-w) \\
&= -\Big(\lambda\gamma_n(\gamma^m\gamma^p+\gamma^p\gamma^m)\lambda\Big)t^n s_m s_p(w)/(z-w) \\
&= -2\eta^{mp}s_m s_p(\lambda\gamma_n\lambda)t^n(w)/(z-w),
\end{aligned} \tag{3.79}$$

$$z\eta^{mn}s_m s_n\rho_{(z)}\eta\big[(\lambda\gamma_m\lambda)t^m\big]_{(w)} \xrightarrow[(\eta,\, \rho)]{} z\eta^{mn}s_m s_n\big[(\lambda\gamma_m\lambda)t^m\big](w)/(z-w). \tag{3.80}$$

From which we find $z = 2$. ∎

This finishes the full bosonic computation, which can be summarized in the BRST generator

$$\begin{aligned}
q =& g_\alpha F^\alpha + s_m(\lambda\gamma^m f) + \eta(\lambda\gamma_m\lambda)t^m \\
&+ 2(\lambda g)(gf) - 2(\lambda\gamma_n\gamma^m)^\alpha g_\alpha s_m t^n + 4(\lambda g)\eta\rho + 2\eta^{mn}s_m s_n\rho.
\end{aligned} \tag{3.81}$$

In accordance to our strategy, we now proceed to consider supersymmetric terms.

Supersymmetric case. We start making three observations. First, the constraint algebra of the F^α is the same as the constraint algebra of the G^α. Secondly, no w_α appear in $q_{(z)}$ besides in F^α. Finally, the term over which G^α and F^α differ does not contain w. These observations imply that the previous calculation remains valid and the corresponding terms gives rise to a nilpotent contribution (at least as simple poles are concerned).

The previous remark suggest we concentrate only in the new terms due to supersymmetry. There is one term we must add due to reducibility. We also add one further term to cancel the OPE between G^α and the fermionic constraint T_F. The new ansatz is therefore:

$$\begin{aligned} q_{ss} =& g_\alpha G^\alpha + \gamma T_F + s_m\big[(\lambda\gamma^m f) + 2\psi^m\beta\big] + \eta(\lambda\gamma_m\lambda)t^m + 2(\lambda g)(gf) \\ &+ a(\lambda g)(\gamma\beta) - 2(\lambda\gamma_n\gamma^m)^\alpha g_\alpha s_m t^n + 4(\lambda g)\eta\rho + 2\eta^{mn}s_m s_n\rho + \bar{q}_{ss}, \end{aligned} \tag{3.82}$$

where $\bar{q}_{ss}$ includes terms with at least two ghosts and one antighost, or one ghost and two antighosts, in such a way that the full BRST generator is nilpotent. Unlike the bosonic case, $\bar{q}_{ss}$ is not a priori constrained to take any form since the constraints are no longer linear in the momenta.

First, we fix the coefficient a:

$$g_\alpha G^\alpha_{(z)}\gamma T_{F(w)} \to -\frac{2}{(z-w)}(\lambda g)\gamma T_F(w), \tag{3.83}$$

where we used the corresponding OPE in (3.45). On the other hand:

$$a\gamma T_{F(z)}(\lambda g)\gamma\beta_{(w)} \xrightarrow[(\beta,\,\gamma)]{} -\frac{a}{(z-w)}(\lambda g)\gamma T_F(w). \tag{3.84}$$

Thus, in order to cancel both contributions we must choose $a = -2$. There is only one more antighost number 0 calculation with the terms so far introduced, but it vanishes by the construction of the BRST generator:

$$g_\alpha G^\alpha_{(z)} s_m(\lambda\gamma^m f)_{(w)} \xrightarrow[(g,\,f)]{} \frac{1}{(z-w)} s_m(\lambda\gamma^m G)(w) = \frac{2}{(z-w)}(s_m\psi^m)T_F(w), \tag{3.85}$$

$$2\gamma T_{F(z)} s_m\psi^m\beta_{(w)} \xrightarrow[(\gamma,\,\beta)]{} -\frac{2}{(z-w)} s_m\psi^m T_F(w). \tag{3.86}$$

The cancellation is then direct. Similarly, one can check that all antighost number 1 OPEs considering the terms so far introduced cancel in a similar way. The new supersymmetric terms do introduce, however, OPEs at antighost number 2 that do not cancel as just shown. For example:

$$4s_m\psi^m\beta_{(z)} s_n\psi^n\beta_{(w)} \xrightarrow[(\psi,\,\psi)]{} \frac{4}{(z-w)} s_n s_m\eta^{nm}\beta^2(w). \tag{3.87}$$

We must then introduce more terms through $\bar{q}_{ss}$ to cancel these contributions. Specifically, we introduce $\bar{q}_{ss} = -\eta\beta^2$. Computing directly to see the cancellation:

$$-2\eta\beta^2_{(z)}\eta^{mn}s_m s_n \rho_{(w)} \times 2 \xrightarrow[(\eta,\,\rho)]{} -\frac{4}{(z-w)}\eta^{mn}s_m s_n \beta^2(w), \tag{3.88}$$

where we introduced a factor of 2 recalling that when computing the OPE we have a cross-term, unlike (3.87) which corresponds to a "square term".

With the last term added we are actually finished. Indeed, the newly added term gives rise to one further antighost number 2 contribution:

$$-4\eta\beta^2_{(z)}(\lambda g)\eta\rho_{(w)} \xrightarrow[(\eta,\,\rho)]{} -\frac{4}{(z-w)}(\lambda g)\eta\beta^2(w), \tag{3.89}$$

$$2(\lambda g)(\gamma\beta)_{(z)}\eta\beta^2_{(2)} \xrightarrow[(\beta,\,\gamma)]{} \frac{4}{(z-w)}(\lambda g)\eta\beta^2(w), \tag{3.90}$$

which directly cancels, and one further antighost number 1 contribution that also cancels directly:

$$-\gamma T_{F(z)}\eta\beta^2_{(w)} \xrightarrow[(\beta,\,\gamma)]{} -\frac{2}{(z-w)}T_F\eta\beta(w), \tag{3.91}$$

$$2(s_m\psi^m)\beta_{(z)}\eta(\lambda\gamma_n\lambda)t^n(w) \xrightarrow[(s,\,t)]{} \frac{2}{(z-w)}\psi^m\beta\eta(\lambda\gamma_m\lambda)(w) = \frac{2}{(z-w)}T_F\eta\beta(w). \tag{3.92}$$

We are therefore finished with the corrections due to the supersymmetric case, at least at the order of simple poles. ■

The Stringy Case. We must now consider new contributions to the the BRST generator because in the stringy case we are considering a two-dimensional conformal field theory. First, we must consider the Virasoro constraint, and secondly we must consider double pole contributions in the OPE (i.e. the "quantum corrections"). Of course, these new terms might spoil the nilpotency so far achieved. Let us first handle the double pole contributions, which we do by adding new terms with derivatives to our previous prescription (3.82). Noticing that the BRST current must have conformal weight one, we have the freedom to add just one more term of the form $\lambda^\alpha \partial g_\alpha$:

$$
\begin{aligned}
q'(z) = \Big[& g_\alpha G^\alpha + \gamma T_F + s_m\big[(\lambda\gamma^m f) + 2\psi^m\beta\big] + \eta(\lambda\gamma_m\lambda)t^m \\
& +2(\lambda g)(gf) - 2(\lambda g)(\gamma\beta) - 2(\lambda\gamma_n\gamma^m)^\alpha g_\alpha s_m t^n \\
& +4(\lambda g)\eta\rho + 2\eta^{mn}s_m s_n\rho - \eta\beta^2 + A\lambda^\alpha\partial g_\alpha \Big](z).
\end{aligned} \tag{3.93}
$$

We compute the double poles using (3.69) up to second order, after which we expand z around w to obtain the corresponding contribution to the simple poles. We focus first in the terms containing two g ghosts. A explicit check gives:

$$4(\lambda g)(gf)_{(z)}(\lambda g)(gf)_{(w)} \underset{\substack{(g,\,f)\\(g,f)}}{\longrightarrow} \frac{\big[52(\partial\lambda g)(\lambda g) + 60(\lambda\partial g)(\lambda g)\big](w)}{(z-w)}, \tag{3.94}$$

$$4(\lambda g)(\gamma\beta)_{(z)}(\lambda g)(\gamma\beta)_{(w)} \underset{\substack{(\gamma,\,\beta)\\(\gamma,\beta)}}{\longrightarrow} -4\frac{\big[(\partial\lambda g)(\lambda g) + (\lambda\partial g)(\lambda g)\big](w)}{(z-w)}, \tag{3.95}$$

$$4(\lambda g)(\eta\rho)_{(z)}(\lambda g)(\eta\rho)_{(w)} \underset{\substack{(\eta,\,\rho)\\(\eta,\rho)}}{\longrightarrow} 16\frac{\big[(\partial\lambda g)(\lambda g) + (\lambda\partial g)(\lambda g)\big](w)}{(z-w)}, \tag{3.96}$$

$$4(\lambda\gamma_n\gamma^m g)s_m t^n_{(z)}(\lambda\gamma_p\gamma^q g)s_q t^p_{(w)} \underset{\substack{(s,\,t)\\(s,t)}}{\longrightarrow} \frac{\big[16(\lambda\gamma^m\lambda)(\partial g\gamma_m g) - 16(\partial\lambda)(\lambda g) - 80(\lambda\partial g)(\lambda g)\big](w)}{(z-w)}, \tag{3.97}$$

and the contribution from the double poles in the $g_\alpha G^\alpha(z) g_\beta G^\beta(w)$ OPE can be read from the $G^\alpha G^\beta$ OPE and expanding z around w:

$$\frac{16(\partial\lambda g)(\lambda g)}{(z-w)} - \frac{56(\lambda\partial g)(\lambda g)}{(z-w)} + \frac{16(\lambda\gamma^m\lambda)(\partial g\gamma_m g)}{(z-w)}. \tag{3.98}$$

There is also a $g_\alpha G^\alpha_{(z)}(\lambda g)(gf)_{(w)}$ OPE, which however gives a vanishing double pole contribution.

Thus, adding all the previous OPEs one gets:

$$\frac{32(\lambda\gamma^m\lambda)(\partial g\gamma_m g) + 64(\partial\lambda g)(\lambda g) - 64(\lambda\partial g)(\lambda g)}{(z-w)}. \tag{3.99}$$

To cancel this contribution we must use the new term in (3.93) and hope everything will go in our favor (notice that the algorithm given before from chapter 10 in [73] regards the classical analysis. It is not rigorously known if we can find a nilpotent BRST generator under the full OPE, which is important for us as now we are considering double poles containing the quantum contributions). If we calculate the corresponding OPEs:

$$A\lambda^\alpha\partial g_{\alpha(z)}g_b G^b_{(w)} + \text{cross-term} \xrightarrow[(\lambda,\, w)]{} \frac{2A(\lambda\gamma^m\lambda)(\partial g\gamma_m g) - 4A(\lambda\partial g)(\lambda g)}{(z-w)}, \tag{3.100}$$

$$2A\lambda^\alpha\partial g_{\alpha(z)}(\lambda g)(gf)_{(w)} + \text{cross-term} \xrightarrow[(\lambda,\, w)]{} \frac{4A(\partial\lambda g)(\lambda g)}{(z-w)}, \tag{3.101}$$

and adding both:

$$\frac{2A(\lambda\gamma^m\lambda)(\partial g\gamma_m g) + 4A(\partial\lambda g)(\lambda g) - 4A(\lambda\partial g)(\lambda g)}{(z-w)}. \tag{3.102}$$

Thus, we see that quite beautifully with the single new contribution in q', the terms arrange themselves so that with $A = -16$ all three independent terms in (3.99) are cancelled.

There are still terms giving double poles which are proportional to the s_m ghost-for-ghost. Since we already chose $A = -16$, we have no remaining freedom and we must hope that we get vanishing contributions. The corresponding OPEs give:

$$2\gamma(\lambda\gamma^m\lambda)\psi_{m(z)}s_n\psi^n\beta_{(w)} + \text{cross-term} \xrightarrow[\substack{(\psi,\, \psi)\\(\beta,\gamma)}]{} \frac{[4(\partial\lambda\gamma^m\lambda)s_m - 2(\lambda\gamma^m\lambda)\partial s_m](w)}{(z-w)}, \tag{3.103}$$

$$-16\lambda\partial g_\alpha s_m(\lambda\gamma^m f)_{(w)} + \text{cross-term} \xrightarrow[(g,\, f)]{} -\frac{16(\lambda\gamma^m\lambda)\partial s_m(w)}{(z-w)}, \tag{3.104}$$

$$-2(\lambda\gamma_n\gamma^m g)s_m t^n_{(z)}s_p(\lambda\gamma^p f)_{(w)} + \text{cross-term} \xrightarrow[\substack{(s,\, t)\\(g,f)}]{} -\frac{16(\lambda\gamma^m\lambda)\partial s_m(w)}{(z-w)}, \tag{3.105}$$

$$g_\alpha G^\alpha_{(z)} s_m(\lambda\gamma^m f)_{(w)} + \text{cross-term} \underset{\substack{(\lambda,\, w)\\ (g, f)}}{\longrightarrow} \frac{[14(\lambda\gamma^m\lambda)\partial s_m - 28(\partial\lambda\gamma^m\lambda)s_m](w)}{(z-w)}, \tag{3.106}$$

$$2\eta^{mn} s_m s_n \rho_{(z)}\eta(\lambda\gamma^p\lambda)t^p_{(w)} + \text{cross-term} \underset{\substack{(s,\, t)\\ (\eta, \rho)}}{\longrightarrow} \frac{[4(\lambda\gamma^m\lambda)\partial s_m - 8(\partial\lambda\gamma^m\lambda)s_m](w)}{(z-w)}. \tag{3.107}$$

Adding all these contributions, one gets:

$$\frac{-16(\lambda\gamma^m\lambda)\partial s_m - 32(\partial\lambda\gamma^m\lambda)s_m}{(z-w)} = -16\frac{\partial[(\lambda\gamma^m\lambda)s_m](w)}{(z-w)}. \tag{3.108}$$

Although the total derivative is not zero, is enough for our purposes, since in a 2D-CFT we can compute the commutator through

$$[Q_1, Q_2](C_2) = \oint_{C_2} \frac{dz_2}{2\pi i}\text{Res}_{z_1\to z_2} j_{(z_1)} j_{(z_2)}, \tag{3.109}$$

and therefore the total derivative leads to an overall vanishing contribution to the double poles. Thus, we have properly handled one of the tasks given by the stringy case.

We still need to consider the Virasoro constraint. In the following, $c(z)$ is the conformal ghost, and $T(z)$ is the stress energy contribution coming from matter, ghosts other than c, and ghosts for ghosts:

$$\begin{aligned} T(z) = \Big[&\frac{1}{2}\lambda^\alpha\partial w_\alpha - \frac{1}{2}w_\alpha\partial\lambda^\alpha - \frac{1}{2}\psi^m\partial\psi_m + \frac{1}{2}g_\alpha\partial f^\alpha - \frac{3}{2}f^\alpha\partial g_\alpha \\ &-\frac{1}{2}(\partial\beta)\gamma - \frac{3}{2}\beta(\partial\gamma) - s_m\partial t^m - 2t^m\partial s_m + 2\eta\partial\rho - 3\rho\partial\eta\Big](z). \end{aligned} \tag{3.110}$$

We must consider this term in the BRST generator in order to consider the Virasoro constraint. Dealing with this new term is not complicated when we realize that $q'(z)$ has a conformal weight equal to one. This implies the OPE

$$\begin{aligned} c(z)T(z)q'(w) &= (c(w) + (z-w)\partial c(w))\left(\frac{q'_{sp}(w)}{(z-w)^2} + \frac{\partial q'_{sp}(w)}{(z-w)}\right) \\ &= ... + \frac{\partial(cq'_{sp})}{(z-w)} + ..., \end{aligned} \tag{3.111}$$

so the simple pole term is again just a total derivative, which implies that the corresponding anticommutators vanish due to (3.109).

Finally, we add a term for the conformal ghosts in order to cancel the OPE between $cT_{(z)}$ with itself. This is given by the usual $c\partial cb$ term. This means that, all contributions considered, the BRST current for the twistor-ambitwistor string is given by:

$$
\begin{aligned}
q(z) = \Big[& cT + c\partial cb + g_\alpha G^\alpha + \gamma T_F + s_m(\lambda\gamma^m f) + s_m(2\psi^m\beta) + \eta(\lambda\gamma_m\lambda)t^m \\
& +2(\lambda^\alpha g_\alpha)(g_\beta f^\beta) - 2(\lambda^\alpha g_\alpha)(\gamma\beta) - 2\big[\lambda^\alpha(\gamma_n)_{\alpha\beta}(\gamma^m)^{\beta\gamma}g_\gamma\big]s_m t^n \\
& +4(\lambda^\alpha g_\alpha)\eta\rho + 2\eta^{nm}s_n s_m\rho - \eta\beta^2 - 16\lambda^\alpha\partial g_\alpha\Big](z)
\end{aligned}
\tag{3.112}
$$

Chapter IV

The $\mathbb{CP}^{k-1}$ Extension of the CHY Formalism and the Tropical Grassmannian

In [36], a surprising connection was found between the CHY formalism (which is in principle just physically motivated), and a geometric structure known as the tropical Grassmannian [37] (which is in principle just mathematically motivated). Such relation was first found by generalizing the CHY formalism to higher dimensional complex projective spaces $\mathbb{CP}^{k-1}$, thus effectively replacing in the formula (2.18) the integration over a punctured Riemann sphere by an integration over punctured "world-volumes". In the next chapter we shall study the soft behavior of the generalized amplitudes defined by this procedure, and since our work extensively employs the extension of CHY just mentioned, we review such extension in this chapter. We start by defining the generalization of CHY to $\mathbb{CP}^{k-1}$ along similar lines to the ones presented in chapter 2. Afterwards, we present the tropical Grassmannian $\mathrm{Tr}\,\mathrm{G}(3,6)$ with a focus on what will be strictly necessary for us to relate it to the CHY formalism. Further details can be found in the original paper [37].

4.1 Scattering Equations over $\mathbb{CP}^{k-1}$ and Grassmannian Duality

In the extended CHY formalism, we define objects $\mathbf{s}_{a_1a_2\ldots a_k}$ as a completely symmetric "higher-dimensional matrix" components. The corresponding scattering equations can then be obtained as the conditions for the extrema of the following potential function [36, 64, 65]:

$$\mathcal{S}_k = \sum_{1\leq a_1<a_2<\ldots<a_k\leq n} \mathbf{s}_{a_1a_2\ldots a_k} \log{(a_1a_2\ldots a_k)}, \tag{4.1}$$

where $(a_1a_2...a_k)$ are SL(k, $\mathbb{C}$) invariant determinants of the homogeneous coordinates of the points a_i. Inhomogeneous coordinates are defined by rescaling the first homogeneous coordinate to be one, so inhomogeneous coordinates can be written as $(1, x^1, \ldots, x^{k-1})$.

In order for the potential $\mathcal{S}_k$ to be well defined, the following condition must be satisfied:

$$\mathbf{s}_{a_1a_2...bb} = 0. \tag{4.2}$$

The projective invariance of $\mathcal{S}_k$ implies

$$\sum_{\substack{a_2,a_3,\ldots,a_k=1 \\ a_i\neq a_j}}^{n} \mathbf{s}_{a_1a_2...a_k} = 0, \qquad \forall a_1. \tag{4.3}$$

If we apply this procedure for $k = 2$ we find that (4.2) corresponds to the condition for a particle to be massless, and (4.3) corresponds to the momentum conservation condition for kinematic invariants in QFT. Just as in subsection 2.1, since the abstract objects $\mathbf{s}_{a_1a_2...a_k}$ satisfy these properties, we identify them with kinematic invariants and we refer to their indices as particle labels from now on.

The configuration space over which the scattering equations are defined is denoted as $X(k, n)$, and it can be written as the Grassmannian G(k,n), modded out by a n-torus action

$$X(k, n) = G(k, n)/(\mathbb{C}^*)^n. \tag{4.4}$$

We have then all the ingredients to define the generalization of the CHY-amplitudes to higher-k:

$$A_n^{(k)} = \int d\mu_n^{(k)}\, \mathcal{I}, \tag{4.5}$$

where in the inhomogeneous coordinates x_a^i of $\mathbb{CP}^{k-1}$ ($i = 1, \ldots, k-1$, $a = 1, \ldots, (n-k-1)$), the measure takes the form

$$d\mu_n^{(k)} = \left(\frac{1}{\mathrm{Vol}[\mathrm{SL}(k,\mathbb{C})]} \prod_{a=1}^{n}\prod_{i=1}^{k-1} dx_a^i\right) \prod_{a=1}^{n}{\prod_{i=1}^{k-1}}' \delta\left(\frac{\partial \mathcal{S}_k}{\partial x_a^i}\right). \tag{4.6}$$

As each puncture lives in $\mathbb{CP}^{k-1}$ the integral must be SL(k, $\mathbb{C}$) invariant. We can then fix the location of $k+1$ punctures to generic positions (i.e. non-vanishing brackets), and similarly remove $k+1$ delta functions with a corresponding jacobian.

Under the same argument as in $k = 2$, $\mathrm{SL}(k, \mathbb{C})$ invariance let us fix the positions of $k+1$ points, so we need to take this into account before performing the integral. The fixing happens in an analogous way, and it amounts to divide by the jacobian associated with the matrix Φ whose elements are given by the second derivatives of the potential

$$\Phi_{(a,i),(b,j)} = \frac{\partial^2 \mathcal{S}_k}{\partial x_a^i \partial x_b^j}. \tag{4.7}$$

Again, this matrix is singular and we have to remove rows and columns associated with the overcounting of scattering equations, and fixing of points over $\mathbb{CP}^{k-1}$. After all this procedure is done the measure is covariant under $\mathrm{SL}(k, \mathbb{C})$ with weight $-2k$. An integrand is then constrained to have weight $2k$.

At this point, we must clarify that we have an abuse of terminology when we refer to the object constructed as "amplitude" in that we do not know if the objects considered actually correspond to scattering amplitudes of a theory having a Lagrangian description, or even a theory satisfying mild physical conditions. Notwithstanding this fact, the mathematical structure generalizing the CHY amplitudes does exist, and motivated by the fact that for $k = 2$ this generalization provides an alternative viewpoint to that of an integration over a punctured Riemann sphere (and for that matter, to that of ambitwistor strings) for the computation of biadjoint amplitudes, we refer to these objects simply as amplitudes.

Now, it was observed in [36] that the potential (4.1) inherits Grassmannian duality $G(k, n) \sim G(n-k, n)$. The argument is as follows. Consider invariance of this function under $\mathrm{SL}(k, \mathbb{C})$, and also use the rescaling of individual points to perform a $\mathrm{GL}(k, \mathbb{C})$ transformation over the $(a_1 a_2 \ldots a_k)$'s. We can gauge fix in such a way that the first $k \times k$ submatrix for the $k \times n$ matrix defined by the columns of the points is set to the identity. Once this is done, all the maximal minors of the matrix can be re-interpreted as the minors of a $(n-k) \times n$ matrix in which a $\mathrm{GL}(n-k, \mathbb{C})$ transformation has been used to set the maximal minor of the last $n-k$ columns to the identity. Let us illustrate this in the following (somewhat trivial) example for $k = 3$, $n = 5$:

$$\begin{pmatrix} 1 & 0 & 0 & a & b \\ 0 & 1 & 0 & c & d \\ 0 & 0 & 1 & e & f \end{pmatrix} \iff \begin{pmatrix} a & c & e & 1 & 0 \\ b & d & f & 0 & 1 \end{pmatrix}$$

The previous map identifies the $k \times k$ minor $(a_1, a_2, \ldots, a_k)$ with the $(n-k) \times (n-k)$ minor $(b_1, b_2, \ldots, b_{n-k})$, where the set $\{b_1, b_2, \ldots, b_{n-k}\}$ is the complement of $\{a_1, a_2, \ldots, a_k\}$ in $\{1, 2, \ldots, n\}$. We denote the complement as $\overline{a_1, a_2, \ldots, a_k}$.

We can apply this map to $\mathcal{S}_k$:

$$\mathcal{S}_k = \sum_{1 \leq a_1 < a_2 < \ldots < a_k \leq n} \mathbf{s}_{a_1 a_2 \ldots a_k} \log\left(\overline{a_1 a_2 \ldots a_k}\right), \tag{4.8}$$

and re-express $\mathcal{S}_k$ as a sum over the complementary sets noticing that the number of terms $\binom{n}{k}$ equals $\binom{n}{n-k}$. We can write:

$$\mathcal{S}_k = \sum_{1 \leq b_1 < b_2 < \ldots < b_{n-k} \leq n} \mathbf{s}_{b_1 b_2 \ldots b_{n-k}} \log{(b_1 b_2 \ldots b_{n-k})}, \tag{4.9}$$

where we have mapped the $\mathbf{s}_{a_1 a_2 \ldots a_k}$'s to corresponding variables with complementary labels $\mathbf{s}_{b_1 b_2 \ldots b_{n-k}}$. We still need to show that, from the point of view of the complementary sets:

$$\sum_{b_2, b_3, \ldots, b_{n-k}} \mathbf{s}_{b_1 b_2 \ldots a_{n-k}} = 0 \quad \forall b_1 \in \{1, 2, \ldots, n\}. \tag{4.10}$$

Without loss of generality, let us take the case $b_1 = 1$. Proving the previous is then equivalent to prove that

$$\sum_{1 < a_1 < a_2 < \ldots < a_k \leq n} \mathbf{s}_{a_1 a_2 \ldots a_k} = 0, \tag{4.11}$$

from the original (k, n) point of view. This is straightforward defining the quantities

$$D_{a_1} = \sum_{\substack{a_2, a_3, \ldots, a_k = 1 \\ a_i \neq a_j}}^{n} \mathbf{s}_{a_1 a_2 \ldots a_k} = 0 \qquad \forall a_1. \tag{4.12}$$

Then:

$$D_2 + D_3 + \ldots + D_n = D_1 + n \left(\sum_{1 < a_1 < a_2 < \ldots < a_k \leq n} \mathbf{s}_{a_1 a_2 \ldots a_k} = 0 \right). \tag{4.13}$$

Since all the D_i's are zero by (generalized) momentum conservation, the desired property is proved. This concludes the argument that $\mathcal{S}_k$, with the aforementioned identifications over the $\mathbf{s}$'s, defines also a valid potential function $\mathcal{S}_{n-k}$ over $\mathbb{CP}^{n-k-1}$. Finally, let us stress that this already shows that the standard scattering equations defined on the space of n points in $\mathbb{CP}^1$ are also equations for n points on $\mathbb{CP}^{n-3}$, which we take as an indication of the importance to fill the gap for projective spaces in between.

In the following, we shall consider the generalization of the Parke-Taylor half-integrand and the corresponding generalized biadjoint amplitudes, given by:

$$\mathrm{PT}_n^{(k)}[1, 2, \ldots, n] = \frac{1}{|12 \cdots k||23 \cdots (k+1)| \ldots |n\ 1 \cdots (k-1)|}, \tag{4.14}$$

$$m_n^{(k)}[\alpha|\beta] = \int d\mu_n^{(k)}\ \mathrm{PT}_n^{(k)}[\alpha]\ \mathrm{PT}_n^{(k)}[\beta]. \tag{4.15}$$

For a canonical chart of $X(k,n)$, $X(n-k,n)$, we can identify the coordinates

$$\begin{aligned} s_{a_1\ldots a_{n-k}} &= s_{b_1\ldots b_k}, \\ (a_1\ldots a_{n-k}) &= (b_1\ldots b_k), \end{aligned} \tag{4.16}$$

for complement labels, i.e. $\{b_i\}\cup\{a_i\} = \{1,\ldots,n\}$. In this case the Parke-Taylor factors satisfy

$$\mathrm{PT}_n^{(k)}[1,2,...,n] = \mathrm{PT}_n^{(n-k)}[1,2,...,n]. \tag{4.17}$$

Finally, as the CHY formula is independent of the $\mathrm{SL}(k,\mathbb{C})$ fixing, it must be that the generalized biadjoint amplitudes satisfy

$$m_n^{(k)}[\alpha|\beta] = m_n^{(n-k)}[\alpha|\beta]. \tag{4.18}$$

In particular $m_{k+2}^{(2)} = m_{k+2}^{(k)}$, which will prove to be a key insight in constructing the soft factor in the next chapter.

4.2 Review of Tr G(3,6)

The tropical Grassmannian Tr G(3,6) [37] is a polyhedral fan consisting of 65 vertices, 550 edges, 1395 triangles and 1035 tetrahedra. The set of vertices is obtained from solutions to "tropical planes" and are given by vectors (or rays) of the form

$$W = (w_{123}, w_{124}, \ldots, w_{456}) \in \mathbb{R}^{\binom{6}{3}}. \tag{4.19}$$

The full set is given in [37]. In particular, the set includes the basis vectors

$$\hat{e}_{ijk} = (0,\ldots,1,\ldots,0), \tag{4.20}$$

with three unordered and unrepeated indices $i,j,k \in \{1,...,6\}$. Considering the coordinate vector $S = (\mathbf{s}_{123}, \mathbf{s}_{124}, \ldots)$, we can define:

$$S\cdot W = \sum_{i<j<k} \mathbf{s}_{ijk} w_{ijk}. \tag{4.21}$$

We can now identify $S\cdot\hat{e}_{ijk}=\mathbf{s}_{ijk}$ as vertices. Since in the rest of this work we consider only the latter objects $\mathbf{s}_{ijk}$, and therefore no confusion should arise, we use the name vertices for them from now on. Now, the vectors (4.19) have a shift symmetry $w_{ijk}\to w_{ijk}+c_i+c_j+c_k$ inherited from their tropical planes. This implies the relation

$$0=\sum_{\substack{j<k\\ j,k\neq i}}\mathbf{s}_{ijk}. \tag{4.22}$$

Besides $\mathbf{s}_{ijk}$, other vertices of $\operatorname{Tr}\mathrm{G}(3,6)$ are mapped to the following combinations:

$$\mathbf{t}_{abcd}=\mathbf{s}_{abc}+\mathbf{s}_{abd}+\mathbf{s}_{acd}+\mathbf{s}_{bcd}, \tag{4.23}$$

$$R_{abcdef}=\mathbf{t}_{abcd}+\mathbf{s}_{cde}+\mathbf{s}_{cdf}. \tag{4.24}$$

The following relations hold:

$$R_{12,34,56}+R_{34,12,56}=\mathbf{t}_{1234}+\mathbf{t}_{3456}+\mathbf{t}_{5612}, \tag{4.25}$$

$$R=R_{12,34,56}=R_{34,56,12}=R_{56,12,34}. \tag{4.26}$$

There are 20 vertices corresponding to $\mathbf{s}$ (forming a set E), 15 vertices corresponding to $\mathbf{t}$ (in a set F), and 30 vertices corresponding to R (in a set G). Starting from the vertices $v_i\in E\cup F\cup G$ one can then build the edges of the $\operatorname{Tr}\mathrm{G}(3,6)$ as particular pairs $\{v_{i_h},v_{j_h}\}$. There are 550 such edges in $\operatorname{Tr}\mathrm{G}(3,6)$.

As we have made obvious through our notation, we have drawn upon the conjecture arising from the work [36] to call the coordinate functions of $\operatorname{Tr}\mathrm{G}(k,n)$ [37] in the same way as the kinematic invariants $\mathbf{s}_{a_1a_2\ldots a_k}$. Along these lines, we notice that the momentum conservation condition (4.3) corresponds to the modding-out condition (4.22) that defines the tropical Grassmannian.

As shown in [36], the generalized biadjoint amplitudes $m_6^{(3)}$ defined in the previous subsection through the generalization of CHY (4.15) can be computed from the facets (i.e. sets of edges) of the tropical Grassmannian $\operatorname{Tr}\mathrm{G}(3,6)$ associated to the given planar orderings. Let us then describe the full set of facets of $\operatorname{Tr}\mathrm{G}(3,6)$.

The facets come in six different classifications and 15 four-simplices. The six classifications correspond to tetrahedral facets[8]' given by all possible relabelings of the representatives shown in Table 1. The 15 four-simplices are given by permutations of labels of:

$$\{\mathbf{t}_{1234},\mathbf{t}_{3456},\mathbf{t}_{5612},R_{12,34,56},R_{12,56,34}\}, \tag{4.27}$$

[8]A tetrahedral facet is defined as a set of four vertices $\{v_1,v_2,v_3,v_4\}\subset E\cup F\cup G$, where each pair of vertices is an edge.

EEEE	$\{\mathbf{s}_{123},\, \mathbf{s}_{345},\, \mathbf{s}_{561}\ ,\mathbf{s}_{246}\}$
EEFF1	$\{\mathbf{s}_{123},\, \mathbf{s}_{456},\, \mathbf{t}_{1234},\, \mathbf{t}_{3456}\}$
EEFF2	$\{\mathbf{s}_{123},\, \mathbf{s}_{345},\, \mathbf{t}_{3456},\, \mathbf{t}_{1236}\}$
EFFG	$\{\mathbf{s}_{345},\, \mathbf{t}_{1256},\, \mathbf{t}_{3456},\, R_{12,34,56}\}$
EEEG	$\{\mathbf{s}_{123},\, \mathbf{s}_{561},\, \mathbf{s}_{345},\, R_{45,23,61}\}$
EEFG	$\{\mathbf{s}_{123},\, \mathbf{t}_{3456},\, \mathbf{s}_{345},\, R_{12,34,56}\}$

Table 1: Representatives of facets of the tropical Grassmannian Tr G(3,6)

and we associate to them a representative contribution

$$\Theta = \frac{\mathbf{t}_{1234} + \mathbf{t}_{3456} + \mathbf{t}_{5612}}{\mathbf{t}_{1234}\mathbf{t}_{3456}\mathbf{t}_{5612} R\tilde{R}}, \qquad \tilde{R} = R_{(12)\leftrightarrow(34)}. \tag{4.28}$$

These contributions must be considered separately from the facets given by Table 1, and we must also consider them together with contributions $\tilde{\Theta}$, arising from (4.27) under a cyclic shift of the labels. Notice that we can use (4.25) to rewrite (4.28) as a sum of either two or three terms,

$$\begin{aligned}\Theta &= \frac{1}{\mathbf{t}_{1234}\mathbf{t}_{3456}\mathbf{t}_{5612} R} + \frac{1}{\mathbf{t}_{1234}\mathbf{t}_{3456}\mathbf{t}_{5612} \tilde{R}} \\ &= \frac{1}{\mathbf{t}_{1234}\mathbf{t}_{3456} R\tilde{R}} + \frac{1}{\mathbf{t}_{1234}\mathbf{t}_{5612} R\tilde{R}} + \frac{1}{\mathbf{t}_{3456}\mathbf{t}_{5612} R\tilde{R}}.\end{aligned} \tag{4.29}$$

As explained in [36] (see also [46]), these correspond to two different ways to split a bipyramid in terms of tetrahedra. Eq. (4.28) makes manifest the fact that the fundamental object is the bipyramid, and not the tetrahedra in which it can be split (and that are not facets of Tr G(3,6)).

With this information one can give an algorithm to construct $m_6^{(3)}[\alpha|\beta]$ solely from Tr G(3,6), where α and β are two permutations of the cyclic ordering $\mathbb{I}_6 = (123456)$. One takes the subset of all vertices in $E \cup F \cup G$ consistent with a given permutation α. This defines an ordered part of the tropical Grassmannian which we refer to as $\mathrm{Tr}\,\mathrm{G}_\alpha(3,6)$. The corresponding facets are the subset of facets of Tr G(3,6) made with the vertices of $\mathrm{Tr}\,\mathrm{G}_\alpha(3,6)$. Following [36] we denote their contribution by $J(\alpha)$, where each facet is represented by the inverse power of its vertices. For instance

$$J(\mathbb{I}_6) = \{\frac{1}{\mathbf{s}_{123}\mathbf{s}_{345}\mathbf{s}_{561}\mathbf{s}_{246}}, \frac{1}{\mathbf{s}_{123}\mathbf{s}_{456}\mathbf{t}_{1234}\mathbf{t}_{3456}}, \frac{1}{\mathbf{s}_{345}\mathbf{t}_{1256}\mathbf{t}_{3456} R_{12,34,56}}, \ldots, \Theta_{123456}, \tilde{\Theta}_{123456}\} \tag{4.30}$$

has 48 elements. Then

$$m_6^{(3)}[\alpha|\beta] = \sum_{\Upsilon\in J(\alpha)\bigcap J(\beta)} \Upsilon\,. \tag{4.31}$$

We will be mostly interested in the full amplitude of $\mathrm{Tr}\,\mathrm{G}_{\mathbb{I}}(3{,}6)$, given by the set of all facets consistent with the canonical ordering. This is given by

$$m_6^{(3)}[\mathbb{I}_6|\mathbb{I}_6] = \sum_{\Upsilon\in J(\mathbb{I}_6)} \Upsilon\,. \tag{4.32}$$

Let us exemplify the procedure just described using the orderings $\alpha = \mathbb{I}_6$, $\beta = 125643$. It is straightforward to see that these orderings are consistent with the vertices (4.27) defining a four-simplex, so we must consider a Θ contribution (4.28), i.e. $\Theta_{123456} = \Theta_{125634}$. One can check that there are only three more facets that are consistent with the orderings α and β. One of these facets belong to the EEFF1 category, and the two remaining facets correspond to the EFFG category. The algorithm gives (up to an overall sign determined in [36]):

$$m_6^{(3)}[\mathbb{I}_6|125643] = \Theta + \frac{1}{\mathbf{t}_{1234}\mathbf{t}_{3456}}\left(\frac{1}{\mathbf{s}_{456}\tilde{R}} + \frac{1}{\mathbf{s}_{123}R} + \frac{1}{\mathbf{s}_{123}\mathbf{s}_{456}}\right), \tag{4.33}$$

which can be checked is correct by direct computation of the CHY integral (4.5) with two half-integrands (4.14) [36]. Thus, we have found a highly non-trivial relation between pure mathematics and physics by relating the tropical Grassmannian $\mathrm{Tr}\,\mathrm{G}_{\mathbb{I}}(3{,}6)$ with the CHY formalism.

Chapter V

A Soft Theorem for the Tropical Grassmannian

In the previous chapter we have outlined the connection between the tropical Grassmannian and generalized biadjoint scalar amplitudes (defined through a suitable generalization of the CHY formalism) first found in [36]. Given the definition of these "amplitudes" through CHY as an integration over higher-dimensional punctured manifolds, and motivated by their geometrical relation to the tropical Grassmannian, we may ask ourselves about a suitable realization of unitarity or locality for these generalized amplitudes. In this chapter we take a first step in this direction studying the soft limit associated to these amplitudes, which provide the simplest possible example of factorization for these newly defined objects. Notably, we find that the soft factor associated to the higher-k amplitudes is actually given in terms of the standard $k = 2$ amplitudes. Having already outlined the connection between CHY and $\mathrm{Tr}\,\mathrm{G}_{\mathbb{I}}(3, 6)$ in the previous chapter, we start studying the soft theorem in purely geometric terms for the specific case of $\mathrm{Tr}\,\mathrm{G}_{\mathbb{I}}(3, 6)$, where in-passing, we shall find also a "hard theorem" which can be understood as the Grassmannian-dual of the soft theorem. In order to gain intuition about the full soft theorem, we continue with a direct proof of the soft theorem for $k = 3$ amplitudes via the GRT, thereby recovering the result obtained geometrically for $\mathrm{Tr}\,\mathrm{G}_{\mathbb{I}}(3, 6)$ and extending it to all multiplicity. We then proceed to show the soft theorem for arbitrary (k, n). We conclude this chapter with some discussions and ideas for further research.

5.1 The Soft Limit for Tr G(3,6)

Although in the previous chapter we have provided the geometric structure of the tropical Grassmannian Tr G(3,6), it is not yet clear what is the interpretation for a single label $i = 1, \ldots, 6$. Since for $k = 2$ these labels correspond to the particles involved in a scattering process, let us begin this section with an attempt to extend such notion for higher values of k, which will lead to the soft theorem. With this in mind, and with the benefit of hindsight, let us consider the following set of vertices in the canonically ordered part of the tropical Grassmannian, $\mathrm{Tr}\,\mathrm{G}_{\mathbb{I}}(3{,}6)$:

$$\left\{\mathbf{s}_{123},\mathbf{s}_{156},\mathbf{s}_{126},\mathbf{t}_{2345},\mathbf{t}_{3456}\right\}. \tag{5.1}$$

Consider now the edges inside $\mathrm{Tr}\,\mathrm{G}_{\mathbb{I}}(3{,}6)$ constructed from this set of vertices. One finds that there are five such edges, and they turn out to form a smaller object, $\mathrm{Tr}\,\mathrm{G}_{\mathbb{I}}(3{,}5)$! The facets of this object are constructed from two vertices [37], and they are characterized by the following contributions:

$$S_{1,23456}=\left\{\frac{1}{\mathbf{s}_{123}\mathbf{s}_{156}},\frac{1}{\mathbf{t}_{2345}\mathbf{s}_{156}},\frac{1}{\mathbf{t}_{2345}\mathbf{s}_{126}},\frac{1}{\mathbf{t}_{3456}\mathbf{s}_{123}},\frac{1}{\mathbf{t}_{3456}\mathbf{s}_{126}}\right\}. \tag{5.2}$$

For later convenience let us also define

$$S^{(3)}_{1,23456}=\sum_{\Upsilon\in S_{1,23456}}\Upsilon=\frac{1}{\mathbf{s}_{123}\mathbf{s}_{156}}+\frac{1}{\mathbf{t}_{2345}}\left(\frac{1}{\mathbf{s}_{156}}+\frac{1}{\mathbf{s}_{126}}\right)+\frac{1}{\mathbf{t}_{3456}}\left(\frac{1}{\mathbf{s}_{123}}+\frac{1}{\mathbf{s}_{126}}\right). \tag{5.3}$$

Since $S_{1,23456}$ is identified with a subset of edges of $\mathrm{Tr}\,\mathrm{G}_{\mathbb{I}}(3{,}6)$, the corresponding edges will naturally appear in the facets of $\mathrm{Tr}\,\mathrm{G}_{\mathbb{I}}(3{,}6)$. In fact, each facet turns out to have at most one edge of the set $S_{1,23456}$. A way to show this explicitly is to write down the amplitude (4.32), which contains the following terms:

$$\begin{aligned}
m_6^{(3)}[\mathbb{I}_6|\mathbb{I}_6]=&\frac{1}{\mathbf{s}_{123}\mathbf{s}_{561}}\left[\frac{1}{R_{45,23,61}\mathbf{s}_{345}}+\frac{\frac{1}{\mathbf{s}_{345}}+\frac{1}{\mathbf{t}_{1234}}}{R_{12,34,56}}+\frac{\frac{1}{R_{45,23,61}}+\frac{1}{\mathbf{t}_{1234}}}{\mathbf{t}_{4561}}\right]\\
&+\frac{1}{\mathbf{t}_{2345}\mathbf{s}_{561}}\left[\frac{1}{R_{45,23,61}\mathbf{s}_{345}}+\frac{\frac{1}{R_{45,23,61}}+\frac{1}{\mathbf{s}_{234}}}{\mathbf{t}_{4561}}+\frac{\frac{1}{\mathbf{s}_{234}}+\frac{1}{\mathbf{s}_{345}}}{\mathbf{t}_{5612}}\right]\\
&+\frac{1}{\mathbf{t}_{2345}\mathbf{s}_{612}}\left[\frac{1}{R_{23,45,61}s_{234}}+\frac{\frac{1}{\mathbf{s}_{234}}+\frac{1}{\mathbf{s}_{345}}}{\mathbf{t}_{5612}}+\frac{\frac{1}{R_{23,45,61}}+\frac{1}{\mathbf{s}_{345}}}{\mathbf{t}_{6123}}\right]\\
&+\frac{1}{\mathbf{t}_{3456}\mathbf{s}_{123}}\left[\frac{1}{R_{12,34,56}\mathbf{s}_{345}}+\frac{\frac{1}{R_{12,34,56}}+\frac{1}{\mathbf{s}_{456}}}{\mathbf{t}_{1234}}+\frac{\frac{1}{\mathbf{s}_{345}}+\frac{1}{\mathbf{s}_{456}}}{\mathbf{t}_{6123}}\right]\\
&+\frac{1}{\mathbf{t}_{3456}\mathbf{s}_{612}}\left[\frac{1}{R_{34,12,56}\mathbf{s}_{456}}+\frac{\frac{1}{R_{34,12,56}}+\frac{1}{\mathbf{s}_{345}}}{\mathbf{t}_{5612}}+\frac{\frac{1}{\mathbf{s}_{345}}+\frac{1}{\mathbf{s}_{456}}}{\mathbf{t}_{6123}}\right]+\ldots
\end{aligned} \tag{5.4}$$

We find that there are five groups of five terms, each group containing a given edge in (5.2). Quite non-trivially, it turns out that these also contain the structure of (5.3). This means each group has the structure of $\mathrm{Tr}\,\mathrm{G}_{\mathbb{I}}(3{,}5)$ but it is constructed from a different set of vertices. In fact, these groups are nothing but copies of the smaller amplitude $m_5^{(3)}$! To see this more clearly we introduce the following limit, which we call the soft limit associated to the label 1:

$$\mathbf{s}_{1ab} = \tau \hat{\mathbf{s}}_{1ab}, \quad \tau \to 0, \quad \hat{\mathbf{s}}_{1ab} \text{ fixed}. \tag{5.5}$$

From all the vertices in $\operatorname{Tr} \mathrm{G}_{\mathbb{I}}(3,6)$, only the set of vertices (5.1) vanish at $\tau = 0$ (which is the reason why we considered this set to begin with). We refer to the vertices in this set as the "soft vertices" associated to the label 1 and to the canonical ordering. The edges inside $S_{1,23456}$ are then called "soft edges". The remaining "hard" vertices are deformed and possibly degenerate into each other. For example, $\mathbf{t}_{1234} \to \mathbf{s}_{234}$, $R_{23,45,61} \to \mathbf{s}_{456}$, etc...

Now, we note that in the limit (5.5) the square brackets in (5.4) collapse to the same quantity. Note further that the terms in $m_6^{(3)}[\mathbb{I}_6|\mathbb{I}_6]$ not containing a soft edge will be subleading as $\tau \to 0$. Hence, the amplitude exhibits the remarkable factorization

$$\begin{aligned} m_6^{(3)}[\mathbb{I}_6|\mathbb{I}_6] = \; & \frac{1}{\tau^2}\left[\frac{1}{\hat{\mathbf{s}}_{123}\hat{\mathbf{s}}_{156}} + \frac{1}{\hat{\mathbf{t}}_{2345}}\left(\frac{1}{\hat{\mathbf{s}}_{156}} + \frac{1}{\hat{\mathbf{s}}_{126}}\right) + \frac{1}{\hat{\mathbf{t}}_{3456}}\left(\frac{1}{\hat{\mathbf{s}}_{123}} + \frac{1}{\hat{\mathbf{s}}_{126}}\right)\right] \\ & \times\left[\frac{1}{\mathbf{s}_{236}\mathbf{s}_{345}} + \frac{1}{\mathbf{s}_{256}}\left(\frac{1}{\mathbf{s}_{345}} + \frac{1}{\mathbf{s}_{234}}\right) + \frac{1}{\mathbf{s}_{456}}\left(\frac{1}{\mathbf{s}_{236}} + \frac{1}{\mathbf{s}_{234}}\right)\right] + \mathcal{O}\left(\frac{1}{\tau}\right) \\ = \; & S^{(3)}_{1,23456} \times m_5^{(3)}[23456|23456] + \mathcal{O}\left(\frac{1}{\tau}\right), \end{aligned} \tag{5.6}$$

where we used that the second factor also corresponds to a $k = 3$ amplitude, independent of label 1. This is a generalization of the $k = 2$ soft theorem (1.5) to the case of $\operatorname{Tr} \mathrm{G}_{\mathbb{I}}(3,6)$. Considering this generalization, we refer to $S^{(3)}_{1,23456}$ in (5.3) as a soft factor. We are going to prove that this factorization holds for all $m_n^{(3)}$ amplitudes in subsection 5.3. Then, we extend the theorem to arbitrary values of k in subsection 5.4. Note that in contrast to the $k = 2$ case, the leading divergence here behaves as $\sim \frac{1}{\tau^2}$.

In terms of the facets, the soft theorem can be interpreted as an embedding

$$J(\mathbb{I}_6) \supset J_{\text{soft}}(\mathbb{I}_6) \sim S_{1,23456} \times J(23456), \tag{5.7}$$

where $J_{\text{soft}}(\mathbb{I}_6)$ denotes the facets in $J(\mathbb{I}_6)$ containing a soft edge and $\times$ is here the cartesian product. This means each facet in $J(23456)$ can be mapped to a facet of $J(\mathbb{I}_6)$ in five different ways, one for each soft edge. Note that the two factors in (5.7) correspond to two instances of $\operatorname{Tr} \mathrm{G}_{\mathbb{I}}(3,5)$.

Note that the formula (5.6) hints that the terms can be grouped as in (5.4). This is because the soft factor $S^{(3)}_{1,23456}$ is invariant under the soft limit, whereas for each soft edge the amplitude $m_5^{(3)}[23456|23456]$ will undergo a particular deformation as we depart from the $\tau \to 0$ limit. Additionally, the fact that two or more soft edges cannot appear together in a facet follows here from the leading order of $m_6^{(3)}$ being τ^{-2}, which is true for all $m_n^{(3)}$ amplitudes (see below subsection 5.2). On the other hand, one should be careful when analyzing facets with non-trivial numerators such as Θ in (4.28) (see [45] for $n = 7$ examples), which here turned out to not contribute to the soft limit as they do not contain soft edges.

The factorization (5.7) is not a property of $m_6^{(3)}[\mathbb{I}_6|\mathbb{I}_6]$ only. Rather, it is a property shown by any $m_6^{(3)}[\alpha|\beta]$, with α and β any orderings. The factorization for arbitrary orderings is given by the following compatibility rule. For each ordering (1γ), there is a set

$$J_{\text{soft}}(1\gamma) \sim S_{1,\gamma} \times J(\gamma). \tag{5.8}$$

The left hand side of (5.8) can be defined as the set of all facets in $\operatorname{Tr} \mathrm{G}_{\mathbb{I}}(3{,}6)$ of order τ^{-2}, and can be thought as evaluated at leading order. According to the definition (4.31) the order τ^{-2} facets of $m_6^{(3)}[\alpha|\beta]$ will be given by

$$\begin{aligned} J_{\text{soft}}(1\alpha) \cap J_{\text{soft}}(1\beta) \sim & \Big(S_{1,\alpha} \times J(\alpha)\Big) \cap \Big(S_{1,\beta} \times J(\beta)\Big) \\ = & \Big(S_{1,\alpha} \cap S_{1,\beta}\Big) \times \Big(J(\alpha) \cap J(\beta)\Big). \end{aligned} \tag{5.9}$$

This means that the leading order of $m_6^{(3)}[1\alpha|1\beta]$ also satisfies a soft theorem (5.6) with the soft factor

$$S^{(3)}_{1\alpha|1\beta} = \sum_{\Upsilon \in S_{1\alpha} \cap S_{1\beta}} \Upsilon \,, \tag{5.10}$$

i.e. only considering soft edges compatible with both orderings. This rule can be trivially extended to all $m_n^{(k)}$ amplitudes, once we provide the corresponding soft theorems.

Now, let us provide a short example and apply the rule to find the soft factorization associated to the amplitude $m_6^{(3)}[\mathbb{I}_6|125643]$ obtained in (4.33). It is straightforward to see from (5.3) that there is only one soft edge consistent with the 125643 ordering, namely, the $(\mathbf{t}_{3456}\mathbf{s}_{123})^{-1}$ term. On the other hand, we have that

$$m_5^{(3)}[\mathbb{I}_5|25643] = \frac{1}{\mathbf{s}_{234}} \left[\frac{1}{\mathbf{s}_{256}} + \frac{1}{\mathbf{s}_{456}} \right], \tag{5.11}$$

which is easily found dualizing the corresponding biadjoint amplitude for $k = 2$. Then, we can check from (4.33) that the $n = 6$ amplitude indeed satisfies the claimed soft theorem:

$$m_6^{(3)}[\mathbb{I}_6|125643] \longrightarrow \frac{1}{\tau^2} \left[\frac{1}{\hat{\mathbf{t}}_{3456}\hat{\mathbf{s}}_{123}} \right] \times \frac{1}{\mathbf{s}_{234}} \left[\frac{1}{\mathbf{s}_{256}} + \frac{1}{\mathbf{s}_{456}} \right]. \tag{5.12}$$

Before closing this section, let us study a different soft factorization. Consider the deformation

$$\mathbf{s}_{abc} = \tau \hat{\mathbf{s}}_{abc}, \quad \tau \to 0, \quad \hat{\mathbf{s}}_{abc} \text{ fixed}, \quad a, b, c \neq 1, \tag{5.13}$$

which we can interpret as a "hard limit" associated to the label 1, since now all vertices corresponding to this label remain finite while other vertices tend to zero. We find that in this limit the amplitude also factorizes, leading to

$$\begin{aligned} m_6^{(3)}[\mathbb{I}_6|\mathbb{I}_6] = \; & \frac{1}{\tau^2}\left[\frac{1}{\hat{\mathbf{s}}_{456}\hat{\mathbf{s}}_{234}} + \frac{1}{\hat{\mathbf{t}}_{2345}}\left(\frac{1}{\hat{\mathbf{s}}_{234}} + \frac{1}{\hat{\mathbf{s}}_{345}}\right) + \frac{1}{\hat{\mathbf{t}}_{3456}}\left(\frac{1}{\hat{\mathbf{s}}_{456}} + \frac{1}{\hat{\mathbf{s}}_{345}}\right)\right] \\ & \times \left[\frac{1}{\mathbf{s}_{145}\mathbf{s}_{126}} + \frac{1}{\mathbf{s}_{134}}\left(\frac{1}{\mathbf{s}_{126}} + \frac{1}{\mathbf{s}_{156}}\right) + \frac{1}{\mathbf{s}_{123}}\left(\frac{1}{\mathbf{s}_{145}} + \frac{1}{\mathbf{s}_{156}}\right)\right] + \mathcal{O}\Big(\frac{1}{\tau}\Big). \end{aligned} \tag{5.14}$$

We call this a "hard theorem" for $\operatorname{Tr} \mathrm{G}_{\mathbb{I}}(3{,}6)$. Even though it looks unrelated at first to the soft theorem, it corresponds to a different soft limit: The limit of $m_6^{(3)}$ when dualized in the Grassmannian sense. We will come back to this remarkable property later. For now we just point out that the fact that we find again two factors with the structure of $\operatorname{Tr} \mathrm{G}_{\mathbb{I}}(2{,}5)$ is a consequence of the self-duality of $\operatorname{Tr} \mathrm{G}(3{,}6)$.

5.2 Soft Limit Setup

Before presenting and proving the general soft theorem, we set the grounds by defining the soft limit in terms of the CHY formula. Throughout this subsection, and in the rest of this thesis, we take particle 1 as the soft particle.

In the ordinary $k = 2$ case, we define the soft limit by taking the low energy-momentum limit of a particle; that is, we define the soft momentum by $p_1^\mu = \tau \hat{p}_1^\mu$, (together with the corresponding deformations from momentum conservation), and then take $\tau \to 0$. For a scalar theory this is equivalent to set $\mathbf{s}_{1a} = \tau \hat{\mathbf{s}}_{1a}$. For arbitrary k one defines the soft limit as the natural generalization of the latter procedure: We take $\mathbf{s}_{1a_2\ldots a_k} = \tau \hat{\mathbf{s}}_{1a_2\ldots a_k}$, with $\tau \to 0$ and $\hat{\mathbf{s}}_{1a_2\ldots a_k}$ fixed, as done in subsection 5.1 for $k = 3$.

Before taking the soft limit in the CHY amplitude, let us review what happens to the scattering equations and momentum conservation in the soft limit. To leading order in the soft limit, the momentum conservation condition for the soft particle decouples from the momentum conservation condition for the other particles, and we obtain the following two separate relations:

$$\sum_{\substack{a_2,a_3,\ldots,a_k=2\\ a_i\neq a_j}}^{n} \mathbf{s}_{a_1a_2\ldots a_k} = 0, \qquad a_1 \neq 1. \tag{5.15}$$

$$\sum_{\substack{a_2,a_3,\ldots,a_k=2\\ a_i\neq a_j}}^{n} \mathbf{s}_{1a_2\ldots a_k} = 0. \tag{5.16}$$

Observe that to obtain these we take the variables $\mathbf{s}_{1a_2\ldots a_k}$ as independent (up to the condition (5.16)), which will be relevant when defining the hard limit. In a similar way, the scattering equations for the soft particle separate to leading order from the scattering equations for the other particles:

$$E_1^{x^i} \to \tau \hat{E}_1^{x^i}, \qquad \hat{E}_1^{x^i} = \sum_{\substack{a_2,a_3,\ldots,a_k=2\\ a_i\neq a_j}}^{n} \frac{\hat{\mathbf{s}}_{1a_2\ldots a_k}}{|1a_2\ldots a_k|}\frac{\partial}{\partial x^i}|1a_2\ldots a_k| = 0, \tag{5.17}$$

$$E_{a_1}^{x^i} \to \sum_{\substack{a_2,a_3,\ldots,a_k=2\\ a_i\neq a_j}}^{n} \frac{\mathbf{s}_{a_1a_2\ldots a_k}}{|a_1a_2\ldots a_k|}\frac{\partial}{\partial x^i}|a_1a_2\ldots a_k| = 0, \qquad a_1 \neq 1, \tag{5.18}$$

where we have called $E_a^{x^i} = \partial_{x_a^i}\mathcal{S}_k$. Note that in general for $k>2$ the scattering equations possess singular solutions [36,45] for which some invariants $|1a_2\ldots a_k|$ may vanish with τ and the expansion (5.17) may not be valid. Here, we assume that such solutions only contribute at subleading orders in the τ expansion of $m_n^{(k)}$, but we note that it has been numerically checked in [63] and further worked out in [52] that such assumption is indeed correct.

We can now use the CHY formula and the factorization properties just described to isolate the integral over the soft particle in the CHY formula (4.15). Using the canonical ordering $1\alpha = (12\cdots n)$:

$$m_n^{(k)}[1\alpha|1\alpha] = \int d\mu_n^{(k)} \left[\mathrm{PT}_n^{(k)}[1\alpha]\right]^2 \sim \frac{1}{\tau^{k-1}}\int d\mu_{n-1}^{(k)}\left[\mathrm{PT}_{n-1}^{(k)}[\alpha]\right]^2 S_n^{(k)}(1\alpha|1\alpha), \tag{5.19}$$

where

$$S_n^{(k)}(1\alpha|1\alpha) = \int\left[\frac{d^{k-1}x_1^i}{\prod_i \hat{E}_1^{x^i}}\right]\left(\frac{|n2\ldots k|\ldots|(n-k+2)(n-k+3)\ldots n2|}{|12\cdot\cdot k||n1\cdot\cdot k-1|\ldots|(n-k+2)(n-k+3)\cdot\cdot 1|}\right)^2, \tag{5.20}$$

and the contour in this soft factor integral is taken to enclose the zeros of the soft scattering equations $|\hat{E}_1^{x^i}| = \epsilon$ for fixed hard punctures x_a^i, $a > 1$. This integral can be computed in the standard way, summing over all solutions of the soft scattering equations:

$$\sum_{\hat{E}_1^{x^i}=0} \left(\frac{|n2...k|...|(n-k+2)(n-k+3)...n|}{|12\cdot\cdot k||n1\cdot\cdot k-1|...|(n-k+2)(n-k+3)\cdot\cdot 1|} \right)^2 \left| \frac{d\hat{E}_1^{x^j}}{dx_1^i} \right|^{-1} . \tag{5.21}$$

From now on we will omit the subindex n and the orderings in the soft factor, as it will be implicitly assumed that we are working with identity amplitudes and a n-particle scattering process. That is, we write $S_n^{(k)}(\mathbb{I}_n|\mathbb{I}_n) = S^{(k)}$. Given the results we have found for $\mathrm{Tr}\,\mathrm{G}_{\mathbb{I}}(3,6)$ in subsection 5.1, it is natural to expect the following soft factorization:

$$m_n^{(k)}[1\alpha|1\alpha] = \frac{1}{\tau^{k-1}} \left(S^{(k)} m_{n-1}^{(k)}[\alpha|\alpha] + \mathcal{O}(\tau) \right). \tag{5.22}$$

In particular, we have just shown, via the CHY representation, that the leading order of the amplitude is $\tau^{-(k-1)}$, e.g. τ^{-2} for $k = 3$. The difficulty that one now faces with this relation is that (5.20) apparently depends on the position of the punctures of the hard particles. If this was truly the case, one could not factorize the soft factor out of the $n-1$ particles CHY integral in (5.19). This translates into the condition for the existence of a soft theorem for tropical Grassmannian amplitudes: if one can show that the integral (5.20) is independent of all the punctures of the hard particles, then a soft theorem is guaranteed to exist, even when the explicit form of the soft factor would not be necessarily known.

Let us close this subsection remarking an important consequence of our definition of soft limit: It is not respected by the duality (4.16): From the perspective of $\mathrm{Tr}\,\mathrm{G}_{\mathbb{I}}(n-k,n)$ the above limit is equivalent to

$$\mathbf{s}_{b_1\ldots b_{n-k}} = \tau\hat{\mathbf{s}}_{b_1\ldots b_{n-k}}\,, \;\; b_i \neq 1, \quad \tau \to 0, \tag{5.23}$$

which means we can drop $\mathbf{s}_{b_1\ldots b_{n-k}}$ with respect to invariants containing label 1, $\mathbf{s}_{1b_2\ldots b_{n-k}}$. Thus, this is nothing but the CHY statement of the "hard limit" of particle 1 that we already encountered in subsection 5.1 for the $\mathrm{Tr}\,\mathrm{G}_{\mathbb{I}}(3,6)$ case. Importantly, in this case we have taken the quantities in (5.23) as independent up to the condition

$$\sum_{b_1\ldots b_{n-k}\neq 1}^{n} \mathbf{s}_{b_1\ldots b_{n-k}} = 0, \tag{5.24}$$

which is the condition (5.16) from the perspective of the dual Grassmannian.

5.3 Direct Proof of the Soft Theorem for Tr $\mathrm{G}_{\mathbb{I}}$(3,n)

Let us here prove and provide the soft theorem for $k = 3$ at all multiplicity via a direct application of the GRT. This will provide the insight that we will build on in subsection 5.4. In order to make the derivation transparent we will adopt homogeneous coordinates on $X(3,n)$, which we denote as (abc) instead of $|abc|$. Then, we can write the soft factor integral (5.20) in homogeneous form:

$$S^{(3)} = \int \frac{(\sigma d^2\sigma)(XY\sigma)}{\underline{\left(\sum_{b,c}\frac{\mathbf{s}_{1bc}(Xbc)}{(\sigma bc)}\right)}\,\underline{\left(\sum_{b,c}\frac{\mathbf{s}_{1bc}(Ybc)}{(\sigma bc)}\right)}}\left(\frac{(n-1,n,2)(n,2,3)}{(n-1,n,\sigma)(n,\sigma,2)(\sigma,2,3)}\right)^2, \tag{5.25}$$

where $X, Y \in \mathbb{CP}^2$ are arbitrary reference vectors (which essentially encode our choice of setting the first coordinate to 1 in inhomogeneous coordinates above), and σ is the puncture associated to the soft particle (i.e. particle 1). For brevity, we also used and we will use underline notation to denote the corresponding contours of integration. As we have not picked a specific chart of $\mathbb{CP}^2$, the GRT as stated in [62] will hold without the need of boundary terms, commonly understood as residues at infinity for a given chart.

In order to use the GRT note that the locus of each of the scattering equations is discontinuous where two lines $(\sigma ab) = 0$ intersect (see Figure 1). We can soften some of these points by defining $\bar{E}_X = (n-1,n,\sigma)(n,\sigma,2)(\sigma,2,3)E_X$ and the same for $\bar{E}_Y$. The curves $\bar{E}_X$ and $\bar{E}_Y$ have the same zeroes as the scattering equations, but also include three additional points defined by the pairings $(n-1,n,\sigma) = (n,\sigma,2) = 0$, $(n,\sigma,2) = (\sigma,2,3) = 0$, and $(n-1,n,\sigma) = (\sigma,2,3) = 0$. Note also that $\{\bar{E}_X, \bar{E}_Y\}$ include all possible singularities in the integration and hence can be thought as divisors on $\mathbb{CP}^2$ (see appendix B for definitions), which means we can implement the GRT as stated in [62]. The statement becomes

$$0 = \int \frac{(\sigma d^2\sigma)(XY\sigma)}{\underline{\bar{E}_X} \times \underline{\bar{E}_Y}}\left((n-1,n,2)(n,2,3)\right)^2 , \tag{5.26}$$

where we sum over residues associated to solutions of $\bar{E}_X = \bar{E}_Y = 0$. This implies $S^{(3)}$ can be computed from the residues at 1) the original zeroes of the scattering equations as in (5.25) or 2) the three new zeroes defined by the lines (σab) in the CHY integrand. This generalizes the contour deformation argument reviewed in the introduction for $\mathbb{CP}^1$.

Consider first the contribution coming from the first pairing $(n-1,n,\sigma) = (n,\sigma,2) = 0$. To evaluate this contribution we change variables

$$\sigma = \sigma_n + \epsilon A, \tag{5.27}$$

where ϵ is one of our new variables. This means the variable $A \in \mathbb{CP}^2$ only carries one independent component. This change of variables makes manifest the fact that puncture 1 is colliding with

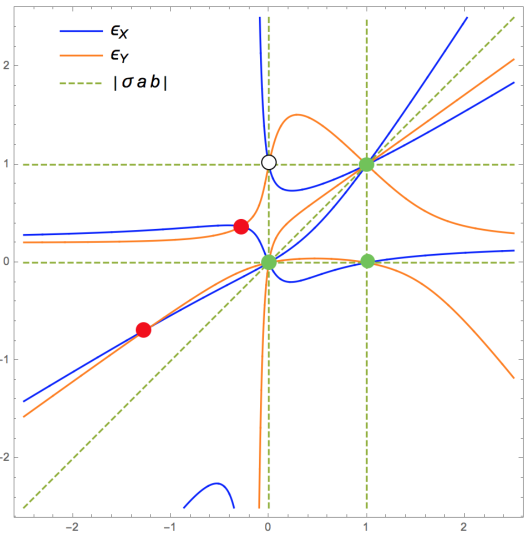

Figure 1: Real slice with the scattering equations for k=3, n=5 and generic $X, Y \in \mathbb{CP}^2$, the two solutions are shown as red blobs. Both curves $E_X = 0$ and $E_Y = 0$ are discontinuous at the white and green blobs: The scattering equations diverge where two lines $|\sigma ab| = 0$ intersect but are forced to pass through a neighborhood of these points. On the other hand, the green blobs are included in the locus of $\bar{E}_X$ and $\bar{E}_Y$. The white blob could also be included but yields no residue. Hence, via the GRT, the new contour will enclose only the green blobs instead of the red ones.

puncture n, as forced by the pairing $(n-1, n, \sigma) = (n, \sigma, 2) = 0$. The corresponding singularity is obtained by $\epsilon \to 0$, and either $(n-1, n, A) \to 0$, or $(n, A, 2) \to 0$. The measure and the integrand change in the following way:

$$(\sigma d^2\sigma) = (n, A, dA)\epsilon d\epsilon, \tag{5.28}$$

$$\left(\frac{(n-1,n,2)(n,2,3)}{(n-1,n,\sigma)(n,\sigma,2)(\sigma,2,3)}\right)^2 = \left(\frac{(n-1,n,2)}{\epsilon^2(n-1,n,A)(n,A,2)}\right)^2 + \mathcal{O}\Big(\frac{1}{\epsilon^3}\Big). \tag{5.29}$$

The behaviour of one of the scattering equations as $\epsilon \to 0$ can be simplified by setting $X = n$. This also yields $(XY\sigma) = \epsilon(nYA)$. It is now straightforward to evaluate the $\epsilon \to 0$ residue:

$$\begin{aligned}
&\int \frac{(n,A,dA)\epsilon^3 d\epsilon (n,Y,A)}{\left[\sum_{b,c\neq n} \frac{\mathbf{s}_{1bc}(nbc)}{(nbc)}\right]\left[\sum_c \frac{\mathbf{s}_{1nc}(Ync)}{(Anc)}\right]}\left(\frac{(n-1,n,2)}{\epsilon^2\,(n-1,n,A)(n,A,2)}\right)^2 \\
&= \frac{1}{\mathbf{t}_{2\ldots n-1}}\int \frac{(n,A,dA)(n,Y,A)}{\left[\sum_c \frac{\mathbf{s}_{1nc}(Ync)}{(Anc)}\right]}\left(\frac{(n-1,n,2)}{(n-1,n,A)(n,A,2)}\right)^2 \\
&= \frac{1}{\mathbf{t}_{2\ldots n-1}}\int \frac{(A,dA)(Y,A)}{\left[\sum_c \frac{\mathbf{s}_{1nc}(Yc)}{(Ac)}\right]}\left(\frac{(n-1,2)}{(n-1,A)(A,2)}\right)^2,
\end{aligned} \tag{5.30}$$

where we have defined the planar kinematic invariant $\mathbf{t}_{2\ldots n-1}=\sum_a \mathbf{s}_{1an}$. In the second equality we have also defined the two-bracket $(PQ)\equiv(PnQ)$, since the only role that n plays in such case is to project out one component.

In the last line of (5.30) we recognize precisely the $k=2$ integral performed in the introduction of this thesis. To see this more clearly set $Y=(0,1)$ and pick the chart $A\to(1,\sigma),\sigma_c\to(1,\sigma_c)$ in $\mathbb{CP}^1$. Very nicely, the remaining integral then turns into a $k=2$ soft factor,

$$
\begin{aligned}
&\int \frac{(A,dA)(Y,A)}{\left[\sum_c \frac{\mathbf{s}_{1nc}(Yc)}{(Ac)}\right]}\left(\frac{(n-1,2)}{\underline{(n-1,A)(A,2)}}\right)^2\\
&=\int \frac{d\sigma}{\sum_c \frac{\mathbf{s}_{1nc}}{\sigma-\sigma_c}}\left(\frac{\sigma_{n-1}-\sigma_2}{\underline{(\sigma_{n-1}-\sigma)(\sigma-\sigma_2)}}\right)^2=\frac{1}{\mathbf{s}_{1nn-1}}+\frac{1}{\mathbf{s}_{1n2}}.
\end{aligned}
\tag{5.31}
$$

We finish this computation remarking that the last observation regarding a recursive structure in the soft factor will show to be instrumental for our proof of a soft factor at arbitrary k.

One can show that the pairing given by $(n,\sigma,2)=(\sigma,2,3)=0$ leads to an answer similar to the last one. The corresponding contribution is

$$
\frac{1}{\mathbf{t}_{3\ldots n}}\left(\frac{1}{\mathbf{s}_{123}}+\frac{1}{\mathbf{s}_{1n2}}\right).
\tag{5.32}
$$

The last contribution is given by the pairing $(n-1,n,\sigma)=(\sigma,2,3)=0$. Geometrically, this contribution is interpreted as the soft puncture standing at the intersection point between the lines defined by punctures $\sigma_2-\sigma_3$ and $\sigma_(n-1)-\sigma_n$. We call this intersection point ξ. Changing variables $\sigma=\alpha\sigma_n+\beta\sigma_2+\xi$, we note that the two lines are reached as $\alpha\to0$ and $\beta\to0$ respectively. We also have

$$
(\sigma d^2\sigma)=d\alpha d\beta(2,n,\xi),
\tag{5.33}
$$

$$
\left(\frac{(n-1,n,2)(n,2,3)}{(n-1,n,\sigma)(n,\sigma,2)(\sigma,2,3)}\right)^2=\left(\frac{1}{\alpha\beta(2,n,\xi)}\right)^2.
\tag{5.34}
$$

Working out (5.25) one can check that the corresponding pole contribution is:

$$
\int \frac{d\alpha d\beta(2n\xi)(XY\sigma)}{\left[\frac{s_{123}(X23)}{\alpha(n23)}+\frac{s_{1n-1n}(Xn-1,n)}{\beta(2,n-1,n)}\right]\left[\frac{s_{123}(Y23)}{\alpha(n23)}+\frac{s_{1n-1n}(Yn-1,n)}{\beta(2,n-1,n)}\right]}\left(\frac{1}{\underline{\alpha}\,\underline{\beta}(2n\xi)}\right)^2,
\tag{5.35}
$$

where again $X,Y\in\mathbb{CP}^2$ are arbitrary reference vectors, which we can take to be $X=\sigma_2$, and

$Y = \sigma_n$. It is straightforward to evaluate the integral with this choice, which leads to a contribution

$$\frac{1}{\mathbf{s}_{123}\mathbf{s}_{1n-1n}}. \tag{5.36}$$

That is, by direct evaluation of the CHY integral, we have proven the existence of a soft theorem for $\mathrm{Tr}\,\mathrm{G}_{\mathbb{I}}(3,n)$ amplitudes. Adding all the contributions, the $k = 3$ soft factor is given by

$$S^{(3)} = \frac{1}{\hat{\mathbf{s}}_{n-1n1}\hat{\mathbf{s}}_{123}} + \frac{1}{\hat{\mathbf{t}}_{3\cdots n}}\left(\frac{1}{\hat{\mathbf{s}}_{n12}} + \frac{1}{\hat{\mathbf{s}}_{123}}\right) + \frac{1}{\hat{\mathbf{t}}_{2\cdots n-1}}\left(\frac{1}{\hat{\mathbf{s}}_{n12}} + \frac{1}{\hat{\mathbf{s}}_{n-1n1}}\right). \tag{5.37}$$

5.4 The Soft Theorem at Arbitrary (k, n)

Let us begin our treatment at arbitrary (k, n) with a simple observation. It is straightforward to see that for arbitrary k, the $n = k+1$ amplitude is equal to one as there are no scattering equations left where to localize the CHY integral. This already hints that the soft factor for $n = k+2$ is the amplitude itself. Under Grassmannian duality, the $m^{(k)}_{k+2}$ amplitude can be expressed as a $m^{(2)}_{k+2}$ amplitude.

We first make explicit the relation between the $\mathrm{Tr}\,\mathrm{G}_{\mathbb{I}}(3,n)$ soft factor (5.37) already derived and $m^{(2)}_5$ amplitudes. It is known from the standard biadjoint scalar theory that

$$m^{(2)}_5(\mathbb{I}_5, \mathbb{I}_5) = \frac{1}{s_{12}s_{34}} + \frac{1}{s_{23}s_{45}} + \frac{1}{s_{34}s_{51}} + \frac{1}{s_{45}s_{12}} + \frac{1}{s_{51}s_{23}}. \tag{5.38}$$

Under duality, $s_{12} \to \mathbf{s}_{345}$, $s_{23} \to \mathbf{s}_{145}$, $s_{34} \to \mathbf{s}_{125}$, $s_{45} \to \mathbf{s}_{123}$, $s_{51} \to \mathbf{s}_{234}$, from which we obtain the dual amplitude:

$$m^{(3)}_5(\mathbb{I}_5, \mathbb{I}_5) = \frac{1}{\hat{\mathbf{s}}_{451}\hat{\mathbf{s}}_{123}} + \frac{1}{\hat{\mathbf{s}}_{345}}\left(\frac{1}{\hat{\mathbf{s}}_{512}} + \frac{1}{\hat{\mathbf{s}}_{123}}\right) + \frac{1}{\hat{\mathbf{s}}_{234}}\left(\frac{1}{\hat{\mathbf{s}}_{512}} + \frac{1}{\hat{\mathbf{s}}_{451}}\right). \tag{5.39}$$

As expected, this amplitude precisely agrees with the soft factor (5.37). This is because for $k = 3$, $m^{(3)}_4(\mathbb{I}_4, \mathbb{I}_4) = 1$ and the τ-scaling is indeed homogeneous. Considering the soft factor (5.37), we see that it is just a slight modification from (5.39) when we take higher values of n, with $k = 3$

fixed. In more specific words, the latter is evaluated at a shifted value of the kinematic invariants:

$$S_n^{(3)}(\mathbb{I}|\mathbb{I}) = m_5^{(3)}(\mathbb{I}_5,\mathbb{I}_5)\Big(\hat{\mathbf{s}}_{234}\to\hat{\mathbf{t}}_{2\cdots n-1},\hat{\mathbf{s}}_{345}\to\hat{\mathbf{t}}_{3\cdots n},\hat{\mathbf{s}}_{451}\to\hat{\mathbf{s}}_{n-1n1},\hat{\mathbf{s}}_{512}\to\hat{\mathbf{s}}_{n12}\Big) \tag{5.40}$$

$$= \frac{1}{\hat{\mathbf{s}}_{n-1n1}\hat{\mathbf{s}}_{123}}+\frac{1}{\hat{\mathbf{t}}_{3\cdots n}}\left(\frac{1}{\hat{\mathbf{s}}_{n12}}+\frac{1}{\hat{\mathbf{s}}_{123}}\right)+\frac{1}{\hat{\mathbf{t}}_{2\cdots n-1}}\left(\frac{1}{\hat{\mathbf{s}}_{n12}}+\frac{1}{\hat{\mathbf{s}}_{n-1n1}}\right) \tag{5.41}$$

$$= \frac{1}{\hat{\mathbf{S}}^{(1)}\hat{\mathbf{S}}^{(3)}}+\frac{1}{\hat{\mathbf{T}}^{(1,2)}}\left(\frac{1}{\hat{\mathbf{S}}^{(2)}}+\frac{1}{\hat{\mathbf{S}}^{(3)}}\right)+\frac{1}{\hat{\mathbf{T}}^{(1,1)}}\left(\frac{1}{\hat{\mathbf{S}}^{(2)}}+\frac{1}{\hat{\mathbf{S}}^{(1)}}\right), \tag{5.42}$$

where in the last line we have defined new planar variables that are going to be useful for us later:

$$\hat{\mathbf{T}}_n^{(r,q)} := \sum_{a_1,..,a_r=1}^{n}\hat{\mathbf{s}}_{12..qa_1..a_r(n-k+q+r+1)..n-1n}, \tag{5.43}$$

where $0\le r\le k-2$, $1\le q\le k-r$. Here r denotes the number of summed indices. For $r=0$ we specialize our notation to:

$$\hat{\mathbf{T}}_n^{(0,q)} = \hat{\mathbf{S}}_n^{(q)} = \hat{\mathbf{s}}_{12..q(n-k+q+1)...n}, \qquad 1\le q\le k. \tag{5.44}$$

The usefulness of this result is twofold. On the one hand, it yields an insight into the geometrical interpretation of the soft limit in similar grounds to the ones covered in subsection 5.1 as an embedding of a smaller tropical Grassmannian into a bigger one. On the other hand, it very directly hints the form that a soft factor for arbitrary k could take.

Our proof of the $k=3$ case, given in section 5.3 using the GRT can be extended into a recursion formula in k that finally leads to $S^{(k)}$ in closed form. We present and provide the proof for this recursion formula and its solution in the next subsections 5.4.1 and 5.4.2. The result is:

$$\boxed{S^{(k)}(1\ldots n) = \sum_{p=2}^{k+1}S_p^{(k)}(1\ldots n)} \tag{5.45}$$

The different terms in the previous sum are given as follows. For $2<p<k+1$:

$$S_p^{(k)}(1\ldots n) = \frac{1}{\hat{\mathbf{T}}_n^{(k-p,p)}\hat{\mathbf{T}}_n^{(p-3,1)}}\Big(S^{(k-p+1)}(p+1\ldots n1)(s_{1a_1\ldots a_{k-p}}\to s_{12\ldots p,a_1\cdots a_{k-p}})\Big)$$
$$\times\Big(S^{(p-2)}(1\ldots n-k+p-2)(s_{1a_1\ldots a_{p-3}}\to s_{n-k+p-1,\ldots,n,1,a_1\cdots a_{p-3}})\Big). \tag{5.46}$$

Meanwhile, for $p = 2$ and $p = k + 2$:

$$S_2^{(k)}(1 \dots n) = \frac{1}{\hat{\mathbf{T}}_n^{(k-2,2)}} \times S^{(k-1)}(3 \dots n1)(s_{1a_1 \dots a_{k-2}} \to s_{12,a_1 \cdots a_{k-2}}), \tag{5.47}$$

$$S_{k+1}^{(k)}(1 \dots n) = \frac{1}{\hat{\mathbf{T}}_n^{(k-2,1)}} \times S^{(k-1)}(1 \dots n-1)(s_{1a_1 \dots a_{k-2}} \to s_{n1,a_1 \cdots a_{k-2}}). \tag{5.48}$$

It is straightforward to see that we recover the correct $k = 2$ soft factor when we take $S^{(1)} = 1$ as the seed of our recursion relation. For instance, for the case $k = 3$:

$$S^{(3)}(1 \dots n) = \sum_{p=2}^{4} S_p^{(3)}(1 \dots n). \tag{5.49}$$

When $p = 3$, (5.46) gives a contribution

$$S_3^{(3)}(1 \dots n) = \frac{1}{\mathbf{s}_{123}\mathbf{s}_{1n-1n}}, \tag{5.50}$$

while the $p = 2$ and $p = 4$ contributions can be read off from (5.47) and (5.48), giving:

$$S_2^{(3)}(1 \dots n) = \frac{1}{\sum \mathbf{s}_{12a_1}} S^{(2)}(3 \dots n1)(\mathbf{s}_{1a_1} \to \mathbf{s}_{12a_1}) = \frac{1}{\mathbf{t}_{3 \dots n}} \left(\frac{1}{\mathbf{s}_{123}} + \frac{1}{\mathbf{s}_{12n}} \right), \tag{5.51}$$

and

$$S_4^{(3)}(1 \dots n) = \frac{1}{\sum \mathbf{s}_{n1a_1}} S^{(2)}(1 \dots n-1)(\mathbf{s}_{1a_1} \to \mathbf{s}_{n1a_1}) = \frac{1}{\mathbf{t}_{2 \dots n-1}} \left(\frac{1}{\mathbf{s}_{12n}} + \frac{1}{\mathbf{s}_{1n-1n}} \right). \tag{5.52}$$

Adding these three contributions obtained through the recursion relation, we precisely get the expression (5.42) for the $k = 3$ soft factor.

The solution for the recursion relation for arbitrary values of k can be given in the grounds of our previous observations. That is, we expect the solution to be a $m_{k+2}^{(k)}(\mathbb{I}, \mathbb{I}) \sim m_{k+2}^{(2)}(\mathbb{I}, \mathbb{I})$ amplitude, evaluated at planar combinations of kinematic invariants just as in equation (5.42). Concretely, the closed form expression that solves the recursion relation is given by:

$$\boxed{S^{(k)} = m_{k+2}^{(k)}(\mathbb{I}|\mathbb{I}) \Big(\hat{\mathbf{T}}_{k+2}^{(p,q)} \longrightarrow \hat{\mathbf{T}}_n^{(p,q)} \Big)} \tag{5.53}$$

where $\hat{\mathbf{T}}_n^{(p,q)}$ are the planar kinematic invariants defined in (5.43).

This closed formula for the soft factor works because the recursion (5.45) is nothing else than the recursion relation presented in [61] for the planar cubic theory, which amplitudes correspond as well to the identity amplitudes of the biadjoint scalar theory. The latter amplitudes are dual to the $m_{k+2}^{(k)}$ amplitudes proposed here.

To conclude this section let us provide a new result by applying formula (5.53) to $\mathrm{Tr}\,\mathrm{G}_{\mathbb{I}}(4,n)$ amplitudes. For this specific case, let us rewrite the kinematic invariants $\hat{\mathbf{T}}_n^{(p,q)}$ in the following way:

$$\hat{\mathbf{t}}_{4..n} = \sum_a \hat{\mathbf{s}}_{123a}, \qquad \hat{\mathbf{t}}_{3..n-1} = \sum_a \hat{\mathbf{s}}_{12an}, \qquad \hat{\mathbf{t}}_{2..n-2} = \sum_a \hat{\mathbf{s}}_{1an-1n}, \tag{5.54}$$

$$\hat{\mathbf{u}}_{3\ldots n} = \sum_{a,b} \hat{\mathbf{s}}_{12ab}, \qquad \hat{\mathbf{u}}_{2\ldots n-1} = \sum_{a,b} \hat{\mathbf{s}}_{1abn}. \tag{5.55}$$

The soft factor for $\mathrm{Tr}\,\mathrm{G}_{\mathbb{I}}(4,n)$ amplitudes is:

$$\begin{aligned} S^{(4)} = {} & \frac{1}{\hat{\mathbf{u}}_{3\ldots n}}\left\{\frac{1}{\hat{\mathbf{s}}_{12n-1n}\hat{\mathbf{s}}_{1234}} + \frac{1}{\hat{\mathbf{t}}_{3..n-1}}\left[\frac{1}{\hat{\mathbf{s}}_{12n-1n}} + \frac{1}{\hat{\mathbf{s}}_{n123}}\right] + \frac{1}{\hat{\mathbf{t}}_{4..n}}\left[\frac{1}{\hat{\mathbf{s}}_{1234}} + \frac{1}{\hat{\mathbf{s}}_{n123}}\right]\right\} \\ & + \frac{1}{\hat{\mathbf{u}}_{2\ldots n-1}}\left\{\frac{1}{\hat{\mathbf{s}}_{123n}\hat{\mathbf{s}}_{1n-2n-1n}} + \frac{1}{\hat{\mathbf{t}}_{3..n-1}}\left[\frac{1}{\hat{\mathbf{s}}_{123n}} + \frac{1}{\hat{\mathbf{s}}_{12n-1n}}\right] + \frac{1}{\hat{\mathbf{t}}_{2..n-2}}\left[\frac{1}{\hat{\mathbf{s}}_{n-2n-1n1}} + \frac{1}{\hat{\mathbf{s}}_{n-1n12}}\right]\right\} \\ & + \frac{1}{\hat{\mathbf{s}}_{1n-2n-1n}\hat{\mathbf{t}}_{4..n}}\left[\frac{1}{\hat{\mathbf{s}}_{1234}} + \frac{1}{\hat{\mathbf{s}}_{123n}}\right] + \frac{1}{\hat{\mathbf{s}}_{1234}\hat{\mathbf{t}}_{2..n-2}}\left[\frac{1}{\hat{\mathbf{s}}_{12n-1n}} + \frac{1}{\hat{\mathbf{s}}_{1n-2n-1n}}\right]. \end{aligned} \tag{5.56}$$

5.4.1 Derivation of the Recursion

In this subsection we use the GRT to prove the recursion relation (5.45):

$$S^{(k)}(1\ldots n) = \sum_{p=2}^{k+1} S_p^{(k)}(1\ldots n)\,, \tag{5.57}$$

where, for $2 < p < k+1$:

$$\begin{aligned} S_p^{(k)}(1\ldots n) = {} & \frac{1}{\sum s_{12\ldots p,a_1\cdots a_{k-p}}}\Big(S^{(k-p+1)}(p+1\ldots n1)(s_{1a_1\ldots a_{k-p}} \to s_{12\ldots p,a_1\cdots a_{k-p}})\Big) \\ \times & \frac{1}{\sum s_{n-k+p-1,\ldots n,1,a_1\cdots a_{p-3}}}\Big(S^{(p-2)}(1\ldots n-k+p-2)(s_{1a_1\ldots a_{p-3}} \to s_{n-k+p-1,\ldots,n,1,a_1\cdots a_{p-3}})\Big), \end{aligned} \tag{5.58}$$

and for $p = 2$ and $p = k+2$:

$$S_2^{(k)}(1\dots n) = \frac{1}{\sum s_{12,a_1\cdots a_{k-2}}} \times S^{(k-1)}(3\dots n1)(s_{1a_1\dots a_{k-2}} \to s_{12,a_1\cdots a_{k-2}}), \tag{5.59}$$

$$S_{k+1}^{(k)}(1\dots n) = \frac{1}{\sum s_{n1,a_1\cdots a_{k-2}}} \times S^{(k-1)}(1\dots n-1)(s_{1a_1\dots a_{k-2}} \to s_{n1,a_1\cdots a_{k-2}}). \tag{5.60}$$

We take $S^{(1)} = 1$, since with this value we recover the correct soft factor for $k = 2$ (which can be found by direct computation). We find the explicit solution to the recursion in the next subsection, where we also find that it can be written in terms of the biadjoint amplitudes $m_{k+2}^{(2)}[\mathbb{I}|\mathbb{I}]$. To prove the recursion formula, we recall the integral representation of the soft factor,

$$S^{(k)} = \int \left[\frac{d^{k-1}x^i}{\prod_i \underline{\hat{E}_i}}\right] \left(\frac{|n2...k|...|(n-k+2)(n-k+3)...n2|}{|12\cdot\cdot k||n1\cdot\cdot k-1|...|(n-k+2)(n-k+3)\cdot\cdot 1|}\right)^2, \tag{5.61}$$

where we have underlined the scattering equations defining the integration contour. The implementation of the GRT is done in analogous manner to section 5.3. In fact, during section 5.3 we have found that the only singularities of the integration, besides the scattering equations, can arise from the consecutive vanishing of determinant-brackets in the CHY integrand. Let us then deform the contour in an analogous way to enclose all these singularities. More precisely, the GRT applied to (5.61) will yield (5.57), with the p-th term

$$\begin{aligned} S_p^{(k)} = \int \frac{d^{k-1}x^i}{\prod_i \hat{E}_i} \Bigg(& \underline{\frac{|n-k+2\dots n2|\cdots|n-k-1+p\dots n2\dots p-1|}{|n-k+2\dots n1|\cdots|n-k-1+p\dots n12\dots p-2|}} \times \\ & \frac{|n-k+p\dots n2\dots p|}{|n-k+p\dots n12\dots p-1|} \times \underline{\frac{|n-k+p+1\dots n2\dots p+1|\cdots|n2\dots k|}{|n-k+p+1\dots n12\dots p|\cdots|1\dots k|}} \Bigg)^2 . \end{aligned} \tag{5.62}$$

To compute this residue we observe that the second line of (5.62) forces the soft particle to lie in the plane generated by particles $23\dots p$, hence we perform the blow up on this configuration (we could also blow up the complement instead, namely the plane defined by the first line of (5.62)). We write

$$\sigma = X_{2\dots p} + \epsilon A, \tag{5.63}$$

where $X_{2\dots p}$ (X for short) has $p-2$ coordinates χ^R and A has $k-p$ coordinates α^r. Here we assumed $p \leq k$ which excludes the case $p = k+1$ in (5.45), in which we are forced to blow up the complement plane instead. Now, the analysis of the ϵ- integration was already outlined in subsection 4.1, hence we jump straight to the result in the strict $\epsilon = 0$ limit. In homogeneous coordinates

$$S_p = \frac{1}{\sum \hat{s}_{12\ldots p,a_1\cdots a_{k-p}}} \int \frac{(AXd^{k-p}Ad^{p-2}X)}{\prod_r \frac{\partial S}{\partial \alpha^r} \prod_R \frac{\partial S}{\partial \chi^R}} (AXA^{(1)}\cdots A^{(k-p)}X^{(1)}\cdots X^{(p-2)})$$
$$\left(\frac{(n-k+2\ldots n2)\cdots(n-k-1+p\ldots n2\ldots p-1)}{\underline{(n-k+2\ldots nX)\cdots(n-k-1+p\ldots nX2\ldots p-2)}} \times \frac{(n-k+p\ldots n2\ldots p)}{(n-k+p\ldots nX2\ldots p-1)} \right.$$
$$\left. \frac{(n-k+p+1\ldots n2\ldots p+1)\cdots(n2\ldots k)}{\underline{(n-k+p+1\ldots nA2\ldots p)\cdots(A\ldots k)}} \right)^2, \tag{5.64}$$

where the vectors are defined as $A^{(r)} = \frac{\partial A}{\partial \alpha^r}$ and $X^{(R)} = \frac{\partial X}{\partial \chi^R}$, with the latter living in the plane $23\ldots p$. Now, at strict $\epsilon = 0$:

$$\frac{\partial S}{\partial \alpha^r} = \sum_a \frac{s_{12\ldots p,a_1\cdots a_{k-p}}}{(A2\ldots pa_1\cdots a_{k-p})}(A^{(r)}2\ldots pa_1\cdots a_{k-p}) \tag{5.65}$$
$$\frac{\partial S}{\partial \chi^R} = \sum_a \frac{s_{1,n-k-1+p\ldots n,a_1\cdots a_{p-3}}}{(X,n-k-1+p\ldots n,a_1\cdots a_{p-3})}(X^{(R)},n-k-1+p\ldots n,a_1\cdots a_{p-3}) + \ldots \tag{5.66}$$

where in $\frac{\partial S}{\partial \chi^R}$ we have only written the terms that will contribute in the contour marked for X, given by the second line of (5.64), which forces X to approach the plane $n-k-1+p\ldots n$. Note that this fixes X to be the point in the intersection of such $(k-p+2)$-plane and the $(p-1)$-plane $2\ldots p$. Also note that the sum over a in the second line is not constrained by the fact that $X^{(R)}$ is in $2\ldots p$ as the labels $a_1\ldots a_{p-3}$ will never contain such plane.

As the plane $2\ldots p$ is spanned by $X, X^{(1)}, \ldots, X^{(p-2)}$ we can use the following Schouten identity:

$$\begin{aligned}(AXd^{k-p}Ad^{p-2}X)(AXA^{(1)}\cdots A^{(k-p)}X^{(1)}\cdots X^{(p-2)})(n-k+p\ldots n2\ldots p)^2 \\ = (Ad^{k-p}A,2\ldots p)(AA^{(1)}\cdots A^{(k-p)},2\ldots p) \\ \times(n-k+p\ldots nXX^{(1)}\cdots X^{(p-2)})(n-k+p\ldots nXd^{p-2}X).\end{aligned} \tag{5.67}$$

(to see this identity we can write $dX = d\chi_R X^{(R)}$). We can now factor the integral in $S_p^{(k)}$ in two parts. In order to write them let us define the short brackets

$$[T_1.\dots.T_{p-1}] := (n-k+p\dots n, T_1.\dots.T_{p-1}), \tag{5.68}$$
$$\{T_1\dots T_{k-p+1}\} := (T_1\dots T_{k-p+1}, 2\dots p). \tag{5.69}$$

The first integral is over X and can be now written

$$\int \frac{[Xd^{p-2}X]}{\prod_R \sum_a \frac{s_{1,n-k-1+p\dots n,a_1\dots a_{p-3}}}{[X,n-k-1+p,a_1\cdots a_{p-3}]}[X^{(R)},n-k-1+p,a_{i_1}\cdots a_{i_{p-3}}]}[XX^{(1)}\cdots X^{(p-2)}]\times$$
$$\left[\frac{[n-k+2\dots n-k+p-1,2]\cdots[n-k-1+p2\dots p-1]}{\underline{[n-k+2\dots n-k+p-1X]\cdots[n-k-1+pX2\dots p-2]}[X2\dots p-1]}\right]^2, \tag{5.70}$$

which then becomes

$$S_p^{(p-1)}(12\dots n-k+p-1)(s_{1a_1\dots a_{p-2}} \to s_{1n-k+p,\dots,n,a_1\dots a_{p-2}}), \tag{5.71}$$

with the first argument means this piece must be evaluated for the canonical ordering with particles $n-k+p,\dots,n$ removed and the second argument indicates a replacement. Observe that as the residue is localized over "planar" brackets (w.r.t the aforementioned ordering) only planar invariants among $s_{1n-k+p,\dots,n,a_1\dots a_{p-2}}$ will appear in $S_p^{(p-1)}$. Hence there is no need to impose momentum conservation on them since they are regarded as independent.

The second integral is over A and reads

$$\int \frac{\{Ad^{k-p}A\}\{AA^{(1)}\cdots A^{(k-p)}\}}{\prod_r \sum_a \frac{s_{12\dots p,a_1\cdots a_{k-p}}}{\{Aa_1\cdots a_{k-p}\}}\{A^{(r)}a_1\cdots a_{k-p}\}}$$
$$\times\left(\frac{\{n-k+p+1\dots n,p+1\}\cdots\{n,p+1\dots k\}}{\underline{\{n-k+p+1\dots nA\}\cdots\{Ap+1\dots k\}}}\right)^2, \tag{5.72}$$

which gives the full soft factor

$$S^{(k-p+1)}(p+1\dots n1)(s_{1a_1\dots a_{k-p}} \to s_{12\dots p,a_1\cdots a_{k-p}}), \tag{5.73}$$

where particles $2,\dots,p$ are removed from the ordering.

We can write our full result as

$$S_p^{(k)}(1\dots n) = \frac{1}{\sum s_{12\dots p,a_1\cdots a_{k-p}}} \times S^{(k-p+1)}(p+1\dots n1)(s_{1a_1\dots a_{k-p}} \to s_{12\dots p,a_1\cdots a_{k-p}})$$
$$\times S_p^{(p-1)}(12\dots n-k+p-1)(s_{1a_1\dots a_{p-2}} \to s_{1n-k+p,\dots,n,a_1\dots a_{p-2}}). \tag{5.74}$$

Recall that this holds for $k \geq p$. Let us provide now an expression for $S_p^{(p-1)}$. Specializing (5.74) to $p=2$ (recalling that $S_p^{(1)} = 1$) we get

$$S_2^{(k)}(1\dots n) = \frac{1}{\sum s_{12,a_1\cdots a_{k-2}}} \times S^{(k-1)}(3\dots n1)(s_{1a_1\dots a_{k-2}} \to s_{12,a_1\cdots a_{k-2}}), \tag{5.75}$$

but we can revert the canonical ordering in the definition (5.62) to write this as

$$S_{k+1}^{(k)}(1\dots n) = \frac{1}{\sum s_{n1,a_1\cdots a_{k-2}}} \times S^{(k-1)}(1\dots n-1)(s_{1a_1\dots a_{k-2}} \to s_{n1,a_1\cdots a_{k-2}}), \tag{5.76}$$

which implies

$$S_p^{(p-1)}(12\dots n-k+p-1) = \frac{1}{\sum s_{n-k+p-1,1,a_1\cdots a_{p-3}}} \times$$
$$S^{(p-2)}(1\dots n-k+p-2)(s_{1a_1\dots a_{p-3}} \to s_{n-k+p-1,1,a_1\cdots a_{p-3}}). \tag{5.77}$$

Using this equation, together with equations (5.75), (5.74), and shifting as indicated, we finally obtain the recursion relation (5.57).

5.4.2 Solving the Recursion: The Soft Factor as a Biadjoint Amplitude

In this subsection we use the recursion relation in [61] for planar cubic theory amplitudes, which coincide with the identity amplitudes for the biadjoint scalar theory. We need the recursion for the specific case $n = k+2$:

$$A_{k+2}(k_1, ..., k_{k+2}) = \frac{1}{(k_3 + \ldots + k_{k+2})^2} A_{k+1}(k_1 + k_2, k_3, \ldots, k_{k+2})$$
$$+ \frac{1}{(k_2 + \ldots + k_{k+1})^2} A_{k+1}(k_{k+2} + k_1, k_2, \ldots, k_{k+1})$$
$$+ \sum_{p=3}^{k} \frac{A_p(\pi_p, k_2, \ldots, k_p) A_{k-p+3}(\bar{\pi}_p, k_{p+1}, \ldots, k_{k+2})}{(k_2 + \ldots + k_p)^2 (k_{p+1} + \ldots + k_{k+2})^2}, \tag{5.78}$$

where

$$\pi_m = -k_2 - \ldots - k_m, \qquad \bar{\pi}_m = -k_m - \ldots - k_{k+2}, \tag{5.79}$$

where k_i correspond to the momentum of the i-th particle.

It is straightforward to see that under duality the propagators transform as:

$$(k_3 + \ldots + k_{k+2})^2 = \sum_{\substack{\text{pairs}(a,b) \\ \in \{3,..,k+2\}}} s_{ab} \longrightarrow \sum_{a_1,\ldots,a_{k-2}=1}^{k+2} \mathbf{s}_{12a_1\ldots a_{k-2}} = \hat{\mathbf{T}}_{k+2}^{(k-2,2)}, \tag{5.80}$$

where the variables $\hat{\mathbf{T}}_n^{(a,b)}$ were defined in (5.43).

$$\hat{\mathbf{T}}_n^{(p,q)} = \sum_{a_1,..,a_p=1}^{n} \hat{\mathbf{s}}_{12..qa_1..a_p(n-k+q+p+1)..n-1n}. \tag{5.81}$$

Similarly one can check that:

$$(k_2 + \ldots + k_{k+1})^2 \longrightarrow \hat{\mathbf{T}}_{k+2}^{(k-2,1)}, \tag{5.82}$$
$$(k_{p+1} + \ldots + k_{k+2})^2 \longrightarrow \hat{\mathbf{T}}_{k+2}^{(k-p,p)}, \tag{5.83}$$
$$(k_2 + \ldots + k_p)^2 \longrightarrow \hat{\mathbf{T}}_{k+2}^{(p-3,1)}. \tag{5.84}$$

Let us study now how the amplitudes factors behave under duality. Take, for example, the $A_{k+1}(k_1 + k_2, k_3, \ldots, k_{k+2})$ term. By momentum conservation $k_1 + k_2 = -k_3 - \ldots - k_{k+2}$, and therefore the amplitude can be regarded as a function $A_{k+1}(s_{a_1\ldots a_m})$ of planar kinematic invariants only, with the indices satisfying a planar order and in the set $\{3, \ldots, k+2\}$. Under duality we get the same amplitude, but evaluated in the corresponding dual kinematic invariants:

$$A_{k+1}(k_1+k_2,k_3,\ldots,k_{k+2}) = A_{k+1}\big(s_{a_1\ldots a_m}\big) \longrightarrow A_{k+1}\Big(\hat{\mathbf{T}}_{k+2}^{(m-2,a_1-1)}\Big). \tag{5.85}$$

We can now follow the same procedure over all amplitude factors appearing in (5.78). We first express them in terms of the planar basis of kinematic invariants, and therefore under duality they can all be written in terms of the variables $\hat{\mathbf{T}}_{k+2}^{(a,b)}$:

$$A_{k+1}(k_{k+2}+k_1,k_2,\ldots,k_{k+1}) = A_{k+1}\big(s_{a_1\ldots a_m}\big) \longrightarrow A_{k+1}\Big(\hat{\mathbf{T}}_{k+2}^{(m-2,a_1-1)}\Big), \tag{5.86}$$

$$A_{k-p+3}(\bar{\pi}_p,k_{p+1},\ldots,k_{k+2}) = A_{k-p+3}\big(s_{a_1\ldots a_m}\big) \longrightarrow A_{k+1}\Big(\hat{\mathbf{T}}_{k+2}^{(m-2,a_1-1)}\Big), \tag{5.87}$$

$$A_p(\pi_p,k_2,\ldots,k_p) = A_p\big(s_{a_1\ldots a_m}\big) \longrightarrow A_{k+1}\Big(\hat{\mathbf{T}}_{k+2}^{(m-2,a_1-1)}\Big), \tag{5.88}$$

where in the previous we have used a condensed notation $A_n\big(s_{a_1\ldots a_m}\big)$ to mean A_n as a function of the planar kinematic invariants $s_{a_1\ldots a_m}$ that can be constructed from the momenta appearing in the corresponding left hand side.

The recursion relation (5.57) for the soft factor takes the same form as the dualized version of (5.78), up to one modification: the summation indices get promoted from the set $\{1,\ldots,k+2\}$ to the set $\{1,\ldots,n\}$. That is, we must promote

$$\hat{\mathbf{T}}_{k+2}^{(a,b)} \longrightarrow \hat{\mathbf{T}}_{n}^{(a,b)}, \tag{5.89}$$

in all instances where a $\hat{\mathbf{T}}_{k+2}^{(a,b)}$ appears. We conclude that the soft factor satisfies the same recursion relation as (5.78), considering duality and the replacements (5.89).

We know by direct computation that the base case

$$S^{(2)} = \left(\frac{1}{\hat{s}_{n1}}+\frac{1}{\hat{s}_{12}}\right), \tag{5.90}$$

satisfies (5.53). Since both expressions in (5.53) are equal in the base case $k=2$, and satisfy the same recursion relation, by induction we can conclude that (5.53) must be true:

$$\boxed{S^{(k)} = m_{k+2}^{(k)}(\mathbb{I}|\mathbb{I})\Big(\hat{\mathbf{T}}_{k+2}^{(p,q)} \longrightarrow \hat{\mathbf{T}}_{n}^{(p,q)}\Big)} \tag{5.91}$$

Furthermore, it has been numerically checked in [63] and further analyzed in [52] that this result is indeed correct.

Chapter VI

Conclusions, Discussions, and Further Directions

In this thesis we have reviewed the CHY formalism of scattering amplitudes in detail, starting from the scattering equations and the statement of the formalism in general, passing by the construction of the CHY expression for the amplitudes of several theories, and ending with the application of the formalism to KLT relations and soft theorems. We discussed how ambitwistor strings provide a physical origin for the formulae given by CHY. In particular, we worked in a recently introduced ambitwistor string model which, as the original twistor string, uses twistor variables in its construction. We found the BRST generator for this model, from which in principle we can find the corresponding BRST-closed vertex operators. Despite a challenge, further work on trying to find the vertex operators seem worthwhile, as doing so could lead us into connected formulae for $d = 10$ super Yang-Mills states. In the second half of this thesis, we have reviewed a recently introduced generalization of the CHY formalism to complex projective spaces of higher dimension. We saw how studying the $\mathbb{CP}^2$ extension of the biadjoint scalar theory revealed striking connections between the "amplitudes" of such theory and a geometrical structure known as the tropical Grassmannian. In the last chapter we extended this connection by studying the soft limit of these newly defined "amplitudes" and found that such theorem indeed exists, and furthermore, we found that the soft theorem allows us to see the standard $k = 2$ biadjoint amplitudes as the soft factor for the higher k amplitudes, at all multiplicities.

Clearly, the most pressing question stemming from chapter 3 is to find the covariant vertex operators that are annihilated by the BRST generator and the corresponding prescription to compute amplitudes so as to arrive to connected formulae for the amplitudes. Other twistorial-ambitwistorial models have been worked out in [27, 29, 30] in 4,6 and 10/11 dimensions and are based on an extension of the standard scattering equations known as the polarized scattering equations, which incorporate polarization data into the constituent spinors. In the vertex operators, these equations are imposed as integration over "auxiliary variables". For example, in four dimensions:

$$\mathcal{V}'_a = \int \frac{ds_a}{s_a} \bar{\delta}^2(\lambda_a - s_a\lambda) e^{is_a([\mu\tilde{\lambda}_a] + \chi^r \tilde{\eta}_{ar})} J \cdot t_a. \tag{6.1}$$

See [27] for precise definitions. In particular, after using the corresponding incidence relation, one gets back the standard bosonic $e^{ik_a\cdot X}$ factor (after using the delta function). As we can see, vertex operators are defined through integration over an additional variable s_a. Let us be highly speculative and attempt something similar in our case. Assume λ can be solved as (similar to [29, 30]):

$$\lambda^\alpha(z) = \sum_{a=1}^{n} \frac{\mu_a^\alpha}{z - z_a}. \tag{6.2}$$

Asking for $P = (\lambda\gamma\lambda)$, and assuming P takes the form (2.2), one can conclude that

$$(\mu_a \gamma^m \mu_a) = 0, \tag{6.3}$$

$$(\lambda\, \gamma^m\, \mu) - k^m = 0, \tag{6.4}$$

which would be the analog of the polarized scattering equations in [29, 30]. Let us stress however that additionally μ_a must be a pure-spinor. If whether these equations can be used inside a well-defined integral as in (6.1) so as to arrive to the $e^{ik_a\cdot X}$ factor and correct vertex operators remain an open question. However, the fact that this construction resembles other contexts seen before, and considering the connection to pure spinors, makes it an idea worth of consideration.

In chapter 5 of this thesis we have presented original research results [63] regarding the soft limit of the generalization of the biadjoint scalar theory to $\mathbb{CP}^{k-1}$. The theorem was derived through explicit use of the CHY formula defining the generalization of the biadjoint scalar theory. We note that the theorem holds for all generalized amplitudes in the sense that

$$m_{n+1}^{(k)}[1\alpha|1\beta] = \; S^{(k)}_{1,\alpha|1,\beta} \times \, m_n^{(k)}[\alpha|\beta] + \mathcal{O}\Big(\frac{1}{\tau^{k-2}}\Big)\,, \tag{6.5}$$

assuming that the intersection rule we provided in subsection 5.1, namely

$$m_n^{(k)}[\alpha|\beta] = \sum_{\Upsilon \in J(\alpha) \bigcap J(\beta)} \Upsilon\,, \tag{6.6}$$

holds for all k, n. This rule is a natural corollary of the conjectured relation between CHY and the geometric object arising as certain ordered restriction of the tropical Grassmannian, i.e. $\mathrm{Tr}\,\mathrm{G}_{\alpha,\beta}(k,n)$.

With the previous assumption the soft theorem presented in this work implies that the soft factors $S^{(k)}_{1,\alpha|1,\beta}$ for $\mathrm{Tr}\,\mathrm{G}_{\alpha,\beta}(k,n)$ amplitudes can be written in terms of ordinary amplitudes $m^{(2)}_{k+2}[1\alpha|1\beta]$ evaluated at specific combinations of generalized kinematic invariants. For the canonical ordering, the latter correspond to the variables $\mathbf{T}_n^{(r,q)}$, in direct bijection to the vertices of $\mathrm{Tr}\,\mathrm{G}_{\mathbb{I}}(2,k+2)$.

This last remark is interesting, for suppose we ask the following question: Is there any theory such that its amplitudes can be interpreted, at any multiplicity, as the soft factor for some other (bigger) structure? We have shown that the answer to such question is affirmative, provided we consider expressions that allow a CHY-form. For instance, in a situation based in what has already been seen in other remarkable contexts such as [81–87] we could define a theory in terms of the (e.g. tropical) geometry, rather than in terms of principles of unitarity and locality. On the other hand, from the perspective of $m_n^{(k)}$ amplitudes it would be interesting to understand general factorization for all k, n in geometrical sense. In this direction an instance of geometrical factorization was studied in [78] via the "delta algebra" $D_{k,n}$ defined for all k, n.

An important consequence can be deduced from the soft theorem introduced in this work when considered along with the duality $\mathrm{Tr}\,\mathrm{G}_{\mathbb{I}}(k,n) \sim \mathrm{Tr}\,\mathrm{G}_{\mathbb{I}}(n-k,n)$. This consequence corresponds to the hard theorem that was already sketched in subsections 5.1 and 5.2. In particular, the limit entails that the objects $S^{(k)}$ provided throughout the thesis can be thought not only as soft factors, but also as "hard factors" emerging in the limit defined by (5.23). Schematically and with appropriate variables, this gives

$$m_n^{(k)} \longrightarrow \begin{cases} S^{(k)} \times m_{n-1}^{(k)} + \mathcal{O}(1/\tau^{k-2}) & \text{(Soft Theorem)} \\ H^{(k)} \times m_{n-1}^{(k-1)} + \mathcal{O}(1/\tau^{n-k-2}) & \text{(Hard Theorem)} \end{cases} \tag{6.7}$$

where $S^{(k)}$ is given by $m_{k+2}^{(2)}$ whereas $H^{(k)}$ is given by $m_{n-k+2}^{(2)}$. Note that while the soft theorem gives a "k-preserving" factorization, the hard theorem is "k-decreasing" in resemblance to other contexts [88]. Expectedly, for $k = 2$ the hard limit does not lead to any non-trivial factorization of the ordinary amplitudes since $H^{(2)} = m_n^{(2)}$: The hard limit (5.23) becomes $s_{ab} \sim \tau$ for $a, b \neq 1$, but ordinary momentum conservation implies that all kinematic invariants tend to zero. Hence these amplitudes scale homogeneously in τ. Note that a non-trivial instance of hard limit in quantum field theories has recently been studied in [75].

The soft theorem that we have presented in this work is only valid at the leading order in the soft expansion. As such, it presents limitations to explore the full structure of the tropical Grassmannian. This structure includes vertices such as R_{abcdef} and its extensions [45]. Even though they are invisible to the soft factor they appear non-trivially in the hard amplitude $m_{n-1}^{(k)}$. Given that identifying the geometry of the general $\mathrm{Tr}\,\mathrm{G}(k, n)$ remains a current mathematical problem, it would be interesting to apply "soft bootstrap" techniques (see e.g. [76, 77, 79]) to tackle this challenge.

In this direction, a natural step to consider is the study of subleading, and next-to-subleading soft theorems in the generalized biadjoint scalar theory. Indeed, we have found that identity amplitudes for the ordinary biadjoint scalar theory satisfy a new next-to-subleading soft theorem. It can be compactly written as

$$S^{(2)}(\mathbb{I}|\mathbb{I}) = \frac{e^{\tau \sum_{a=3}^{n-2} \hat{s}_{1a} \frac{\partial}{\partial s_{an}}}}{\tau \hat{s}_{1n}} + \frac{e^{\tau \sum_{a=3}^{n-2} \hat{s}_{1a} \frac{\partial}{\partial s_{2a}}}}{\tau \hat{s}_{12}} + \frac{\tau}{2} \sum_{l=3}^{n-2} \hat{s}_{1l} \left[\frac{\partial}{\partial s_{2l}} - \frac{\partial}{\partial s_{ln}} \right]^2 + \mathcal{O}(\tau^2). \tag{6.8}$$

This expression contains the subleading soft theorem presented in [80] and can be deduced by performing the soft expansion in the CHY formula, as in [17] (choosing as a basis of kinematic invariants the set $\{s_{ab}, s_{a2}, s_{an}\}$, where $a \in \{3, ..., n-2\}$). It would be interesting to see if any of these relations has an analog for the generalized biadjoint scalar theory and a corresponding dual theorem. If affirmative, it would also be interesting to see if any of these subleading theorems has a realization in terms of the tropical Grassmannian geometry. Notice, however, that application of the same procedure is complicated for higher values of k, as singular solutions to the scattering equations arise [36, 45, 52]. Further scrutiny is necessary in order to find such subleading soft theorems.

As we showed, the soft factors satisfy a recursion relation that matches the one presented in [61] for the planar cubic theory (i.e. identity amplitudes for the biadjoint scalar theory). It would be fascinating to study if the $\mathrm{Tr}\,\mathrm{G}_{\mathbb{I}}(k,n)$ amplitudes themselves satisfy some kind of recursion formula that reduces to the recursion formula in [61] for the ordinary $\mathrm{Tr}\,\mathrm{G}_{\mathbb{I}}(2,n)$ case. Another possibility left for future study is to establish a Berends-Giele like recursion for the generalized biadjoint scalar theory, as the one worked out in [89] for the ordinary biadjoint scalar theory. This could be possible if a "self-similarity" structure is present in the tropical Grassmannians. Such a result would be interesting, as it could allows us to understand better the factorization structure of the generalized amplitudes defined by the $\mathbb{CP}^{k-1}$ extension of the CHY formalism.

Finally, as we showed in chapter 3, the standard scattering equations over $\mathbb{CP}^1$ naturally arise from ambitwistor strings and the amplitude prescription integrating over a n-punctured Riemann sphere. A natural idea to explore then would be to consider further constructions that would give rise to the $\mathbb{CP}^{k-1}$ scattering equations and the corresponding amplitudes (say, for biadjoint scalar theory) as an integration over the higher-dimensional n-punctured "world-volumes". Hopefully, the future will see an interesting resolution to all our current questions and challenges!

Appendix A

A Mini-Introduction to the Theory of Reducible Constrained Systems

In this section we outline some generalities behind the construction of the BRST generator for reducible constrained systems. We have the specific goal in mind of reviewing the necessary procedure to follow the construction of the BRST generator for the twistor-ambitwistor string model described in section 4. The main reference for this appendix is [73], and in this appendix we use the conventions there described.

A constrained system is characterized by the presence of a number of relations constraining the value of the canonical variables describing the system (for example, the (q, p)'s in Hamiltonian mechanics). Such *constraints* take the form:

$$G_{a_0}(q, p) = 0, \qquad a_0 = 1, \ldots, m_0. \tag{A.1}$$

The constraints (A.1) do not have to be all independent. When this happens, by definition, there are functions $Z_{a_1}{}^{a_0}$ of the canonical variables for which

$$Z_{a_1}{}^{a_0} G_{a_0} = 0, \qquad a_1 = 1, \ldots, m_1. \tag{A.2}$$

In a similar way, the reducibility functions $Z_{a_1}{}^{a_0}$ may themselves not be all independent, in which case there are functions $Z_{a_2}{}^{a_1}$ for which

$$Z_{a_2}{}^{a_1} Z_{a_1}{}^{a_0} = (-)^{\varepsilon_0} C_{a_2}{}^{a_0 b_0} G_{b_0}, \quad a_2 = 1, \ldots, m_2, \tag{A.3}$$

where we allow now for the right hand side to be a linear combination of the constraints (since describing the system assumes (A.1)). As it should be clear, this procedure may in principle continue *ad infinitum*, so we define several levels of reducibility:

$$Z_{a_k}^{\ a_{k-1}} Z_{a_{k-1}}^{\ a_{k-2}} = (-)^{\varepsilon_{a_{k-2}}} C_{a_k}^{a_{k-2}a_0} G_{a_0}, \quad k = 1, \ldots, L, \quad a_k = 1, \ldots, m_k, \tag{A.4}$$

where L can be arbitrarily large or even infinite. The number of independent constraints is then $m = \sum_{i=0}^{L} (-)^i m_i$.

We say a constraint is first-class if its Poisson bracket (OPE) with every constraint vanishes on the surface defined by (A.1): $[G_{a_0}, G_{b_0}] = C_{a_0 b_0}^{c_0} G_{c_0}$. When a set of constraints is such that $[G_{a_0}, G_{b_0}] \neq 0$ on the constraint surface, we say such constraints are second class. From now on we assume our constraints are all first-class.

At this point, one may ask oneself why not to work directly with the independent constraints, or for that matter, why not to solve the constraints once and for all and working with an unconstrained theory from the very beginning. In principle, one can always solve (locally) for the dependent constraints in terms of a set of independent ones. Notwithstanding this fact, performing this separation in the set of constraints between independent and dependent ones, or just solving the constraints, may be too complicated to perform in practice, it might spoil a desired manifest invariance under some important or useful symmetry (such as manifest Lorentz invariance), or it might just be downright impossible to perform globally due to topological obstructions.

Under these considerations, it is desirable to have a formulation to treat constrained systems on their own right (i.e. "without solving the constraints"). The BRST construction provides such a treatment, where one enlarges the original phase space by new variables -"the ghosts"- and, in a sense, the new redundancy cancels the one given by keeping (A.1) or the reducibilities unsolved. The BRST procedure replaces the original local symmetry generated by the constraints by a global symmetry -the BRST symmetry- acting on the enlarged space. The key property of such symmetry is that it is nilpotent, and therefore one can consider the corresponding cohomology groups $H^{(k)}(Q)$. The BRST generator Q is constructed specifically so that the condition

$$H^{(0)}(Q) = \{\text{gauge invariant functions}\} \tag{A.5}$$

is fulfilled. Thus, finding the BRST generator allows us to recover the original physical information by passing to its cohomology, avoiding in this way solving either (A.1) or the reducibilities. In the case of reducible constraints, one can find the BRST generator without having to resort to the "awkward" splitting between dependent and independent constraints. In the following, we describe how to construct the BRST generator.

For each constraint (A.1) we enlarge our space by a ghost-antighost pair (η_{a_0}, ρ_{a_0}) of opposite parity to that of the corresponding constraint, and for each reducibility level we introduce a ghost-for-ghost pair (η_{a_k}, ρ_{a_k}). We associate a total ghost number $(k+1, -(k+1))$, and antighost numbers $(0, (k+1))$ to (η_{a_k}, ρ_{a_k}) (including $k = 0$). The BRST generator is constructed in the following way:

$$
\begin{aligned}
(i) \qquad & \mathrm{gh}(Q) = 1, \quad \varepsilon(Q) = 1, \quad Q^* = Q, && \text{(A.6)}\\
(ii) \qquad & Q = \eta^{a_0} G_{a_0} + \eta^{a_k} Z_{a_k}^{\;\;a_{k-1}} \rho_{a_{k-1}} + \bar{Q}, && \text{(A.7)}\\
(iii) \qquad & [Q, Q] = 0, && \text{(A.8)}
\end{aligned}
$$

where $\bar{Q}$ stands for further terms in Q containing at least two η's and one ρ, or two ρ's and one η, and that are constructed such that the (A.8) is fulfilled.

Out of these properties, the only "non-trivial" one to satisfy corresponds to the nilpotency condition (A.8). We can develop a systematic method to fulfill such condition by expanding Q in antighost number:

$$Q = \sum_{p \geq 0} \Omega^{(p)}, \tag{A.9}$$

$$\mathrm{antigh}(\Omega^{(p)}) = p. \tag{A.10}$$

We can then perform an analogous expansion in antighost number when we compute $[Q, Q]$:

$$[Q, Q] = \sum_{p \geq 0} B^{(p)}, \tag{A.11}$$

$$\mathrm{antigh}(B^{(p)}) = p, \tag{A.12}$$

where the $B^{(p)}$ terms are given by

$$B^{(p)} = \sum_{k=0}^{p} [\Omega^{(p-k)}, \Omega^{(k)}]_{\mathrm{orig}} + \sum_{k=0}^{p+1} \sum_{s=0}^{k} [\Omega^{(p-k+s+1)}, \Omega^{(k)}]_{\eta_{a_s}, \rho_{a_s}}. \tag{A.13}$$

Here, $[,]_{\mathrm{orig}}$ denotes the Poisson bracket in the original phase space of the theory (i.e. no ghosts), which leaves the antighost number invariant. Meanwhile $[,]_{\eta_{a_s}, \rho_{a_s}}$ denotes the bracket with respect to the pair η_{a_s}, ρ_{a_s} only. This latter bracket cancels one ρ_{a_s} and therefore reduces the antighost number by $s + 1$.

Since terms with different antighost number cannot cancel between each other, the previous expansion gives a simplifying systematic procedure to find the $\bar{Q}$ terms so that the nilpotency condition is fulfilled. The nilpotency condition (A.8) is therefore equivalent to the following set of simpler equations:

$$B^{(p)} = 0, \qquad p = 0, 1, 2, \ldots, \tag{A.14}$$

where $B^{(p)}$ involves the functions $\Omega^{(i)}$ up to order $p+1$. The term involving $\Omega^{(p+1)}$ in B^p is explicitly

$$2\sum_{s=0}^{p}[\Omega^{(p+1)},\Omega^s]_{\eta_{a_s},\rho_{a_s}}, \tag{A.15}$$

and the sum over s ends at p since the term

$$[\Omega^{(p+1)},\Omega^{(p+1)}]_{\eta_{a_{p+1}},\rho_{a_{p+1}}}$$

in (A.13) is zero, since Ω_{p+1} with antighost number $p+1$ cannot contain $\rho_{a_{p+1}}$ which has ghost number $p+2$.

The BRST operator constructed in this way exists and it is unique up to canonical transformations in the extended phase space (see chapter 10 in [73]).

The previous construction does not provide any further information or insight into the structure of the $\bar{Q}$ terms. Although the previous remark states that the BRST generator exists, it may do so in such a way that $\bar{Q}$ consists of a formal sum of infinite terms. A simplifying result that is fundamental in our calculation of the BRST generator in section 4 is the following:

Proposition 3.4.1. *When the constraints are linear in the momenta, the BRST generator can be taken to be linear in the extended phase space momenta.*

Proof. We take the original proof from [74]. Define

$$\Delta^{(p)}=\frac{1}{2}\sum_{k=0}^{p}[\Omega^{(p-k)},\Omega^{(k)}]_{\text{orig}}+\frac{1}{2}\sum_{k=1}^{p}\sum_{s=0}^{k-1}[\Omega^{(p-k+s+1)},\Omega^{(k)}]_{\eta_{a_s},\rho_{a_s}}. \tag{A.16}$$

Then, $\Omega^{(p+1)}$ satisfies (see subsection 10.5.4 in [73] for more details)

$$\delta\Omega^{(p+1)}+\Delta^{(p)}=0, \tag{A.17}$$

where δ is a differential operator ("the Koszul–Tate differential", acting on the extended phase space) defined as follows:

$$\delta z=0,\quad \delta\eta_{a_0,a_k}=0,\quad \delta\rho_{a_0}=-G_{a_0},\quad \delta\rho_{a_k}=-Z_{a_k}^{\ a_{k-1}}\rho_{a_{k-1}},\quad k=1,\dots,L, \tag{A.18}$$

and z is shorthand for the original phase space variables. One has

$$\Delta^{(0)} = \frac{1}{2}\eta_{a_0}\eta_{b_0}C^{c_0}_{a_0b_0}G_{c_0}. \Longrightarrow \Omega^{(1)} = \frac{1}{2}\eta_{a_0}\eta_{b_0}C^{c_0}_{a_0b_0}\rho_{c_0}. \tag{A.19}$$

Assume now that $\Omega^{(k)}$ for $k \leq p$ are linear in the momenta (we have just shown this is true for $k = 1$). Then, by its definition, $\Delta^{(p)}$ is linear in the momenta. Moreover, by the definition of δ, we see from (A.17) that $\Omega^{(p+1)}$ can be taken to be linear in the momenta. ■

Appendix B

The Global Residue Theorem

In this appendix we make a brief review of residues in many complex variables due to its use in the proof of the soft theorem in section 6 of this thesis. We provide all the necessary definitions for a proper statement of the mathematical tools we use, but we do not provide any proofs. We rather focus on the computation of multivariate residues, and specially on the global residue theorem. The main reference for this appendix is chapter 5 in the book by Griffiths and Harris [62].

We start with some definitions: We say a subset V of an open set $U \in \mathbb{C}^n$ is an *analytic variety* in U if, for any $p \in U$, there exists a neighborhood U' of p in U such that $V \cap U'$ is the common zero locus of a finite collection of holomorphic functions $f_1, \ldots, f_k$ on U'. More importantly for our purposes, V is called an *analytic hypersurface* if V is locally the zero-locus of a single nonzero holomorphic function f. An analytic variety $V \subset U \subset \mathbb{C}^n$ is said to be *irreducible* if V cannot be written as the union of two analytic varieties $V_1, V_2 \subset U$, with $V_1, V_2 \neq V$.

We further define: A *divisor* D on M is a locally finite formal linear combination

$$D = \sum a_i V_i \tag{B.1}$$

of irreducible analytic hypersurfaces of M. "Locally finite" means that for any $p \in M$ there is a neighborhood of p meeting only a finite number of the V_i defining D. When M is compact, this is just the condition that the sum above is finite.

Now, after such arid but necessary definitions, consider n complex variables $z \equiv (z_1, \ldots, z_n)$, and let U be the ball $\{z \in \mathbb{C}^n : \|z\| < \epsilon\}$. Take $f_1, \ldots, f_n \in \mathcal{O}(\bar{U})$ to be holomorphic functions in a neighborhood of the closure $\bar{U}$ of U. In the following we consider the theory around the origin, so we allow ϵ to be as small as necessary. We assume the origin to be an isolated common zero of the functions $f_i(z)$.

We are interested in residues associated to a meromorphic n-form:

$$\omega = \frac{g(z) dz_1 \wedge dz_2 \wedge \cdots \wedge dz_n}{f_1(z) \cdots f_n(z)}, \qquad \Big(g(z) \in \mathcal{O}(\bar{U})\Big). \tag{B.2}$$

The residue at the origin is then defined as

$$\mathrm{Res}_{\{0\}}\ \omega = \left(\frac{1}{2\pi i}\right)^n \oint_\Gamma \omega, \tag{B.3}$$

where the contour Γ is the real n-cycle defined as $\Gamma = \{z : f_i(z) = \epsilon_i\}$, with the orientation given by

$$d(\mathrm{arg} f_1) \wedge d(\mathrm{arg} f_2) \wedge \cdots \wedge d(\mathrm{arg} f_n) \geq 0.$$

An analogous definition is valid when we have an isolated pole ξ different than the origin.

In the trivial case where $g(z)$ is locally generated by $\{f_1, \ldots, f_n\}$:

$$g(z) = \alpha_1(z) f_1(z) + \cdots + \alpha_n f_n(z), \tag{B.4}$$

with α_i holomorphic in a neighborhood of ξ, we have that the residue is zero:

$$\mathrm{Res}_\xi\ \omega = 0. \tag{B.5}$$

We call a residue *non-degenerate* when the Jacobian of $\{f_1, \ldots f_n\}$ at ξ is non-zero:

$$J(\xi) \equiv \det\Big(\frac{\partial f_i}{\partial z_j}(\xi) \neq 0\Big). \tag{B.6}$$

In this case the residue can be computed as

$$\mathrm{Res}_\xi\ \omega = \frac{g(\xi)}{J(\xi)}. \tag{B.7}$$

Another situation in which a residue can be computed straightforwardly corresponds to the case where each f_i is a univariate polynomial $f_i(z) = f_i(z_i)$. As one may expect, in this case the residue formula factorizes:

$$\mathrm{Res}_{\{0\}}\ \omega = \left(\frac{1}{2\pi i}\right)^n \oint_{|f_1(z_1)|=\epsilon_1} \frac{dz_1}{f_1(z_1)} \cdots \oint_{|f_n(z_n)|=\epsilon} \frac{dz_n}{f_n(z_n)} g(z), \tag{B.8}$$

so we can use the univariate residue theorem n times and obtain the residue.

There is a number of approaches to compute a multivariate residue when the residue is degenerate or the functions f_i are not all univariate polynomials. It may happen as well that the formula

(B.7) is too complicated to use in practice (e.g. when finding the zero ξ). All these situations call for considerations of further methods of computation when dealing with integrals of the form (B.3). One of these methods is the Global Residue Theorem, stated next.

Theorem (Global Residue Theorem, GRT): *Let M be a n-dimensional compact complex manifold, and $D_1, D_2, ..., D_n$ be divisors of M. Assume that the intersection $S = D_1 \cap D_2 \cap \cdots \cap D_n$ only consists of a discrete set of points, and let ω be a holomorphic n-form on $M - D_1 \cup D_2 \cup \cdots \cup D_n$.* Then:

$$\sum_{P \in S} \mathrm{Res}_P \, \omega = 0. \tag{B.9}$$

A proof of this theorem can be found in chapter 5 in [62]. Notice that unlike the univariate residue theorem, there are many global residue theorem identities of the form (B.9). There is one such identity for each different choice of divisors.

The GRT can be generalized to non-compact complex manifolds, and notice that when applying the formula for the compact case we must be aware that there may be residues at infinity. Fortunately for us, none of these cases appear in our application of the GRT. To conclude, let us provide an example to illustrate how the GRT works in the case of concern for us:

Example. Let x and y be inhomogeneous coordinates over $\mathbb{CP}^2$, and consider the following differential form:

$$\omega = \frac{1}{xy(1+x+y)(2-y)} dx \wedge dy. \tag{B.10}$$

Let us evaluate the following integral:

$$\mathrm{Res}_{(1,0,0)} \, \omega = \left(\frac{1}{2\pi i}\right)^2 \int_{\substack{|x|<\epsilon \\ |y|<\epsilon}} \omega. \tag{B.11}$$

The answer is clearly $1/2$, but let us evaluate this integral through the GRT for pedagogical purposes. It is straightforward to see that there are no poles at infinity, so ω is holomorphic except at the points where $x = 0 \vee y = 0 \vee 1 + x + y = 0 \vee 2 - y = 0$. Thus, we can take the divisors $D_1 = \{x(1 + x + y) = 0\}$, $D_2 = \{y(2 - y) = 0\}$. The intersection S is then given by the following points in $\mathbb{CP}^2$: $S = \{(1, 0, 0), (1, 0, 2), (1, -1, 0), (1, -3, 2)\}$. Since the denominator only has linear functions we can easily evaluate:

$$\mathrm{Res}_{(1,0,0)} \, \omega = -\mathrm{Res}_{(1,0,2)} \, \omega - \mathrm{Res}_{(1,-1,0)} \, \omega - \mathrm{Res}_{(1,-3,2)} \, \omega = -\Big(-\frac{1}{6}\Big) - \Big(-\frac{1}{2}\Big) - \Big(\frac{1}{6}\Big) = \frac{1}{2}. \tag{B.12}$$

As expected. Now, let us take another choice of divisors in order to present explicitly our previous remark that there is indeed more than one global residue identity. Take the divisors $D_1' = \{x(2 - y)\}$, $D_2' = \{y(1 + x + y)\}$. The intersection is now given by $S' = \{(1, 0, 0), (1, 0, -1), (1, -3, 2)\}$.

Evaluating (noticing that by our current choice of divisors the pole at $(1,-3,2)$ now has a reverse orientation compared to the first case), we find again that

$$\operatorname{Res}_{(1,0,0)} \omega = -\operatorname{Res}_{(1,0,-1)} \omega - \operatorname{Res}_{(1,-3,2)} \omega = -\Big(-\frac{1}{3}\Big) - \Big(-\frac{1}{6}\Big) = \frac{1}{2}. \tag{B.13}$$

$\blacksquare$

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