

SEDENIONS OF CAYLEY-DICKSON AND
THE CAUCHY-RIEMANN LIKE RELATIONS

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Abstract: This work is an extension to sedenions of the Cauchy-Riemann relations, following a similar earlier construction made by one of the authors (M. Borges) to quaternions and octonions, see [1], [2], [3].

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1. Introduction

Sedenions, like quaternions or even octonions, are well known as hypercomplex numbers. In algebra, sedenions whose set is denoted by S , form a 16-dimensional algebra over the reals. They are currently classified in 2 types: (i) sedenions obtained by applying the Cayley-Dickson construction; (ii) conic sedenions (M-algebra 16 dimensional) or sedenions of Charles Musès, as they are also known, belonging to a part of a hypernumber concept, see [4], [5] [6], [7]. As well as octonions numbers, multiplication of Cayley-Dickson sedenions is neither commutative nor associative. They do have the property of power associativity,

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meaning that if S is a sedenion, $s^n.S^m = S^{n+m}$, but in contrast to the octonions, the sedenions do not even have the property of being alternative. Every sedenion may be regarded as a real linear combination of the unit sedenions $1, i, j, h, e', i', j', k', e'', i'', j'', k'', e''', i''', j''', k'''$. The sedenions have a multiplicative identity element 1 and multiplicative inverse, but they are not a division algebra, as they have zero divisors. This means that two nonzero numbers can be multiplied to obtain a zero result. Let us now define a sedenionic function on a domain in S . Suppose that I and I' are two domains in the 16 dimensional space $R^{16}, I \subset S$ and $I' \subset S$ a function $f : I \rightarrow I'$ is called a sedenionic function if f is a mapping such that for each

$$\mathbf{x} = (a_0, a_1, a_2, \dots, a_{15}) \in S$$

we have $y - f(\mathbf{x}) y \in I' \subset S$. Being f a sedenionic function it can be decomposed in a scalar part, $f_0(\mathbf{x}) = \phi(\mathbf{x})$ and a vectorial part given by $if_1(x) + jf_2(x) + kf_3(x) + e'f_4(x) + i'f_5(x) + j'f_6(x) + k'f_7(x) + e''f_8(x) + i''f_9(x) + j''f_{10}(x) + k''f_{11}(x) + e'''f_{12}(x) + i'''f_{13}(x) + j'''f_{14}(x) + k'''f_{15}(x) = \varphi(x)$, i.e $f(x) = f_0(x) + if_1(x) + jf_2(x) + kf_3(x) + e'f_4(x) + i'f_5(x) + j'f_6(x) + k'f_7(x) + e''f_8(x) + i''f_9(x) + j''f_{10}(x) + k''f_{11}(x) + e'''f_{12}(x) + i'''f_{13}(x) + j'''f_{14}(x) + k'''f_{15}(x) = \varphi(x)$, then $f(x) = f_0(x) + \xi(x) = \phi(x) + \varphi(x) = f_0(x) + if_1(x) + jf_2(x) + kf_3(x) + e'f_4(x) + i'f_5(x) + j'f_6(x) + k'f_7(x) + e''f_8(x) + i''f_9(x) + j''f_{10}(x) + k''f_{11}(x) + e'''f_{12}(x) + i'''f_{13}(x) + j'''f_{14}(x) + k'''f_{15}(x)$ where the $f_i : R^{16} \rightarrow R$ the coordinates functions of real values to $i = 0, 1, 2, \dots, 15$. Consequently, a norm of f may be defined as:

$$|f| = \left(f_0^2 + f_1^2 + f_2^2 + f_3^2 + f_4^2 + f_5^2 + f_6^2 + f_7^2 + f_8^2 + f_9^2 + f_{10}^2 + f_{11}^2 + f_{12}^2 + f_{13}^2 + f_{14}^2 + f_{15}^2 \right)^{\frac{1}{2}}. \quad (1)$$

Nowadays, sedenions are also useful in applications to physics. For instance, the Dirac equation in physics can be expressed on hyperbolic octonion algebra and then transformed into a counterpart on circular octonions, by means of conic sedenions as unifying number concept, see [6], [7]. In this work, with the purpose of setting geometrical features for sedenions, we present remarks on differentiability and integrability of hypercomplex functions of sedenionic type. A set of Cauchy-Riemann like relations is obtained for Cayley-Dickson construction.

*	1	i	j	k	e'	i'	j'	k'	e''	i''	j''	k''	e'''	i'''	j'''	k'''
1	1	i	j	k	e'	i'	j'	k'	e''	i''	j''	k''	e'''	i'''	j'''	k'''
i	i	-1	k	-j	i'	-e'	-k'	j'	i''	-e''	-k''	j''	-i'''	e'''	k'''	-j'''
j	j	-k	-1	i	j'	k'	-e'	-i'	j''	k''	-e''	-i''	-j'''	-k'''	e'''	i'''
k	k	j	-i	-1	k'	-j'	i'	-e'	k''	-j''	i''	-e''	-k'''	j'''	-i'''	e'''
e'	e'	-i'	-j'	-k'	-1	i	j	k	e''	i''	j''	k''	-e'''	-i'''	-j'''	-k'''
i'	i'	e'	-k'	j'	-i	-1	-k	j	i''	-e''	k''	-j''	i'''	-e'''	k'''	-j'''
j'	j'	k'	e'	-i'	-j	k	-1	-i	j''	-k''	-e''	i''	-j'''	-k'''	-e'''	i'''
k'	k'	-j'	i'	e'	-k	-j	i	-1	k''	j''	-i''	-e''	k'''	j'''	-i'''	-e'''
e''	e''	-i''	-j''	-k''	-e'''	-i'''	-j'''	-k'''	-1	i	j	k	e'	i'	j'	k'
i''	i''	e''	-k''	j''	-i'''	e'''	k'''	-j'''	-i	-1	-k	j	-i'	e'	k'	-j'
j''	j''	k''	e''	-i''	-j'''	-k'''	e'''	i'''	-j	k	-1	-i	-j'	-k'	e'	i'
k''	k''	-j''	i''	e''	-k'''	j'''	-i'''	e'''	-k	-j	i	-1	-k'	j'	-i'	e'
e'''	e'''	i'''	j'''	k'''	e''	-i''	-j''	-k''	-e'	i'	j'	k'	-1	-i	-j	-k
i'''	i'''	-e'''	k'''	-j'''	i''	e''	k''	-j''	-i'	-e'	k'	-j'	i	-1	k	-j
j'''	j'''	-k'''	-e'''	i'''	j''	-k''	e''	i''	-j'	-k'	-e'	i'	j	-k	-1	i
k'''	k'''	j'''	-i'''	-e'''	k''	j''	-i''	e''	-k'	j'	-i'	-e'	k	j	-i	-1

Table 1: Multiplication table for sedenions

2. On Cauchy-Riemann Like Equations for Sedenions

Recent works [1], [2], [3], have shown as hypercomplex regular functions satisfying integrability conditions. In this paper we use the concept of integrability the left and to the right to expand Cauchy-Riemann like equations for sedenions. Let f be a function, $R^{16} \rightarrow R$. As sedenions are non-commutative we define two integrals: Either f function $R^{16} \mapsto S$, of ownership the fact that sedenions to be born comutatives, defines two integrals (i) $\int f dz = \int (f_0 + f_1 \mathbf{i} + f_2 \mathbf{j} + f_3 \mathbf{k} + f_4 \mathbf{e}' + f_5 \mathbf{i}' + f_6 \mathbf{j}' + f_7 \mathbf{k}' + f_8 \mathbf{e}'' + f_9 \mathbf{i}'' + f_{10} \mathbf{j}'' + f_{11} \mathbf{k}'' + f_{12} \mathbf{e}''' + f_{13} \mathbf{i}''' + f_{14} \mathbf{j}''' + f_{15} \mathbf{k}''') (da_0 + da_1 \mathbf{i} + da_2 \mathbf{j} + da_3 \mathbf{k} + da_4 \mathbf{e}' + da_5 \mathbf{i}' + da_6 \mathbf{j}' + da_7 \mathbf{k}' + da_8 \mathbf{e}'' + da_9 \mathbf{i}'' + da_{10} \mathbf{j}'' + da_{11} \mathbf{k}'' + da_{12} \mathbf{e}''' + da_{13} \mathbf{i}''' + da_{14} \mathbf{j}''' + da_{15} \mathbf{k}''') = \int (f_0 da_0 - f_1 da_1 - f_2 da_2 - f_3 da_3 - f_4 da_4 - f_5 da_5 - f_6 da_6 - f_7 da_7 - f_8 da_8 - f_9 da_9 - f_{10} da_{10} - f_{11} da_{11} - f_{12} da_{12} - f_{13} da_{13} - f_{14} da_{14} - f_{15} da_{15}) + \int (f_1 da_0 + f_0 da_1 - f_3 da_2 + f_2 da_3 - f_5 da_4 + f_4 da_5 + f_7 da_6 - f_6 da_7 - f_9 da_8 + f_8 da_9 + f_{11} da_{10} - f_{10} da_{11} + f_{13} da_{12} - f_{12} da_{13} - f_{15} da_{14} + f_{14} da_{15}) \mathbf{i} + \int (f_2 da_0 + f_3 da_1 + f_0 da_2 - f_1 da_3 - f_6 da_4 - f_7 da_5 + f_4 da_6 + f_5 da_7 - f_{10} da_8 - f_{11} da_9 + f_8 da_{10} + f_9 da_{11} + f_{14} da_{12} + f_{15} da_{13} - f_{12} da_{14} - f_{13} da_{15}) \mathbf{j} + \int (f_3 da_0 - f_2 da_1 + f_1 da_2 + f_0 da_3 - f_7 da_4 + f_6 da_5 - f_5 da_6 + f_4 da_7 - f_{11} da_8 + f_{10} da_9 - f_9 da_{10} + f_8 da_{11} + f_{15} da_{12} - f_{14} da_{13} + f_{13} da_{14} - f_{12} da_{15}) \mathbf{k} + \int (f_4 da_0 + f_5 da_1 + f_6 da_2 + f_7 da_3 + f_0 da_4 - f_1 da_5 - f_2 da_6 - f_3 da_7 - f_{12} da_8 - f_{13} da_9 - f_{14} da_{10} - f_{15} da_{11} + f_8 da_{12} + f_9 da_{13} + f_{10} da_{14} + f_{11} da_{15}) \mathbf{e}' + \int (f_5 da_0 - f_4 da_1 + f_7 da_2 - f_6 da_3 + f_1 da_4 + f_0 da_5 + f_3 da_6 - f_2 da_7 - f_{13} da_8 + f_{12} da_9 - f_{15} da_{10} + f_{14} da_{11} - f_9 da_{12} + f_8 da_{13} - f_{11} da_{14} + f_{10} da_{15}) \mathbf{i}' + \int (f_6 da_0 - f_7 da_1 - f_4 da_2 + f_5 da_3 + f_2 da_4 - f_3 da_5 + f_0 da_6 + f_1 da_7 - f_{14} da_8 + f_{15} da_9 + f_{12} da_{10} - f_{13} da_{11} - f_{10} da_{12} + f_{11} da_{13} + f_8 da_{14} - f_9 da_{15}) \mathbf{j}' + \int (f_7 da_0 + f_6 da_1 - f_5 da_2 - f_4 da_3 + f_3 da_4 + f_2 da_5 - f_1 da_6 + f_0 da_7 - f_{15} da_8 - f_{14} da_9 + f_{13} da_{10} + f_{12} da_{11} - f_{11} da_{12} - f_{10} da_{13} + f_9 da_{14} + f_8 da_{15}) \mathbf{k}' + \int (f_8 da_0 + f_9 da_1 + f_{10} da_2 + f_{11} da_3 + f_{12} da_4 + f_{13} da_5 + f_{14} da_6 + f_{15} da_7 + f_0 da_8 - f_1 da_9 - f_2 da_{10} - f_3 da_{11} - f_4 da_{12} - f_5 da_{13} - f_6 da_{14} - f_7 da_{15}) \mathbf{e}'' + \int (f_9 da_0 - f_8 da_1 + f_{11} da_2 - f_{10} da_3 + f_{13} da_4 - f_{12} da_5 - f_{15} da_6 + f_{14} da_7 + f_1 da_8 + f_0 da_9 + f_3 da_{10} - f_2 da_{11} + f_5 da_{12} - f_4 da_{13} - f_7 da_{14} + f_6 da_{15}) \mathbf{i}'' + \int (f_{10} da_0 - f_{11} da_1 - f_8 da_2 + f_9 da_3 + f_{14} da_4 + f_{15} da_5 - f_{12} da_6 - f_{13} da_7 + f_2 da_8 - f_3 da_9 + f_0 da_{10} + f_1 da_{11} + f_6 da_{12} + f_7 da_{13} - f_4 da_{14} - f_5 da_{15}) \mathbf{j}'' + \int (f_{11} da_0 + f_{10} da_1 - f_9 da_2 - f_8 da_3 + f_{15} da_4 - f_{14} da_5 + f_{13} da_6 - f_{12} da_7 + f_3 da_8 + f_2 da_9 - f_1 da_{10} + f_0 da_{11} + f_7 da_{12} - f_6 da_{13} + f_5 da_{14} - f_4 da_{15}) \mathbf{k}'' + \int (f_{12} da_0 - f_{13} da_1 - f_{14} da_2 - f_{15} da_3 - f_8 da_4 + f_9 da_5 + f_{10} da_6 + f_{11} da_7 + f_4 da_8 - f_5 da_9 - f_6 da_{10} - f_7 da_{11} + f_0 da_{12} + f_1 da_{13} + f_2 da_{14} + f_3 da_{15}) \mathbf{e}''' + \int (f_{13} da_0 + f_{12} da_1 - f_{15} da_2 + f_{14} da_3 - f_9 da_4 - f_8 da_5 - f_{11} da_6 + f_{10} da_7 + f_5 da_8 + f_4 da_9 - f_7 da_{10} + f_6 da_{11} - f_1 da_{12} + f_0 da_{13} - f_3 da_{14} + f_2 da_{15}) \mathbf{i}''' + \int (f_{14} da_0 + f_{15} da_1 + f_{12} da_2 - f_{13} da_3 - f_{10} da_4 + f_{11} da_5 - f_8 da_6 - f_9 da_7 + f_6 da_8 + f_7 da_9 + f_4 da_{10} - f_5 da_{11} - f_2 da_{12} + f_3 da_{13} + f_0 da_{14} - f_1 da_{15}) \mathbf{j}''' + \int (f_{15} da_0 - f_{14} da_1 + f_{13} da_2 + f_{12} da_3 - f_{11} da_4 - f_{10} da_5 + f_9 da_6 - f_8 da_7 +$

$$\begin{aligned}
 & f_7 da_8 - f_6 da_9 + f_5 da_{10} + f_4 da_{11} - f_3 da_{12} - f_2 da_{13} + f_1 da_{14} + f_0 da_{15} \mathbf{k}''' \text{ and (ii)} \\
 \int dz f &= \int (da_0 + da_1 \mathbf{i} + da_2 \mathbf{j} + da_3 \mathbf{k} + da_4 \mathbf{e}' + da_5 \mathbf{i}'' + da_6 \mathbf{j}''' + da_7 \mathbf{k}'' + da_8 \mathbf{e}'' + da_9 \mathbf{i}' + \\
 & da_{10} \mathbf{j}'' + da_{11} \mathbf{k}'' + da_{12} \mathbf{e}''' + da_{13} \mathbf{i}''' + da_{14} \mathbf{j}''' + da_{15} \mathbf{k}''') (f_0 + f_1 \mathbf{i} + f_2 \mathbf{j} + f_3 \mathbf{k} + f_4 \mathbf{e}' + \\
 & f_5 \mathbf{i}'' + f_6 \mathbf{j}'' + f_7 \mathbf{k}'' + f_8 \mathbf{e}'' + f_9 \mathbf{i}''' + f_{10} \mathbf{j}''' + f_{11} \mathbf{k}''' + f_{12} \mathbf{e}''' + f_{13} \mathbf{i}''' + f_{14} \mathbf{j}''' + f_{15} \mathbf{k}''') = \\
 & \int (da_0 f_0 - da_1 f_1 - da_2 f_2 - da_3 f_3 - da_4 f_4 - da_5 f_5 - da_6 f_6 - da_7 f_7 - da_8 f_8 - da_9 f_9 - \\
 & da_{10} f_{10} - da_{11} f_{11} - da_{12} f_{12} - da_{13} f_{13} - da_{14} f_{14} - da_{15} f_{15}) + \int (da_0 f_1 + da_1 f_0 + \\
 & da_2 f_3 - da_3 f_2 + da_4 f_5 - da_5 f_4 - da_6 f_7 + da_7 f_6 + da_8 f_9 - da_9 f_8 - da_{10} f_{11} + da_{11} f_{10} - \\
 & da_{12} f_{13} + da_{13} f_{12} + da_{14} f_{15} - da_{15} f_{14}) \mathbf{i} + \int (da_0 f_2 - da_1 f_3 + da_2 f_0 + da_3 f_1 + \\
 & da_4 f_6 + da_5 f_7 - da_6 f_4 - da_7 f_5 + da_8 f_{10} + da_9 f_{11} - da_{10} f_8 - da_{11} f_9 - da_{12} f_{14} - \\
 & da_{13} f_{15} + da_{14} f_{12} + da_{15} f_{13}) \mathbf{j} + \int (da_0 f_3 + da_1 f_2 - da_2 f_1 + da_3 f_0 + da_4 f_7 - da_5 f_6 + \\
 & da_6 f_5 - da_7 f_4 + da_8 f_{11} - da_9 f_{10} + da_{10} f_9 - da_{11} f_8 - da_{12} f_{15} + da_{13} f_{14} - da_{14} f_{13} + \\
 & da_{15} f_{12}) \mathbf{k} + \int (da_0 f_4 - da_1 f_5 - da_2 f_6 - da_3 f_7 + da_4 f_0 + da_5 f_1 + da_6 f_2 + da_7 f_3 + \\
 & da_8 f_{12} + da_9 f_{13} + da_{10} f_{14} + da_{11} f_{15} - da_{12} f_8 - da_{13} f_9 - da_{14} f_{10} - da_{15} f_{11}) \mathbf{e}' + \\
 & \int (da_0 f_5 + da_1 f_4 - da_2 f_7 + da_3 f_6 - da_4 f_1 + da_5 f_0 - da_6 f_3 + da_7 f_2 + da_8 f_{13} - da_9 f_{12} + \\
 & da_{10} f_{15} - da_{11} f_{14} + da_{12} f_9 - da_{13} f_8 + da_{14} f_{11} - da_{15} f_{10}) \mathbf{i}'' + \int (da_0 f_6 + da_1 f_7 + \\
 & da_2 f_4 - da_3 f_5 - da_4 f_2 + da_5 f_3 + da_6 f_0 - da_7 f_1 + da_8 f_{14} - da_9 f_{15} - da_{10} f_{12} + da_{11} f_{13} + \\
 & da_{12} f_{10} - da_{13} f_{11} - da_{14} f_8 + da_{15} f_9) \mathbf{j}'' + \int (da_0 f_7 - da_1 f_6 + da_2 f_5 + da_3 f_4 - da_4 f_3 - \\
 & da_5 f_2 + da_6 f_1 + da_7 f_0 + da_8 f_{15} + da_9 f_{14} - da_{10} f_{13} - da_{11} f_{12} + da_{12} f_{11} + da_{13} f_{10} - \\
 & da_{14} f_9 - da_{15} f_8) \mathbf{k}'' + \int (da_0 f_8 - da_1 f_9 - da_2 f_{10} - da_3 f_{11} - da_4 f_{12} - da_5 f_{13} - da_6 f_{14} - \\
 & da_7 f_{15} + da_8 f_0 + da_9 f_1 + da_{10} f_2 + da_{11} f_3 + da_{12} f_4 + da_{13} f_5 + da_{14} f_6 + da_{15} f_7) \mathbf{e}'' + \\
 & \int (da_0 f_9 + da_1 f_8 - da_2 f_{11} + da_3 f_{10} - da_4 f_{13} + da_5 f_{12} + da_6 f_{15} - da_7 f_{14} - da_8 f_1 + \\
 & da_9 f_0 - da_{10} f_3 + da_{11} f_2 - da_{12} f_5 + da_{13} f_4 + da_{14} f_7 - da_{15} f_6) \mathbf{i}'' + \int (da_0 f_{10} + da_1 f_{11} + \\
 & da_2 f_8 - da_3 f_9 - da_4 f_{14} - da_5 f_{15} + da_6 f_{12} + da_7 f_{13} - da_8 f_2 + da_9 f_3 + da_{10} f_0 - \\
 & da_{11} f_1 - da_{12} f_6 - da_{13} f_7 + da_{14} f_4 + da_{15} f_5) \mathbf{j}'' + \int (da_0 f_{11} - da_1 f_{10} + da_2 f_9 + \\
 & da_3 f_8 - da_4 f_{15} + da_5 f_{14} - da_6 f_{13} + da_7 f_{12} - da_8 f_3 - da_9 f_2 + da_{10} f_1 + da_{11} f_0 - \\
 & da_{12} f_7 + da_{13} f_6 - da_{14} f_5 + da_{15} f_4) \mathbf{k}'' + \int (da_0 f_{12} + da_1 f_{13} + da_2 f_{14} + da_3 f_{15} + \\
 & da_4 f_8 - da_5 f_9 - da_6 f_{10} - da_7 f_{11} - da_8 f_4 + da_9 f_5 + da_{10} f_6 + da_{11} f_7 + da_{12} f_0 - \\
 & da_{13} f_1 - da_{14} f_2 - da_{15} f_3) \mathbf{e}''' + \int (da_0 f_{13} - da_1 f_{12} + da_2 f_{15} - da_3 f_{14} + da_4 f_9 + \\
 & da_5 f_8 + da_6 f_{11} - da_7 f_{10} - da_8 f_5 - da_9 f_4 + da_{10} f_7 - da_{11} f_6 + da_{12} f_1 + da_{13} f_0 + \\
 & da_{14} f_3 - da_{15} f_2) \mathbf{i}''' + \int (da_0 f_{14} - da_1 f_{15} - da_2 f_{12} + da_3 f_{13} + da_4 f_{10} - da_5 f_{11} + \\
 & da_6 f_8 + da_7 f_9 - da_8 f_6 - da_9 f_7 - da_{10} f_4 + da_{11} f_5 + da_{12} f_2 - da_{13} f_3 + da_{14} f_0 + \\
 & da_{15} f_1) \mathbf{j}''' + \int (da_0 f_{15} + da_1 f_{14} - da_2 f_{13} - da_3 f_{12} + da_4 f_{11} + da_5 f_{10} - da_6 f_9 + \\
 & da_7 f_8 - da_8 f_7 + da_9 f_6 - da_{10} f_5 - da_{11} f_4 + da_{12} f_3 + da_{13} f_2 - da_{14} f_1 + da_{15} f_0) \mathbf{k}'''.
 \end{aligned}$$

Assuming the function $f_l : R^{16} \mapsto S$, and a path with extremities in $x = (x_0, x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{15})$ and $y = (y_0, y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8, y_9, y_{10}, y_{11}, y_{12}, y_{13}, y_{14}, y_{15})$ in a connected region in a space of 16-dimensions, the integrals $\int f dz$ e $\int dz f$, will be independent of the integration way if they satisfy the conditions of the following theorems.

3. Cauchy-Like Theorems for Sedenions

Theorem 1. For all pair of points x and y any path way binding them in a 16-dimensional simply connected space, the integral $\int_x^y fdz$ does not depend of the given path if, and only if, there exists a function $F = F_0 + F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k} + F_4\mathbf{e}' + F_5\mathbf{i}' + F_6\mathbf{j}' + F_7\mathbf{k}' + F_8\mathbf{e}'' + F_9\mathbf{i}'' + F_{10}\mathbf{j}'' + F_{11}\mathbf{k}'' + F_{12}\mathbf{e}''' + F_{13}\mathbf{i}''' + F_{14}\mathbf{j}''' + F_{15}\mathbf{k}'''$ with $\int_x^y fdz = F(y) - F(x)$ and that it satisfies the following conditions

$$f_0 = \frac{\partial F_0}{\partial a_0} = \frac{\partial F_1}{\partial a_1} = \frac{\partial F_2}{\partial a_2} = \frac{\partial F_3}{\partial a_3} = \frac{\partial F_4}{\partial a_4} = \frac{\partial F_5}{\partial a_5} = \frac{\partial F_6}{\partial a_6} = \frac{\partial F_7}{\partial a_7} = \frac{\partial F_8}{\partial a_8} \\ = \frac{\partial F_9}{\partial a_9} = \frac{\partial F_{10}}{\partial a_{10}} = \frac{\partial F_{11}}{\partial a_{11}} = \frac{\partial F_{12}}{\partial a_{12}} = \frac{\partial F_{13}}{\partial a_{13}} = \frac{\partial F_{14}}{\partial a_{14}} = \frac{\partial F_{15}}{\partial a_{15}}, \quad (2)$$

$$f_1 = \frac{\partial F_1}{\partial a_0} = -\frac{\partial F_0}{\partial a_1} = \frac{\partial F_3}{\partial a_2} = -\frac{\partial F_2}{\partial a_3} = \frac{\partial F_5}{\partial a_4} = -\frac{\partial F_4}{\partial a_5} = -\frac{\partial F_7}{\partial a_6} = \frac{\partial F_6}{\partial a_7} = \frac{\partial F_9}{\partial a_8} \\ = -\frac{\partial F_8}{\partial a_9} = -\frac{\partial F_{11}}{\partial a_{10}} = \frac{\partial F_{10}}{\partial a_{11}} = -\frac{\partial F_{13}}{\partial a_{12}} = \frac{\partial F_{12}}{\partial a_{13}} = \frac{\partial F_{15}}{\partial a_{14}} = -\frac{\partial F_{14}}{\partial a_{15}}, \quad (3)$$

$$f_2 = \frac{\partial F_2}{\partial a_0} = -\frac{\partial F_3}{\partial a_1} = -\frac{\partial F_0}{\partial a_2} = \frac{\partial F_1}{\partial a_3} = \frac{\partial F_6}{\partial a_4} = \frac{\partial F_7}{\partial a_5} = -\frac{\partial F_4}{\partial a_6} = -\frac{\partial F_5}{\partial a_7} = \frac{\partial F_{10}}{\partial a_8} \\ = \frac{\partial F_{11}}{\partial a_9} = -\frac{\partial F_8}{\partial a_{10}} = -\frac{\partial F_9}{\partial a_{11}} = -\frac{\partial F_{14}}{\partial a_{12}} = -\frac{\partial F_{15}}{\partial a_{13}} = \frac{\partial F_{12}}{\partial a_{14}} = \frac{\partial F_{13}}{\partial a_{15}}, \quad (4)$$

$$f_3 = \frac{\partial F_3}{\partial a_0} = \frac{\partial F_2}{\partial a_1} = -\frac{\partial F_1}{\partial a_2} = -\frac{\partial F_0}{\partial a_3} = \frac{\partial F_7}{\partial a_4} = -\frac{\partial F_6}{\partial a_5} = \frac{\partial F_5}{\partial a_6} = -\frac{\partial F_4}{\partial a_7} = \frac{\partial F_{11}}{\partial a_8} \\ = -\frac{\partial F_{10}}{\partial a_9} = \frac{\partial F_9}{\partial a_{10}} = -\frac{\partial F_8}{\partial a_{11}} = -\frac{\partial F_{15}}{\partial a_{12}} = \frac{\partial F_{14}}{\partial a_{13}} = -\frac{\partial F_{13}}{\partial a_{14}} = \frac{\partial F_{12}}{\partial a_{15}}, \quad (5)$$

$$f_4 = \frac{\partial F_4}{\partial a_0} = -\frac{\partial F_5}{\partial a_1} = -\frac{\partial F_6}{\partial a_2} = -\frac{\partial F_7}{\partial a_3} = -\frac{\partial F_0}{\partial a_4} = \frac{\partial F_1}{\partial a_5} = \frac{\partial F_2}{\partial a_6} = \frac{\partial F_3}{\partial a_7} = \frac{\partial F_{12}}{\partial a_8} \\ = \frac{\partial F_{13}}{\partial a_9} = \frac{\partial F_{14}}{\partial a_{10}} = \frac{\partial F_{15}}{\partial a_{11}} = -\frac{\partial F_8}{\partial a_{12}} = -\frac{\partial F_9}{\partial a_{13}} = -\frac{\partial F_{10}}{\partial a_{14}} = -\frac{\partial F_{11}}{\partial a_{15}}, \quad (6)$$

$$f_5 = \frac{\partial F_5}{\partial a_0} = \frac{\partial F_4}{\partial a_1} = -\frac{\partial F_7}{\partial a_2} = \frac{\partial F_6}{\partial a_3} = -\frac{\partial F_1}{\partial a_4} = -\frac{\partial F_0}{\partial a_5} = -\frac{\partial F_3}{\partial a_6} = \frac{\partial F_2}{\partial a_7} = \frac{\partial F_{13}}{\partial a_8}$$

$$= -\frac{\partial F_{12}}{\partial a_9} = \frac{\partial F_{15}}{\partial a_{10}} = -\frac{\partial F_{14}}{\partial a_{11}} = \frac{\partial F_9}{\partial a_{12}} = -\frac{\partial F_8}{\partial a_{13}} = \frac{\partial F_{11}}{\partial a_{14}} = -\frac{\partial F_{10}}{\partial a_{15}}, \quad (7)$$

$$\begin{aligned} f_6 &= \frac{\partial F_6}{\partial a_0} = \frac{\partial F_7}{\partial a_1} = \frac{\partial F_4}{\partial a_2} = -\frac{\partial F_5}{\partial a_3} = -\frac{\partial F_2}{\partial a_4} = \frac{\partial F_3}{\partial a_5} = -\frac{\partial F_0}{\partial a_6} = -\frac{\partial F_1}{\partial a_7} = \frac{\partial F_{14}}{\partial a_8} \\ &= -\frac{\partial F_{15}}{\partial a_9} = -\frac{\partial F_{12}}{\partial a_{10}} = \frac{\partial F_{13}}{\partial a_{11}} = \frac{\partial F_{10}}{\partial a_{12}} = -\frac{\partial F_{11}}{\partial a_{13}} = -\frac{\partial F_8}{\partial a_{14}} = \frac{\partial F_9}{\partial a_{15}}, \quad (8) \end{aligned}$$

$$\begin{aligned} f_7 &= \frac{\partial F_7}{\partial a_0} = -\frac{\partial F_6}{\partial a_1} = \frac{\partial F_5}{\partial a_2} = \frac{\partial F_4}{\partial a_3} = -\frac{\partial F_3}{\partial a_4} = -\frac{\partial F_2}{\partial a_5} = \frac{\partial F_1}{\partial a_6} = -\frac{\partial F_0}{\partial a_7} = \frac{\partial F_{15}}{\partial a_8} \\ &= \frac{\partial F_{14}}{\partial a_9} = -\frac{\partial F_{13}}{\partial a_{10}} = -\frac{\partial F_{12}}{\partial a_{11}} = \frac{\partial F_{11}}{\partial a_{12}} = \frac{\partial F_{10}}{\partial a_{13}} = -\frac{\partial F_9}{\partial a_{14}} = -\frac{\partial F_8}{\partial a_{15}}, \quad (9) \end{aligned}$$

$$\begin{aligned} f_8 &= \frac{\partial F_8}{\partial a_0} = -\frac{\partial F_9}{\partial a_1} = -\frac{\partial F_{10}}{\partial a_2} = -\frac{\partial F_{11}}{\partial a_3} = -\frac{\partial F_{12}}{\partial a_4} = -\frac{\partial F_{13}}{\partial a_5} = -\frac{\partial F_{14}}{\partial a_6} \\ &= -\frac{\partial F_{15}}{\partial a_7} = -\frac{\partial F_0}{\partial a_8} = \frac{\partial F_1}{\partial a_9} = \frac{\partial F_2}{\partial a_{10}} = \frac{\partial F_3}{\partial a_{11}} = \frac{\partial F_4}{\partial a_{12}} = \frac{\partial F_5}{\partial a_{13}} = \frac{\partial F_6}{\partial a_{14}} = \frac{\partial F_7}{\partial a_{15}}, \quad (10) \end{aligned}$$

$$\begin{aligned} f_9 &= \frac{\partial F_9}{\partial a_0} = \frac{\partial F_8}{\partial a_1} = -\frac{\partial F_{11}}{\partial a_2} = \frac{\partial F_{10}}{\partial a_3} = -\frac{\partial F_{13}}{\partial a_4} = \frac{\partial F_{12}}{\partial a_5} = \frac{\partial F_{15}}{\partial a_6} = -\frac{\partial F_{14}}{\partial a_7} \\ &= -\frac{\partial F_1}{\partial a_8} = -\frac{\partial F_0}{\partial a_9} = -\frac{\partial F_3}{\partial a_{10}} = \frac{\partial F_2}{\partial a_{11}} = -\frac{\partial F_5}{\partial a_{12}} = \frac{\partial F_4}{\partial a_{13}} = \frac{\partial F_7}{\partial a_{14}} = -\frac{\partial F_6}{\partial a_{15}}, \quad (11) \end{aligned}$$

$$\begin{aligned} f_{10} &= \frac{\partial F_{10}}{\partial a_0} = \frac{\partial F_{11}}{\partial a_1} = \frac{\partial F_8}{\partial a_2} = -\frac{\partial F_9}{\partial a_3} = -\frac{\partial F_{14}}{\partial a_4} = -\frac{\partial F_{15}}{\partial a_5} = \frac{\partial F_{12}}{\partial a_6} = \frac{\partial F_{13}}{\partial a_7} \\ &= -\frac{\partial F_2}{\partial a_8} = \frac{\partial F_3}{\partial a_9} = -\frac{\partial F_0}{\partial a_{10}} = -\frac{\partial F_1}{\partial a_{11}} = -\frac{\partial F_6}{\partial a_{12}} = -\frac{\partial F_7}{\partial a_{13}} = \frac{\partial F_4}{\partial a_{14}} = \frac{\partial F_5}{\partial a_{15}}, \quad (12) \end{aligned}$$

$$\begin{aligned} f_{11} &= \frac{\partial F_{11}}{\partial a_0} = -\frac{\partial F_{10}}{\partial a_1} = \frac{\partial F_9}{\partial a_2} = \frac{\partial F_8}{\partial a_3} = -\frac{\partial F_{15}}{\partial a_4} = \frac{\partial F_{14}}{\partial a_5} = -\frac{\partial F_{13}}{\partial a_6} = \frac{\partial F_{12}}{\partial a_7} \\ &= -\frac{\partial F_3}{\partial a_8} = -\frac{\partial F_2}{\partial a_9} = \frac{\partial F_1}{\partial a_{10}} = -\frac{\partial F_0}{\partial a_{11}} = -\frac{\partial F_7}{\partial a_{12}} = \frac{\partial F_6}{\partial a_{13}} = -\frac{\partial F_5}{\partial a_{14}} = \frac{\partial F_4}{\partial a_{15}}, \quad (13) \end{aligned}$$

$$\begin{aligned}
 f_{12} &= \frac{\partial F_{12}}{\partial a_0} = \frac{\partial F_{13}}{\partial a_1} = \frac{\partial F_{14}}{\partial a_2} = \frac{\partial F_{15}}{\partial a_3} = \frac{\partial F_8}{\partial a_4} = -\frac{\partial F_9}{\partial a_5} = -\frac{\partial F_{10}}{\partial a_6} = -\frac{\partial F_{11}}{\partial a_7} \\
 &= -\frac{\partial F_4}{\partial a_8} = \frac{\partial F_5}{\partial a_9} = \frac{\partial F_6}{\partial a_{10}} = \frac{\partial F_7}{\partial a_{11}} = -\frac{\partial F_0}{\partial a_{12}} = -\frac{\partial F_1}{\partial a_{13}} = -\frac{\partial F_2}{\partial a_{14}} = -\frac{\partial F_3}{\partial a_{15}}, \quad (14)
 \end{aligned}$$

$$\begin{aligned}
 f_{13} &= \frac{\partial F_{13}}{\partial a_0} = -\frac{\partial F_{12}}{\partial a_1} = \frac{\partial F_{15}}{\partial a_2} = -\frac{\partial F_{14}}{\partial a_3} = \frac{\partial F_9}{\partial a_4} = \frac{\partial F_8}{\partial a_5} = \frac{\partial F_{11}}{\partial a_6} = -\frac{\partial F_{10}}{\partial a_7} \\
 &= -\frac{\partial F_5}{\partial a_8} = -\frac{\partial F_4}{\partial a_9} = \frac{\partial F_7}{\partial a_{10}} = -\frac{\partial F_6}{\partial a_{11}} = \frac{\partial F_1}{\partial a_{12}} = -\frac{\partial F_0}{\partial a_{13}} = \frac{\partial F_3}{\partial a_{14}} = -\frac{\partial F_2}{\partial a_{15}}, \quad (15)
 \end{aligned}$$

$$\begin{aligned}
 f_{14} &= \frac{\partial F_{14}}{\partial a_0} = -\frac{\partial F_{15}}{\partial a_1} = -\frac{\partial F_{12}}{\partial a_2} = \frac{\partial F_{13}}{\partial a_3} = \frac{\partial F_{10}}{\partial a_4} = -\frac{\partial F_{11}}{\partial a_5} = \frac{\partial F_8}{\partial a_6} = \frac{\partial F_9}{\partial a_7} \\
 &= -\frac{\partial F_6}{\partial a_8} = -\frac{\partial F_7}{\partial a_9} = -\frac{\partial F_4}{\partial a_{10}} = \frac{\partial F_5}{\partial a_{11}} = \frac{\partial F_2}{\partial a_{12}} = -\frac{\partial F_3}{\partial a_{13}} = -\frac{\partial F_0}{\partial a_{14}} = \frac{\partial F_1}{\partial a_{15}}, \quad (16)
 \end{aligned}$$

$$\begin{aligned}
 f_{15} &= \frac{\partial F_{15}}{\partial a_0} = \frac{\partial F_{14}}{\partial a_1} = -\frac{\partial F_{13}}{\partial a_2} = -\frac{\partial F_{12}}{\partial a_3} = \frac{\partial F_{11}}{\partial a_4} = \frac{\partial F_{10}}{\partial a_5} = -\frac{\partial F_9}{\partial a_6} = \frac{\partial F_8}{\partial a_7} \\
 &= -\frac{\partial F_7}{\partial a_8} = \frac{\partial F_6}{\partial a_9} = -\frac{\partial F_5}{\partial a_{10}} = -\frac{\partial F_4}{\partial a_{11}} = \frac{\partial F_3}{\partial a_{12}} = \frac{\partial F_2}{\partial a_{13}} = -\frac{\partial F_1}{\partial a_{14}} = -\frac{\partial F_0}{\partial a_{15}}. \quad (17)
 \end{aligned}$$

Proof. The integral $\int_x^y f dz$, will not depend on the path if there exists a function G such that $\int_x^y f dz = \int_x^y dF = \int_x^y d(F_0 + F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k} + F_4\mathbf{e}' + F_5\mathbf{i}' + F_6\mathbf{j}' + F_7\mathbf{k}' + F_8\mathbf{e}'' + F_9\mathbf{i}'' + F_{10}\mathbf{j}'' + F_{11}\mathbf{k}'' + F_{12}\mathbf{e}''' + F_{13}\mathbf{i}''' + F_{14}\mathbf{j}''' + F_{15}\mathbf{k}''') = F(y) - F(x)$, in way that the value of this difference will depend solely on the extreme points G , such that

$$\begin{aligned}
 dF_0 &= \frac{\partial F_0}{\partial a_0} da_0 + \frac{\partial F_0}{\partial a_1} da_1 + \frac{\partial F_0}{\partial a_2} da_2 + \frac{\partial F_0}{\partial a_3} da_3 + \frac{\partial F_0}{\partial a_4} da_4 + \frac{\partial F_0}{\partial a_5} da_5 \\
 &\quad + \frac{\partial F_0}{\partial a_6} da_6 + \frac{\partial F_0}{\partial a_7} da_7 + \frac{\partial F_0}{\partial a_8} da_8 + \frac{\partial F_0}{\partial a_9} da_9 + \frac{\partial F_0}{\partial a_{10}} da_{10} \\
 &\quad + \frac{\partial F_0}{\partial a_{11}} da_{11} + \frac{\partial F_0}{\partial a_{12}} da_{12} + \frac{\partial F_0}{\partial a_{13}} da_{13} + \frac{\partial F_0}{\partial a_{14}} da_{14} + \frac{\partial F_0}{\partial a_{15}} da_{15} \\
 &= f_0 da_0 - f_1 da_1 - f_2 da_2 - f_3 da_3 - f_4 da_4 - f_5 da_5 - f_6 da_6 - f_7 da_7 - f_8 da_8 \\
 &\quad - f_9 da_9 - f_{10} da_{10} - f_{11} da_{11} - f_{12} da_{12} - f_{13} da_{13} - f_{14} da_{14} - f_{15} da_{15},
 \end{aligned}$$

$$\begin{aligned}
dF_1 &= \frac{\partial F_1}{\partial a_0} da_0 + \frac{\partial F_1}{\partial a_1} da_1 + \frac{\partial F_1}{\partial a_2} da_2 + \frac{\partial F_1}{\partial a_3} da_3 + \frac{\partial F_1}{\partial a_4} da_4 + \frac{\partial F_1}{\partial a_5} da_5 \\
&\quad + \frac{\partial F_1}{\partial a_6} da_6 + \frac{\partial F_1}{\partial a_7} da_7 + \frac{\partial F_1}{\partial a_8} da_8 + \frac{\partial F_1}{\partial a_9} da_9 + \frac{\partial F_1}{\partial a_{10}} da_{10} \\
&\quad + \frac{\partial F_1}{\partial a_{11}} da_{11} + \frac{\partial F_1}{\partial a_{12}} da_{12} + \frac{\partial F_1}{\partial a_{13}} da_{13} + \frac{\partial F_1}{\partial a_{14}} da_{14} + \frac{\partial F_1}{\partial a_{15}} da_{15} \\
&= f_1 da_0 + f_0 da_1 - f_3 da_2 + f_2 da_3 - f_5 da_4 + f_4 da_5 + f_7 da_6 - f_6 da_7 - f_9 da_8 \\
&\quad + f_8 da_9 + f_{11} da_{10} - f_{10} da_{11} + f_{13} da_{12} - f_{12} da_{13} - f_{15} da_{14} + f_{14} da_{15},
\end{aligned}$$

$$\begin{aligned}
dF_2 &= \frac{\partial F_2}{\partial a_0} da_0 + \frac{\partial F_2}{\partial a_1} da_1 + \frac{\partial F_2}{\partial a_2} da_2 + \frac{\partial F_2}{\partial a_3} da_3 + \frac{\partial F_2}{\partial a_4} da_4 + \frac{\partial F_2}{\partial a_5} da_5 \\
&\quad + \frac{\partial F_2}{\partial a_6} da_6 + \frac{\partial F_2}{\partial a_7} da_7 + \frac{\partial F_2}{\partial a_8} da_8 + \frac{\partial F_2}{\partial a_9} da_9 + \frac{\partial F_2}{\partial a_{10}} da_{10} \\
&\quad + \frac{\partial F_2}{\partial a_{11}} da_{11} + \frac{\partial F_2}{\partial a_{12}} da_{12} + \frac{\partial F_2}{\partial a_{13}} da_{13} + \frac{\partial F_2}{\partial a_{14}} da_{14} + \frac{\partial F_2}{\partial a_{15}} da_{15} \\
&= f_2 da_0 + f_3 da_1 + f_0 da_2 - f_1 da_3 - f_6 da_4 - f_7 da_5 + f_4 da_6 + f_5 da_7 - f_{10} da_8 \\
&\quad - f_{11} da_9 + f_8 da_{10} + f_9 da_{11} + f_{14} da_{12} + f_{15} da_{13} - f_{12} da_{14} - f_{13} da_{15},
\end{aligned}$$

$$\begin{aligned}
dF_3 &= \frac{\partial F_3}{\partial a_0} da_0 + \frac{\partial F_3}{\partial a_1} da_1 + \frac{\partial F_3}{\partial a_2} da_2 + \frac{\partial F_3}{\partial a_3} da_3 + \frac{\partial F_3}{\partial a_4} da_4 + \frac{\partial F_3}{\partial a_5} da_5 \\
&\quad + \frac{\partial F_3}{\partial a_6} da_6 + \frac{\partial F_3}{\partial a_7} da_7 + \frac{\partial F_3}{\partial a_8} da_8 + \frac{\partial F_3}{\partial a_9} da_9 + \frac{\partial F_3}{\partial a_{10}} da_{10} \\
&\quad + \frac{\partial F_3}{\partial a_{11}} da_{11} + \frac{\partial F_3}{\partial a_{12}} da_{12} + \frac{\partial F_3}{\partial a_{13}} da_{13} + \frac{\partial F_3}{\partial a_{14}} da_{14} + \frac{\partial F_3}{\partial a_{15}} da_{15} \\
&= f_3 da_0 - f_2 da_1 + f_1 da_2 + f_0 da_3 - f_7 da_4 + f_6 da_5 - f_5 da_6 + f_4 da_7 - f_{11} da_8 \\
&\quad + f_{10} da_9 - f_9 da_{10} + f_8 da_{11} + f_{15} da_{12} - f_{14} da_{13} + f_{13} da_{14} - f_{12} da_{15},
\end{aligned}$$

$$\begin{aligned}
dF_4 &= \frac{\partial F_4}{\partial a_0} da_0 + \frac{\partial F_4}{\partial a_1} da_1 + \frac{\partial F_4}{\partial a_2} da_2 + \frac{\partial F_4}{\partial a_3} da_3 + \frac{\partial F_4}{\partial a_4} da_4 + \frac{\partial F_4}{\partial a_5} da_5 \\
&\quad + \frac{\partial F_4}{\partial a_6} da_6 + \frac{\partial F_4}{\partial a_7} da_7 + \frac{\partial F_4}{\partial a_8} da_8 + \frac{\partial F_4}{\partial a_9} da_9 + \frac{\partial F_4}{\partial a_{10}} da_{10} \\
&\quad + \frac{\partial F_4}{\partial a_{11}} da_{11} + \frac{\partial F_4}{\partial a_{12}} da_{12} + \frac{\partial F_4}{\partial a_{13}} da_{13} + \frac{\partial F_4}{\partial a_{14}} da_{14} + \frac{\partial F_4}{\partial a_{15}} da_{15} \\
&= f_4 da_0 + f_5 da_1 + f_6 da_2 + f_7 da_3 + f_0 da_4 - f_1 da_5 - f_2 da_6 - f_3 da_7 - f_{12} da_8 \\
&\quad - f_{13} da_9 - f_{14} da_{10} - f_{15} da_{11} + f_8 da_{12} + f_9 da_{13} + f_{10} da_{14} + f_{11} da_{15},
\end{aligned}$$

$$\begin{aligned}
dF_5 &= \frac{\partial F_5}{\partial a_0} da_0 + \frac{\partial F_5}{\partial a_1} da_1 + \frac{\partial F_5}{\partial a_2} da_2 + \frac{\partial F_5}{\partial a_3} da_3 + \frac{\partial F_5}{\partial a_4} da_4 + \frac{\partial F_5}{\partial a_5} da_5 \\
&+ \frac{\partial F_5}{\partial a_6} da_6 + \frac{\partial F_5}{\partial a_7} da_7 + \frac{\partial F_5}{\partial a_8} da_8 + \frac{\partial F_5}{\partial a_9} da_9 + \frac{\partial F_5}{\partial a_{10}} da_{10} \\
&+ \frac{\partial F_5}{\partial a_{11}} da_{11} + \frac{\partial F_5}{\partial a_{12}} da_{12} + \frac{\partial F_5}{\partial a_{13}} da_{13} + \frac{\partial F_5}{\partial a_{14}} da_{14} + \frac{\partial F_5}{\partial a_{15}} da_{15} \\
&= f_5 da_0 - f_4 da_1 + f_7 da_2 - f_6 da_3 + f_1 da_4 + f_0 da_5 + f_3 da_6 - f_2 da_7 - f_{13} da_8 \\
&+ f_{12} da_9 - f_{15} da_{10} + f_{14} da_{11} - f_9 da_{12} + f_8 da_{13} - f_{11} da_{14} + f_{10} da_{15},
\end{aligned}$$

$$\begin{aligned}
dF_6 &= \frac{\partial F_6}{\partial a_0} da_0 + \frac{\partial F_6}{\partial a_1} da_1 + \frac{\partial F_6}{\partial a_2} da_2 + \frac{\partial F_6}{\partial a_3} da_3 + \frac{\partial F_6}{\partial a_4} da_4 + \frac{\partial F_6}{\partial a_5} da_5 \\
&+ \frac{\partial F_6}{\partial a_6} da_6 + \frac{\partial F_6}{\partial a_7} da_7 + \frac{\partial F_6}{\partial a_8} da_8 + \frac{\partial F_6}{\partial a_9} da_9 + \frac{\partial F_6}{\partial a_{10}} da_{10} \\
&+ \frac{\partial F_6}{\partial a_{11}} da_{11} + \frac{\partial F_6}{\partial a_{12}} da_{12} + \frac{\partial F_6}{\partial a_{13}} da_{13} + \frac{\partial F_6}{\partial a_{14}} da_{14} + \frac{\partial F_6}{\partial a_{15}} da_{15} \\
&= f_6 da_0 - f_7 da_1 - f_4 da_2 + f_5 da_3 + f_2 da_4 - f_3 da_5 + f_0 da_6 + f_1 da_7 - f_{14} da_8 \\
&+ f_{15} da_9 + f_{12} da_{10} - f_{13} da_{11} - f_{10} da_{12} + f_{11} da_{13} + f_8 da_{14} - f_9 da_{15},
\end{aligned}$$

$$\begin{aligned}
dF_7 &= \frac{\partial F_7}{\partial a_0} da_0 + \frac{\partial F_7}{\partial a_1} da_1 + \frac{\partial F_7}{\partial a_2} da_2 + \frac{\partial F_7}{\partial a_3} da_3 + \frac{\partial F_7}{\partial a_4} da_4 + \frac{\partial F_7}{\partial a_5} da_5 \\
&+ \frac{\partial F_7}{\partial a_6} da_6 + \frac{\partial F_7}{\partial a_7} da_7 + \frac{\partial F_7}{\partial a_8} da_8 + \frac{\partial F_7}{\partial a_9} da_9 + \frac{\partial F_7}{\partial a_{10}} da_{10} \\
&+ \frac{\partial F_7}{\partial a_{11}} da_{11} + \frac{\partial F_7}{\partial a_{12}} da_{12} + \frac{\partial F_7}{\partial a_{13}} da_{13} + \frac{\partial F_7}{\partial a_{14}} da_{14} + \frac{\partial F_7}{\partial a_{15}} da_{15} \\
&= f_7 da_0 + f_6 da_1 - f_5 da_2 - f_4 da_3 + f_3 da_4 + f_2 da_5 - f_1 da_6 + f_0 da_7 - f_{15} da_8 \\
&- f_{14} da_9 + f_{13} da_{10} + f_{12} da_{11} - f_{11} da_{12} - f_{10} da_{13} + f_9 da_{14} + f_8 da_{15},
\end{aligned}$$

$$\begin{aligned}
dF_8 &= \frac{\partial F_8}{\partial a_0} da_0 + \frac{\partial F_8}{\partial a_1} da_1 + \frac{\partial F_8}{\partial a_2} da_2 + \frac{\partial F_8}{\partial a_3} da_3 + \frac{\partial F_8}{\partial a_4} da_4 + \frac{\partial F_8}{\partial a_5} da_5 \\
&+ \frac{\partial F_8}{\partial a_6} da_6 + \frac{\partial F_8}{\partial a_7} da_7 + \frac{\partial F_8}{\partial a_8} da_8 + \frac{\partial F_8}{\partial a_9} da_9 + \frac{\partial F_8}{\partial a_{10}} da_{10} \\
&+ \frac{\partial F_8}{\partial a_{11}} da_{11} + \frac{\partial F_8}{\partial a_{12}} da_{12} + \frac{\partial F_8}{\partial a_{13}} da_{13} + \frac{\partial F_8}{\partial a_{14}} da_{14} + \frac{\partial F_8}{\partial a_{15}} da_{15} \\
&= f_8 da_0 + f_9 da_1 + f_{10} da_2 + f_{11} da_3 + f_{12} da_4 + f_{13} da_5 + f_{14} da_6 + f_{15} da_7 \\
&+ f_0 da_8 - f_1 da_9 - f_2 da_{10} - f_3 da_{11} - f_4 da_{12} - f_5 da_{13} - f_6 da_{14} - f_7 da_{15},
\end{aligned}$$

$$\begin{aligned}
dF_9 &= \frac{\partial F_9}{\partial a_0} da_0 + \frac{\partial F_9}{\partial a_1} da_1 + \frac{\partial F_9}{\partial a_2} da_2 + \frac{\partial F_9}{\partial a_3} da_3 + \frac{\partial F_9}{\partial a_4} da_4 + \frac{\partial F_9}{\partial a_5} da_5 \\
&\quad + \frac{\partial F_9}{\partial a_6} da_6 + \frac{\partial F_9}{\partial a_7} da_7 + \frac{\partial F_9}{\partial a_8} da_8 + \frac{\partial F_9}{\partial a_9} da_9 + \frac{\partial F_9}{\partial a_{10}} da_{10} \\
&\quad + \frac{\partial F_9}{\partial a_{11}} da_{11} + \frac{\partial F_9}{\partial a_{12}} da_{12} + \frac{\partial F_9}{\partial a_{13}} da_{13} + \frac{\partial F_9}{\partial a_{14}} da_{14} + \frac{\partial F_9}{\partial a_{15}} da_{15} \\
&= f_9 da_0 - f_8 da_1 + f_{11} da_2 - f_{10} da_3 + f_{13} da_4 - f_{12} da_5 - f_{15} da_6 + f_{14} da_7 \\
&\quad + f_1 da_8 + f_0 da_9 + f_3 da_{10} - f_2 da_{11} + f_5 da_{12} - f_4 da_{13} - f_7 da_{14} + f_6 da_{15},
\end{aligned}$$

$$\begin{aligned}
dF_{10} &= \frac{\partial F_{10}}{\partial a_0} da_0 + \frac{\partial F_{10}}{\partial a_1} da_1 + \frac{\partial F_{10}}{\partial a_2} da_2 + \frac{\partial F_{10}}{\partial a_3} da_3 + \frac{\partial F_{10}}{\partial a_4} da_4 + \frac{\partial F_{10}}{\partial a_5} da_5 \\
&\quad + \frac{\partial F_{10}}{\partial a_6} da_6 + \frac{\partial F_{10}}{\partial a_7} da_7 + \frac{\partial F_{10}}{\partial a_8} da_8 + \frac{\partial F_{10}}{\partial a_9} da_9 + \frac{\partial F_{10}}{\partial a_{10}} da_{10} \\
&\quad + \frac{\partial F_{10}}{\partial a_{11}} da_{11} + \frac{\partial F_{10}}{\partial a_{12}} da_{12} + \frac{\partial F_{10}}{\partial a_{13}} da_{13} + \frac{\partial F_{10}}{\partial a_{14}} da_{14} + \frac{\partial F_{10}}{\partial a_{15}} da_{15} \\
&= f_{10} da_0 - f_{11} da_1 - f_8 da_2 + f_9 da_3 + f_{14} da_4 + f_{15} da_5 - f_{12} da_6 - f_{13} da_7 \\
&\quad + f_2 da_8 - f_3 da_9 + f_0 da_{10} + f_1 da_{11} + f_6 da_{12} + f_7 da_{13} - f_4 da_{14} - f_5 da_{15},
\end{aligned}$$

$$\begin{aligned}
dF_{11} &= \frac{\partial F_{11}}{\partial a_0} da_0 + \frac{\partial F_{11}}{\partial a_1} da_1 + \frac{\partial F_{11}}{\partial a_2} da_2 + \frac{\partial F_{11}}{\partial a_3} da_3 + \frac{\partial F_{11}}{\partial a_4} da_4 + \frac{\partial F_{11}}{\partial a_5} da_5 \\
&\quad + \frac{\partial F_{11}}{\partial a_6} da_6 + \frac{\partial F_{11}}{\partial a_7} da_7 + \frac{\partial F_{11}}{\partial a_8} da_8 + \frac{\partial F_{11}}{\partial a_9} da_9 + \frac{\partial F_{11}}{\partial a_{10}} da_{10} \\
&\quad + \frac{\partial F_{11}}{\partial a_{11}} da_{11} + \frac{\partial F_{11}}{\partial a_{12}} da_{12} + \frac{\partial F_{11}}{\partial a_{13}} da_{13} + \frac{\partial F_{11}}{\partial a_{14}} da_{14} + \frac{\partial F_{11}}{\partial a_{15}} da_{15} \\
&= f_{11} da_0 + f_{10} da_1 - f_9 da_2 - f_8 da_3 + f_{15} da_4 - f_{14} da_5 + f_{13} da_6 - f_{12} da_7 \\
&\quad + f_3 da_8 + f_2 da_9 - f_1 da_{10} + f_0 da_{11} + f_7 da_{12} - f_6 da_{13} + f_5 da_{14} - f_4 da_{15},
\end{aligned}$$

$$\begin{aligned}
dF_{12} &= \frac{\partial F_{12}}{\partial a_0} da_0 + \frac{\partial F_{12}}{\partial a_1} da_1 + \frac{\partial F_{12}}{\partial a_2} da_2 + \frac{\partial F_{12}}{\partial a_3} da_3 + \frac{\partial F_{12}}{\partial a_4} da_4 + \frac{\partial F_{12}}{\partial a_5} da_5 \\
&\quad + \frac{\partial F_{12}}{\partial a_6} da_6 + \frac{\partial F_{12}}{\partial a_7} da_7 + \frac{\partial F_{12}}{\partial a_8} da_8 + \frac{\partial F_{12}}{\partial a_9} da_9 + \frac{\partial F_{12}}{\partial a_{10}} da_{10} \\
&\quad + \frac{\partial F_{12}}{\partial a_{11}} da_{11} + \frac{\partial F_{12}}{\partial a_{12}} da_{12} + \frac{\partial F_{12}}{\partial a_{13}} da_{13} + \frac{\partial F_{12}}{\partial a_{14}} da_{14} + \frac{\partial F_{12}}{\partial a_{15}} da_{15} \\
&= f_{12} da_0 - f_{13} da_1 - f_{14} da_2 - f_{15} da_3 - f_8 da_4 + f_9 da_5 + f_{10} da_6 + f_{11} da_7 \\
&\quad + f_4 da_8 - f_5 da_9 - f_6 da_{10} - f_7 da_{11} + f_0 da_{12} + f_1 da_{13} + f_2 da_{14} + f_3 da_{15},
\end{aligned}$$

$$\begin{aligned}
dF_{13} &= \frac{\partial F_{13}}{\partial a_0} da_0 + \frac{\partial F_{13}}{\partial a_1} da_1 + \frac{\partial F_{13}}{\partial a_2} da_2 + \frac{\partial F_{13}}{\partial a_3} da_3 + \frac{\partial F_{13}}{\partial a_4} da_4 + \frac{\partial F_{13}}{\partial a_5} da_5 \\
&\quad + \frac{\partial F_{13}}{\partial a_6} da_6 + \frac{\partial F_{13}}{\partial a_7} da_7 + \frac{\partial F_{13}}{\partial a_8} da_8 + \frac{\partial F_{13}}{\partial a_9} da_9 + \frac{\partial F_{13}}{\partial a_{10}} da_{10} \\
&\quad + \frac{\partial F_{13}}{\partial a_{11}} da_{11} + \frac{\partial F_{13}}{\partial a_{12}} da_{12} + \frac{\partial F_{13}}{\partial a_{13}} da_{13} + \frac{\partial F_{13}}{\partial a_{14}} da_{14} + \frac{\partial F_{13}}{\partial a_{15}} da_{15} \\
&= f_{13} da_0 + f_{12} da_1 - f_{15} da_2 + f_{14} da_3 - f_9 da_4 - f_8 da_5 - f_{11} da_6 + f_{10} da_7 \\
&\quad + f_5 da_8 + f_4 da_9 - f_7 da_{10} + f_6 da_{11} - f_1 da_{12} + f_0 da_{13} - f_3 da_{14} + f_2 da_{15},
\end{aligned}$$

$$\begin{aligned}
dF_{14} &= \frac{\partial F_{14}}{\partial a_0} da_0 + \frac{\partial F_{14}}{\partial a_1} da_1 + \frac{\partial F_{14}}{\partial a_2} da_2 + \frac{\partial F_{14}}{\partial a_3} da_3 + \frac{\partial F_{14}}{\partial a_4} da_4 + \frac{\partial F_{14}}{\partial a_5} da_5 \\
&\quad + \frac{\partial F_{14}}{\partial a_6} da_6 + \frac{\partial F_{14}}{\partial a_7} da_7 + \frac{\partial F_{14}}{\partial a_8} da_8 + \frac{\partial F_{14}}{\partial a_9} da_9 + \frac{\partial F_{14}}{\partial a_{10}} da_{10} \\
&\quad + \frac{\partial F_{14}}{\partial a_{11}} da_{11} + \frac{\partial F_{14}}{\partial a_{12}} da_{12} + \frac{\partial F_{14}}{\partial a_{13}} da_{13} + \frac{\partial F_{14}}{\partial a_{14}} da_{14} + \frac{\partial F_{14}}{\partial a_{15}} da_{15} \\
&= f_{14} da_0 + f_{15} da_1 + f_{12} da_2 - f_{13} da_3 - f_{10} da_4 + f_{11} da_5 - f_8 da_6 - f_9 da_7 \\
&\quad + f_6 da_8 + f_7 da_9 + f_4 da_{10} - f_5 da_{11} - f_2 da_{12} + f_3 da_{13} + f_0 da_{14} - f_1 da_{15},
\end{aligned}$$

$$\begin{aligned}
dF_{15} &= \frac{\partial F_{15}}{\partial a_0} da_0 + \frac{\partial F_{15}}{\partial a_1} da_1 + \frac{\partial F_{15}}{\partial a_2} da_2 + \frac{\partial F_{15}}{\partial a_3} da_3 + \frac{\partial F_{15}}{\partial a_4} da_4 + \frac{\partial F_{15}}{\partial a_5} da_5 \\
&\quad + \frac{\partial F_{15}}{\partial a_6} da_6 + \frac{\partial F_{15}}{\partial a_7} da_7 + \frac{\partial F_{15}}{\partial a_8} da_8 + \frac{\partial F_{15}}{\partial a_9} da_9 + \frac{\partial F_{15}}{\partial a_{10}} da_{10} \\
&\quad + \frac{\partial F_{15}}{\partial a_{11}} da_{11} + \frac{\partial F_{15}}{\partial a_{12}} da_{12} + \frac{\partial F_{15}}{\partial a_{13}} da_{13} + \frac{\partial F_{15}}{\partial a_{14}} da_{14} + \frac{\partial F_{15}}{\partial a_{15}} da_{15} \\
&= f_{15} da_0 - f_{14} da_1 + f_{13} da_2 + f_{12} da_3 - f_{11} da_4 - f_{10} da_5 + f_9 da_6 - f_8 da_7 \\
&\quad + f_7 da_8 - f_6 da_9 + f_5 da_{10} + f_4 da_{11} - f_3 da_{12} - f_2 da_{13} + f_1 da_{14} + f_0 da_{15}.
\end{aligned}$$

Consequently the following relations will appear directly:

$$\begin{aligned}
f_0 &= \frac{\partial F_0}{\partial a_0} = \frac{\partial F_1}{\partial a_1} = \frac{\partial F_2}{\partial a_2} = \frac{\partial F_3}{\partial a_3} = \frac{\partial F_4}{\partial a_4} = \frac{\partial F_5}{\partial a_5} = \frac{\partial F_6}{\partial a_6} = \frac{\partial F_7}{\partial a_7} = \frac{\partial F_8}{\partial a_8} = \frac{\partial F_9}{\partial a_9} \\
&= \frac{\partial F_{10}}{\partial a_{10}} = \frac{\partial F_{11}}{\partial a_{11}} = \frac{\partial F_{12}}{\partial a_{12}} = \frac{\partial F_{13}}{\partial a_{13}} = \frac{\partial F_{14}}{\partial a_{14}} = \frac{\partial F_{15}}{\partial a_{15}},
\end{aligned}$$

$$\begin{aligned}
f_1 &= \frac{\partial F_1}{\partial a_0} = -\frac{\partial F_0}{\partial a_1} = \frac{\partial F_3}{\partial a_2} = -\frac{\partial F_2}{\partial a_3} = \frac{\partial F_5}{\partial a_4} = -\frac{\partial F_4}{\partial a_5} = -\frac{\partial F_7}{\partial a_6} = \frac{\partial F_6}{\partial a_7} = \frac{\partial F_9}{\partial a_8} \\
&= -\frac{\partial F_8}{\partial a_9} = -\frac{\partial F_{11}}{\partial a_{10}} = \frac{\partial F_{10}}{\partial a_{11}} = -\frac{\partial F_{13}}{\partial a_{12}} = \frac{\partial F_{12}}{\partial a_{13}} = \frac{\partial F_{15}}{\partial a_{14}} = -\frac{\partial F_{14}}{\partial a_{15}},
\end{aligned}$$

$$f_2 = \frac{\partial F_2}{\partial a_0} = -\frac{\partial F_3}{\partial a_1} = -\frac{\partial F_0}{\partial a_2} = \frac{\partial F_1}{\partial a_3} = \frac{\partial F_6}{\partial a_4} = \frac{\partial F_7}{\partial a_5} = -\frac{\partial F_4}{\partial a_6} = -\frac{\partial F_5}{\partial a_7} = \frac{\partial F_{10}}{\partial a_8}$$

$$= \frac{\partial F_{11}}{\partial a_9} = -\frac{\partial F_8}{\partial a_{10}} = -\frac{\partial F_9}{\partial a_{11}} = -\frac{\partial F_{14}}{\partial a_{12}} = -\frac{\partial F_{15}}{\partial a_{13}} = \frac{\partial F_{12}}{\partial a_{14}} = \frac{\partial F_{13}}{\partial a_{15}},$$

$$f_3 = \frac{\partial F_3}{\partial a_0} = \frac{\partial F_2}{\partial a_1} = -\frac{\partial F_1}{\partial a_2} = -\frac{\partial F_0}{\partial a_3} = \frac{\partial F_7}{\partial a_4} = -\frac{\partial F_6}{\partial a_5} = \frac{\partial F_5}{\partial a_6} = -\frac{\partial F_4}{\partial a_7} = \frac{\partial F_{11}}{\partial a_8}$$

$$= -\frac{\partial F_{10}}{\partial a_9} = \frac{\partial F_9}{\partial a_{10}} = -\frac{\partial F_8}{\partial a_{11}} = -\frac{\partial F_{15}}{\partial a_{12}} = \frac{\partial F_{14}}{\partial a_{13}} = -\frac{\partial F_{13}}{\partial a_{14}} = \frac{\partial F_{12}}{\partial a_{15}},$$

$$f_4 = \frac{\partial F_4}{\partial a_0} = -\frac{\partial F_5}{\partial a_1} = -\frac{\partial F_6}{\partial a_2} = -\frac{\partial F_7}{\partial a_3} = -\frac{\partial F_0}{\partial a_4} = \frac{\partial F_1}{\partial a_5} = \frac{\partial F_2}{\partial a_6} = \frac{\partial F_3}{\partial a_7} = \frac{\partial F_{12}}{\partial a_8}$$

$$= \frac{\partial F_{13}}{\partial a_9} = \frac{\partial F_{14}}{\partial a_{10}} = \frac{\partial F_{15}}{\partial a_{11}} = -\frac{\partial F_8}{\partial a_{12}} = -\frac{\partial F_9}{\partial a_{13}} = -\frac{\partial F_{10}}{\partial a_{14}} = -\frac{\partial F_{11}}{\partial a_{15}},$$

$$f_5 = \frac{\partial F_5}{\partial a_0} = \frac{\partial F_4}{\partial a_1} = -\frac{\partial F_7}{\partial a_2} = \frac{\partial F_6}{\partial a_3} = -\frac{\partial F_1}{\partial a_4} = -\frac{\partial F_0}{\partial a_5} = -\frac{\partial F_3}{\partial a_6} = \frac{\partial F_2}{\partial a_7}$$

$$= \frac{\partial F_{13}}{\partial a_8} = -\frac{\partial F_{12}}{\partial a_9} = \frac{\partial F_{15}}{\partial a_{10}} = -\frac{\partial F_{14}}{\partial a_{11}} = \frac{\partial F_9}{\partial a_{12}} = -\frac{\partial F_8}{\partial a_{13}} = \frac{\partial F_{11}}{\partial a_{14}} = -\frac{\partial F_{10}}{\partial a_{15}},$$

$$f_6 = \frac{\partial F_6}{\partial a_0} = \frac{\partial F_7}{\partial a_1} = \frac{\partial F_4}{\partial a_2} = -\frac{\partial F_5}{\partial a_3} = -\frac{\partial F_2}{\partial a_4} = \frac{\partial F_3}{\partial a_5} = -\frac{\partial F_0}{\partial a_6} = -\frac{\partial F_1}{\partial a_7} = \frac{\partial F_{14}}{\partial a_8}$$

$$= -\frac{\partial F_{15}}{\partial a_9} = -\frac{\partial F_{12}}{\partial a_{10}} = \frac{\partial F_{13}}{\partial a_{11}} = \frac{\partial F_{10}}{\partial a_{12}} = -\frac{\partial F_{11}}{\partial a_{13}} = -\frac{\partial F_8}{\partial a_{14}} = \frac{\partial F_9}{\partial a_{15}},$$

$$f_7 = \frac{\partial F_7}{\partial a_0} = -\frac{\partial F_6}{\partial a_1} = \frac{\partial F_5}{\partial a_2} = \frac{\partial F_4}{\partial a_3} = -\frac{\partial F_3}{\partial a_4} = -\frac{\partial F_2}{\partial a_5} = \frac{\partial F_1}{\partial a_6} = -\frac{\partial F_0}{\partial a_7} = \frac{\partial F_{15}}{\partial a_8}$$

$$= \frac{\partial F_{14}}{\partial a_9} = -\frac{\partial F_{13}}{\partial a_{10}} = -\frac{\partial F_{12}}{\partial a_{11}} = \frac{\partial F_{11}}{\partial a_{12}} = \frac{\partial F_{10}}{\partial a_{13}} = -\frac{\partial F_9}{\partial a_{14}} = -\frac{\partial F_8}{\partial a_{15}},$$

$$f_8 = \frac{\partial F_8}{\partial a_0} = -\frac{\partial F_9}{\partial a_1} = -\frac{\partial F_{10}}{\partial a_2} = -\frac{\partial F_{11}}{\partial a_3} = -\frac{\partial F_{12}}{\partial a_4} = -\frac{\partial F_{13}}{\partial a_5} = -\frac{\partial F_{14}}{\partial a_6} = -\frac{\partial F_{15}}{\partial a_7}$$

$$= -\frac{\partial F_0}{\partial a_8} = \frac{\partial F_1}{\partial a_9} = \frac{\partial F_2}{\partial a_{10}} = \frac{\partial F_3}{\partial a_{11}} = \frac{\partial F_4}{\partial a_{12}} = \frac{\partial F_5}{\partial a_{13}} = \frac{\partial F_6}{\partial a_{14}} = \frac{\partial F_7}{\partial a_{15}},$$

$$\begin{aligned}
 f_9 &= \frac{\partial F_9}{\partial a_0} = \frac{\partial F_8}{\partial a_1} = -\frac{\partial F_{11}}{\partial a_2} = \frac{\partial F_{10}}{\partial a_3} = -\frac{\partial F_{13}}{\partial a_4} = \frac{\partial F_{12}}{\partial a_5} = \frac{\partial F_{15}}{\partial a_6} = -\frac{\partial F_{14}}{\partial a_7} \\
 &= -\frac{\partial F_1}{\partial a_8} = -\frac{\partial F_0}{\partial a_9} = -\frac{\partial F_3}{\partial a_{10}} = \frac{\partial F_2}{\partial a_{11}} = -\frac{\partial F_5}{\partial a_{12}} = \frac{\partial F_4}{\partial a_{13}} = \frac{\partial F_7}{\partial a_{14}} \\
 &= -\frac{\partial F_6}{\partial a_{15}},
 \end{aligned}$$

$$\begin{aligned}
 f_{10} &= \frac{\partial F_{10}}{\partial a_0} = \frac{\partial F_{11}}{\partial a_1} = \frac{\partial F_8}{\partial a_2} = -\frac{\partial F_9}{\partial a_3} = -\frac{\partial F_{14}}{\partial a_4} = -\frac{\partial F_{15}}{\partial a_5} = \frac{\partial F_{12}}{\partial a_6} \\
 &= \frac{\partial F_{13}}{\partial a_7} = -\frac{\partial F_2}{\partial a_8} = \frac{\partial F_3}{\partial a_9} = -\frac{\partial F_0}{\partial a_{10}} = -\frac{\partial F_1}{\partial a_{11}} = -\frac{\partial F_6}{\partial a_{12}} = -\frac{\partial F_7}{\partial a_{13}} = \frac{\partial F_4}{\partial a_{14}} = \frac{\partial F_5}{\partial a_{15}},
 \end{aligned}$$

$$\begin{aligned}
 f_{11} &= \frac{\partial F_{11}}{\partial a_0} = -\frac{\partial F_{10}}{\partial a_1} = \frac{\partial F_9}{\partial a_2} = \frac{\partial F_8}{\partial a_3} = -\frac{\partial F_{15}}{\partial a_4} = \frac{\partial F_{14}}{\partial a_5} = -\frac{\partial F_{13}}{\partial a_6} = \frac{\partial F_{12}}{\partial a_7} \\
 &= -\frac{\partial F_3}{\partial a_8} = -\frac{\partial F_2}{\partial a_9} = \frac{\partial F_1}{\partial a_{10}} = -\frac{\partial F_0}{\partial a_{11}} = -\frac{\partial F_7}{\partial a_{12}} = \frac{\partial F_6}{\partial a_{13}} = -\frac{\partial F_5}{\partial a_{14}} = \frac{\partial F_4}{\partial a_{15}},
 \end{aligned}$$

$$\begin{aligned}
 f_{12} &= \frac{\partial F_{12}}{\partial a_0} = \frac{\partial F_{13}}{\partial a_1} = \frac{\partial F_{14}}{\partial a_2} = \frac{\partial F_{15}}{\partial a_3} = \frac{\partial F_8}{\partial a_4} = -\frac{\partial F_9}{\partial a_5} = -\frac{\partial F_{10}}{\partial a_6} = -\frac{\partial F_{11}}{\partial a_7} \\
 &= -\frac{\partial F_4}{\partial a_8} = \frac{\partial F_5}{\partial a_9} = \frac{\partial F_6}{\partial a_{10}} = \frac{\partial F_7}{\partial a_{11}} = -\frac{\partial F_0}{\partial a_{12}} = -\frac{\partial F_1}{\partial a_{13}} = -\frac{\partial F_2}{\partial a_{14}} = -\frac{\partial F_3}{\partial a_{15}},
 \end{aligned}$$

$$\begin{aligned}
 f_{13} &= \frac{\partial F_{13}}{\partial a_0} = -\frac{\partial F_{12}}{\partial a_1} = \frac{\partial F_{15}}{\partial a_2} = -\frac{\partial F_{14}}{\partial a_3} = \frac{\partial F_9}{\partial a_4} = \frac{\partial F_8}{\partial a_5} = \frac{\partial F_{11}}{\partial a_6} \\
 &= -\frac{\partial F_{10}}{\partial a_7} = -\frac{\partial F_5}{\partial a_8} = -\frac{\partial F_4}{\partial a_9} = \frac{\partial F_7}{\partial a_{10}} = -\frac{\partial F_6}{\partial a_{11}} = \frac{\partial F_1}{\partial a_{12}} = -\frac{\partial F_0}{\partial a_{13}} = \frac{\partial F_3}{\partial a_{14}} \\
 &= -\frac{\partial F_2}{\partial a_{15}},
 \end{aligned}$$

$$\begin{aligned}
 f_{14} &= \frac{\partial F_{14}}{\partial a_0} = -\frac{\partial F_{15}}{\partial a_1} = -\frac{\partial F_{12}}{\partial a_2} = \frac{\partial F_{13}}{\partial a_3} = \frac{\partial F_{10}}{\partial a_4} = -\frac{\partial F_{11}}{\partial a_5} = \frac{\partial F_8}{\partial a_6} = \frac{\partial F_9}{\partial a_7} \\
 &= -\frac{\partial F_6}{\partial a_8} = -\frac{\partial F_7}{\partial a_9} = -\frac{\partial F_4}{\partial a_{10}} = \frac{\partial F_5}{\partial a_{11}} = \frac{\partial F_2}{\partial a_{12}} = -\frac{\partial F_3}{\partial a_{13}} = -\frac{\partial F_0}{\partial a_{14}} = \frac{\partial F_1}{\partial a_{15}},
 \end{aligned}$$

$$\begin{aligned}
 f_{15} &= \frac{\partial F_{15}}{\partial a_0} = \frac{\partial F_{14}}{\partial a_1} = -\frac{\partial F_{13}}{\partial a_2} = -\frac{\partial F_{12}}{\partial a_3} = \frac{\partial F_{11}}{\partial a_4} = \frac{\partial F_{10}}{\partial a_5} = -\frac{\partial F_9}{\partial a_6} = \frac{\partial F_8}{\partial a_7} \\
 &= -\frac{\partial F_7}{\partial a_8} = \frac{\partial F_6}{\partial a_9} = -\frac{\partial F_5}{\partial a_{10}} = -\frac{\partial F_4}{\partial a_{11}} = \frac{\partial F_3}{\partial a_{12}} = \frac{\partial F_2}{\partial a_{13}} = -\frac{\partial F_1}{\partial a_{14}} = -\frac{\partial F_0}{\partial a_{15}}. \quad \square
 \end{aligned}$$

Theorem 2. For all pair points x and y any path does not binding them in a 16-dimensional simply connected space, the integral $\int_x^y dzf$, depend on the given path if, and only if, there exists a function $G = G_0 + G_1\mathbf{i} + G_2\mathbf{j} + G_3\mathbf{k} + G_4\mathbf{e}' + G_5\mathbf{i}' + G_6\mathbf{j}' + G_7\mathbf{k}' + G_8\mathbf{e}'' + G_9\mathbf{i}'' + G_{10}\mathbf{j}'' + G_{11}\mathbf{k}'' + G_{12}\mathbf{e}''' + G_{13}\mathbf{i}''' + G_{14}\mathbf{j}''' + G_{15}\mathbf{k}'''$ with $\int_x^y dzf = G(y) - G(x)$ and that satisfies the following condition

$$\begin{aligned} \frac{\partial G_0}{\partial a_0} &= \frac{\partial G_1}{\partial a_1} = \frac{\partial G_2}{\partial a_2} = \frac{\partial G_3}{\partial a_3} = \frac{\partial G_4}{\partial a_4} = \frac{\partial G_5}{\partial a_5} = \frac{\partial G_6}{\partial a_6} = \frac{\partial G_7}{\partial a_7} = \frac{\partial G_8}{\partial a_8} \\ &= \frac{\partial G_9}{\partial a_9} = \frac{\partial G_{10}}{\partial a_{10}} = \frac{\partial G_{11}}{\partial a_{11}} = \frac{\partial G_{12}}{\partial a_{12}} = \frac{\partial G_{13}}{\partial a_{13}} = \frac{\partial G_{14}}{\partial a_{14}} = \frac{\partial G_{15}}{\partial a_{15}}, \end{aligned} \tag{18}$$

$$\begin{aligned} \frac{\partial G_1}{\partial a_0} &= -\frac{\partial G_0}{\partial a_1} = -\frac{\partial G_3}{\partial a_2} = \frac{\partial G_2}{\partial a_3} = -\frac{\partial G_5}{\partial a_4} = \frac{\partial G_4}{\partial a_5} = \frac{\partial G_7}{\partial a_6} = -\frac{\partial G_6}{\partial a_7} = -\frac{\partial G_9}{\partial a_8} \\ &= \frac{\partial G_8}{\partial a_9} = \frac{\partial G_{11}}{\partial a_{10}} = -\frac{\partial G_{10}}{\partial a_{11}} = \frac{\partial G_{13}}{\partial a_{12}} = -\frac{\partial G_{12}}{\partial a_{13}} = -\frac{\partial G_{15}}{\partial a_{14}} = \frac{\partial G_{14}}{\partial a_{15}}, \end{aligned} \tag{19}$$

$$\begin{aligned} \frac{\partial G_2}{\partial a_0} &= \frac{\partial G_3}{\partial a_1} = -\frac{\partial G_0}{\partial a_2} = -\frac{\partial G_1}{\partial a_3} = -\frac{\partial G_6}{\partial a_4} = -\frac{\partial G_7}{\partial a_5} = \frac{\partial G_4}{\partial a_6} = \frac{\partial G_5}{\partial a_7} = \frac{\partial G_{10}}{\partial a_8} \\ &= -\frac{\partial G_{11}}{\partial a_9} = \frac{\partial G_8}{\partial a_{10}} = \frac{\partial G_9}{\partial a_{11}} = \frac{\partial G_{14}}{\partial a_{12}} = \frac{\partial G_{15}}{\partial a_{13}} = -\frac{\partial G_{12}}{\partial a_{14}} = -\frac{\partial G_{13}}{\partial a_{15}}, \end{aligned} \tag{20}$$

$$\begin{aligned} \frac{\partial G_3}{\partial a_0} &= -\frac{\partial G_2}{\partial a_1} = \frac{\partial G_1}{\partial a_2} = -\frac{\partial G_0}{\partial a_3} = -\frac{\partial G_7}{\partial a_4} = \frac{\partial G_6}{\partial a_5} = -\frac{\partial G_5}{\partial a_6} = \frac{\partial G_4}{\partial a_7} = -\frac{\partial G_{11}}{\partial a_8} \\ &= \frac{\partial G_{10}}{\partial a_9} = -\frac{\partial G_9}{\partial a_{10}} = \frac{\partial G_8}{\partial a_{11}} = \frac{\partial G_{15}}{\partial a_{12}} = -\frac{\partial G_{14}}{\partial a_{13}} = \frac{\partial G_{13}}{\partial a_{14}} = -\frac{\partial G_{12}}{\partial a_{15}}, \end{aligned} \tag{21}$$

$$\begin{aligned} \frac{\partial G_4}{\partial a_0} &= \frac{\partial G_5}{\partial a_1} = \frac{\partial G_6}{\partial a_2} = \frac{\partial G_7}{\partial a_3} = -\frac{\partial G_0}{\partial a_4} = -\frac{\partial G_1}{\partial a_5} = -\frac{\partial G_2}{\partial a_6} = -\frac{\partial G_3}{\partial a_7} = -\frac{\partial G_{12}}{\partial a_8} \\ &= -\frac{\partial G_{13}}{\partial a_9} = -\frac{\partial G_{14}}{\partial a_{10}} = -\frac{\partial G_{15}}{\partial a_{11}} = \frac{\partial G_8}{\partial a_{12}} = \frac{\partial G_9}{\partial a_{13}} = \frac{\partial G_{10}}{\partial a_{14}} = \frac{\partial G_{11}}{\partial a_{15}}, \end{aligned} \tag{22}$$

$$\begin{aligned} \frac{\partial G_5}{\partial a_0} &= -\frac{\partial G_4}{\partial a_1} = \frac{\partial G_7}{\partial a_2} = -\frac{\partial G_6}{\partial a_3} = \frac{\partial G_1}{\partial a_4} = -\frac{\partial G_0}{\partial a_5} = \frac{\partial G_3}{\partial a_6} = -\frac{\partial G_2}{\partial a_7} = -\frac{\partial G_{13}}{\partial a_8} \\ &= \frac{\partial G_{12}}{\partial a_9} = -\frac{\partial G_{15}}{\partial a_{10}} = \frac{\partial G_{14}}{\partial a_{11}} = -\frac{\partial G_9}{\partial a_{12}} = \frac{\partial G_8}{\partial a_{13}} = -\frac{\partial G_{11}}{\partial a_{14}} = \frac{\partial G_{10}}{\partial a_{15}}, \end{aligned} \tag{23}$$

$$\begin{aligned} \frac{\partial G_6}{\partial a_0} &= -\frac{\partial G_7}{\partial a_1} = -\frac{\partial G_4}{\partial a_2} = \frac{\partial G_5}{\partial a_3} = \frac{\partial G_2}{\partial a_4} = -\frac{\partial G_3}{\partial a_5} = -\frac{\partial G_0}{\partial a_6} = \frac{\partial G_1}{\partial a_7} = -\frac{\partial G_{14}}{\partial a_8} \\ &= \frac{\partial G_{15}}{\partial a_9} = \frac{\partial G_{12}}{\partial a_{10}} = -\frac{\partial G_{13}}{\partial a_{11}} = -\frac{\partial G_{10}}{\partial a_{12}} = \frac{\partial G_{11}}{\partial a_{13}} = \frac{\partial G_8}{\partial a_{14}} = -\frac{\partial G_9}{\partial a_{15}}, \end{aligned} \quad (24)$$

$$\begin{aligned} \frac{\partial G_7}{\partial a_0} &= \frac{\partial G_6}{\partial a_1} = -\frac{\partial G_5}{\partial a_2} = -\frac{\partial G_4}{\partial a_3} = \frac{\partial G_3}{\partial a_4} = \frac{\partial G_2}{\partial a_5} = -\frac{\partial G_1}{\partial a_6} = -\frac{\partial G_0}{\partial a_7} = -\frac{\partial G_{15}}{\partial a_8} \\ &= -\frac{\partial G_{14}}{\partial a_9} = \frac{\partial G_{13}}{\partial a_{10}} = \frac{\partial G_{12}}{\partial a_{11}} = -\frac{\partial G_{11}}{\partial a_{12}} = -\frac{\partial G_{10}}{\partial a_{13}} = \frac{\partial G_9}{\partial a_{14}} = \frac{\partial G_8}{\partial a_{15}}, \end{aligned} \quad (25)$$

$$\begin{aligned} \frac{\partial G_8}{\partial a_0} &= \frac{\partial G_9}{\partial a_1} = \frac{\partial G_{10}}{\partial a_2} = \frac{\partial G_{11}}{\partial a_3} = \frac{\partial G_{12}}{\partial a_4} = \frac{\partial G_{13}}{\partial a_5} = \frac{\partial G_{14}}{\partial a_6} = \frac{\partial G_{15}}{\partial a_7} = -\frac{\partial G_0}{\partial a_8} \\ &= -\frac{\partial G_1}{\partial a_9} = -\frac{\partial G_2}{\partial a_{10}} = -\frac{\partial G_3}{\partial a_{11}} = -\frac{\partial G_4}{\partial a_{12}} = -\frac{\partial G_5}{\partial a_{13}} = -\frac{\partial G_6}{\partial a_{14}} = -\frac{\partial G_7}{\partial a_{15}}, \end{aligned} \quad (26)$$

$$\begin{aligned} \frac{\partial G_9}{\partial a_0} &= -\frac{\partial G_8}{\partial a_1} = \frac{\partial G_{11}}{\partial a_2} = -\frac{\partial G_{10}}{\partial a_3} = \frac{\partial G_{13}}{\partial a_4} = -\frac{\partial G_{12}}{\partial a_5} = -\frac{\partial G_{15}}{\partial a_6} = \frac{\partial G_{14}}{\partial a_7} \\ &= \frac{\partial G_1}{\partial a_8} = -\frac{\partial G_0}{\partial a_9} = \frac{\partial G_3}{\partial a_{10}} = -\frac{\partial G_2}{\partial a_{11}} = \frac{\partial G_5}{\partial a_{12}} = -\frac{\partial G_4}{\partial a_{13}} = -\frac{\partial G_7}{\partial a_{14}} = \frac{\partial G_6}{\partial a_{15}}, \end{aligned} \quad (27)$$

$$\begin{aligned} \frac{\partial G_{10}}{\partial a_0} &= -\frac{\partial G_{11}}{\partial a_1} = -\frac{\partial G_8}{\partial a_2} = \frac{\partial G_9}{\partial a_3} = \frac{\partial G_{14}}{\partial a_4} = \frac{\partial G_{15}}{\partial a_5} = -\frac{\partial G_{12}}{\partial a_6} = -\frac{\partial G_{13}}{\partial a_7} \\ &= \frac{\partial G_2}{\partial a_8} = -\frac{\partial G_3}{\partial a_9} = -\frac{\partial G_0}{\partial a_{10}} = \frac{\partial G_1}{\partial a_{11}} = \frac{\partial G_6}{\partial a_{12}} = \frac{\partial G_7}{\partial a_{13}} = -\frac{\partial G_4}{\partial a_{14}} = -\frac{\partial G_5}{\partial a_{15}}, \end{aligned} \quad (28)$$

$$\begin{aligned} \frac{\partial G_{11}}{\partial a_0} &= \frac{\partial G_{10}}{\partial a_1} = -\frac{\partial G_9}{\partial a_2} = -\frac{\partial G_8}{\partial a_3} = \frac{\partial G_{15}}{\partial a_4} = -\frac{\partial G_{14}}{\partial a_5} = \frac{\partial G_{13}}{\partial a_6} = -\frac{\partial G_{12}}{\partial a_7} \\ &= \frac{\partial G_3}{\partial a_8} = \frac{\partial G_2}{\partial a_9} = -\frac{\partial G_1}{\partial a_{10}} = -\frac{\partial G_0}{\partial a_{11}} = \frac{\partial G_7}{\partial a_{12}} = -\frac{\partial G_6}{\partial a_{13}} = \frac{\partial G_5}{\partial a_{14}} = -\frac{\partial G_4}{\partial a_{15}}, \end{aligned} \quad (29)$$

$$\begin{aligned} \frac{\partial G_{12}}{\partial a_0} &= -\frac{\partial G_{13}}{\partial a_1} = -\frac{\partial G_{14}}{\partial a_2} = -\frac{\partial G_{15}}{\partial a_3} = -\frac{\partial G_8}{\partial a_4} = \frac{\partial G_9}{\partial a_5} = \frac{\partial G_{10}}{\partial a_6} = \frac{\partial G_{11}}{\partial a_7} \\ &= \frac{\partial G_4}{\partial a_8} = -\frac{\partial G_5}{\partial a_9} = -\frac{\partial G_6}{\partial a_{10}} = -\frac{\partial G_7}{\partial a_{11}} = -\frac{\partial G_0}{\partial a_{12}} = \frac{\partial G_1}{\partial a_{13}} = \frac{\partial G_2}{\partial a_{14}} = \frac{\partial G_3}{\partial a_{15}}, \end{aligned} \quad (30)$$

$$\begin{aligned} \frac{\partial G_{13}}{\partial a_0} &= \frac{\partial G_{12}}{\partial a_1} = -\frac{\partial G_{15}}{\partial a_2} = \frac{\partial G_{14}}{\partial a_3} = -\frac{\partial G_9}{\partial a_4} = -\frac{\partial G_8}{\partial a_5} = -\frac{\partial G_{11}}{\partial a_6} = \frac{\partial G_{10}}{\partial a_7} \\ &= \frac{\partial G_5}{\partial a_8} = \frac{\partial G_4}{\partial a_9} = -\frac{\partial G_7}{\partial a_{10}} = \frac{\partial G_6}{\partial a_{11}} = -\frac{\partial G_1}{\partial a_{12}} = -\frac{\partial G_0}{\partial a_{13}} = -\frac{\partial G_3}{\partial a_{14}} = \frac{\partial G_2}{\partial a_{15}}, \end{aligned} \tag{31}$$

$$\begin{aligned} \frac{\partial G_{14}}{\partial a_0} &= \frac{\partial G_{15}}{\partial a_1} = \frac{\partial G_{12}}{\partial a_2} = -\frac{\partial G_{13}}{\partial a_3} = -\frac{\partial G_{10}}{\partial a_4} = \frac{\partial G_{11}}{\partial a_5} = -\frac{\partial G_8}{\partial a_6} = -\frac{\partial G_9}{\partial a_7} \\ &= \frac{\partial G_6}{\partial a_8} = \frac{\partial G_7}{\partial a_9} = \frac{\partial G_4}{\partial a_{10}} = -\frac{\partial G_5}{\partial a_{11}} = -\frac{\partial G_2}{\partial a_{12}} = \frac{\partial G_3}{\partial a_{13}} = -\frac{\partial G_0}{\partial a_{14}} = -\frac{\partial G_1}{\partial a_{15}}, \end{aligned} \tag{32}$$

$$\begin{aligned} \frac{\partial G_{15}}{\partial a_0} &= -\frac{\partial G_{14}}{\partial a_1} = \frac{\partial G_{13}}{\partial a_2} = \frac{\partial G_{12}}{\partial a_3} = -\frac{\partial G_{11}}{\partial a_4} = -\frac{\partial G_{10}}{\partial a_5} = \frac{\partial G_9}{\partial a_6} = -\frac{\partial G_8}{\partial a_7} \\ &= \frac{\partial G_7}{\partial a_8} = -\frac{\partial G_6}{\partial a_9} = \frac{\partial G_5}{\partial a_{10}} = \frac{\partial G_4}{\partial a_{11}} = -\frac{\partial G_3}{\partial a_{12}} = -\frac{\partial G_2}{\partial a_{13}} = \frac{\partial G_1}{\partial a_{14}} = -\frac{\partial G_0}{\partial a_{15}}. \end{aligned} \tag{33}$$

Proof. The integral $\int_x^y dz f$ will not depend on the path if there exists a function G such that $\int_x^y dz f = G_0 + G_1 \mathbf{i} + G_2 \mathbf{j} + G_3 \mathbf{k} + G_4 \mathbf{e}' + G_5 \mathbf{i}' + G_6 \mathbf{j}' + G_7 \mathbf{k}' + G_8 \mathbf{e}'' + G_9 \mathbf{i}'' + G_{10} \mathbf{j}'' + G_{11} \mathbf{k}'' + G_{12} \mathbf{e}''' + G_{13} \mathbf{i}''' + G_{14} \mathbf{j}''' + G_{15} \mathbf{k}''' = G(y) - G(x)$, in way that value of this difference will depend solely on the extreme points G , such that

$$\begin{aligned} dG_0 &= \frac{\partial G_0}{\partial a_0} da_0 + \frac{\partial G_0}{\partial a_1} da_1 + \frac{\partial G_0}{\partial a_2} da_2 + \frac{\partial G_0}{\partial a_3} da_3 + \frac{\partial G_0}{\partial a_4} da_4 + \frac{\partial G_0}{\partial a_5} da_5 \\ &\quad + \frac{\partial G_0}{\partial a_6} da_6 + \frac{\partial G_0}{\partial a_7} da_7 + \frac{\partial G_0}{\partial a_8} da_8 + \frac{\partial G_0}{\partial a_9} da_9 + \frac{\partial G_0}{\partial a_{10}} da_{10} \\ &\quad + \frac{\partial G_0}{\partial a_{11}} da_{11} + \frac{\partial G_0}{\partial a_{12}} da_{12} + \frac{\partial G_0}{\partial a_{13}} da_{13} + \frac{\partial G_0}{\partial a_{14}} da_{14} + \frac{\partial G_0}{\partial a_{15}} da_{15} \\ &= da_0 f_0 - da_1 f_1 - da_2 f_2 - da_3 f_3 - da_4 f_4 - da_5 f_5 - da_6 f_6 - da_7 f_7 - da_8 f_8 \\ &\quad - da_9 f_9 - da_{10} f_{10} - da_{11} f_{11} - da_{12} f_{12} - da_{13} f_{13} - da_{14} f_{14} - da_{15} f_{15}, \end{aligned}$$

$$\begin{aligned} dG_1 &= \frac{\partial G_1}{\partial a_0} da_0 + \frac{\partial G_1}{\partial a_1} da_1 + \frac{\partial G_1}{\partial a_2} da_2 + \frac{\partial G_1}{\partial a_3} da_3 + \frac{\partial G_1}{\partial a_4} da_4 + \frac{\partial G_1}{\partial a_5} da_5 \\ &\quad + \frac{\partial G_1}{\partial a_6} da_6 + \frac{\partial G_1}{\partial a_7} da_7 + \frac{\partial G_1}{\partial a_8} da_8 + \frac{\partial G_1}{\partial a_9} da_9 + \frac{\partial G_1}{\partial a_{10}} da_{10} \\ &\quad + \frac{\partial G_1}{\partial a_{11}} da_{11} + \frac{\partial G_1}{\partial a_{12}} da_{12} + \frac{\partial G_1}{\partial a_{13}} da_{13} + \frac{\partial G_1}{\partial a_{14}} da_{14} + \frac{\partial G_1}{\partial a_{15}} da_{15} \\ &= da_0 f_1 + da_1 f_0 + da_2 f_3 - da_3 f_2 + da_4 f_5 - da_5 f_4 - da_6 f_7 + da_7 f_6 + da_8 f_9 \\ &\quad - da_9 f_8 - da_{10} f_{11} + da_{11} f_{10} - da_{12} f_{13} + da_{13} f_{12} + da_{14} f_{15} - da_{15} f_{14}, \end{aligned}$$

$$\begin{aligned}
dG_2 &= \frac{\partial G_2}{\partial a_0} da_0 + \frac{\partial G_2}{\partial a_1} da_1 + \frac{\partial G_2}{\partial a_2} da_2 + \frac{\partial G_2}{\partial a_3} da_3 + \frac{\partial G_2}{\partial a_4} da_4 + \frac{\partial G_2}{\partial a_5} da_5 \\
&\quad + \frac{\partial G_2}{\partial a_6} da_6 + \frac{\partial G_2}{\partial a_7} da_7 + \frac{\partial G_2}{\partial a_8} da_8 + \frac{\partial G_2}{\partial a_9} da_9 + \frac{\partial G_2}{\partial a_{10}} da_{10} \\
&\quad + \frac{\partial G_2}{\partial a_{11}} da_{11} + \frac{\partial G_2}{\partial a_{12}} da_{12} + \frac{\partial G_2}{\partial a_{13}} da_{13} + \frac{\partial G_2}{\partial a_{14}} da_{14} + \frac{\partial G_2}{\partial a_{15}} da_{15} \\
&= da_0 f_2 - da_1 f_3 + da_2 f_0 + da_3 f_1 + da_4 f_6 + da_5 f_7 - da_6 f_4 - da_7 f_5 + da_8 f_{10} \\
&\quad + da_9 f_{11} - da_{10} f_8 - da_{11} f_9 - da_{12} f_{14} - da_{13} f_{15} + da_{14} f_{12} + da_{15} f_{13},
\end{aligned}$$

$$\begin{aligned}
dG_3 &= \frac{\partial G_3}{\partial a_0} da_0 + \frac{\partial G_3}{\partial a_1} da_1 + \frac{\partial G_3}{\partial a_2} da_2 + \frac{\partial G_3}{\partial a_3} da_3 + \frac{\partial G_3}{\partial a_4} da_4 + \frac{\partial G_3}{\partial a_5} da_5 \\
&\quad + \frac{\partial G_3}{\partial a_6} da_6 + \frac{\partial G_3}{\partial a_7} da_7 + \frac{\partial G_3}{\partial a_8} da_8 + \frac{\partial G_3}{\partial a_9} da_9 + \frac{\partial G_3}{\partial a_{10}} da_{10} \\
&\quad + \frac{\partial G_3}{\partial a_{11}} da_{11} + \frac{\partial G_3}{\partial a_{12}} da_{12} + \frac{\partial G_3}{\partial a_{13}} da_{13} + \frac{\partial G_3}{\partial a_{14}} da_{14} + \frac{\partial G_3}{\partial a_{15}} da_{15} \\
&= da_0 f_3 + da_1 f_2 - da_2 f_1 + da_3 f_0 + da_4 f_7 - da_5 f_6 + da_6 f_5 - da_7 f_4 + da_8 f_{11} \\
&\quad - da_9 f_{10} + da_{10} f_9 - da_{11} f_8 - da_{12} f_{15} + da_{13} f_{14} - da_{14} f_{13} + da_{15} f_{12},
\end{aligned}$$

$$\begin{aligned}
dG_4 &= \frac{\partial G_4}{\partial a_0} da_0 + \frac{\partial G_4}{\partial a_1} da_1 + \frac{\partial G_4}{\partial a_2} da_2 + \frac{\partial G_4}{\partial a_3} da_3 + \frac{\partial G_4}{\partial a_4} da_4 + \frac{\partial G_4}{\partial a_5} da_5 \\
&\quad + \frac{\partial G_4}{\partial a_6} da_6 + \frac{\partial G_4}{\partial a_7} da_7 + \frac{\partial G_4}{\partial a_8} da_8 + \frac{\partial G_4}{\partial a_9} da_9 + \frac{\partial G_4}{\partial a_{10}} da_{10} \\
&\quad + \frac{\partial G_4}{\partial a_{11}} da_{11} + \frac{\partial G_4}{\partial a_{12}} da_{12} + \frac{\partial G_4}{\partial a_{13}} da_{13} + \frac{\partial G_4}{\partial a_{14}} da_{14} + \frac{\partial G_4}{\partial a_{15}} da_{15} \\
&= da_0 f_4 - da_1 f_5 - da_2 f_6 - da_3 f_7 + da_4 f_0 + da_5 f_1 + da_6 f_2 + da_7 f_3 + da_8 f_{12} \\
&\quad + da_9 f_{13} + da_{10} f_{14} + da_{11} f_{15} - da_{12} f_8 - da_{13} f_9 - da_{14} f_{10} - da_{15} f_{11},
\end{aligned}$$

$$\begin{aligned}
dG_5 &= \frac{\partial G_5}{\partial a_0} da_0 + \frac{\partial G_5}{\partial a_1} da_1 + \frac{\partial G_5}{\partial a_2} da_2 + \frac{\partial G_5}{\partial a_3} da_3 + \frac{\partial G_5}{\partial a_4} da_4 + \frac{\partial G_5}{\partial a_5} da_5 \\
&\quad + \frac{\partial G_5}{\partial a_6} da_6 + \frac{\partial G_5}{\partial a_7} da_7 + \frac{\partial G_5}{\partial a_8} da_8 + \frac{\partial G_5}{\partial a_9} da_9 + \frac{\partial G_5}{\partial a_{10}} da_{10} \\
&\quad + \frac{\partial G_5}{\partial a_{11}} da_{11} + \frac{\partial G_5}{\partial a_{12}} da_{12} + \frac{\partial G_5}{\partial a_{13}} da_{13} + \frac{\partial G_5}{\partial a_{14}} da_{14} + \frac{\partial G_5}{\partial a_{15}} da_{15} \\
&= da_0 f_5 + da_1 f_4 - da_2 f_7 + da_3 f_6 - da_4 f_1 + da_5 f_0 - da_6 f_3 + da_7 f_2 + da_8 f_{13} \\
&\quad - da_9 f_{12} + da_{10} f_{15} - da_{11} f_{14} + da_{12} f_9 - da_{13} f_8 + da_{14} f_{11} - da_{15} f_{10},
\end{aligned}$$

$$\begin{aligned}
dG_6 &= \frac{\partial G_6}{\partial a_0} da_0 + \frac{\partial G_6}{\partial a_1} da_1 + \frac{\partial G_6}{\partial a_2} da_2 + \frac{\partial G_6}{\partial a_3} da_3 + \frac{\partial G_6}{\partial a_4} da_4 + \frac{\partial G_6}{\partial a_5} da_5 \\
&\quad + \frac{\partial G_6}{\partial a_6} da_6 + \frac{\partial G_6}{\partial a_7} da_7 + \frac{\partial G_6}{\partial a_8} da_8 + \frac{\partial G_6}{\partial a_9} da_9 + \frac{\partial G_6}{\partial a_{10}} da_{10} \\
&\quad + \frac{\partial G_6}{\partial a_{11}} da_{11} + \frac{\partial G_6}{\partial a_{12}} da_{12} + \frac{\partial G_6}{\partial a_{13}} da_{13} + \frac{\partial G_6}{\partial a_{14}} da_{14} + \frac{\partial G_6}{\partial a_{15}} da_{15} \\
&= da_0 f_6 + da_1 f_7 + da_2 f_4 - da_3 f_5 - da_4 f_2 + da_5 f_3 + da_6 f_0 - da_7 f_1 + da_8 f_{14} \\
&\quad - da_9 f_{15} - da_{10} f_{12} + da_{11} f_{13} + da_{12} f_{10} - da_{13} f_{11} - da_{14} f_8 + da_{15} f_9,
\end{aligned}$$

$$\begin{aligned}
dG_7 &= \frac{\partial G_7}{\partial a_0} da_0 + \frac{\partial G_7}{\partial a_1} da_1 + \frac{\partial G_7}{\partial a_2} da_2 + \frac{\partial G_7}{\partial a_3} da_3 + \frac{\partial G_7}{\partial a_4} da_4 + \frac{\partial G_7}{\partial a_5} da_5 \\
&\quad + \frac{\partial G_7}{\partial a_6} da_6 + \frac{\partial G_7}{\partial a_7} da_7 + \frac{\partial G_7}{\partial a_8} da_8 + \frac{\partial G_7}{\partial a_9} da_9 + \frac{\partial G_7}{\partial a_{10}} da_{10} \\
&\quad + \frac{\partial G_7}{\partial a_{11}} da_{11} + \frac{\partial G_7}{\partial a_{12}} da_{12} + \frac{\partial G_7}{\partial a_{13}} da_{13} + \frac{\partial G_7}{\partial a_{14}} da_{14} + \frac{\partial G_7}{\partial a_{15}} da_{15} \\
&= da_0 f_7 - da_1 f_6 + da_2 f_5 + da_3 f_4 - da_4 f_3 - da_5 f_2 + da_6 f_1 + da_7 f_0 + da_8 f_{15} \\
&\quad + da_9 f_{14} - da_{10} f_{13} - da_{11} f_{12} + da_{12} f_{11} + da_{13} f_{10} - da_{14} f_9 - da_{15} f_8,
\end{aligned}$$

$$\begin{aligned}
dG_8 &= \frac{\partial G_8}{\partial a_0} da_0 + \frac{\partial G_8}{\partial a_1} da_1 + \frac{\partial G_8}{\partial a_2} da_2 + \frac{\partial G_8}{\partial a_3} da_3 + \frac{\partial G_8}{\partial a_4} da_4 + \frac{\partial G_8}{\partial a_5} da_5 \\
&\quad + \frac{\partial G_8}{\partial a_6} da_6 + \frac{\partial G_8}{\partial a_7} da_7 + \frac{\partial G_8}{\partial a_8} da_8 + \frac{\partial G_8}{\partial a_9} da_9 + \frac{\partial G_8}{\partial a_{10}} da_{10} \\
&\quad + \frac{\partial G_8}{\partial a_{11}} da_{11} + \frac{\partial G_8}{\partial a_{12}} da_{12} + \frac{\partial G_8}{\partial a_{13}} da_{13} + \frac{\partial G_8}{\partial a_{14}} da_{14} + \frac{\partial G_8}{\partial a_{15}} da_{15} \\
&= da_0 f_8 - da_1 f_9 - da_2 f_{10} - da_3 f_{11} - da_4 f_{12} - da_5 f_{13} - da_6 f_{14} - da_7 f_{15} \\
&\quad + da_8 f_0 + da_9 f_1 + da_{10} f_2 + da_{11} f_3 + da_{12} f_4 + da_{13} f_5 + da_{14} f_6 + da_{15} f_7,
\end{aligned}$$

$$\begin{aligned}
dG_9 &= \frac{\partial G_9}{\partial a_0} da_0 + \frac{\partial G_9}{\partial a_1} da_1 + \frac{\partial G_9}{\partial a_2} da_2 + \frac{\partial G_9}{\partial a_3} da_3 + \frac{\partial G_9}{\partial a_4} da_4 + \frac{\partial G_9}{\partial a_5} da_5 \\
&\quad + \frac{\partial G_9}{\partial a_6} da_6 + \frac{\partial G_9}{\partial a_7} da_7 + \frac{\partial G_9}{\partial a_8} da_8 + \frac{\partial G_9}{\partial a_9} da_9 + \frac{\partial G_9}{\partial a_{10}} da_{10} \\
&\quad + \frac{\partial G_9}{\partial a_{11}} da_{11} + \frac{\partial G_9}{\partial a_{12}} da_{12} + \frac{\partial G_9}{\partial a_{13}} da_{13} + \frac{\partial G_9}{\partial a_{14}} da_{14} + \frac{\partial G_9}{\partial a_{15}} da_{15} \\
&= da_0 f_9 + da_1 f_8 - da_2 f_{11} + da_3 f_{10} - da_4 f_{13} + da_5 f_{12} + da_6 f_{15} - da_7 f_{14} \\
&\quad - da_8 f_1 + da_9 f_0 - da_{10} f_3 + da_{11} f_2 - da_{12} f_5 + da_{13} f_4 + da_{14} f_7 - da_{15} f_6,
\end{aligned}$$

$$\begin{aligned}
dG_{10} &= \frac{\partial G_{10}}{\partial a_0} da_0 + \frac{\partial G_{10}}{\partial a_1} da_1 + \frac{\partial G_{10}}{\partial a_2} da_2 + \frac{\partial G_{10}}{\partial a_3} da_3 + \frac{\partial G_{10}}{\partial a_4} da_4 + \frac{\partial G_{10}}{\partial a_5} da_5 \\
&\quad + \frac{\partial G_{10}}{\partial a_6} da_6 + \frac{\partial G_{10}}{\partial a_7} da_7 + \frac{\partial G_{10}}{\partial a_8} da_8 + \frac{\partial G_{10}}{\partial a_9} da_9 + \frac{\partial G_{10}}{\partial a_{10}} da_{10} \\
&\quad + \frac{\partial G_{10}}{\partial a_{11}} da_{11} + \frac{\partial G_{10}}{\partial a_{12}} da_{12} + \frac{\partial G_{10}}{\partial a_{13}} da_{13} + \frac{\partial G_{10}}{\partial a_{14}} da_{14} + \frac{\partial G_{10}}{\partial a_{15}} da_{15} \\
&= da_0 f_{10} + da_1 f_{11} + da_2 f_8 - da_3 f_9 - da_4 f_{14} - da_5 f_{15} + da_6 f_{12} + da_7 f_{13} \\
&\quad - da_8 f_2 + da_9 f_3 + da_{10} f_0 - da_{11} f_1 - da_{12} f_6 - da_{13} f_7 + da_{14} f_4 + da_{15} f_5,
\end{aligned}$$

$$\begin{aligned}
dG_{11} &= \frac{\partial G_{11}}{\partial a_0} da_0 + \frac{\partial G_{11}}{\partial a_1} da_1 + \frac{\partial G_{11}}{\partial a_2} da_2 + \frac{\partial G_{11}}{\partial a_3} da_3 + \frac{\partial G_{11}}{\partial a_4} da_4 + \frac{\partial G_{11}}{\partial a_5} da_5 \\
&\quad + \frac{\partial G_{11}}{\partial a_6} da_6 + \frac{\partial G_{11}}{\partial a_7} da_7 + \frac{\partial G_{11}}{\partial a_8} da_8 + \frac{\partial G_{11}}{\partial a_9} da_9 + \frac{\partial G_{11}}{\partial a_{10}} da_{10} \\
&\quad + \frac{\partial G_{11}}{\partial a_{11}} da_{11} + \frac{\partial G_{11}}{\partial a_{12}} da_{12} + \frac{\partial G_{11}}{\partial a_{13}} da_{13} + \frac{\partial G_{11}}{\partial a_{14}} da_{14} + \frac{\partial G_{11}}{\partial a_{15}} da_{15} \\
&= da_0 f_{11} - da_1 f_{10} + da_2 f_9 + da_3 f_8 - da_4 f_{15} + da_5 f_{14} - da_6 f_{13} + da_7 f_{12} \\
&\quad - da_8 f_3 - da_9 f_2 + da_{10} f_1 + da_{11} f_0 - da_{12} f_7 + da_{13} f_6 - da_{14} f_5 + da_{15} f_4,
\end{aligned}$$

$$\begin{aligned}
dG_{12} &= \frac{\partial G_{12}}{\partial a_0} da_0 + \frac{\partial G_{12}}{\partial a_1} da_1 + \frac{\partial G_{12}}{\partial a_2} da_2 + \frac{\partial G_{12}}{\partial a_3} da_3 + \frac{\partial G_{12}}{\partial a_4} da_4 + \frac{\partial G_{12}}{\partial a_5} da_5 \\
&\quad + \frac{\partial G_{12}}{\partial a_6} da_6 + \frac{\partial G_{12}}{\partial a_7} da_7 + \frac{\partial G_{12}}{\partial a_8} da_8 + \frac{\partial G_{12}}{\partial a_9} da_9 + \frac{\partial G_{12}}{\partial a_{10}} da_{10} \\
&\quad + \frac{\partial G_{12}}{\partial a_{11}} da_{11} + \frac{\partial G_{12}}{\partial a_{12}} da_{12} + \frac{\partial G_{12}}{\partial a_{13}} da_{13} + \frac{\partial G_{12}}{\partial a_{14}} da_{14} + \frac{\partial G_{12}}{\partial a_{15}} da_{15} \\
&= da_0 f_{12} + da_1 f_{13} + da_2 f_{14} + da_3 f_{15} + da_4 f_8 - da_5 f_9 - da_6 f_{10} - da_7 f_{11} \\
&\quad - da_8 f_4 + da_9 f_5 + da_{10} f_6 + da_{11} f_7 + da_{12} f_0 - da_{13} f_1 - da_{14} f_2 - da_{15} f_3,
\end{aligned}$$

$$\begin{aligned}
dG_{13} &= \frac{\partial G_{13}}{\partial a_0} da_0 + \frac{\partial G_{13}}{\partial a_1} da_1 + \frac{\partial G_{13}}{\partial a_2} da_2 + \frac{\partial G_{13}}{\partial a_3} da_3 + \frac{\partial G_{13}}{\partial a_4} da_4 + \frac{\partial G_{13}}{\partial a_5} da_5 \\
&\quad + \frac{\partial G_{13}}{\partial a_6} da_6 + \frac{\partial G_{13}}{\partial a_7} da_7 + \frac{\partial G_{13}}{\partial a_8} da_8 + \frac{\partial G_{13}}{\partial a_9} da_9 + \frac{\partial G_{13}}{\partial a_{10}} da_{10} \\
&\quad + \frac{\partial G_{13}}{\partial a_{11}} da_{11} + \frac{\partial G_{13}}{\partial a_{12}} da_{12} + \frac{\partial G_{13}}{\partial a_{13}} da_{13} + \frac{\partial G_{13}}{\partial a_{14}} da_{14} + \frac{\partial G_{13}}{\partial a_{15}} da_{15} \\
&= da_0 f_{13} - da_1 f_{12} + da_2 f_{15} - da_3 f_{14} + da_4 f_9 + da_5 f_8 + da_6 f_{11} - da_7 f_{10} \\
&\quad - da_8 f_5 - da_9 f_4 + da_{10} f_7 - da_{11} f_6 + da_{12} f_1 + da_{13} f_0 + da_{14} f_3 - da_{15} f_2,
\end{aligned}$$

$$\begin{aligned}
 dG_{14} &= \frac{\partial G_{14}}{\partial a_0} da_0 + \frac{\partial G_{14}}{\partial a_1} da_1 + \frac{\partial G_{14}}{\partial a_2} da_2 + \frac{\partial G_{14}}{\partial a_3} da_3 + \frac{\partial G_{14}}{\partial a_4} da_4 + \frac{\partial G_{14}}{\partial a_5} da_5 \\
 &+ \frac{\partial G_{14}}{\partial a_6} da_6 + \frac{\partial G_{14}}{\partial a_7} da_7 + \frac{\partial G_{14}}{\partial a_8} da_8 + \frac{\partial G_{14}}{\partial a_9} da_9 + \frac{\partial G_{14}}{\partial a_{10}} da_{10} \\
 &+ \frac{\partial G_{14}}{\partial a_{11}} da_{11} + \frac{\partial G_{14}}{\partial a_{12}} da_{12} + \frac{\partial G_{14}}{\partial a_{13}} da_{13} + \frac{\partial G_{14}}{\partial a_{14}} da_{14} + \frac{\partial G_{14}}{\partial a_{15}} da_{15} \\
 &= da_0 f_{14} - da_1 f_{15} - da_2 f_{12} + da_3 f_{13} + da_4 f_{10} - da_5 f_{11} + da_6 f_8 + da_7 f_9 \\
 &- da_8 f_6 - da_9 f_7 - da_{10} f_4 + da_{11} f_5 + da_{12} f_2 - da_{13} f_3 + da_{14} f_0 + da_{15} f_1,
 \end{aligned}$$

$$\begin{aligned}
 dG_{15} &= \frac{\partial G_{15}}{\partial a_0} da_0 + \frac{\partial G_{15}}{\partial a_1} da_1 + \frac{\partial G_{15}}{\partial a_2} da_2 + \frac{\partial G_{15}}{\partial a_3} da_3 + \frac{\partial G_{15}}{\partial a_4} da_4 + \frac{\partial G_{15}}{\partial a_5} da_5 \\
 &+ \frac{\partial G_{15}}{\partial a_6} da_6 + \frac{\partial G_{15}}{\partial a_7} da_7 + \frac{\partial G_{15}}{\partial a_8} da_8 + \frac{\partial G_{15}}{\partial a_9} da_9 + \frac{\partial G_{15}}{\partial a_{10}} da_{10} \\
 &+ \frac{\partial G_{15}}{\partial a_{11}} da_{11} + \frac{\partial G_{15}}{\partial a_{12}} da_{12} + \frac{\partial G_{15}}{\partial a_{13}} da_{13} + \frac{\partial G_{15}}{\partial a_{14}} da_{14} + \frac{\partial G_{15}}{\partial a_{15}} da_{15} \\
 &= da_0 f_{15} + da_1 f_{14} - da_2 f_{13} - da_3 f_{12} + da_4 f_{11} + da_5 f_{10} - da_6 f_9 + da_7 f_8 \\
 &- da_8 f_7 + da_9 f_6 - da_{10} f_5 - da_{11} f_4 + da_{12} f_3 + da_{13} f_2 - da_{14} f_1 + da_{15} f_0.
 \end{aligned}$$

Consequently, the following relations will appear immediatly:

$$\begin{aligned}
 \frac{\partial G_0}{\partial a_0} &= \frac{\partial G_1}{\partial a_1} = \frac{\partial G_2}{\partial a_2} = \frac{\partial G_3}{\partial a_3} = \frac{\partial G_4}{\partial a_4} = \frac{\partial G_5}{\partial a_5} = \frac{\partial G_6}{\partial a_6} = \frac{\partial G_7}{\partial a_7} = \frac{\partial G_8}{\partial a_8} \\
 &= \frac{\partial G_9}{\partial a_9} = \frac{\partial G_{10}}{\partial a_{10}} = \frac{\partial G_{11}}{\partial a_{11}} = \frac{\partial G_{12}}{\partial a_{12}} = \frac{\partial G_{13}}{\partial a_{13}} = \frac{\partial G_{14}}{\partial a_{14}} = \frac{\partial G_{15}}{\partial a_{15}},
 \end{aligned}$$

$$\begin{aligned}
 \frac{\partial G_1}{\partial a_0} &= -\frac{\partial G_0}{\partial a_1} = -\frac{\partial G_3}{\partial a_2} = \frac{\partial G_2}{\partial a_3} = -\frac{\partial G_5}{\partial a_4} = \frac{\partial G_4}{\partial a_5} = \frac{\partial G_7}{\partial a_6} = -\frac{\partial G_6}{\partial a_7} = -\frac{\partial G_9}{\partial a_8} \\
 &= \frac{\partial G_8}{\partial a_9} = \frac{\partial G_{11}}{\partial a_{10}} = -\frac{\partial G_{10}}{\partial a_{11}} = \frac{\partial G_{13}}{\partial a_{12}} = -\frac{\partial G_{12}}{\partial a_{13}} = -\frac{\partial G_{15}}{\partial a_{14}} = \frac{\partial G_{14}}{\partial a_{15}},
 \end{aligned}$$

$$\begin{aligned}
 \frac{\partial G_2}{\partial a_0} &= \frac{\partial G_3}{\partial a_1} = -\frac{\partial G_0}{\partial a_2} = -\frac{\partial G_1}{\partial a_3} = -\frac{\partial G_6}{\partial a_4} = -\frac{\partial G_7}{\partial a_5} = \frac{\partial G_4}{\partial a_6} = \frac{\partial G_5}{\partial a_7} = \frac{\partial G_{10}}{\partial a_8} \\
 &= -\frac{\partial G_{11}}{\partial a_9} = \frac{\partial G_8}{\partial a_{10}} = \frac{\partial G_9}{\partial a_{11}} = \frac{\partial G_{14}}{\partial a_{12}} = \frac{\partial G_{15}}{\partial a_{13}} = -\frac{\partial G_{12}}{\partial a_{14}} = -\frac{\partial G_{13}}{\partial a_{15}},
 \end{aligned}$$

$$\begin{aligned}
 \frac{\partial G_3}{\partial a_0} &= -\frac{\partial G_2}{\partial a_1} = \frac{\partial G_1}{\partial a_2} = -\frac{\partial G_0}{\partial a_3} = -\frac{\partial G_7}{\partial a_4} = \frac{\partial G_6}{\partial a_5} = -\frac{\partial G_5}{\partial a_6} = \frac{\partial G_4}{\partial a_7} = -\frac{\partial G_{11}}{\partial a_8} \\
 &= \frac{\partial G_{10}}{\partial a_9} = -\frac{\partial G_9}{\partial a_{10}} = \frac{\partial G_8}{\partial a_{11}} = \frac{\partial G_{15}}{\partial a_{12}} = -\frac{\partial G_{14}}{\partial a_{13}} = \frac{\partial G_{13}}{\partial a_{14}} = -\frac{\partial G_{12}}{\partial a_{15}},
 \end{aligned}$$

$$\begin{aligned} \frac{\partial G_4}{\partial a_0} &= \frac{\partial G_5}{\partial a_1} = \frac{\partial G_6}{\partial a_2} = \frac{\partial G_7}{\partial a_3} = -\frac{\partial G_0}{\partial a_4} = -\frac{\partial G_1}{\partial a_5} = -\frac{\partial G_2}{\partial a_6} = -\frac{\partial G_3}{\partial a_7} = -\frac{\partial G_{12}}{\partial a_8} \\ &= -\frac{\partial G_{13}}{\partial a_9} = -\frac{\partial G_{14}}{\partial a_{10}} = -\frac{\partial G_{15}}{\partial a_{11}} = \frac{\partial G_8}{\partial a_{12}} = \frac{\partial G_9}{\partial a_{13}} = \frac{\partial G_{10}}{\partial a_{14}} = \frac{\partial G_{11}}{\partial a_{15}}, \end{aligned}$$

$$\begin{aligned} \frac{\partial G_5}{\partial a_0} &= -\frac{\partial G_4}{\partial a_1} = \frac{\partial G_7}{\partial a_2} = -\frac{\partial G_6}{\partial a_3} = \frac{\partial G_1}{\partial a_4} = -\frac{\partial G_0}{\partial a_5} = \frac{\partial G_3}{\partial a_6} = -\frac{\partial G_2}{\partial a_7} = -\frac{\partial G_{13}}{\partial a_8} \\ &= \frac{\partial G_{12}}{\partial a_9} = -\frac{\partial G_{15}}{\partial a_{10}} = \frac{\partial G_{14}}{\partial a_{11}} = -\frac{\partial G_9}{\partial a_{12}} = \frac{\partial G_8}{\partial a_{13}} = -\frac{\partial G_{11}}{\partial a_{14}} = \frac{\partial G_{10}}{\partial a_{15}}, \end{aligned}$$

$$\begin{aligned} \frac{\partial G_6}{\partial a_0} &= -\frac{\partial G_7}{\partial a_1} = -\frac{\partial G_4}{\partial a_2} = \frac{\partial G_5}{\partial a_3} = \frac{\partial G_2}{\partial a_4} = -\frac{\partial G_3}{\partial a_5} = -\frac{\partial G_0}{\partial a_6} = \frac{\partial G_1}{\partial a_7} = -\frac{\partial G_{14}}{\partial a_8} \\ &= \frac{\partial G_{15}}{\partial a_9} = \frac{\partial G_{12}}{\partial a_{10}} = -\frac{\partial G_{13}}{\partial a_{11}} = -\frac{\partial G_{10}}{\partial a_{12}} = \frac{\partial G_{11}}{\partial a_{13}} = \frac{\partial G_8}{\partial a_{14}} = -\frac{\partial G_9}{\partial a_{15}}, \end{aligned}$$

$$\begin{aligned} \frac{\partial G_7}{\partial a_0} &= \frac{\partial G_6}{\partial a_1} = -\frac{\partial G_5}{\partial a_2} = -\frac{\partial G_4}{\partial a_3} = \frac{\partial G_3}{\partial a_4} = \frac{\partial G_2}{\partial a_5} = -\frac{\partial G_1}{\partial a_6} = -\frac{\partial G_0}{\partial a_7} = -\frac{\partial G_{15}}{\partial a_8} \\ &= -\frac{\partial G_{14}}{\partial a_9} = \frac{\partial G_{13}}{\partial a_{10}} = \frac{\partial G_{12}}{\partial a_{11}} = -\frac{\partial G_{11}}{\partial a_{12}} = -\frac{\partial G_{10}}{\partial a_{13}} = \frac{\partial G_9}{\partial a_{14}} = \frac{\partial G_8}{\partial a_{15}}, \end{aligned}$$

$$\begin{aligned} \frac{\partial G_8}{\partial a_0} &= \frac{\partial G_9}{\partial a_1} = \frac{\partial G_{10}}{\partial a_2} = \frac{\partial G_{11}}{\partial a_3} = \frac{\partial G_{12}}{\partial a_4} = \frac{\partial G_{13}}{\partial a_5} = \frac{\partial G_{14}}{\partial a_6} = \frac{\partial G_{15}}{\partial a_7} = -\frac{\partial G_0}{\partial a_8} \\ &= -\frac{\partial G_1}{\partial a_9} = -\frac{\partial G_2}{\partial a_{10}} = -\frac{\partial G_3}{\partial a_{11}} = -\frac{\partial G_4}{\partial a_{12}} = -\frac{\partial G_5}{\partial a_{13}} = -\frac{\partial G_6}{\partial a_{14}} = -\frac{\partial G_7}{\partial a_{15}}, \end{aligned}$$

$$\begin{aligned} \frac{\partial G_9}{\partial a_0} &= -\frac{\partial G_8}{\partial a_1} = \frac{\partial G_{11}}{\partial a_2} = -\frac{\partial G_{10}}{\partial a_3} = \frac{\partial G_{13}}{\partial a_4} = -\frac{\partial G_{12}}{\partial a_5} = -\frac{\partial G_{15}}{\partial a_6} = \frac{\partial G_{14}}{\partial a_7} \\ &= \frac{\partial G_1}{\partial a_8} = -\frac{\partial G_0}{\partial a_9} = \frac{\partial G_3}{\partial a_{10}} = -\frac{\partial G_2}{\partial a_{11}} = \frac{\partial G_5}{\partial a_{12}} = -\frac{\partial G_4}{\partial a_{13}} = -\frac{\partial G_7}{\partial a_{14}} = \frac{\partial G_6}{\partial a_{15}}, \end{aligned}$$

$$\begin{aligned} \frac{\partial G_{10}}{\partial a_0} &= -\frac{\partial G_{11}}{\partial a_1} = -\frac{\partial G_8}{\partial a_2} = \frac{\partial G_9}{\partial a_3} = \frac{\partial G_{14}}{\partial a_4} = \frac{\partial G_{15}}{\partial a_5} = -\frac{\partial G_{12}}{\partial a_6} = -\frac{\partial G_{13}}{\partial a_7} \\ &= \frac{\partial G_2}{\partial a_8} = -\frac{\partial G_3}{\partial a_9} = -\frac{\partial G_0}{\partial a_{10}} = \frac{\partial G_1}{\partial a_{11}} = \frac{\partial G_6}{\partial a_{12}} = \frac{\partial G_7}{\partial a_{13}} = -\frac{\partial G_4}{\partial a_{14}} = -\frac{\partial G_5}{\partial a_{15}}, \end{aligned}$$

$$\begin{aligned} \frac{\partial G_{11}}{\partial a_0} &= \frac{\partial G_{10}}{\partial a_1} = -\frac{\partial G_9}{\partial a_2} = -\frac{\partial G_8}{\partial a_3} = \frac{\partial G_{15}}{\partial a_4} = -\frac{\partial G_{14}}{\partial a_5} = \frac{\partial G_{13}}{\partial a_6} = -\frac{\partial G_{12}}{\partial a_7} \\ &= \frac{\partial G_3}{\partial a_8} = \frac{\partial G_2}{\partial a_9} = -\frac{\partial G_1}{\partial a_{10}} = -\frac{\partial G_0}{\partial a_{11}} = \frac{\partial G_7}{\partial a_{12}} = -\frac{\partial G_6}{\partial a_{13}} = \frac{\partial G_5}{\partial a_{14}} = -\frac{\partial G_4}{\partial a_{15}}, \end{aligned}$$

$$\begin{aligned} \frac{\partial G_{12}}{\partial a_0} &= -\frac{\partial G_{13}}{\partial a_1} = -\frac{\partial G_{14}}{\partial a_2} = -\frac{\partial G_{15}}{\partial a_3} = -\frac{\partial G_8}{\partial a_4} = \frac{\partial G_9}{\partial a_5} = \frac{\partial G_{10}}{\partial a_6} = \frac{\partial G_{11}}{\partial a_7} \\ &= \frac{\partial G_4}{\partial a_8} = -\frac{\partial G_5}{\partial a_9} = -\frac{\partial G_6}{\partial a_{10}} = -\frac{\partial G_7}{\partial a_{11}} = -\frac{\partial G_0}{\partial a_{12}} = \frac{\partial G_1}{\partial a_{13}} = \frac{\partial G_2}{\partial a_{14}} = \frac{\partial G_3}{\partial a_{15}}, \end{aligned}$$

$$\begin{aligned} \frac{\partial G_{13}}{\partial a_0} &= \frac{\partial G_{12}}{\partial a_1} = -\frac{\partial G_{15}}{\partial a_2} = \frac{\partial G_{14}}{\partial a_3} = -\frac{\partial G_9}{\partial a_4} = -\frac{\partial G_8}{\partial a_5} = -\frac{\partial G_{11}}{\partial a_6} = \frac{\partial G_{10}}{\partial a_7} \\ &= \frac{\partial G_5}{\partial a_8} = \frac{\partial G_4}{\partial a_9} = -\frac{\partial G_7}{\partial a_{10}} = \frac{\partial G_6}{\partial a_{11}} = -\frac{\partial G_1}{\partial a_{12}} = -\frac{\partial G_0}{\partial a_{13}} = -\frac{\partial G_3}{\partial a_{14}} = \frac{\partial G_2}{\partial a_{15}}, \end{aligned}$$

$$\begin{aligned} \frac{\partial G_{14}}{\partial a_0} &= \frac{\partial G_{15}}{\partial a_1} = \frac{\partial G_{12}}{\partial a_2} = -\frac{\partial G_{13}}{\partial a_3} = -\frac{\partial G_{10}}{\partial a_4} = \frac{\partial G_{11}}{\partial a_5} = -\frac{\partial G_8}{\partial a_6} = -\frac{\partial G_9}{\partial a_7} \\ &= \frac{\partial G_6}{\partial a_8} = \frac{\partial G_7}{\partial a_9} = \frac{\partial G_4}{\partial a_{10}} = -\frac{\partial G_5}{\partial a_{11}} = -\frac{\partial G_2}{\partial a_{12}} = \frac{\partial G_3}{\partial a_{13}} = -\frac{\partial G_0}{\partial a_{14}} = -\frac{\partial G_1}{\partial a_{15}}, \end{aligned}$$

$$\begin{aligned} \frac{\partial G_{15}}{\partial a_0} &= -\frac{\partial G_{14}}{\partial a_1} = \frac{\partial G_{13}}{\partial a_2} = \frac{\partial G_{12}}{\partial a_3} = -\frac{\partial G_{11}}{\partial a_4} = -\frac{\partial G_{10}}{\partial a_5} = \frac{\partial G_9}{\partial a_6} = -\frac{\partial G_8}{\partial a_7} = \frac{\partial G_7}{\partial a_8} \\ &= -\frac{\partial G_6}{\partial a_9} = \frac{\partial G_5}{\partial a_{10}} = \frac{\partial G_4}{\partial a_{11}} = -\frac{\partial G_3}{\partial a_{12}} = -\frac{\partial G_2}{\partial a_{13}} = \frac{\partial G_1}{\partial a_{14}} = -\frac{\partial G_0}{\partial a_{15}}. \quad \square \end{aligned}$$

4. Conclusion

In this work we obtain a like wise extension of results and concepts of the classical complex variable theory, see [8]. Concepts such as left and right derivation on independent paths have been all well discussed through the paper.

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