UNIVERSIDADE ESTADUAL PAULISTA
"JÚLIO DE MESQUITA FILHO"
Câmpus de São José do Rio Preto

Caroline de Arruda Signorini

One-dimensional cutting stock problems with multiple periods applied to the precast slab industry

São José do Rio Preto
2022

Caroline de Arruda Signorini

One-dimensional cutting stock problems with multiple periods applied to the precast slab industry

Tese apresentada como parte dos requisitos para obtenção do título de Doutora em Matemática, junto ao Programa de Pós-Graduação em Matemática, do Instituto de Biociências, Letras e Ciências Exatas da Universidade Estadual Paulista "Júlio de Mesquita Filho", Câmpus de São José do Rio Preto.

Financiadora: FAPESP 2018/10284-0
CAPES

Orientador: Prof. Dr. Silvio Alexandre de Araujo
Co-orientadora: Profa. Dra. Gislaine Mara Melega

São José do Rio Preto
2022

S578o

Signorini, Caroline de Arruda

One-dimensional cutting stock problems with multiple periods applied to the precast slab industry / Caroline de Arruda Signorini. -- São José do Rio Preto, 2022

134 p. : il., tabs.

Tese (doutorado) - Universidade Estadual Paulista (Unesp), Instituto de Biociências Letras e Ciências Exatas, São José do Rio Preto

Orientador: Silvio Alexandre de Araujo

Coorientadora: Gislaine Mara Melega

1. Matemática. 2. Pesquisa Operacional. 3. Otimização Matemática. 4. Programação inteira-mista. 5. Problemas de corte de estoque multiperíodo. I. Título.

Sistema de geração automática de fichas catalográficas da Unesp. Biblioteca do Instituto de Biociências Letras e Ciências Exatas, São José do Rio Preto. Dados fornecidos pelo autor(a).

Caroline de Arruda Signorini

One-dimensional cutting stock problems with multiple periods
applied to the precast slab industry

Tese apresentada como parte dos requisitos para obtenção do título de Doutora em Matemática, junto ao Programa de Pós-Graduação em Matemática, do Instituto de Biociências, Letras e Ciências Exatas da Universidade Estadual Paulista "Júlio de Mesquita Filho", Câmpus de São José do Rio Preto.

Financiadora: FAPESP 2018/10284-0
CAPES

Comissão Examinadora

Prof. Dr. Silvio Alexandre de Araujo
Orientador

Profa. Dra. Sônia Cristina Poltroniere Silva
UNESP - Câmpus de Bauru

Profa. Dra. Adriana Cristina Cherri Nicola
UNESP - Câmpus de Bauru

Prof. Dr. Reinaldo Morabito Neto
UFSCar - Câmpus de São Carlos

Prof. Dr. Bruno de Athayde Prata
UFC - Fortaleza

São José do Rio Preto
10 de dezembro de 2021

Ao Nenê

AGRADECIMENTOS

A conquista de elaborar esta tese de doutorado não seria possível sem a motivação de meus pais e a priorização que eles deram em minha educação. A convivência entre gerações diferentes nem sempre é fácil, mas a aprendizagem é inevitável. Agradeço à minha mãe, Patricia, e meu pai, Antonio Carlos, por terem apoiado minhas decisões e possibilitado minha progressão acadêmica.

Os anos de doutorado não foram fáceis: cobrança, ansiedade e comparação fizeram parte deste percurso e me confundiram, acrescentando mais barreiras a serem contornadas e até superadas. Uma grande parte da força para continuar obtive por cuidar do meu cachorrinho e irmão. Agradeço ao Nenê por ter me dado a oportunidade de amadurecer sendo sua irmã/mãe e ter me conduzido a valorizar as coisas simples que me trazem alegria.

Ao meu parceiro-namorado Bruno, agradeço por ter me apoiado desde o primeiro dia de aula na graduação, tanto nas disciplinas quanto nas grandes decisões de vida e carreira. Seu apoio foi essencial para eu conseguir trilhar o caminho da graduação ao fim do doutorado.

Agradeço a minhas amigas Ju, Bea, Flavinha, Isa, Mari, Amanda e Bia, e meus amigos Gui, Dan e Cebola pelo suporte emocional, pelos momentos e risos compartilhados. A amizade de vocês faz minha vida ser mais leve.

Fizeram parte dos últimos anos de meu doutorado meus amigos à distância, especialmente Igor, Markin e Tynho, que me acolheram no ambiente online, onde compartilhamos histórias, aprendizado e risadas por meio de conversas e partidas em nossas noites de jogatina.

Sou muito grata aos meus amigos de Ibilce e de vida, em especial André Oak, Lucas Aracato, Jess, Kumon, Will, Ronísio, Luiz, Samuel, Eduardo, Jennifer, Helô, Junior, Ana, Otávio, Jarnin e Plínio. Pela companhia do dia-a-dia, estudos juntos, almoços no RU, cafés na salinha e rolês tão memoráveis.

Muito importantes nessa jornada foram também os servidores do Ibilce. Agradeço em especial ao Getúlio e ao Thiago pelo suporte técnico e brincadeiras descontraídas, à Michele, Eliani, Netinha e Eldes do Restaurante Universitário (RU) que tratam a todos com tanto carinho, à Eliane e ao André da STAEPE que me auxiliaram com paciência a encaminhar documentos para a FAPESP, e ao Mauro, Silvia e Alíria da Seção de Pós-Graduação pelo excelente atendimento. Prestativos, atenciosos e acolhedores, auxiliaram sempre que preciso e tornam o ambiente da universidade uma casa.

Agradeço aos meus professores, que cumpriram papel essencial em minha formação. Em especial, Gorete, por ter me acolhido em meus três anos de PET, onde aprendi que a universidade é mais do que apenas cursar disciplinas. Hermínia, por ter acompanhado meus primeiros passos como pesquisadora. Hermes, que me inspira como pessoa e professora atenciosa. Valeriano e Socorro, que exigiram sempre meu melhor. Tim, pelas correções que me ensinaram matemática além do inglês.

Ao Silvio e Gi, sou grata pelo amparo durante esses mais de quatro anos com tanta paciência e confiança em meu trabalho. Como meu orientador e co-orientadora no papel, atuaram como pais me guiando nessa aventura que é o doutorado.

Agradeço aos membros da banca, Sônia, Bruno, Reinaldo, Adriana, Fernando, Valeriano e Kelly, por terem se disponibilizado a participarem de minha defesa.

O presente trabalho foi realizado com apoio da Coordenação de Aperfeiçoamento de Pessoal de Nível Superior - Brasil (CAPES) - Código de Financiamento 001. Agradeço a FAPESP pela concessão da bolsa de pesquisa, sob o processo 2018/10284-0, Fundação de Amparo à Pesquisa do Estado de São Paulo (FAPESP).

‘All we have to decide is what to do with the time that is given us’,
John Ronald Reuel Tolkien

RESUMO

Este estudo analisa o planejamento da produção de lajes pré-moldadas decorrente de fábricas brasileiras especializadas. A primeira parte desta pesquisa considera o processo de produção de lajes alveolares, para o qual são propostos dois modelos matemáticos baseados no problema de corte de estoque unidimensional multiperíodo com aspectos inovadores relativos a múltiplos modos de manufatura que podem ser usados para produzir as lajes. Visando a minimização de custos de produção e de estoque, os dois modelos são resolvidos por métodos heurísticos. Com base em dados fornecidos por uma fábrica, resultados computacionais indicam a relevância de tais modelos como ferramentas auxiliares na tomada de decisão. A segunda parte deste estudo trata o processo de produção de lajes treliçadas, o qual compreende dois estágios: corte de treliças de aço e concretagem nos moldes. O modelo matemático proposto consiste em um problema de corte de estoque unidimensional multiperíodo de dois estágios de modo a minimizar custos de *setup*, produção e estoque sob restrições de capacidade e balanceamento de estoque. Diferentes estratégias de solução baseadas no método de geração de colunas são aplicadas a este modelo para gerar os padrões de corte que serão utilizados na produção, e um pacote computacional é utilizado para identificar soluções inteiras. As instâncias são baseadas em informações obtidas da fábrica e os resultados computacionais ilustram a aplicação do modelo proposto em um contexto realista.

Palavras-chave: Problemas de corte de estoque multiperíodo, Produção de laje alveolar, Produção de laje treliçada, Relax-and-fix, Geração de colunas.

ABSTRACT

This study looks at the production planning of precast slabs derived from specialized Brazilian factories. The first part of this research considers the production process of hollow-core slabs. For this, two mathematical models are proposed for the arising problem, which consists of a multi-period one-dimensional cutting stock problem with innovative aspects regarding the multiple manufacturing modes that can be used to produce the slabs. These models are solved using heuristic methods, both models aiming at the minimization of production and inventory costs. Based on real data provided by a company, the computational results indicate the relevance of such models as auxiliary tools in decision-making. The second part of this study addresses the production of lattice slabs, which comprises a process in two stages: cutting steel trusses and concreting them in the molds. The proposed mathematical model is expressed by a two-stage multi-period one-dimensional cutting stock problem and aims at the minimization of setup, production and inventory costs, subject to capacity and demand balance constraints. To solve this model, different strategies based on the column generation procedure are implemented to generate the cutting patterns used in the production and then different rounding strategies are applied to identify integer solutions. The instances are based on information obtained from the factory and computational results are used to evaluate the application of the proposed model in a realistic context.

Keywords: Multi-period cutting stock problems, Hollow-core slab production, Lattice slab production, Relax-and-fix, Column generation.

List of Figures

List of Tables

List of Abbreviations

KMPCSP	Kantorovich-based multi-period cutting stock problem
GGMPCSP	Gilmore and Gomory-based multi-period cutting stock problem
DWMPCSP	Dantzig-Wolfe-based multi-period cutting stock problem
MILP	Mixed-integer linear programming
RF	Relax-and-fix
CG	Column generation
RMP	Restricted master problem
O	Optimal solution
F	Feasible solution
NF	Non-feasibility
U	Unknown solution status
OF	Objective function
ICG	Integrated column generation
SCG	Separated column generation

Contents

1 Introduction

The advances in infrastructure conquered by humanity in the last two centuries are clear. In civil construction, industrial processes ease and speed up building while offering durability, sustainability and quality control. In the second industrial age, precast concrete combined with steel emerged as new technology to compose structural elements of buildings, including joists, beams, columns, modular boxes, stairways and walls, among other components. The use of precast elements enhances technological and social development since their application combines more sophisticated equipment, better working conditions and more efficient quality management. Consequently, an increasing use of precast concrete has been seen (Debs, 2000; Steinle et al., 2019). However, the construction industry in Brazil still has many challenges, since there is widescale waste of materials, low productivity, low quality, slow processes and an informal and low-skilled workforce. To face these challenges, this sector has been increasingly integrating innovative technologies and management tools into its production processes, aiming to increase productivity, quality, safety and client satisfaction (Debs, 2000; Barbosa and Vilnītis, 2017).

In the light of growing competitiveness, environmental responsibility and lack of resources, the industrial sector has had to develop decision-making tools intending to both design and produce materials economically and efficiently. As a scientific tool, mathematical modeling applied to industrial processes provides objective and measurable support for decision-making. In these circumstances, the cutting stock problem (CSP) can be a helpful approach to solve arising problems.

The cutting stock problem arises in many industries as a fundamental subproblem of production planning and establishes the best way to cut standard length objects (material units such as aluminum or paper coils, metal or wood sheets, and steel bars, among others) into smaller items with specified dimensions, minimizing material waste while meeting the demand (Dyckhoff, 1990, Wang and Wäscher, 2002, Wäscher et al., 2007, Morabito et al., 2009, Gomes et al., 2016). There are many studies of the CSP from different perspectives. For example, the residual length of a cutting pattern can be stocked for future use, i.e. handling the usable leftovers (Cui et al., 2017). A multi-objective approach can also be considered, aiming, for example, to minimize the frequency of different cutting patterns to be used and, simultaneously, to minimize the number of cut objects (Aliano Filho et al., 2018, İhsan Erozan and Çalışkan, 2020, Oliveira et al., 2021). Another perspective in production planning is the approach of multiple periods, where the decision can be made looking not only at the demand of a single period but considering the demand of future periods of a planning horizon (Aktin and Özdemir, 2009).

The main objective of this thesis is to study mixed-integer linear programs applied to the production planning of hollow-core slabs and lattice slabs. According to the classification in Melega et al. (2018), the models presented here consist of a *Multi-Period Cutting Stock Problems* (MPCSP), also denoted by (- /L2/ - /M), since they do not presuppose more than one production level and allow integration of periods by carrying inventory items from one period to another. Here, the one-dimensional case of the cutting process is considered.

In Chapter 2, there is a literature review on the cutting stock problems, including papers with similar applications to those addressed in this research, along with a discussion of the mathematical formulations and papers that address mixed-integer mathematical models with cutting processes of two stages. The next two chapters of this work are based on the production planning of precast slabs derived from two different companies. Chapter 3 addresses the hollow-core slab production planning problem of a plant, and Chapter 4 is based on the production process of a factory specialized in lattice slabs.

More specifically, Chapter 3 is based on the published paper by Signorini et al. (2021) that focuses on the production planning of hollow-core slabs at a specific plant, which is briefly described next. First, an order form is considered. It contains the client order specifications: prestressed wire configurations, the length of the slabs and the due date. Based on this information, the production manager plans weekly when, which and how many hollow-core slabs (items) should be made in each available mold. Following these instructions, the assembly sector conducts the preparation of the mold, prestressing, casting the concrete, curing, post-tensioning, wire release and item removal. A more detailed description of this process is presented in Section 3.1. Considering the previously-described production process and focusing on the weekly planning decision on which hollow-core slabs will be produced in each available mold, the main contribution of Chapter 3 is the proposal of two mathematical models based on the cutting stock problem literature, aiming at the minimization of inventory and production costs (mainly related to the use of steel wires) subject to demand balance and capacity constraints, as presented in Section 3.2. There is one aspect of these models which, to the best of our knowledge, has not appeared in other studies in the literature about cutting stock problems. In this study, the handling of wire configurations is more comprehensive, as it contemplates the combination of items with different wire configurations in the same production line (same mold) under an unusual restriction: to reduce the total costs, an item might be produced with a wire configuration whose load is equal to or higher than that required (multiple manufacturing modes). Moreover, a theoretical analysis of the compact and extended formulations is discussed in Section 3.3. The solution methods are described in Section 3.4. The first proposed model is solved by an exact method and heuristic methods, based on decomposition approaches, are used to solve both models. Considering a data set based on real information, Section 3.5 analyzes the computational results, which provides an interesting introspection for the industry. Section 3.6 summarizes the outcomes of Chapter 3 and some directions to amend this study.

Looking upon the lattice slab production process of a Brazilian factory, the main contribution of Chapter 4 is the proposal of a mathematical model for the production planning of lattice joist based on the one-dimensional multi-period cutting stock problem with two stages. In the joists' production process, a limited number of molds is used, which are divided into sections of equal length. The production planning of lattice slabs is also weekly and the production process starts with the preparation of the molds. Next, each mold is filled with concrete. The steel trusses previously cut into the length of the items are used as reinforcement, being inserted into the still liquid concrete base. Subsequently, the items are removed from the molds, resetting the production cycle. A more detailed description of the production process of lattice joists is presented in Section 4.1. The main concerns of the factory are the waste of steel trusses during the cutting process, the minimization of the setup and inventory costs and the increase in productivity. Considering a cutting process with two stages, Section 4.2 introduces a mathematical model based on the cutting pattern formulation that matches the factory objectives while meeting demand and satisfying capacity constraints. The first stage is the cutting of steel trusses of standard length into pieces of specific lengths, and the second stage consists of dividing the molds with separators that will outline the allocation of each item. Different cutting patterns in the multiple sections of each mold are allowed, which entails a setup cost related to this use. To the best of our knowledge, this approach has not been handled by any other studies in the literature on cutting stock problems, since we optimize the multi-period cutting stock decisions in a two-stage scenario simultaneously. Another contribution of this research is the proposition of different column generation-based methods with innovative aspects arising from the two-stage problem, where two types of subproblems have to be solved. Exploring decomposition by stages, different rounding strategies are also proposed. Details are specified in Section 4.3. To evaluate the application of the proposed model in a realistic context, Section 4.4 presents the computational results based on information collected from the factory. Conclusions and opportunities for further research are discussed in Section 4.5.

Finally, Chapter 5 revises the contributions of this thesis and points out some perspectives for extending the research presented here. Other contributions of this thesis are presented in the appendices. Appendix A illustrate by a numerical example the application of the solution methods proposed in Chapter 3. Tables with more detailed computational results of Chapter 3 can be found in Appendix B. Variations of the solution strategies proposed in Chapter 4 are in Appendix C. Lastly, Appendix D contains a general example to illustrate the model proposed in Chapter 4 and the application of one of the proposed solution strategies to a numerical example.

2 Literature review

This chapter reviews papers that are related to the topics studied in this thesis. To do so, the revision includes studies addressing (i) the Multi-Period Cutting Stock Problem, as well as its integration with other production levels and other problems, in different applications; (ii) studies applying cutting stock problems (not restricted to multi-period cutting stock problems) to the concrete industry; and (iii) papers that tackle mixed-integer mathematical models with cutting processes of two stages.

2.1 Multi-Period Cutting Stock Problems

The mathematical models proposed in this thesis to represent the production planning of precast slabs consist of one-dimensional *Multi-Period Cutting Stock Problems*, (- /L2/ - /M) according to the classification in Melega et al. (2018), since only one production level (cutting objects) is considered and there is the integration of periods by carrying inventory items from one period to another. The one-dimensional multi-period cutting stock problems can be seen in a variety of industrial settings. Respício and Captivo (2002) presented an adaptation of the Gilmore and Gomory (1961) Column Generation (CG) formulation, also called extended formulation due to the exponential number of variables, for the cutting stock problem found in an application process of the paper industry. The mathematical model considers cumulative demand, initial inventory variables and capacity constraints, which are modeled in terms of the total processing time of the objects in each period. Gramani and França (2006) approached a problem from the furniture industry. They modeled an extended capacitated multi-period cutting stock problem with setup costs, in which the goal was to minimize the trim loss, the inventory and setup costs. Trkman and Gradisar (2007) presented a compact (number of variables and constraints is bounded by some polynomial function) formulation for a general application of the one-dimensional multi-period cutting stock problem considering the use of objects/leftovers in stock. Aktin and Özdemir (2009) developed and implemented a solution method at a medical apparatus manufacturer, proposing an extended model to meet the demand of items, where the cutting patterns were generated with minimum waste. The resulting cutting patterns were used as input in a capacitated cutting stock problem with setup time and costs for each cutting pattern. The number of objects used in the cutting process was limited by the number available. In a general context, Poldi and Arenales (2010) proposed an extended multi-period cutting stock problem, implementing the simplex method with CG to solve the linear relaxation and developed two rounding heuristics to find integer solutions to the problem.

It is worth noticing that, when addressing a cutting stock problem with multiple periods integrated to other production levels, e.g., planning of the acquisition of the objects (level 1), or the assembly of the final products using the pieces as components (level 3), Melega et al. (2018) classify the resulting problem as the Integrated Lot-sizing and Cutting Stock Problem. Recently, many papers have applied this integrated problem to several industrial contexts, such as: corrugated box factory (Savsar and Cogun, 1994), furniture industry (Morabito and Arenales, 2000, Vanzela et al., 2017), sheet metal operations (Verlinden et al., 2007), window frame manufacturer (Kim et al., 2016), paper industry (Leao et al., 2017, Campello et al., 2020, Melega et al., 2020), mattress industry (Christofoletti et al., 2020), wood products (Kokten and Sel, 2020), and in more general contexts (Gilmore and Gomory, 1961, 1963, 1965; Dyckhoff, 1981, Foerster and Wascher, 2000; Yanasse and Morabito, 2006; Wang et al., 2019). Nascimento et al. (2020) addressed the integrated lot-sizing and one-dimensional cutting stock problem with usable leftovers. Considering the production process of automotive springs, Andrade et al. (2021) proposed an integrated lot-sizing and cutting stock model to reduce inventory costs and losses in the steel bar cutting process, taking into account parallel machines and assembly of final products in multiple time periods.

2.2 Cutting Stock Problems in the concrete industry

Recalling that the focus of this research is in the context of civil construction, some related papers are discussed here. Prata et al. (2015) used the compact formulation of the one-dimensional multi-period cutting stock problem to model the precast concrete beam production problem, to minimize the production waste, while satisfying capacity constraints. Vassoler et al. (2016) proposed an extended formulation of the one-dimensional multi-period cutting stock problem model for the beam production planning problem in the lattice slab industry. Lemos et al. (2021) proposed an extended one-dimensional cutting stock problem model based on the concrete pole production process with the goal of minimizing the cost of the objects while meeting the demand for final products with different configurations. Introducing a novel linear programming model similar to the classical model for one dimension, Araújo et al. (2021a) and Araújo et al. (2021b) integrate the production planning problem of heterogeneous prestressed precast beams and the problem of cutting steel bars. In their papers, the focus is on minimizing mold idle capacity, total completion time (makespan) and/or waste of materials while meeting the demand and fulfilling constraints over the use of cutting patterns corresponding to specific characteristics of the application.

We could not find any mathematical model that contemplates the same aspects (multi-manufacturing modes in a multi-period setting) that are being addressed in Chapter 3 of this thesis. Even so, importantly, in Lemos et al. (2021), distinct alternatives of steel modes are acceptable to compose a concrete pole. However, the authors consider a single period cutting stock problem applied to a totally different context when compared to the contents of Chapter 3.

2.3 Two-stage cutting processes

In a similar approach to that developed in Chapter 4, there are papers tackling mixed-integer mathematical models with two stages of cutting processes in different contexts. Correia et al. (2004) reported a supporting system to the cutting planning at a Portuguese paper mill, proposing two models to minimize waste satisfying the production order. In this study, they approach the concept of an auxiliary reel, related to the technological process involved, requiring the generation of sub-patterns. Reinertsen and Vossen (2010) proposed a general cutting stock problem that integrated the generation of cutting patterns with due date restrictions on the orders. They also presented extensions applied to the cutting of reinforcing steel bars. Santos et al. (2018) proposed a two-stage one-dimensional cutting stock problem that arises from the steel industry, in which ingots (objects in stock) are cut into billets that will be heated and molded into the shape of a bar, which is then cut into smaller bars, the final products. Muter and Sezer (2018) introduced an exact column-and-row generation algorithm to a two-stage extension of one-dimensional cutting stock problem in a general context.

Regarding the lattice slab production planning problem addressed in Chapter 4, where two stages of production are considered (cutting steel trusses and dividing the molds outlining the allocation of each item), the proposed model evaluates the use of different cutting patterns in the multiple sections of each mold. To the best of our knowledge, this approach has not been investigated by any other studies in the literature on cutting stock problems with two stages.

3 The multi-period cutting stock problem applied to hollow-core slab production

This chapter approaches the study on the production planning of hollow-core slabs, where two models for the multi-period cutting stock problem are proposed to minimize the inventory and preparation costs, while demand balance, capacity and some industry's specifications are met. A novel feature incorporated in these models is the handling of wire configurations, where it is possible to merge items with different wire configurations in the same production line (same mold). Also, an atypical restriction is considered: an item must be produced with a wire configuration whose load is equal to or higher than the required one. Exact and heuristic solution methods for these models are proposed and computational results are presented. In the end of this chapter, the results reviewed and directions for extending this study are discussed. This chapter is based on the paper "One-dimensional multi-period cutting stock problems in the concrete industry" by C. A. Signorini, S. A. de Araujo and G. M. Melega that was published in *International Journal of Production Research*, 09 Mar 2021, and is available on `https://www.tandfonline.com/doi/10.1080/00207543.2021.1890261`.

3.1 Production process

In this section, the production process of prestressed concrete structures, such as hollow-core slabs, is described as illustrated in Figure 3.1.[1] Initially, the client order is specified along with the due date on a order form. The order (demand) describes the required quantity of items of a certain length and wire configuration by a fixed date. Here, the term "wire configuration" means the number of steel wires that compose the slab loads (objects). This information goes to the production planning department, where the process managers make a weekly plan, deciding when, which and how many items must be produced, making decisions according to their knowledge and giving preference to the items that appear in contracts with a shorter due date. Note that it is possible to produce items for different clients in the same production line (a mold,

[1]Images retrieved from Elecosoft PLC (2010) and CBS Precast Limited/3D Warehouse (2015).

in this case). Generally, client orders are considered about three months before their due date. However, urgent orders may appear, and in this case, daily reprogramming must be performed.

Production planning goes to the assembly sector which consists of the production of the items, from the preparation of molds, prestressing, casting concrete to item removal. Generally, the company uses limited number of identical molds in the same period. At first, each mold is cleaned and a release agent is applied in it to avoid damage to the mold or the items to be produced. Then, concrete separators are placed according to the planning (Figure 3.1(a)), and steel wires are stretched from the top to the bottom of the mold (Figure 3.1(b)). The next step is casting the concrete: with no interruption, the concrete is placed into the mold (Figure 3.1(c)) and allowed to set. After the curing process, the steel in the concrete is stretched (Figure 3.1(d)). When the concrete reaches the desired compression strength, the connection between the wires is released (Figure 3.1(e)) and items are made from that mold. After the concrete reaches sufficient strength, each item is removed from the mold carefully so as not to damage its edges (Figure 3.1(f)) and transported to storage.[2]

Here are some points to note about this production process:

- concrete separators are placed along with the mold to define the beginning and the ending of each item, as illustrated in Figure 3.1 (a) Mold preparation. The only limitation to the placement of the separators is the mold length so that it is possible to use the whole mold in the production;

- the production planning manages the assorted items for each period, in which items of various sizes with different wire configurations can be placed in the same mold;

- in each period, the quantity of steel wires placed in a specific mold corresponds to the highest number requested among all demanded items that will be produced in that mold, and the loads are readjusted manually in each part. The leading wire configuration refers to the highest number of steel wires in a mold. The extra wires are not removed from an item, they are prevented from being stretched after the concrete is cured (see Figure 3.1 (d) Cure and post-tensioning). It is worth mentioning that, in the case of mixing different wire configurations in the same mold, there is a waste related to the excess of wires in the items of lower loads;

[2] See more about prestressed concrete structures production in Husein and Softić (2013), The Prestressed Group (2020) and International Prestressed Hollowcore Association (2020).

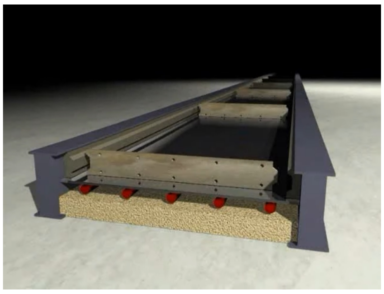
Order form

↓

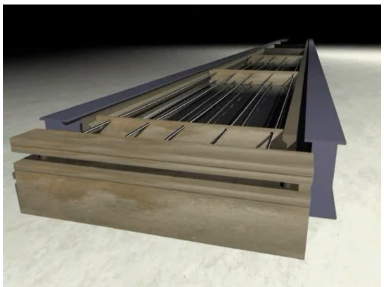
Production planning

↓

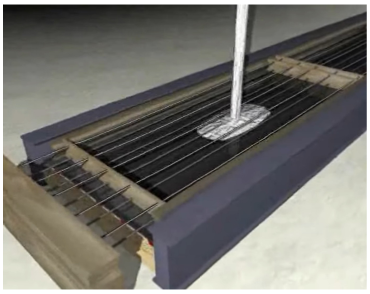
Assembly sector:



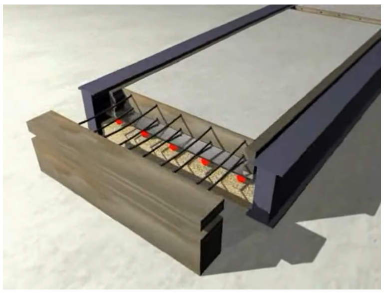
(a) Mold preparation.



(b) Prestressing.

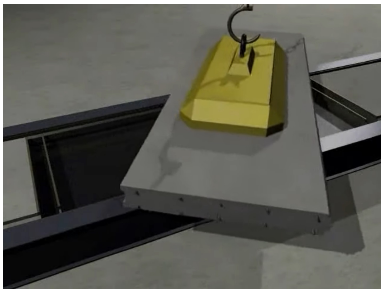


(c) Casting concrete.



(d) Cure and post-tensioning.

(e) Wires release.



(f) Item removal.

(g) Finished items.

Figure 3.1: The production process of prestressed concrete structures.

- any unused wire length in a part of the mold where there is no casting of concrete is considered a loss. Prestressed steel wires are of a high cost in the production of prestressed concrete structures;

- considering that the capacity of the production is limited to the number of available molds in each period and that the concrete cure takes about a day, the production bottleneck consists of the use of these molds;

- related to the storage management, the plant do not appraise a fixed value for stocking items. But since it is a task to be handled, a inventory penalty cost is attributed to each unit of item brought between periods.

3.2 Mathematical models

Two mathematical models are proposed for the production process of hollow-core slabs. The problem is the one-dimensional multi-period cutting stock problem, consisting of cutting large objects (molds) available in inventory into smaller items (slabs) to minimize inventory penalty and production costs, satisfying demand balance and capacity constraints.

In the literature about cutting stock problems, the mathematical models can have compact or extended formulations, according to the number of constraints and variables involved. Some compact models use the ideas presented by Kantorovich (1960), that describe how to cut each object in stock via assignment variables and additional constraints that represent the physical limitations of the cutting process, in terms of object size/length/area/volume. Other compact formulations are based on the arc flow model of Valério de Carvalho (1999, 2002). Cutting stock problems modeled by extended formulations are derived from the well-known idea of cutting patterns (Gilmore and Gomory, 1961, 1963, 1965), in which the decision variables represent the corresponding frequencies of the cutting patterns which are composed of feasible combinations of cutting items from each object type.

In the practical application addressed in this chapter, a low demand rate of items arises, which is closely related to bin-packing problems. Some researchers argue that Kantorovich (1960) based models should be considered by studies where the average demand for items is low, as occurs in bin packing problems, and yield a weak lower bound to the integer problem (Valério de Carvalho, 2002), as well as possibly presenting symmetrical structures, causing branch-and-bound methods to be less efficient. On the other hand, models based on Gilmore and Gomory (1961, 1963) perform better in cases where the average demand for items is high as can be seen in cutting stock problems.

The solution from these models offers a tight linear relaxation, producing stronger lower bounds, since it eliminates some fractional solutions (Savelsbergh, 2008) and manages to remove symmetries. In this chapter, both types of formulation are addressed to emphasize the main advantages and disadvantages of each approach to the cutting stock problem being studied here. Next, these models are described in more detail.

3.2.1 A Kantorovich-based multi-period cutting stock problem (KMPCSP)

The first model is based on the formulation resulting from the description of the problem presented in Kantorovich (1960) for the one-dimensional cutting stock problem. It consists of the assignment of items to objects in order to be cut. We consider setup, inventory penalty, and production variables. The cutting patterns (item arrangement along with the mold) are explicitly assigned in the formulation. In this model, the following sets, parameters, and decision variables are necessary:

Sets:
I: set of items, $\{1, \ldots, |I|\}$ (index i);
T: set of wire configurations, $\{1, \ldots, |T|\}$ (indices τ and β);
K_p: set of molds available in period p, $\{1, \ldots, |K_p|\}$ (index k);
P: set of time periods, $\{1, \ldots, |P|\}$ (index p);

Parameters:
W: mold length;
c: cost of a meter of steel wire;
w_i: length of item i;
$h_{i,p}$: inventory penalty cost of item i in period p;
f_τ: number of steel wires in the wire configuration τ;
$d_{i,\beta,p}$: demand of item i with wire configuration β in period p;

Decision variables:

$Y_{\tau,k,p}$: binary variable that takes value 1 if the leading wire configuration τ is used in mold k on period p;

$S_{i,\beta,p}$: inventory level of item i with wire configuration β at the end of period p;

$Z_{i,\beta,\tau,k,p}$: quantity of item i with wire configuration β produced in mold k with leading wire configuration τ in period p.

Mathematical model KMPCSP:

$$\text{minimize} \quad \sum_{i \in I} \sum_{\beta \in T} \left(\sum_{\substack{p \in P \\ p \neq |P|}} h_{i,p} \cdot S_{i,\beta,p} + c \cdot h_{i,|P|} \cdot S_{i,\beta,|P|} \right) + \sum_{\tau \in T} \sum_{p \in P} \sum_{k \in K_p} W \cdot c \cdot (f_\tau + 1) \cdot Y_{\tau,k,p} \tag{3.1}$$

subject to

$$\sum_{\tau \geqslant \beta} \sum_{k \in K_p} Z_{i,\beta,\tau,k,p} + S_{i,\beta,p-1} - S_{i,\beta,p} = d_{i,\beta,p} \qquad \forall i,\ \forall \beta,\ \forall p \tag{3.2}$$

$$\sum_{i \in I} \sum_{\beta \leqslant \tau} w_i \cdot Z_{i,\beta,\tau,k,p} \leqslant W \cdot Y_{\tau,k,p} \qquad \forall \tau,\ \forall k,\ \forall p \tag{3.3}$$

$$\sum_{\tau \in T} Y_{\tau,k,p} \leqslant 1 \qquad \forall k,\ \forall p \tag{3.4}$$

$$S_{i,\beta,0} = 0 \qquad \forall i,\ \forall \beta, \tag{3.5}$$

$$Y_{\tau,k,p} \in \{0,1\}, \quad S_{i,\beta,p} \in \mathbb{R}_+, \quad Z_{i,\beta,\tau,k,p} \in \mathbb{Z}_+ \qquad \forall i,\ \forall \beta,\ \forall \tau,\ \forall k,\ \forall p. \tag{3.6}$$

The objective function (3.1) minimizes the total cost. The first term is the inventory cost and the extra penalization cost to products left as inventory items in the last period of the planning horizon. This last cost is based on current company politics: hollow-core slabs are produced only on demand since each client's construction project is unique and the use of the building and the loads applied will determine the slab characteristics. The second term of the objective function is the mold cost related to the preparation of the molds, which demands time, use of cleaning products and workforce. In each period, a mold is cleaned and used at most once. In a real-world problem observed in the plant, some items did not require the use of steel wires, hence it is used the term $f_\tau + 1$ instead of f_τ to handle null coefficients for variables Y.

Constraints (3.2) are the demand balance and constraints (3.3) impose the capacity limits of the production process in terms of the length of the mold. Constraints (3.4) limit to one the number of leading wire configurations for each mold k in each period p, regarding the number of steel wires to be placed along with the mold - this condition is detailed in Section 3.1. Constraints (3.5) ensure that production starts with no initial inventory and constraints (3.6) provide the integrality and non-negativity requirements.

Observe that, for each period p, the limit to the number of available molds is implicitly included in the model, and is represented by the label k belonging to the finite set K_p.

3.2.2 A Gilmore and Gomory-based multi-period cutting stock problem (GGMPCSP)

Another formulation presented in this section for the hollow-core production planning problem is based on Gilmore and Gomory (1961, 1963). The structure of this model uses the idea of cutting patterns: the possible combinations of ordered items satisfying the mold length limit. Let $J_{\tau,p}$ be the collection of cutting patterns with leading wire configuration $\tau \in T$ in period $p \in P$, also described by the set $\{1, \ldots, |J_{\tau,p}|\}$. The cutting patterns are represented by columns of the type

$$A_{j,\tau,p} = \left(a_{1,1,j,\tau,p}, a_{2,1,j,\tau,p}, \ldots, a_{i,\beta,j,\tau,p}, \ldots, a_{|I|,|T|,j,\tau,p}\right)^T, \tag{3.7}$$

where $j \in J_{\tau,p}$, and $a_{i,\beta,j,\tau,p}$ is number of items i with wire configuration β in the cutting pattern j. So a valid cutting pattern $A_{j,\tau,p}$ satisfies

$$\sum_{i \in I} \sum_{\beta \leqslant \tau} a_{i,\beta,j,\tau,p} \cdot w_i \leqslant W, \tag{3.8}$$

$$a_{i,\beta,j,\tau,p} \in \mathbb{Z}_+ \qquad \forall i,\ \forall \beta. \tag{3.9}$$

Recalling that an item can be produced with the number of steel wires equal or greater than that which it demands, a cutting pattern $A_{j,\tau,p}$ is only valid if it also satisfies

$$\sum_{i \in I} \sum_{\beta > \tau} a_{i,\beta,j,\tau,p} = 0. \tag{3.10}$$

In this model, it is assumed that all possible cutting patterns are known *a priori*. Thus, a large number of variables related to the cutting patterns is involved, which generally makes computation an infeasible task. To deal with this problem, a heuristic method using the CG procedure, described in Section 3.4.2, is used. The following parameters and variables are considered:

Parameters:

Cap_p: number of available molds in period p;

Variables:

$X_{j,\tau,p}$: number of molds used according to cutting pattern j with the leading wire configuration τ in period p;

Mathematical model GGMPCSP:

$$\begin{aligned}
\text{minimize} \quad & \sum_{i\in I}\sum_{\beta\in T}\left(\sum_{\substack{p\in P\\ p\neq|P|}} h_{i,p}\cdot S_{i,\beta,p} + c\cdot h_{i,|P|}\cdot S_{i,\beta,|P|}\right) \\
& + \sum_{\tau\in T}\sum_{p\in P}\sum_{j\in J_{\tau,p}} W\cdot c\cdot(f_\tau+1)\cdot X_{j,\tau,p}
\end{aligned} \tag{3.11}$$

subject to

$$\sum_{\tau\geq\beta}\sum_{j\in J_{\tau,p}} a_{i,\beta,j,\tau,p}\cdot X_{j,\tau,p} + S_{i,\beta,p-1} - S_{i,\beta,p} = d_{i,\beta,p} \qquad \forall i,\ \forall\beta,\ \forall p \tag{3.12}$$

$$\sum_{\tau\in T}\sum_{j\in J_{\tau,p}} X_{j,\tau,p} \leqslant Cap_p \qquad \forall p \tag{3.13}$$

$$S_{i,\beta,0} = 0 \qquad \forall i,\ \forall\beta \tag{3.14}$$

$$S_{i,\beta,p}\in\mathbb{R}_+,\quad X_{j,\tau,p}\in\mathbb{Z}_+ \qquad \forall i,\ \forall\beta,\ \forall\tau,\ \forall j,\ \forall p. \tag{3.15}$$

As in the previous mathematical model, in GGMPCSP, the objective function (3.11) minimizes the total cost: inventory cost with extra penalization costs to the last period, and mold cost, regarding the number of used wires. The term $f_\tau + 1$ is used to contour null coefficients for variables X since we observed that in a real production context some items might not require the use of steel wires.

Constraints (3.12) establish that the production of slabs meets the client's demand, whereas constraints (3.13) impose the capacity limits of molds availability. Constraints (3.14) determine that no initial inventory is considered and variables domain is expressed by constraints (3.15).

We observe that the parameters Cap_p correspond to the same number as $|K_p|$, i.e., Cap_p is equal to the size of set K_p of the KMPCSP, limiting the number of used molds to a given available quantity in each period p.

3.3 A theoretical analysis of the formulations

In this section, the models GGMPCSP and KMPCSP are shown to be alternative formulations to the studied problem and, also, that the linear relaxation bound provided by the GGMPCSP formulation is stronger than the KMPCSP one. The proof consists of applying Dantzig-Wolfe decomposition to the compact KMPCSP model in order to obtain the extended GGMPCSP model. The proof here presented is based on the arguments presented in Vance (1998) and Valério de Carvalho (2002). This approach is generally used to get stronger lower bounds for the cutting stock problem. It cuts off fractional extreme points of the knapsack polytope, which are feasible solutions to the linear relaxation, i.e., it considers only the non-negative linear combinations of the integer solutions to the knapsack problem.

For each leading wire configuration $\tau \in T$, period $p \in P$ and mold label $k \in K_p$, the Dantzig-Wolfe decomposition applied to the knapsack constraints (3.3) implies that these constraints can be replaced by a convex combination of the extreme points of their convex hull, mathematically described by the bounded and convex polytope

$$\mathcal{C}^{\tau,k,p} = \text{conv}\left\{ \mathbf{Z}_{\tau,k,p} \in \mathbb{Z}_+^{|I|\times|T|} : \sum_{i\in I}\sum_{\beta\leqslant\tau} w_i \cdot Z_{i,\beta,\tau,k,p} \leqslant W,\ \sum_{i\in I}\sum_{\beta>\tau} Z_{i,\beta,\tau,k,p} = 0 \right\},$$

considering the knapsack integer solutions. This fact is based on the Minkowski Theorem (which can be found in Jünger et al. (2009), p. 435) since $\mathcal{C}^{\tau,k,p}$ has a finite number of extreme points but no extreme rays. In other words, the polytope $\mathcal{C}^{\tau,k,p}$ can be represented in the form

$$\mathcal{C}^{\tau,k,p} = \left\{ \mathbf{Z}_{\tau,k,p} \in \mathbb{Z}_+^{|I|\times|T|} : \mathbf{Z}_{\tau,k,p} = \sum_{j\in N} \mu^j_{\tau,k,p} \cdot A^j_{\tau,k,p},\ \sum_{j\in N_{\tau,k,p}} \mu^j_{\tau,k,p} = 1,\ \mu^j_{\tau,k,p} \geqslant 0 \right\}.$$

where $A^j_{\tau,k,p} = \left(a^j_{1,1,\tau,k,p}, \ldots, a^j_{i,\beta,\tau,k,p}, \ldots, a^j_{|I|,|T|,\tau,k,p}\right) \in \mathbb{Z}^{|I|\times|T|}$, with $j \in N_{\tau,k,p}$, represent the extreme points of $\mathcal{C}^{\tau,k,p}$ and $N_{\tau,k,p}$ is a finite set of indices. Considering this notation, we obtain the following Dantzig-Wolfe-based multi-period cutting stock formulation (DWMPCSP):

$$\begin{aligned} \text{minimize} \quad & \sum_{i\in I}\sum_{\beta\in T}\left(\sum_{\substack{p\in P\\ p\neq|P|}} h_{i,p}\cdot S_{i,\beta,p} + c\cdot h_{i,|P|}\cdot S_{i,\beta,|P|} \right) \\ & + \sum_{\tau\in T}\sum_{p\in P}\sum_{j\in N_{\tau,k,p}} W\cdot c\cdot(f_\tau+1)\cdot\left(\sum_{k\in K_p}\mu^j_{\tau,k,p}\right) \end{aligned} \tag{3.16}$$

subject to

$$S_{i,\beta,p-1} + \sum_{\tau\geqslant\beta}\sum_{j\in N_{\tau,k,p}}\sum_{k\in K_p} \mu^j_{\tau,k,p}\cdot a^j_{i,\beta,\tau,k,p} - S_{i,\beta,p} = d_{i,\beta,p} \quad \forall i,\ \forall\beta,\ \forall p \tag{3.17}$$

$$\sum_{\tau\in T}\sum_{j\in N_{\tau,k,p}} \mu^j_{\tau,k,p} \leqslant 1 \qquad \forall k,\ \forall p \tag{3.18}$$

$$\sum_{j\in N_{\tau,k,p}} \mu^j_{\tau,k,p} \leqslant 1 \qquad \forall \tau,\ \forall k,\ \forall p \tag{3.19}$$

$$\sum_{j\in N_{\tau,k,p}} \mu^j_{\tau,k,p} \cdot a^j_{i,\beta,\tau,k,p} \in \mathbb{Z}_+ \qquad \forall i,\ \forall \beta,\ \forall \tau,\ \forall k,\ \forall p \tag{3.20}$$

$$S_{i,\beta,0} = 0 \qquad \forall i,\ \forall \beta \tag{3.21}$$

$$\mu^j_{\tau,k,p},\ S_{i,\beta,p} \in \mathbb{R}_+, \qquad \forall i,\ \forall \beta,\ \forall \tau,\ \forall k,\ \forall p,\ \forall j. \tag{3.22}$$

In the previous formulation, the parameters $a^j_{i,\beta,\tau,k,p}$ represent the number of items i with wire configuration β in the cutting pattern j in mold k with leading wire configuration τ in period p. Constraints (3.17) ensure that the chosen patterns attend the items demand and constraints (3.18) limit to one the number of cutting patterns and leading wire configurations to each mold k and period p. For each mold k with leading wire configuration τ in period p, the convexity constraints (3.19) determine that only cutting patterns in the convex hull of the integer solutions to the knapsack constraint must be considered. Note that the inequality in this constraint is justified by the validation of the null solution to the knapsack problems. Constraints (3.20) guarantee that each convex combination result in an integer cutting pattern. Constraints (3.21) indicate that no initial inventory is available, and the variables domain is described by constraints (3.22).

Note that the sets $\mathcal{C}^{\tau,k,p}$ can be differentiated one from the other only if the leading wire configurations are distinct since all molds have the same length. This unique condition is due to the load limitation in the production process: an item can be produced only with a load equal to or higher than the one described by its requested wire configuration. Hence, there is a single set of feasible cutting patterns to all molds with leading wire configuration τ, in other words, $\mathcal{C}^{\tau,p}$ can be denoted as an alternative to $\mathcal{C}^{\tau,k,p}$.

The previous formulation can be simplified. Note that constraints (3.19) and (3.20) imply that each cutting pattern j is a feasible integer solution to the knapsack constraint, thus, we can look upon all integer points of $\mathcal{C}^{\tau,p}$ instead of its extreme points only. Let $A_{j,\tau,p} = \big(a_{1,1,j,\tau,p}, \ldots, a_{i,\beta,j,\tau,p}, \ldots, a_{|I|,|T|,j,\tau,p}\big)$, with $j \in J_{\tau,p}$, be the feasible integer solutions to the knapsack constraint to a mold with leading wire configuration τ and period $p \in P$. Without eliminating any feasible solution, variables $\mu^j_{\tau,k,p}$ can be set to be binary and the convexity (3.19) and integrality (3.20) constraints can be dismissed from the problem. Considering this, integer variables $X_{j,\tau,p}$ can be defined as the number of molds using cutting pattern j with leading wire configuration τ in period p, which can be also described as

$$X_{j,\tau,p} = \sum_{k \in K_p} \mu^j_{\tau,k,p} \quad \forall j,\ \forall \tau,\ \forall p.$$

Denoting by $Cap_p = |K_p|$ the number of available molds in period p (capacity), constraints (3.18) imply

$$\sum_{\tau \in T} \sum_{j \in J_{\tau,p}} X_{j,\tau,p} = \sum_{\tau \in T} \sum_{j \in J_{\tau,p}} \sum_{k \in K_p} \mu^j_{\tau,k,p} \leqslant \sum_{k \in K_p} 1 = |K_p| = Cap_p \quad \forall p.$$

Hence, we recover the Gilmore and Gomory-based model (3.11)-(3.15).

The formulations DWMPCSP ((3.16)-(3.22)) and GGMPCSP ((3.11)-(3.15)) provide the same lower bound by linear relaxation. First, it is proved that a solution from GGMPCSP can be derived from a DWMPCSP solution. Note that $N_{\tau,k,p} \subset J_{\tau,p}$ for all $\tau \in T, k \in K$ and $p \in P$ since $N_{\tau,k,p}$ is the index set of integer extreme points of $\mathcal{C}^{\tau,p}$ and $J_{\tau,p}$ is composed of indices that represent all the integer points of $\mathcal{C}^{\tau,p}$. In other words, each feasible cutting pattern represented by a column $A^j_{\tau,k,p}$ of DWMPCSP can be also described as a column $A_{j,\tau,p}$ of GGMPCSP. Let $\hat{\mu}^j_{\tau,k,p} \in \mathbb{R}_+$ and $\hat{S}_{i,\beta,p} \in \mathbb{R}_+$ be solutions to DWMPCSP. For each $\tau \in T$ and $p \in P$, define

$$\hat{X}_{j,\tau,p} = \begin{cases} \sum\limits_{k \in K} \hat{\mu}^j_{\tau,k,p} & \text{if } \ j \in N_{\tau,k,p}, \\ 0 & \text{if } \ j \in J_{\tau,p} \setminus N_{\tau,k,p}. \end{cases}$$

Then

$$\begin{aligned} F_{GG}(\hat{X}) &= \sum_{i \in I} \sum_{\beta \in T} \left(\sum_{\substack{p \in P \\ p \neq |P|}} h_{i,p} \cdot \hat{S}_{i,\beta,p} + c \cdot h_{i,|P|} \cdot \hat{S}_{i,\beta,|P|} \right) \\ &\quad + \sum_{\tau \in T} \sum_{p \in P} \sum_{j \in N_{\tau,k,p}} W \cdot c \cdot (f_\tau + 1) \cdot \hat{X}_{j,\tau,p} \\ &= \sum_{i \in I} \sum_{\beta \in T} \left(\sum_{\substack{p \in P \\ p \neq |P|}} h_{i,p} \cdot \hat{S}_{i,\beta,p} + c \cdot h_{i,|P|} \cdot \hat{S}_{i,\beta,|P|} \right) \\ &\quad + \sum_{\tau \in T} \sum_{p \in P} \sum_{j \in N_{\tau,k,p}} W \cdot c \cdot (f_\tau + 1) \cdot \left(\sum_{k \in K_p} \hat{\mu}^j_{\tau,k,p} \right) = F_{DW}(\hat{\mu}), \end{aligned}$$

i.e., a GGMPCSP solution can be derived from a DWMPCSP solution and their objective values coincide.

It remains to show the reverse, i.e., that a solution from the DWMPCSP can be derived from a GGMPCSP solution. To simplify the notation, let $N_{\tau,p}$ denote $N_{\tau,k,p}$ (as explained previously, if τ and p are fixed, the sets $N_{\tau,k,p}$ are identical regardless of the mold label k) and write

$$N_{\tau,p} = \{1, 2, \dots, |N_{\tau,p}|\} \subseteq \{1, 2, \dots, |N_{\tau,p}|, |N_{\tau,p}|+1, \dots, |J_{\tau,p}|\} = J_{\tau,p},$$

so that the first $|N_{\tau,p}|$ indices of $N_{\tau,p}$ and $J_{\tau,p}$ correspond to the same columns, which represent the extreme points of the convex hull of the integer solutions to the knapsack problem with leading wire configuration $\tau \in T$ and period $p \in P$.

Let $\tilde{X}_{j,\tau,p} \in \mathbb{R}_+$ and $\tilde{S}_{i,\beta,p} \in \mathbb{R}_+$ be solutions to the linear relaxation of GGMPCSP. The idea to determine a solution of DWMPCSP from this GGMPCSP solution consists of the allocation of the molds according to the number needed to cut the patterns. For each $\tau \in T, p \in P$ and $j \in J_{\tau,p}$, consider

$$M_{0,\tau,p} = 0 \quad \text{and} \quad M_{j,\tau,p} = \sum_{\ell=1}^{j} \lceil \tilde{X}_{\ell,\tau,p} \rceil$$

where $M_{j,\tau,p}$ represents the cumulative amount of molds used to cut patterns $\ell = 1, \dots, j$ considering the rounding up of each fractional pattern, which composed the relaxed solution, to obtain an integer solution. In this notation, molds $k = M_{j-1,\tau,p} + 1, \dots, M_{j,\tau,p}$ are assigned to the production of pattern j with leading wire configuration τ in period p.

Note that a convex combination of the first $|N_{\tau,p}|$ columns (the extreme points) expresses each cutting pattern assigned to the remaining indices of $J_{\tau,p}$, which are indicated by $|N_{\tau,p}|+1, \dots, |J_{\tau,p}|$. In other words, for each $\tau, \in T, p \in P$ and $j \in J_{\tau,p}$, there are $t_j^\ell \geqslant 0$, $\ell \in N_{\tau,p}$, such that

$$\sum_{\ell \in N_{\tau,p}} t_j^\ell = 1 \quad \text{and} \quad A_{j,\tau,p} = \sum_{\ell \in N_{\tau,p}} t_j^\ell \cdot A_{\tau,k,p}^\ell,$$

which is valid for any index k, since all molds are identical. The columns $A_{j,\tau,p}$ and $A_{\tau,k,p}^\ell$ belong to formulations GGMPCSP and DWMPCSP respectively.

Hereupon, for each $\tau \in T$, $p \in P$ and $j, \ell \in N_{\tau,p}$, we define $\tilde{\mu}_{\tau,k,p}^j$ as follows

$$\tilde{\mu}_{\tau,k,p}^j = \begin{cases} 1 & \text{if } k = M_{j-1,\tau,p} + 1, \dots, M_{j,\tau,p} - 1, \\ \tilde{X}_{j,\tau,p} - \lfloor \tilde{X}_{j,\tau,p} \rfloor & \text{if } k = M_{j,\tau,p}, \\ 0 & \text{if } k = M_{\ell-1,\tau,p} + 1, \dots, M_{\ell,\tau,p}, \quad j \neq \ell. \end{cases}$$

Given $\tau \in T$, $p \in P$, $\ell \in N_{\tau,p}$ and $j \in J_{\tau,p} \setminus N_{\tau,p}$, we set

$$\tilde{\mu}^{\ell}_{\tau,k,p} = \begin{cases} t^{\ell}_{j} & \text{if } \ k = M_{j-1,\tau,p}+1,\ldots,M_{j,\tau,p}-1, \\ t^{\ell}_{j}\cdot\left(\tilde{X}_{j,\tau,p} - \lfloor \tilde{X}_{j,\tau,p} \rfloor\right) & \text{if } \ k = M_{j,\tau,p}. \end{cases}$$

If there are $k \in K_p$ bigger than $M_{|J_{\tau,p}|,\tau,p}$, we set $\tilde{\mu}^{j}_{\tau,k,p} = 0$.

Hence, for each $\tau \in T$ and $p \in P$,

$$\begin{aligned}
\sum_{j\in N_{\tau,p}} \sum_{k=1}^{M_{|N_{\tau,p}|,\tau,p}} \tilde{\mu}^{j}_{\tau,k,p} &= \sum_{j\in N_{\tau,p}} \left(M_{j,\tau,p} - M_{j-1,\tau,p} + 1 + \tilde{X}_{j,\tau,p} - \lfloor \tilde{X}_{j,\tau,p} \rfloor\right) \\
&= \sum_{j\in N_{\tau,p}} \left(\sum_{\ell=1}^{j} \lceil \tilde{X}_{\ell,\tau,p} \rceil - \sum_{\ell=1}^{j-1} \lceil \tilde{X}_{\ell,\tau,p} \rceil + 1 + \tilde{X}_{j,\tau,p} - \lfloor \tilde{X}_{j,\tau,p} \rfloor\right) \\
&= \sum_{j\in N_{\tau,p}} \left(\lceil \tilde{X}_{j,\tau,p} \rceil + \sum_{\ell=1}^{j-1} \lceil \tilde{X}_{\ell,\tau,p} \rceil - \sum_{\ell=1}^{j-1} \lceil \tilde{X}_{\ell,\tau,p} \rceil + 1 + \tilde{X}_{j,\tau,p} - \lfloor \tilde{X}_{j,\tau,p} \rfloor\right) \\
&= \sum_{j\in N_{\tau,p}} \left(\lceil \tilde{X}_{j,\tau,p} \rceil + 1 + \tilde{X}_{j,\tau,p} - \lfloor \tilde{X}_{j,\tau,p} \rfloor\right) \\
&= \sum_{j\in N_{\tau,p}} \tilde{X}_{j,\tau,p},
\end{aligned}$$

and

$$\begin{aligned}
\sum_{j\in N_{\tau,p}} \sum_{k>M_{|N_{\tau,p}|,\tau,p}} \tilde{\mu}^{j}_{\tau,k,p} &= \sum_{j\in N_{\tau,p}} \sum_{\ell>|N_{\tau,p}|} t^{j}_{\ell}\cdot\left(M_{\ell,\tau,p} - M_{\ell-1,\tau,p} + 1 + \tilde{X}_{\ell,\tau,p} - \lfloor \tilde{X}_{\ell,\tau,p} \rfloor\right) \\
&= \sum_{j\in N_{\tau,p}} \sum_{\ell>|N_{\tau,p}|} t^{j}_{\ell}\cdot \tilde{X}_{\ell,\tau,p} = \sum_{\ell>|N_{\tau,p}|} \tilde{X}_{\ell,\tau,p}\cdot\left(\sum_{j\in N_{\tau,p}} t^{j}_{\ell}\right) \\
&= \sum_{\ell>|N_{\tau,p}|} \tilde{X}_{\ell,\tau,p},
\end{aligned}$$

i.e., a DWMPCSP solution $(\tilde{\mu}^{j}_{\tau,k,p})$ was obtained from a GGMPCSP solution $(\tilde{X}_{j,\tau,p})$ and it was proved that their objective values coincide. Therefore, it can be concluded that DWMPCSP and GGMPCSP are alternative formulations. Considering that DWMPCSP was obtained from the Dantzig-Wolfe decomposition applied to KMPCSP, it can be deduced that GGMPCSP and KMPCSP are alternative formulations as well and the linear relaxation lower bound provided by the GGMPCSP formulation is stronger than the one from KMPCSP.

Note the relevance of using the extended formulation (GGMPCSP) for the cutting stock problem, since it yields stronger lower bounds eliminating the fractional extreme points of the knapsack polytope, which are feasible solutions to the linear relaxation of the compact model (KMPCSP), through the non-negative linear combination of integer solutions to the knapsack problem.

3.4 Solution methods

In this section, the solution methods used to solve each model are presented. In order to solve the KMPCSP, it was used an exact solution method (MILP), which consists of a general-purpose optimization package. Since the MILP presented difficulties in solving all the instances being tested within the given time, a Relax-and-Fix (RF) heuristic was also applied to solve KMPCSP, which is described in Section 3.4.1. Combinations of the RF with a fix-and-optimize procedure were also tested, as an improvement heuristic in an attempt to obtain better solutions. However, long processing times for the tested instances made its use nonviable.

Due to the elevated number of variables in the GGMPCSP model, a heuristic solution method based on the CG procedure was used. First, integrality constraints are relaxed. A Restricted Master Problem (RMP) with a reduced number of columns is solved successively, being updated with columns generated by the knapsack subproblems. When there are no more attractive cutting patterns left, the CG is discontinued and integrality constraints are added to the last obtained RMP, which is then solved by an optimization package. A more detailed description of this method is included in Section 3.4.2.

3.4.1 Relax-and-fix procedure

The RF method is known in the literature as a MILP-based heuristic, since it uses preceding knowledge of the problem and decomposes it, showing the way integer/binary variables connect to each other. This type of heuristic has been used to solve production planning problems, obtaining positive results in terms of solution quality and computational time, as can be seen in a variety of papers with uses in different industries: an animal-feed plant (Toso et al., 2009), soft drinks plants (Ferreira et al., 2009, 2010), brewing (Baldo et al., 2014), the paper industry (Leao et al., 2017), food industry (Soler et al., 2019), and in more general contexts (Mercé and Fontan, 2003; Stadtler, 2003; Akartunalı and Miller, 2009; de Araujo and Clark, 2013; Melo and Ribeiro, 2017; Fiorotto et al., 2017, 2020; Melega et al., 2020).

The RF heuristic is an iterative procedure, which consists of partitioning the integer/binary set of variables into disjoints sets that will initially be relaxed. Then, subset by subset, it is converted to integer/binary, i.e., the integrality of the variables related to this set is inserted into the subproblem, at each iteration. This kind of method generates smaller subproblems that are simpler to deal with and more likely to yield an optimal solution. Note that partitioning can be performed to one or more types of variables. For example, for our problem it can chosen to partition variables Y and Z separately or together, using different or similar intervals of partitioning. Moreover, overlapping intervals can be considered.

After converting a subset of variables, the resulting model is solved. Next, the last converted set of variables is fixed to their current integer value. A new iteration begins with a new subset of variables being converted. This procedure ends when all subsets of the decomposed variables are converted.

To build a partition to the variables in the RF procedure, the following parameters must be determined: window size (P_{window}), overlap ($P_{overlap}$) and extremes (P_{begin} and P_{end}) of the interval in which the variables will be converted to integers. In this study, these parameters are related to the period index p, as this is a period decomposition. At each iteration, the parameters P_{begin} and P_{end} are updated according to $P_{begin} = P_{end} - P_{overlap}$ and $P_{end} = P_{begin} + P_{window}$. Hence, the chosen variables with indices in the half-open interval $[P_{begin},\ P_{end})$ are converted to integer, the resultant problem is solved and the half-open interval $[P_{begin},\ P_{end} - P_{overlap})$ indicates the indices of the variables that will be fixed to their current integer values.

In this study, all variables partitions are performed by period and the following strategies to decompose the variables were tested:

I) keep variables Z as integer and decompose variables Y;

II) keep variables Z relaxed and decompose variables Y - in the last RF iteration, the integrality constraints for all variables Z are added to the resulting model;

III) decompose variables Y and Z together.

To each one of these three decomposition strategies, different values of the RF parameters were tested. Their results were compared in terms of solution quality and computational time, and for this study, the RF results based on the third strategy are given, together with the following set of parameters:

$$\begin{aligned} P_{window} &= 1 \quad \text{if } |P| = 5, \\ P_{window} &= 3 \quad \text{if } |P| = 10, \\ P_{window} &= 4 \quad \text{if } |P| = 15, \end{aligned}$$

and $P_{overlap} = 1$ regardless of the total number of periods in the instance. The decomposition strategies are illustrated in Table 3.1.

Table 3.1: RF decomposition strategies per number of periods.

(a) RF partitioning considering $|P| = 5$.

<table>
<tr><th rowspan="2">Periods</th><th colspan="4">Iterations</th></tr>
<tr><th>1</th><th>2</th><th>3</th><th>4</th></tr>
<tr><td>1</td><td rowspan="2">Integer</td><td>Fixed</td><td rowspan="2">Fixed</td><td rowspan="3">Fixed</td></tr>
<tr><td>2</td><td rowspan="2">Integer</td></tr>
<tr><td>3</td><td rowspan="3">Relaxed</td><td rowspan="2">Integer</td></tr>
<tr><td>4</td><td rowspan="2">Relaxed</td><td rowspan="2">Integer</td></tr>
<tr><td>5</td><td>Relaxed</td></tr>
</table>

(b) RF partitioning considering $|P| = 10$.

<table>
<tr><th rowspan="2">Periods</th><th colspan="5">Iterations</th></tr>
<tr><th>1</th><th>2</th><th>3</th><th>4</th><th>5</th></tr>
<tr><td>1</td><td rowspan="3">Integer</td><td rowspan="2">Fixed</td><td rowspan="4">Fixed</td><td rowspan="6">Fixed</td><td rowspan="8">Fixed</td></tr>
<tr><td>2</td></tr>
<tr><td>3</td><td rowspan="3">Integer</td></tr>
<tr><td>4</td><td rowspan="7">Relaxed</td></tr>
<tr><td>5</td><td rowspan="3">Integer</td></tr>
<tr><td>6</td><td rowspan="5">Relaxed</td></tr>
<tr><td>7</td><td rowspan="3">Integer</td></tr>
<tr><td>8</td><td rowspan="3">Relaxed</td></tr>
<tr><td>9</td><td rowspan="2">Integer</td></tr>
<tr><td>10</td><td>Relaxed</td></tr>
</table>

(c) RF partitioning considering $|P| = 15$

<table>
<tr><th rowspan="2">Periods</th><th colspan="5">Iterations</th></tr>
<tr><th>1</th><th>2</th><th>3</th><th>4</th><th>5</th></tr>
<tr><td>1</td><td rowspan="4">Integer</td><td rowspan="3">Fixed</td><td rowspan="6">Fixed</td><td rowspan="9">Fixed</td><td rowspan="12">Fixed</td></tr>
<tr><td>2</td></tr>
<tr><td>3</td></tr>
<tr><td>4</td><td rowspan="4">Integer</td></tr>
<tr><td>5</td><td rowspan="11">Relaxed</td></tr>
<tr><td>6</td></tr>
<tr><td>7</td><td rowspan="4">Integer</td></tr>
<tr><td>8</td><td rowspan="8">Relaxed</td></tr>
<tr><td>9</td></tr>
<tr><td>10</td><td rowspan="4">Integer</td></tr>
<tr><td>11</td><td rowspan="5">Relaxed</td></tr>
<tr><td>12</td></tr>
<tr><td>13</td><td rowspan="3">Integer</td></tr>
<tr><td>14</td><td rowspan="2">Relaxed</td></tr>
<tr><td>15</td></tr>
</table>

The results from the RF approach are compared with those obtained by the MILP (optimization package).

3.4.2 Column generation procedure

In the GGMPCSP model, assuming that all possible cutting patterns are known, a large number of variables related to the frequency of each cutting pattern is involved, making computation an impractical task. One of the most well-known methods to get round this issue and solve the linear relaxation of the GGMPCSP model is the CG procedure (Gilmore and Gomory, 1961, 1963). This method applied to the GGMPCSP is detailed below.

Initially, RMP is defined taking into account only homogeneous cutting patterns, where each element is given by

$$a_{i,\beta,j,\tau,p} = \begin{cases} \min\left\{\dfrac{W}{w_i};\ \max\left\{1; \bar{d}_{i,\beta,p}\right\}\right\} & \text{if } i = j \text{ and } \beta = \tau, \\ \\ 0 & \text{otherwise,} \end{cases}$$

where $\bar{d}_{i,\beta,p}$ is the sum of demands from period p to the last period of item i with wire configuration γ, whose load is lower than or equal to the one in the wire configuration β, i.e.,

$$\bar{d}_{i,\beta,p} = \sum_{\gamma \leqslant \beta} \sum_{r=p}^{|P|} d_{i,\gamma,r} \quad \forall i,\ \forall \beta,\ \forall p.$$

Given the possibility that $\bar{d}_{i,\beta,p}$ can be zero for some indices, which would result in the generation of null columns, the term '$\max\left\{1; \bar{d}_{i,\beta,p}\right\}$' was included in the homogeneous cutting patterns definition. Consequently, for each $\tau \in T$ and $p \in P$, the initial value $|J_{\tau,p}|$ is equal to the number of items $|I|$, so that the number of cutting patterns in the first restricted master problem is the union of sets $J_{\tau,p}$, with the total of $|I| \times |T| \times |P|$ cutting patterns.

After solving the RMP, the optimal dual solutions related to the constraints (3.12) and (3.13) are recovered and the knapsack subproblem (3.23)-(3.26) is solved to determine if there are attractive cutting patterns $A_{j,\tau,p}$ that can improve the current RMP solution.

Let $[\lambda_{i,\beta,p} \ \ \mu_p]$ be the dual variables associated to the inventory balance (3.12) and capacity (3.13) constraints respectively, and let variable $\alpha_{i,\beta}$ represent the number of items i with wire configuration β in the new cutting pattern. For each period p and leading wire configuration τ, the knapsack subproblem (3.23)-(3.26) is solved in order to find whether there is an attractive cutting pattern for the RMP.

$$\text{minimize} \quad W \cdot c \cdot (f_\tau + 1) - \sum_{i \in I} \sum_{\beta \in T} (\lambda_{i,\beta,p} \cdot \alpha_{i,\beta}) - \mu_p \tag{3.23}$$

subject to

$$\sum_{i \in I} \sum_{\beta \leqslant \tau} w_i \cdot \alpha_{i,\beta} \leqslant W \tag{3.24}$$

$$\sum_{i \in I} \sum_{\beta > \tau} \alpha_{i,\beta} = 0; \tag{3.25}$$

$$\alpha_{i,\beta} \in \mathbb{Z}_+ \qquad \forall i,\ \forall \beta. \tag{3.26}$$

The subproblem objective (3.23) is to minimize the columns reduced cost. Constraints (3.24) impose the capacity limits concerning the mold length, and constraints (3.25) establish that only items of wire configuration β composed of a number of steel wires lower than or equal to the quantity in the configuration τ (which indicates the leading wire configuration placed along with the mold) are allowed to incorporate the new cutting pattern. Constraints (3.26) describe the variables domain.

If the reduced cost related to the subproblem solution $\hat{\alpha}_{i,\beta}$ is negative, i.e.,

$$W \cdot c \cdot (f_\tau + 1) - \sum_{i \in I} \sum_{\beta \in T} (\lambda_{i,\beta,p} \cdot \hat{\alpha}_{i,\beta}) - \mu_p < 0,$$

then the column $\hat{A}_{(|J|+1),\tau,p} = [\hat{\alpha}_{i,\beta}]$ and its associated frequency variable $X_{(|J|+1),\tau,p}$ are added to the current RMP. Note that the inventory variables are kept in the RMP and are not generated during the CG procedure.

Each CG iteration is performed for each period $p \in P$ and leading wire configuration $\tau \in T$. When the generated columns no longer price out or if there are 50 consecutive iterations without improvement in the objective function of the restricted master problem, the CG procedure is discontinued, and a reduced variant of model GGMPCSP (3.11)-(3.15) is built considering the initial and all the generated cutting patterns, jointly with the integrality constraints over variables $X_{j,\tau,p}$. Then, a feasible solution to the model GGMPCSP can be obtained by solving this reduced mixed-integer problem by an optimization package. It is worth mentioning that this approach is classified as a heuristic method, since not all optimal cutting patterns are considered when solving the mixed-integer problem. Figure 3.2 summarizes the column generation procedure applied to the GGMPCSP.

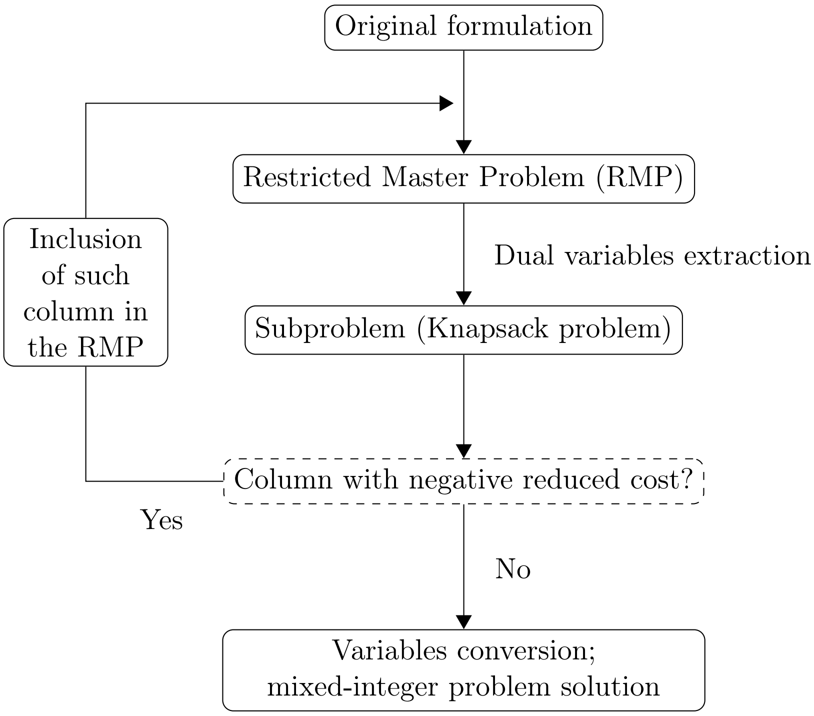


Figure 3.2: Steps of the column generation procedure with a rounding strategy applied to the GGMPCSP.

3.5 Computational results

In this section, the tests performed for the KMPCSP and GGMPCSP models are given. Initially, in Section 3.5.1, the data set based on a real precast plant is described. In Section 3.5.2, the results obtained using the MILP are compared with the ones from RF heuristic, both applied to the KMPCSP model. Section 3.5.3 gives the CG results for GGMPCSP, comparing those results with the results obtained by the MILP applied to the KMPCSP.

3.5.1 Data set

A data set of 180 randomly generated instances was built based on real data provided by the company being studied, which is organized as follows: the standard mold length W is 90 meters in all instances; the lengths of the items vary between 1 and 11 meters; seven types of wire configurations are considered, being composed of 0, 8, 10, 12, 14, 16 and 18 steel wires; inventory penalty cost of items are defined by $h_{i,p} = w_i/2$ for all $i \in I,\ p \in P$; the cost c is based on real values and estimated by 10 units of currency per meter of steel wire; the demand ranges from 0 up to 20 units per item type, in which only 3% of the demand matrix is non-null. These instances are organized in 36 classes characterized by the number of item types: 10, 25, 40 or 55; the number of

Table 3.2: Data set configuration.

Class	**No. of items**	**No. of periods**	**F**	**Class**	**No. of items**	**No. of periods**	**F**
1			1	19			1
2		5	1.5	20		5	1.5
3			2	21			2
4			1	22			1
5	10	10	1.5	23	40	10	1.5
6			2	24			2
7			1	25			1
8		15	1.5	26		15	1.5
9			2	27			2
10			1	28			1
11		5	1.5	29		5	1.5
12			2	30			2
13			1	31			1
14	25	10	1.5	32	55	10	1.5
15			2	33			2
16			1	34			1
17		15	1.5	35		15	1.5
18			2	36			2

periods: 5, 10 or 15; and the number of available molds, which is equal for all periods, calculated using the following formula:

$$\text{Number of molds } = |K_{p'}| = Cap_{p'} = \underset{p\in P}{max}\left\lceil \text{F} \times \frac{\displaystyle\sum_{i\in I}\sum_{\beta\in T} w_i \cdot d_{i,\beta,p}}{W} \right\rceil, \quad p' \in P,$$

where factor F is a parameter that assumes the values 1, 1.5, and 2. Moreover, we generated 5 examples for each of these 36 classes, totalling 180 instances. The data set configuration for each class is illustrated in Table 3.2.[3]

The proposed models and solution approaches were implemented in Microsoft Visual Studio Express 2017 for Windows Desktop using IBM CPLEX version 12.9 with default settings and Concert technology in a C++ project. The machine used was a Dell 64 bit desktop with Intel Core i7-8700 CPU @ 3.20 GHz, 16 GB RAM, Windows 10 Pro. The time limit for CPLEX execution was 1800 seconds for each instance, considering the linear relaxation and mixed-integer problem.

In Tables 3.3 and 3.4 presented as follows, and in Tables B.1, B.2 and B.3 in the Appendix, some specific solution information is given for each class and solution approach. The displayed values are the average between the results of the feasible solutions among the five instances of each class. Column "Solution status" shows the number of instances from each class for which an optimal solution (O), a feasible

[3]All tested instances are available at: `https://github.com/gislainemelega/DataSet_KMPCSP-GGMPCSP`.

solution (F), non-feasibility (NF), or an unknown solution status (U) (the solver did not find a feasible solution in the given time) was obtained. In the "Processing time" column, the values, expressed in seconds, are the average processing time of the five instances of each class. For each feasible solution of each class, the GAP1, in percentage, is calculated by the formula:

$$GAP1 = 100 \times \frac{UB - LB}{UB},$$

where UB (upper bound) and LB (lower bound) are the objective values retrieved from the mixed-integer and linear relaxation (at root node) solutions, respectively. Considering this notation, column "GAP1" presents the average between the gaps of each class. The parameter GAP2 is calculated similarly to GAP1:

$$GAP2 = 100 \times \frac{UB - BLB}{UB},$$

where UB (upper bound) is the mixed-integer objective value and BLB is the best lower bound provided by CPLEX. The values in columns "Mixed-Integer objective value" and "Linear relaxation objective value" are expressed in currency units, and in the columns "Stock cost" and "Mold cost" the percentages of the mixed-integer objective function related to each type of cost are displayed.

3.5.2 KMPCSP: MILP and heuristic results

Figure 3.3 illustrates the solution status distribution of the MILP and RF solution methods applied to the KMPCSP model. It is observed that MILP found 15 optimal solutions, 164 feasible solutions and only one unknown solution status, that was obtained in Class 34 (55 types of items and 15 periods). On the other hand, RF heuristic identified feasible solutions for all 180 instances, from which only two solutions were identified as optimal to the original problem, as their objectives values were equal to the optimal values obtained by MILP. The ease with which RF heuristic obtains a feasible solution is due to the decomposition strategy, which is the main characteristic of this method.

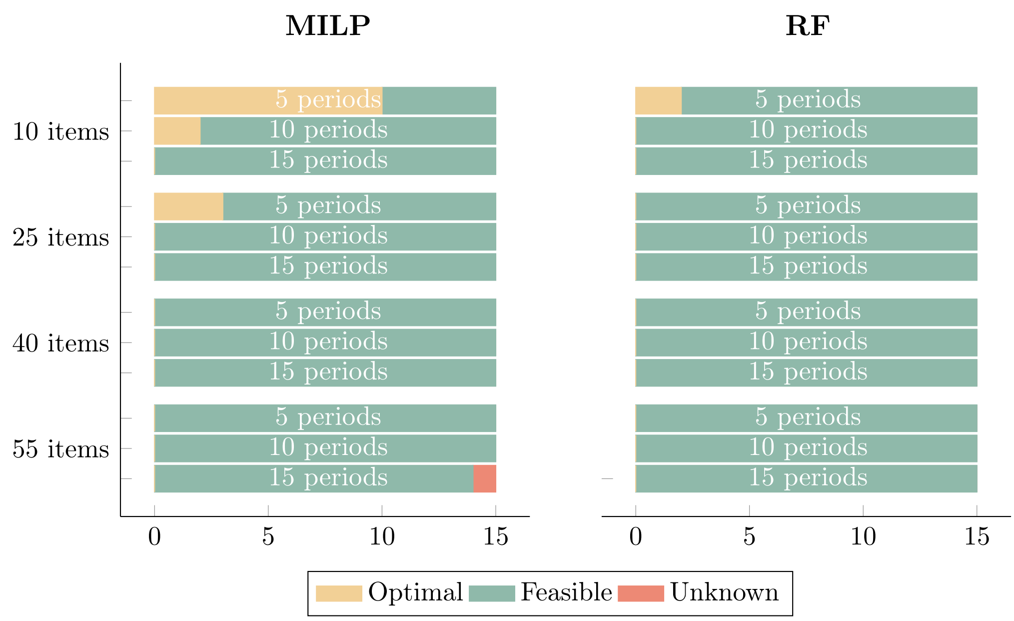


Figure 3.3: Bar chart representing the MILP and RF solution status distribution.

Table 3.3 presents a comparison between the exact and the heuristic methods regarding the processing time and GAPs. In this table, only instances that are feasible for both methods were considered, and the values shown are the average between the results of the feasible examples of each class. Observe that MILP performed with an average processing time of 1,513.00 seconds, which is about 61% longer than the average computational time spent by the RF heuristic, that was 939.12 seconds. In 23 of the 36 classes, RF yielded average values for GAP1 lower than the ones provided by MILP. The average GAP1 for the RF heuristic was of 4.51%, which is about 93% of MILP average GAP of 4.83%. Regarding the best lower bound provided by CPLEX, MILP provided an average GAP2 of 3.35%, which corresponds to approximately 113% of the average yielded by the heuristic, which is 2.96%.

In the Appendix, there are tables with more detailed information about the results. Tables B.1 and B.2 in the Appendix show the averages of the objective values and their distribution between inventory penalty cost and mold cost for the exact and the heuristic methods. It is observed that, for both solution methods of KMPCSP, in all classes, the proportion of the objective value related to the mold cost is higher than to the inventory penalty cost: an average of more than 99% of the total costs were committed to mold preparation. Note that, along with performing in less time than the MILP on average, the RF heuristic produces the better upper bounds compared to those obtained by the MILP, in 22 out of 36 classes, whose values are shown in **bold**.

Hence, the RF approach used in this problem provided better results than the MILP with a greater performance mainly in the larger instances, yielding better upper bounds in 61% of all classes and performing faster in all classes.

Table 3.3: Comparison between MILP and RF heuristic results: processing time and GAPs.

Class	Processing time		GAP1 (%)		GAP2 (%)	
	MILP	RF	MILP	RF	MILP	RF
1	366.24	23.85	12.35	15.43	0.30	3.67
2	717.50	120.58	10.07	13.32	0.59	3.34
3	718.06	242.35	10.07	13.33	1.15	3.89
4	1,479.37	361.67	5.35	8.13	2.95	5.74
5	1,457.72	333.47	5.47	7.75	3.32	5.48
6	1,792.53	364.21	5.35	7.75	3.23	5.56
7	1,796.77	320.45	5.85	7.54	4.46	6.00
8	1,789.47	614.59	5.80	7.42	4.51	5.96
9	1,755.54	811.35	5.87	7.94	4.55	6.51
10	1,447.82	432.96	4.37	6.30	1.75	3.44
11	944.22	574.49	4.36	5.69	1.77	2.85
12	1,662.62	904.82	4.38	6.06	2.59	3.95
13	1,497.84	1,015.52	3.21	3.45	2.55	2.83
14	1,453.03	775.23	3.80	3.30	3.56	3.01
15	1,490.46	892.01	3.89	2.46	3.64	2.24
16	1,486.57	1,017.11	3.51	2.32	3.27	2.05
17	1,481.61	793.93	3.44	2.39	3.21	2.16
18	1,468.77	1,211.65	4.22	2.39	3.98	2.17
19	1,416.21	919.12	3.51	3.31	2.95	2.76
20	1,511.15	1,062.49	3.61	3.36	3.23	2.89
21	1,529.22	746.28	3.63	3.74	3.26	3.43
22	1,659.38	753.63	3.54	2.22	3.33	2.02
23	1,606.35	1,017.53	3.19	2.27	3.08	2.05
24	1,660.97	1,048.19	3.38	2.32	3.32	2.17
25	1,676.48	1,525.75	4.12	1.76	3.98	1.62
26	1,727.72	1,535.70	4.34	1.70	4.24	1.57
27	1,719.69	1,494.95	4.90	1.68	4.80	1.57
28	1,464.89	1,229.82	3.53	2.54	2.92	1.95
29	1,598.94	1,410.10	3.37	2.47	3.24	2.38
30	1,621.64	1,192.19	3.02	2.42	2.90	2.33
31	1,697.68	1,284.72	3.88	1.71	3.80	1.57
32	1,695.38	1,391.07	3.88	1.63	3.82	1.57
33	1,696.71	1,410.44	3.79	1.77	3.75	1.70
34	1,796.04	1,629.15	4.86	1.46	4.81	1.44
35	1,788.74	1,690.69	5.79	1.38	5.73	1.33
36	1,794.59	1,656.18	6.08	1.48	6.14	1.43
Average	1,513.00	939.12	4.83	4.51	3.35	2.96

3.5.3 GGMPCSP: CG procedure results and a comparison with the KMPCSP results using the MILP

In this section, we present the results obtained with the CG procedure for the GGMPCSP formulation.

For this model, only the first 27 classes, totalling 135 different instances, were tested. The limit was set due to the poor quality of the upper bound and the significant difficulty to solve instances during the CG procedure. This difficulty is a result of the large cutting pattern matrix and the high number of generated columns, which caused an insufficient memory failure. Considering that all instances from classes 22 to 27 yielded unknown solution status, these lines were omitted from the tables. All values shown in the tables are the average between the feasible instances of each class.

For the generated columns, this method did not obtain a feasible solution for 69 out of 135 instances, of which 13 were declared infeasible problems due to the limited number of available molds and the restricted assortment of cutting patterns. The other 56 instances indicated unknown solutions due to insufficient memory, including 11 out of 15 instances in classes 13 to 15 (25 items and 10 periods), all instances in classes 16 to 18 (25 items and 15 periods) and classes 22 to 27 (40 items and 10/15 periods).

In the Appendix, Table B.3 shows the total number of cutting patterns in the demand matrix at the end of the CG procedure, the processing time in seconds and the distribution of objective values between inventory penalty cost and mold preparation costs.

For simple comparison, Figure 3.4 shows, side by side, solution status and Table 3.4 presents linear relaxations, mixed-integer objective values and GAP1 to KMPCSP solved by the MILP and GGMPCSP solved by the CG procedure. The MILP for KMPCSP obtained 39 feasible solutions more than GGMPCSP, apart from the other 30 instances from classes 22 to 27 in which feasible solutions were recovered for KMPCSP but not for GGMPCSP.

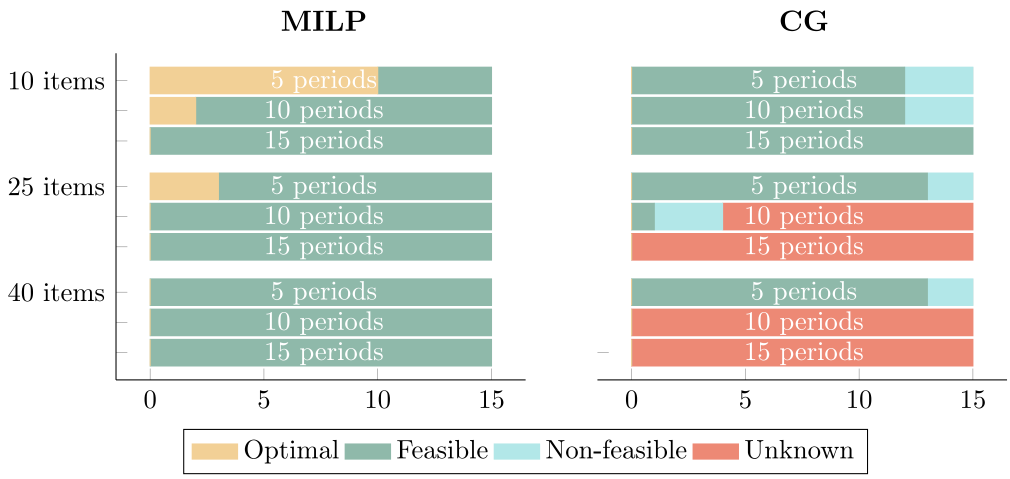


Figure 3.4: Bar chart representing the MILP and CG solution status distribution.

It is observed that the linear relaxations obtained for GGMPCSP are, on average, 0.56% larger than KMPCSP. Although these linear relaxations are of better quality, the GGMPCSP upper bounds are 20.78% greater than KMPCSP. This fact produced the high GAP values for GGMPCSP, whose average was 21.94%, is about four times the average GAP obtained by the MILP for KMPCSP, which was 5.54%. Accordingly, GGMPCSP with a CG procedure can provide good lower bounds, even though it is a difficult strategy to deal with. Regarding the complexity in generating interesting columns, the large demand matrix and the high number of generated columns demanded more from the hardware, causing insufficient memory failures.

Table 3.4: Comparison between KMPCSP MILP and GGMPCSP CG procedure results: linear relaxations, objective values and GAP.

Class	**Linear relaxation objective value**		**Mixed-Integer objective value**		**GAP1**	
	MILP	**CG**	**MILP**	**CG**	**MILP**	**CG**
1	118,526.00	120,143.50	128,802.00	157,135.00	7.90	23.60
2	93,184.18	94,340.36	103,480.80	127,561.66	10.07	26.46
3	93,184.18	94,333.56	103,480.42	127,561.66	10.07	26.46
4	183,869.00	185,485.33	193,158.00	237,741.67	4.72	22.58
5	185,222.75	186,838.25	195,563.00	232,107.50	5.17	19.92
6	180,167.20	181,575.40	190,402.20	232,503.60	5.35	22.16
7	220,745.80	222,365.60	234,321.20	291,568.20	5.85	24.09
8	220,745.80	222,359.00	234,194.80	291,565.40	5.80	24.09
9	220,745.80	222,359.00	234,359.00	291,266.40	5.87	24.01
10	253,716.67	254,843.33	263,994.67	312,654.33	3.98	18.73
11	218,186.20	219,130.40	227,799.60	281,349.80	4.36	23.07
12	218,186.20	219,130.40	227,851.60	279,665.80	4.38	22.57
13	-	-	-	-	-	-
14	-	-	-	-	-	-
15	351,682.00	352,596.00	365,900.00	426,946.00	3.89	17.41
16	-	-	-	-	-	-
17	-	-	-	-	-	-
18	-	-	-	-	-	-
19	291,327.33	292,143.67	303,529.00	355,748.33	4.01	18.37
20	314,618.80	315,348.40	326,391.80	387,319.60	3.61	18.77
21	314,618.80	315,348.40	326,347.80	387,319.60	3.63	18.77
Average	217,420.42	218,646.29	228,723.49	276,250.91	5.54	21.94

3.6 Conclusion and future research

In this chapter, two mathematical models are proposed based on compact and extended formulations of the one-dimensional multi-period cutting stock problem applied to the production planning of hollow-core slabs in the concrete industry. Motivated by the production planning system of a real plant, these models enable multiple manufacturing modes to designate the products to meet client demand, considering that several wire configurations are suitable to meet the demand for an item, provided that some restrictions are satisfied. This specific aspect is related to the idea of product substitution present in Operations Management literature, since it offers alternatives to meet the demand for a product within a category of items. To the best of our knowledge, these models carry a novel feature concerning the way products can be made. In addition to the modeling, theoretical proofs are presented showing that both models produce the same optimal mixed-integer solution and that one model provides stronger linear relaxation than the other. Computational results were obtained using instances based on real data provided by the plant.

Unlike some other papers that approach the extended formulation, in this study, the results showed that MILP applied to our compact model performed better than the CG procedure applied to our extended model due to some particularities of the problem being studied. The demand matrix is based on specific characteristics of items that vary in each construction planning project, which operates differently from a large scale production system. In the mathematical models, these item characteristics are represented by different indices which attribute higher dimensions to the demand matrix and, consequently, to the cutting patterns matrix, increasing the complexity in generating interesting columns and requiring more from the hardware. Despite this fact, the CG procedure utilized in our extended model yielded better lower bounds than MILP applied to the compact model. Regarding the two solution methods applied to the compact model, the RF heuristic found more feasible solutions and produced better upper bounds than MILP mostly in the larger instances, generating better quality solutions while performing in a shorter time.

As future research, the research group intends to study alternatives to obtain good results for our extended model, through the proposition of alternative mathematical models and the implementation of different ways of generating attractive columns. The application of the models in practice is also an interesting area for future work since it could bring interesting insights that can then be used in the decision-making process.

4 The multi-period cutting stock problem applied to lattice slab production

This chapter considers lattice slab production planning, where a mathematical model is proposed, based on the multi-period cutting stock problem with two stages, that minimizes setup, production and inventory costs while meeting demand and satisfying capacity constraints. The first stage is the cutting of steel trusses of standard length into pieces of specific lengths and the second stage consists of dividing the molds with separators that will outline the allocation of each item. The proposed model contemplates the possibility of using different cutting patterns in the multiple sections of each mold, which entails a setup cost related to this use. To the best of our knowledge, this strategy has not been dealt with in any other studies in the literature on cutting stock problems with two stages. Furthermore, different solution approaches are described for the proposed model and the computational results are given. At the end of this chapter, the conclusions and some prospects for extending this study are presented.

4.1 Production process

In civil construction, a lattice joist (called 'item' in this study) is a precast concrete structure composed of a plank base of concrete reinforced with a steel truss, which is composed of steel rebars welded to form trusses joined by a vertex in the shape of a triangle (Figure 4.1). The production planning of lattice joists starts with the client order (demand), which specifies the quantity and length of the items together with the due dates. This information is organized in a weekly plan by the process manager, who decides which and how many items will be produced in each period of the planning horizon.

The production process of a lattice joist is described as illustrated in Figure 4.3. The first stage of production is the steel truss cutting, where steel trusses of standard length are cut into the length of the items. The second stage is the manufacturing of the concrete planks, where a limited number of molds are used and they are divided into sections of identical dimensions. Some companies may present a set of molds with heterogeneous sizes, however, the studied factory uses identical molds. Initially, the molds are prepared, being cleaned and wet with a release agent. In each section of the mold, separators are used according to the length of each item, composing a cutting pattern (Figure 4.2). Then, the mold is mechanically filled with a mixture of cement, water, aggregates and additives to obtain the concrete base of the joists (plank). The steel trusses cut in the length of the items are inserted into the still liquid concrete base, which is vibrated to settle. Following the curing process, the items (finished lattice joists) are removed from the molds, resetting the production cycle. The stocked items are organized in layers, overlapped and supported on wooden slats.

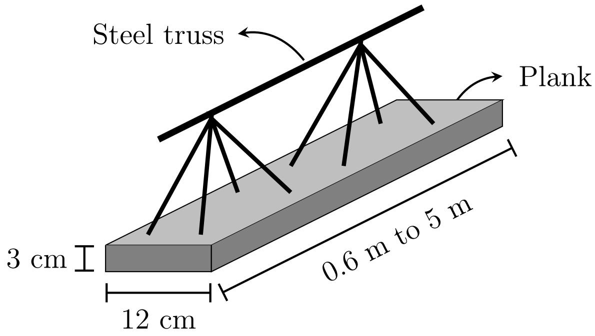


Figure 4.1: Representation of a lattice joist.

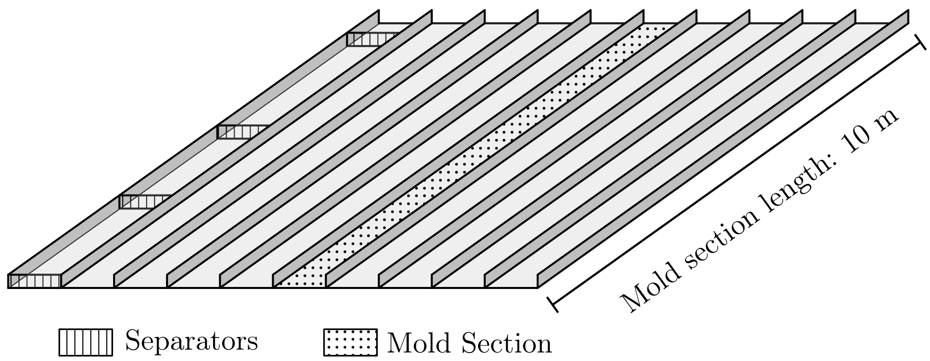


Figure 4.2: Representation of a mold used in the lattice joist production.

Considering the lattice joist production process, the main concerns of the factory are the waste of steel trusses during the cutting process, the minimization of the setup and inventory costs and the increase in productivity during the production periods (better use of the workforce is desirable, avoiding consuming time in the placement of different cutting patterns in sections of the same mold).

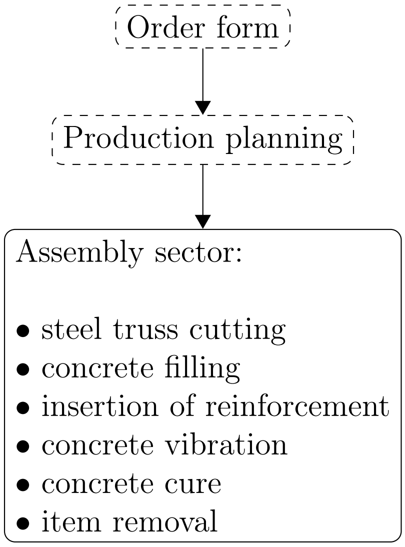


Figure 4.3: The production process of lattice joist.

4.2 Mathematical model

Based on the production process described in Section 4.1, we present a two-stage multi-period cutting stock model to minimize setup, production and inventory costs, subject to capacity and demand balance constraints, without back-order. The following sets, parameters and decision variables are necessary for this model.

Sets:

I: set of items, $\{1, \ldots, |I|\}$ (index i);

P: set of time periods, $\{1, \ldots, |P|\}$ (index p);

K_p: set of available molds in period p, $\{1, \ldots, |K_p|\}$ (index k);

J_p^C: set of cutting patterns for concreting the mold sections for period p, $\{1, \ldots, |J_p^C|\}$ (index j_C);

J_p^S: set of cutting patterns to the steel trusses for period p, $\{1, \ldots, |J_p^S|\}$ (index j_S);

Parameters:

W^C: length of molds (objects);

W^S: standard length of steel trusses (objects);

N: number of sections in a mold;

w_i: length of item i;

d_{ip}: demand of item i in period p;

a_{ij_Cp}: number of joists of length w_i in the cutting pattern j_C, which composes a mold section configuration, in period p;

b_{ij_Sp}: number of steel trusses of length w_i in the cutting pattern j_S in period p;

sc_p^{mold}: setup cost of a mold in period p;

sc_p^{pat}: setup cost for using a cutting pattern in a mold section in period p;

c_p^C: mold section production cost in period p;

c_p^S: steel truss production cost in period p;

h_{ip}^C: inventory penalty cost of item i in period p;

h_{ip}^S: inventory penalty cost of steel truss of length w_i in period p.

Decision variables:

Y_{kp}^{mold}: binary variable that takes value 1 if mold k is used in period p, or 0 otherwise;

$Y_{j_C kp}^{pat}$: binary variable that takes value 1 if the cutting pattern (mold section configuration) j_C is used in mold k in period p, or 0 otherwise;

$X_{j_C kp}^C$: frequency of cutting pattern j_C in mold k in period p;

$X_{j_S p}^S$: frequency of steel trusses cutting pattern j_S in period p;

S_{ip}^C: inventory level of item i in period p;

S_{ip}^S: inventory level of steel truss of length w_i in period p.

Mathematical model

$$\begin{aligned}\text{minimize} \quad & \sum_{p\in P}\sum_{k\in K_p} sc_p^{mold}\cdot Y_{kp}^{mold} + \sum_{p\in P}\sum_{k\in K_p}\sum_{j_C\in J_p^C} sc_p^{pat}\cdot Y_{j_C kp}^{pat} \\ & + \sum_{p\in P}\sum_{k\in K_p}\sum_{j_C\in J_p^C} c_p^C\cdot W^C\cdot X_{j_C kp}^C + \sum_{p\in P}\sum_{j_S\in J_p^S} c_p^S\cdot W^S\cdot X_{j_S p}^S \\ & + \sum_{p\in P}\sum_{i\in I} h_{ip}^C\cdot S_{ip}^C + \sum_{p\in P}\sum_{i\in I} h_{ip}^S\cdot S_{ip}^S\end{aligned} \tag{4.1}$$

subject to

$$\sum_{k\in K_p}\sum_{j_C\in J_p^C} a_{ij_C p}\cdot X_{j_C kp}^C + S_{i,(p-1)}^C - S_{i,p}^C = d_{ip}, \qquad i\in I,\ p\in P \tag{4.2}$$

$$\begin{aligned}\sum_{j_S\in J_p^S} b_{ij_S p}\cdot X_{j_S p}^S + S_{i,(p-1)}^S - S_{i,p}^S & \\ = \sum_{k\in K_p}\sum_{j_C\in J_p^C} a_{ij_C p}\cdot X_{j_C kp}^C, &\end{aligned} \qquad i\in I,\ p\in P \tag{4.3}$$

$$\sum_{j_C\in J_p^C} X_{j_C kp}^C \leqslant N\cdot Y_{kp}^{mold}, \qquad p\in P,\ k\in K_p \tag{4.4}$$

$$X_{j_C kp}^C \leqslant N\cdot Y_{j_C kp}^{pat}, \qquad p\in P,\ j_C\in J_p^C,\ k\in K_p \tag{4.5}$$

$$S_{ip}^C,\ S_{ip}^S\in\mathbb{R}_+, \qquad i\in I,\ p\in P \tag{4.6}$$

$$X_{j_C kp}^C\in\mathbb{Z}_+, \qquad p\in P,\ j_C\in J_p^C,\ k\in K_p \tag{4.7}$$

$$X_{j_S p}^S\in\mathbb{Z}_+, \qquad p\in P, j_S\in J_p^S \tag{4.8}$$

$$Y_{kp}^{mold}\in\mathbb{B}, \qquad p\in P,\ k\in K_p \tag{4.9}$$

$$Y^{pat}_{j_C kp} \in \mathbb{B}, \qquad p \in P,\ j_C \in J^C_p,\ k \in K_p. \qquad (4.10)$$

The objective function (4.1) minimizes the mold setup in the first term, which consists of preparing a mold (a complete mold and not only a specific section), cleaning and wetting it with the release agent. The cutting pattern setup costs in the second term involves placing the separators according to a mold section configuration to determine the allocation of the items inside the mold section, penalizing the utilization of different cutting patterns in the sections of the same mold in each period. The third term of the objective function represents the concrete planks production cost in the mold sections, while the fourth term is the steel truss cutting cost. The last two terms of the objective function compute the inventory penalty costs of finished items (lattice joists) and cut steel trusses respectively.

Constraints (4.2) require that the lattice joist demands are met, while constraints (4.3) establish the integration between the steel trusses cutting and the production of the concrete planks. The steel trusses are cut considering the lattice joists production and there is the possibility of stocking to be used in subsequent periods. Constraints (4.4) ensure that, in each period, the sum of the frequencies of the cutting patterns used in each mold does not exceed the number of sections of the mold. Constraints (4.5), together with the objective function (4.1), compute the use of different cutting patterns in the same mold. In the production process of the lattice joist, the placement of separators in the sections of the molds requires a considerable amount of time and workforce, being related to the number of different configurations (cutting patterns) used in the same mold. Consequently, the smaller the number of different cutting patterns used in the same mold, the faster this activity is performed. The variable domains are described by constraints (4.6), (4.7), (4.8), (4.9) and (4.10).

Chapter 3 proposed two mathematical models for the production process of hollow-core slabs, one based on the compact formulation resultant from the Kantorovich (1960) description for the one-dimensional cutting stock problem (KMPCSP), and the other based on the extended formulation reasoned by the papers Gilmore and Gomory (1961, 1963). Unlike the hollow-core slabs plant, the demand profile of the lattice joist factory is non-degenerate, which makes it unprofitable to apply the compact formulation to this production process.

4.3 Solution methods

Cutting stock problem models based on the Gilmore and Gomory (1961, 1963) formulation are generally solved using column generation procedures since these models contain a large number of variables associated with all the possible cutting patterns (in total, there are $\sum_{p \in P} \left(2 \cdot |J_p^C| \cdot |K_p| + |J_p^S|\right)$ decision variables related to the cutting patterns). In this strategy, a restricted number of cutting patterns (columns) is selected to compose the initial basis for the linear relaxation. The dual solution related to this Restricted Master Problem (RMP) is used to build the pricing problem, which will identify, via reduced costs, attractive columns to be part of the new relaxed problem. This procedure continues until interesting columns are no longer found or if another stop criterion formulated by the user is reached.

This section presents solution methods based on the column generation procedure used to solve the proposed model. Different strategies are used to explore the two-stage feature and are classified according to their approach to the column generation. The classification depends on whether the column generation is performed in an *integrated* set, which considers the two cutting processes together, or in a *separated* configuration, where two different models are used to investigate the best columns for each stage one at a time. Besides that, in the integrated column generation approach, a similar strategy using the separated configuration is applied to obtain an integer solution. Other variations of the integrated and separated configurations were tested, however, some of their aspects made their use impractical. More details of these tests are in Appendix C. The proposed methods are detailed in Subsections 4.3.1, 4.3.2 and 4.3.3.

4.3.1 Integrated column generation procedure (ICG)

In this section, the RMP is initially defined taking into account only homogeneous cutting patterns, given by

$$a_{i,j_C,p} = \begin{cases} \min\left\{\dfrac{W^C}{w_i};\ \max\left\{1; \gamma_{i,p}\right\}\right\} & \text{if } i = j_C \\ 0 & \text{otherwise,} \end{cases}$$

and

$$b_{i,j_S,p} = \begin{cases} \min\left\{\dfrac{W^S}{w_i};\ \max\left\{1; \gamma_{i,p}\right\}\right\} & \text{if } i = j_S \\ 0 & \text{otherwise,} \end{cases}$$

where $\gamma_{i,p}$ is the sum of demands of item i from period p to the last period, i.e.,

$$\gamma_{i,p} = \sum_{r=p}^{|P|} d_{i,r} \quad i \in I,\ p \in P.$$

After solving the RMP, the dual optimal solutions $[\lambda_{ip},\ \sigma_{ip},\ \mu_{kp}]$ respectively related to constraints (4.2), (4.3) and (4.4) are recovered and the knapsack subproblems related to the two stages of the production are solved to verify if there are any columns that would enhance the RMP solution. For each period $p \in P$, the knapsack subproblems (4.11)-(4.13) and (4.14)-(4.16) are solved to check if there are attractive cutting patterns for the RMP.

- Stage 1: Steel truss cutting subproblem

$$\mathrm{OF}^{\mathrm{S}}_{\mathrm{SUB}}(\mathrm{p}) = \min \quad c^S_p \cdot W^S - \sum_{i \in I} \sigma_{ip} \cdot \alpha^S_i \tag{4.11}$$

$$\text{s. t.}$$

$$\sum_{i \in I} w_i \cdot \alpha^S_i \leqslant W^S \tag{4.12}$$

$$\alpha^S_i \in \mathbb{Z}_+ \quad i \in I. \tag{4.13}$$

- Stage 2: Mold section concrete filling subproblem

$$\mathrm{OF}^{\mathrm{C}}_{\mathrm{SUB}}(\mathrm{p}) = \min_{k \in K_p} \quad c^C_p \cdot W^C - \sum_{i \in I} (\lambda_{ip} - \sigma_{ip}) \cdot \alpha^C_i + \mu_{kp} \tag{4.14}$$

$$\text{s. t.}$$

$$\sum_{i \in I} w_i \cdot \alpha^C_i \leqslant W^C \tag{4.15}$$

$$\alpha^C_i \in \mathbb{Z}_+ \quad i \in I. \tag{4.16}$$

For each period p, a cutting pattern is attractive and hence added to the RMP if $\mathrm{OF}^{\mathrm{S}}_{\mathrm{SUB}}(p) < 0$ or $\mathrm{OF}^{\mathrm{C}}_{\mathrm{SUB}}(p) < 0$.

Observe that, since all molds are identical, it wouldn't be coherent to associate different sets of columns to each mold. For this reason, in Stage 2 subproblems, the minimum is taken over all molds $k \in K_p$ and only the generated column associated with the lowest reduced cost is added to the RMP, where the cutting pattern can be used with all molds. After solving the $\sum_{p \in P} (1 + |K_p|)$ subproblems, a new column generation iteration is started. When there are no attractive columns for both subproblems or if the improvement of the RMP objective value is lower than the tolerance (0.001) in 3 consecutive iterations (tailing off stop criterion), the method is interrupted and integrality constraints on variables $X^C_{j_C kp}$, X^S_{jsp}, Y^{mold}_{kp} and $Y^{pat}_{j_C kp}$ are included in the last built RMP.

Note that the dual variables related to constraints (4.5) are not taken into account in the column generation procedure, since these constraints only exist after the cutting pattern generation. Moreover, this variable absence does not restrain the method of recovering a lower bound to the problem (a similar case is addressed by Melega et al., 2020). Indeed, let $[\pi_{j_C kp}]$ be the dual variables related to constraints (4.5). The corresponding subproblem including all dual variables related to the constraints associated to variable $X^C_{j_C kp}$ is

$$\overline{\mathrm{OF}^{\mathrm{C}}_{\mathrm{SUB}}(\mathrm{p})} = \min_{k \in K_p} \quad c^C_p \cdot W^C - \sum_{i \in I} (\lambda_{ip} - \sigma_{ip}) \cdot \alpha^C_i + \mu_{kp} - \pi_{j_C kp} \tag{4.17}$$

$$\text{s. t.}$$

$$\sum_{i \in I} w_i \cdot \alpha^C_i \leqslant W^C \tag{4.18}$$

$$\alpha^C_i \in \mathbb{Z}_+ \quad i \in I. \tag{4.19}$$

Because dual variables $[\pi_{j_C kp}]$ are non-positive by duality theory, they produce a non-negative increment to the subproblem reduced cost. Hence, comparing the objective functions (4.14) and (4.17), one obtains $\mathrm{OF}^{\mathrm{C}}_{\mathrm{SUB}}(\mathrm{p}) \leqslant \overline{\mathrm{OF}^{\mathrm{C}}_{\mathrm{SUB}}(\mathrm{p})}$, which assures that every column generated by subproblem (4.17)-(4.19) can be generated by subproblem (4.14)-(4.16). Thus, in the procedure, the similar generation of columns by both subproblems might lead to similar values of objective functions of the restricted master problems at optimality, both being lower bounds to the original problem.

After the column generation procedure, a feasible solution to the original problem can be obtained from the reduced mixed-integer linear problem using an optimization package. Observe that this is a heuristic procedure since the mixed-integer linear problem solved is restricted to the generated columns from the relaxed problem. It is worth mentioning that, in this strategy, the difference between the last built RMP and the mixed-integer problem used to obtain the feasible solution is the presence of the integrality constraints in the latter.

4.3.2 Integrated column generation with two steps (ICG with two steps)

In this strategy, the column generation procedure is performed as described in Section 4.3.1. Regarding the resultant mixed-integer linear problem, the previous ICG approach deals with a unique integrated problem, adding integrality constraints considering the two stages simultaneously, which can be computationally costly to solve. On the other hand, the ICG with a two-step approach adds integrality constraints and solves each cutting stage separately, as will be detailed as follows.

In this method, the first step after the column generation procedure is to add integrality constraints and solve the RMP sub-model regarding Stage 2 (Model (4.20)-

(4.27)):

$$\begin{aligned} \min \quad & \sum_{p\in P}\sum_{k\in K_p} sc_p^{mold}\cdot Y_{kp}^{mold} + \sum_{p\in P}\sum_{k\in K_p}\sum_{j_C\in J_p^C} sc_p^{pat}\cdot Y_{j_Ckp}^{pat} \\ & + \sum_{p\in P}\sum_{k\in K_p}\sum_{j_C\in J_p^C} c_p^C\cdot W^C\cdot X_{j_Ckp}^C + \sum_{p\in P}\sum_{i\in I} h_{ip}^C\cdot S_{ip}^C \end{aligned} \tag{4.20}$$

s. t.

$$\sum_{k\in K_p}\sum_{j_C\in J_p^C} a_{ij_Cp}\cdot X_{j_Ckp}^C + S_{i,(p-1)}^C - S_{i,p}^C = d_{ip}, \qquad i\in I,\ p\in P \tag{4.21}$$

$$\sum_{j_C\in J_p^C} X_{j_Ckp}^C \leqslant N\cdot Y_{kp}^{mold}, \qquad p\in P,\ k\in K_p \tag{4.22}$$

$$X_{j_Ckp}^C \leqslant N\cdot Y_{j_Ckp}^{pat}, \qquad p\in P,\ j_C\in J_p^C,\ k\in K_p \tag{4.23}$$

$$S_{ip}^C, \in \mathbb{R}_+, \qquad i\in I,\ p\in P \tag{4.24}$$

$$X_{j_Ckp}^C \in \mathbb{Z}_+, \qquad p\in P,\ j_C\in J_p^C,\ k\in K_p \tag{4.25}$$

$$Y_{kp}^{mold} \in \mathbb{B}, \qquad p\in P,\ k\in K_p \tag{4.26}$$

$$Y_{j_Ckp}^{pat} \in \mathbb{B}, \qquad p\in P,\ j_C\in J_p^C,\ k\in K_p. \tag{4.27}$$

In the second step, values of the variables $X_{j_Ckp}^C$ are recovered from Model (4.20)-(4.27) solution and are used as parameters $\bar{X}_{j_Ckp}^C$ to formulate the sub-model related to Stage 1 (Model (4.28)-(4.31)):

$$\min \quad \sum_{p\in P}\sum_{j_S\in J_p^S} c_p^S\cdot W^S\cdot X_{jsp}^S + \sum_{p\in P}\sum_{i\in I} h_{ip}^S\cdot S_{ip}^S \tag{4.28}$$

s. t.

$$\sum_{j_S\in J_p^S} b_{ijsp}\cdot X_{jsp}^S + S_{i,(p-1)}^S - S_{i,p}^S = \sum_{k\in K_p}\sum_{j_C\in J_p^C} a_{ij_Cp}\cdot \bar{X}_{j_Ckp}^C, \qquad i\in I,\ p\in P \tag{4.29}$$

$$S_{ip}^S \in \mathbb{R}_+, \qquad i\in I,\ p\in P \tag{4.30}$$

$$X_{jsp}^S \in \mathbb{Z}_+, \qquad p\in P, j_S\in J_p^S. \tag{4.31}$$

Note that the solutions of Models (4.20)-(4.27) and (4.28)-(4.31) compose a feasible solution to the integrated problem, since the original constraints are directly satisfied.

Due to the separation of the integrated mixed-integer linear problem into two distinct models using different variables in the two-step approach, the ICG with the two-step strategy is likely to perform faster than the traditional ICG.

4.3.3 Separated column generation procedure (SCG)

The SCG approach, as the name suggests, splits the column generation procedure. Based on the cutting stages of the proposed model, two separated column generations are performed. Therefore, a two-step strategy is likewise applied. First, the column generation is applied to the linear relaxation of the Stage 2 sub-model (Model (4.20)-(4.27)) and then, after adding the integrality constraints, the resultant mixed-integer linear problem RMP is solved. Subsequently, the integer solution of the second stage is used to define the linear relaxation of the Stage 1 sub-model (Model (4.28)-(4.31)), which is solved by column generation. Finally, the integrality constraints are added to the last RMP and the resultant mixed-integer linear RMP is solved. The solutions of both stages compose a feasible solution to the original problem. When comparing this strategy to the integrated column generation procedure, some straightforward adaptations for the separated column generation procedure become apparent. For reasons of simplicity, such adjustments have been omitted.

Note that the difference between the SCG and ICG with two-step approaches lies in the time to apply the separation: before or after the column generation procedure.

4.4 Computational results

To validate the proposed model and compare the proposed solution methods (ICG, ICG with two steps and SCG) to the two-stage multi-period cutting stock problem, computational tests were performed using a randomly generated set of instances based on real data, obtained in a Brazilian lattice slab factory.

The proposed model and solution methods were implemented in Microsoft Visual Studio Express 2017 for Windows Desktop using IBM CPLEX version 12.9 with default settings and Concert technology in a C++ project. The apparatus used was a Dell 64 bit desktop with Intel Core i7-8700 CPU @ 3.20 GHz, 16 GB RAM, Windows 10 Pro. The time limit for CPLEX execution was 1800 seconds for each instance, considering the linear relaxation and mixed-integer problem and an optimality gap tolerance of 0.1% for each instance. Considering the ICG with two steps and the SCG approach, the time limits were adapted: 900 seconds for each step.

4.4.1 Data set

Table 4.1 presents how the 16 classes were generated based on the number of items and periods, which assume the values 5 or 10; the length variation establishes the interval in meters in which the length of the items will be generated: "Small" indicates the interval $[0.6, 2.8]$ and "Varied" indicates $[0.6, 5.0]$. Since the capacity is a tolerance parameter being tested, every two instances (1 and 2, 3 and 4, ...) are identical except for the number of available molds, calculated from expression (4.32), where there is a multiplier factor $\mathcal{F}$ assuming the values 1.5 or 2.

$$\text{Number of molds } = \underset{p \in P}{max} \left\lceil \mathcal{F} \times \frac{\sum_{i \in I} w_i \cdot d_{i,p}}{N \cdot W^C} \right\rceil . \tag{4.32}$$

Briefly, equation (4.32) indicates that, for a certain instance, the number of available molds in each period is given by the maximum number of molds required for the production of items in any period of the planning horizon, a number which is multiplied by factor $\mathcal{F}$. For each class, 5 examples are generated, yielding a total of 80 instances.

Table 4.1: Data set distribution.

Class	**No. of items**	**No. of periods**	**Length variation of the items**		$\mathcal{F}$
1	5	5	Small	$[0.6, 2.8]$	1.5
2					2
3			Varied	$[0.6, 5]$	1.5
4					2
5		10	Small	$[0.6, 2.8]$	1.5
6					2
7			Varied	$[0.6, 5]$	1.5
8					2
9	10	5	Small	$[0.6, 2.8]$	1.5
10					2
11			Varied	$[0.6, 5]$	1.5
12					2
13		10	Small	$[0.6, 2.8]$	1.5
14					2
15			Varied	$[0.6, 5]$	1.5
16					2

Considering data retrieved from a Brazilian lattice slab factory, for all instances, the following parameters are set: the length of the mold W^C is 10 meters; the standard length of the steel truss W^S is 12 meters; each mold is divided in $N = 10$ identical sections, measuring 10 m $\times$ 0.12 m $\times$ 0.03 m; and the demand of each item randomly varies between 50 and 150 units in every period.

The following information is based on a search over Brazilian specialized websites and online stores accessed on April 30, 2021. The estimated average cost of concrete was BRL 440.00 $/m^3$, while the cost of a steel truss of 12 meters in length was approximately BRL 132.00 per unit. This information was used to determine the variable costs. Representing the cost of the concrete volume necessary to fill a meter of a mold section length, we define $c_p^C = 1.6$, and the cost per meter of steel truss is $c_p^S = 11$. The setup costs are predefined according to $sc_p^{mold} = 2 \cdot W^C = 20$ and $sc_p^{pat} = 0.1 \cdot sc_p^{mold} = 2$ for each $p \in P$. Since storage is limited, the inventory penalty costs in this study represent a percentage of the total length of each produced item or cut steel truss: $h_{ip}^C = 0.5 \cdot w_i$, and $h_{ip}^S = 0.25 \cdot w_i$ for $i \in I$ and $p \in P$.

In Tables 4.2, 4.3 and 4.2, specific solution information is presented for each class. Table 4.2 presents the number of instances from each class for which a feasible solution (F) or an unknown solution status (U) (the solver did not find a feasible solution in the given time) was obtained.

It is worth mentioning that, in all 80 instances and all solution methods applied to the model studied here, the column generation procedure was stopped after not achieving a significant improvement in the relaxed objective value for three consecutive iterations (tailing-off). For this reason, no solution obtained can be considered optimal for the relaxed problem. This condition generally occurs when there are degenerate solutions. Since the reduced cost tolerance criterion is satisfied, the column generation procedure continues, but the RMP objective value scarcely improves, causing tailing-off. A mechanism based on duality was applied to get through the tailing-off difficulty. According to Valério de Carvalho (2005), the insertion of a polynomial number of dual cuts in the problem when applying a column generation-based solution strategy can reduce the number of columns generated, degenerate iterations and processing time. Inopportunely, the model and data structures used in our research inhibited the operation of the dual cuts strategy and another approach was chosen. In this case, a stop criterion was implemented to discontinue the column generation procedure when the RMP objective function improvement was lower than a tolerance (0.001) in three consecutive iterations.

Moreover, considering the linear relaxation solution OF^r in an iteration r (RMP at iteration r), a lower bound (LB) was calculated based on the approach presented by Degraeve and Jans (2007) (see also Desaulniers et al., 2005):

$$LB = OF^r + \sum_{p \in P} \left(OF_{SUB}^S(p) + OF_{SUB}^C(p) \right). \tag{4.33}$$

Also, for each mixed-integer solution (UB, upper bound), the 'GAP' is calculated by the following formula:

$$GAP = 100 \cdot \frac{UB - LB}{UB}\%. \tag{4.34}$$

In Tables 4.2 and 4.3, the processing times are expressed in seconds, the lower and upper bounds are in currency units and the GAPs are percentages. More details are described according to necessity in the next section.

4.4.2 General analysis of the results

This section presents an overall analysis of the 3 strategies applied to the studied model. Figure 4.4 shows the number of instances each method found a feasible solution for. The ICG obtained feasible solutions for 78 instances while its two-step version found feasible solutions for 71 instances. On the other hand, SCG obtained feasible solutions for all 80 instances. The efficiency with which SCG obtains a feasible solution is a consequence of the separation of stages to apply the column generation, which makes the resultant problems simpler to be solved. Observe that the unknown solution status can be seen in Figure 4.4 from classes in the data set distribution that manages 10 different item lengths. This solution status becomes more common where the number of periods in the time horizon is 10.

Note that in Tables 4.2 and 4.3, only the solution of instances that are feasible in all proposed methods are used to calculate the average for each class and the final result is presented for every class. Since all instances of Class 14 presented an unknown solution status either in ICG or its two-step version, no averages can be presented to this class. The values in bold in Table 4.2 indicate the lowest processing time and the lowest average GAPs of each class between the applied methods. This table also shows the average processing time performed by each solution method: ICG reached the predetermined time limit of 1,800 seconds in most instances, while its two-step version performed in 1070.17 seconds on average, and SCG operated in 1171.06 seconds on average. Note that, in SCG strategy, the Stage 1 master problem is not a relaxed formulation of the original problem, since this method restrains its feasible space using the Stage 2 solution. For this reason, the Equation (4.33) is not valid to calculate a lower bound using the SCG values. In this case, the GAPs are calculated using the ICG lower bounds (see Equation (4.34)), which are presented in Table 4.2. Observe that, once again, the traditional ICG approach obtained the largest number of the lowest GAPs of each class. In addition, the overall average GAPs floated from 0.21% to 1.80%.

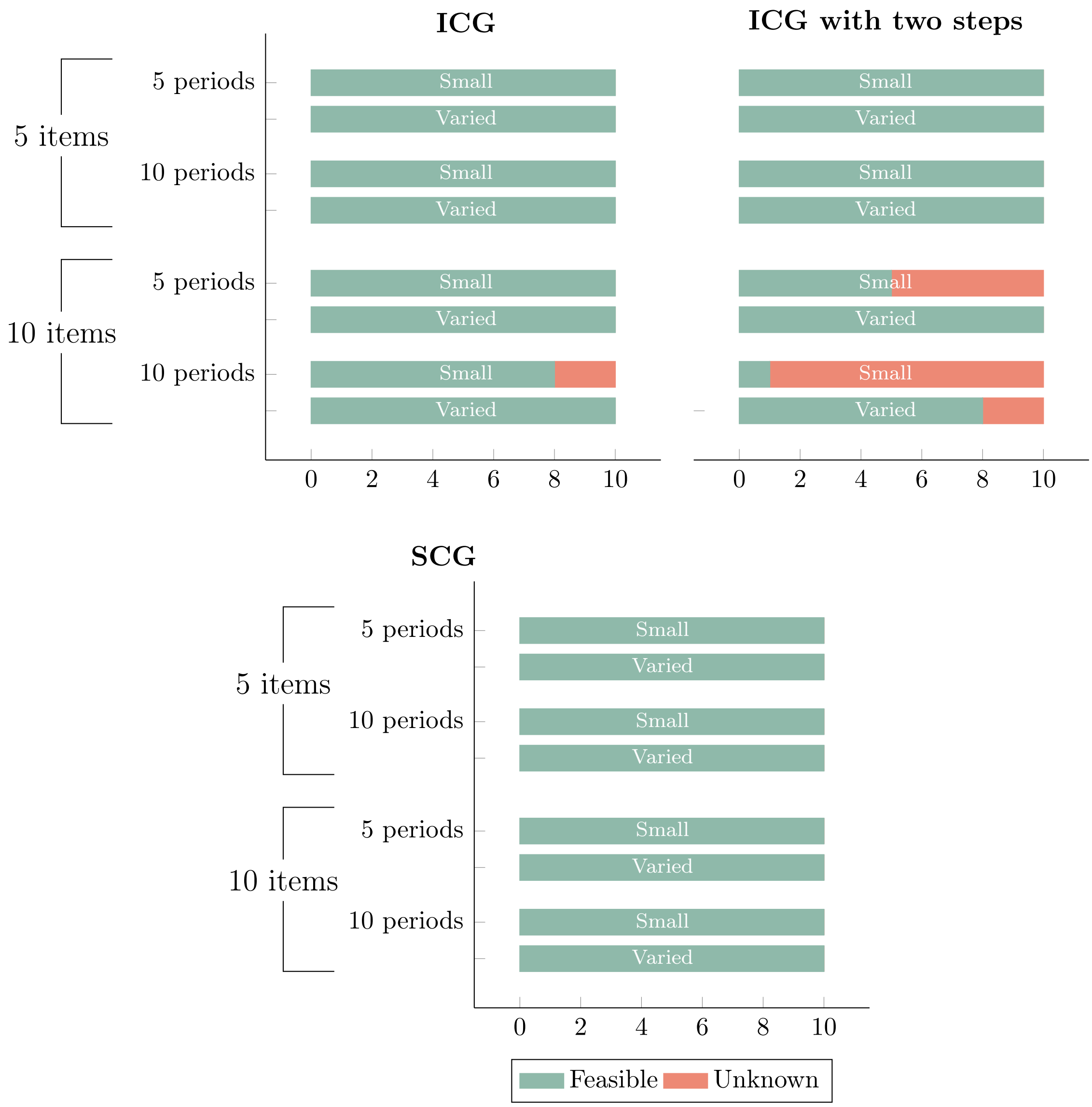


Figure 4.4: Bar chart representing the solution status distribution of the ICG, ICG with two steps and SCG strategies.

Table 4.3 presents the lower bounds obtained by ICG strategy and the upper bounds of each method. The values in bold correspond to the lowest upper bounds of each class. Observe that the traditional ICG obtained the largest number of the lowest upper bound values in each class, although SCG obtained the lowest upper bound average.

Summarizing the results, one can infer that the SCG strategy was the best for obtaining feasible solutions, while the ICG with two steps performed in a shorter time on average. Although the two-step method operated faster than the traditional ICG, the latter obtained the best upper bounds and GAPs. Note that the ICG strategies clearly provide solutions of higher quality, while the SCG approach is better at obtaining feasible solutions, mostly in larger (or more complex) instances. In general, the column generation procedure applied to the studied model was certainly profitable, yielding high-quality feasible solutions.

Table 4.2: General analysis of the results: processing time and GAP.

Class	**Processing Time**			**GAP**		
	ICG	**ICG with two steps**	**SCG**	**ICG**	**ICG with two steps**	**SCG**
1	1798.65	1616.94	**1259.24**	**0.40**	0.44	0.56
2	1798.87	**1303.86**	1618.82	0.42	**0.41**	0.60
3	1799.83	**1079.02**	1083.69	**0.24**	0.25	0.41
4	1799.45	**1067.48**	1259.07	**0.28**	**0.28**	0.42
5	1799.98	1077.94	**915.76**	**0.37**	0.38	0.39
6	1800.00	**1081.35**	1088.35	**0.38**	0.40	0.42
7	1800.00	900.83	**899.45**	**0.21**	0.22	0.28
8	1800.00	900.16	**900.06**	**0.23**	0.24	0.28
9	1800.00	**929.33**	1798.45	0.63	0.75	**0.49**
10	1800.00	**1133.04**	1162.40	0.64	0.67	**0.39**
11	1800.00	904.04	**900.36**	0.24	**0.21**	0.22
12	1800.00	**1076.50**	1080.27	0.27	**0.26**	0.27
13	1800.00	**1168.25**	1798.97	1.35	1.80	**0.45**
14	-	-	-	-	-	-
15	1800.00	906.14	**900.48**	**0.22**	0.33	0.23
16	1800.00	907.61	**900.59**	0.42	**0.25**	**0.25**
Averages	1799.78	1070.17	1171.06	0.42	0.46	0.38

Table 4.3: General analysis of the results: lower and upper bounds.

Class	**Lower bound**	**Upper bound**		
	ICG	**ICG**	**ICG with two steps**	**SCG**
1	64224.62	**64484.38**	64513.22	64583.10
2	64219.22	64492.00	**64485.32**	64606.96
3	92084.10	**92304.84**	92314.44	92463.40
4	92077.20	92339.46	**92339.36**	92461.00
5	110710.66	**111104.74**	111130.56	111138.44
6	110698.96	**111110.26**	111132.16	111153.02
7	175024.80	**175382.40**	175406.40	175500.20
8	175010.00	**175397.60**	175426.60	175494.00
9	93949.40	94546.50	94656.80	**94408.50**
10	109121.35	109805.15	109840.33	**109544.60**
11	175830.60	176244.40	**176199.60**	176208.40
12	175816.20	176282.00	**176269.20**	176289.60
13	192509.00	195138.00	196042.00	**193375.00**
14	-	-	-	-
15	373022.75	**373808.25**	374204.50	373884.25
16	379855.25	381372.25	**380789.75**	380805.00
Average	158943.61	159587.48	159650.02	159461.03

4.4.3 Additional analysis of the results

Further information was obtained in the computational tests and is displayed in Tables 4.4 and 4.5 and Figure 4.5. Unlike Subsection 4.4.2, where the values are presented for each class, in this subsection, the values represent the average between all classes, considering only instances in which all 3 methods obtained feasible solutions. Additionally, in each of these tables there is a 'Overall average' column containing the average between all classes and solution method results, providing an overview of the results.

Table 4.4 presents, for each solution strategy, the percentages of each type of cost that composes the mixed-integer objective value: item and steel truss inventory penalty costs, mold section and steel truss production costs, mold and cutting pattern setup costs. Figure 4.5 illustrates the overall average of distribution of costs. It is observed that the overall average of 85.52% of the objective function is allocated to steel truss production cost, since the cost of a steel truss of standard length is very high compared to the other costs. About 12.53% of the objective function is directed to mold section production cost, while 1.61% is used in mold setup cost, and only 0.34% of the objective function is assigned to costs related to item inventory penalty, steel truss inventory and cutting patterns setup.

Table 4.5 gives the values of inventory level, and mold and steel truss usage that help to understand the percentages indicated in Table 4.4. A relatively homogeneous distribution of costs is seen in Table 4.4.

However, there is a contrasting quantity concerning the inventory levels in Table 4.5. The SCG strategy obtained feasible solutions producing half or less than half the number of inventory items yielded in ICG strategies. Table 4.5 also presents the mold availability and usage per period, together with a space metric comparison of available mold, used mold and production. The total space of the available molds (column 'Available space'), the space of the molds used in production (column 'Used mold space'), and the total space of the concrete planks production (column 'Production space') are expressed in meters. On average, about half of the total number of available molds per period were used. Related to the used mold space, the average productive space was 98.27%, indicating very satisfactory space management by the proposed solution methods.

Table 4.4: Additional analysis of the results: distribution of costs.

	Methods		
	ICG	**ICG with two steps**	**SCG**
Item inventory penalty cost (%)	0.08	0.15	0.04
Steel truss inventory penalty cost (%)	0.05	0.05	0.05
Steel truss production cost (%)	85.52	85.43	85.60
Mold section production cost (%)	12.52	12.55	12.52
Mold setup cost (%)	1.62	1.61	1.60
Cutting patterns setup cost (%)	0.20	0.21	0.20

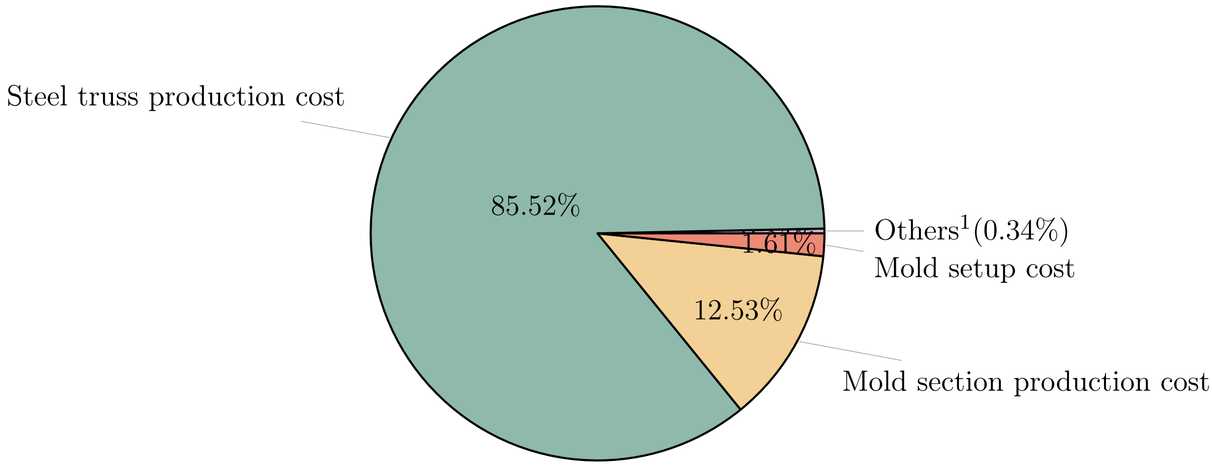


Figure 4.5: Distribution of costs: overall averages between all results of ICG, ICG with two steps and SCG strategies.

[1]Item inventory penalty cost, steel truss inventory penalty cost and cutting patterns setup cost.

Table 4.5: Additional analysis of the results: inventory level, use of steel truss, mold availability and usage.

	Methods ICG	ICG with two steps	SCG	Overall average
No. of inventory items	173.03	275.90	67.48	172.14
No. of inventory steel trusses	205.26	152.19	126.14	161.20
No. of used steel trusses	1033.50	1032.97	1032.96	1033.14
Average No. of used steel trusses per period	140.98	140.91	140.91	140.94
No. of mold sections used	1251.58	1253.67	1250.38	1251.88
No. of mold setups	129.10	128.41	128.02	128.51
No. of cutting pattern setups	160.60	162.40	156.41	159.80
Average No. of available molds per period	34.28	34.28	34.28	34.28
Average No. of used molds per period	17.60	17.50	17.43	17.51
Available space (AS)	30442	30442	30442	30442
Used mold space (UMS)	12515.83	12536.73	12503.77	12518.78
Production space (PS)	12280.75	12277.27	12276.96	12278.33
PS/AS (%)	41.18	41.16	41.16	41.17
PS/UMS (%)	98.34	98.08	98.38	98.27

4.5 Conclusions and future research for this model

In this chapter, a mathematical model is presented for multi-period cutting stock problems with two stages, representing the cutting of steel trusses and the production of concrete planks, looking at the production planning of lattice joists in a Brazilian factory. Based on the extended formulation of the cutting stock problem and considering the characteristics of the factory, the model contemplates the availability of different cutting patterns to be used in multiple sections of the same mold and period. To satisfy the main concerns of the factory, the proposed model minimizes the setup and production costs, while managing inventory. Another contribution of this chapter is the proposal of different solution strategies based on the column generation procedure. These strategies explore the two-stage feature that led to the use of two sets of subproblems, which provide cutting patterns for each cutting stage. This allowed analysis of the applicability of each approach to the model. Based on the information retrieved from the factory, the computational results suggest a very positive evaluation of the column generation procedure for solving the model, providing interesting information, like high-quality GAPs and low rates of waste, that encourage its application in the factory.

A possible opportunity for extending this specific study is to incorporate other production processes, such as packing and distribution of final products or acquisition of raw materials since steel trusses represent the largest portion of the production cost. For this, it may be necessary to apply other modeling approaches, for example, an exact formulation based on the arc flow model presented by Wolsey (1977) and Valério de Carvalho (1999, 2002). The implementation of alternative ways of generating interesting columns and the use of stabilization techniques are also possibilities for future work. Applying this model in practice could provide an extensive understanding to be used in decision-making.

5 Conclusion and future research

The main objective of this research was to develop mathematical-computational tools that assist the decision making for the production planning of hollow-core slabs and lattice slabs motivated by partnership with precast concrete producers in Brazil.

Considering the production planning of hollow-core slabs at a specific plant, two models based on compact and extended formulations of the one-dimensional multi-period cutting stock problems were proposed in the first part of this thesis. Both models aim at the minimization of inventory penalty and production costs (mainly related to the use of steel wires while satisfying demand balance and capacity constraints). Also, both models handle with wire configurations more comprehensively, since they foresee the combination of items with different wire configurations in the same production line provided that the wire configuration of a produced item is equal or higher than that required. To the best of our knowledge, this feature has not appeared in other studies. To solve the compact model KMPCSP, the MILP approach and the Relax-and-fix heuristic were applied, while a strategy based on the column generation procedure was used to solve the extended model GGMPCSP. The computational results used fictional instances generated based on real data from the plant. A better efficiency of the compact model compared to the extended model was observed. This unusual result arises from some particularities of this problem that produces a demand matrix and cutting pattern matrix of higher dimensions, which increases the complexity in generating interesting columns, forcing the hardware. It was also noted that bin-packing based models (compact formulation) are generally considered by studies where the average demand for items is low and a weak lower bound to the integer problem is produced, along with symmetrical structures, decreasing the efficiency of branch-and-bound methods. On the other hand, models based on the cutting stock problems (extended formulation) perform better in cases where the average demand for items is high, granting a tight linear relaxation and stronger lower bounds through the elimination of some fractional solutions and removal of symmetries. In this study, the low demand rate in the instances tested caused a drawback in the performance of the column generation procedure used to solve the extended formulation, provoking its low efficiency. Furthermore, a theoretical analysis of the compact and extended formulations was argued.

Another widely used precast slab in Brazil is the lattice slab, the focus of the second part of this research. Considering the production planning of lattice joists (which compose the slab) and keeping in view the central interests of the factory (the minimization of setup and production costs and waste of steel trusses, while managing the inventory level and increasing productivity), a model based on the compact formulation for the multi-period cutting stock problem with two stages was proposed. The first stage comprehends the cutting of steel trusses of standard length into smaller pieces, while the second stage involve the division of the molds with separators, outlining the placement of each item. Since this model optimizes cutting stock decisions in a multi-period and two-stage scenario concomitantly, to the best of our knowledge, it provides an innovative addition to the related literature. An additional contribution of this research is the proposal of three different solution strategies based on the column generation procedure that explore the two-stage feature, where two groups of subproblems are used to deal with the two cutting processes involved. The computational results used fictional instances based on existing orders from the factory in focus, and provide a very positive evaluation that supports the application of such research in the real-world problem.

Opportunities for further research include the incorporation of other production processes, e.g., the acquisition of raw material, packing, and distribution of final products. Readjustments in the proposed models and the use of alternative mathematical models, such as the arc-flow formulation, might be necessary. Other possibilities to extend this research are improving the solution methods addressed here, while comparing their results to others existent in the literature, implementing different techniques to generate interesting columns, and exploring combinations of different heuristic approaches.

References

K. Akartunalı and A. J. Miller. A heuristic approach for big bucket multi-level production planning problems. *European Journal of Operational Research*, 193(2):396–411, 2009. doi: 10.1016/j.ejor.2007.11.033.

T. Aktin and R. G. Özdemir. An integrated approach to the one-dimensional cutting stock problem in coronary stent manufacturing. *European Journal of Operational Research*, 196(2):737–743, 2009. doi: 10.1016/j.ejor.2008.04.005.

A. Aliano Filho, A. C. Moretti, and M. V. Pato. A comparative study of exact methods for the bi-objective integer one-dimensional cutting stock problem. *Journal of the Operational Research Society*, 69:91–107, 2018. doi: 10.1057/s41274-017-0214-7.

P. R. d. L. Andrade, S. A. de Araujo, A. C. Cherri, and F. K. Lemos. The integrated lot sizing and cutting stock problem in an automotive spring factory. *Applied Mathematical Modelling*, 91:1023 – 1036, 2021. ISSN 0307-904X. doi: 10.1016/j.apm.2020.10.033.

K. A. G. Araújo, T. O. Bonates, and B. A. Prata. The integrated cutting and packing heterogeneous precast beams multiperiod production planning problem. *RAIRO-Oper. Res.*, 55(4):2491–2524, 2021a. doi: 10.1051/ro/2021107.

K. A. G. Araújo, T. O. Bonates, B. A. Prata, and A. R. Pitombeira-Neto. Heterogeneous prestressed precast beams multiperiod production planning problem: modeling and solution methods. *TOP*, 29:660–693, 2021b. doi: 10.1007/s11750-020-00589-4.

T. A. Baldo, M. O. Santos, B. Almada-Lobo, and R. Morabito. An optimization approach for the lot sizing and scheduling problem in the brewery industry. *Computers & Industrial Engineering*, 72:58–71, 2014. doi: 10.1016/j.cie.2014.02.008.

A. A. R. Barbosa and M. Vilnītis. Innovation and construction management in brazil: Challenges of companies in times of quality and productivity. In *IOP Conference Series: Materials Science and Engineering*, volume 251, page 012040. IOP Publishing, 2017. doi: 10.1088/1757-899X/251/1/012040.

B. S. C. Campello, C. T. L. S. Ghidini, A. O. C. Ayres, and W. A. Oliveira. A multiobjective integrated model for lot sizing and cutting stock problems. *Journal of the Operational Research Society*, 71(9):1466–1478, 2020. doi: 10.1080/01605682.2019.1619892.

CBS Precast Limited/3D Warehouse. 150mm standard hollowcore plank, 2015. URL `<https://3dwarehouse.sketchup.com/model/ue741ed05-16f8-43c7-9669-7efc74f9742b/150mm-standard-Hollowcore-plank>`.

M. M. Christofoletti, S. A. de Araujo, and A. C. Cherri. Integrated lot-sizing and cutting stock problem applied to the mattress industry. *Journal of the Operational Research Society*, 0(0):1–15, 2020. doi: 10.1080/01605682.2020.1718013.

M. Correia, J. F. Oliveira, and J. Ferreira. Reel and sheet cutting at a paper mill. *Computers Operations Research*, 31(8):1223–1243, 2004. ISSN 0305-0548. doi: 10.1016/S0305-0548(03)00076-5.

Y. Cui, X. Song, Y. Chen, and Y.-P. Cui. New model and heuristic solution approach for one-dimensional cutting stock problem with usable leftovers. *Journal of the Operational Research Society*, 68(3):269–280, 2017. doi: 10.1057/s41274-016-0098-y.

S. A. de Araujo and A. Clark. A priori reformulations for joint rolling-horizon scheduling of materials processing and lot-sizing problem. *Computers Industrial Engineering*, 65(4):577 – 585, 2013. ISSN 0360-8352. doi: 10.1016/j.cie.2013.04.003.

M. K. Debs. *Concreto Pré-Moldado: Fundamentos e Aplicações. Oficina de Textos*. EESC-USP, 2000.

Z. Degraeve and R. Jans. A new dantzig-wolfe reformulation and branch-and-price algorithm for the capacitated lot-sizing problem with setup times. *Operations research*, 55(5):909–920, 2007. doi: 10.1287/opre.1070.0404.

G. Desaulniers, J. Desrosiers, and M. M. Solomon. *Column Generation*. Springer, Boston, MA, 2005. doi: 10.1007/b135457.

H. Dyckhoff. A new linear programming approach to the cutting stock problem. *Operations research*, 29(6):1092–1104, 1981. doi: 10.1287/opre.29.6.1092.

H. Dyckhoff. A typology of cutting and packing problems. *European Journal of Operational Research*, 44(2):145–159, 1990. ISSN 0377-2217. doi: https://doi.org/10.1016/0377-2217(90)90350-K. Cutting and Packing.

Elecosoft PLC. Prestressed Concrete - Milbury Systems, 2010. URL <`https://youtu.be/0z6gjjrSn0M`>.

D. Ferreira, R. Morabito, and S. Rangel. Solution approaches for the soft drink integrated production lot sizing and scheduling problem. *European Journal of Operational Research*, 196(2):697–706, 2009. doi: 10.1016/j.ejor.2008.03.035.

D. Ferreira, R. Morabito, and S. Rangel. Relax and fix heuristics to solve one-stage one-machine lot-scheduling models for small-scale soft drink plants. *Computers & Operations Research*, 37(4):684–691, 2010. doi: 10.1016/j.cor.2009.06.007.

D. J. Fiorotto, R. Jans, and S. A. de Araujo. An analysis of formulations for the capacitated lot sizing problem with setup crossover. *Computers & Industrial Engineering*, 106:338–350, 2017. doi: 10.1016/j.cie.2016.12.037.

D. J. Fiorotto, J. d. C. Huaccha Neyra, and S. A. de Araujo. Impact analysis of setup carryover and crossover on lot sizing problems. *International Journal of Production Research*, 58(20):6350–6369, 2020. doi: 10.1080/00207543.2019.1680892.

H. Foerster and G. Wascher. Pattern reduction in one-dimensional cutting stock problems. *International Journal of Production Research*, 38(7):1657–1676, 2000. doi: 10.1080/002075400188780.

P. C. Gilmore and R. E. Gomory. A linear programming approach to the cutting-stock problem. *Operations research*, 9(6):849–859, 1961. doi: 10.1287/opre.9.6.849.

P. C. Gilmore and R. E. Gomory. A linear programming approach to the cutting stock problem – Part II. *Operations research*, 11(6):863–888, 1963. doi: 10.1287/opre.11.6.863.

P. C. Gilmore and R. E. Gomory. Multistage cutting stock problems of two and more dimensions. *Operations research*, 13(1):94–120, 1965. doi: 10.1287/opre.13.1.94.

A. M. Gomes, J. F. Gonçalves, R. Alvarez-Valdés, and J. V. de Carvalho. Preface to the special issue on cutting and packing. *International Transactions in Operational Research*, 23(1-2):3–4, 2016. doi: https://doi.org/10.1111/itor.12208.

M. C. N. Gramani and P. França. The combined cutting stock and lot sizing problem in industrial process. *European Journal of Operational Research*, 174(1):509–521, 2006. doi: 10.1016/j.ejor.2004.12.019.

O. Husein and E. Softić. Prestressed concrete beams production process. pages 351–356, 2013. URL `https://pdfs.semanticscholar.org/deff/e7fc3d65ad8ebc608686606741ac96edaf33.pdf`.

International Prestressed Hollowcore Association. Hollowcore slab production, 2020. URL `<https://hollowcore.org/hollowcore/production/>`.

M. Jünger, T. M. Liebling, D. Naddef, G. L. Nemhauser, W. R. Pulleyblank, G. Reinelt, G. Rinaldi, and L. A. Wolsey. *50 Years of Integer Programming 1958-2008: From the Early Years to the State-of-the-Art*. Springer Science & Business Media, 2009. doi: 10.1007/978-3-540-68279-0.

L. V. Kantorovich. Mathematical methods of organizing and planning production. *Management Science*, 6(4):366–422, 1960. doi: 10.1287/mnsc.6.4.366.

B.-I. Kim, Y. Ki, D. Son, B. Bae, and J.-S. Park. An algorithm for a cutting problem in window frame production. *International Journal of Production Research*, 54(14): 4327–4339, 2016. doi: 10.1080/00207543.2016.1148279.

E. S. Kokten and Ç. Sel. A cutting stock problem in the wood products industry: a two-stage solution approach. *International Transactions in Operational Research*, 00:1–29, 2020. doi: 10.1111/itor.12802.

A. A. Leao, M. M. Furlan, and F. M. Toledo. Decomposition methods for the lot-sizing and cutting-stock problems in paper industries. *Applied Mathematical Modelling*, 48: 250–268, 2017. doi: 10.1016/j.apm.2017.04.010.

F. K. Lemos, A. C. Cherri, and S. A. de Araujo. The cutting stock problem with multiple manufacturing modes applied to a construction industry. *International Journal of Production Research*, 59(4):1088–1106, 2021. doi: 10.1080/00207543.2020.1720923.

G. M. Melega, S. A. de Araujo, and R. Jans. Classification and literature review of integrated lot-sizing and cutting stock problems. *European Journal of Operational Research*, 271(1):1–19, 2018. doi: 10.1016/j.ejor.2018.01.002.

G. M. Melega, S. A. de Araujo, and R. Morabito. Mathematical model and solution approaches for integrated lot-sizing, scheduling and cutting stock problems. *Annals of Operations Research*, 295:695–736, 2020. doi: 10.1007/s10479-020-03764-9.

R. A. Melo and C. C. Ribeiro. Formulations and heuristics for the multi-item uncapacitated lot-sizing problem with inventory bounds. *International Journal of Production Research*, 55(2):576–592, 2017. doi: 10.1080/00207543.2016.1215567.

C. Mercé and G. Fontan. Mip-based heuristics for capacitated lotsizing problems. *International Journal of Production Economics*, 85(1):97–111, 2003. doi: 10.1016/S0925-5273(03)00090-2.

R. Morabito and M. Arenales. Optimizing the cutting of stock plates in a furniture company. *International Journal of Production Research*, 38(12):2725–2742, 2000. doi: 10.1080/002075400411457.

R. Morabito, M. N. Arenales, and H. H. Yanasse. Special issue on cutting, packing and related problems. *International Transactions in Operational Research*, 16(6): 659–659, 2009. doi: 10.1111/j.1475-3995.2009.00739.x.

I. Muter and Z. Sezer. Algorithms for the one-dimensional two-stage cutting stock problem. *European Journal of Operational Research*, 271(1):20–32, 2018. doi: 10.1016/j.ejor.2018.04.04.

D. N. Nascimento, A. N. Cherri, and S. A. de Araujo. Integrated lot sizing and one-dimensional cutting stock problem with usable leftovers. *Annals of Operations Research (accepted)*, 2020. doi: 10.1007/s10479-020-03772-9.

W. A. Oliveira, D. J. Fiorotto, X. Song, and D. F. Jones. An extended goal programming model for the multiobjective integrated lot-sizing and cutting stock problem. *European Journal of Operational Research*, 2021. ISSN 0377-2217. doi: https://doi.org/10.1016/j.ejor.2021.03.049.

K. C. Poldi and M. N. Arenales. O problema de corte de estoque unidimensional multiperíodo. *Pesquisa Operacional*, 30(1):153–174, 2010. doi: 10.1590/S0101-74382010000100008.

B. A. Prata, A. R. Pitombeira-Neto, and C. J. M. Sales. An integer linear programming model for the multiperiod production planning of precast concrete beams. *Journal of Construction Engineering and Management*, 141(10):04015029, 2015. doi: 10.1061/(ASCE)CO.1943-7862.0000991.

H. Reinertsen and T. W. Vossen. The one-dimensional cutting stock problem with due dates. *European Journal of Operational Research*, 201(3):701–711, 2010. ISSN 0377-2217. doi: 10.1016/j.ejor.2009.03.042.

A. Respício and M. E. Captivo. Integrating the cutting stock problem in capacity planning. *Department of Informatics and Centre of Operational Research, University of Lisbon, Portugal (2002)*, 2002.

J. L. Santos, J. Santos, M. J. Ferreira, N. Alves, and M. Guevara. Application of the two-stage one-dimensional cutting stock problem in the steel industry. In *2018 IEEE 27th International Symposium on Industrial Electronics (ISIE)*, pages 683–690, 2018. doi: 10.1109/ISIE.2018.8433734.

M. W. P. Savelsbergh. *Branch and price: Integer programming with column generation*. Springer US, Boston, MA, 2008. doi: 10.1007/978-0-387-74759-0_58.

M. Savsar and C. Cogun. Analysis and modelling of a production line in a corrugated box factory. *International Journal of Production Research*, 32(7):1571–1589, 1994. doi: 10.1080/00207549408957023.

C. A. Signorini, S. A. de Araujo, and G. M. Melega. One-dimensional multi-period cutting stock problems in the concrete industry. *International Journal of Production Research*, 0(0):1–18, 2021. doi: 10.1080/00207543.2021.1890261.

W. A. O. Soler, M. O. Santos, and K. Akartunalı. Mip approaches for a lot sizing and scheduling problem on multiple production lines with scarce resources, temporary workstations, and perishable products. *Journal of the Operational Research Society*, 0(0):1–16, 2019. doi: 10.1080/01605682.2019.1640588.

H. Stadtler. Multilevel lot sizing with setup times and multiple constrained resources: Internally rolling schedules with lot-sizing windows. *Operations Research*, 51(3): 487–502, 2003. doi: 10.1287/opre.51.3.487.14949.

A. Steinle, H. Bachmann, and M. Tillmann. *Precast concrete structures*. Wiley Online Library, 2019. doi: 10.1002/9783433609064.

The Prestressed Group. Prestressed and precast concrete, 2020. URL <http://www.theprecaster.com/prestressed-concrete-precast-concrete/>.

E. A. Toso, R. Morabito, and A. R. Clark. Lot sizing and sequencing optimisation at an animal-feed plant. *Computers & Industrial Engineering*, 57(3):813–821, 2009. doi: 10.1016/j.cie.2009.02.011.

P. Trkman and M. Gradisar. One-dimensional cutting stock optimization in consecutive time periods. *European Journal of Operational Research*, 179(2):291–301, 2007. doi: 10.1016/j.ejor.2006.03.027.

J. M. V. Valério de Carvalho. Exact solution of bin-packing problems using column generation and branch-and-bound. *Annals of Operations Research*, 86:629–659, 1999. doi: 10.1023/A:1018952112615.

J. M. V. Valério de Carvalho. LP models for bin packing and cutting stock problems. *European Journal of Operational Research*, 41(2):253–273, 2002. doi: 10.1016/S0377-2217(02)00124-8.

J. M. V. Valério de Carvalho. Using extra dual cuts to accelerate column generation. *INFORMS Journal on Computing*, 17(2):175–182, 2005. doi: 10.1287/ijoc.1030.0060.

P. H. Vance. Branch-and-price algorithms for the one-dimensional cutting stock problem. *Computational Optimization and Applications*, 9(3):211–228, 1998. ISSN 0926-6003. doi: 10.1023/A:1018346107246.

M. Vanzela, G. M. Melega, S. Rangel, and S. A. de Araujo. The integrated lot sizing and cutting stock problem with saw cycle constraints applied to furniture production. *Computers Operations Research*, 79:148 – 160, 2017. ISSN 0305-0548. doi: 10.1016/j.cor.2016.10.015.

A. H. D. Vassoler, S. C. Poltroniere, and S. A. de Araujo. Modelagem matemática para o problema de produção de vigotas na indústria de lajes treliçadas. *C. Q. D. Revista Eletrônica Paulista de Matemática*, 7:68–77, 2016. doi: 10.21167/cqdvol7ermac201623169664ahdvscpsar6877.

B. Verlinden, D. Cattrysse, and D. V. Oudheusden. Integrated sheet-metal production planning for laser cutting and bending. *International Journal of Production Research*, 45(2):369–383, 2007. doi: 10.1080/00207540600658062.

P. Y. Wang and G. Wäscher. Editorial on cutting and packing. *European Journal of Operational Research*, 141(2):239–240, 2002.

W. Wang, Z. Shi, L. Shi, and Q. Zhao. Integrated optimisation on flow-shop production with cutting stock. *International Journal of Production Research*, 57(19):5996–6012, 2019. doi: 10.1080/00207543.2018.1556823.

G. Wäscher, H. Haußner, and H. Schumann. An improved typology of cutting and packing problems. *European Journal of Operational Research*, 183(3):1109–1130, 2007. doi: 10.1016/j.ejor.2005.12.047.

L. A. Wolsey. Valid inequalities, covering problems and discrete dynamic programs. In P. Hammer, E. Johnson, B. Korte, and G. Nemhauser, editors, *Studies in Integer Programming*, volume 1 of *Annals of Discrete Mathematics*, pages 527–538. Elsevier, 1977. doi: 10.1016/S0167-5060(08)70758-1.

H. H. Yanasse and R. Morabito. Linear models for 1-group two-dimensional guillotine cutting problems. *International Journal of Production Research*, 44(17):3471–3491, 2006. doi: 10.1080/00207540500478603.

İhsan Erozan and E. Çalışkan. A multi-objective genetic algorithm for a special type of 2d orthogonal packing problems. *Applied Mathematical Modelling*, 77:66–81, 2020. ISSN 0307-904X. doi: 10.1016/j.apm.2019.07.010.

Appendix A - Numerical example for the hollow-core slab production planning

In this appendix, an algebraic example is presented to illustrate the application of the MILP and RF strategies to the KMPCSP, and the CG procedure to the GGMPCSP proposed in Chapter 3.

This instance is based on the information presented in a fictitious order form described in Table A.1, where there are $|I|= 2$ items with $|T|= 3$ types of wire configuration that must be produced in $|P|= 5$ periods. In addition, suppose that:

i. in each period $p \in P$, there are available $|K_p|= 3$ molds to produce hollow-core slabs;

ii. each available mold measures $W = 13$ meters in length;

iii. the number of steel wires in each wire configuration τ is given by

$$f_\tau = \begin{cases} 1, & \text{if } \tau = 1, \\ 3, & \text{if } \tau = 2, \\ 8, & \text{if } \tau = 3; \end{cases}$$

iv. the cost of a meter of steel wire is fixed in $c = 100$ currency units;

v. the cost of stocking a hollow-core slab/item in currency units is calculated based on the length of the item:

$$h_{ip} = 0.5 \cdot w_i = \begin{cases} 2, & \text{if } i = 1, \\ 1.5, & \text{if } i = 2 \end{cases} \qquad \forall p.$$

Table A.1: Order form for the hollow-core slabs production planning numerical example.

Items length		$w_1 = 4$	$w_2 = 3$
Demanded quantities ($d_{i\beta p}$)	**Period 1**	$d_{111} = 1$	$d_{211} = 2$
		$d_{121} = 0$	$d_{221} = 1$
		$d_{131} = 1$	$d_{231} = 0$
	Period 2	$d_{112} = 0$	$d_{212} = 3$
		$d_{122} = 2$	$d_{222} = 2$
		$d_{132} = 0$	$d_{232} = 1$
	Period 3	$d_{113} = 3$	$d_{213} = 0$
		$d_{123} = 0$	$d_{223} = 0$
		$d_{133} = 3$	$d_{233} = 0$
	Period 4	$d_{114} = 0$	$d_{214} = 2$
		$d_{124} = 2$	$d_{224} = 1$
		$d_{134} = 0$	$d_{234} = 2$
	Period 5	$d_{115} = 3$	$d_{215} = 4$
		$d_{125} = 0$	$d_{225} = 6$
		$d_{135} = 2$	$d_{235} = 0$

Applying the example to the KMPCSP

Considering the previous information, the mathematical model KMPCSP for this example is presented as follows:

$$
\begin{aligned}
\text{minimize}\quad & 2 \cdot S_{111} + 2 \cdot S_{112} + 2 \cdot S_{113} + 2 \cdot S_{114} + 200 \cdot S_{115} + 2 \cdot S_{121} + 2 \cdot S_{122} \\
& + 2 \cdot S_{123} + 2 \cdot S_{124} + 200 \cdot S_{125} + 2 \cdot S_{131} + 2 \cdot S_{132} + 2 \cdot S_{133} + 2 \cdot S_{134} \\
& + 200 \cdot S_{135} + 1.5 \cdot S_{211} + 1.5 \cdot S_{212} + 1.5 \cdot S_{213} + 1.5 \cdot S_{214} + 150 \cdot S_{215} + 1.5 \cdot S_{221} + 1.5 \cdot S_{222} \\
& + 1.5 \cdot S_{223} + 1.5 \cdot S_{224} + 150 \cdot S_{225} + 1.5 \cdot S_{231} + 1.5 \cdot S_{232} + 1.5 \cdot S_{233} + 1.5 \cdot S_{234} + 150 \cdot S_{235} \\
& + 2600 \cdot Y_{111} + 2600 \cdot Y_{112} + 2600 \cdot Y_{113} + 2600 \cdot Y_{114} + 2600 \cdot Y_{115} + 2600 \cdot Y_{121} \\
& + 2600 \cdot Y_{122} + 2600 \cdot Y_{123} + 2600 \cdot Y_{124} + 2600 \cdot Y_{125} + 2600 \cdot Y_{131} + 2600 \cdot Y_{132} \\
& + 2600 \cdot Y_{133} + 2600 \cdot Y_{134} + 2600 \cdot Y_{135} + 5200 \cdot Y_{211} + 5200 \cdot Y_{212} + 5200 \cdot Y_{213} \\
& + 5200 \cdot Y_{214} + 5200 \cdot Y_{215} + 5200 \cdot Y_{221} + 5200 \cdot Y_{222} + 5200 \cdot Y_{223} + 5200 \cdot Y_{224} \\
& + 5200 \cdot Y_{225} + 5200 \cdot Y_{231} + 5200 \cdot Y_{232} + 5200 \cdot Y_{233} + 5200 \cdot Y_{234} + 5200 \cdot Y_{235} \\
& + 11700 \cdot Y_{311} + 11700 \cdot Y_{312} + 11700 \cdot Y_{313} + 11700 \cdot Y_{314} + 11700 \cdot Y_{315} \\
& + 11700 \cdot Y_{321} + 11700 \cdot Y_{322} + 11700 \cdot Y_{323} + 11700 \cdot Y_{324} + 11700 \cdot Y_{325} \\
& + 11700 \cdot Y_{331} + 11700 \cdot Y_{332} + 11700 \cdot Y_{333} + 11700 \cdot Y_{334} + 11700 \cdot Y_{335}
\end{aligned}
$$

subject to

$$Z_{11111} + Z_{11211} + Z_{11311} + Z_{11121} + Z_{11221} + Z_{11321} + Z_{11131} + Z_{11231} + Z_{11331} - S_{111} = 1$$

$$Z_{21111} + Z_{21211} + Z_{21311} + Z_{21121} + Z_{21221} + Z_{21321} + Z_{21131} + Z_{21231} + Z_{21331} - S_{211} = 2$$

$Z_{12211} + Z_{12311} + Z_{12221} + Z_{12321} + Z_{12231} + Z_{12331} - S_{121} = 0$

$Z_{22211} + Z_{22311} + Z_{22221} + Z_{22321} + Z_{22231} + Z_{22331} - S_{221} = 1$

$Z_{13311} + Z_{13321} + Z_{13331} - S_{131} = 1$

$Z_{23311} + Z_{23321} + Z_{23331} - S_{231} = 0$

$Z_{11112} + Z_{11212} + Z_{11312} + Z_{11122} + Z_{11222} + Z_{11322} + Z_{11132} + Z_{11232} + Z_{11332} + S_{111} - S_{112} = 0$

$Z_{21112} + Z_{21212} + Z_{21312} + Z_{21122} + Z_{21222} + Z_{21322} + Z_{21132} + Z_{21232} + Z_{21332} + S_{211} - S_{212} = 3$

$Z_{12212} + Z_{12312} + Z_{12222} + Z_{12322} + Z_{12232} + Z_{12332} + S_{121} - S_{122} = 2$

$Z_{22212} + Z_{22312} + Z_{22222} + Z_{22322} + Z_{22232} + Z_{22332} + S_{221} - S_{222} = 2$

$Z_{13312} + Z_{13322} + Z_{13332} + S_{131} - S_{132} = 0$

$Z_{23312} + Z_{23322} + Z_{23332} + S_{231} - S_{232} = 1$

$Z_{11113} + Z_{11213} + Z_{11313} + Z_{11123} + Z_{11223} + Z_{11323} + Z_{11133} + Z_{11233} + Z_{11333} + S_{112} - S_{113} = 3$

$Z_{21113} + Z_{21213} + Z_{21313} + Z_{21123} + Z_{21223} + Z_{21323} + Z_{21133} + Z_{21233} + Z_{21333} + S_{212} - S_{213} = 0$

$Z_{12213} + Z_{12313} + Z_{12223} + Z_{12323} + Z_{12233} + Z_{12333} + S_{122} - S_{123} = 0$

$Z_{22213} + Z_{22313} + Z_{22223} + Z_{22323} + Z_{22233} + Z_{22333} + S_{222} - S_{223} = 0$

$Z_{13313} + Z_{13323} + Z_{13333} + S_{132} - S_{133} = 3$

$Z_{23313} + Z_{23323} + Z_{23333} + S_{232} - S_{233} = 0$

$Z_{11114} + Z_{11214} + Z_{11314} + Z_{11124} + Z_{11224} + Z_{11324} + Z_{11134} + Z_{11234} + Z_{11334} + S_{113} - S_{114} = 0$

$Z_{21114} + Z_{21214} + Z_{21314} + Z_{21124} + Z_{21224} + Z_{21324} + Z_{21134} + Z_{21234} + Z_{21334} + S_{213} - S_{214} = 2$

$Z_{12214} + Z_{12314} + Z_{12224} + Z_{12324} + Z_{12234} + Z_{12334} + S_{123} - S_{124} = 4$

$Z_{22214} + Z_{22314} + Z_{22224} + Z_{22324} + Z_{22234} + Z_{22334} + S_{223} - S_{224} = 1$

$Z_{13314} + Z_{13324} + Z_{13334} + S_{133} - S_{134} = 0$

$Z_{23314} + Z_{23324} + Z_{23334} + S_{233} - S_{234} = 2$

$Z_{11115} + Z_{11215} + Z_{11315} + Z_{11125} + Z_{11225} + Z_{11325} + Z_{11135} + Z_{11235} + Z_{11335} + S_{114} - S_{115} = 5$

$Z_{21115} + Z_{21215} + Z_{21315} + Z_{21125} + Z_{21225} + Z_{21325} + Z_{21135} + Z_{21235} + Z_{21335} + S_{214} - S_{215} = 6$

$Z_{12215} + Z_{12315} + Z_{12225} + Z_{12325} + Z_{12235} + Z_{12335} + S_{124} - S_{125} = 0$

$Z_{22215} + Z_{22315} + Z_{22225} + Z_{22325} + Z_{22235} + Z_{22335} + S_{224} - S_{225} = 6$

$Z_{13315} + Z_{13325} + Z_{13335} + S_{134} - S_{135} = 5$

$Z_{23315} + Z_{23325} + Z_{23335} + S_{234} - S_{235} = 0$

$3 \cdot Z_{11111} + 4 \cdot Z_{21111} \leqslant 13 \cdot Y_{111}$

$3 \cdot Z_{11112} + 4 \cdot Z_{21112} \leqslant 13 \cdot Y_{112}$

$3 \cdot Z_{11113} + 4 \cdot Z_{21113} \leqslant 13 \cdot Y_{113}$

$3 \cdot Z_{11114} + 4 \cdot Z_{21114} \leqslant 13 \cdot Y_{114}$

$3 \cdot Z_{11115} + 4 \cdot Z_{21115} \leqslant 13 \cdot Y_{115}$

$3 \cdot Z_{11121} + 4 \cdot Z_{21121} \leqslant 13 \cdot Y_{121}$

$$3 \cdot Z_{11122} + 4 \cdot Z_{21122} \leqslant 13 \cdot Y_{122}$$

$$3 \cdot Z_{11123} + 4 \cdot Z_{21123} \leqslant 13 \cdot Y_{123}$$

$$3 \cdot Z_{11124} + 4 \cdot Z_{21124} \leqslant 13 \cdot Y_{124}$$

$$3 \cdot Z_{11125} + 4 \cdot Z_{21125} \leqslant 13 \cdot Y_{125}$$

$$3 \cdot Z_{11131} + 4 \cdot Z_{21131} \leqslant 13 \cdot Y_{131}$$

$$3 \cdot Z_{11132} + 4 \cdot Z_{21132} \leqslant 13 \cdot Y_{132}$$

$$3 \cdot Z_{11133} + 4 \cdot Z_{21133} \leqslant 13 \cdot Y_{133}$$

$$3 \cdot Z_{11134} + 4 \cdot Z_{21134} \leqslant 13 \cdot Y_{134}$$

$$3 \cdot Z_{11135} + 4 \cdot Z_{21135} \leqslant 13 \cdot Y_{135}$$

$$3 \cdot Z_{11211} + 3 \cdot Z_{12211} + 4 \cdot Z_{21211} + 4 \cdot Z_{22211} \leqslant 13 \cdot Y_{211}$$

$$3 \cdot Z_{11212} + 3 \cdot Z_{12212} + 4 \cdot Z_{21212} + 4 \cdot Z_{22212} \leqslant 13 \cdot Y_{212}$$

$$3 \cdot Z_{11213} + 3 \cdot Z_{12213} + 4 \cdot Z_{21213} + 4 \cdot Z_{22213} \leqslant 13 \cdot Y_{213}$$

$$3 \cdot Z_{11214} + 3 \cdot Z_{12214} + 4 \cdot Z_{21214} + 4 \cdot Z_{22214} \leqslant 13 \cdot Y_{214}$$

$$3 \cdot Z_{11215} + 3 \cdot Z_{12215} + 4 \cdot Z_{21215} + 4 \cdot Z_{22215} \leqslant 13 \cdot Y_{215}$$

$$3 \cdot Z_{11221} + 3 \cdot Z_{12221} + 4 \cdot Z_{21221} + 4 \cdot Z_{22221} \leqslant 13 \cdot Y_{221}$$

$$3 \cdot Z_{11222} + 3 \cdot Z_{12222} + 4 \cdot Z_{21222} + 4 \cdot Z_{22222} \leqslant 13 \cdot Y_{222}$$

$$3 \cdot Z_{11223} + 3 \cdot Z_{12223} + 4 \cdot Z_{21223} + 4 \cdot Z_{22223} \leqslant 13 \cdot Y_{223}$$

$$3 \cdot Z_{11224} + 3 \cdot Z_{12224} + 4 \cdot Z_{21224} + 4 \cdot Z_{22224} \leqslant 13 \cdot Y_{224}$$

$$3 \cdot Z_{11225} + 3 \cdot Z_{12225} + 4 \cdot Z_{21225} + 4 \cdot Z_{22225} \leqslant 13 \cdot Y_{225}$$

$$3 \cdot Z_{11231} + 3 \cdot Z_{12231} + 4 \cdot Z_{21231} + 4 \cdot Z_{22231} \leqslant 13 \cdot Y_{231}$$

$$3 \cdot Z_{11232} + 3 \cdot Z_{12232} + 4 \cdot Z_{21232} + 4 \cdot Z_{22232} \leqslant 13 \cdot Y_{232}$$

$$3 \cdot Z_{11233} + 3 \cdot Z_{12233} + 4 \cdot Z_{21233} + 4 \cdot Z_{22233} \leqslant 13 \cdot Y_{233}$$

$$3 \cdot Z_{11234} + 3 \cdot Z_{12234} + 4 \cdot Z_{21234} + 4 \cdot Z_{22234} \leqslant 13 \cdot Y_{234}$$

$$3 \cdot Z_{11235} + 3 \cdot Z_{12235} + 4 \cdot Z_{21235} + 4 \cdot Z_{22235} \leqslant 13 \cdot Y_{235}$$

$$3 \cdot Z_{11311} + 3 \cdot Z_{12311} + 3 \cdot Z_{13311} + 4 \cdot Z_{21311} + 4 \cdot Z_{22311} + 4 \cdot Z_{23311} \leqslant 13 \cdot Y_{311}$$

$$3 \cdot Z_{11312} + 3 \cdot Z_{12312} + 3 \cdot Z_{13312} + 4 \cdot Z_{21312} + 4 \cdot Z_{22312} + 4 \cdot Z_{23312} \leqslant 13 \cdot Y_{312}$$

$$3 \cdot Z_{11313} + 3 \cdot Z_{12313} + 3 \cdot Z_{13313} + 4 \cdot Z_{21313} + 4 \cdot Z_{22313} + 4 \cdot Z_{23313} \leqslant 13 \cdot Y_{313}$$

$$3 \cdot Z_{11314} + 3 \cdot Z_{12314} + 3 \cdot Z_{13314} + 4 \cdot Z_{21314} + 4 \cdot Z_{22314} + 4 \cdot Z_{23314} \leqslant 13 \cdot Y_{314}$$

$$3 \cdot Z_{11315} + 3 \cdot Z_{12315} + 3 \cdot Z_{13315} + 4 \cdot Z_{21315} + 4 \cdot Z_{22315} + 4 \cdot Z_{23315} \leqslant 13 \cdot Y_{315}$$

$$3 \cdot Z_{11321} + 3 \cdot Z_{12321} + 3 \cdot Z_{13321} + 4 \cdot Z_{21321} + 4 \cdot Z_{22321} + 4 \cdot Z_{23321} \leqslant 13 \cdot Y_{321}$$

$$3 \cdot Z_{11322} + 3 \cdot Z_{12322} + 3 \cdot Z_{13322} + 4 \cdot Z_{21322} + 4 \cdot Z_{22322} + 4 \cdot Z_{23322} \leqslant 13 \cdot Y_{322}$$

$$3 \cdot Z_{11323} + 3 \cdot Z_{12323} + 3 \cdot Z_{13323} + 4 \cdot Z_{21323} + 4 \cdot Z_{22323} + 4 \cdot Z_{23323} \leqslant 13 \cdot Y_{323}$$

$$3 \cdot Z_{11324} + 3 \cdot Z_{12324} + 3 \cdot Z_{13324} + 4 \cdot Z_{21324} + 4 \cdot Z_{22324} + 4 \cdot Z_{23324} \leqslant 13 \cdot Y_{324}$$

$$3 \cdot Z_{11325} + 3 \cdot Z_{12325} + 3 \cdot Z_{13325} + 4 \cdot Z_{21325} + 4 \cdot Z_{22325} + 4 \cdot Z_{23325} \leqslant 13 \cdot Y_{325}$$

$3 \cdot Z_{11331} + 3 \cdot Z_{12331} + 3 \cdot Z_{13331} + 4 \cdot Z_{21331} + 4 \cdot Z_{22331} + 4 \cdot Z_{23331} \leqslant 13 \cdot Y_{331}$

$3 \cdot Z_{11332} + 3 \cdot Z_{12332} + 3 \cdot Z_{13332} + 4 \cdot Z_{21332} + 4 \cdot Z_{22332} + 4 \cdot Z_{23332} \leqslant 13 \cdot Y_{332}$

$3 \cdot Z_{11333} + 3 \cdot Z_{12333} + 3 \cdot Z_{13333} + 4 \cdot Z_{21333} + 4 \cdot Z_{22333} + 4 \cdot Z_{23333} \leqslant 13 \cdot Y_{333}$

$3 \cdot Z_{11334} + 3 \cdot Z_{12334} + 3 \cdot Z_{13334} + 4 \cdot Z_{21334} + 4 \cdot Z_{22334} + 4 \cdot Z_{23334} \leqslant 13 \cdot Y_{334}$

$3 \cdot Z_{11335} + 3 \cdot Z_{12335} + 3 \cdot Z_{13335} + 4 \cdot Z_{21335} + 4 \cdot Z_{22335} + 4 \cdot Z_{23335} \leqslant 13 \cdot Y_{335}$

$Y_{111} + Y_{211} + Y_{311} \leqslant 1$

$Y_{121} + Y_{221} + Y_{321} \leqslant 1$

$Y_{131} + Y_{231} + Y_{331} \leqslant 1$

$Y_{112} + Y_{212} + Y_{312} \leqslant 1$

$Y_{122} + Y_{222} + Y_{322} \leqslant 1$

$Y_{132} + Y_{232} + Y_{332} \leqslant 1$

$Y_{113} + Y_{213} + Y_{313} \leqslant 1$

$Y_{123} + Y_{223} + Y_{323} \leqslant 1$

$Y_{133} + Y_{233} + Y_{333} \leqslant 1$

$Y_{114} + Y_{214} + Y_{314} \leqslant 1$

$Y_{124} + Y_{224} + Y_{324} \leqslant 1$

$Y_{134} + Y_{234} + Y_{334} \leqslant 1$

$Y_{115} + Y_{215} + Y_{315} \leqslant 1$

$Y_{125} + Y_{225} + Y_{325} \leqslant 1$

$Y_{135} + Y_{235} + Y_{335} \leqslant 1$

$S_{110} = 0$

$S_{210} = 0$

$S_{120} = 0$

$S_{220} = 0$

$S_{130} = 0$

$S_{230} = 0$

$Y_{\tau,k,p} \in \{0,1\},\ S_{i,\beta,p} \in \mathbb{R}_+,\ Z_{i,\beta,\tau,k,p} \in \mathbb{Z}_+ \qquad \forall i,\ \forall \beta,\ \forall \tau,\ \forall k,\ \forall p.$

In the MILP approach, first the linear relaxation of the problem is solved, then the integrality constraints are added to the problem and the optimization package CPLEX, in its default settings, is called to solve the resultant mixed-integer problem. This is done to assess the quality of the upper bound obtained by the application through the GAP calculation.

Applying the RF approach to this example, the decomposition strategy portrayed in Table 3.1(a) will be used. In the first RF iteration, the integrality constraints over all variables are relaxed with the exception of variables $Y_{\tau,k,1}$, $Y_{\tau,k,2}$, $Z_{i,\beta,\tau,k,1}$ and $Z_{i,\beta,\tau,k,2}$ for all indices $i,\ \beta,\ \tau,\ k$, which are maintained as integer, and the resultant problem is solved by the optimization package. Next, the values of the variables $Y_{\tau,k,1}$ and $Z_{i,\beta,\tau,k,1}$ are retrieved from the solution and are fixed at these values for the next iteration.

In the second RF iteration, the integrality constraints to variables $Y_{\tau,k,3}$ and $Z_{i,\beta,\tau,k,3}$ are added to the problem together with variables $Y_{\tau,k,2}$ and $Z_{i,\beta,\tau,k,2}$, while the integrality constraints over variables $Y_{\tau,k,4}$, $Y_{\tau,k,5}$, $Z_{i,\beta,\tau,k,4}$ and $Z_{i,\beta,\tau,k,5}$ are maintained relaxed for all indices $i,\ \beta,\ \tau,\ k$.

In the third RF iteration, for all indices $i,\ \beta,\ \tau,\ k$, variables $Y_{\tau,k,2}$ and $Z_{i,\beta,\tau,k,2}$ are fixed in their integer values, the integrality constraints over variables $Y_{\tau,k,3}$, $Y_{\tau,k,4}$, $Z_{i,\beta,\tau,k,3}$ and $Z_{i,\beta,\tau,k,4}$ are added to the problem, and only variables $Y_{\tau,k,5}$ and $Z_{i,\beta,\tau,k,5}$ are maintained relaxed.

In the fourth and last RF iteration of this example, variables $Y_{\tau,k,3}$ and $Z_{i,\beta,\tau,k,3}$ are fixed, and the integrality constraints over variables $Y_{\tau,k,4}$, $Y_{\tau,k,5}$, $Z_{i,\beta,\tau,k,4}$ and $Z_{i,\beta,\tau,k,5}$ are added to the problem. These variable conditions, at each iteration, are illustrated in Table A.2. Note that this procedure works until a feasible solution to the original problem is obtained or an infeasible problem is detected. Then the procedure is stopped.

Table A.2: Numerical example for KMPCSP - variable conditions in each RF iteration.

Variables conditions	**RF iterations 1**	**2**	**3**	**4**
Fixed	-	$Y_{\tau,k,\mathbf{1}}$ $Z_{i,\beta,\tau,k,\mathbf{1}}$	$Y_{\tau,k,\mathbf{1}}$ $Y_{\tau,k,\mathbf{2}}$ $Z_{i,\beta,\tau,k,\mathbf{1}}$ $Z_{i,\beta,\tau,k,\mathbf{2}}$	$Y_{\tau,k,\mathbf{1}}$ $Y_{\tau,k,\mathbf{2}}$ $Y_{\tau,k,\mathbf{3}}$ $Z_{i,\beta,\tau,k,\mathbf{1}}$ $Z_{i,\beta,\tau,k,\mathbf{2}}$ $Z_{i,\beta,\tau,k,\mathbf{3}}$
Integer	$Y_{\tau,k,\mathbf{1}}$ $Y_{\tau,k,\mathbf{2}}$ $Z_{i,\beta,\tau,k,\mathbf{1}}$ $Z_{i,\beta,\tau,k,\mathbf{2}}$	$Y_{\tau,k,\mathbf{2}}$ $Y_{\tau,k,\mathbf{3}}$ $Z_{i,\beta,\tau,k,\mathbf{2}}$ $Z_{i,\beta,\tau,k,\mathbf{3}}$	$Y_{\tau,k,\mathbf{3}}$ $Y_{\tau,k,\mathbf{4}}$ $Z_{i,\beta,\tau,k,\mathbf{3}}$ $Z_{i,\beta,\tau,k,\mathbf{4}}$	$Y_{\tau,k,\mathbf{4}}$ $Y_{\tau,k,\mathbf{5}}$ $Z_{i,\beta,\tau,k,\mathbf{4}}$ $Z_{i,\beta,\tau,k,\mathbf{5}}$
Relaxed	$Y_{\tau,k,\mathbf{3}}$ $Y_{\tau,k,\mathbf{4}}$ $Y_{\tau,k,\mathbf{5}}$ $Z_{i,\beta,\tau,k,\mathbf{3}}$ $Z_{i,\beta,\tau,k,\mathbf{4}}$ $Z_{i,\beta,\tau,k,\mathbf{5}}$	$Y_{\tau,k,\mathbf{4}}$ $Y_{\tau,k,\mathbf{5}}$ $Z_{i,\beta,\tau,k,\mathbf{4}}$ $Z_{i,\beta,\tau,k,\mathbf{5}}$	$Y_{\tau,k,\mathbf{5}}$ $Z_{i,\beta,\tau,k,\mathbf{5}}$	-

In comparison to the original problem, in the RF solution method, a smaller number of integrality constraints is considered at each iteration, which generally makes the resultant problem easier to be solved.

The results of the MILP and RF strategies are presented in Tables A.3, A.4 and A.5, where processing time is given in seconds, the objective values and lower bounds are expressed in currency units, and in columns containing the term '(%)', the values are percentages. Besides that, two types of GAP are considered. The first, GAP1, is calculated as a percentage by the formula:

$$GAP1 = 100 \times \frac{UB - LR}{UB},$$

where UB (upper bound) and LR (linear relaxation) are the objective values retrieved from the mixed-integer and linear relaxation (at root node) solutions, respectively. The parameter GAP2 is calculated in a similar way to GAP1:

$$GAP2 = 100 \times \frac{UB - BLB}{UB},$$

where UB (upper bound) is the mixed-integer objective value and BLB is the best lower bound provided by CPLEX in the MILP strategy.

Table A.3: Numerical example for KMPCSP - MILP and RF results: solution status, processing time, objective value and distribution of costs.

Solution status		**Processing time**		**Mixed-Integer objective value**		**Inventory penalty cost (%)**		**Mold cost (%)**	
MILP	**RF**	**MILP**	**RF**	**MILP**	**RF**	**MILP**	**RF**	**MILP**	**RF**
Optimal	Feasible	1.75	0.08	66332.00	66351.50	0.05	0.08	99.95	99.92

Table A.4: Numerical example for KMPCSP - MILP and RF results: lower bounds and GAPs.

Linear relaxation objective value	**Best lower bound provided by CPLEX**	**GAP1 (%)**		**GAP2 (%)**	
		MILP	**RF**	**MILP**	**RF**
60305.50	66267.50	9.09	9.11	0.10	0.13

Table A.5: Numerical example for KMPCSP - MILP and RF results: mold usage.

Average no. of available molds per period	**Average no. of used molds per period**		**Used mold space (UMS)**		**Production space (PS)**		**PS/UMS (%)**	
	MILP	**RF**	**MILP**	**RF**	**MILP**	**RF**	**MILP**	**RF**
3.00	2.20	2.20	143.00	143.00	140.00	140.00	97.90	97.90

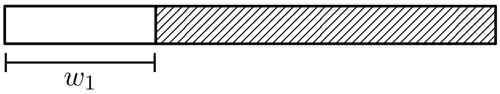
Applying the example to the GGMPCSP

Here, a numerical example of solving the GGMPCSP is presented considering the same instance used to exemplify the KMPCSP solution methods.

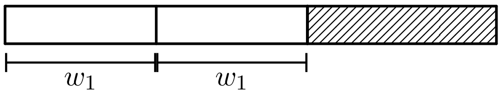
Writing in a matrix all possible combinations (cutting patterns) to allocate the ordered items (hollow-core slabs) in a mold, one obtains:

$$\left[\mathrm{a}_{ij}\right] = \begin{bmatrix} \mathrm{a}_{1j} \\ \mathrm{a}_{2j} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 0 & 0 & 0 & 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 & 3 & 4 & 1 & 1 & 2 & 3 \end{bmatrix}$$

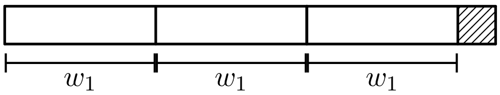
which is illustrated by Figure A.1.



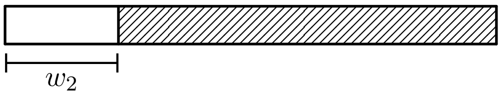
(a) Cutting pattern $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.



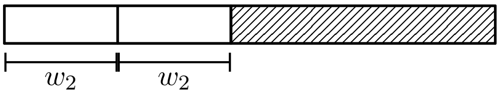
(b) Cutting pattern $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$.



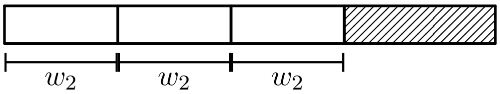
(c) Cutting pattern $\begin{bmatrix} 3 \\ 0 \end{bmatrix}$.



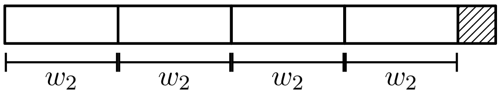
(d) Cutting pattern $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.



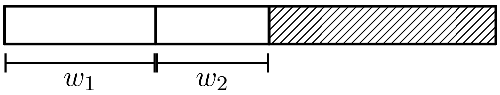
(e) Cutting pattern $\begin{bmatrix} 0 \\ 2 \end{bmatrix}$.



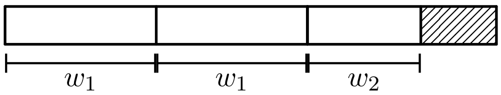
(f) Cutting pattern $\begin{bmatrix} 0 \\ 3 \end{bmatrix}$.



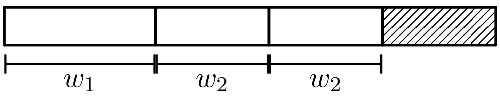
(g) Cutting pattern $\begin{bmatrix} 0 \\ 4 \end{bmatrix}$.



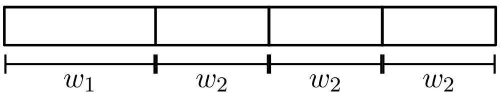
(h) Cutting pattern $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.



(i) Cutting pattern $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$.



(j) Cutting pattern $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.



(k) Cutting pattern $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

Figure A.1: Cutting patterns for the GGMPCSP numerical example.

Considering all possible cutting patterns to formulate this problem, one would have to solve a linear problem with 195 variables. To get round this laborious task, the

Table A.6: Initial cutting patterns for the GGMPCSP numerical example.

			Periods 1, 2, 3 and 4						**Period 5**					
		RHS	X_{11p}	X_{21p}	X_{12p}	X_{22p}	X_{13p}	X_{23p}	X_{115}	X_{215}	X_{125}	X_{225}	X_{135}	X_{235}
Demand balance constraints	$=$	d_{11p}	3	0										
	$=$	d_{21p}	0	4										
	$=$	d_{12p}			3	0								
	$=$	d_{22p}			0	4								
	$=$	d_{13p}					3	0						
	$=$	d_{23p}					0	4						
	$=$	d_{115}							3	0				
	$=$	d_{215}							0	4				
	$=$	d_{125}									1	0		
	$=$	d_{225}									0	4		
	$=$	d_{135}											2	0
	$=$	d_{235}											0	1

selection of a smaller number of available cutting patterns enables the formulation of a restricted master problem. Thus, removing the integrality constraints of the variables, a column generation procedure can be applied as a solution method. In this study, the RMP is initially defined only taking into account homogeneous cutting patterns, given by

$$a_{i,\beta,j,\tau,p} = \begin{cases} \min\left\{\dfrac{W}{w_i};\ \max\left\{1; \bar{d}_{i,\beta,p}\right\}\right\} & \text{if } i = j \text{ and } \beta = \tau, \\ 0 & \text{otherwise,} \end{cases}$$

where

$$\bar{d}_{i,\beta,p} = \sum_{\gamma \leqslant \beta} \sum_{r=p}^{|P|} d_{i,\gamma,r} \quad \forall i,\ \forall \beta,\ \forall p.$$

Table A.6 contains the initial cutting patterns used in the first iteration of the column generation procedure applied to this example. This strategy yields a total of 60 variables, which represents less than a third of the total possible variables, making the resultant problem more straightforward to be solved. To elucidate the operation of the column generation procedure, the models and solutions of each iteration are presented.

Iteration 1

The first column generation iteration starts with solving the RMP with homogeneous cutting patterns as follows:

$$\begin{aligned} \text{minimize} \quad & 2 \cdot S_{111} + 2 \cdot S_{112} + 2 \cdot S_{113} + 2 \cdot S_{114} + 200 \cdot S_{115} + 2 \cdot S_{121} + 2 \cdot S_{122} + 2 \cdot S_{123} \\ & + 2 \cdot S_{124} + 200 \cdot S_{125} + 2 \cdot S_{131} + 2 \cdot S_{132} + 2 \cdot S_{133} + 2 \cdot S_{134} + 200 \cdot S_{135} + 1.5 \cdot S_{211} \\ & + 1.5 \cdot S_{212} + 1.5 \cdot S_{213} + 1.5 \cdot S_{214} + 150 \cdot S_{215} + 1.5 \cdot S_{221} + 1.5 \cdot S_{222} + 1.5 \cdot S_{223} \end{aligned}$$

$$+ 1.5 \cdot S_{224} + 150 \cdot S_{225} + 1.5 \cdot S_{231} + 1.5 \cdot S_{232} + 1.5 \cdot S_{233} + 1.5 \cdot S_{234} + 150 \cdot S_{235}$$
$$+ 2600 \cdot X_{111} + 2600 \cdot X_{112} + 2600 \cdot X_{113} + 2600 \cdot X_{114} + 2600 \cdot X_{115} + 5200 \cdot X_{121}$$
$$+ 5200 \cdot X_{122} + 5200 \cdot X_{123} + 5200 \cdot X_{124} + 5200 \cdot X_{125} + 11700 \cdot X_{131} + 11700 \cdot X_{132}$$
$$+ 11700 \cdot X_{133} + 11700 \cdot X_{134} + 11700 \cdot X_{135} + 2600 \cdot X_{211} + 2600 \cdot X_{212} + 2600 \cdot X_{213}$$
$$+ 2600 \cdot X_{214} + 2600 \cdot X_{215} + 5200 \cdot X_{221} + 2600 \cdot X_{122} + 5200 \cdot X_{223} + 5200 \cdot X_{224}$$
$$+ 5200 \cdot X_{225} + 11700 \cdot X_{231} + 11700 \cdot X_{232} + 11700 \cdot X_{233} + 11700 \cdot X_{234} + 11700 \cdot X_{235}$$

subject to

$$3 \cdot X_{111} + 3 \cdot X_{121} + 3 \cdot X_{131} - S_{111} = 1$$
$$4 \cdot X_{211} + 4 \cdot X_{221} + 4 \cdot X_{231} - S_{211} = 2$$
$$3 \cdot X_{121} + 3 \cdot X_{131} - S_{121} = 0$$
$$4 \cdot X_{221} + 4 \cdot X_{231} - S_{221} = 1$$
$$3 \cdot X_{131} - S_{131} = 1$$
$$4 \cdot X_{231} - S_{231} = 0$$
$$3 \cdot X_{112} + 3 \cdot X_{122} + 3 \cdot X_{132} + S_{111} - S_{112} = 0$$
$$4 \cdot X_{212} + 4 \cdot X_{222} + 4 \cdot X_{232} + S_{211} - S_{212} = 3$$
$$3 \cdot X_{122} + 3 \cdot X_{132} + S_{121} - S_{122} = 2$$
$$4 \cdot X_{222} + 4 \cdot X_{232} + S_{221} - S_{222} = 2$$
$$3 \cdot X_{132} + S_{131} - S_{132} = 0$$
$$4 \cdot X_{232} + S_{231} - S_{232} = 1$$
$$3 \cdot X_{113} + 3 \cdot X_{123} + 3 \cdot X_{133} + S_{112} - S_{113} = 3$$
$$4 \cdot X_{213} + 4 \cdot X_{223} + 4 \cdot X_{233} + S_{212} - S_{213} = 0$$
$$3 \cdot X_{123} + 3 \cdot X_{133} + S_{122} - S_{123} = 0$$
$$4 \cdot X_{223} + 4 \cdot X_{233} + S_{222} - S_{223} = 0$$
$$3 \cdot X_{133} + S_{132} - S_{133} = 3$$
$$4 \cdot X_{233} + S_{232} - S_{233} = 0$$
$$3 \cdot X_{113} + 3 \cdot X_{123} + 3 \cdot X_{133} + S_{113} - S_{114} = 0$$
$$4 \cdot X_{213} + 4 \cdot X_{223} + 4 \cdot X_{233} + S_{213} - S_{214} = 2$$
$$3 \cdot X_{123} + 3 \cdot X_{133} + S_{123} - S_{124} = 4$$
$$4 \cdot X_{223} + 4 \cdot X_{233} + S_{223} - S_{224} = 1$$
$$3 \cdot X_{133} + S_{133} - S_{134} = 0$$
$$4 \cdot X_{233} + S_{233} - S_{234} = 2$$
$$3 \cdot X_{113} + 1 \cdot X_{123} + 2 \cdot X_{133} + S_{114} - S_{115} = 5$$
$$4 \cdot X_{213} + 4 \cdot X_{223} + 1 \cdot X_{233} + S_{214} - S_{215} = 6$$
$$1 \cdot X_{123} + 2 \cdot X_{133} + S_{124} - S_{125} = 0$$

$$4 \cdot X_{223} + 1 \cdot X_{233} + S_{224} - S_{225} = 6$$

$$2 \cdot X_{133} + S_{134} - S_{135} = 5$$

$$1 \cdot X_{233} + S_{234} - S_{235} = 0$$

$$X_{111} + X_{211} + X_{121} + X_{221} + X_{131} + X_{231} \leqslant 3$$

$$X_{112} + X_{212} + X_{122} + X_{222} + X_{132} + X_{232} \leqslant 3$$

$$X_{113} + X_{213} + X_{123} + X_{223} + X_{133} + X_{233} \leqslant 3$$

$$X_{114} + X_{214} + X_{124} + X_{224} + X_{134} + X_{234} \leqslant 3$$

$$X_{115} + X_{215} + X_{125} + X_{225} + X_{135} + X_{235} \leqslant 3$$

$$S_{110} = 0$$

$$S_{210} = 0$$

$$S_{120} = 0$$

$$S_{220} = 0$$

$$S_{130} = 0$$

$$S_{230} = 0$$

$$S_{i,\beta,p},\ X_{j,\tau,p} \in \mathbb{Z}_+ \qquad \forall i,\ \forall \beta,\ \forall \tau,\ \forall j,\ \forall p.$$

After solving it with CPLEX using the default settings, the following results are obtained:

Relaxed RMP Objective Value = 65332.5.

The initial solution is given by:

$$\begin{bmatrix} X_{111} & X_{112} & X_{113} & X_{114} & X_{115} \\ X_{121} & X_{122} & X_{123} & X_{124} & X_{125} \\ X_{131} & X_{132} & X_{133} & X_{134} & X_{135} \\ X_{211} & X_{212} & X_{213} & X_{214} & X_{215} \\ X_{221} & X_{222} & X_{223} & X_{224} & X_{225} \\ X_{231} & X_{232} & X_{233} & X_{234} & X_{235} \end{bmatrix} = \begin{bmatrix} 0.33 & 0 & 1.08 & 0.92 & 0 \\ 0 & 0.67 & 0 & 0.67 & 0 \\ 0.33 & 0 & 1 & 0 & 0.67 \\ 0.5 & 0.75 & 0 & 0.67 & 0.83 \\ 0.25 & 0.5 & 0 & 0.25 & 1.5 \\ 0 & 0.25 & 0 & 0.5 & 0 \end{bmatrix}$$

$$\begin{bmatrix} S_{111} & S_{112} & S_{113} & S_{114} & S_{115} \\ S_{121} & S_{122} & S_{123} & S_{124} & S_{125} \\ S_{131} & S_{132} & S_{133} & S_{134} & S_{135} \\ S_{211} & S_{212} & S_{213} & S_{214} & S_{215} \\ S_{221} & S_{222} & S_{223} & S_{224} & S_{225} \\ S_{231} & S_{232} & S_{233} & S_{234} & S_{235} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0.25 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.67 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The dual optimal variables $\lambda_{i,\beta,p}$ (related to the demand balance constraints) and

μ_p (related to the capacity constraints) are given below.

$$\left[\begin{array}{ccccc} \lambda_{111} & \lambda_{112} & \lambda_{113} & \lambda_{114} & \lambda_{115} \\ \lambda_{121} & \lambda_{122} & \lambda_{123} & \lambda_{124} & \lambda_{125} \\ \lambda_{131} & \lambda_{132} & \lambda_{133} & \lambda_{134} & \lambda_{135} \\ \lambda_{211} & \lambda_{212} & \lambda_{213} & \lambda_{214} & \lambda_{215} \\ \lambda_{221} & \lambda_{222} & \lambda_{223} & \lambda_{224} & \lambda_{225} \\ \lambda_{231} & \lambda_{232} & \lambda_{233} & \lambda_{234} & \lambda_{235} \\ \hdashline \mu_1 & \mu_2 & \mu_3 & \mu_4 & \mu_5 \end{array}\right] = \left[\begin{array}{ccccc} 866.67 & 864.67 & 866.67 & 868.67 & 870.67 \\ 1733.33 & 1733.33 & 1733.33 & 1735.33 & -200 \\ 3900 & 3898 & 3900 & 3902 & 3904 \\ 650 & 650 & 650 & 651.5 & 653 \\ 1300 & 1300 & 1300 & 1301.5 & 1303 \\ 2925 & 2925 & 2925 & 2926.5 & -150 \\ \hdashline 0 & 0 & 0 & -6 & -12 \end{array}\right]$$

Now, let variable $\alpha_{i,\beta}$ represent the number of items i with wire configuration β in the new cutting pattern. For each period p and leading wire configuration τ, the following knapsack subproblems are solved in order to find whether there is an attractive cutting pattern for the RMP.

- $p =$ 1

$\tau = 1$:

Reduced cost: − 216.67

$$\begin{array}{ll} \min & 2600 - 866.67 \cdot \alpha_{11} - 650 \cdot \alpha_{21} \\ \text{s. t.} & 4 \cdot \alpha_{11} + 3 \cdot \alpha_{21} \leqslant 13 \\ & \alpha_{12} + \alpha_{13} + \alpha_{22} + \alpha_{23} = 0, \\ & \alpha_{i,\beta} \in \mathbb{Z}_+ \quad \forall i,\ \forall \beta. \end{array} \xrightarrow{\text{solution}} \begin{bmatrix} \alpha_{11} \\ \alpha_{21} \\ \alpha_{12} \\ \alpha_{22} \\ \alpha_{13} \\ \alpha_{23} \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\tau = 2$:

Reduced cost: − 433.33

$$\begin{array}{ll} \min & 5200 - 866.67 \cdot \alpha_{11} - 1733.33 \cdot \alpha_{12} \\ & \qquad -650 \cdot \alpha_{21} - 1300 \cdot \alpha_{22} \\ \text{s. t.} & 4 \cdot \alpha_{11} + 4 \cdot \alpha_{12} + 3 \cdot \alpha_{21} + 3 \cdot \alpha_{22} \leqslant 13 \\ & \alpha_{13} + \alpha_{23} = 0, \\ & \alpha_{i,\beta} \in \mathbb{Z}_+ \quad \forall i,\ \forall \beta. \end{array} \xrightarrow{\text{solution}} \begin{bmatrix} \alpha_{11} \\ \alpha_{21} \\ \alpha_{12} \\ \alpha_{22} \\ \alpha_{13} \\ \alpha_{23} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 3 \\ 0 \\ 0 \end{bmatrix}$$

$\tau = 3$:

$$\begin{aligned} \min \quad & 11700 - 866.67 \cdot \alpha_{11} - 1733.33 \cdot \alpha_{12} - 3900 \cdot \alpha_{13} \\ & \quad -650 \cdot \alpha_{21} - 1300 \cdot \alpha_{22} - 2925 \cdot \alpha_{23} \\ \text{s. t.} \quad & 4 \cdot \alpha_{11} + 4 \cdot \alpha_{12} + 4 \cdot \alpha_{13} + 3 \cdot \alpha_{21} + 3 \cdot \alpha_{22} \\ & \qquad +3 \cdot \alpha_{23} \leqslant 13 \\ & \alpha_{i,\beta} \in \mathbb{Z}_+ \quad \forall i,\ \forall \beta. \end{aligned} \quad \xrightarrow{\text{solution}} \quad \begin{array}{c} \text{Reduced cost: } -975 \\ \begin{bmatrix} \alpha_{11} \\ \alpha_{21} \\ \alpha_{12} \\ \alpha_{22} \\ \alpha_{13} \\ \alpha_{23} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 3 \end{bmatrix} \end{array}$$

- $p = \ 2$

$\tau = 1$:

$$\begin{aligned} \min \quad & 2600 - 864.67 \cdot \alpha_{11} - 650 \cdot \alpha_{21} \\ \text{s. t.} \quad & 4 \cdot \alpha_{11} + 3 \cdot \alpha_{21} \leqslant 13 \\ & \alpha_{12} + \alpha_{13} + \alpha_{22} + \alpha_{23} = 0, \\ & \alpha_{i,\beta} \in \mathbb{Z}_+ \quad \forall i,\ \forall \beta. \end{aligned} \quad \xrightarrow{\text{solution}} \quad \begin{array}{c} \text{Reduced cost: } -214.67 \\ \begin{bmatrix} \alpha_{11} \\ \alpha_{21} \\ \alpha_{12} \\ \alpha_{22} \\ \alpha_{13} \\ \alpha_{23} \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \end{array}$$

$\tau = 2$:

$$\begin{aligned} \min \quad & 5200 - 864.67 \cdot \alpha_{11} - 1733.33 \cdot \alpha_{12} \\ & \quad -650 \cdot \alpha_{21} - 1300 \cdot \alpha_{22} \\ \text{s. t.} \quad & 4 \cdot \alpha_{11} + 4 \cdot \alpha_{12} + 3 \cdot \alpha_{21} + 3 \cdot \alpha_{22} \leqslant 13 \\ & \alpha_{13} + \alpha_{23} = 0, \\ & \alpha_{i,\beta} \in \mathbb{Z}_+ \quad \forall i,\ \forall \beta. \end{aligned} \quad \xrightarrow{\text{solution}} \quad \begin{array}{c} \text{Reduced cost: } -433.33 \\ \begin{bmatrix} \alpha_{11} \\ \alpha_{21} \\ \alpha_{12} \\ \alpha_{22} \\ \alpha_{13} \\ \alpha_{23} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 3 \\ 0 \\ 0 \end{bmatrix} \end{array}$$

$\tau = 3$:

$$\begin{aligned} \min \quad & 11700 - 864.67 \cdot \alpha_{11} - 1733.33 \cdot \alpha_{12} - 3898 \cdot \alpha_{13} \\ & \quad -650 \cdot \alpha_{21} - 1300 \cdot \alpha_{22} - 2925 \cdot \alpha_{23} \\ \text{s. t.} \quad & 4 \cdot \alpha_{11} + 4 \cdot \alpha_{12} + 4 \cdot \alpha_{13} + 3 \cdot \alpha_{21} + 3 \cdot \alpha_{22} \\ & \qquad +3 \cdot \alpha_{23} \leqslant 13 \\ & \alpha_{i,\beta} \in \mathbb{Z}_+ \quad \forall i,\ \forall \beta. \end{aligned} \quad \xrightarrow{\text{solution}} \quad \begin{array}{c} \text{Reduced cost: } -973 \\ \begin{bmatrix} \alpha_{11} \\ \alpha_{21} \\ \alpha_{12} \\ \alpha_{22} \\ \alpha_{13} \\ \alpha_{23} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 3 \end{bmatrix} \end{array}$$

- $p =\ 3$

$\tau = 1$:

Reduced cost: $-\,216.67$

$$\begin{aligned} \min \quad & 2600 - 866.67 \cdot \alpha_{11} - 650 \cdot \alpha_{21} \\ \text{s. t.} \quad & 4 \cdot \alpha_{11} + 3 \cdot \alpha_{21} \leqslant 13 \\ & \alpha_{12} + \alpha_{13} + \alpha_{22} + \alpha_{23} = 0, \\ & \alpha_{i,\beta} \in \mathbb{Z}_+ \quad \forall i,\ \forall \beta. \end{aligned} \xrightarrow{\text{solution}} \begin{bmatrix} \alpha_{11} \\ \alpha_{21} \\ \alpha_{12} \\ \alpha_{22} \\ \alpha_{13} \\ \alpha_{23} \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\tau = 2$:

Reduced cost: $-\,433.33$

$$\begin{aligned} \min \quad & 5200 - 866.67 \cdot \alpha_{11} - 1733.33 \cdot \alpha_{12} \\ & \qquad -650 \cdot \alpha_{21} - 1300 \cdot \alpha_{22} \\ \text{s. t.} \quad & 4 \cdot \alpha_{11} + 4 \cdot \alpha_{12} + 3 \cdot \alpha_{21} + 3 \cdot \alpha_{22} \leqslant 13 \\ & \alpha_{13} + \alpha_{23} = 0, \\ & \alpha_{i,\beta} \in \mathbb{Z}_+ \quad \forall i,\ \forall \beta. \end{aligned} \xrightarrow{\text{solution}} \begin{bmatrix} \alpha_{11} \\ \alpha_{21} \\ \alpha_{12} \\ \alpha_{22} \\ \alpha_{13} \\ \alpha_{23} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 3 \\ 0 \\ 0 \end{bmatrix}$$

$\tau = 3$:

Reduced cost: $-\,975$

$$\begin{aligned} \min \quad & 11700 - 866.67 \cdot \alpha_{11} - 1733.33 \cdot \alpha_{12} - 3900 \cdot \alpha_{13} \\ & \qquad -650 \cdot \alpha_{21} - 1300 \cdot \alpha_{22} - 2925 \cdot \alpha_{23} \\ \text{s. t.} \quad & 4 \cdot \alpha_{11} + 4 \cdot \alpha_{12} + 4 \cdot \alpha_{13} + 3 \cdot \alpha_{21} + 3 \cdot \alpha_{22} \\ & \qquad\qquad\qquad\qquad +3 \cdot \alpha_{23} \leqslant 13 \\ & \alpha_{i,\beta} \in \mathbb{Z}_+ \quad \forall i,\ \forall \beta. \end{aligned} \xrightarrow{\text{solution}} \begin{bmatrix} \alpha_{11} \\ \alpha_{21} \\ \alpha_{12} \\ \alpha_{22} \\ \alpha_{13} \\ \alpha_{23} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 3 \end{bmatrix}$$

- $p =\ 4$

$\tau = 1$:

Reduced cost: $-\,217.17$

$$\begin{aligned} \min \quad & 2606 - 868.67 \cdot \alpha_{11} - 651.5 \cdot \alpha_{21} \\ \text{s. t.} \quad & 4 \cdot \alpha_{11} + 3 \cdot \alpha_{21} \leqslant 13 \\ & \alpha_{12} + \alpha_{13} + \alpha_{22} + \alpha_{23} = 0, \\ & \alpha_{i,\beta} \in \mathbb{Z}_+ \quad \forall i,\ \forall \beta. \end{aligned} \xrightarrow{\text{solution}} \begin{bmatrix} \alpha_{11} \\ \alpha_{21} \\ \alpha_{12} \\ \alpha_{22} \\ \alpha_{13} \\ \alpha_{23} \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\tau = 2$:

$$\begin{aligned} \min \quad & 5206 - 868.67 \cdot \alpha_{11} - 1735.33 \cdot \alpha_{12} \\ & \quad -651.5 \cdot \alpha_{21} - 1301.5 \cdot \alpha_{22} \\ \text{s. t.} \quad & 4 \cdot \alpha_{11} + 4 \cdot \alpha_{12} + 3 \cdot \alpha_{21} + 3 \cdot \alpha_{22} \leqslant 13 \\ & \alpha_{13} + \alpha_{23} = 0, \\ & \alpha_{i,\beta} \in \mathbb{Z}_+ \quad \forall i,\ \forall \beta. \end{aligned} \xrightarrow{\text{solution}} \overset{\text{Reduced cost: } -433.83}{\begin{bmatrix} \alpha_{11} \\ \alpha_{21} \\ \alpha_{12} \\ \alpha_{22} \\ \alpha_{13} \\ \alpha_{23} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 3 \\ 0 \\ 0 \end{bmatrix}}$$

$\tau = 3$:

$$\begin{aligned} \min \quad & 11706 - 868.67 \cdot \alpha_{11} - 1735.33 \cdot \alpha_{12} - 3902 \cdot \alpha_{13} \\ & \quad -651.5 \cdot \alpha_{21} - 1301.5 \cdot \alpha_{22} - 2926.5 \cdot \alpha_{23} \\ \text{s. t.} \quad & 4 \cdot \alpha_{11} + 4 \cdot \alpha_{12} + 4 \cdot \alpha_{13} + 3 \cdot \alpha_{21} + 3 \cdot \alpha_{22} \\ & \qquad + 3 \cdot \alpha_{23} \leqslant 13 \\ & \alpha_{i,\beta} \in \mathbb{Z}_+ \quad \forall i,\ \forall \beta. \end{aligned} \xrightarrow{\text{solution}} \overset{\text{Reduced cost: } -975.5}{\begin{bmatrix} \alpha_{11} \\ \alpha_{21} \\ \alpha_{12} \\ \alpha_{22} \\ \alpha_{13} \\ \alpha_{23} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 3 \end{bmatrix}}$$

- $p = 5$

$\tau = 1$:

$$\begin{aligned} \min \quad & 2612 - 870.67 \cdot \alpha_{11} - 653 \cdot \alpha_{21} \\ \text{s. t.} \quad & 4 \cdot \alpha_{11} + 3 \cdot \alpha_{21} \leqslant 13 \\ & \alpha_{12} + \alpha_{13} + \alpha_{22} + \alpha_{23} = 0, \\ & \alpha_{i,\beta} \in \mathbb{Z}_+ \quad \forall i,\ \forall \beta. \end{aligned} \xrightarrow{\text{solution}} \overset{\text{Reduced cost: } -217.67}{\begin{bmatrix} \alpha_{11} \\ \alpha_{21} \\ \alpha_{12} \\ \alpha_{22} \\ \alpha_{13} \\ \alpha_{23} \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}$$

$\tau = 2$:

$$\begin{aligned} \min \quad & 5212 - 870.67 \cdot \alpha_{11} + 200 \cdot \alpha_{12} \\ & \quad -653 \cdot \alpha_{21} - 1303 \cdot \alpha_{22} \\ \text{s. t.} \quad & 4 \cdot \alpha_{11} + 4 \cdot \alpha_{12} + 3 \cdot \alpha_{21} + 3 \cdot \alpha_{22} \leqslant 13 \\ & \alpha_{13} + \alpha_{23} = 0, \\ & \alpha_{i,\beta} \in \mathbb{Z}_+ \quad \forall i,\ \forall \beta. \end{aligned} \xrightarrow{\text{solution}} \overset{\text{Reduced cost:} 0}{\begin{bmatrix} \alpha_{11} \\ \alpha_{21} \\ \alpha_{12} \\ \alpha_{22} \\ \alpha_{13} \\ \alpha_{23} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 4 \\ 0 \\ 0 \end{bmatrix}}$$

$\tau = 3$:

$$\begin{aligned}
\min \quad & 11712 - 870.67 \cdot \alpha_{11} + 200 \cdot \alpha_{12} - 3904 \cdot \alpha_{13} \\
& \qquad -653 \cdot \alpha_{21} - 1303 \cdot \alpha_{22} + 150 \cdot \alpha_{23} \\
\text{s. t.} \quad & 4 \cdot \alpha_{11} + 4 \cdot \alpha_{12} + 4 \cdot \alpha_{13} + 3 \cdot \alpha_{21} + 3 \cdot \alpha_{22} \\
& \qquad\qquad\qquad\qquad +3 \cdot \alpha_{23} \leqslant 13 \\
& \alpha_{i,\beta} \in \mathbb{Z}_+ \quad \forall i,\ \forall \beta.
\end{aligned}
\xrightarrow{\text{solution}}
\begin{array}{c}
\text{Reduced cost:0} \\
\begin{bmatrix} \alpha_{11} \\ \alpha_{21} \\ \alpha_{12} \\ \alpha_{22} \\ \alpha_{13} \\ \alpha_{23} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 3 \\ 0 \end{bmatrix}
\end{array}$$

For each of these subproblems, the columns are evaluated whether their reduced costs are lower than or equal to -0.001. In these cases, the subproblems solutions are added as new cutting patterns followed by new variables in the RMP. Table A.7 contains the initial cutting patterns and the columns generated in the first iteration of the column generation procedure. After updating the RMP with the new cutting patterns, a new iteration is done.

Table A.7: Cutting patterns for the GGMPCSP numerical example - end of the first iteration.

			Periods 1, 2, 3 and 4									**Period 5**					
		RHS	X_{11p}	X_{21p}	$\mathbf{X_{31p}}$	X_{12p}	X_{22p}	$\mathbf{X_{32p}}$	X_{13p}	X_{23p}	$\mathbf{X_{33p}}$	X_{115}	X_{215}	$\mathbf{X_{315}}$	X_{125}	X_{225}	X_{135}	X_{235}
Demand balance constraints	=	d_{11p}	3	0	**1**													
	=	d_{21p}	0	4	**3**													
	=	d_{12p}				3	0	**1**										
	=	d_{22p}				0	4	**3**										
	=	d_{13p}							3	0	**1**							
	=	d_{23p}							0	4	**3**							
	=	d_{115}										3	0	**1**				
	=	d_{215}										0	4	**3**				
	=	d_{125}													1	0		
	=	d_{225}													0	4		
	=	d_{135}															2	0
	=	d_{235}															0	1

Iteration 2

Then, for the RMP with the new columns:

$$\begin{aligned}
\text{minimize}\quad & 2 \cdot S_{111} + 2 \cdot S_{112} + 2 \cdot S_{113} + 2 \cdot S_{114} + 200 \cdot S_{115} + 2 \cdot S_{121} + 2 \cdot S_{122} + 2 \cdot S_{123} \\
& + 2 \cdot S_{124} + 200 \cdot S_{125} + 2 \cdot S_{131} + 2 \cdot S_{132} + 2 \cdot S_{133} + 2 \cdot S_{134} + 200 \cdot S_{135} + 1.5 \cdot S_{211} \\
& + 1.5 \cdot S_{212} + 1.5 \cdot S_{213} + 1.5 \cdot S_{214} + 150 \cdot S_{215} + 1.5 \cdot S_{221} + 1.5 \cdot S_{222} + 1.5 \cdot S_{223} \\
& + 1.5 \cdot S_{224} + 150 \cdot S_{225} + 1.5 \cdot S_{231} + 1.5 \cdot S_{232} + 1.5 \cdot S_{233} + 1.5 \cdot S_{234} + 150 \cdot S_{235} \\
& + 2600 \cdot X_{111} + 2600 \cdot X_{112} + 2600 \cdot X_{113} + 2600 \cdot X_{114} + 2600 \cdot X_{115} + 5200 \cdot X_{121} \\
& + 5200 \cdot X_{122} + 5200 \cdot X_{123} + 5200 \cdot X_{124} + 5200 \cdot X_{125} + 11700 \cdot X_{131} + 11700 \cdot X_{132} \\
& + 11700 \cdot X_{133} + 11700 \cdot X_{134} + 11700 \cdot X_{135} + 2600 \cdot X_{211} + 2600 \cdot X_{212} + 2600 \cdot X_{213} \\
& + 2600 \cdot X_{214} + 2600 \cdot X_{215} + 5200 \cdot X_{221} + 2600 \cdot X_{122} + 5200 \cdot X_{223} + 5200 \cdot X_{224} \\
& + 5200 \cdot X_{225} + 11700 \cdot X_{231} + 11700 \cdot X_{232} + 11700 \cdot X_{233} + 11700 \cdot X_{234} \\
& + 11700 \cdot X_{235} + \mathbf{2600} \cdot \mathbf{X_{311}} + \mathbf{2600} \cdot \mathbf{X_{312}} + \mathbf{2600} \cdot \mathbf{X_{313}} + \mathbf{2600} \cdot \mathbf{X_{314}} \\
& + \mathbf{2600} \cdot \mathbf{X_{315}} + \mathbf{5200} \cdot \mathbf{X_{321}} + \mathbf{5200} \cdot \mathbf{X_{322}} + \mathbf{5200} \cdot \mathbf{X_{323}} + \mathbf{5200} \cdot \mathbf{X_{324}} \\
& + \mathbf{11700} \cdot \mathbf{X_{331}} + \mathbf{11700} \cdot \mathbf{X_{332}} + \mathbf{11700} \cdot \mathbf{X_{333}} + \mathbf{11700} \cdot \mathbf{X_{334}}
\end{aligned}$$

subject to

$$\begin{aligned}
& 3 \cdot X_{111} + 3 \cdot X_{121} + 3 \cdot X_{131} + \mathbf{1} \cdot \mathbf{X_{311}} + \mathbf{1} \cdot \mathbf{X_{321}} + \mathbf{1} \cdot \mathbf{X_{331}} - S_{111} = 1 \\
& 4 \cdot X_{211} + 4 \cdot X_{221} + 4 \cdot X_{231} + \mathbf{3} \cdot \mathbf{X_{311}} + \mathbf{3} \cdot \mathbf{X_{321}} + \mathbf{3} \cdot \mathbf{X_{331}} - S_{211} = 2 \\
& 3 \cdot X_{121} + 3 \cdot X_{131} + \mathbf{1} \cdot \mathbf{X_{321}} + \mathbf{1} \cdot \mathbf{X_{331}} - S_{121} = 0 \\
& 4 \cdot X_{221} + 4 \cdot X_{231} + \mathbf{3} \cdot \mathbf{X_{321}} + \mathbf{3} \cdot \mathbf{X_{331}} - S_{221} = 1 \\
& 3 \cdot X_{131} + \mathbf{1} \cdot \mathbf{X_{331}} - S_{131} = 1 \\
& 4 \cdot X_{231} + \mathbf{3} \cdot \mathbf{X_{331}} - S_{231} = 0 \\
& 3 \cdot X_{112} + 3 \cdot X_{122} + 3 \cdot X_{132} + \mathbf{1} \cdot \mathbf{X_{312}} + \mathbf{1} \cdot \mathbf{X_{322}} + \mathbf{1} \cdot \mathbf{X_{332}} + S_{111} - S_{112} = 0 \\
& 4 \cdot X_{212} + 4 \cdot X_{222} + 4 \cdot X_{232} + \mathbf{3} \cdot \mathbf{X_{312}} + \mathbf{3} \cdot \mathbf{X_{322}} + \mathbf{3} \cdot \mathbf{X_{332}} + S_{211} - S_{212} = 3 \\
& 3 \cdot X_{122} + 3 \cdot X_{132} + \mathbf{1} \cdot \mathbf{X_{322}} + \mathbf{1} \cdot \mathbf{X_{332}} + S_{121} - S_{122} = 2 \\
& 4 \cdot X_{222} + 4 \cdot X_{232} + \mathbf{3} \cdot \mathbf{X_{322}} + \mathbf{3} \cdot \mathbf{X_{332}} + S_{221} - S_{222} = 2 \\
& 3 \cdot X_{132} + \mathbf{1} \cdot \mathbf{X_{332}} + S_{131} - S_{132} = 0 \\
& 4 \cdot X_{232} + \mathbf{3} \cdot \mathbf{X_{332}} + S_{231} - S_{232} = 1 \\
& 3 \cdot X_{113} + 3 \cdot X_{123} + 3 \cdot X_{133} + \mathbf{1} \cdot \mathbf{X_{313}} + \mathbf{1} \cdot \mathbf{X_{323}} + \mathbf{1} \cdot \mathbf{X_{333}} + S_{112} - S_{113} = 3 \\
& 4 \cdot X_{213} + 4 \cdot X_{223} + 4 \cdot X_{233} + \mathbf{3} \cdot \mathbf{X_{313}} + \mathbf{3} \cdot \mathbf{X_{323}} + \mathbf{3} \cdot \mathbf{X_{333}} + S_{212} - S_{213} = 0 \\
& 3 \cdot X_{123} + 3 \cdot X_{133} + \mathbf{1} \cdot \mathbf{X_{323}} + \mathbf{1} \cdot \mathbf{X_{333}} + S_{122} - S_{123} = 0 \\
& 4 \cdot X_{223} + 4 \cdot X_{233} + \mathbf{3} \cdot \mathbf{X_{323}} + \mathbf{3} \cdot \mathbf{X_{333}} + S_{222} - S_{223} = 0 \\
& 3 \cdot X_{133} + \mathbf{1} \cdot \mathbf{X_{333}} + S_{132} - S_{133} = 3 \\
& 4 \cdot X_{233} + \mathbf{3} \cdot \mathbf{X_{333}} + S_{232} - S_{233} = 0 \\
& 3 \cdot X_{113} + 3 \cdot X_{123} + 3 \cdot X_{133} + \mathbf{1} \cdot \mathbf{X_{314}} + \mathbf{1} \cdot \mathbf{X_{324}} + \mathbf{1} \cdot \mathbf{X_{334}} + S_{113} - S_{114} = 0
\end{aligned}$$

$$4 \cdot X_{213} + 4 \cdot X_{223} + 4 \cdot X_{233} + \mathbf{3} \cdot \mathbf{X_{314}} + \mathbf{3} \cdot \mathbf{X_{324}} + \mathbf{3} \cdot \mathbf{X_{334}} + S_{213} - S_{214} = 2$$

$$3 \cdot X_{123} + 3 \cdot X_{133} + \mathbf{1} \cdot \mathbf{X_{324}} + \mathbf{1} \cdot \mathbf{X_{334}} + S_{123} - S_{124} = 4$$

$$4 \cdot X_{223} + 4 \cdot X_{233} + \mathbf{3} \cdot \mathbf{X_{324}} + \mathbf{3} \cdot \mathbf{X_{334}} + S_{223} - S_{224} = 1$$

$$3 \cdot X_{133} + \mathbf{1} \cdot \mathbf{X_{334}} + S_{133} - S_{134} = 0$$

$$4 \cdot X_{233} + \mathbf{3} \cdot \mathbf{X_{334}} + S_{233} - S_{234} = 2$$

$$3 \cdot X_{113} + 1 \cdot X_{123} + 2 \cdot X_{133} + \mathbf{1} \cdot \mathbf{X_{315}} + S_{114} - S_{115} = 5$$

$$4 \cdot X_{213} + 4 \cdot X_{223} + 1 \cdot X_{233} + \mathbf{3} \cdot \mathbf{X_{315}} + S_{214} - S_{215} = 6$$

$$1 \cdot X_{123} + 2 \cdot X_{133} + S_{124} - S_{125} = 0$$

$$4 \cdot X_{223} + 1 \cdot X_{233} + S_{224} - S_{225} = 6$$

$$2 \cdot X_{133} + S_{134} - S_{135} = 5$$

$$1 \cdot X_{233} + S_{234} - S_{235} = 0$$

$$X_{111} + X_{211} + \mathbf{X_{311}} + X_{121} + X_{221} + \mathbf{X_{321}} + X_{131} + X_{231} + \mathbf{X_{331}} \leqslant 3$$

$$X_{112} + X_{212} + \mathbf{X_{312}} + X_{122} + X_{222} + \mathbf{X_{322}} + X_{132} + X_{232} + \mathbf{X_{332}} \leqslant 3$$

$$X_{113} + X_{213} + \mathbf{X_{313}} + X_{123} + X_{223} + \mathbf{X_{323}} + X_{133} + X_{233} + \mathbf{X_{333}} \leqslant 3$$

$$X_{114} + X_{214} + \mathbf{X_{314}} + X_{124} + X_{224} + \mathbf{X_{324}} + X_{134} + X_{234} + \mathbf{X_{324}} \leqslant 3$$

$$X_{115} + X_{215} + \mathbf{X_{315}} + X_{125} + X_{225} + X_{135} + X_{235} \leqslant 3$$

$$S_{110} = 0$$

$$S_{210} = 0$$

$$S_{120} = 0$$

$$S_{220} = 0$$

$$S_{130} = 0$$

$$S_{230} = 0$$

$$S_{i,\beta,p},\ X_{j,\tau,p} \in \mathbb{R}_+ \qquad \forall i,\ \forall \beta,\ \forall \tau,\ \forall j,\ \forall p.$$

After solving it with CPLEX using the default settings, the following results are obtained:

Relaxed RMP Objective Value $= 62129.9$.

The solution for the RMP in the second iteration is given by:

$$\left[\begin{array}{ccccc}
X_{111} & X_{112} & X_{113} & X_{114} & X_{115} \\
X_{121} & X_{122} & X_{123} & X_{124} & X_{125} \\
X_{131} & X_{132} & X_{133} & X_{134} & X_{135} \\
X_{211} & X_{212} & X_{213} & X_{214} & X_{215} \\
X_{221} & X_{222} & X_{223} & X_{224} & X_{225} \\
X_{231} & X_{232} & X_{233} & X_{234} & X_{235} \\
X_{311} & X_{312} & X_{313} & X_{314} & X_{315} \\
X_{321} & X_{322} & X_{323} & X_{324} & \\
X_{331} & X_{332} & X_{333} & X_{334} & \\
\hdashline
S_{111} & S_{112} & S_{113} & S_{114} & S_{115} \\
S_{121} & S_{122} & S_{123} & S_{124} & S_{125} \\
S_{131} & S_{132} & S_{133} & S_{134} & S_{135} \\
S_{211} & S_{212} & S_{213} & S_{214} & S_{215} \\
S_{221} & S_{222} & S_{223} & S_{224} & S_{225} \\
S_{231} & S_{232} & S_{233} & S_{234} & S_{235}
\end{array}\right] = \left[\begin{array}{ccccc}
0.11 & 0 & 0.67 & 0 & 0.33 \\
0 & 0.22 & 0 & 0 & 0 \\
0.33 & 0 & 0.78 & 0 & 0.56 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0.67 & 1 & 0 & 0.67 & 1.33 \\
0.33 & 1 & 0 & 2 & \\
0 & 0.33 & 0.33 & 0.33 & \\
\hdashline
0 & 1 & 0 & 0.67 & 0 \\
0.33 & 0 & 0 & 0 & 0 \\
0 & 0.33 & 0 & 0.33 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 6 & 0 \\
0 & 0 & 1 & 0 & 0
\end{array}\right]$$

The dual optimal solutions $\lambda_{i,\beta,p}$ (related to the demand balance constraints) and μ_p (related to the capacity constraints) are the following:

$$\left[\begin{array}{ccccc}
\lambda_{111} & \lambda_{112} & \lambda_{113} & \lambda_{114} & \lambda_{115} \\
\lambda_{121} & \lambda_{122} & \lambda_{123} & \lambda_{124} & \lambda_{125} \\
\lambda_{131} & \lambda_{132} & \lambda_{133} & \lambda_{134} & \lambda_{135} \\
\lambda_{211} & \lambda_{212} & \lambda_{213} & \lambda_{214} & \lambda_{215} \\
\lambda_{221} & \lambda_{222} & \lambda_{223} & \lambda_{224} & \lambda_{225} \\
\lambda_{231} & \lambda_{232} & \lambda_{233} & \lambda_{234} & \lambda_{235} \\
\hdashline
\mu_1 & \mu_2 & \mu_3 & \mu_4 & \mu_5
\end{array}\right] = \left[\begin{array}{ccccc}
866.67 & 864.67 & 866.67 & 864.67 & 866.67 \\
1731.33 & 1733.33 & 1728.83 & 1726.83 & -200 \\
3900 & 3898 & 3900 & 3898 & 3900 \\
577.78 & 578.44 & 577.78 & 579.28 & 577.78 \\
1156.22 & 1155.56 & 1157.06 & 1158.56 & 1160.06 \\
2600 & 2600.67 & 2600 & 2601.5 & -150 \\
\hdashline
0 & 0 & 0 & -2.5 & 0
\end{array}\right]$$

Now, let variable $\alpha_{i,\beta}$ represent the number of items i with wire configuration β in the new cutting pattern. For each period p and leading wire configuration τ, the following knapsack subproblems are solved in order to find whether there is an attractive cutting pattern for the RMP.

(p = 1)

$$\begin{array}{ll} \min & 2600 - 866.67 \cdot \alpha_{11} - 577.78 \cdot \alpha_{21} \\ \text{s. t.} & 4 \cdot \alpha_{11} + 3 \cdot \alpha_{21} \leqslant 13 \\ & \alpha_{12} + \alpha_{13} + \alpha_{22} + \alpha_{23} = 0, \\ & \alpha_{i,\beta} \in \mathbb{Z}_+ \quad \forall i,\ \forall \beta. \end{array} \xrightarrow{\text{solution}} \begin{bmatrix} \alpha_{11} \\ \alpha_{21} \\ \alpha_{12} \\ \alpha_{22} \\ \alpha_{13} \\ \alpha_{23} \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Reduced cost: 0

$$\begin{array}{ll} \min & 5200 - 866.67 \cdot \alpha_{11} - 1731.33 \cdot \alpha_{12} \\ & \quad -577.78 \cdot \alpha_{21} - 1156.22 \cdot \alpha_{22} \\ \text{s. t.} & 4 \cdot \alpha_{11} + 4 \cdot \alpha_{12} + 3 \cdot \alpha_{21} + 3 \cdot \alpha_{22} \leqslant 13 \\ & \alpha_{13} + \alpha_{23} = 0, \\ & \alpha_{i,\beta} \in \mathbb{Z}_+ \quad \forall i,\ \forall \beta. \end{array} \xrightarrow{\text{solution}} \begin{bmatrix} \alpha_{11} \\ \alpha_{21} \\ \alpha_{12} \\ \alpha_{22} \\ \alpha_{13} \\ \alpha_{23} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 3 \\ 0 \\ 0 \end{bmatrix}$$

Reduced cost: 0

$$\begin{array}{ll} \min & 11700 - 866.67 \cdot \alpha_{11} - 1731.33 \cdot \alpha_{12} - 3900 \cdot \alpha_{13} \\ & \quad -577.78 \cdot \alpha_{21} - 1156.22 \cdot \alpha_{22} - 2600 \cdot \alpha_{23} \\ \text{s. t.} & 4 \cdot \alpha_{11} + 4 \cdot \alpha_{12} + 4 \cdot \alpha_{13} + 3 \cdot \alpha_{21} + 3 \cdot \alpha_{22} \\ & \qquad +3 \cdot \alpha_{23} \leqslant 13 \\ & \alpha_{i,\beta} \in \mathbb{Z}_+ \quad \forall i,\ \forall \beta. \end{array} \xrightarrow{\text{solution}} \begin{bmatrix} \alpha_{11} \\ \alpha_{21} \\ \alpha_{12} \\ \alpha_{22} \\ \alpha_{13} \\ \alpha_{23} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 3 \\ 0 \end{bmatrix}$$

Reduced cost: 0

(p = 2)

$$\begin{array}{ll} \min & 2600 - 864.67 \cdot \alpha_{11} - 578.44 \cdot \alpha_{21} \\ \text{s. t.} & 4 \cdot \alpha_{11} + 3 \cdot \alpha_{21} \leqslant 13 \\ & \alpha_{12} + \alpha_{13} + \alpha_{22} + \alpha_{23} = 0, \\ & \alpha_{i,\beta} \in \mathbb{Z}_+ \quad \forall i,\ \forall \beta. \end{array} \xrightarrow{\text{solution}} \begin{bmatrix} \alpha_{11} \\ \alpha_{21} \\ \alpha_{12} \\ \alpha_{22} \\ \alpha_{13} \\ \alpha_{23} \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Reduced cost: 0

$$\begin{aligned}
\min \quad & 5200 - 864.67 \cdot \alpha_{11} - 1733.33 \cdot \alpha_{12} \\
& \quad -578.44 \cdot \alpha_{21} - 1155.56 \cdot \alpha_{22} \\
\text{s. t.} \quad & 4 \cdot \alpha_{11} + 4 \cdot \alpha_{12} + 3 \cdot \alpha_{21} + 3 \cdot \alpha_{22} \leqslant 13 \\
& \alpha_{13} + \alpha_{23} = 0, \\
& \alpha_{i,\beta} \in \mathbb{Z}_+ \quad \forall i,\ \forall \beta.
\end{aligned}
\xrightarrow{\text{solution}}
\begin{bmatrix}\alpha_{11}\\\alpha_{21}\\\alpha_{12}\\\alpha_{22}\\\alpha_{13}\\\alpha_{23}\end{bmatrix} = \begin{bmatrix}0\\0\\3\\0\\0\\0\end{bmatrix}$$

Reduced cost: 0

$$\begin{aligned}
\min \quad & 11700 - 864.67 \cdot \alpha_{11} - 1733.33 \cdot \alpha_{12} - 3898 \cdot \alpha_{13} \\
& \quad -578.44 \cdot \alpha_{21} - 1155.56 \cdot \alpha_{22} - 2600.67 \cdot \alpha_{23} \\
\text{s. t.} \quad & 4 \cdot \alpha_{11} + 4 \cdot \alpha_{12} + 4 \cdot \alpha_{13} + 3 \cdot \alpha_{21} + 3 \cdot \alpha_{22} \\
& \qquad\qquad +3 \cdot \alpha_{23} \leqslant 13 \\
& \alpha_{i,\beta} \in \mathbb{Z}_+ \quad \forall i,\ \forall \beta.
\end{aligned}
\xrightarrow{\text{solution}}
\begin{bmatrix}\alpha_{11}\\\alpha_{21}\\\alpha_{12}\\\alpha_{22}\\\alpha_{13}\\\alpha_{23}\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\\1\\3\end{bmatrix}$$

Reduced cost: 0

(p = 3)

$$\begin{aligned}
\min \quad & 2600 - 866.67 \cdot \alpha_{11} - 577.78 \cdot \alpha_{21} \\
\text{s. t.} \quad & 4 \cdot \alpha_{11} + 3 \cdot \alpha_{21} \leqslant 13 \\
& \alpha_{12} + \alpha_{13} + \alpha_{22} + \alpha_{23} = 0, \\
& \alpha_{i,\beta} \in \mathbb{Z}_+ \quad \forall i,\ \forall \beta.
\end{aligned}
\xrightarrow{\text{solution}}
\begin{bmatrix}\alpha_{11}\\\alpha_{21}\\\alpha_{12}\\\alpha_{22}\\\alpha_{13}\\\alpha_{23}\end{bmatrix} = \begin{bmatrix}3\\0\\0\\0\\0\\0\end{bmatrix}$$

Reduced cost: 0

$$\begin{aligned}
\min \quad & 5200 - 866.67 \cdot \alpha_{11} - 1728.83 \cdot \alpha_{12} \\
& \quad -577.78 \cdot \alpha_{21} - 1157.06 \cdot \alpha_{22} \\
\text{s. t.} \quad & 4 \cdot \alpha_{11} + 4 \cdot \alpha_{12} + 3 \cdot \alpha_{21} + 3 \cdot \alpha_{22} \leqslant 13 \\
& \alpha_{13} + \alpha_{23} = 0, \\
& \alpha_{i,\beta} \in \mathbb{Z}_+ \quad \forall i,\ \forall \beta.
\end{aligned}
\xrightarrow{\text{solution}}
\begin{bmatrix}\alpha_{11}\\\alpha_{21}\\\alpha_{12}\\\alpha_{22}\\\alpha_{13}\\\alpha_{23}\end{bmatrix} = \begin{bmatrix}0\\0\\1\\3\\0\\0\end{bmatrix}$$

Reduced cost: 0

$$\begin{aligned} \min \quad & 11700 - 866.67 \cdot \alpha_{11} - 1728.83 \cdot \alpha_{12} - 3900 \cdot \alpha_{13} \\ & \quad -577.78 \cdot \alpha_{21} - 1157.06 \cdot \alpha_{22} - 2600 \cdot \alpha_{23} \\ \text{s. t.} \quad & 4 \cdot \alpha_{11} + 4 \cdot \alpha_{12} + 4 \cdot \alpha_{13} + 3 \cdot \alpha_{21} + 3 \cdot \alpha_{22} \\ & \qquad\qquad +3 \cdot \alpha_{23} \leqslant 13 \\ & \alpha_{i,\beta} \in \mathbb{Z}_+ \quad \forall i, \ \forall \beta. \end{aligned} \xrightarrow{\text{solution}} \begin{array}{c} \text{Reduced cost: } 0 \\ \begin{bmatrix} \alpha_{11} \\ \alpha_{21} \\ \alpha_{12} \\ \alpha_{22} \\ \alpha_{13} \\ \alpha_{23} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 3 \\ 0 \end{bmatrix} \end{array}$$

(p = 4)

$$\begin{aligned} \min \quad & 2602.5 - 864.67 \cdot \alpha_{11} - 579.28 \cdot \alpha_{21} \\ \text{s. t.} \quad & 4 \cdot \alpha_{11} + 3 \cdot \alpha_{21} \leqslant 13 \\ & \alpha_{12} + \alpha_{13} + \alpha_{22} + \alpha_{23} = 0, \\ & \alpha_{i,\beta} \in \mathbb{Z}_+ \quad \forall i, \ \forall \beta. \end{aligned} \xrightarrow{\text{solution}} \begin{array}{c} \text{Reduced cost: } 0 \\ \begin{bmatrix} \alpha_{11} \\ \alpha_{21} \\ \alpha_{12} \\ \alpha_{22} \\ \alpha_{13} \\ \alpha_{23} \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \end{array}$$

$$\begin{aligned} \min \quad & 5202.5 - 864.67 \cdot \alpha_{11} - 1726.83 \cdot \alpha_{12} \\ & \quad -579.28.5 \cdot \alpha_{21} - 1158.56 \cdot \alpha_{22} \\ \text{s. t.} \quad & 4 \cdot \alpha_{11} + 4 \cdot \alpha_{12} + 3 \cdot \alpha_{21} + 3 \cdot \alpha_{22} \leqslant 13 \\ & \alpha_{13} + \alpha_{23} = 0, \\ & \alpha_{i,\beta} \in \mathbb{Z}_+ \quad \forall i, \ \forall \beta. \end{aligned} \xrightarrow{\text{solution}} \begin{array}{c} \text{Reduced cost: } 0 \\ \begin{bmatrix} \alpha_{11} \\ \alpha_{21} \\ \alpha_{12} \\ \alpha_{22} \\ \alpha_{13} \\ \alpha_{23} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 3 \\ 0 \\ 0 \end{bmatrix} \end{array}$$

$$\begin{aligned} \min \quad & 11702.5 - 864.67 \cdot \alpha_{11} - 1726.83 \cdot \alpha_{12} - 3898 \cdot \alpha_{13} \\ & \quad -579.28 \cdot \alpha_{21} - 1158.56 \cdot \alpha_{22} - 2601.5 \cdot \alpha_{23} \\ \text{s. t.} \quad & 4 \cdot \alpha_{11} + 4 \cdot \alpha_{12} + 4 \cdot \alpha_{13} + 3 \cdot \alpha_{21} + 3 \cdot \alpha_{22} \\ & \qquad\qquad +3 \cdot \alpha_{23} \leqslant 13 \\ & \alpha_{i,\beta} \in \mathbb{Z}_+ \quad \forall i, \ \forall \beta. \end{aligned} \xrightarrow{\text{solution}} \begin{array}{c} \text{Reduced cost: } 0 \\ \begin{bmatrix} \alpha_{11} \\ \alpha_{21} \\ \alpha_{12} \\ \alpha_{22} \\ \alpha_{13} \\ \alpha_{23} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 3 \end{bmatrix} \end{array}$$

(p = 5)

$$\begin{aligned} \min \quad & 2600 - 866.67 \cdot \alpha_{11} - 577.78 \cdot \alpha_{21} \\ \text{s. t.} \quad & 4 \cdot \alpha_{11} + 3 \cdot \alpha_{21} \leqslant 13 \\ & \alpha_{12} + \alpha_{13} + \alpha_{22} + \alpha_{23} = 0, \\ & \alpha_{i,\beta} \in \mathbb{Z}_+ \quad \forall i,\ \forall \beta. \end{aligned} \xrightarrow{\text{solution}} \begin{bmatrix} \alpha_{11} \\ \alpha_{21} \\ \alpha_{12} \\ \alpha_{22} \\ \alpha_{13} \\ \alpha_{23} \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Reduced cost: 0

$$\begin{aligned} \min \quad & 5200 - 866.67 \cdot \alpha_{11} + 200 \cdot \alpha_{12} \\ & \quad -577.78 \cdot \alpha_{21} - 1160.06 \cdot \alpha_{22} \\ \text{s. t.} \quad & 4 \cdot \alpha_{11} + 4 \cdot \alpha_{12} + 3 \cdot \alpha_{21} + 3 \cdot \alpha_{22} \leqslant 13 \\ & \alpha_{13} + \alpha_{23} = 0, \\ & \alpha_{i,\beta} \in \mathbb{Z}_+ \quad \forall i,\ \forall \beta. \end{aligned} \xrightarrow{\text{solution}} \begin{bmatrix} \alpha_{11} \\ \alpha_{21} \\ \alpha_{12} \\ \alpha_{22} \\ \alpha_{13} \\ \alpha_{23} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 4 \\ 0 \\ 0 \end{bmatrix}$$

Reduced cost: 559.78

$$\begin{aligned} \min \quad & 11700 - 866.67 \cdot \alpha_{11} + 200 \cdot \alpha_{12} - 3900 \cdot \alpha_{13} \\ & \quad -577.78 \cdot \alpha_{21} - 1160.06 \cdot \alpha_{22} + 150 \cdot \alpha_{23} \\ \text{s. t.} \quad & 4 \cdot \alpha_{11} + 4 \cdot \alpha_{12} + 4 \cdot \alpha_{13} + 3 \cdot \alpha_{21} + 3 \cdot \alpha_{22} \\ & \qquad\qquad +3 \cdot \alpha_{23} \leqslant 13 \\ & \alpha_{i,\beta} \in \mathbb{Z}_+ \quad \forall i,\ \forall \beta. \end{aligned} \xrightarrow{\text{solution}} \begin{bmatrix} \alpha_{11} \\ \alpha_{21} \\ \alpha_{12} \\ \alpha_{22} \\ \alpha_{13} \\ \alpha_{23} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 3 \\ 0 \end{bmatrix}$$

Reduced cost: 0

Since all subproblems yielded non-negative reduced costs, there are no columns that would improve the RMP solution and the column generation procedure is discontinued. The integrality constraints related to the variables $X_{j\tau p}$ are inserted to the latter RMP, and the optimization package is applied to solve the resultant mixed-integer problem. The results of this heuristic applied to solve this example problem follow. First, the variables are given. Tables A.8, A.9 and A.10 present specific solution information. The processing time is expressed in seconds, the objective values are in currency units. And, in the columns containing the term '(%)', the values are percentages.

Variables:

$$
\left[\begin{array}{ccccc}
X_{111} & X_{112} & X_{113} & X_{114} & X_{115} \\
X_{121} & X_{122} & X_{123} & X_{124} & X_{125} \\
X_{131} & X_{132} & X_{133} & X_{134} & X_{135} \\
X_{211} & X_{212} & X_{213} & X_{214} & X_{215} \\
X_{221} & X_{222} & X_{223} & X_{224} & X_{225} \\
X_{231} & X_{232} & X_{233} & X_{234} & X_{235} \\
X_{311} & X_{312} & X_{313} & X_{314} & X_{315} \\
X_{321} & X_{322} & X_{323} & X_{324} & \\
X_{331} & X_{332} & X_{333} & X_{334} & \\
\hdashline
S_{111} & S_{112} & S_{113} & S_{114} & S_{115} \\
S_{121} & S_{122} & S_{123} & S_{124} & S_{125} \\
S_{131} & S_{132} & S_{133} & S_{134} & S_{135} \\
S_{211} & S_{212} & S_{213} & S_{214} & S_{215} \\
S_{221} & S_{222} & S_{223} & S_{224} & S_{225} \\
S_{231} & S_{232} & S_{233} & S_{234} & S_{235}
\end{array}\right]
=
\left[\begin{array}{ccccc}
0 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & \\
0 & 0 & 0 & 0 & \\
\hdashline
0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 \\
2 & 2 & 2 & 2 & 0 \\
1 & 2 & 2 & 0 & 0 \\
3 & 1 & 1 & 3 & 1 \\
0 & 3 & 3 & 1 & 1
\end{array}\right]
$$

Table A.8: Numerical example for GGMPCSP - CG results: solution status, processing time, objective value and distribution of costs.

Solution status	**Processing time**	**Mixed-Integer objective value**	**Inventory penalty cost (%)**	**Mold cost (%)**
Feasible	0.00	69250.00	0.51	99.49

Table A.9: Numerical example for GGMPCSP - CG results: lower bounds and GAPs.

Linear relaxation objective value	**GAP1 (%)**	**CG stop criterium**	**Total columns**	**No. of CG iterations**
62129.90	10.28	Optimality	43	1

Table A.10: Numerical example for GGMPCSP - CG results: mold usage.

Average no. of available molds per period	**Average no. of used molds per period**	**Used mold space (UMS)**	**Production space (PS)**	**PS/UMS (%)**
3	2.40	156.00	146.00	93.59

Appendix B - Hollow-core slab production planning detailed results

This appendix presents tables of more detailed computational results of Chapter 3. Tables B.1 and B.2 show the averages of the objective values and their distribution between inventory and mold costs for the MILP and RF heuristic applied to the KM-PCSP. Table B.3 shows the total number of cutting patterns in the demand matrix at the end of the CG procedure applied to the GGMPCSP, the processing time in seconds and the distribution of objective values between inventory and mold preparation costs.

Table B.1: Comparison between MILP and the RF heuristic results: objective values.

Class	**Linear relaxation objective value**	**Best lower bound provided by CPLEX**	**Mixed-Integer objective value**	
			MILP	**RF**
1	93,184.18	104,933.58	**105,246.34**	108,928.22
2	93,184.18	102,873.20	**103,480.80**	106,431.50
3	93,184.18	102,293.66	**103,480.42**	106,431.80
4	180,167.20	184,782.40	**190,399.60**	196,031.80
5	180,167.20	184,395.00	**190,723.60**	195,095.80
6	180,167.20	184,254.60	**190,402.20**	195,093.00
7	220,745.80	223,865.60	**234,321.20**	238,143.60
8	220,745.80	223,634.60	**234,194.80**	237,817.60
9	220,745.80	223,695.40	**234,359.00**	239,259.40
10	218,186.20	223,815.60	**227,808.40**	231,788.20
11	218,186.20	223,777.40	**227,799.60**	230,330.80
12	218,186.20	221,951.60	**227,851.60**	231,068.60
13	385,977.80	388,806.80	**398,962.00**	400,142.80
14	385,977.80	386,885.40	401,168.40	**398,890.40**
15	385,977.80	386,823.00	401,455.00	**395,684.00**
16	519,975.00	521,433.80	539,077.40	**532,334.20**
17	519,975.00	521,178.60	538,488.00	**532,671.60**
18	519,975.00	521,072.60	542,686.40	**532,622.80**
19	314,618.80	316,251.40	325,880.60	**325,231.20**
20	314,618.80	315,861.40	326,391.80	**325,248.60**
21	314,618.80	315,711.40	**326,347.80**	326,917.20
22	561,871.20	562,658.80	582,047.00	**574,246.00**
23	561,871.20	562,537.80	580,417.20	**574,298.40**
24	561,871.20	562,456.80	581,746.60	**574,913.80**
25	967,603.60	968,787.00	1,008,973.40	**984,728.80**
26	967,603.60	968,769.80	1,011,613.40	**984,200.40**
27	967,603.60	968,575.80	1,017,449.60	**983,989.80**
28	429,029.80	431,771.20	444,758.40	**440,346.00**
29	429,029.80	429,626.00	443,992.80	**440,107.00**
30	429,029.80	429,532.40	442,369.80	**439,765.20**
31	880,640.20	881,812.60	916,616.60	**895,887.80**
32	880,640.20	881,133.20	916,120.00	**895,166.00**
33	880,640.20	881,163.80	915,457.80	**896,421.40**
34	1,389,592.50	1,390,467.50	1,460,742.50	**1,410,727.50**
35	1,342,360.00	1,343,040.00	1,424,642.00	**1,361,106.00**
36	1,342,360.00	1,342,856.00	1,430,642.00	**1,362,386.00**
Average	510,842.00	513,430.16	534,669.84	**525,123.70**

Table B.2: Comparison between MILP and the RF heuristic results: distribution of costs.

Class	**Inventory penalty cost (%)**		**Mold cost (%)**	
	MILP	**RF**	**MILP**	**RF**
1	0.13	0.18	99.87	99.82
2	0.18	0.21	99.82	99.79
3	0.18	0.21	99.82	99.79
4	0.25	0.36	99.75	99.64
5	0.24	0.35	99.76	99.65
6	0.25	0.35	99.75	99.65
7	0.29	0.61	99.71	99.39
8	0.39	0.64	99.61	99.36
9	0.31	0.63	99.69	99.37
10	0.13	0.21	99.87	99.79
11	0.12	0.20	99.88	99.80
12	0.14	0.21	99.86	99.79
13	0.16	0.27	99.84	99.73
14	0.17	0.32	99.83	99.68
15	0.19	0.33	99.81	99.67
16	0.33	0.42	99.67	99.58
17	0.36	0.42	99.64	99.58
18	0.37	0.41	99.63	99.59
19	0.08	0.16	99.92	99.84
20	0.13	0.17	99.87	99.83
21	0.11	0.18	99.89	99.82
22	0.20	0.30	99.80	99.70
23	0.24	0.31	99.76	99.69
24	0.27	0.29	99.73	99.71
25	0.30	0.32	99.70	99.68
26	0.32	0.34	99.68	99.66
27	0.34	0.36	99.66	99.64
28	0.08	0.14	99.92	99.86
29	0.11	0.17	99.89	99.83
30	0.11	0.17	99.89	99.83
31	0.16	0.25	99.84	99.75
32	0.17	0.25	99.83	99.75
33	0.25	0.30	99.75	99.70
34	0.19	0.30	99.81	99.70
35	0.25	0.31	99.75	99.69
36	0.40	0.36	99.60	99.64
Average	0.22	0.31	99.78	99.69

Table B.3: GGMPCSP CG procedure results: total number of columns, processing time and distribution of costs.

Class	**Total number of columns**	**Processing time**	**Inventory penalty cost (%)**	**Mold cost (%)**
1	401.00	0.30	0.63	99.37
2	399.20	0.20	0.82	99.18
3	399.20	0.20	0.82	99.18
4	859.00	1,795.67	0.98	99.02
5	858.20	1,035.94	0.96	99.04
6	857.60	827.22	1.00	99.00
7	1,344.20	1,702.94	1.55	98.45
8	1,343.40	1,670.20	1.55	98.45
9	1,343.40	1,627.97	1.58	98.42
10	1,039.20	1,793.11	0.50	99.50
11	1,036.60	1,091.03	0.73	99.27
12	1,036.80	1,081.29	0.79	99.21
13	-	-	-	-
14	-	-	-	-
15	2,178.00	1,798.22	1.13	98.87
16	-	-	-	-
17	-	-	-	-
18	-	-	-	-
19	1,644.00	1,274.83	0.84	99.16
20	1,639.00	1,796.72	0.84	99.16
21	1,639.00	1,797.16	0.84	99.16
Average	**1,126.11**	**1,205.81**	**0.97**	**99.03**

Appendix C - Variations of ICG with two steps and SCG strategies

This appendix discusses the variations V-ICG and V-SCG of the strategies ICG with two steps and SCG, respectively, proposed and applied in Chapter 4, as well as explains the lack of success when applying these variants to the proposed model in Section 4.2.

C.1 Variant of the Integrated column generation with two steps (V-ICG)

In the V_ICG strategy, the column generation procedure is performed as described in Section 4.3.1. The first step after the column generation procedure is to add integrality constraints and solve the RMP sub-model regarding Stage 1 (Model (C.1)-(C.4)):

$$\min \sum_{p\in P}\sum_{j_S\in J_p^S} c_p^S \cdot W^S \cdot X_{jsp}^S + \sum_{p\in P}\sum_{i\in I} h_{ip}^S \cdot S_{ip}^S \tag{C.1}$$

s. t.

$$\sum_{j_S\in J_p^S} b_{ijsp} \cdot X_{jsp}^S + S_{i,(p-1)}^S - S_{i,p}^S = d_{ip}, \qquad i\in I,\ p\in P \tag{C.2}$$

$$S_{ip}^S \in \mathbb{R}_+, \qquad i\in I,\ p\in P \tag{C.3}$$

$$X_{jsp}^S \in \mathbb{Z}_+, \qquad p\in P, j_S\in J_p^S. \tag{C.4}$$

Next, the second step consists of recovering the values of the variables $X_{j_Ckp}^S$ from Model (C.1)-(C.4) solution that are used as parameters $\bar{X}_{j_Ckp}^S$ to formulate the sub-model related to Stage 2 (Model (C.5)-(C.12)):

$$\begin{aligned}\min \ & \sum_{p\in P}\sum_{k\in K_p} sc_p^{mold} \cdot Y_{kp}^{mold} + \sum_{p\in P}\sum_{k\in K_p}\sum_{j_C\in J_p^C} sc_p^{pat} \cdot Y_{j_Ckp}^{pat} \\ & + \sum_{p\in P}\sum_{k\in K_p}\sum_{j_C\in J_p^C} c_p^C \cdot W^C \cdot X_{j_Ckp}^C + \sum_{p\in P}\sum_{i\in I} h_{ip}^C \cdot S_{ip}^C\end{aligned} \tag{C.5}$$

s. t.

$$\sum_{k \in K_p} \sum_{j_C \in J_p^C} a_{ij_Cp} \cdot X_{j_Ckp}^C + S_{i,(p-1)}^C - S_{i,p}^C = \sum_{j_S \in J_p^S} b_{ij_Sp} \cdot \bar{X}_{j_Sp}^S, \qquad i \in I,\ p \in P \tag{C.6}$$

$$\sum_{j_C \in J_p^C} X_{j_Ckp}^C \leqslant N \cdot Y_{kp}^{mold}, \qquad p \in P,\ k \in K_p \tag{C.7}$$

$$X_{j_Ckp}^C \leqslant N \cdot Y_{j_Ckp}^{pat}, \qquad p \in P,\ j_C \in J_p^C,\ k \in K_p \tag{C.8}$$

$$S_{ip}^C, \in \mathbb{R}_+, \qquad i \in I,\ p \in P \tag{C.9}$$

$$X_{j_Ckp}^C \in \mathbb{Z}_+, \qquad p \in P,\ j_C \in J_p^C,\ k \in K_p \tag{C.10}$$

$$Y_{kp}^{mold} \in \mathbb{B}, \qquad p \in P,\ k \in K_p \tag{C.11}$$

$$Y_{j_Ckp}^{pat} \in \mathbb{B}, \qquad p \in P,\ j_C \in J_p^C,\ k \in K_p. \tag{C.12}$$

C.2 Variant of the Separated column generation procedure (V-SCG)

Likewise the SCG approach, in the V-SCG strategy, two separated column generations are performed. At first, the column generation is applied to the linear relaxation of the Stage 1 sub-model (Model (C.1)-(C.2)) and then, after adding the integrality constraints, the resultant mixed-integer linear problem RMP is solved. Thereafter, the integer solution of the Stage 1 is used to define the linear relaxation of the Stage 2 sub-model (Model (C.5)-(C.9)), which is solved by column generation. Finally, the integrality constraints are added to the last RMP and the resultant mixed-integer linear RMP is solved.

C.3 Explanation of the non-applicability

Note that *it is not guaranteed that* the solutions of Models (C.1)-(C.4) and (C.5)-(C.12) compose a feasible solution to the integrated problem, since the original constraints of lattice joist demand balance (4.2) and two stages integration (4.3) were modified and, hence, are *not* directly satisfied.

In practice, a lattice joist of length w_i is made only if there is available a steel truss cut in w_i length. Since constraints (C.6) allow that a lattice joist can be produced without a steel truss, leading to an incongruity.

Following, an example of a feasible solution that turns these variations impractical. Fixed an item $i = 1$ and a period $p = 1$, suppose that $w_1 = 2$, $W^S = 12$, $b_{1,1,1} = 6$ and $d_{11} = 22$, a feasible solution to Stage 1 sub-model (Model (C.1)-(C.4)) is given by $X^S_{1,1} = 4$, $S^S_{1,0} = 0$ and $S^S_{1,1} = 2$, since

$$\begin{array}{lccccccccc} & \sum\limits_{j_S \in J^S_p} b_{ij_Sp} & \cdot & X^S_{j_Sp} & + & S^S_{i,(p-1)} & - & S^S_{i,p} & = & d_{ip} \\ \Longrightarrow & 6 & \cdot & 4 & + & 0 & - & 2 & = & 22, \end{array}$$

that is, constraint (C.2) is satisfied. Using $\bar{X}^S_{1,1} = 4$ to define the sub-model related to Stage 2 (Model (C.5)-(C.12)) and considering $a_{1,1,1} = 5$ and $N = 10$ a feasible solution is $Y^{mold}_{11} = 1$, $Y^{pat}_{111} = 1$, $X^C_{1,1,1} = 6$, $S^C_{1,0} = 0$ and $S^C_{1,1} = 6$, given that constraints (C.6):

$$\begin{array}{lccccccccccc} & \sum\limits_{k \in K_p} \sum\limits_{j_C \in J^C_p} a_{ij_Cp} & \cdot & X^C_{j_Ckp} & + & S^C_{i,(p-1)} & - & S^C_{i,p} & = & \sum\limits_{j_S \in J^S_p} b_{ij_Sp} & \cdot & \bar{X}^S_{j_Sp} \\ \Longrightarrow & 5 & \cdot & 6 & + & 0 & - & 6 & = & 6 & \cdot & 4 \end{array},$$

constraints (C.7):

$$\begin{array}{lccccc} & \sum\limits_{j_C \in J^C_p} X^C_{j_Ckp} & \leqslant & N & \cdot & Y^{mold}_{kp} \\ \Longrightarrow & 6 & \leqslant & 10 & \cdot & 1 \end{array}$$

and constraints (C.8):

$$\begin{array}{lccccc} & X^C_{j_Ckp} & \leqslant & N & \cdot & Y^{pat}_{j_Ckp} \\ \Longrightarrow & 6 & \leqslant & 10 & \cdot & 1 \end{array}$$

are all satisfied. Note that, to produce $5 \cdot 6 = 30$ lattice joists with $w_1 = 2$, it is necessary that at least 30 units of steel trusses with $w_1 = 2$ were available in the same period. However, in this example, only $6 \cdot 4 = 24$ were cut in Stage 1, turning these solutions impractical to the factory.

Appendix D - General and numerical example for the lattice slab production planning

In this appendix, we present a general example in Section D.1 to illustrate the model proposed in Section 4.2 and to explain in more detail how the subproblems are formulated according to the ICG solution strategy. In addition, Section D.2 shows how this method is applied in a numerical example.

D.1 General example

Suppose that there are $|I|= 2$ different items that must be produced in $|P|= 3$ periods in $|K_p|= 2$ molds for each $p \in P$. In addition, for each period $p \in P$, consider that $|J_p^C|= 2$ and $|J_p^S|= 2$ are the number of cutting patterns available at the first column generation procedure. The RMP of the first ICG iteration is given as follows (note that all terms involving variables were adapted to be in the left hand side of the equations and inequalities).

$$\begin{aligned}
\min \quad & sc_1^{mold} \cdot Y_{11}^{mold} + sc_1^{mold} \cdot Y_{21}^{mold} + sc_2^{mold} \cdot Y_{12}^{mold} + sc_2^{mold} \cdot Y_{22}^{mold} \\
& + sc_3^{mold} \cdot Y_{13}^{mold} + sc_3^{mold} \cdot Y_{23}^{mold} + sc_1^{pat} \cdot Y_{111}^{pat} + sc_1^{pat} \cdot Y_{211}^{pat} + sc_1^{pat} \cdot Y_{121}^{pat} \\
& + sc_1^{pat} \cdot Y_{221}^{pat} + sc_2^{pat} \cdot Y_{112}^{pat} + sc_2^{pat} \cdot Y_{212}^{pat} + sc_2^{pat} \cdot Y_{122}^{pat} + sc_2^{pat} \cdot Y_{222}^{pat} \\
& + sc_3^{pat} \cdot Y_{113}^{pat} + sc_3^{pat} \cdot Y_{213}^{pat} + sc_3^{pat} \cdot Y_{123}^{pat} + sc_3^{pat} \cdot Y_{223}^{pat} + c_1^C \cdot W^C \cdot X_{111}^C \\
& + c_1^C \cdot W^C \cdot X_{211}^C + c_1^C \cdot W^C \cdot X_{121}^C + c_1^C \cdot W^C \cdot X_{221}^C + c_2^C \cdot W^C \cdot X_{112}^C \\
& + c_2^C \cdot W^C \cdot X_{212}^C + c_2^C \cdot W^C \cdot X_{122}^C + c_2^C \cdot W^C \cdot X_{222}^C + c_3^C \cdot W^C \cdot X_{113}^C \\
& + c_3^C \cdot W^C \cdot X_{213}^C + c_3^C \cdot W^C \cdot X_{123}^C + c_3^C \cdot W^C \cdot X_{223}^C + c_1^S \cdot W^S \cdot X_{11}^S \\
& + c_1^S \cdot W^S \cdot X_{21}^S + c_2^S \cdot W^S \cdot X_{12}^S + c_2^S \cdot W^S \cdot X_{22}^S + c_3^S \cdot W^S \cdot X_{13}^S \\
& + c_3^S \cdot W^S \cdot X_{23}^S + h_{11}^C \cdot S_{11}^C + h_{21}^C \cdot S_{21}^C + h_{12}^C \cdot S_{12}^C + h_{22}^C \cdot S_{22}^C + h_{13}^C \cdot S_{13}^C \\
& + h_{23}^C \cdot S_{23}^C + h_{11}^S \cdot S_{11}^S + h_{21}^S \cdot S_{21}^S + h_{12}^S \cdot S_{12}^S + h_{22}^S \cdot S_{22}^S + h_{13}^S \cdot S_{13}^S + h_{23}^S \cdot S_{23}^S
\end{aligned} \tag{D.1}$$

s. t.

$$a_{111} \cdot X_{111}^C + a_{111} \cdot X_{121}^C + a_{121} \cdot X_{211}^C + a_{121} \cdot X_{221}^C - S_{11}^C = d_{11} \tag{D.2}$$

$$a_{211} \cdot X^C_{111} + a_{211} \cdot X^C_{121} + a_{221} \cdot X^C_{211} + a_{221} \cdot X^C_{221} - S^C_{21} = d_{21} \tag{D.3}$$

$$a_{112} \cdot X^C_{112} + a_{112} \cdot X^C_{122} + a_{122} \cdot X^C_{212} + a_{122} \cdot X^C_{222} + S^C_{11} - S^C_{12} = d_{12} \tag{D.4}$$

$$a_{212} \cdot X^C_{112} + a_{212} \cdot X^C_{122} + a_{222} \cdot X^C_{212} + a_{222} \cdot X^C_{222} + S^C_{21} - S^C_{22} = d_{22} \tag{D.5}$$

$$a_{113} \cdot X^C_{113} + a_{113} \cdot X^C_{123} + a_{123} \cdot X^C_{213} + a_{123} \cdot X^C_{223} + S^C_{12} - S^C_{13} = d_{13} \tag{D.6}$$

$$a_{213} \cdot X^C_{113} + a_{213} \cdot X^C_{123} + a_{223} \cdot X^C_{213} + a_{223} \cdot X^C_{223} + S^C_{22} - S^C_{23} = d_{23} \tag{D.7}$$

$$\begin{split} b_{111} \cdot X^S_{11} + b_{121} \cdot X^S_{21} - S^S_{11} - a_{111} \cdot X^C_{111} - a_{111} \cdot X^C_{121} - a_{121} \cdot X^C_{211} \\ - a_{121} \cdot X^C_{221} = 0 \end{split} \tag{D.8}$$

$$\begin{split} b_{211} \cdot X^S_{11} + b_{221} \cdot X^S_{21} - S^S_{21} - a_{211} \cdot X^C_{111} - a_{211} \cdot X^C_{121} - a_{221} \cdot X^C_{211} \\ - a_{221} \cdot X^C_{221} = 0 \end{split} \tag{D.9}$$

$$\begin{split} b_{112} \cdot X^S_{12} + b_{122} \cdot X^S_{22} + S^S_{11} - S^S_{12} - a_{112} \cdot X^C_{112} - a_{112} \cdot X^C_{122} - a_{122} \cdot X^C_{212} \\ - a_{122} \cdot X^C_{222} = 0 \end{split} \tag{D.10}$$

$$\begin{split} b_{212} \cdot X^S_{12} + b_{222} \cdot X^S_{22} + S^S_{21} - S^S_{22} - a_{212} \cdot X^C_{112} - a_{212} \cdot X^C_{122} - a_{222} \cdot X^C_{212} \\ - a_{222} \cdot X^C_{222} = 0 \end{split} \tag{D.11}$$

$$\begin{split} b_{113} \cdot X^S_{13} + b_{123} \cdot X^S_{23} + S^S_{12} - S^S_{13} - a_{113} \cdot X^C_{113} - a_{113} \cdot X^C_{123} - a_{123} \cdot X^C_{213} \\ - a_{123} \cdot X^C_{223} = 0 \end{split} \tag{D.12}$$

$$\begin{split} b_{213} \cdot X^S_{13} + b_{223} \cdot X^S_{23} + S^S_{22} - S^S_{23} - a_{213} \cdot X^C_{113} - a_{213} \cdot X^C_{123} - a_{223} \cdot X^C_{213} \\ - a_{223} \cdot X^C_{223} = 0 \end{split} \tag{D.13}$$

$$X^C_{111} + X^C_{211} - N \cdot Y^{mold}_{11} \leqslant 0 \tag{D.14}$$

$$X^C_{121} + X^C_{221} - N \cdot Y^{mold}_{21} \leqslant 0 \tag{D.15}$$

$$X^C_{112} + X^C_{212} - N \cdot Y^{mold}_{12} \leqslant 0 \tag{D.16}$$

$$X^C_{122} + X^C_{222} - N \cdot Y^{mold}_{22} \leqslant 0 \tag{D.17}$$

$$X^C_{113} + X^C_{213} - N \cdot Y^{mold}_{13} \leqslant 0 \tag{D.18}$$

$$X^C_{123} + X^C_{223} - N \cdot Y^{mold}_{23} \leqslant 0 \tag{D.19}$$

$$X^C_{111} - N \cdot Y^{pat}_{111} \leqslant 0 \tag{D.20}$$

$$X^C_{211} - N \cdot Y^{pat}_{211} \leqslant 0 \tag{D.21}$$

$$X^C_{121} - N \cdot Y^{pat}_{121} \leqslant 0 \tag{D.22}$$

$$X^C_{221} - N \cdot Y^{pat}_{221} \leqslant 0 \tag{D.23}$$

$$X^C_{112} - N \cdot Y^{pat}_{112} \leqslant 0 \tag{D.24}$$

$$X^C_{212} - N \cdot Y^{pat}_{212} \leqslant 0 \tag{D.25}$$

$$X^C_{122} - N \cdot Y^{pat}_{122} \leqslant 0 \tag{D.26}$$

$$X^C_{222} - N \cdot Y^{pat}_{222} \leqslant 0 \tag{D.27}$$

$$X^C_{113} - N \cdot Y^{pat}_{113} \leqslant 0 \tag{D.28}$$

$$X^C_{213} - N \cdot Y^{pat}_{213} \leqslant 0 \tag{D.29}$$

$$X^C_{123} - N \cdot Y^{pat}_{123} \leqslant 0 \tag{D.30}$$

$$X_{223}^{C} - N \cdot Y_{223}^{pat} \leqslant 0 \tag{D.31}$$

$$\begin{aligned}
&S_{ip}^{C},\ S_{ip}^{S} \in \mathbb{R}_{+}, && i \in I,\ p \in P, \\
&X_{j_C kp}^{C},\ X_{j_S p}^{S} \in {}_{+}, && i \in I,\ p \in P, \\
&0 \leqslant Y_{kp}^{mold},\ Y_{j_C kp}^{pat} \leqslant 1, && p \in P,\ j_C \in J_p^C,\ k \in K_p.
\end{aligned} \tag{D.32}$$

Let the dual variables λ_{ip} be associated to the demand balance constraints (D.2)-(D.7), σ_{ip} be related to the integration constraints (D.8)-(D.13), and μ_{kp} be related to the mold capacity constraints (D.14)-(D.19). The corresponding dual problem is given below.

$$\text{maximize} \quad d_{11} \cdot \lambda_{11} + d_{21} \cdot \lambda_{21} + d_{12} \cdot \lambda_{12} + d_{22} \cdot \lambda_{22} + d_{13} \cdot \lambda_{13} + d_{23} \cdot \lambda_{23}$$

subject to

$$\begin{aligned}
&a_{111} \cdot \lambda_{11} + a_{211} \cdot \lambda_{21} - a_{111} \cdot \sigma_{11} - a_{211} \cdot \sigma_{21} + \mu_{11} + \mu_{21} \leqslant c_1^C \\
&a_{121} \cdot \lambda_{11} + a_{221} \cdot \lambda_{21} - a_{121} \cdot \sigma_{11} - a_{221} \cdot \sigma_{21} + \mu_{11} + \mu_{21} \leqslant c_1^C \\
&a_{112} \cdot \lambda_{12} + a_{212} \cdot \lambda_{22} - a_{112} \cdot \sigma_{12} - a_{212} \cdot \sigma_{22} + \mu_{12} + \mu_{22} \leqslant c_2^C \\
&a_{122} \cdot \lambda_{12} + a_{222} \cdot \lambda_{22} - a_{122} \cdot \sigma_{12} - a_{222} \cdot \sigma_{22} + \mu_{12} + \mu_{22} \leqslant c_2^C \\
&a_{113} \cdot \lambda_{13} + a_{213} \cdot \lambda_{23} - a_{113} \cdot \sigma_{13} - a_{213} \cdot \sigma_{23} + \mu_{13} + \mu_{23} \leqslant c_3^C \\
&a_{123} \cdot \lambda_{13} + a_{223} \cdot \lambda_{23} - a_{123} \cdot \sigma_{13} - a_{223} \cdot \sigma_{23} + \mu_{13} + \mu_{23} \leqslant c_3^C \\
&-\lambda_{11} + \lambda_{12} \leqslant h_{11}^C \\
&-\lambda_{21} + \lambda_{22} \leqslant h_{21}^C \\
&-\lambda_{12} + \lambda_{13} \leqslant h_{12}^C \\
&-\lambda_{22} + \lambda_{23} \leqslant h_{22}^C \\
&-\lambda_{13} \leqslant h_{13}^C \\
&-\lambda_{23} \leqslant h_{23}^C \\
&b_{111} \cdot \sigma_{11} + b_{211} \cdot \sigma_{21} \leqslant c_1^S \\
&b_{121} \cdot \sigma_{11} + b_{221} \cdot \sigma_{21} \leqslant c_1^S \\
&b_{112} \cdot \sigma_{12} + b_{212} \cdot \sigma_{22} \leqslant c_2^S \\
&b_{122} \cdot \sigma_{12} + b_{222} \cdot \sigma_{22} \leqslant c_2^S \\
&b_{113} \cdot \sigma_{13} + b_{213} \cdot \sigma_{23} \leqslant c_3^S \\
&b_{123} \cdot \sigma_{13} + b_{223} \cdot \sigma_{23} \leqslant c_3^S \\
&-\sigma_{11} + \sigma_{12} \leqslant h_{11}^S \\
&-\sigma_{21} + \sigma_{22} \leqslant h_{21}^S \\
&-\sigma_{12} + \sigma_{13} \leqslant h_{12}^S \\
&-\sigma_{22} + \sigma_{23} \leqslant h_{22}^S
\end{aligned}$$

$$
\begin{aligned}
&-\sigma_{13} \leqslant h_{13}^S \\
&-\sigma_{23} \leqslant h_{23}^S \\
&-N \cdot \mu_{11} \leqslant sc_1^{mold} \\
&-N \cdot \mu_{21} \leqslant sc_1^{mold} \\
&-N \cdot \mu_{12} \leqslant sc_2^{mold} \\
&-N \cdot \mu_{22} \leqslant sc_2^{mold} \\
&-N \cdot \mu_{13} \leqslant sc_3^{mold} \\
&-N \cdot \mu_{23} \leqslant sc_3^{mold} \\
&\lambda_{ip} \in \mathbb{R},\ \sigma_{ip} \in \mathbb{R},\ \mu_{kp} \leqslant 0 \qquad \forall i,\ \forall p
\end{aligned}
$$

Tables D.1, D.2, D.3, D.4 and D.5 contain the objective functions and constraints coefficients of the primal and dual problems.

The RMP (primal) is solved, the dual optimal solution is retrieved and the dual variables are used to formulate the subproblems. A new column is attractive if the associated reduced cost is negative. In this case, by duality, the first block of dual constraints correspond to the inequalities that must be inactive for a new column j'_C from Stage 2 to be interesting to be added in the RMP:

$$\sum_{i=1}^{2} a_{ij'_Cp} \cdot \lambda_{ip} - \sum_{i=1}^{2} \sum a_{ij'_Cp} \cdot \sigma_{ip} + \sum_{k=1}^{2} \mu_{kp} < c_p^C,$$

and the third block of dual constraints correspond to the inequalities that must be inactive for a new column j'_S from Stage 1 to be attractive:

$$\sum_{i=1}^{2} b_{ij'_Sp} \cdot \sigma_{ip} \leqslant c_p^S,$$

where $p \in P$. Moreover, these columns must also satisfy capacity constraints regarding the mold section length:

$$\sum_{i=1}^{2} w_i \cdot a_{ij'_Sp} \leqslant W^S \text{ in Stage 1,}$$

and

$$\sum_{i=1}^{2} w_i \cdot a_{ij'_Cp} \leqslant W^C \text{ in Stage 2.}$$

Finally, there must also be an integrality constraint over these new columns. Based on these arguments, the subproblems are given as follows.

Stage 1: Steel truss cutting subproblems

$$\begin{aligned} \mathrm{OF}^{\mathrm{S}}_{\mathrm{SUB}}(1) = \min \quad & c^S_1 \cdot W^S - \sigma_{11} \cdot \alpha^S_1 - \sigma_{21} \cdot \alpha^S_2 \\ \text{s. t.} \quad & w_1 \cdot \alpha^S_1 + w_2 \cdot \alpha^S_2 \leqslant W^S \\ & \alpha^S_1, \alpha^S_2 \in \mathbb{Z}_+ \end{aligned}$$

$$\begin{aligned} \mathrm{OF}^{\mathrm{S}}_{\mathrm{SUB}}(2) = \min \quad & c^S_2 \cdot W^S - \sigma_{12} \cdot \alpha^S_1 - \sigma_{22} \cdot \alpha^S_2 \\ \text{s. t.} \quad & w_1 \cdot \alpha^S_1 + w_2 \cdot \alpha^S_2 \leqslant W^S \\ & \alpha^S_1, \alpha^S_2 \in \mathbb{Z}_+ \end{aligned}$$

$$\begin{aligned} \mathrm{OF}^{\mathrm{S}}_{\mathrm{SUB}}(3) = \min \quad & c^S_2 \cdot W^S - \sigma_{13} \cdot \alpha^S_1 - \sigma_{23} \cdot \alpha^S_2 \\ \text{s. t.} \quad & w_1 \cdot \alpha^S_1 + w_2 \cdot \alpha^S_2 \leqslant W^S \\ & \alpha^S_1, \alpha^S_2 \in \mathbb{Z}_+ \end{aligned}$$

Stage 2: Mold section concrete filling subproblems

$$\begin{aligned} \mathrm{OF}^{\mathrm{C}}_{\mathrm{SUB}}(1) = \min_{k \in \{1,2\}} \quad & c^C_1 \cdot W^C - (\lambda_{11} - \sigma_{11}) \cdot \alpha^C_1 - (\lambda_{21} - \sigma_{21}) \cdot \alpha^C_2 - \mu_{k1} \\ \text{s. t.} \quad & w_1 \cdot \alpha^C_1 + w_2 \cdot \alpha^C_2 \leqslant W^C \\ & \alpha^C_1, \alpha^C_2 \in \mathbb{Z}_+ \end{aligned}$$

$$\begin{aligned} \mathrm{OF}^{\mathrm{C}}_{\mathrm{SUB}}(2) = \min_{k \in \{1,2\}} \quad & c^C_2 \cdot W^C - (\lambda_{12} - \sigma_{12}) \cdot \alpha^C_1 - (\lambda_{22} - \sigma_{22}) \cdot \alpha^C_2 - \mu_{k2} \\ \text{s. t.} \quad & w_1 \cdot \alpha^C_1 + w_2 \cdot \alpha^C_2 \leqslant W^C \\ & \alpha^C_1, \alpha^C_2 \in \mathbb{Z}_+ \end{aligned}$$

$$\begin{aligned} \mathrm{OF}^{\mathrm{C}}_{\mathrm{SUB}}(3) = \min_{k \in \{1,2\}} \quad & c^C_3 \cdot W^C - (\lambda_{13} - \sigma_{13}) \cdot \alpha^C_1 - (\lambda_{23} - \sigma_{23}) \cdot \alpha^C_2 - \mu_{k3} \\ \text{s. t.} \quad & w_1 \cdot \alpha^C_1 + w_2 \cdot \alpha^C_2 \leqslant W^C \\ & \alpha^C_1, \alpha^C_2 \in \mathbb{Z}_+ \end{aligned}$$

These subproblems are solved and their objective values (reduced costs) are evaluated to see whether they are negative so that their solutions can be added as new columns to the new RMP. It is important to note that, since all molds are identical, it wouldn't be coherent to associate different sets of columns to each mold. For this reason, in Stage 2 subproblems, the minimum is taken over all molds $k \in K$ and only the generated column associated with the lowest reduced cost is added to the RMP, where the cutting pattern can be used in all molds.

Table D.1: Primal problem: setup variable coefficients.

			$Y^{mold} \in \mathbb{B}$						$Y^{pat} \in \mathbb{B}$											
			Y_{11}^{mold}	Y_{21}^{mold}	Y_{12}^{mold}	Y_{22}^{mold}	Y_{13}^{mold}	Y_{23}^{mold}	Y_{111}^{pat}	Y_{121}^{pat}	Y_{211}^{pat}	Y_{221}^{pat}	Y_{112}^{pat}	Y_{122}^{pat}	Y_{212}^{pat}	Y_{222}^{pat}	Y_{113}^{pat}	Y_{123}^{pat}	Y_{213}^{pat}	Y_{223}^{pat}
Objective function		-	sc_1^{mold}	sc_1^{mold}	sc_2^{mold}	sc_2^{mold}	sc_3^{mold}	sc_3^{mold}	sc_1^{pat}	sc_1^{pat}	sc_1^{pat}	sc_1^{pat}	sc_2^{pat}	sc_2^{pat}	sc_2^{pat}	sc_2^{pat}	sc_3^{pat}	sc_3^{pat}	sc_3^{pat}	sc_3^{pat}
Inventory balance	$=$	d_{11}																		
		d_{21}																		
		d_{12}																		
		d_{22}																		
		d_{13}																		
		d_{23}																		
Cutting processes integration	$=$	0																		
		0																		
		0																		
		0																		
		0																		
		0																		
Mold setup	$\leqslant$	0	$-N$																	
		0		$-N$																
		0			$-N$															
		0				$-N$														
		0					$-N$													
		0						$-N$												
Cutting pattern setup	$\leqslant$	0							$-N$											
		0								$-N$										
		0									$-N$									
		0										$-N$								
		0											$-N$							
		0												$-N$						
		0													$-N$					
		0														$-N$				
		0															$-N$			
		0																$-N$		
		0																	$-N$	
		0																		$-N$

Table D.2: Primal problem: mold section production variable coefficients.

			$X^C \in \mathbb{Z}_+$											
			X^C_{111}	X^C_{121}	X^C_{211}	X^C_{221}	X^C_{112}	X^C_{122}	X^C_{212}	X^C_{222}	X^C_{113}	X^C_{123}	X^C_{213}	X^C_{223}
Objective function		-	$c^C_1 \cdot W^C$	$c^C_1 \cdot W^C$	$c^C_1 \cdot W^C$	$c^C_1 \cdot W^C$	$c^C_2 \cdot W^C$	$c^C_2 \cdot W^C$	$c^C_2 \cdot W^C$	$c^C_2 \cdot W^C$	$c^C_3 \cdot W^C$	$c^C_3 \cdot W^C$	$c^C_3 \cdot W^C$	$c^C_3 \cdot W^C$
Inventory balance	$=$	d_{11}	a_{111}	a_{111}	a_{121}	a_{121}								
		d_{21}	a_{211}	a_{211}	a_{221}	a_{221}								
		d_{12}					a_{112}	a_{112}	a_{122}	a_{122}				
		d_{22}					a_{212}	a_{212}	a_{222}	a_{222}				
		d_{13}									a_{113}	a_{113}	a_{123}	a_{123}
		d_{23}									a_{213}	a_{213}	a_{223}	a_{223}
Cutting processes integration	$=$	0	$-a_{111}$	$-a_{111}$	$-a_{121}$	$-a_{121}$								
		0	$-a_{211}$	$-a_{211}$	$-a_{221}$	$-a_{221}$								
		0					$-a_{112}$	$-a_{112}$	$-a_{122}$	$-a_{122}$				
		0					$-a_{212}$	$-a_{212}$	$-a_{222}$	$-a_{222}$				
		0									$-a_{113}$	$-a_{113}$	$-a_{123}$	$-a_{123}$
		0									$-a_{213}$	$-a_{213}$	$-a_{223}$	$-a_{223}$
Mold setup	$\leqslant$	0	1		1									
		0		1		1								
		0					1		1					
		0						1		1				
		0									1		1	
		0										1		1
Cutting pattern setup	$\leqslant$	0	1											
		0		1										
		0			1									
		0				1								
		0					1							
		0						1						
		0							1					
		0								1				
		0									1			
		0										1		
		0											1	
		0												1

Table D.3: Primal problem: steel truss production variable coefficients.

			$X^S \in \mathbb{Z}_+$					
			X_{11}^C	X_{21}^C	X_{12}^C	X_{22}^C	X_{13}^C	X_{23}^C
Objective function		-	$c_1^S \cdot W^S$	$c_1^S \cdot W^S$	$c_2^S \cdot W^S$	$c_2^S \cdot W^S$	$c_3^S \cdot W^S$	$c_3^S \cdot W^S$
Inventory balance	$=$	d_{11}						
		d_{21}						
		d_{12}						
		d_{22}						
		d_{13}						
		d_{23}						
Cutting processes integration	$=$	0	b_{111}	b_{121}				
		0	b_{211}	b_{221}				
		0			b_{112}	b_{122}		
		0			b_{212}	b_{222}		
		0					b_{113}	b_{123}
		0					b_{213}	b_{223}
Mold setup	$\leqslant$	0						
		0						
		0						
		0						
		0						
		0						
Cutting pattern setup	$\leqslant$	0						
		0						
		0						
		0						
		0						
		0						
		0						
		0						
		0						
		0						
		0						
		0						

Table D.4: Primal problem: inventory variable coefficients.

			$S^C \in \mathbb{R}_+$						$S^S \in \mathbb{R}_+$					
			S^C_{11}	S^C_{21}	S^C_{12}	S^C_{22}	S^C_{13}	S^C_{23}	S^S_{11}	S^S_{21}	S^S_{12}	S^S_{22}	S^S_{13}	S^S_{23}
Objective function		-	hC_1	hC_1	hC_2	hC_2	hC_3	hC_3	hS_1	hS_1	hS_2	hS_2	hS_3	hS_3
Inventory balance	$=$	d_{11}	-1											
		d_{21}		-1										
		d_{12}	1		-1									
		d_{22}		1		-1								
		d_{13}			1		-1							
		d_{23}				1		-1						
Cutting processes integration	$=$	0							-1					
		0								-1				
		0							1		-1			
		0								1		-1		
		0									1		-1	
		0										1		-1
Mold setup	$\leqslant$	0												
		0												
		0												
		0												
		0												
		0												
Cutting pattern setup	$\leqslant$	0												
		0												
		0												
		0												
		0												
		0												
		0												
		0												
		0												
		0												
		0												
		0												

Table D.5: Dual problem: variable coefficients.

				Inventory balance						**Cutting processes integration**						**Mold setup**					
				λ_{11}	λ_{21}	λ_{12}	λ_{22}	λ_{13}	λ_{23}	σ_{11}	σ_{21}	σ_{12}	σ_{22}	σ_{13}	σ_{23}	μ_{11}	μ_{21}	μ_{12}	μ_{22}	μ_{13}	μ_{23}
				$\mathbb{R}$						$\mathbb{R}$						$\mathbb{R}_-$					
Objective coefficients				d_{11}	d_{21}	d_{12}	d_{22}	d_{13}	d_{23}	0	0	0	0	0	0	0	0	0	0	0	0
Constraints	**(10)**	$\leqslant$	c_1^C	a_{111}	a_{211}					$-a_{111}$	$-a_{211}$					1					
			c_1^C	a_{111}	a_{211}					$-a_{111}$	$-a_{211}$						1				
			c_1^C	a_{121}	a_{221}					$-a_{121}$	$-a_{221}$					1					
			c_1^C	a_{121}	a_{221}					$-a_{121}$	$-a_{221}$						1				
			c_2^C			a_{112}	a_{212}					$-a_{112}$	$-a_{212}$					1			
			c_2^C			a_{112}	a_{212}					$-a_{112}$	$-a_{212}$						1		
			c_2^C			a_{122}	a_{222}					$-a_{122}$	$-a_{222}$					1			
			c_2^C			a_{122}	a_{222}					$-a_{122}$	$-a_{222}$						1		
			c_3^C					a_{113}	a_{213}					$-a_{113}$	$-a_{213}$					1	
			c_3^C					a_{113}	a_{213}					$-a_{113}$	$-a_{213}$						1
			c_3^C					a_{123}	a_{223}					$-a_{123}$	$-a_{223}$					1	
			c_3^C					a_{123}	a_{223}					$-a_{123}$	$-a_{223}$						1
	(11)	$\leqslant$	c_1^S							b_{111}	b_{211}										
			c_1^S							b_{121}	b_{221}										
			c_2^S									b_{112}	b_{212}								
			c_2^S									b_{122}	b_{222}								
			c_3^S											b_{113}	b_{213}						
			c_3^S											b_{123}	b_{223}						
	(12a)	$\leqslant$	hC_1	-1		1															
			hC_1		-1		1														
			hC_2			-1		1													
			hC_2				-1		1												
	(12b)		hC_3					-1													
			hC_3						-1												
	(13a)	$\leqslant$	hS_1							-1		1									
			hS_1								-1		1								
			hS_2									-1		1							
			hS_2										-1		1						
	(13b)		hS_3											-1							
			hS_3												-1						
	(14)	$\leqslant$	sc_1^{mold}													$-N$					
			sc_1^{mold}														$-N$				
			sc_2^{mold}															$-N$			
			sc_2^{mold}																$-N$		
			sc_3^{mold}																	$-N$	
			sc_3^{mold}																		$-N$

D.2 Numerical example

Here, a numerical example of solving a multi-period cutting stock problem with two stages is given, which represents the production planning of lattice slabs, based on the following fictional order form:

	Demanded quantities (d_{ip})		
Item length	**Period 1**	**Period 2**	**Period 3**
$w_1 = 4$	$d_{11} = 5$	$d_{12} = 0$	$d_{13} = 13$
$w_2 = 3$	$d_{21} = 15$	$d_{22} = 10$	$d_{23} = 1$

Table D.6: Order form.

Suppose that:

i. in each period $p \in P$, there are available $|K_p|= 2$ molds to produce lattice slabs;

ii. each available mold is divided into $N = 4$ identical sections of $W^C = 10$ meters in length each;

iii. the steel truss' standard length is $W^S = 12$ meters;

iv. the cost of preparing a mold (sc^{mold}) is fixed at 20 currency units;

v. the setup cost of a mold section (sc^{pat}) is fixed at 2 currency units;

vi. the cost of producing in a mold (c^C) is fixed at 1.6 currency units;

vii. the cost of using a steel truss (c^S) is fixed at 11 currency units;

viii. the cost of stocking a lattice slab/item (h^C) in currency units is calculated based on the length of the item:

$$0.5 \cdot w_i = \begin{cases} 2, & \text{if } i = 1, \\ 1.5, & \text{if } i = 2; \end{cases}$$

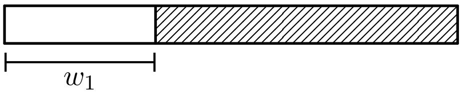
ix. the cost of stocking a steel truss (h^S) in currency units is calculated based on the length of the item:

$$0.25 \cdot w_i = \begin{cases} 1, & \text{if } i = 1, \\ 0.75, & \text{if } i = 2. \end{cases}$$

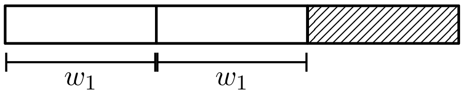
Next, all possible combinations (cutting patterns) to allocate the ordered item lengths in a steel truss of standard length are represented in a matrix:

$$\begin{bmatrix} \mathrm{b}_{ij_S} \end{bmatrix} = \begin{bmatrix} \mathrm{b}_{1j_S} \\ \mathrm{b}_{2j_S} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 0 & 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 2 & 3 & 4 & 1 & 1 & 2 \end{bmatrix}$$

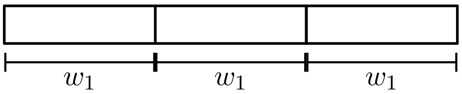
and illustrated by Figure D.1.



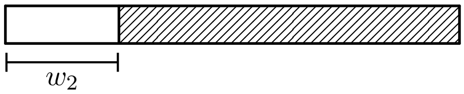
(a) Cutting pattern $\begin{bmatrix}1\\0\end{bmatrix}$.



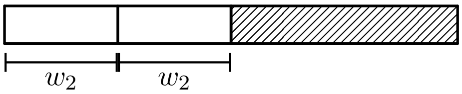
(b) Cutting pattern $\begin{bmatrix}2\\0\end{bmatrix}$.



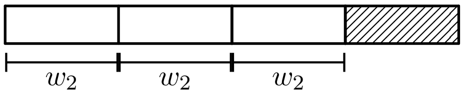
(c) Cutting pattern $\begin{bmatrix}3\\0\end{bmatrix}$.



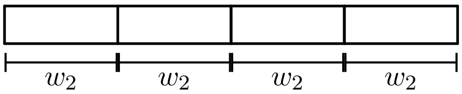
(d) Cutting pattern $\begin{bmatrix}0\\1\end{bmatrix}$.



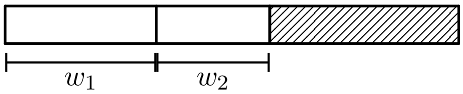
(e) Cutting pattern $\begin{bmatrix}0\\2\end{bmatrix}$.



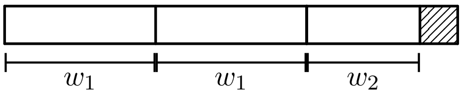
(f) Cutting pattern $\begin{bmatrix}0\\3\end{bmatrix}$.



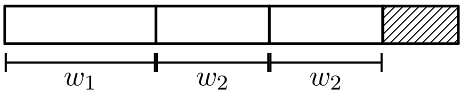
(g) Cutting pattern $\begin{bmatrix}0\\4\end{bmatrix}$.



(h) Cutting pattern $\begin{bmatrix}1\\1\end{bmatrix}$.



(i) Cutting pattern $\begin{bmatrix}2\\1\end{bmatrix}$.



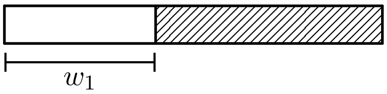
(j) Cutting pattern $\begin{bmatrix}1\\2\end{bmatrix}$.

Figure D.1: Cutting patterns for the steel truss cutting problem.

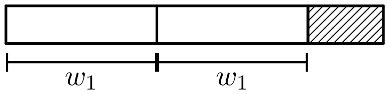
Writing in a matrix all possible combinations (cutting patterns) to allocate the demanded items (lattice slabs) in a mold section, one obtains:

$$\begin{bmatrix}\mathrm{a}_{ij_C}\end{bmatrix} = \begin{bmatrix}\mathrm{a}_{1j_C}\\ \mathrm{a}_{2j_C}\end{bmatrix} = \begin{bmatrix}1 & 2 & 0 & 0 & 0 & 1 & 1\\ 0 & 0 & 1 & 2 & 3 & 1 & 2\end{bmatrix},$$

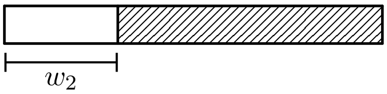
which is illustrated by Figure D.2.



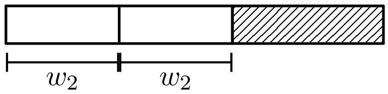
(a) Cutting pattern $\begin{bmatrix}1\\0\end{bmatrix}$.



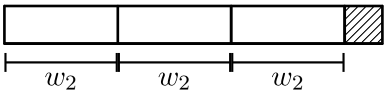
(b) Cutting pattern $\begin{bmatrix}2\\0\end{bmatrix}$.



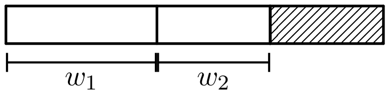
(c) Cutting pattern $\begin{bmatrix}0\\1\end{bmatrix}$.



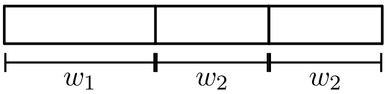
(d) Cutting pattern $\begin{bmatrix}0\\2\end{bmatrix}$.



(e) Cutting pattern $\begin{bmatrix}0\\3\end{bmatrix}$.



(f) Cutting pattern $\begin{bmatrix}1\\1\end{bmatrix}$.



(g) Cutting pattern $\begin{bmatrix}1\\2\end{bmatrix}$.

Figure D.2: Cutting patterns for the concrete filling problem.

Considering all possible cutting patterns to formulate this problem, one would have to solve a linear problem with 132 variables. To get round this laborious task, the selection of a smaller number of available cutting patterns enables the formulation of a restricted master problem. Thus, removing the integrality constraints of the variables, a column generation procedure can be applied as a solution method. In this study, the RMP is initially defined taking into account only homogeneous cutting patterns, given by

$$b_{i,j_S,p} = \begin{cases} \min\left\{\dfrac{W^S}{w_i};\ \max\{1;\gamma_{i,p}\}\right\} & \text{if } i = j_S \\ 0 & \text{otherwise,} \end{cases}$$

and

$$a_{i,j_C,p} = \begin{cases} \min\left\{\dfrac{W^C}{w_i};\ \max\{1;\gamma_{i,p}\}\right\} & \text{if } i = j_C \\ 0 & \text{otherwise,} \end{cases}$$

where $\gamma_{i,p}$ is the sum of demands of item i from period p to the last period, i.e.,

$$\gamma_{i,p} = \sum_{r=p}^{|P|} d_{i,r} \quad i \in I,\ p \in P.$$

In this example, the matrices with the initial columns are

$$B^0 = \begin{bmatrix} 3 & 0 & 3 & 0 & 3 & 0 \\ 0 & 4 & 0 & 4 & 0 & 1 \end{bmatrix}$$

and

$$A^0 = \begin{bmatrix} 2 & 0 & 2 & 0 & 2 & 0 \\ 0 & 3 & 0 & 3 & 0 & 1 \end{bmatrix}$$

Applying the ICG strategy to this example, the models and solutions of each iteration of the column generation procedure are presented to clarify the functionality of this solution method.

Iteration 1

The first column generation iteration starts with solving the Restricted Master Problem (RMP) with homogeneous cutting patterns as follows:

$$\begin{aligned}
\text{minimize}\quad & 20 \cdot Y_{11}^{mold} + 20 \cdot Y_{21}^{mold} + 20 \cdot Y_{12}^{mold} + 20 \cdot Y_{22}^{mold} + 20 \cdot Y_{13}^{mold} + 20 \cdot Y_{23}^{mold} \\
& + 2 \cdot Y_{111}^{pat} + 2 \cdot Y_{211}^{pat} + 2 \cdot Y_{121}^{pat} + 2 \cdot Y_{221}^{pat} + 2 \cdot Y_{112}^{pat} + 2 \cdot Y_{212}^{pat} + 2 \cdot Y_{122}^{pat} + 2 \cdot Y_{222}^{pat} \\
& + 2 \cdot Y_{113}^{pat} + 2 \cdot Y_{213}^{pat} + 2 \cdot Y_{123}^{pat} + 2 \cdot Y_{223}^{pat} + 16 \cdot X_{111}^{C} + 16 \cdot X_{211}^{C} + 16 \cdot X_{121}^{C} \\
& + 16 \cdot X_{221}^{C} + 16 \cdot X_{112}^{C} + 16 \cdot X_{212}^{C} + 16 \cdot X_{122}^{C} + 16 \cdot X_{222}^{C} + 16 \cdot X_{113}^{C} + 16 \cdot X_{213}^{C} \\
& + 16 \cdot X_{123}^{C} + 16 \cdot X_{223}^{C} + 132 \cdot X_{11}^{S} + 132 \cdot X_{21}^{S} + 132 \cdot X_{12}^{S} + 132 \cdot X_{22}^{S} + 132 \cdot X_{13}^{S} \\
& + 132 \cdot X_{23}^{S} + 2 \cdot S_{11}^{C} + 1.5 \cdot S_{21}^{C} + 2 \cdot S_{12}^{C} + 1.5 \cdot S_{22}^{C} + 2 \cdot S_{13}^{C} + 1.5 \cdot S_{23}^{C} + 1 \cdot S_{11}^{S} \\
& + 0.75 \cdot S_{21}^{S} + 1 \cdot S_{12}^{S} + 0.75 \cdot S_{22}^{S} + 1 \cdot S_{13}^{S} + 0.75 \cdot S_{23}^{S}
\end{aligned}$$

subject to

$$\begin{aligned}
& 2 \cdot X_{111}^{C} + 2 \cdot X_{121}^{C} + 0 \cdot X_{211}^{C} + 0 \cdot X_{221}^{C} - S_{11}^{C} = 5 \\
& 0 \cdot X_{111}^{C} + 0 \cdot X_{121}^{C} + 3 \cdot X_{211}^{C} + 3 \cdot X_{221}^{C} - S_{21}^{C} = 15 \\
& 2 \cdot X_{112}^{C} + 2 \cdot X_{122}^{C} + 0 \cdot X_{212}^{C} + 0 \cdot X_{222}^{C} + S_{11}^{C} - S_{12}^{C} = 0 \\
& 0 \cdot X_{112}^{C} + 0 \cdot X_{122}^{C} + 3 \cdot X_{212}^{C} + 3 \cdot X_{222}^{C} + S_{21}^{C} - S_{22}^{C} = 10 \\
& 2 \cdot X_{113}^{C} + 2 \cdot X_{123}^{C} + 0 \cdot X_{213}^{C} + 0 \cdot X_{223}^{C} + S_{12}^{C} - S_{13}^{C} = 13 \\
& 0 \cdot X_{113}^{C} + 0 \cdot X_{123}^{C} + 1 \cdot X_{213}^{C} + 1 \cdot X_{223}^{C} + S_{22}^{C} - S_{23}^{C} = 1 \\
& 3 \cdot X_{11}^{S} + 0 \cdot X_{21}^{S} - S_{11}^{S} - 2 \cdot X_{111}^{C} - 2 \cdot X_{121}^{C} - 0 \cdot X_{211}^{C} - 0 \cdot X_{221}^{C} = 0 \\
& 0 \cdot X_{11}^{S} + 4 \cdot X_{21}^{S} - S_{21}^{S} - 0 \cdot X_{111}^{C} - 0 \cdot X_{121}^{C} - 3 \cdot X_{211}^{C} - 3 \cdot X_{221}^{C} = 0 \\
& 3 \cdot X_{12}^{S} + 0 \cdot X_{22}^{S} + S_{11}^{S} - S_{12}^{S} - 2 \cdot X_{112}^{C} - 2 \cdot X_{122}^{C} - 0 \cdot X_{212}^{C} - 0 \cdot X_{222}^{C} = 0 \\
& 0 \cdot X_{12}^{S} + 4 \cdot X_{22}^{S} + S_{21}^{S} - S_{22}^{S} - 0 \cdot X_{112}^{C} - 0 \cdot X_{122}^{C} - 3 \cdot X_{212}^{C} - 3 \cdot X_{222}^{C} = 0 \\
& 3 \cdot X_{13}^{S} + 0 \cdot X_{23}^{S} + S_{12}^{S} - S_{13}^{S} - 2 \cdot X_{113}^{C} - 2 \cdot X_{123}^{C} - 0 \cdot X_{213}^{C} - 0 \cdot X_{223}^{C} = 0 \\
& 0 \cdot X_{13}^{S} + 1 \cdot X_{23}^{S} + S_{22}^{S} - S_{23}^{S} - 0 \cdot X_{113}^{C} - 0 \cdot X_{123}^{C} - 1 \cdot X_{213}^{C} - 1 \cdot X_{223}^{C} = 0
\end{aligned}$$

$$
\begin{aligned}
&X^C_{111} + X^C_{211} \leqslant 4 \cdot Y^{mold}_{11}\\
&X^C_{121} + X^C_{221} \leqslant 4 \cdot Y^{mold}_{21}\\
&X^C_{112} + X^C_{212} \leqslant 4 \cdot Y^{mold}_{12}\\
&X^C_{122} + X^C_{222} \leqslant 4 \cdot Y^{mold}_{22}\\
&X^C_{113} + X^C_{213} \leqslant 4 \cdot Y^{mold}_{13}\\
&X^C_{123} + X^C_{223} \leqslant 4 \cdot Y^{mold}_{23}\\
&X^C_{111} \leqslant 4 \cdot Y^{pat}_{111}\\
&X^C_{211} \leqslant 4 \cdot Y^{pat}_{211}\\
&X^C_{121} \leqslant 4 \cdot Y^{pat}_{121}\\
&X^C_{221} \leqslant 4 \cdot Y^{pat}_{221}\\
&X^C_{112} \leqslant 4 \cdot Y^{pat}_{112}\\
&X^C_{212} \leqslant 4 \cdot Y^{pat}_{212}\\
&X^C_{122} \leqslant 4 \cdot Y^{pat}_{122}\\
&X^C_{222} \leqslant 4 \cdot Y^{pat}_{222}\\
&X^C_{113} \leqslant 4 \cdot Y^{pat}_{113}\\
&X^C_{213} \leqslant 4 \cdot Y^{pat}_{213}\\
&X^C_{123} \leqslant 4 \cdot Y^{pat}_{123}\\
&X^C_{223} \leqslant 4 \cdot Y^{pat}_{223}\\
&0 \leqslant Y^{mold}_{kp},\ Y^{pat}_{j_C kp} \leqslant 1 && \forall p,\ \forall k,\ \forall j_C,\\
&X^C_{j_C kp},\ X^S_{j_S p},\ S^C_{ip},\ S^S_{ip},\ \in \mathbb{R}_+ && \forall i,\ \forall p,\ \forall k,\ \forall j_C,\ \forall j_S.
\end{aligned}
$$

After solving it with CPLEX using the default settings, the following results are obtained:

Relaxed RMP Objective Value = 2031.33.

The initial solution is given by:

$$
\left[\begin{array}{cccccc}
Y^{mold}_{11} & Y^{mold}_{12} & Y^{mold}_{13} & Y^{mold}_{21} & Y^{mold}_{22} & Y^{mold}_{23}\\ \hline
Y^{pat}_{111} & Y^{pat}_{112} & Y^{pat}_{113} & Y^{pat}_{121} & Y^{pat}_{122} & Y^{pat}_{123}\\
Y^{pat}_{211} & Y^{pat}_{212} & Y^{pat}_{213} & Y^{pat}_{221} & Y^{pat}_{222} & Y^{pat}_{223}\\ \hline
X^C_{111} & X^C_{112} & X^C_{113} & X^C_{121} & X^C_{122} & X^C_{123}\\
X^C_{211} & X^C_{212} & X^C_{213} & X^C_{221} & X^C_{222} & X^C_{223}\\ \hline
X^S_{11} & X^S_{12} & X^S_{13} & X^S_{21} & X^S_{22} & X^S_{23}\\ \hline
S^C_{11} & S^C_{12} & S^C_{13} & S^C_{21} & S^C_{22} & S^C_{23}\\ \hline
S^S_{11} & S^S_{12} & S^S_{13} & S^S_{21} & S^S_{22} & S^S_{23}
\end{array}\right]
=
\left[\begin{array}{cccccc}
1 & 0.92 & 0.63 & 0.88 & 0 & 1\\ \hline
0.63 & 0 & 0.63 & 0 & 0 & 1\\
0.38 & 0.92 & 0 & 0.88 & 0 & 0\\ \hline
2.5 & 0 & 2.5 & 0 & 0 & 4\\
1.5 & 3.67 & 0 & 3.5 & 0 & 0\\ \hline
1.67 & 0 & 4.33 & 3.75 & 2.75 & 0\\ \hline
0 & 0 & 0 & 0 & 1 & 0\\ \hline
0 & 0 & 00 & 0 & 0 &
\end{array}\right]
$$

The dual optimal solutions λ_{ip} (related to the item demand balance constraints), σ_{ip}

(related to the steel truss demand balance constraints) and μ_{kp} (related to the capacity constraints) are the following:

$$\begin{bmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} & \lambda_{21} & \lambda_{22} & \lambda_{23} \\ \hdashline \sigma_{11} & \sigma_{12} & \sigma_{13} & \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \hdashline \mu_{11} & \mu_{12} & \mu_{13} & \mu_{21} & \mu_{22} & \mu_{23} \end{bmatrix} = \begin{bmatrix} 54.75 & 52.75 & 54.75 & 40.17 & 40.17 & 41.67 \\ \hdashline 44 & 44 & 44 & 33 & 33 & 33.75 \\ \hdashline -5 & -5 & -5 & -5 & -5 & -5 \end{bmatrix}$$

Next, we use the optimal dual solution values to build two sets of subproblems associated to the two stages of production.

Stage 1: Steel truss cutting subproblems

$$\begin{array}{lll} \mathrm{OF}^{\mathrm{S}}_{\mathrm{SUB}}(1) = \min & 132 - 44 \cdot \alpha_1^S - 33 \cdot \alpha_2^S & \text{Reduced cost:}0 \\ \text{s. t.} & 4 \cdot \alpha_1^S + 3 \cdot \alpha_2^S \leqslant 12 & \xrightarrow{\text{solution}} \begin{bmatrix} \alpha_1^S \\ \alpha_2^S \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} \\ & \alpha_1^S, \alpha_2^S \in \mathbb{Z}_+ & \end{array}$$

$$\begin{array}{lll} \mathrm{OF}^{\mathrm{S}}_{\mathrm{SUB}}(2) = \min & 132 - 44 \cdot \alpha_1^S - 33 \cdot \alpha_2^S & \text{Reduced cost:}0 \\ \text{s. t.} & 4 \cdot \alpha_1^S + 3 \cdot \alpha_2^S \leqslant 12 & \xrightarrow{\text{solution}} \begin{bmatrix} \alpha_1^S \\ \alpha_2^S \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} \\ & \alpha_1^S, \alpha_2^S \in \mathbb{Z}_+ & \end{array}$$

$$\begin{array}{lll} \mathrm{OF}^{\mathrm{S}}_{\mathrm{SUB}}(3) = \min & 132 - 44 \cdot \alpha_1^S - 33.75 \cdot \alpha_2^S & \text{Reduced cost:} -3 \\ \text{s. t.} & 4 \cdot \alpha_1^S + 3 \cdot \alpha_2^S \leqslant 12 & \xrightarrow{\text{solution}} \begin{bmatrix} \alpha_1^S \\ \alpha_2^S \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \end{bmatrix} \\ & \alpha_1^S, \alpha_2^S \in \mathbb{Z}_+ & \end{array}$$

Stage 2: Mold section concrete filling subproblem[1]

$$\begin{array}{lll} \mathrm{OF}^{\mathrm{C}}_{\mathrm{SUB}}(1) = \min & 21 - 10.75 \cdot \alpha_1^C - 7.17 \cdot \alpha_2^C & \text{Reduced cost:} -4.08 \\ \text{s. t.} & 4 \cdot \alpha_1^C + 3 \cdot \alpha_2^C \leqslant 10 & \xrightarrow{\text{solution}} \begin{bmatrix} \alpha_1^C \\ \alpha_2^C \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ & \alpha_1^C, \alpha_2^C \in \mathbb{Z}_+ & \end{array}$$

$$\begin{array}{lll} \mathrm{OF}^{\mathrm{C}}_{\mathrm{SUB}}(2) = \min & 21 - 8.75 \cdot \alpha_1^C - 7.17 \cdot \alpha_2^C & \text{Reduced cost:} -2.08 \\ \text{s. t.} & 4 \cdot \alpha_1^C + 3 \cdot \alpha_2^C \leqslant 10 & \xrightarrow{\text{solution}} \begin{bmatrix} \alpha_1^C \\ \alpha_2^C \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ & \alpha_1^C, \alpha_2^C \in \mathbb{Z}_+ & \end{array}$$

$$\begin{array}{lll} \mathrm{OF}^{\mathrm{C}}_{\mathrm{SUB}}(3) = \min & 21 - 10.75 \cdot \alpha_1^C - 7.92 \cdot \alpha_2^C & \text{Reduced cost:} -5.58 \\ \text{s. t.} & 4 \cdot \alpha_1^C + 3 \cdot \alpha_2^C \leqslant 10 & \xrightarrow{\text{solution}} \begin{bmatrix} \alpha_1^C \\ \alpha_2^C \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ & \alpha_1^C, \alpha_2^C \in \mathbb{Z}_+ & \end{array}$$

[1]Observe that $\mu_{kp} = -5$ regardless k and p values.

For each of these subproblems, the columns are evaluated whether their reduced costs are lower than or equal to -0.001. In the positive case, the subproblems solutions are added as new cutting patterns followed by new variables in the RMP.

The updated cutting pattern matrices are

$$B^1 = \begin{bmatrix} b_{ikp} \end{bmatrix} = \begin{bmatrix} b_{111} & b_{121} & b_{112} & b_{122} & b_{113} & b_{123} & \mathbf{b_{133}} \\ b_{211} & b_{221} & b_{212} & b_{222} & b_{213} & b_{223} & \mathbf{b_{233}} \end{bmatrix}$$
$$= \begin{bmatrix} 3 & 0 & 3 & 0 & 3 & 0 & \mathbf{0} \\ 0 & 4 & 0 & 4 & 0 & 1 & \mathbf{4} \end{bmatrix}$$

and

$$A^1 = \begin{bmatrix} a_{ijp} \end{bmatrix} = \begin{bmatrix} a_{111} & a_{121} & \mathbf{a_{131}} & a_{112} & a_{122} & \mathbf{a_{132}} & a_{113} & a_{123} & \mathbf{a_{133}} \\ a_{211} & a_{221} & \mathbf{a_{231}} & a_{212} & a_{222} & \mathbf{a_{232}} & a_{213} & a_{223} & \mathbf{a_{233}} \end{bmatrix}$$
$$= \begin{bmatrix} 2 & 0 & \mathbf{1} & 2 & 0 & \mathbf{1} & 2 & 0 & \mathbf{1} \\ 0 & 3 & \mathbf{2} & 0 & 3 & \mathbf{2} & 0 & 1 & \mathbf{2} \end{bmatrix}.$$

After updating the RMP with the new cutting patterns, a new iteration is realized.

Iteration 2

Then, for the RMP with the new columns:

$$\begin{aligned}
\text{minimize}\quad & 20 \cdot Y_{11}^{mold} + 20 \cdot Y_{21}^{mold} + 20 \cdot Y_{12}^{mold} + 20 \cdot Y_{22}^{mold} + 20 \cdot Y_{13}^{mold} + 20 \cdot Y_{23}^{mold} \\
& + 2 \cdot Y_{111}^{pat} + 2 \cdot Y_{211}^{pat} + 2 \cdot Y_{121}^{pat} + 2 \cdot Y_{221}^{pat} + 2 \cdot Y_{112}^{pat} + 2 \cdot Y_{212}^{pat} + 2 \cdot Y_{122}^{pat} + 2 \cdot Y_{222}^{pat} \\
& + 2 \cdot Y_{113}^{pat} + 2 \cdot Y_{213}^{pat} + 2 \cdot Y_{123}^{pat} + 2 \cdot Y_{223}^{pat} \mathbf{+ 2 \cdot Y_{311}^{pat} + 2 \cdot Y_{321}^{pat} + 2 \cdot Y_{312}^{pat} + 2 \cdot Y_{322}^{pat}} \\
& \mathbf{+2 \cdot Y_{313}^{pat} + 2 \cdot Y_{323}^{pat}} + 16 \cdot X_{111}^{C} + 16 \cdot X_{211}^{C} + 16 \cdot X_{121}^{C} + 16 \cdot X_{221}^{C} + 16 \cdot X_{112}^{C} \\
& + 16 \cdot X_{212}^{C} + 16 \cdot X_{122}^{C} + 16 \cdot X_{222}^{C} + 16 \cdot X_{113}^{C} + 16 \cdot X_{213}^{C} + 16 \cdot X_{123}^{C} + 16 \cdot X_{223}^{C} \\
& \mathbf{+16 \cdot X_{311}^{C} + 16 \cdot X_{321}^{C} + 16 \cdot X_{312}^{C} + 16 \cdot X_{322}^{C} + 16 \cdot X_{313}^{C} + 16 \cdot X_{323}^{C}} \\
& + 132 \cdot X_{11}^{S} + 132 \cdot X_{21}^{S} + 132 \cdot X_{12}^{S} + 132 \cdot X_{22}^{S} + 132 \cdot X_{13}^{S} + 132 \cdot X_{23}^{S} \mathbf{+ 132 \cdot X_{33}^{S}} \\
& + 2 \cdot S_{11}^{C} + 1.5 \cdot S_{21}^{C} + 2 \cdot S_{12}^{C} + 1.5 \cdot S_{22}^{C} + 2 \cdot S_{13}^{C} + 1.5 \cdot S_{23}^{C} + 1 \cdot S_{11}^{S} + 0.75 \cdot S_{21}^{S} \\
& + 1 \cdot S_{12}^{S} + 0.75 \cdot S_{22}^{S} + 1 \cdot S_{13}^{S} + 0.75 \cdot S_{23}^{S}
\end{aligned}$$

subject to

$$\begin{aligned}
& 2 \cdot X_{111}^{C} + 2 \cdot X_{121}^{C} + 0 \cdot X_{211}^{C} + 0 \cdot X_{221}^{C} \mathbf{+ 1 \cdot X_{311}^{C} + 1 \cdot X_{321}^{C}} - S_{11}^{C} = 5 \\
& 0 \cdot X_{111}^{C} + 0 \cdot X_{121}^{C} + 3 \cdot X_{211}^{C} + 3 \cdot X_{221}^{C} \mathbf{+ 2 \cdot X_{311}^{C} + 2 \cdot X_{321}^{C}} - S_{21}^{C} = 15 \\
& 2 \cdot X_{112}^{C} + 2 \cdot X_{122}^{C} + 0 \cdot X_{212}^{C} + 0 \cdot X_{222}^{C} \mathbf{+ 1 \cdot X_{312}^{C} + 1 \cdot X_{322}^{C}} + S_{11}^{C} - S_{12}^{C} = 0 \\
& 0 \cdot X_{112}^{C} + 0 \cdot X_{122}^{C} + 3 \cdot X_{212}^{C} + 3 \cdot X_{222}^{C} \mathbf{+ 2 \cdot X_{312}^{C} + 2 \cdot X_{322}^{C}} + S_{21}^{C} - S_{22}^{C} = 10 \\
& 2 \cdot X_{113}^{C} + 2 \cdot X_{123}^{C} + 0 \cdot X_{213}^{C} + 0 \cdot X_{223}^{C} \mathbf{+ 1 \cdot X_{313}^{C} + 1 \cdot X_{323}^{C}} + S_{12}^{C} - S_{13}^{C} = 13
\end{aligned}$$

$$0 \cdot X_{113}^C + 0 \cdot X_{123}^C + 1 \cdot X_{213}^C + 1 \cdot X_{223}^C + \mathbf{2} \cdot \mathbf{X_{313}^C} + \mathbf{2} \cdot \mathbf{X_{323}^C} + S_{22}^C - S_{23}^C = 1$$

$$3 \cdot X_{11}^S + 0 \cdot X_{21}^S - S_{11}^S - 2 \cdot X_{111}^C - 2 \cdot X_{121}^C - 0 \cdot X_{211}^C - 0 \cdot X_{221}^C - \mathbf{1} \cdot \mathbf{X_{311}^C} - \mathbf{1} \cdot \mathbf{X_{321}^C} = 0$$

$$0 \cdot X_{11}^S + 4 \cdot X_{21}^S - S_{21}^S - 0 \cdot X_{111}^C - 0 \cdot X_{121}^C - 3 \cdot X_{211}^C - 3 \cdot X_{221}^C - \mathbf{2} \cdot \mathbf{X_{311}^C} - \mathbf{2} \cdot \mathbf{X_{321}^C} = 0$$

$$3 \cdot X_{12}^S + 0 \cdot X_{22}^S + S_{11}^S - S_{12}^S - 2 \cdot X_{112}^C - 2 \cdot X_{122}^C - 0 \cdot X_{212}^C - 0 \cdot X_{222}^C - \mathbf{1} \cdot \mathbf{X_{312}^C} - \mathbf{1} \cdot \mathbf{X_{322}^C} = 0$$

$$0 \cdot X_{12}^S + 4 \cdot X_{22}^S + S_{21}^S - S_{22}^S - 0 \cdot X_{112}^C - 0 \cdot X_{122}^C - 3 \cdot X_{212}^C - 3 \cdot X_{222}^C - \mathbf{2} \cdot \mathbf{X_{312}^C} - \mathbf{2} \cdot \mathbf{X_{322}^C} = 0$$

$$3 \cdot X_{13}^S + 0 \cdot X_{23}^S + \mathbf{0} \cdot \mathbf{X_{33}^S} + S_{12}^S - S_{13}^S - 2 \cdot X_{113}^C - 2 \cdot X_{123}^C - 0 \cdot X_{213}^C - 0 \cdot X_{223}^C - \mathbf{1} \cdot \mathbf{X_{313}^C} - \mathbf{1} \cdot \mathbf{X_{323}^C} = 0$$

$$0 \cdot X_{13}^S + 1 \cdot X_{23}^S + \mathbf{4} \cdot \mathbf{X_{33}^S} + S_{22}^S - S_{23}^S - 0 \cdot X_{113}^C - 0 \cdot X_{123}^C - 1 \cdot X_{213}^C - 1 \cdot X_{223}^C - \mathbf{2} \cdot \mathbf{X_{313}^C} - \mathbf{2} \cdot \mathbf{X_{323}^C} = 0$$

$$X_{111}^C + X_{211}^C + \mathbf{X_{311}^C} \leqslant 4 \cdot Y_{11}^{mold}$$

$$X_{121}^C + X_{221}^C + \mathbf{X_{321}^C} \leqslant 4 \cdot Y_{21}^{mold}$$

$$X_{112}^C + X_{212}^C + \mathbf{X_{312}^C} \leqslant 4 \cdot Y_{12}^{mold}$$

$$X_{122}^C + X_{222}^C + \mathbf{X_{322}^C} \leqslant 4 \cdot Y_{22}^{mold}$$

$$X_{113}^C + X_{213}^C + \mathbf{X_{313}^C} \leqslant 4 \cdot Y_{13}^{mold}$$

$$X_{123}^C + X_{223}^C + \mathbf{X_{323}^C} \leqslant 4 \cdot Y_{23}^{mold}$$

$$X_{111}^C \leqslant 4 \cdot Y_{111}^{pat}$$

$$X_{211}^C \leqslant 4 \cdot Y_{211}^{pat}$$

$$\mathbf{X_{311}^C} \leqslant \mathbf{4} \cdot \mathbf{Y_{311}^{pat}}$$

$$X_{121}^C \leqslant 4 \cdot Y_{121}^{pat}$$

$$X_{221}^C \leqslant 4 \cdot Y_{221}^{pat}$$

$$\mathbf{X_{321}^C} \leqslant \mathbf{4} \cdot \mathbf{Y_{321}^{pat}}$$

$$X_{112}^C \leqslant 4 \cdot Y_{112}^{pat}$$

$$X_{212}^C \leqslant 4 \cdot Y_{212}^{pat}$$

$$\mathbf{X_{312}^C} \leqslant \mathbf{4} \cdot \mathbf{Y_{312}^{pat}}$$

$$X_{122}^C \leqslant 4 \cdot Y_{122}^{pat}$$

$$X_{222}^C \leqslant 4 \cdot Y_{222}^{pat}$$

$$\mathbf{X_{322}^C} \leqslant \mathbf{4} \cdot \mathbf{Y_{322}^{pat}}$$

$$X_{113}^C \leqslant 4 \cdot Y_{113}^{pat}$$

$$X_{213}^C \leqslant 4 \cdot Y_{213}^{pat}$$

$$\mathbf{X_{313}^C} \leqslant \mathbf{4} \cdot \mathbf{Y_{313}^{pat}}$$

$$X_{123}^C \leqslant 4 \cdot Y_{123}^{pat}$$

$$
\begin{aligned}
&X^C_{223} \leqslant 4 \cdot Y^{pat}_{223} \\
&\mathbf{X^C_{323} \leqslant 4 \cdot Y^{pat}_{323}} \\
&0 \leqslant Y^{mold}_{kp},\ Y^{pat}_{j_C kp} \leqslant 1 && \forall p,\ \forall k,\ \forall j_C, \\
&X^C_{j_C kp},\ X^S_{j_S p} \in \mathbb{R}_+ && \forall p,\ \forall k,\ \forall j_C,\ \forall j_S \\
&S^C_{ip},\ S^S_{ip}, \in \mathbb{R}_+ && \forall i,\ \forall p.
\end{aligned}
$$

After solving it with CPLEX, the following results are obtained:

Relaxed RMP Objective Value = 2002.21.

The solution is given by:

$$
\left[\begin{array}{ccc}
Y^{mold}_{11} & Y^{mold}_{12} & Y^{mold}_{13} \\
Y^{mold}_{21} & Y^{mold}_{22} & Y^{mold}_{23} \\
\hdashline
Y^{pat}_{111} & Y^{pat}_{112} & Y^{pat}_{113} \\
Y^{pat}_{121} & Y^{pat}_{122} & Y^{pat}_{123} \\
Y^{pat}_{211} & Y^{pat}_{212} & Y^{pat}_{213} \\
Y^{pat}_{221} & Y^{pat}_{222} & Y^{pat}_{223} \\
\mathbf{Y^{pat}_{311}} & \mathbf{Y^{pat}_{312}} & \mathbf{Y^{pat}_{313}} \\
\mathbf{Y^{pat}_{321}} & \mathbf{Y^{pat}_{322}} & \mathbf{Y^{pat}_{323}} \\
\hdashline
X^C_{111} & X^C_{112} & X^C_{113} \\
X^C_{121} & X^C_{122} & X^C_{123} \\
X^C_{211} & X^C_{212} & X^C_{213} \\
X^C_{221} & X^C_{222} & X^C_{223} \\
\mathbf{X^C_{311}} & \mathbf{X^C_{312}} & \mathbf{X^C_{313}} \\
\mathbf{X^C_{321}} & \mathbf{X^C_{322}} & \mathbf{X^C_{323}} \\
\hdashline
X^S_{11} & X^S_{12} & X^S_{13} \\
X^S_{21} & X^S_{22} & X^S_{23} \\
 & & \mathbf{X^S_{33}} \\
\hdashline
S^C_{11} & S^C_{12} & S^C_{13} \\
S^C_{21} & S^C_{22} & S^C_{23} \\
\hdashline
S^S_{11} & S^S_{12} & S^S_{13} \\
S^S_{21} & S^S_{22} & S^S_{23}
\end{array}\right]
=
\left[\begin{array}{ccc}
1 & 1 & 0.13 \\
0.67 & 0.25 & 0.94 \\
\hdashline
0 & 0 & 0 \\
0 & 0 & 0.94 \\
0 & 0 & 0 \\
0.42 & 0 & 0 \\
\mathbf{1} & \mathbf{1} & \mathbf{0.13} \\
\mathbf{0.25} & \mathbf{0.25} & 0 \\
\hdashline
0 & 0 & 0 \\
0 & 0 & 3.75 \\
0 & 0 & 0 \\
1.68 & 0 & 0 \\
\mathbf{4} & \mathbf{4} & \mathbf{0.5} \\
\mathbf{1} & \mathbf{1} & 0 \\
\hdashline
1.67 & 1.67 & 2.67 \\
3.75 & 2.5 & 0 \\
 & & \mathbf{0.25} \\
\hdashline
0 & 5 & 0 \\
0 & 0 & 0 \\
\hdashline
0 & 0 & 0 \\
0 & 0 & 0
\end{array}\right]
$$

The dual optimal solutions λ_{ip} (related to the item demand balance constraints), σ_{ip} (related to the integration between stages constraints) and μ_{kp} (related to the capacity constraints) are the following:

$$\left[\begin{array}{ccc}\lambda_{11} & \lambda_{12} & \lambda_{13}\\ \lambda_{21} & \lambda_{22} & \lambda_{23}\\ \hdashline \sigma_{11} & \sigma_{12} & \sigma_{13}\\ \sigma_{21} & \sigma_{22} & \sigma_{23}\\ \hdashline \mu_{11} & \mu_{12} & \mu_{13}\\ \mu_{21} & \mu_{22} & \mu_{23}\end{array}\right] = \left[\begin{array}{ccc}51.17 & 52.75 & 54.75\\ 40.17 & 39.38 & 38.38\\ \hdashline 44 & 44 & 44\\ 33 & 33 & 33\\ \hdashline -5 & -5 & -5\\ -5 & -5 & -5\end{array}\right]$$

These optimal dual solution values are used to build the subproblems.

Stage 1: Steel truss cutting subproblems

$$\begin{aligned} \mathrm{OF}^{\mathrm{S}}_{\mathrm{SUB}}(1) = \min \quad & 132 - 44 \cdot \alpha_1^S - 33 \cdot \alpha_2^S & & \text{Reduced cost: } 0\\ \text{s. t.} \quad & 4 \cdot \alpha_1^S + 3 \cdot \alpha_2^S \leqslant 12 & \xrightarrow{\text{solution}} \; & \begin{bmatrix}\alpha_1^S\\ \alpha_2^S\end{bmatrix} = \begin{bmatrix}3\\ 0\end{bmatrix}\\ & \alpha_1^S, \alpha_2^S \in \mathbb{Z}_+ \end{aligned}$$

$$\begin{aligned} \mathrm{OF}^{\mathrm{S}}_{\mathrm{SUB}}(2) = \min \quad & 132 - 44 \cdot \alpha_1^S - 33 \cdot \alpha_2^S & & \text{Reduced cost: } 0\\ \text{s. t.} \quad & 4 \cdot \alpha_1^S + 3 \cdot \alpha_2^S \leqslant 12 & \longrightarrow \; & \begin{bmatrix}\alpha_1^S\\ \alpha_2^S\end{bmatrix} = \begin{bmatrix}3\\ 0\end{bmatrix}\\ & \alpha_1^S, \alpha_2^S \in \mathbb{Z}_+ \end{aligned}$$

$$\begin{aligned} \mathrm{OF}^{\mathrm{S}}_{\mathrm{SUB}}(3) = \min \quad & 132 - 44 \cdot \alpha_1^S - 33 \cdot \alpha_2^S & & \text{Reduced cost: } 0\\ \text{s. t.} \quad & 4 \cdot \alpha_1^S + 3 \cdot \alpha_2^S \leqslant 12 & \longrightarrow \; & \begin{bmatrix}\alpha_1^S\\ \alpha_2^S\end{bmatrix} = \begin{bmatrix}3\\ 0\end{bmatrix}\\ & \alpha_1^S, \alpha_2^S \in \mathbb{Z}_+ \end{aligned}$$

Stage 2: Mold section concrete filling subproblem[2]

$$\begin{aligned} \mathrm{OF}^{\mathrm{C}}_{\mathrm{SUB}}(1) = \min \quad & 21 - 7.17 \cdot \alpha_1^C - 7.17 \cdot \alpha_2^C & & \text{Reduced cost: } -0.5\\ \text{s. t.} \quad & 4 \cdot \alpha_1^C + 3 \cdot \alpha_2^C \leqslant 10 & \xrightarrow{\text{solution}} \; & \begin{bmatrix}\alpha_1^C\\ \alpha_2^C\end{bmatrix} = \begin{bmatrix}0\\ 3\end{bmatrix}\\ & \alpha_1^C, \alpha_2^C \in \mathbb{Z}_+ \end{aligned}$$

$$\begin{aligned} \mathrm{OF}^{\mathrm{C}}_{\mathrm{SUB}}(2) = \min \quad & 21 - 8.75 \cdot \alpha_1^C - 6.38 \cdot \alpha_2^C & & \text{Reduced cost: } -0.5\\ \text{s. t.} \quad & 4 \cdot \alpha_1^C + 3 \cdot \alpha_2^C \leqslant 10 & \longrightarrow \; & \begin{bmatrix}\alpha_1^C\\ \alpha_2^C\end{bmatrix} = \begin{bmatrix}1\\ 2\end{bmatrix}\\ & \alpha_1^C, \alpha_2^C \in \mathbb{Z}_+ \end{aligned}$$

[2]Observe that $\mu_{kp} = -5$ regardless k and p values.

$$\begin{array}{lll} \mathrm{OF}^{\mathrm{C}}_{\mathrm{SUB}}(3) = \min & 21 - 10.75 \cdot \alpha_1^C - 5.38 \cdot \alpha_2^C & \text{Reduced cost: } -0.5 \\ \text{s. t.} & 4 \cdot \alpha_1^C + 3 \cdot \alpha_2^C \leqslant 10 & \longrightarrow \begin{bmatrix} \alpha_1^C \\ \alpha_2^C \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \\ & \alpha_1^C, \alpha_2^C \in \mathbb{Z}_+ & \end{array}$$

Note that all generated columns are already included in the cutting pattern matrices. Even the Stage 2 columns have their associated reduced costs lower than the tolerance of -0.001, so they don't need to be added to the related matrix. This condition generally occurs when there are degenerate solutions. Since the reduced cost tolerance criterion is satisfied, the column generation procedure is continued but the RMP objective value scarcely improves, causing tailing off. To resolve this situation, a stop criterion based on the solution upgrade is inserted in the heuristic. The improvement in the RMP objective function is evaluated and compared to that previously obtained. The column generation procedure is discontinued when the improvement is lower than the tolerance (0.001) in 3 consecutive iterations.

End of column generation procedure: solving the mixed-integer problem

By adding integrality constraints to the resultant RMP, one obtains the following mixed-integer problem:

$$\begin{aligned}
\text{minimize}\quad & 20 \cdot Y_{11}^{mold} + 20 \cdot Y_{21}^{mold} + 20 \cdot Y_{12}^{mold} + 20 \cdot Y_{22}^{mold} + 20 \cdot Y_{13}^{mold} + 20 \cdot Y_{23}^{mold} \\
& + 2 \cdot Y_{111}^{pat} + 2 \cdot Y_{211}^{pat} + 2 \cdot Y_{121}^{pat} + 2 \cdot Y_{221}^{pat} + 2 \cdot Y_{112}^{pat} + 2 \cdot Y_{212}^{pat} + 2 \cdot Y_{122}^{pat} + 2 \cdot Y_{222}^{pat} \\
& + 2 \cdot Y_{113}^{pat} + 2 \cdot Y_{213}^{pat} + 2 \cdot Y_{123}^{pat} + 2 \cdot Y_{223}^{pat} \boldsymbol{+ 2 \cdot \mathbf{Y}_{311}^{\mathbf{pat}} + 2 \cdot \mathbf{Y}_{321}^{\mathbf{pat}} + 2 \cdot \mathbf{Y}_{312}^{\mathbf{pat}} + 2 \cdot \mathbf{Y}_{322}^{\mathbf{pat}}} \\
& \boldsymbol{+ 2 \cdot \mathbf{Y}_{313}^{\mathbf{pat}} + 2 \cdot \mathbf{Y}_{323}^{\mathbf{pat}}} + 16 \cdot X_{111}^C + 16 \cdot X_{211}^C + 16 \cdot X_{121}^C + 16 \cdot X_{221}^C + 16 \cdot X_{112}^C \\
& + 16 \cdot X_{212}^C + 16 \cdot X_{122}^C + 16 \cdot X_{222}^C + 16 \cdot X_{113}^C + 16 \cdot X_{213}^C + 16 \cdot X_{123}^C + 16 \cdot X_{223}^C \\
& \boldsymbol{+ 16 \cdot \mathbf{X}_{311}^{\mathbf{C}} + 16 \cdot \mathbf{X}_{321}^{\mathbf{C}} + 16 \cdot \mathbf{X}_{312}^{\mathbf{C}} + 16 \cdot \mathbf{X}_{322}^{\mathbf{C}} + 16 \cdot \mathbf{X}_{313}^{\mathbf{C}} + 16 \cdot \mathbf{X}_{323}^{\mathbf{C}}} \\
& + 132 \cdot X_{11}^S + 132 \cdot X_{21}^S + 132 \cdot X_{12}^S + 132 \cdot X_{22}^S + 132 \cdot X_{13}^S + 132 \cdot X_{23}^S \boldsymbol{+ 132 \cdot \mathbf{X}_{33}^{\mathbf{S}}} \\
& + 2 \cdot S_{11}^C + 1.5 \cdot S_{21}^C + 2 \cdot S_{12}^C + 1.5 \cdot S_{22}^C + 2 \cdot S_{13}^C + 1.5 \cdot S_{23}^C + 1 \cdot S_{11}^S + 0.75 \cdot S_{21}^S \\
& + 1 \cdot S_{12}^S + 0.75 \cdot S_{22}^S + 1 \cdot S_{13}^S + 0.75 \cdot S_{23}^S
\end{aligned}$$

subject to

$$\begin{aligned}
& 2 \cdot X_{111}^C + 2 \cdot X_{121}^C + 0 \cdot X_{211}^C + 0 \cdot X_{221}^C \boldsymbol{+ 1 \cdot \mathbf{X}_{311}^{\mathbf{C}} + 1 \cdot \mathbf{X}_{321}^{\mathbf{C}}} - S_{11}^C = 5 \\
& 0 \cdot X_{111}^C + 0 \cdot X_{121}^C + 3 \cdot X_{211}^C + 3 \cdot X_{221}^C \boldsymbol{+ 2 \cdot \mathbf{X}_{311}^{\mathbf{C}} + 2 \cdot \mathbf{X}_{321}^{\mathbf{C}}} - S_{21}^C = 15 \\
& 2 \cdot X_{112}^C + 2 \cdot X_{122}^C + 0 \cdot X_{212}^C + 0 \cdot X_{222}^C \boldsymbol{+ 1 \cdot \mathbf{X}_{312}^{\mathbf{C}} + 1 \cdot \mathbf{X}_{322}^{\mathbf{C}}} + S_{11}^C - S_{12}^C = 0 \\
& 0 \cdot X_{112}^C + 0 \cdot X_{122}^C + 3 \cdot X_{212}^C + 3 \cdot X_{222}^C \boldsymbol{+ 2 \cdot \mathbf{X}_{312}^{\mathbf{C}} + 2 \cdot \mathbf{X}_{322}^{\mathbf{C}}} + S_{21}^C - S_{22}^C = 10 \\
& 2 \cdot X_{113}^C + 2 \cdot X_{123}^C + 0 \cdot X_{213}^C + 0 \cdot X_{223}^C \boldsymbol{+ 1 \cdot \mathbf{X}_{313}^{\mathbf{C}} + 1 \cdot \mathbf{X}_{323}^{\mathbf{C}}} + S_{12}^C - S_{13}^C = 13 \\
& 0 \cdot X_{113}^C + 0 \cdot X_{123}^C + 1 \cdot X_{213}^C + 1 \cdot X_{223}^C \boldsymbol{+ 2 \cdot \mathbf{X}_{313}^{\mathbf{C}} + 2 \cdot \mathbf{X}_{323}^{\mathbf{C}}} + S_{22}^C - S_{23}^C = 1
\end{aligned}$$

$$3 \cdot X_{11}^S + 0 \cdot X_{21}^S - S_{11}^S - 2 \cdot X_{111}^C - 2 \cdot X_{121}^C - 0 \cdot X_{211}^C - 0 \cdot X_{221}^C \mathbf{-1 \cdot X_{311}^C} \mathbf{-1 \cdot X_{321}^C} = 0$$

$$0 \cdot X_{11}^S + 4 \cdot X_{21}^S - S_{21}^S - 0 \cdot X_{111}^C - 0 \cdot X_{121}^C - 3 \cdot X_{211}^C - 3 \cdot X_{221}^C \mathbf{-2 \cdot X_{311}^C} \mathbf{-2 \cdot X_{321}^C} = 0$$

$$3 \cdot X_{12}^S + 0 \cdot X_{22}^S + S_{11}^S - S_{12}^S - 2 \cdot X_{112}^C - 2 \cdot X_{122}^C - 0 \cdot X_{212}^C - 0 \cdot X_{222}^C \mathbf{-1 \cdot X_{312}^C} \mathbf{-1 \cdot X_{322}^C} = 0$$

$$0 \cdot X_{12}^S + 4 \cdot X_{22}^S + S_{21}^S - S_{22}^S - 0 \cdot X_{112}^C - 0 \cdot X_{122}^C - 3 \cdot X_{212}^C - 3 \cdot X_{222}^C \mathbf{-2 \cdot X_{312}^C} \mathbf{-2 \cdot X_{322}^C} = 0$$

$$3 \cdot X_{13}^S + 0 \cdot X_{23}^S \mathbf{+0 \cdot X_{33}^S} + S_{12}^S - S_{13}^S - 2 \cdot X_{113}^C - 2 \cdot X_{123}^C - 0 \cdot X_{213}^C - 0 \cdot X_{223}^C \mathbf{-1 \cdot X_{313}^C - 1 \cdot X_{323}^C} = 0$$

$$0 \cdot X_{13}^S + 1 \cdot X_{23}^S \mathbf{+4 \cdot X_{33}^S} + S_{22}^S - S_{23}^S - 0 \cdot X_{113}^C - 0 \cdot X_{123}^C - 1 \cdot X_{213}^C - 1 \cdot X_{223}^C \mathbf{-2 \cdot X_{313}^C - 2 \cdot X_{323}^C} = 0$$

$$X_{111}^C + X_{211}^C \mathbf{+X_{311}^C} \leqslant 4 \cdot Y_{11}^{mold}$$
$$X_{121}^C + X_{221}^C \mathbf{+X_{321}^C} \leqslant 4 \cdot Y_{21}^{mold}$$
$$X_{112}^C + X_{212}^C \mathbf{+X_{312}^C} \leqslant 4 \cdot Y_{12}^{mold}$$
$$X_{122}^C + X_{222}^C \mathbf{+X_{322}^C} \leqslant 4 \cdot Y_{22}^{mold}$$
$$X_{113}^C + X_{213}^C \mathbf{+X_{313}^C} \leqslant 4 \cdot Y_{13}^{mold}$$
$$X_{123}^C + X_{223}^C \mathbf{+X_{323}^C} \leqslant 4 \cdot Y_{23}^{mold}$$
$$X_{111}^C \leqslant 4 \cdot Y_{111}^{pat}$$
$$X_{211}^C \leqslant 4 \cdot Y_{211}^{pat}$$
$$\mathbf{X_{311}^C \leqslant 4 \cdot Y_{311}^{pat}}$$
$$X_{121}^C \leqslant 4 \cdot Y_{121}^{pat}$$
$$X_{221}^C \leqslant 4 \cdot Y_{221}^{pat}$$
$$\mathbf{X_{321}^C \leqslant 4 \cdot Y_{321}^{pat}}$$
$$X_{112}^C \leqslant 4 \cdot Y_{112}^{pat}$$
$$X_{212}^C \leqslant 4 \cdot Y_{212}^{pat}$$
$$\mathbf{X_{312}^C \leqslant 4 \cdot Y_{312}^{pat}}$$
$$X_{122}^C \leqslant 4 \cdot Y_{122}^{pat}$$
$$X_{222}^C \leqslant 4 \cdot Y_{222}^{pat}$$
$$\mathbf{X_{322}^C \leqslant 4 \cdot Y_{322}^{pat}}$$
$$X_{113}^C \leqslant 4 \cdot Y_{113}^{pat}$$
$$X_{213}^C \leqslant 4 \cdot Y_{213}^{pat}$$
$$\mathbf{X_{313}^C \leqslant 4 \cdot Y_{313}^{pat}}$$
$$X_{123}^C \leqslant 4 \cdot Y_{123}^{pat}$$
$$X_{223}^C \leqslant 4 \cdot Y_{223}^{pat}$$

$$\mathbf{X^C_{323} \leqslant 4 \cdot Y^{pat}_{323}}$$

$$\begin{array}{lr}
Y^{mold}_{kp},\ Y^{pat}_{j_C kp} \in \mathbb{B} & \forall p,\ \forall k,\ \forall j_C \\
X^C_{j_C kp},\ X^S_{j_S p} \in \mathbb{Z}_+ & \forall p,\ \forall k,\ \forall j_C,\ \forall j_S \\
S^C_{ip},\ S^S_{ip},\ \in \mathbb{R}_+ & \forall i,\ \forall p.
\end{array}$$

The CPLEX optimization software is used to solve the resultant mixed-integer problem. The results of this heuristic applied to solving this example problem follow. First, the variables are given. Tables D.7, D.8, D.9 and D.10 present specific solution information. The processing time is expressed in seconds, the objective values are in currency units and, in the columns containing the term '(%)', the values are percentages. Besides, the GAP is calculated using the Equation (4.34).

Variables:

$$\left[\begin{array}{ccccccc}
Y^{mold}_{11} & Y^{mold}_{12} & Y^{mold}_{13} & Y^{mold}_{21} & Y^{mold}_{22} & Y^{mold}_{23} & \\
\hline
Y^{pat}_{111} & Y^{pat}_{112} & Y^{pat}_{113} & Y^{pat}_{121} & Y^{pat}_{122} & Y^{pat}_{123} & \\
Y^{pat}_{211} & Y^{pat}_{212} & Y^{pat}_{213} & Y^{pat}_{221} & Y^{pat}_{222} & Y^{pat}_{223} & \\
Y^{pat}_{311} & Y^{pat}_{312} & Y^{pat}_{313} & Y^{pat}_{321} & Y^{pat}_{322} & Y^{pat}_{323} & \\
\hline
X^C_{111} & X^C_{112} & X^C_{113} & X^C_{121} & X^C_{122} & X^C_{123} & \\
X^C_{211} & X^C_{212} & X^C_{213} & X^C_{221} & X^C_{222} & X^C_{223} & \\
X^C_{311} & X^C_{312} & X^C_{313} & X^C_{321} & X^C_{322} & X^C_{323} & \\
\hline
X^S_{11} & X^S_{12} & X^S_{13} & X^S_{21} & X^S_{22} & X^S_{23} & X^S_{33} \\
\hline
S^C_{11} & S^C_{12} & S^C_{13} & S^C_{21} & S^C_{22} & S^C_{23} & \\
\hline
S^S_{11} & S^S_{12} & S^S_{13} & S^S_{21} & S^S_{22} & S^S_{23} &
\end{array}\right]
=
\left[\begin{array}{ccccccc}
1 & 1 & 0 & 1 & 0 & 1 & \\
\hline
0 & 0 & 0 & 0 & 0 & 1 & \\
0 & 0 & 0 & 1 & 0 & 0 & \\
1 & 1 & 0 & 1 & 0 & 0 & \\
\hline
0 & 0 & 0 & 0 & 0 & 4 & \\
0 & 0 & 0 & 2 & 0 & 0 & \\
4 & 4 & 0 & 2 & 0 & 0 & \\
\hline
2 & 2 & 2 & 5 & 2 & 0 & 0 \\
\hline
1 & 5 & 0 & 3 & 1 & 0 & \\
\hline
0 & 2 & 0 & 2 & 2 & 2 &
\end{array}\right]$$

Table D.7: Numerical example results: solution status, processing time, lower and upper-bound and GAP.

Solution status	**Processing time**	**Linear relaxation objective value**	**Lower bound value (4.33)**	**Mixed-integer objective value**	**GAP (%)**
Feasible	0.02	2002.21	1999.21	2086.50	4.04

Table D.8: Numerical example results: distribution of costs.

Item inventory penalty cost (%)	**Steel truss inventory penalty cost (%)**	**Steel truss production cost (%)**	**Mold section production cost (%)**	**Mold setup cost (%)**	**Cutting pattern setup cost (%)**
0.86	0.31	82.24	12.27	3.83	0.48

Table D.9: Numerical example results: mold and steel truss inventory level and usage.

No. of inventory items	**No. of inventory steel trusses**	**No. of steel trusses used**	**Average no. of steel trusses used per period**	**No. of mold sections used**	**No. of mold setups**	**No. of cutting pattern setups**
10	8	13	4.33	16	4	5

Table D.10: Numerical example results: mold availability and usage.

Average no. of available molds per period	**Average no. of molds used per period**	**Available space (AS)**	**Used mold space (UMS)**	**Production space (PS)**	**PS/AS (%)**	**PS/UMS (%)**
2.00	1.33	288.00	160.00	150.00	52.08	93.75

Analyzing these numbers, one can observe that, despite the tailing off issue, a good solution was obtained by this column-generation-based heuristic, since it provided a low GAP (4.04%) and a reasonable mold space management (93.75% of productivity over the molds effectively used in the production).