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New negative grade solitonic sector for supersymmetric KdV and mKdV hierarchies

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ABSTRACT

A systematic construction for supersymmetric negative graded (non-local) flows for mKdV and KdV based on $sl(2,1)$ with a principal gradation is proposed in this paper. We show that smKdV and sKdV can be mapped onto each other through a gauge super Miura transformation, together with an additional condition for the negative flows, which ensure the supersymmetry of the negative sKdV flow. In addition, we classify both smKdV and sKdV flows with respect to the vacuum (boundary) solutions. These are classified according to zero or non-zero constant vacuum. Each vacuum solution is used to derive both soliton solutions and the corresponding Heisenberg subalgebra for the smKdV hierarchy. We present the new solutions corresponding to non-zero bosonic and fermionic vacuum by constructing the deformed vertex operators. Finally, the gauge Miura transformation is employed to obtain the sKdV solutions, which exhibit a rich degeneracy due to both multiple gauge super Miura transformations and multiple vacuum possibilities.

1. Introduction

Integrable field theories are very peculiar models admitting infinite number of conservation laws. These are, in turn responsible for the stability of soliton solutions. The Korteweg–de Vries (KdV) [1–3] and modified KdV (mKdV) [4,5] equations are for instance, typical examples of such a class of models and have acted as prototypes for many new developments in the subject. More recently integrable hierarchies appear in many areas of theoretical physics, as in 2D CFTs [6–9] and string theory [10,11]. In fact, apart from single equations, say mKdV (or KdV) a series of other (higher/lower grade) evolution equations of motion (flows) can be systematically constructed given rise to an *Integrable Hierarchy* [12–14]. It has been realized that these classes of field theoretical models can be described and classified in terms of an affine algebraic structure [15–27] within the framework of zero curvature representation. The zero curvature representation appears as a very powerful method in constructing systematically flow equations. It has the virtue of being gauge invariant and this property has been explored in order to construct soliton solutions from a vacuum solution and moreover, to generate Backlund and Miura transformations for generalized A_r- mKdV hierachy [28,29]. The key ingredient in the construction involves the construction of an object of algebraic origin, the Lax operator, universal to all flows of the hierarchy. It depends upon the decomposition of the affine Lie algebra $\mathcal{G}$ according to a grading operator denoted by Q. The Lax operator is further specified by a second decomposition according to a choice of a constant grade one generator $E^{(1)}$. The construction of (multi)

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soliton solutions consists in mapping the zero curvature for a specific vacuum configuration into a non-trivial solution by gauge transformation. It therefore follows that a hierarchy is defined and classified by the following Lie algebraic ingredients, namely, i) the affine Lie algebra $\mathcal{G}$, ii) the grading operator Q, iii) the constant grade one generator $E^{(1)}$ and iv) a vacuum orbit.[1] The $\mathcal{N}=1$ supersymmetric version of KdV and mKdV equations (sKdV and smKdV) was originally introduced in [30], and generalized to odd sKdV hierarchy in [31]. Following the algebraic approach of constructing integrable hierarchies based on superalgebras [32], the sKdV equation has been studied both under a construction based upon the $osp(2|2)$ [33,34] or $sl(2,1)$ affine algebra [35,36]. In addition, the smKdV and the supersymmetric sinh-Gordon (sshG) equations have also been largely explored [37–41], and it turns out that both equations belong to the same integrable hierarchy when negative odd flows are taken into account [36]. Soliton solutions within the zero vacuum orbit were also explored for both smKdV and sshG equations using the dressing method [42], as well as integrable defects and Bäcklund transformation for such systems, by using both the Lagrangian and Gauge-Bäcklund transformations formalisms [43–47]. More recently, novel results on negative even flows for the bosonic mKdV hierarchy required non-zero vacuum orbit in order to construct soliton solutions [4,48]. It follows that the structure of the vacuum and the commutativity of the flows are directly connected. Two sub-hierarchies can be considered, namely, mKdV-I (positive and negative odd flows with a zero vacuum) and mKdV-II (positive odd and negative even flows with non-zero vacuum) [49]. Similarly, the same splitting in terms of different vacuum structures can be observed within the KdV hierarchy. This can be accomplished from the Miura transformation realized as a gauge transformation [29]. In fact it has been shown that the KdV hierarchy also splits in two sub-hierarchies, both having the same flows, positive and negative odd, but with different vacuum orbits. Consistency requires additional conditions upon the temporal derivative of the KdV fields for negative flows, the so-called temporal Miura relations. This paper aims to fill some gaps concerning the supersymmetric versions of mKdV and KdV equations, namely:

- To explore the negative even flows of smKdV hierarchy and determine whether they are supersymmetric. As expected, in order to do that, we analyse the hierarchy in terms of its orbit vacuum solution.
- To obtain the super Miura transformation as a gauge transformation, and use it to study the negative flows of sKdV hierarchy, as well as its implications.
- To construct new soliton solutions for the smKdV and sKdV equations by combining both the new non-zero dressing orbits and super Miura transformation.

This paper is organized as follows. In section 2, we review the algebraic formulation for the Lax pair of smKdV and sKdV hierachies, and exhibit some examples of flows, including a new negative even non-local flow for smKdV, and a negative odd for sKdV. In section 3, smKdV and sKdV hierarchies are connected by a gauge transformation leading to four types of super Miura transformations together with additional conditions for the time derivatives for the negative flows. In section 4, we introduce the dressing method for $sl(2,1)$ in order to prove, in section 5, the existence of different sub-hierarchies and its supersymmetry. Finally, we calculate the smKdV solution considering different vacuum orbits in section 6. In 7 we discuss the super Miura transformation to determine sKdV solutions.

2. Lax pair for the smKdV and sKdV flows

In this section, we shall employ the algebraic formalism to construct integrable smKdV and sKdV hierarchies. The general structure for a given flow t_N, with $N \in \mathbb{Z}$, is encoded in the zero curvature condition (ZCC),

$$\left[\partial_x + A_x,\, \partial_{t_N} + A_{t_N}\right] = 0, \tag{2.1}$$

for the universal spatial Lax potential, $\mathcal{L}_x = \partial_x + A_x$, and the temporal Lax potential, $\mathcal{L}_N = \partial_{t_N} + A_{t_N}$ which is specified for each integer N. Both smKdV and sKdV hierarchies are based upon the affine $sl(2,1)$ superalgebra endowed with a principal grading operator Q (see Appendix A). For the smKdV spatial gauge potential, we have

$$A_x^{\text{smKdV}} = \mathcal{E}^{(1)} + A_0 + A_{\frac{1}{2}} = \begin{pmatrix} \sqrt{\lambda}+v & 1 & -\bar{\psi} \\ \lambda & \sqrt{\lambda}-v & \sqrt{\lambda}\,\bar{\psi} \\ \sqrt{\lambda}\,\bar{\psi} & -\bar{\psi} & 2\sqrt{\lambda} \end{pmatrix}, \tag{2.2}$$

where

$$\mathcal{E}^{(1)} = K_1^{(1)} + K_2^{(1)} \in \mathcal{G}_1, \tag{2.3}$$

is a constant grade one element,

$$A_0 = v\, M_1^{(0)} \in \mathcal{G}_0, \tag{2.4}$$

contains the bosonic field $v = v(x, t_N)$ and

[1] See for instance [4] for a review.

$$A_{\frac{1}{2}} = \bar{\psi}\, G_2^{(\frac{1}{2})} \in \mathcal{G}_{\frac{1}{2}}, \tag{2.5}$$

contains the fermionic field $\bar{\psi} = \bar{\psi}(x, t_N)$ and the generators are given in Appendix A. On the other hand, the spatial gauge potential for sKdV hierarchy differs from smKdV in the algebraic elements associated to the fields,

$$A_x^{\text{sKdV}} = \mathcal{E}^{(1)} + A_{-1} + A_{-\frac{1}{2}} = \begin{pmatrix} \sqrt{\lambda} & 1 & 0 \\ \lambda + J & \sqrt{\lambda} & \bar{\chi}/\sqrt{\lambda} \\ 0 & -\bar{\chi}/\sqrt{\lambda} & 2\sqrt{\lambda} \end{pmatrix}, \tag{2.6}$$

with

$$A_{-1} = \frac{1}{2}\, J \left(K_1^{(-1)} - M_2^{(-1)} \right) \in \mathcal{G}_{-1} \tag{2.7}$$

and

$$A_{-\frac{1}{2}} = \frac{1}{2}\, \bar{\chi} \left(F_2^{(-\frac{1}{2})} + G_1^{(-\frac{1}{2})} \right) \in \mathcal{G}_{-\frac{1}{2}}, \tag{2.8}$$

which contain the bosonic $J = J(x, t_N)$ and the fermionic $\bar{\chi} = \bar{\chi}(x, t_N)$ fields respectively. Now, regarding the temporal gauge potential, it is possible to propose a different ansatz for each integer flow:

- For the smKdV positive sub-hierarchy

$$A_{t_N}^{\text{smKdV}} = D_N^{(N)} + D_N^{(N-\frac{1}{2})} + \cdots + D_N^{(\frac{1}{2})} + D_N^{(0)}, \qquad (D_N^{(a)} \in \mathcal{G}_a). \tag{2.9}$$

- For the sKdV positive sub-hierarchy

$$A_{t_N}^{\text{sKdV}} = \mathcal{D}_N^{(N)} + \mathcal{D}_N^{(N-\frac{1}{2})} + \mathcal{D}_N^{(N-1)} + \cdots + \mathcal{D}_N^{(0)} + \mathcal{D}_N^{(-\frac{1}{2})} + \mathcal{D}_N^{(-1)}, \qquad (\mathcal{D}_N^{(a)} \in \mathcal{G}_a). \tag{2.10}$$

The elements $D_N^{(a)}$ and $\mathcal{D}_N^{(a)}$ can be determined recursively using grade by grade decomposition of ZCC (for details see Appendix B). This leads to a constraint upon the highest grade elements for the associated positive sub-hierarchies, i.e.,

$$\left[\mathcal{E}^{(1)}, D_N^{(N)}\right] = \left[\mathcal{E}^{(1)}, \mathcal{D}_N^{(N)}\right] = 0. \tag{2.11}$$

Consequently both $D_N^{(N)}$ and $\mathcal{D}_N^{(N)}$ belong to the kernel of $\mathcal{E}^{(1)}$ (see more details in (A.6)), implying that $N = 2m + 1$, with $m \in \mathbb{Z}$. In both positive sub-hierarchies only odd flows are allowed.

Considering now the temporal gauge potentials for the negative flows, we propose the following structure:

- For the smKdV negative sub-hierarchy

$$A_{t_{-N}}^{\text{smKdV}} = D_{-N}^{(-N)} + D_{-N}^{(-N+\frac{1}{2})} + \cdots + D_{-N}^{(-\frac{1}{2})}, \qquad (D_{-N}^{(a)} \in \mathcal{G}_a). \tag{2.12}$$

- For the sKdV negative sub-hierarchy

$$A_{t_{-N}}^{\text{sKdV}} = \mathcal{D}_{-N}^{(-N-2)} + \mathcal{D}_{-N}^{(-N-\frac{3}{2})} + \mathcal{D}_{-N}^{(-N-1)} + \cdots + \mathcal{D}_{-N}^{(-1)} + \mathcal{D}_{-N}^{(-\frac{1}{2})}, \qquad (\mathcal{D}_{-N}^{(a)} \in \mathcal{G}_a), \tag{2.13}$$

where, analogously to the positive case, we can decompose the ZCC and determine each component $D_{-N}^{(a)}$ or $\mathcal{D}_{-N}^{(a)}$. From the smKdV lowest grade, we find the non-local equation for $D_{-N}^{(-N)}$,

$$\partial_x D_{-N}^{(-N)} + \left[A_0, D_{-N}^{(-N)}\right] = 0. \tag{2.14}$$

Now, unlike the smKdV negative case (2.14), we do obtain a constraint for the negative sub-hierarchy of sKdV,

$$\left[A_{-1}, \mathcal{D}_{-N}^{(-N-2)}\right] = 0, \tag{2.15}$$

implying that $\mathcal{D}_{-N}^{(-N-2)}$ is proportional to $A_{-1} = \frac{1}{2}\, J(K_1^{(-1)} - M_2^{(-1)})$, and restricting the negative flows to $N = 2m + 1$. We can therefore conclude that *the negative sub-hierarchy of sKdV admits only odd flows*, while there are no restrictions on the negative smKdV sub-hierarchy.[2]

Let us now consider the smKdV hierarchy and present the first few flows (the Lax Pairs are given in Appendix C):

[2] In Sect. 3 we shall demonstrate how equations can be mapped from one hierarchy to another through super Miura transformations. The fact that there are fewer temporal flows in sKdV side leads to a coalescence of smKdV flows into sKdV flows.

- $N = 3$

$$4\partial_{t_3} v = \partial_x^3 v - 6v^2\partial_x v - 3\bar{\psi}\partial_x\left(v\partial_x\bar{\psi}\right), \tag{2.16a}$$

$$4\partial_{t_3}\bar{\psi} = \partial_x^3\bar{\psi} - 3v\partial_x\left(v\bar{\psi}\right), \tag{2.16b}$$

leads to the super mKdV equation, which names the whole hierarchy.

- $N = 5$

$$\begin{aligned}16\partial_{t_5} v = \partial_x \Big(&\partial_x^4 v + 6v^5 - 10v\left(\partial_x v\right)^2 - 10v^2\partial_x^2 v - 5v\bar{\psi}\partial_x^3\bar{\psi}\\ &- 5\partial_x^2 v\bar{\psi}\partial_x\bar{\psi} - 5\partial_x v\bar{\psi}\partial_x^2\bar{\psi} + 20v^3\bar{\psi}\partial_x\bar{\psi}\Big),\end{aligned} \tag{2.17a}$$

$$\begin{aligned}16\partial_{t_5}\bar{\psi} = {}&\partial_x^5\bar{\psi} - 5v^2\partial_x^3\bar{\psi} - 15v\partial_x v\partial_x^2\bar{\psi} - 10(\partial_x v)^2\partial_x\bar{\psi} + 10v^4\partial_x\bar{\psi}\\ &- 10\partial_x v\partial_x^2 v\bar{\psi} - 5v\partial_x^3 v\bar{\psi} - 15v\partial_x^2 v\partial_x\bar{\psi} + 20v^3\partial_x v\bar{\psi}.\end{aligned} \tag{2.17b}$$

- $N = -1$

$$\partial_x\partial_{t_{-1}}\phi = 2\sinh 2\phi - 2\bar{\psi}\psi\sinh\phi, \tag{2.18a}$$

$$\partial_{t_{-1}}\bar{\psi} = 2\psi\cosh\phi, \tag{2.18b}$$

$$\partial_x\psi = 2\bar{\psi}\cosh\phi, \tag{2.18c}$$

leads to the super sinh-Gordon equation. Here, we have introduced a convenient reparametrization $v(x,t_N) = \partial_x\phi(x,t_N)$ (or $\phi(x,t_N) = \partial_x^{-1}v(x,t_N)$).

- $N = -2$

$$\partial_{t_{-2}}\partial_x\phi = -2\left(a_- + a_+\right) + 2\bar{\psi}\left(\Omega_- + \Omega_+\right), \tag{2.19a}$$

$$\partial_{t_{-2}}\bar{\psi} = -2\left(\Omega_+ - \Omega_-\right), \tag{2.19b}$$

with

$$\psi_\pm = \partial_x^{-1}\left(e^{\pm\phi}\bar{\psi}\right), \tag{2.20a}$$

$$a_\pm = e^{\pm 2\phi}\partial_x^{-1}\left[e^{\mp 2\phi}\left(1 + \psi_\mp\partial_x\psi_\pm\right)\right], \tag{2.20b}$$

and

$$\Omega_\pm = \frac{e^{\pm\phi}}{2}\partial_x^{-1}\left[e^{\mp 2\phi}\psi_\pm - \psi_\mp \mp \left(\partial_x\psi_\mp\right)\left(1 + \psi_-\psi_+ \mp 2a_\pm\right)\right]. \tag{2.21}$$

Here we have used the anti-derivative operator, defined as $\partial_x^{-1} f = \int^x f(y)\mathrm{d}y$.

Taking the limit $\bar{\psi} \to 0$, we recover the mKdV bosonic hierarchy, which is based on the $sl(2)$ affine algebra [49]. Now, if we consider the $N = \frac{1}{2}$ case, we obtain

$$\delta v \equiv \partial_{t_{\frac{1}{2}}} v = \xi\,\partial_x\bar{\psi}, \tag{2.22a}$$

$$\delta\bar{\psi} \equiv \partial_{t_{\frac{1}{2}}}\bar{\psi} = -\xi\, v, \tag{2.22b}$$

the supersymmetry transformation relating the bosonic and fermionic fields of smKdV. Here ξ is a Grassmann constant parameter.

For the sKdV hierarchy the first temporal non-trivial flows leads to the following equations (the Lax Pairs are given in Appendix C):

- $N = 3$

$$4\partial_{t_3} J = \partial_x^3 J - 6J\partial_x J - 3\bar{\chi}\partial_x^2\bar{\chi}, \tag{2.23a}$$

$$4\partial_{t_3}\bar{\chi} = \partial_x^3\bar{\chi} - 3\partial_x\left(J\bar{\chi}\right), \tag{2.23b}$$

which is the supersymmetric KdV equation and name the entire hierarchy.

- $N = 5$ leads to the supersymmetric Sawada-Kotera equation,

$$\begin{aligned}16\partial_{t_5} J = {}&\partial_x^5 J - 10\partial_x J\partial_x^2 J - 10J\partial_x^3 J + 30J^2\partial_x J\\ &+ 26J\bar{\chi}\partial_x^2\bar{\chi} + 32\partial_x J\bar{\chi}\partial_x\bar{\chi} - 8\bar{\chi}\partial_x^4\bar{\chi} - 4\partial_x\bar{\chi}\partial_x^3\bar{\chi},\end{aligned} \tag{2.24a}$$

$$\begin{aligned}16\partial_{t_5}\bar{\chi} = {}&\partial_x^5\bar{\chi} - 5\partial_x^3 J\bar{\chi} - 10\partial_x^2 J\partial_x\bar{\chi} - 10\partial_x J\partial_x^2\bar{\chi}\\ &- 5J\partial_x^3\bar{\chi} + 20J\partial_x J\bar{\chi} + 10J^2\partial_x\bar{\chi}.\end{aligned} \tag{2.24b}$$

- $N=-1$

$$\partial_{t_{-1}}\partial_x^3\eta - 2\,\partial_x^2\eta\left(\partial_{t_{-1}}\eta+\gamma\right) - 4\,\partial_{t_{-1}}\partial_x\eta\,\partial_x\eta - \partial_x\left(\partial_{t_{-1}}\bar{\gamma}\,\partial_x^2\bar{\gamma}\right)$$
$$- \partial_x\eta\,\partial_x\bar{\gamma}\,\partial_{t_{-1}}\bar{\gamma} + \partial_x^2\bar{\gamma}\left(\bar{\nu}_- - \bar{\nu}_+ + 2\bar{\gamma}\right) + 2\,\partial_{t_{-1}}\partial_x^2\bar{\gamma}\,\partial_x\bar{\gamma} \;=\; 0, \tag{2.25a}$$
$$\partial_{t_{-1}}\left(\partial_x\eta\,\partial_x\bar{\gamma}\right) + \partial_x^2\left(\bar{\nu}_+ + \bar{\nu}_-\right) + \partial_x\left(\partial_{t_{-1}}\partial_x^2\bar{\gamma} + 2\partial_x\bar{\gamma} + 2\,\partial_x\bar{\gamma}\,\gamma\right)$$
$$- \partial_x\eta(\bar{\nu}_- - \bar{\nu}_+ + 2\bar{\gamma}) \;=\; 0, \tag{2.25b}$$

here we have used the following reparametrization,

$$J(x,t_N)=\partial_x\eta(x,t_N), \qquad \bar{\chi}(x,t_N)=\partial_x\bar{\gamma}(x,t_N), \qquad \partial_x\gamma=\partial_x\bar{\gamma}\partial_{t_{-1}}\bar{\gamma}, \tag{2.26a}$$
$$\partial_x\bar{\nu}_+=\partial_{t_{-1}}\eta\partial_x\bar{\gamma}, \qquad \partial_x\bar{\nu}_-=\partial_x\eta\partial_{t_{-1}}\bar{\gamma}. \tag{2.26b}$$

- $N=\frac{1}{2}$

$$\delta J=\partial_{t_{\frac{1}{2}}}J=-\xi\,\partial_x\bar{\chi}, \tag{2.27a}$$
$$\delta\bar{\chi}=\partial_{t_{\frac{1}{2}}}\bar{\chi}=\xi J, \tag{2.27b}$$

provides the supersymmetry transformation for sKdV model.

The sKdV(-1) equations (2.25) were derived in [50]. Notice that these equations present a very remarkable feature compared to negative flows of smKdV (2.18) and (2.19), since they do not depend on exponential terms and are mainly composed of temporal and spatial derivatives.

Regarding supersymmetry property, it was already proven in [36] that the positive flows of sKdV were supersymmetric, as well as the positive and odd negative flows for the smKdV hierarchy. In this work, we extend this verification to the novel negative flows of sKdV and smKdV exhibited here. It is possible to check by direct calculations that the equation of motion for smKdV(-2) is indeed supersymmetric (see Appendix D). For the sKdV(-1), the supersymmetry proof is accomplished by making use of the super Miura transformation (SMT) in section 3.

The supersymmetric mKdV and KdV hierarchies are related through via super Miura transformation (SMT), and it is known that each positive flow of the sKdV hierarchy is mapped (SMT) into a positive flow of smKdV [33]. Now, with the presentation of these novel negative flows for sKdV and even negative flows for smKdV, a discussion opens up on how to map the negative flows from one hierarchy into the other, since that there are more negative flows in smKdV sector than in sKdV. This indicates a coalescence similar to those we have indicated in earlier work for the pure bosonic case [49]. In the next section, we present the super Miura transformation as a gauge transformation and demonstrate how it provides a map between the sKdV and smKdV hierarchies.

3. Gauge super Miura transformation

In this section, we discuss the connection between the supersymmetric versions of mKdV and KdV hierarchies. For the bosonic case, the mKdV and KdV equations can be related through the well-known Miura transformation [1,51,52]. The extension to the complete hierarchy was shown to be connected to the $sl(2)$ affine algebra and considered in [53]. This construction was later generalized to the $sl(r+1)$ case in [29] and more recently, extended to negative flows in [49].

For the supersymmetric case, a connection between smKdV and sKdV equations was established by the super Miura transformation [30]. More recently, we have recovered and extended these results for the complete hierarchies in [50]. The key ingredient is to construct a gauge transformation $\mathcal{S}$ which maps the gauge potentials given by (2.2) and (2.6) as

$$A_x^{\text{sKdV}}=\mathcal{S}A_x^{\text{smKdV}}\mathcal{S}^{-1}+\mathcal{S}\partial_x\mathcal{S}^{-1}. \tag{3.1}$$

Following the algebraic approach proposed in [29], two different graded ansatz can be considered for $\mathcal{S}$:

$$\mathcal{S}_1=s^{(0)}+s^{(-\frac{1}{2})}+s^{(-1)} \qquad \text{or} \qquad \mathcal{S}_2=s^{(-1)}+s^{(-3/2)}+s^{(-2)}, \tag{3.2}$$

where $s^{(n)}$ are given by the following graded elements

$$s^{(2m)}=\lambda^m\begin{pmatrix} a_{11} & 0 & 0\\ 0 & a_{22} & 0\\ 0 & 0 & a_{33}\end{pmatrix}, \quad s^{(2m+1)}=\lambda^m\begin{pmatrix} \sqrt{\lambda}\,b_{11} & a_{12} & 0\\ \lambda\,a_{21} & \sqrt{\lambda}\,b_{22} & 0\\ 0 & 0 & \sqrt{\lambda}\,b_{33}\end{pmatrix}, \tag{3.3}$$

$$s^{(2m+\frac{1}{2})}=\lambda^m\begin{pmatrix} 0 & 0 & a_{13}\\ 0 & 0 & \sqrt{\lambda}\,a_{23}\\ \sqrt{\lambda}\,a_{31} & a_{32} & 0\end{pmatrix}, \quad s^{(2m+\frac{3}{2})}=\lambda^m\begin{pmatrix} 0 & 0 & \sqrt{\lambda}a_{13}\\ 0 & 0 & \lambda a_{23}\\ \lambda\,a_{31} & \sqrt{\lambda}\,a_{32} & 0\end{pmatrix}, \tag{3.4}$$

leading to four different solutions, namely,

$$\mathcal{S}_{1,\pm} = \begin{pmatrix} 1 & 0 & 0 \\ v & 1 & -\bar{\psi} \\ \pm\bar{\psi} & 0 & \pm 1 \end{pmatrix}, \qquad \mathcal{S}_{2,\pm} = \begin{pmatrix} 0 & \frac{1}{\lambda} & 0 \\ 1 & -\frac{v}{\lambda} & \frac{\bar{\psi}}{\sqrt{\lambda}} \\ 0 & \mp\frac{\bar{\psi}}{\lambda} & \frac{\pm 1}{\sqrt{\lambda}} \end{pmatrix}. \tag{3.5}$$

The sKdV and smKdV fields are related by

$$\begin{aligned} J^{(1,\pm)} &= v^2 - \partial_x v + \bar{\psi}\partial_x\bar{\psi}, \\ \bar{\chi}^{(1,\pm)} &= \mp v\bar{\psi} \pm \partial_x\bar{\psi}, \end{aligned} \tag{3.6}$$

for $\mathcal{S}_{1,\pm}$ and by

$$\begin{aligned} J^{(2,\pm)} &= v^2 + \partial_x v + \bar{\psi}\partial_x\bar{\psi}, \\ \bar{\chi}^{(2,\pm)} &= \mp v\bar{\psi} \mp \partial_x\bar{\psi}, \end{aligned} \tag{3.7}$$

for $\mathcal{S}_{2,\pm}$. The existence of several different gauge transformations is nothing more than a manifestation of the symmetry of super mKdV equation under the parity transformation $v \to -v$ and $\bar{\psi} \to -\bar{\psi}$. Also, we might consider the inversion matrix L_- acting upon the fermionic subspace, i.e.,

$$L_- = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \tag{3.8}$$

so we relate

$$\mathcal{S}_{i,-} = L_-\,\mathcal{S}_{i,+} \qquad \text{with} \qquad i = 1, 2.$$

Thus, we can proceed with our analysis assuming that $\mathcal{S} = \mathcal{S}_{1,+}$ without loss of generality. This is indeed highly convenient, as it allows the gauge-Miura $\mathcal{S}_+$ to be expressed in exponential form

$$\mathcal{S} \equiv \mathcal{S}_+ = \mathrm{e}^{\frac{v}{2}\left(K_1^{(-1)} - M_2^{(-1)}\right) - \frac{\bar{\psi}}{2}\left(F_2^{(-\frac{1}{2})} + G_1^{(-\frac{1}{2})}\right)}. \tag{3.9}$$

It therefore follows that it is possible to show that each positive smKdV N flow can be mapped into a one-to-one correspondence with sKdV N flow:

$$t_N^{\mathrm{smKdV}} \xrightarrow{\mathcal{S}} t_N^{\mathrm{sKdV}}. \tag{3.10}$$

In order to extend this analysis to the negative sector, consider (3.9) and a generic negative *odd* flow $A_{t_{-2n+1}}^{\mathrm{smKdV}}$ under the gauge transformation induced by $\mathcal{S}$, yielding the following graded structure

$$\begin{aligned} A_{t_{-2n+1}}^{\mathrm{sKdV}} &\equiv \mathcal{S}\left(D_{-2n+1}^{(-2n+1)} + D_{-2n+1}^{(-2n+\frac{3}{2})} + \cdots + D_{-2n+1}^{(-\frac{1}{2})}\right)\mathcal{S}^{-1} + \mathcal{S}\partial_{t_{-2n+1}}\mathcal{S}^{-1} \\ &= \mathcal{D}_{-2n+1}^{(-2n-1)} + \mathcal{D}_{-2n+1}^{(-2n-\frac{1}{2})} + \mathcal{D}_{-2n+1}^{(-2n)} + \cdots + \mathcal{D}_{-2n+1}^{(-1)} + \mathcal{D}_{-2n+1}^{(-\frac{1}{2})}. \end{aligned} \tag{3.11}$$

Considering the subsequent negative *even* flow $A_{t_{-2n}}^{\mathrm{smKdV}}$, the lower grade operator is proportional to $D_{-2n}^{(-2n)} \sim M_1^{(-2n)}$, leaving the algebraic structure invariant, i.e.,

$$\begin{aligned} \widetilde{A}_{t_{-2n+1}}^{\mathrm{sKdV}} &\equiv \mathcal{S}\left(D_{-2n}^{(-2n)} + D_{-2n}^{(-2n+\frac{1}{2})} + \cdots + D_{-2n}^{(-\frac{1}{2})}\right)\mathcal{S}^{-1} + \mathcal{S}\partial_{t_{-2n}}\mathcal{S}^{-1} \\ &= \widetilde{\mathcal{D}}_{-2n+1}^{(-2n-1)} + \widetilde{\mathcal{D}}_{-2n+1}^{(-2n-\frac{1}{2})} + \widetilde{\mathcal{D}}_{-2n+1}^{(-2n)} + \cdots + \widetilde{\mathcal{D}}_{-2n+1}^{(-1)} + \widetilde{\mathcal{D}}_{-2n+1}^{\left(-\frac{1}{2}\right)}. \end{aligned} \tag{3.12}$$

Since the potentials A_x^{sKdV} and A_x^{smKdV} are universal within the hierarchies, the zero curvature condition for (3.11) and (3.12) must yield the same operator, i.e.,

$$A_{t_{-2n+1}}^{\mathrm{sKdV}} = \widetilde{A}_{t_{-2n+1}}^{\mathrm{sKdV}}, \tag{3.13}$$

and the two gauge potentials provide the same evolution equations. Therefore, a negative even and its subsequent odd flow collapse into the same negative odd sKdV flow, which is consistent with the fact that there is no *even* negative flow within the sKdV hierarchy, i.e.,

$$\begin{array}{lll} t_{-N}^{\mathrm{smKdV}} & & \\ & \searrow^{\mathcal{S}} & \\ & & t_{-N}^{\mathrm{sKdV}} \\ & \nearrow_{\mathcal{S}} & \\ t_{-N-1}^{\mathrm{smKdV}} & & \end{array} \tag{3.14}$$

for $N = 2n-1$, $n \in \mathbb{Z}^+$. Let us consider the explicit example of $A_{t_{-1}}^{\mathrm{smKdV}}$ and $A_{t_{-2}}^{\mathrm{smKdV}}$ given by (C.5) and (C.6) in Appendix C. The mapping

$$A_{t_{-1}}^{\mathrm{sKdV}} = \mathcal{S} A_{t_{-1}}^{\mathrm{smKdV}} \mathcal{S}^{-1} + \mathcal{S}\partial_x \mathcal{S}^{-1}, \tag{3.15}$$

implies the following relations among the smKdV and the sKdV fields,

$$\eta_{t_{-1}} = 2\left(\mathrm{e}^{-2\phi} + \mathrm{e}^{-\phi}\psi\bar{\psi}\right), \tag{3.16a}$$

$$\bar{\gamma}_{t_{-1}} = 2\mathrm{e}^{-\phi}\psi, \tag{3.16b}$$

where $(\eta, \bar{\gamma})$ are a distinct set of parametrized solutions and the smKdV fields in r.h.s. are solutions of t_{-1} equations (2.18). However, if the map follows t_{-2}^{smKdV} into t_{-1}^{sKdV}, i.e.,

$$A_{t_{-1}}^{\mathrm{sKdV}} = \mathcal{S} A_{t_{-2}}^{\mathrm{smKdV}} \mathcal{S}^{-1} + \mathcal{S}\partial_x \mathcal{S}^{-1}, \tag{3.17}$$

this relation requires

$$\eta_{t_{-1}} = 4\left(a_- + \Omega_-\bar{\psi} + \frac{1}{2}\psi_+\psi_-\right), \tag{3.18a}$$

$$\bar{\gamma}_{t_{-1}} = 4\Omega_-, \tag{3.18b}$$

where $\psi_\pm$, $a_\pm$ and $\Omega_\pm$ are given by (2.20a), (2.20b) and (2.21) and the fields satisfy (2.19). Notice that such set of relations involving KdV fields, $(\eta, \bar{\gamma})$ define a distinct set of solutions for equation (2.25). This allows us to determine a larger class of solutions for the negative flows within the sKdV hierarchy and will be quite useful in determining the Heisenberg sKdV subalgebra. Moreover, one of the most remarkable applications of such relations is to allow us to directly verify supersymmetry of (2.25). It is straightforward to show that combining the reparametrization $J = \partial_x \eta$ and $\bar{\chi} = \partial_x \bar{\gamma}$ together with relations (2.27), we have

$$\delta\bar{\chi} = \xi J \qquad \rightarrow \qquad \delta\bar{\gamma} = \xi\eta + f(t), \tag{3.19a}$$

$$\delta J = -\xi\partial_x\bar{\chi} \quad \rightarrow \quad \delta\eta = -\xi\partial_x\bar{\gamma} + g(t). \tag{3.19b}$$

For the positive sKdV equations, $f(t)$ and $g(t)$ have no relevance, since there is always at least one derivative with respect to x acting upon η or $\bar{\gamma}$. Nevertheless, for sKdV(-1) and other negative flow equations, there will always be additional explicit relations involving temporal derivatives like (3.16) or (3.18) to take care of those terms. Consider for instance sKdV(-1) as gauge transformed from the sshG (2.18) and uses (3.19) together with (3.16), it is possible to determine $f(t)$ and $g(t)$ to obtain

$$\delta\bar{\gamma} = \xi\eta - 2\xi t, \tag{3.20a}$$

$$\delta\eta = -\xi\partial_x\bar{\gamma}. \tag{3.20b}$$

Furthermore, in order to compute the supersymmetric transformations of the auxiliary fields γ and $\nu_\pm$ (2.26a) sKdV(-1), we will need to use (3.6) and (3.16) to get the expressions

$$\gamma = 2\mathrm{e}^{-\phi}\bar{\psi}\psi, \tag{3.21a}$$

$$\bar{\nu_-} - \bar{\nu_+} + 2\bar{\gamma} = 2\bar{\psi} - 2(\partial_x\phi)\mathrm{e}^{-\phi}\psi - 2\mathrm{e}^{-2\phi}\bar{\psi}, \tag{3.21b}$$

and finally obtain the variations

$$\delta\gamma = \xi\left(\nu_- - \nu_+ + 2\bar{\gamma}\right), \tag{3.22a}$$

$$\delta\left(\bar{\nu_-} - \bar{\nu_+} + 2\bar{\gamma}\right) = -\xi\partial_x\gamma. \tag{3.22b}$$

It is clear that this pair acts as supersymmetric variables among themselves. Finally, after a straightforward but long calculation, by using (3.20) and (3.22), it is possible to show that the pair of equations (2.25) is supersymmetric invariant. On the other hand, if we had used (3.18), the procedure will follow the same steps, with a minor adjustment in the auxiliary variables, which will now be given by

$$\gamma = 4\bar{\psi}\Omega_- - 2\psi_+\psi_-, \tag{3.23a}$$

$$\bar{\nu_-} - \bar{\nu_+} + 2\bar{\gamma} = 2\bar{\psi} + 2\mathrm{e}^{\phi}\psi_- + 2\mathrm{e}^{-\phi}\psi_+ + 2\bar{\psi}\psi_-\psi_+ - 4\bar{\psi}a_- - 4\partial_x\phi\Omega_-. \tag{3.23b}$$

This however, will not introduce any changes to the supersymmetry relations (3.20) and (3.22), leading to the same result, i.e., the sKdV(−1) flow is supersymmetric. In this way, the importance of the gauge-Miura mapping in determining sKdV properties in terms of smKdV becomes evident once again.

Despite the fact that the temporal Miura is an essential ingredient and highly non-trivial relation, it is possible to obtain a general formula for any flow using both the zero curvature equation for sKdV and the super Miura transformation. Let us consider an arbitrary negative sKdV flow. The ZCC decomposition in Appendix B gives us equations (B.8g) and (B.8f) that set both $\mathcal{D}_{-N}^{\left(-\frac{1}{2}\right)}$ and $\mathcal{D}_{-N}^{(-1)}$ in the *Kernel* of $\mathcal{E}^{(1)}$. Next, one can use the equations (B.8e) and (B.8d) to obtain the following form for the lowest elements $\mathcal{D}_{-N}$ for any N, namely[3]

$$\mathcal{D}_{-N}^{(-\frac{1}{2})}[\bar{\gamma},\eta] = \frac{\partial_{t_{-N}}\bar{\gamma}(x,t)}{2}F_2^{\left(-\frac{1}{2}\right)}, \tag{3.24}$$

and

$$\mathcal{D}_{-N}^{(-1)}[\bar{\gamma},\eta] = \frac{1}{2}\left(\gamma + \partial_{t_{-N}}\eta(x,t)\right)K_1^{(-1)} + \frac{1}{2}(\gamma+2)K_2^{(-1)}. \tag{3.25}$$

Additionally, the super Miura transformation (3.9) gives us the following results for a generic mapping upon smKdV($-M$) flow to sKdV($-N$) flow for the minus one-half graded ($\mathcal{D}_{-N}^{(-\frac{1}{2})}[\bar{\gamma},\eta]$) element

$$\mathcal{D}_{-N}^{(-\frac{1}{2})}[\bar{\gamma},\eta] = D_{-M}^{(-\frac{1}{2})}[\bar{\psi},\phi] + \frac{\partial_{t_{-M}}\bar{\psi}}{2}\left(F_2^{(-\frac{1}{2})} + G_1^{(-\frac{1}{2})}\right). \tag{3.26}$$

Using (B.6d), the equation of motion for $\bar{\psi}$ is determined

$$\partial_{t_{-M}}\bar{\psi} = -2b_{-M}^{(-\frac{1}{2})}, \tag{3.27}$$

where

$$D_{-M}^{(-\frac{1}{2})}[\bar{\psi},\phi] = a_{-M}^{(-\frac{1}{2})}F_2^{(-\frac{1}{2})} + b_{-M}^{(-\frac{1}{2})}G_1^{(-\frac{1}{2})}. \tag{3.28}$$

Combining these two equations, it is easy to see that

$$\mathcal{D}_{-N}^{(-\frac{1}{2})}[\bar{\gamma},\eta] = \left(a_{-M}^{(-\frac{1}{2})} - b_{-M}^{(-\frac{1}{2})}\right)F_2^{(-\frac{1}{2})}, \tag{3.29}$$

and due to the Lax pair uniqueness, one can compare (3.24) and (3.29) to obtain the first temporal super Miura

$$\partial_{t_{-N}}\bar{\gamma}(x,t) = 2\left(a_{-M}^{(-\frac{1}{2})} - b_{-M}^{(-\frac{1}{2})}\right). \tag{3.30}$$

The other one is obtained in the same way, by using the results obtained from the super Miura transformation for the grade minus one element, $\mathcal{D}_{-N}^{(-1)}[\bar{\gamma},\eta]$:

$$\mathcal{D}_{-N}^{(-1)}[\bar{\gamma},\eta] = \left(a_{-M}^{(-1)} + c_{-M}^{(-1)}\right)K_1^{(-1)} + \frac{1}{2}\left(1 + b_{-M}^{(-1)} + \bar{\psi}a_{-M}^{(-\frac{1}{2})} - \bar{\psi}b_{-M}^{(-\frac{1}{2})}\right)K_2^{(-1)}, \tag{3.31}$$

where

$$D_{-M}^{(-1)}[\bar{\psi},\phi] = a_{-M}^{(-1)}K_1^{(-1)} + (1 + b_{-M}^{(-1)})K_2^{(-1)} + c_{-M}^{(-1)}M_2^{(-1)}, \tag{3.32}$$

and when compared with (3.25) yields

$$\partial_{t_{-N}}\eta(x,t) = 2(a_{-M}^{(-1)} + c_{-M}^{(-1)}) + \partial_{t_{-N}}\bar{\gamma}(x,t)\bar{\psi}. \tag{3.33}$$

It therefore follows that, knowing the elements $D_{-M}^{(-1)}[\bar{\psi},\phi]$ and $D_{-M}^{(-\frac{1}{2})}[\bar{\psi},\phi]$ of the smKdV hierarchy we are able to determine the temporal Miura relations (consider the notation $f(x,t_N) \equiv f_N$ for simplicity):

$$\partial_{t_{-2n+1}}\bar{\gamma} = \begin{cases} 2\left(a_{-2n+1}^{(-\frac{1}{2})} - b_{-2n+1}^{(-\frac{1}{2})}\right)\left[\phi_{-2n+1},\bar{\psi}_{-2n+1}\right] & \text{for odd smKdV flows,} \\ 2\left(a_{-2n}^{(-\frac{1}{2})} - b_{-2n}^{(-\frac{1}{2})}\right)\left[\phi_{-2n},\bar{\psi}_{-2n}\right] & \text{for even smKdV flows.} \end{cases} \tag{3.34}$$

[3] Here, we have chosen the integration constant appearing in the K_2 term to be 1. We also recall that $\partial_x\gamma = \partial_x\bar{\gamma}\partial_{t_{-N}}\bar{\gamma}$.

$$\partial_{t_{-2n+1}}\eta = \begin{cases} 2\left(a^{(-1)}_{-2n+1}+c^{(-1)}_{-2n+1}\right)\left[\phi_{-2n+1},\bar{\Psi}_{-2n+1}\right]+\partial_{t_{-2n+1}}\bar{\gamma}\,\bar{\Psi}_{-2n+1} & \text{for odd smKdV flows,} \\ 2\left(a^{(-1)}_{-2n}+c^{(-1)}_{-2n}\right)\left[\phi_{-2n},\bar{\Psi}_{-2n}\right]+\partial_{t_{-2n+1}}\bar{\gamma}\,\bar{\Psi}_{-2n} & \text{for even smKdV flows.} \end{cases} \tag{3.35}$$

As examples, equations (3.16a), (3.16b), (3.18a) and (3.18b), are direct applications of this relation. Now we have established all the structure concerning the equations of motion for both the smKdV and sKdV hierarchy as well as the ingredients to map the equations of motion of one into the other. In next section, we will consider the remaining ingredient, namely, the solution of the field equations.

4. The dressing method and tau functions for $sl(2,1)$

We shall now discuss the construction of soliton solutions for smKdV and sKdV hierarchies. In order to do that, we will introduce the *dressing method* and classify each hierarchy according to its vacuum orbit. Let us start with a brief compilation of the most important ideas of this method. The main point of the dressing method is to start with a simple solution (vacuum solution or vacuum orbit) to obtain the non-trivial (multi)-soliton solutions for the entire hierarchy. Let us define

$$A^{\rm vac}_{\mu} = A_{\mu}\Big|_{v=v_{\rm vac}}, \tag{4.1}$$

where $\mu = x, t_N$, and $v_{\rm vac}$ is the vacuum solution for the zero curvature condition (2.1). We use the fact that ZCC is invariant under a gauge transformation $\Theta_{\pm}$ to map the vacuum into some non-trivial soliton solution, i.e.,

$$A_{\mu} = \Theta_{\pm}(\partial_{\mu} + A^{\rm vac}_{\mu})\Theta^{-1}_{\pm}. \tag{4.2}$$

The plus-minus label upon Θ indicates that it can be decomposed either as a exponential of positive or negative graded elements of the algebra, i.e.,

$$\Theta_{+} = \prod_{n\geq 0} \mathrm{e}^{\theta_{n/2}}, \qquad \Theta_{-} = \prod_{n>0} \mathrm{e}^{\theta_{-n/2}}, \tag{4.3}$$

where $\theta_a \in \mathcal{G}_a$ (see Appendix A). Another important property is the fact that the Lax potentials can be written as a pure gauge potential

$$A^{\rm vac}_{\mu} = T_0\partial_{\mu}T_0^{-1}, \quad A_{\mu} = T\partial_{\mu}T^{-1}, \tag{4.4}$$

where

$$T_0 = e^{-xA^{\rm vac}_x}\, e^{-tA^{\rm vac}_t} \qquad \text{and} \qquad T = \Theta_{\pm}T_0. \tag{4.5}$$

Combining (4.2), (4.4) and (4.5), we can show that the following condition holds,

$$\Theta^{-1}_{-}\Theta_{+} = T_0\, g\, T^{-1}_0, \tag{4.6}$$

where g is a constant group element. Let us introduce highest weight representation (based upon the Kac-Moody algebra $\mathcal{G}$) such that

$$\theta_i|\lambda_j\rangle = 0, \quad \text{with} \quad i > 0. \tag{4.7}$$

Consider the matrix elements $\tau_{ij}(\boldsymbol{v})$

$$\tau_{ij}(\boldsymbol{v},\sigma) \equiv \langle\lambda_i|\mathcal{G}_{\sigma}\Theta^{-1}_{-}\Theta_{+}|\lambda_j\rangle, \tag{4.8}$$

where $\mathcal{G}_{\sigma}$ is a properly chosen algebra element of grade σ. On the other hand, using (4.6) we can also write the following equivalence

$$\tau_{ij}(\boldsymbol{v},\sigma) = \langle\lambda_i|\mathcal{G}_{\sigma}T_0\, g\, T^{-1}_0|\lambda_j\rangle. \tag{4.9}$$

Now, let us consider the constant element g in the following form, where n refers to the n-soliton solution

$$g = \prod_{i=1}^{n} \mathrm{e}^{V(k_i)} \quad \text{such that} \quad \left[A^{\rm vac}_{\mu}, V(k_i)\right] = \omega_{\mu}(k_i)V(k_i), \tag{4.10}$$

where $V(k_i)$ is the so-called *vertex operator* [20,21,25]. It there follows that expression (4.6) became rather simple

$$T_0\, g\, T^{-1}_0 = \prod_{i=1}^{n} \mathrm{e}^{\rho_i(t_N,x)V(k_i)}, \tag{4.11}$$

with

$$\rho_i(t_N,x) \equiv \mathrm{e}^{-\omega_{t_N}(k_i)t_N - \omega_x(k_i)x}. \tag{4.12}$$

Substituting the above expressions in (4.9), we find

$$\tau_{ij}(\boldsymbol{v},\sigma)=\langle\lambda_i|\mathcal{G}_\sigma\prod_{i=1}^{n}\mathrm{e}^{\rho_i(t_N,x)V(k_i)}|\lambda_j\rangle, \tag{4.13}$$

and assuming that $V(k_i)$ is a nilpotent operator, we have

$$\tau_{ij}(\boldsymbol{v},\sigma)=\langle\lambda_i|\prod_{i=1}^{n}\left(\mathcal{G}_\sigma+\rho_i\mathcal{G}_\sigma V(k_i)+\cdots+\frac{1}{m!}\rho_i^m\mathcal{G}_\sigma V^m(k_i)\right)|\lambda_j\rangle, \tag{4.14}$$

where m depends on the vertex operator and n is chosen to match the number of solitons. For instance, for models based on the $sl(2,1)$ superalgebra, $m=1$, and we are mainly interested in one-soliton solution, i.e. $n=1$, so we can simplify the expression above as

$$\tau_{ij}(\boldsymbol{v},\sigma)=\langle\lambda_i|\mathcal{G}_\sigma\Theta_-^{-1}\Theta_+|\lambda_j\rangle=\langle\lambda_i|\mathcal{G}_\sigma|\lambda_j\rangle+\langle\lambda_i|\mathcal{G}_\sigma V(k_i)|\lambda_j\rangle\rho(t,x). \tag{4.15}$$

Now, let us do some practical calculations concerning the $sl(2,1)$ smKdV case. First, we establish the fundamental weight system as follows

$$\begin{aligned}
\hat{\kappa}\,|\lambda_i\rangle&=|\lambda_i\rangle\,, & \langle\lambda_i|\,\hat{\kappa}&=\langle\lambda_i|\,,\\
M_1^{(0)}\,|\lambda_i\rangle&=\delta_{i,1}\,|\lambda_i\rangle\,, & \langle\lambda_i|\,M_1^{(0)}&=\langle\lambda_i|\,\delta_{i,1}, \qquad (n=1,2,3\dots)\\
\mathcal{G}^{(n/2)}\,|\lambda_i\rangle&=0, & \langle\lambda_i|\,\mathcal{G}^{(-\frac{n}{2})}&=0,
\end{aligned} \tag{4.16}$$

and consider the spatial Lax for super mKdV hierarchy[4]:

$$A_x^{\mathrm{smKdV}}=\mathcal{E}^{(1)}+A_0(\phi,\nu)+A_{\frac{1}{2}}(\bar{\psi})=K_1^{(1)}+K_2^{(1)}+\partial_x\phi_1\,M_1^{(0)}+\partial_x\nu\kappa+\bar{\psi}G_2^{(\frac{1}{2})}. \tag{4.17}$$

Consider the most general vacuum projection $(\nu,\phi,\bar{\psi})=(0,\upsilon_0 x,\bar{\psi}_0)$

$$A_{x,\mathrm{vac}}^{\mathrm{smKdV}}=K_1^{(1)}+K_2^{(1)}+\upsilon_0\,M_1^{(0)}+\bar{\psi}_0G_2^{(\frac{1}{2})}. \tag{4.18}$$

Since the field appears in grade 0 and $\frac{1}{2}$ terms, we propose τ functions with projections on these grades. The first obvious choice for the grade zero term is

$$\tau_{00}(\boldsymbol{v},0)\equiv\tau_0=\langle\lambda_0|\Theta_-^{-1}\Theta_+|\lambda_0\rangle=\langle\lambda_0|\mathrm{e}^{\theta_0}|\lambda_0\rangle, \tag{4.19a}$$

$$\tau_{11}(\boldsymbol{v},0)\equiv\tau_1=\langle\lambda_1|\Theta_-^{-1}\Theta_+|\lambda_1\rangle=\langle\lambda_1|\mathrm{e}^{\theta_0}|\lambda_1\rangle. \tag{4.19b}$$

In turn, since the matrix element must have zero grade in general, and it must project on grade $\frac{1}{2}$ in order to extract some information of the fermionic $\bar{\psi}$ field, we will propose the following projections,

$$\begin{aligned}
\langle\lambda_i|\mathcal{G}_{\frac{1}{2}}\Theta_-^{-1}\Theta_+|\lambda_i\rangle&=\langle\lambda_i|\mathcal{G}_{\frac{1}{2}}\left(1-\theta_{-\frac{1}{2}}+O^2(\theta_{-\frac{1}{2}})\right)\mathrm{e}^{\theta_0}|\lambda_i\rangle\\
&=-\langle\lambda_i|\mathcal{G}_{\frac{1}{2}}\theta_{-\frac{1}{2}}\mathrm{e}^{\theta_0}|\lambda_i\rangle,
\end{aligned} \tag{4.20}$$

and define

$$\tau_{00}\left(\boldsymbol{v},\frac{1}{2}\right)\equiv\tau_2=\langle\lambda_0|\mathcal{G}_{\frac{1}{2}}\Theta_-^{-1}\Theta_+|\lambda_0\rangle=-\langle\lambda_0|\mathcal{G}_{\frac{1}{2}}\theta_{-\frac{1}{2}}\mathrm{e}^{\theta_0}|\lambda_0\rangle\quad\text{and} \tag{4.21a}$$

$$\tau_{11}\left(\boldsymbol{v},\frac{1}{2}\right)\equiv\tau_3=\langle\lambda_1|\mathcal{G}_{\frac{1}{2}}\Theta_-^{-1}\Theta_+|\lambda_1\rangle=-\langle\lambda_1|\mathcal{G}_{\frac{1}{2}}\theta_{-\frac{1}{2}}\mathrm{e}^{\theta_0}|\lambda_1\rangle. \tag{4.21b}$$

Now, we use equations (4.2), (4.3) for Θ_+, and (4.18), in order to establish the relation between θ_0, $\theta_{-\frac{1}{2}}$, and the fields of the model. Then, from the grade zero term, we have

$$(\partial_x\phi)M_1^{(0)}+(\partial_x\nu)\hat{\kappa}=-\partial_x\theta_0+\mathrm{e}^{\theta_0}\left(\upsilon_0M_1^{(0)}\right)\mathrm{e}^{-\theta_0}, \tag{4.22}$$

with solution given by

$$\mathrm{e}^{-\theta_0}=\mathrm{e}^{(\phi-\upsilon_0x)M_1^{(0)}+\nu\hat{\kappa}}, \tag{4.23}$$

and then

[4] We have added a central extension contribution $\nu_x\kappa$ in order to complete the model, but the ν field has no physical relevance.

$$\tau_0 = \langle \lambda_0 | \Theta_-^{-1} \Theta_+ | \lambda_0 \rangle = e^{-\nu}, \tag{4.24}$$

$$\tau_1 = \langle \lambda_1 | \Theta_-^{-1} \Theta_+ | \lambda_1 \rangle = e^{-\nu-\phi+v_0 x}, \tag{4.25}$$

from where we find the bosonic field written as

$$\phi(x,t) = v_0 x + \ln\left(\frac{\tau_0(x,t)}{\tau_1(x,t)}\right). \tag{4.26}$$

Of course, we could continue the calculation of the following higher grades terms in order to completely determine Θ_+. However, since we are mainly interested in determining the field components, we now apply the same procedure using equations (4.2), (4.3), (4.17) and (4.18) with Θ_-. The simplest projection is given on the grade $\frac{1}{2}$ term,

$$A_{\frac{1}{2}}(\bar\psi) = [\theta_{-\frac{1}{2}}, \mathcal{E}^{(1)}] + A_{\frac{1}{2},\mathrm{vac}}(\bar\psi_0), \tag{4.27}$$

from which we get

$$\theta_{-\frac{1}{2}} = f_{1,-\frac{1}{2}} F_2^{(-\frac{1}{2})} + \frac{1}{2}(\bar\psi - \bar\psi_0) G_1^{(-\frac{1}{2})}. \tag{4.28}$$

The other projections are slightly more complicated, given by

$$A_0 = [\theta_{-\frac{1}{2}}, A_{\frac{1}{2},\mathrm{vac}}] + \frac{1}{2}[\theta_{-\frac{1}{2}}, [\theta_{-\frac{1}{2}}, \mathcal{E}^{(1)}]] + [\theta_{-1}, \mathcal{E}^{(1)}] + A_{0,\mathrm{vac}}, \tag{4.29}$$

$$\begin{aligned}\partial_x \theta_{-\frac{1}{2}} &= [\theta_{-\frac{1}{2}}, A_{0,\mathrm{vac}}] + [\theta_{-\frac{3}{2}}, \mathcal{E}^{(1)}] + [\theta_{-\frac{1}{2}}, [\theta_{-1}, \mathcal{E}^{(1)}]] \\ &+ \frac{1}{2}[\theta_{-\frac{1}{2}}, [\theta_{-\frac{1}{2}}, A_{\frac{1}{2},\mathrm{vac}}]] + \frac{1}{3!}[\theta_{-\frac{1}{2}}, [\theta_{-\frac{1}{2}}, [\theta_{-\frac{1}{2}}, \mathcal{E}^{(1)}]]] + [\theta_{-1}, A_{\frac{1}{2},\mathrm{vac}}],\end{aligned} \tag{4.30}$$

$$\begin{aligned}\partial_x \theta_{-1} &= [\theta_{-2}, \mathcal{E}^{(1)}] + [\theta_{-1}, A_{0,\mathrm{vac}}] + [\theta_{-\frac{3}{2}}, A_{\frac{1}{2},\mathrm{vac}}] + \frac{1}{2}[\partial_x \theta_{-\frac{1}{2}}, \theta_{-\frac{1}{2}}] + \frac{1}{2}[\theta_{-1}, [\theta_{-1}, \mathcal{E}^{(1)}]] \\ &+ [\theta_{-\frac{1}{2}}, [\theta_{-\frac{3}{2}}, \mathcal{E}^{(1)}]] + \frac{1}{2}[\theta_{-\frac{1}{2}}, [\theta_{-\frac{1}{2}}, A_{0,\mathrm{vac}}]] + [\theta_{-\frac{1}{2}}, [\theta_{-1}, A_{\frac{1}{2},\mathrm{vac}}]] \\ &+ \frac{1}{2}[\theta_{-\frac{1}{2}}, [\theta_{-\frac{1}{2}}, [\theta_{-1}, \mathcal{E}^{(1)}]]] + \frac{1}{3!}[\theta_{-\frac{1}{2}}, [\theta_{-\frac{1}{2}}, [\theta_{-\frac{1}{2}}, A_{\frac{1}{2},\mathrm{vac}}]]] \\ &+ \frac{1}{4!}[\theta_{-\frac{1}{2}}, [\theta_{-\frac{1}{2}}, [\theta_{-\frac{1}{2}}, [\theta_{-\frac{1}{2}}, \mathcal{E}^{(1)}]]]].\end{aligned} \tag{4.31}$$

Thus, by proposing a general form for the other θ components appearing in the equations, namely

$$\begin{aligned}\theta_{-1} &= f_{1,-1} K_1^{(-1)} + f_{2,-1} K_2^{(-1)} + f_{3,-1} M_2^{(-1)}, \\ \theta_{-\frac{3}{2}} &= f_{1,-\frac{3}{2}} F_1^{(-\frac{3}{2})} + f_{2,-\frac{3}{2}} G_2^{(-\frac{3}{2})}, \\ \theta_{-2} &= f_{1,-2} h_1 M_2^{(-2)},\end{aligned} \tag{4.32}$$

we find the following set of equations

$$\partial_x \phi = f_{1,-\frac{1}{2}} \bar\psi + 2 f_{1,-\frac{1}{2}} \bar\psi_0 + 2 f_{3,-1} + v_0, \tag{4.33a}$$

$$\partial_x \nu = -f_{1,-1} + f_{2,-1} - f_{3,-1} - \frac{1}{2} f_{1,-\frac{1}{2}} \bar\psi - f_{1,-\frac{1}{2}} \bar\psi_0 + \frac{1}{2}\bar\psi \bar\psi_0, \tag{4.33b}$$

$$\partial_x f_{1,-\frac{1}{2}} = \bar\psi f_{3,-1} + \frac{v_0}{2}(\bar\psi - \bar\psi_0) - \frac{1}{2} f_{1,-\frac{1}{2}} \bar\psi_0 \bar\psi, \tag{4.33c}$$

$$\partial_x \bar\psi = 2 f_{1,-\frac{1}{2}} v_0 + 4 f_{2,-\frac{3}{2}} + 4 f_{1,-\frac{1}{2}} f_{3,-1} - 2(f_{1,-1} + f_{2,-1})\bar\psi_0. \tag{4.33d}$$

Since we are mainly interested in determining $\theta_{-\frac{1}{2}}$, let us consider equations (4.33a) and (4.33c), to find the following differential equation (where we have renamed $f_{1,-\frac{1}{2}} \equiv \frac{\Xi}{2}$ for simplicity)

$$(\partial_x \phi)\bar\psi = \partial_x \Xi - \bar\psi_0 v_0 + \frac{3\Xi}{2}\bar\psi_0 \bar\psi, \tag{4.34}$$

which completely determines Ξ in terms of the $(\phi, \bar\psi)$ fields for a given vacuum configuration (see Appendix E). Once we have determined $\theta_{-\frac{1}{2}}$, we are able to write the functional form of $\bar\psi$ in terms of τ functions. From equations (4.21a) and (4.21b), we have

$$\tau_2 = -\langle \lambda_0 | \mathcal{G}_{\frac{1}{2}} \theta_{-\frac{1}{2}} | \lambda_0 \rangle e^{-\nu} = -\langle \lambda_0 | \mathcal{G}_{\frac{1}{2}} \theta_{-\frac{1}{2}} | \lambda_0 \rangle \tau_0, \tag{4.35a}$$

$$\tau_3 = -\langle \lambda_1 | \mathcal{G}_{\frac{1}{2}} \theta_{-\frac{1}{2}} | \lambda_1 \rangle e^{-\nu-\phi+v_0 x} = -\langle \lambda_1 | \mathcal{G}_{\frac{1}{2}} \theta_{-\frac{1}{2}} | \lambda_1 \rangle \tau_1, \tag{4.35b}$$

and by choosing $\mathcal{G}_{\frac{1}{2}} = F_1^{(\frac{1}{2})}$, from equation (4.28), we find that the τ functions are given by

$$\tau_2 = \frac{1}{2}\left(\Xi - (\bar{\psi} - \bar{\psi}_0)\right)\tau_0, \tag{4.36}$$

$$\tau_3 = \frac{1}{2}\left(\Xi + (\bar{\psi} - \bar{\psi}_0)\right)\tau_1, \tag{4.37}$$

and the fermionic field is given in the following form

$$\bar{\psi} = \bar{\psi}_0 + \frac{\tau_3}{\tau_1} - \frac{\tau_2}{\tau_0}. \tag{4.38}$$

Having established the relationship between the fields of the model and the τ functions, we can now compute the τ functions using equation (4.15). The next natural step is to determine the vertex operators as described in equation (4.10). However, the vertex operator is highly dependent on the vacuum configuration accepted by the model. Therefore, we will first focus on understanding the possible vacuum states for each flow.

5. Heisenberg subalgebras and the commutativity of flows

Let us now discuss the different vacuum configurations and its implications in the commutativity of the flows. The equations of motion derived so far allow for a direct verification of the following conditions for the vacuum:

- For *positive smKdV* flows, any combination of a constant vacuum $(0,0)$, $(v_0,0)$, $(0,\bar{\psi}_0)$, and $(v_0,\bar{\psi}_0)$ are allowed;
- For *negative even smKdV* flows only two combinations possibilities $(v_0,0)$ and $(v_0,\bar{\psi}_0)$ are allowed;
- For *negative odd smKdV* flows, only the zero vacuum for bosonic and fermionic field is possible, i.e. $(v_0,\bar{\psi}_0)=(0,0)$;
- For *sKdV flows*, any combination of a constant vacuum is possible, $(0,0)$, $(v_0,0)$, $(0,\bar{\psi}_0)$, $(v_0,\bar{\psi}_0)$.

We would like to provide a general algebraic argument for such conditions. In order to do that, consider the zero curvature in the vacuum configuration,

$$\left[A_{x,\,\text{vac}}, A_{t_N,\,\text{vac}}\right] = 0, \tag{5.1}$$

for both smKdV and sKdV hierarchies. Firstly, it is important to remark that the higher graded component (positive flows) and the lowest graded component (negative flows) A_{t_N} for the smKdV case, is always non vanishing (including the vacuum configuration). Indeed, one can use (B.2a) and (B.6a) to show that for each flow the higher graded element is always determined and non-zero, namely

- Positive odd $N=2m+1$:
$$D^{(2m+1)} = \mathcal{E}^{(2m+1)} \in \mathcal{K}_{\mathcal{E}}; \tag{5.2}$$
- Negative even $N=-2m$:
$$D^{(-2m)} = M_1^{(-2m)}; \tag{5.3}$$
- Negative odd $N=-2m+1$:
$$D^{(-2m+1)} = \cosh\phi K_1^{(-2m+1)} + K_2^{(-2m+1)}. \tag{5.4}$$

Starting from the simplest case, i.e., a positive flow projected in a general constant vacuum $\left(v_0,\bar{\psi}_0\right)$ configuration, we find the following relation from equation (5.1),

$$\left[A^{\text{smKdV}}_{x,\,\text{vac}}, A^{\text{smKdV}}_{t_N,\,\text{vac}}\right] = \left[\mathcal{E}^{(1)} + v_0 M_1^{(0)} + \bar{\psi}_0 G_2^{\frac{1}{2}}, D^{(N)}_{N,\text{vac}} + D^{(N-\frac{1}{2})}_{N,\text{vac}} + \cdots + D^{(0)}_{N,\text{vac}}\right] = 0. \tag{5.5}$$

Therefore, we must find the kernel $\mathcal{K}_\Omega$ of the operator $\Omega \equiv \mathcal{E}^{(1)} + v_0 M_1^{(0)} + \bar{\psi}_0 G_2^{\frac{1}{2}}$, and analyze whether it coincides with the vacuum projection for the positive flows. Such kernel is given by:

$$\mathcal{K}_\Omega = \left\{\Omega_{2m+1}, \Gamma_{2m+1}\right\} \quad \text{such that} \quad \left[X, \mathcal{E}^{(1)} + v_0 M_1^{(0)} + \bar{\psi}_0 G_2^{\frac{1}{2}}\right] = 0, \quad X \in \mathcal{K}_\Omega, \tag{5.6}$$

with

$$\Omega_{2m+1} = v_0 M_1^{(2m)} + \bar{\psi}_0 G_2^{(2m+\frac{1}{2})} + K_1^{(2m+1)} + K_2^{(2m+1)}, \tag{5.7}$$

$$\Gamma_{2m+1} = K_2^{(2m+1)} + \frac{\bar{\psi}_0}{v_0} F_2^{(2m+\frac{3}{2})}. \tag{5.8}$$

It is straightforward to show that the vacuum projection of the smKdV Lax operators given in the Appendix C are linear combinations of the $\mathcal{K}_\Omega$ elements, namely

$$A^{\text{smKdV}}_{t_1,\text{vac}} = A^{\text{vac}}_x = \Omega_1, \tag{5.9a}$$

$$A^{\text{smKdV}}_{t_3,\text{vac}} = \Omega_3 + \frac{v_0^2}{2}\left(\Gamma_1 - \Omega_1\right), \tag{5.9b}$$

$$A^{\text{smKdV}}_{t_5,\text{vac}} = \Omega_5 + \frac{v_0^2}{2}\left(\Gamma_3 - \Omega_3\right) + \frac{3v_0^4}{8}\left(\Omega_1 - \Gamma_1\right), \tag{5.9c}$$

$$\vdots$$

$$A^{\text{smKdV}}_{t_{2m+1},\text{vac}} = \Omega_{2m+1} + \sum_{i=0}^{m-1} v_0^{2(m-i)}\left(a_i\Omega_{2i+1} + b_i\Gamma_{2i+1}\right), \tag{5.9d}$$

where a_i, b_i are constant coefficients. The relation (5.5) is valid for any combination of constant vacuum $(0,0)$, $(v_0,0)$, $(0,\bar{\psi}_0)$, and $(v_0,\bar{\psi}_0)$, as stated before. Notice that for the special case of the zero vacuum state $(0,0)$, Γ_{2m+1} vanishes and hence

$$A^{\text{smKdV}}_{t_{2n+1},\text{vac}} = \Omega_{2n+1}\Big|_{(v_0,\bar{\psi}_0)=(0,0)} = \mathcal{E}^{(2n+1)}. \tag{5.10}$$

For negative smKdV flows, the situation is a bit more intricated. If $v_0 \neq 0$, from the lowest grade equation

$$\left[v_0 M_1^{(0)}, D^{(-N)}_{-N,\text{vac}}\right] = 0, \tag{5.11}$$

and we conclude that the only possible value for N is $-2m$, since no odd element commutes with M_1. Then, considering the case where $N = -2m$, we get from equations (5.1) and (5.3), the following relation

$$\left[A^{\text{smKdV}}_{x,\,\text{vac}}, A^{\text{smKdV}}_{t_{-N},\,\text{vac}}\right] = \left[\mathcal{E}^{(1)} + v_0 M_1^{(0)} + \bar{\psi}_0 G_2^{\frac{1}{2}}, M_1^{(-2m)} + D^{(-2m+\frac{1}{2})}_{-2m,\text{vac}} + \cdots + D^{(-1)}_{-2m,\text{vac}}\right] = 0, \tag{5.12}$$

from which we can easily obtain the elements of $\mathcal{K}_\Omega$, as long as we introduce a factor of v_0^{-1}. For instance, in the $N=-2$ case, we have

$$A^{\text{smKdV}}_{t_{-2},\text{vac}} = \frac{1}{v_0}\left(\Omega_{-1} - \Gamma_{-1}\right) + \Gamma_{-1}, \tag{5.13}$$

and for a general vacuum operator, we can write

$$A^{\text{smKdV}}_{t_{-2m},\text{vac}} = \sum_{i=1}^{m} v_0^{-2(m-i)-1}\left(c_i\Omega_{-2i+1} + d_i\Gamma_{-2i+1}\right), \tag{5.14}$$

where c_i, d_i are constant coefficients. Notice that in this case we can not take the limit $v_0 \to 0$, due the pole introduced. Therefore follows that the only vacuum configurations possible are $(v_0,0)$ and $(v_0,\bar{\psi}_0)$.

On the other hand, if $v_0 \equiv 0$, the lowest grade equation is now given by

$$\left[\bar{\psi}_0 G_2^{\frac{1}{2}}, D^{(-N)}_{-N,\text{vac}}\right] = 0, \tag{5.15}$$

which is clearly not satisfied by any even flow ($N=-2m$). However, for each odd flow, we get the following restriction on $D^{(-N)}_{-N,\text{vac}}$

$$D^{(-N)}_{-N,\text{vac}} = a^{-2m+1}_{N,\text{vac}}\left(K_1^{(-2m+1)} - K_2^{(-2m+1)}\right). \tag{5.16}$$

A quick check on equation (5.4) with $v_0 = \phi_0 = 0$ shows that the super sinh-Gordon and other lower flows within the hierarchy do not obey this restriction[5]

$$D^{(-2m+1)}_{\text{vac}} = \left(K_1^{(-2m+1)} + K_2^{(-2m+1)}\right) = \mathcal{E}^{(-2m+1)}, \tag{5.18}$$

and hence $\bar{\psi}_0 = 0$. Therefore, for super sinh-Gordon and other negative odd flows, both the bosonic and fermionic vacuum vanish. In fact, the lowest equation in such case is

[5] By solving the ZCC for odd negative flows of the $N=-1$ case, we can perform some modifications of the constant parameters in such a way that we obtain a distinct temporal Lax operator, associated with a *modified super sinh-Gordon* that allows us to set both $(0,0)$ or $(0,\bar{\psi}_0)$ as a vacuum configuration (restriction (5.16) is completely satisfied), i.e.,

$$\partial_x\partial_{t_{-1}}\phi = 2\sinh 2\phi - 2\bar{\psi}\psi\cosh\phi, \quad \partial_{t_{-1}}\bar{\psi} = 2\psi\sinh\phi, \quad \partial_x\psi = 2\bar{\psi}\sinh\phi. \tag{5.17}$$

$$\left[\mathcal{E}^{(1)}, D^{(-N)}_{-N,\mathrm{vac}}\right] = 0, \tag{5.19}$$

which is automatically satisfied for negative odd flows. In general, we have

$$A^{\mathrm{smKdV}}_{t_{-2n+1},\mathrm{vac}} = \mathcal{E}^{(-2n+1)}, \tag{5.20}$$

and the vacuum structure is such that we have a zero vacuum configuration for both bosonic and fermionic fields.

Finally, the same analysis can be applied for both positive and negative sKdV flows. The conclusion is that all the vacuum combinations are possible (see more details in Appendix F).

Now that we have determined the vacuum structure, it is possible to show which flows are in involution with each other.[6] First, we recall that for a given integrable model described by a general Lax operator

$$\mathcal{L}_N = \partial_{t_N} + A_{t_N}, \tag{5.21}$$

a sufficient condition to prove the commutation of two different flows, say N and M, is to show that their corresponding operators commute with each other, i.e.

$$\left[\mathcal{L}_N, \mathcal{L}_M\right] = 0. \tag{5.22}$$

As a direct consequence of the dressing operator (4.2), this expression can be rewritten as,

$$\left[\mathcal{L}_N, \mathcal{L}_M\right] = \Theta \left[\mathcal{L}_{N,\mathrm{vac}}, \mathcal{L}_{M,\mathrm{vac}}\right] \Theta^{-1} = 0, \tag{5.23}$$

showing that commutation of the flows is equivalent to prove that the corresponding vacuum operators commute with each other. For smKdV, the vacuum operators are defined in terms of three objects $\mathcal{E}^{2m+1}$, Ω_{2m+1} and Γ_{2m+1}. As one can easily verify, two distinct abelian subalgebras can be formed with these objects. The first one is composed of

$$\left[\mathcal{E}^{2m+1}, \mathcal{E}^{2n+1}\right] = 0, \tag{5.24}$$

and the second is

$$\left[\Omega_{2m+1}, \Omega_{2n+1}\right] = 0, \quad \left[\Gamma_{2m+1}, \Omega_{2n+1}\right] = 0, \quad \left[\Gamma_{2m+1}, \Gamma_{2n+1}\right] = 0. \tag{5.25}$$

When a central extension is considered, they correspond to the so-called *Heisenberg subalgebras*. This leads us to conclude that two different hierarchies (in the sense of flows in involution) exist within smKdV. The first one is formed for flows where $\mathcal{L}_{N,\mathrm{vac}} = \mathcal{E}^{(N)}$ as vacuum operator, namely, positive odd (5.10) and negative odd (5.20) flows. Hence, using (5.23) and (5.24), we find that

$$\left[\mathcal{L}^{\mathrm{smKdV}}_{2m+1}, \mathcal{L}^{\mathrm{smKdV}}_{2n+1}\right] = \Theta_I \left[\mathcal{E}^{2m+1}, \mathcal{E}^{2n+1}\right] \Theta_I^{-1} = 0, \tag{5.26}$$

with $n, m \in \mathbb{Z}$. Similarly, we obtain a second hierarchy of vacuum operators formed by the linear combination, $\mathcal{L}_{N,\mathrm{vac}} = \sum_i^N \alpha_i \Omega_{2i+1} + \beta_i \Gamma_{2i+1}$, namely, positive odd (5.9d) and negative even (5.14). Then, equation (5.25) leads to

$$\left[\mathcal{L}^{\mathrm{smKdV}}_{-2m}, \mathcal{L}^{\mathrm{smKdV}}_{2n+1}\right] = \Theta_{II} \left[\sum_i^m \alpha_i \Omega_{-2i+1} + \beta_i \Gamma_{-2i+1}, \sum_j^n \gamma_j \Omega_{2j+1} + \omega_j \Gamma_{2j+1}\right] \Theta_{II}^{-1} = 0, \tag{5.27}$$

with $n, m \in \mathbb{Z}$ and $\alpha_i, \beta_i, \gamma_i, \omega_i$ coefficients.[7] Therefore, we find that are two distinct sets of hierarchies within smKdV:

- Positive and negative odd flows within smKdV commute with each other and form a hierarchy with zero vacuum orbit, *smKdV - I*;
- Positive odd and negative even flows within smKdV commute with each other and form a hierarchy that posses non-zero vacuum orbit, *smKdV - II*.

For the sKdV the procedure is quite similar (see Appendix F). In this case, the Lax operators are constructed based on $\mathcal{E}^{2m+1}$, Λ_{2m+1} and Π_{2m+1} operators (F.3), which also form two different abelian subalgebras:

$$\left[\mathcal{E}^{2m+1}, \mathcal{E}^{2n+1}\right] = 0, \tag{5.28}$$

and

$$\left[\Lambda_{2m+1}, \Lambda_{2n+1}\right] = 0, \quad \left[\Lambda_{2m+1}, \Pi_{2n+1}\right] = 0, \quad \left[\Pi_{2m+1}, \Pi_{2n+1}\right] = 0. \tag{5.29}$$

From them, we can conclude that sKdV also has two distinct sets of hierarchies:

[6] The argument for commuting flows follows the same line of reasoning used in the bosonic case [49,25,5].

[7] For simplicity, we are considering any combination with a non-zero bosonic vacuum orbit $(v_0, 0)$ or $(v_0, \bar{\psi}_0)$, as a non-zero vacuum. This works very well since the commutation relations hold for any of them. The same is valid for sKdV.

- Positive odd and negative odd flows within sKdV commute with each other and form a hierarchy with zero vacuum orbit, *sKdV - I*.
- Positive odd and negative even flows within sKdV commute with each other and form a hierarchy with non-zero vacuum orbit, *sKdV - II*.

At this point, several important remarks can be made:

1. For smKdV, the hierarchy splits in two different classes, both constructed using the same A_x but with different vacuum orbits. Both type I and II share the same positive flows in this case, but distinct negative flows.
2. The smKdV - I and smKdV - II hierarchies are constructed using different dressing operators Θ_I and Θ_{II}, since the operators θ_i depend explicitly on the vacuum chosen. This will lead to different vertex operators and soliton solutions.
3. Although sKdV - I and sKdV - II are constructed using different vacua and dressing operators, they share the same positive and negative flows.
4. The super Miura transformation maps type I/II smKdV hierarchy into type I/II sKdV hierarchy. Although the positive flows coincide in both types for smKdV, the negative ones do not. In this way, it may appear that two different flows lead to the same sKdV equation. However, each flow actually arises from a specific vacuum orbit, either

$$t_N^{\text{smKdV}-I} \xrightarrow{\mathcal{S}} t_N^{\text{sKdV}-I}, \tag{5.30}$$

where $N = 2m+1$ for $m \in \mathbb{Z}$, or

$$t_M^{\text{smKdV}-II} \xrightarrow{\mathcal{S}} t_N^{\text{sKdV}-II}, \tag{5.31}$$

for $M = 2m+1$ or $M = -2m$ with $m \in \mathbb{Z}^+$ and $N = 2l+1$ for $l \in \mathbb{Z}$.

Having concluded our analysis of the vacuum structure and commutativity of flows, in the next section we will obtain the vertex operators, and construct the one-soliton solutions for different vacuum configurations.

6. Vertex operator for smKdV and soliton solutions

We recall that the final step to obtain the soliton solution involves determining vertex operators that depend upon the vacuum structure. The general vacuum configuration, is written in terms of the operators Ω_{2m+1} and Γ_{2m+1} given by (5.7) and (5.8), i.e.,

$$\left[A_{\mu,\text{vac}}^{\text{smKdV}}, V(k_i)\right] = \omega_\mu(k_i)V(k_i). \tag{6.1}$$

Let us consider a generic vertex operator as follows

$$\begin{aligned} V(k_1) = \sum_{j=-\infty}^{\infty} k_1^{-2j} \Big[& a_0\delta_{j,0}\,\hat{\kappa} + a_1 M_1^{(2j)} + a_2 K_1^{(2j+1)} + a_3 K_2^{(2j+1)} + a_4 M_1^{(2j+1)} \\ & + a_5 F_1^{(2j+\frac{1}{2})} + a_6 G_2^{(2j+\frac{1}{2})} + a_7 F_2^{(2j+\frac{3}{2})} + a_8 G_1^{(2j+\frac{3}{2})} \Big], \end{aligned} \tag{6.2}$$

from which we can compute the following matrix elements for a one-soliton solution

$$\tau_0 = 1 + \langle \lambda_0 | V(k_1) | \lambda_0 \rangle \rho = 1 + a_0\rho, \tag{6.3a}$$

$$\tau_1 = 1 + \langle \lambda_1 | V(k_1) | \lambda_1 \rangle \rho = 1 + \left(a_0 + a_1\right)\rho, \tag{6.3b}$$

$$\tau_2 = \langle \lambda_0 | F_1^{\left(\frac{1}{2}\right)} V(k_1) | \lambda_0 \rangle \rho = \left(a_8 - a_7\right) k_1^2 \rho, \tag{6.3c}$$

$$\tau_3 = \langle \lambda_1 | F_1^{\left(\frac{1}{2}\right)} V(k_1) | \lambda_1 \rangle \rho = -\left(a_8 + a_7\right) k_1^2 \rho, \tag{6.3d}$$

where the space-time dependence is given by $\rho = \rho(x, t_N) = e^{-\omega_x x - \omega_N t_N}$. Now, in order to simplify the computations of our solutions, let us summarize the results obtained for vertex operators V_i, $i = 1, \ldots, 5$ and their eigenvalues. We consider two categories, those containing only pure fermionic solutions and those including mixed solutions:

1. *Pure fermionic vertex*

(a) Non-zero fermionic vacuum ($\bar{\psi}_0 \neq 0$, c_1 is a generic bosonic constant)

$$V_1(\bar{k}_1) = \sum_{j=-\infty}^{\infty} \left(\frac{(2\bar{k}_1 - v_0)(2\bar{k}_1 + v_0)}{4\bar{k}_1} \right)^{-2j} c_1 \bar{\psi}_0 \left[-\frac{v_0}{2\bar{k}_1} F_1^{(2j+\frac{1}{2})} + G_2^{(2j+\frac{1}{2})} + \frac{2v_0}{4\bar{k}_1^2 - v_0^2} F_2^{(2j+\frac{3}{2})} - \frac{4\bar{k}_1}{4\bar{k}_1^2 - v_0^2} G_1^{(2j+\frac{3}{2})} \right], \tag{6.4}$$

with eigenvalues

$$\left[\Omega_{2m+1}, V_1(\bar{k}_1)\right] = 2\bar{k}_1 \left(\frac{4\bar{k}_1^2 - v_0^2}{4\bar{k}_1} \right)^{2m} V_1(\bar{k}_1), \tag{6.5a}$$

$$\left[\Gamma_{2m+1}, V_1(\bar{k}_1)\right] = \left(\frac{4\bar{k}_1^2 - v_0^2}{4\bar{k}_1} \right)^{2m+1} V_1(\bar{k}_1). \tag{6.5b}$$

(b) Zero fermionic vacuum ($\bar{\psi}_0 = 0$, ν_1 is a generic fermionic constant)

$$V_2(\bar{k}_1) = \sum_{j=-\infty}^{\infty} \left(\frac{(2\bar{k}_1 - v_0)(2\bar{k}_1 + v_0)}{4\bar{k}_1} \right)^{-2j} \nu_1 \left[-\frac{v_0}{2\bar{k}_1} F_1^{(2j+\frac{1}{2})} + G_2^{(2j+\frac{1}{2})} + \frac{2v_0}{4\bar{k}_1^2 - v_0^2} F_2^{(2j+\frac{3}{2})} - \frac{4\bar{k}_1}{4\bar{k}_1^2 - v_0^2} G_1^{(2j+\frac{3}{2})} \right], \tag{6.6}$$

with eigenvalues

$$\left[\Omega_{2m+1}, V_2(\bar{k}_1)\right] = 2\bar{k}_1 \left(\frac{4\bar{k}_1^2 - v_0^2}{4\bar{k}_1} \right)^{2m} V_2(\bar{k}_1), \tag{6.7a}$$

$$\left[\Gamma_{2m+1}, V_2(\bar{k}_1)\right] = \left(\frac{4\bar{k}_1^2 - v_0^2}{4\bar{k}_1} \right)^{2m+1} V_2(\bar{k}_1). \tag{6.7b}$$

In both cases, there are no problems taking the limit $v_0 \to 0$ to obtain other vacuum configurations, such as $(0, \bar{\psi}_0)$.

2. *Bosonic-fermionic vertex*

(a) Non-zero bosonic vacuum ($v_0 \neq 0$):

$$V_3(\bar{k}_1) = \sum_{j=-\infty}^{\infty} \left(\bar{k}_1^2 - v_0^2\right)^{-j} \left[M_1^{(2j)} - \frac{\left(v_0 + \bar{k}_1\right)}{2\bar{k}_1} \hat{\kappa} \delta_{j,0} + \frac{v_0}{v_0^2 - \bar{k}_1^2} K_1^{(2j+1)} + \frac{\bar{k}_1}{v_0^2 - \bar{k}_1^2} M_2^{(2j+1)} + v_0^{-1} \bar{\psi}_0 G_2^{(2j+\frac{1}{2})} - \frac{\bar{\psi}_0}{v_0^2 - \bar{k}_1^2} F_2^{(2j+\frac{3}{2})} + \frac{\bar{\psi}_0 \bar{k}_1 v_0^{-1}}{v_0^2 - \bar{k}_1^2} G_1^{(2j+\frac{3}{2})} \right], \tag{6.8}$$

with eigenvalues

$$\left[\Omega_{2m+1}, V_3(\bar{k}_1)\right] = 2\bar{k}_1 \left(\bar{k}_1^2 - v_0^2\right)^m V_3(\bar{k}_1), \tag{6.9a}$$

$$\left[\Gamma_{2m+1}, V_3(\bar{k}_1)\right] = 0. \tag{6.9b}$$

By taking the limit of $\bar{\psi}_0 \to 0$ one obtains the vacuum configurations $(v_0, 0)$.

(b) Zero bosonic and non-zero fermionic vacuum ($v_0 = 0$, $\bar{\psi}_0 \neq 0$):

$$V_4(k_1) = \sum_{j=-\infty}^{\infty} k_1^{-2j} \left\{ \left[M_1^{(2j)} - \frac{1}{2} \hat{\kappa} \delta_{j,0} - \frac{1}{k_1} M_2^{(2j+1)} + \frac{\bar{\psi}_0}{2k_1} F_1^{(2j+\frac{1}{2})} + \frac{\bar{\psi}_0}{2k_1^2} F_2^{(2j+\frac{3}{2})} \right] + c_1 \left[\frac{\psi_0}{2k_1} G_2^{(2j+\frac{1}{2})} - \frac{\psi_0}{2k_1^2} G_1^{(2j+\frac{3}{2})} \right] \right\}, \tag{6.10}$$

with eigenvalues

$$\left[\Omega_{2m+1}, V_4(k_1)\right] = 2k_1^{2m+1} V_4(k_1), \tag{6.11a}$$

$$\left[\Gamma_{2m+1}, V_4(k_1)\right] = 0. \tag{6.11b}$$

(c) Zero bosonic and zero fermionic vacuum ($v_0 = 0$, $\bar{\psi}_0 = 0$):

$$V_5(k_1) = \sum_{j=-\infty}^{\infty} k_1^{-2j} \left\{ \left[M_1^{(2j)} - \frac{1}{2}\hat{\kappa}\delta_{j,0} - \frac{1}{k_1} M_2^{(2j+1)} + \frac{\bar{\psi}_0}{2k_1} F_1^{(2j+\frac{1}{2})} \right] + \nu_1 \left[G_2^{(2j+\frac{1}{2})} - \frac{1}{k_1} G_1^{(2j+\frac{3}{2})} \right] \right\}, \tag{6.12}$$

with eigenvalues

$$\left[\Omega_{2m+1}, V_5(k_1)\right] = 2k_1^{2m+1} V_5(k_1), \tag{6.13a}$$

$$\left[\Gamma_{2m+1}, V_5(k_1)\right] = 0. \tag{6.13b}$$

Once we have constructed the vertex operators and its eigenvalues, we can finally obtain one-soliton solutions for the different hierarchies related to smKdV. Here, we will use c_1 to denote a bosonic constant, and ν_1 to denote a fermionic constant. It is also worth mentioning that although we are mainly focusing in one-soliton solutions,[8] we can use the vertex operators to obtain in general multi-soliton solutions for the models [42].

6.1. The smKdV-I (zero vacuum) solutions

For both positive and odd negative flows, we must have a zero vacuum solution such $(v_0, \bar{\psi}_0) = (0, 0)$. In such case, we consider the vertices $V_1(k_1)$ and $V_5(k_1)$ leading to two different solutions:

$$v_2 = \partial_x \ln\left(\frac{\tau_0}{\tau_1}\right) = 0, \qquad \bar{\psi}_2 = \frac{\tau_3}{\tau_1} - \frac{\tau_2}{\tau_0} = 2k_1 \nu_1 \rho_N, \tag{6.14}$$

and

$$v_5 = \partial_x \ln\left(\frac{1 - \frac{1}{2}\rho_N}{1 + \frac{1}{2}\rho_N}\right), \qquad \bar{\psi}_5 = \nu_1\, \partial_x \ln\left(\frac{1 - \frac{1}{2}\rho_N}{1 + \frac{1}{2}\rho_N}\right), \tag{6.15}$$

with $\rho_N = \rho(x, t_N) = \mathrm{e}^{-2k_1 x - \omega_N t_N}$, where the subscript number on the fields corresponds to the vertex that was used. For both cases, the eigenvalues are given by

- The smKdV ($N = 3$)

$$\omega_3 = 2k_1^3; \tag{6.16}$$

- The sshG ($N = -1$)

$$\omega_{-1} = 2k_1^{-1}; \tag{6.17}$$

- General Case ($N = 2m + 1$)

$$\omega_{2n+1} = 2k_1^{2n+1}; \tag{6.18}$$

These solutions are in agreement with the ones reported previously in the literature [42].

6.2. The smKdV-II (non-zero vacuum) solutions

Now, for the flows possessing a wider range of vacuums solutions, we can obtain novel combinations. First of all, for both positive and negative even flows, we must have a non-zero bosonic vacuum solution $(v_0, \bar{\psi}_0) = (v_0, 0)$ or $(v_0, \bar{\psi}_0)$. In that case, we can consider the vertex V_1, V_2 or V_3, leading to three different solutions. For $V_1(\bar{k}_1)$ and $V_2(\bar{k}_1)$:

$$v_1 = v_0, \qquad \bar{\psi}_1 = \bar{\psi}_0 + \left(\frac{4\bar{k}_1^2 - v_0^2}{2\bar{k}_1}\right) c_1 \bar{\psi}_0 \rho_N, \tag{6.19}$$

$$v_2 = v_0, \qquad \bar{\psi}_2 = \left(\frac{4\bar{k}_1^2 - v_0^2}{2\bar{k}_1}\right) \nu_1 \rho_N, \tag{6.20}$$

where $\rho_N = \rho(x, t_N) = \mathrm{e}^{-2k_1 x - \omega_N t_N}$, with eigenvalues given by[9]

[8] All the solutions we obtained were double checked using the *Hirota Method* implemented in *Mathematica*.

[9] A general formula for the eigenvalues can be obtained by using (5.14), (5.9d) and (6.5).

$$\omega_3 = 2\bar{k}_1^3 - \frac{3\bar{k}_1 v_0^2}{2}, \qquad \text{for} \qquad N=3, \tag{6.21}$$

$$\omega_{-2} = \frac{4\bar{k}_1 v_0^2\left(v_0-1\right) + 16\bar{k}_1^3\left(v_0+1\right)}{v_0\left(4\bar{k}_1^2 - v_0^2\right)^2}, \qquad \text{for} \qquad N=-2, \tag{6.22}$$

Regarding $V_3(\bar{k}_1)$, we have a more interesting solution that includes mixed terms

$$v_3 = v_0 + \partial_x \ln\left(\frac{1 - \frac{(\bar{k}_1+v_0)}{2\bar{k}_1}\rho_N}{1 + \frac{(\bar{k}_1-v_0)}{2\bar{k}_1}\rho_N}\right), \qquad \bar{\psi}_3 = \bar{\psi}_0 + \frac{\bar{\psi}_0}{v_0}\,\partial_x \ln\left(\frac{1 - \frac{(\bar{k}_1+v_0)}{2\bar{k}_1}\rho_N}{1 + \frac{(\bar{k}_1-v_0)}{2\bar{k}_1}\rho_N}\right), \tag{6.23}$$

with the eigenvalues

$$\omega_3 = 2\bar{k}_1^3 - 3v_0^2\bar{k}_1, \qquad \text{for} \qquad N=3, \tag{6.24}$$

$$\omega_{-2} = \frac{2\bar{k}_1}{v_0\left(\bar{k}_1^2 - v_0^2\right)}, \qquad \text{for} \qquad N=-2, \tag{6.25}$$

which can also be generalized using (6.9). In addition, we can construct solutions for positive times with a different mixed vacuum $(v_0, \bar{\psi}_0) = (0, \bar{\psi}_0)$, by using the vertices $V_1(k_1)$ and $V_4(k_1)$. We get

$$v_1 = 0, \qquad \bar{\psi}_1 = \bar{\psi}_0 + 2k_1\bar{\psi}_0 c_1 \rho_N, \tag{6.26}$$

and

$$v_4 = \partial_x \ln\left(\frac{1-\frac{1}{2}\rho_N}{1+\frac{1}{2}\rho_N}\right), \qquad \bar{\psi}_4 = \bar{\psi}_0 + \frac{\left(2c_1\bar{\psi}_0 + \bar{\psi}_0\rho_N\right)}{4\bar{k}_1}\partial_x \ln\left(\frac{1-\frac{1}{2}\rho_N}{1+\frac{1}{2}\rho_N}\right), \tag{6.27}$$

both with eigenvalue $\omega_3 = 2k_1^3$. In this case, we notice that v_0 does not contribute to the eigenvalue.

These results show us that introducing different parameters to the vacuum configuration leads to novel and interesting solutions that mix bosonic and fermionic parameters. The possibility of combining vacuum configurations leads to six different solutions for smKdV-II models. In order to visualize the difference in the behavior of the solutions, in Fig. 1 we have plotted the profile for the dynamical solutions $v_3, \bar{\psi}_3$, $v_4, \bar{\psi}_4$ and $v_5, \bar{\psi}_5$, considering $t=0$. For the fermionic fields $\bar{\psi}_i$, we have plotted only the bosonic part of the solution, factorizing the Grassmann constant, namely the *bosonic projection* $P[\bar{\psi}_i]$, such that $\bar{\psi}_i = \bar{\psi}_0 P[\bar{\psi}_i]$, or $\bar{\psi}_i = \nu_1 P[\bar{\psi}_i]$. We see that different kind of solutions, kinks, anti-kinks, dark-solitons and discontinuous solutions, appear by choosing the parameters accordingly.

In the next section, we will combine these solutions with the Gauge-super Miura transformation to obtain new solutions for the sKdV system.

7. The sKdV solutions via super Miura transformation

As highlighted in section 3, one can map the smKdV solutions into the sKdV solutions by using the super Miura transformation (3.6) with the positive sign:

$$J = v^2 - \partial_x v + \bar{\psi}\partial_x\bar{\psi},$$
$$\bar{\chi} = -v\bar{\psi} + \partial_x\bar{\psi}.$$

Due the fact that the smKdV equation is invariant under the parity transformation $v \to -v$ and $\bar{\psi} \to -\bar{\psi}$, it is possible to obtain three other different kinds of super Miura transformations, leading to four different possibilities. Here, we evaluate all of them to obtain the corresponding sKdV solutions.[10]

7.1. The sKdV-I (non-zero vacuum) solutions

For the sKdV-I hierarchy, we must have a zero vacuum solution such $(J_0, \bar{\chi}_0) = (0,0)$. In this case, we get six different solutions:

$$J^{\pm}_{2,-} = J^{\pm}_{2,+} = 0, \qquad \chi^{\pm}_{2,-} = \chi^{\mp}_{2,+} = \pm 4k_1^2\nu_1\rho_N, \tag{7.1}$$

and

$$J^{\pm}_{5,-} = -2\partial_x^2 \ln\left(1 - \frac{1}{2}\rho_N\right), \qquad \bar{\chi}^{\pm}_{5,-} = \pm 2\nu_1\,\partial_x^2 \ln\left(1-\frac{1}{2}\rho_N\right), \tag{7.2}$$

[10] Here, we use the same notation for subscript label of the vertex. The subscript $\pm$ sign denotes if the solution is obtained using either v or $-v$, while the upper $\pm$ sign denotes if the solution is obtained using either $\bar{\psi}$ or $-\bar{\psi}$. In addition ρ_N and ω_N are the same as defined by smKdV in the previous section.

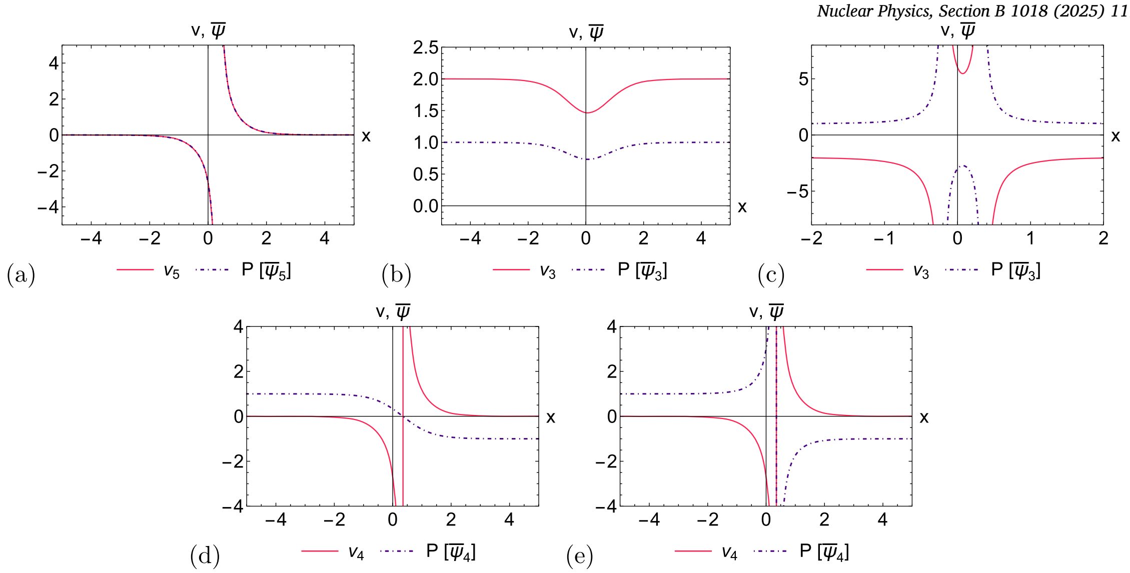

Fig. 1. The smKdV-I and smKdV-II solutions. In all cases, $k_1 = 1$ and $t = 0$. **(a)** Classical profile for the zero vacuum solution $v_5, \bar\psi_5$. In this case, both the bosonic field and the projection of the fermionic field matches. **(b)** Setting the bosonic vacuum background to be positive $v_0 = 2$ for the $v_3, \bar\psi_3$ solution, we obtain a anti-solitonic profile (*dark soliton*). In this case, the projection background for the fermionic field will depend on $\bar\psi_0$ **(c)** On the other hand, choosing a negative vacuum background $v_0 = -2$ for $v_3, \bar\psi_3$ solution leads to a discontinuous solution. Notice that due to the inversion of the vacuum, the fermionic projection will have a overall signed compared to the bosonic solution. **(d - e)** Analysis of the c_1 dependent solution $v_4, \bar\psi_4$. Both share the same bosonic profile since it does not depend on the constant but have different fermionic projection. In (d) we have chosen $c_1 = -1$, leading to anti-kink profile, while in (e) we have chosen $c_1 = 1$, leading to a mirrored standard solution.

$$J^{\pm}_{5,+} = -2\partial_x^2 \ln\left(1+\frac{1}{2}\rho_N\right), \qquad \bar\chi^{\pm}_{5,-} = \pm 2\nu_1\, \partial_x^2 \ln\left(1+\frac{1}{2}\rho_N\right), \tag{7.3}$$

with $\rho_N = \rho(x, t_N) = e^{-2k_1 x - \omega_N t_N}$.

These solutions agree with the well-known format for KdV system, which includes a second-order derivative acting on the logarithmic term [54,26]. Notice that the parity transformation for the bosonic field $v \to -v$ interchanges the τ_0 function with the τ_1 function. This feature will be also present for a non-zero vacuum. On the other hand, the parity transformation for the fermionic field just changes a global sign.

7.2. The sKdV-II (non-zero vacuum) solutions

Now, for the sKdV-II flows, we find again several solutions associated to the different vacuum configurations. For $(J_0, \bar\chi_0) = (J_0, 0)$ or $(J_0, \bar\chi_0)$, we find twelve different solutions. The first set contains eight solutions where only the fermionic field is dynamical, namely

$$J^{\pm}_{1,-} = v_0^2, \qquad \bar\chi^{\pm}_{1,-} = \mp v_0\bar\psi_0 \mp \frac{\left(4\bar k_1^2 - v_0^2\right)\left(v_0 + 2\bar k_1\right)}{2\bar k_1} c_1 \bar\psi_0 \rho_N, \tag{7.4}$$

$$J^{\pm}_{1,+} = v_0^2, \qquad \bar\chi^{\pm}_{1,+} = \mp v_0\bar\psi_0 \mp \frac{\left(4\bar k_1^2 - v_0^2\right)\left(v_0 - 2\bar k_1\right)}{2\bar k_1} c_1 \bar\psi_0 \rho_N, \tag{7.5}$$

$$J^{\pm}_{2,-} = v_0^2, \qquad \bar\chi^{\pm}_{2,-} = \mp \frac{\left(4\bar k_1^2 - v_0^2\right)\left(v_0 + 2\bar k_1\right)}{2\bar k_1} \nu_1 \rho_N, \tag{7.6}$$

$$J^{\pm}_{2,+} = v_0^2, \qquad \bar\chi^{\pm}_{2,+} = \mp \frac{\left(4\bar k_1^2 - v_0^2\right)\left(v_0 - 2\bar k_1\right)}{2\bar k_1} \nu_1 \rho_N. \tag{7.7}$$

Notice that the sKdV and smKdV vacuum are related by $J_0 = v_0^2$, no matter if we are using the positive or negative sign for the bosonic field. Then, for the sKdV systems, the background for the bosonic vacuum is always positive. On the other hand, for the fermionic vacuum is $\chi_0 = \pm v_0 \bar\psi_0$, allowing both positive and negative configurations.

Finally, the remaining solutions contain both dynamical fields, but only a positive bosonic vacuum is possible. They have the following form,

$$J^{\pm}_{3,-} = v_0^2 - 2\partial_x^2 \ln\left(1 - \frac{\left(v_0 + \bar k_1\right)}{2\bar k_1}\rho_N\right), \qquad \bar\chi^{\pm}_{3,-} = \mp v_0\bar\psi_0 \pm \frac{2\bar\psi_0}{v_0}\partial_x^2 \ln\left(1 - \frac{\left(v_0 + \bar k_1\right)}{2\bar k_1}\rho_N\right), \tag{7.8}$$

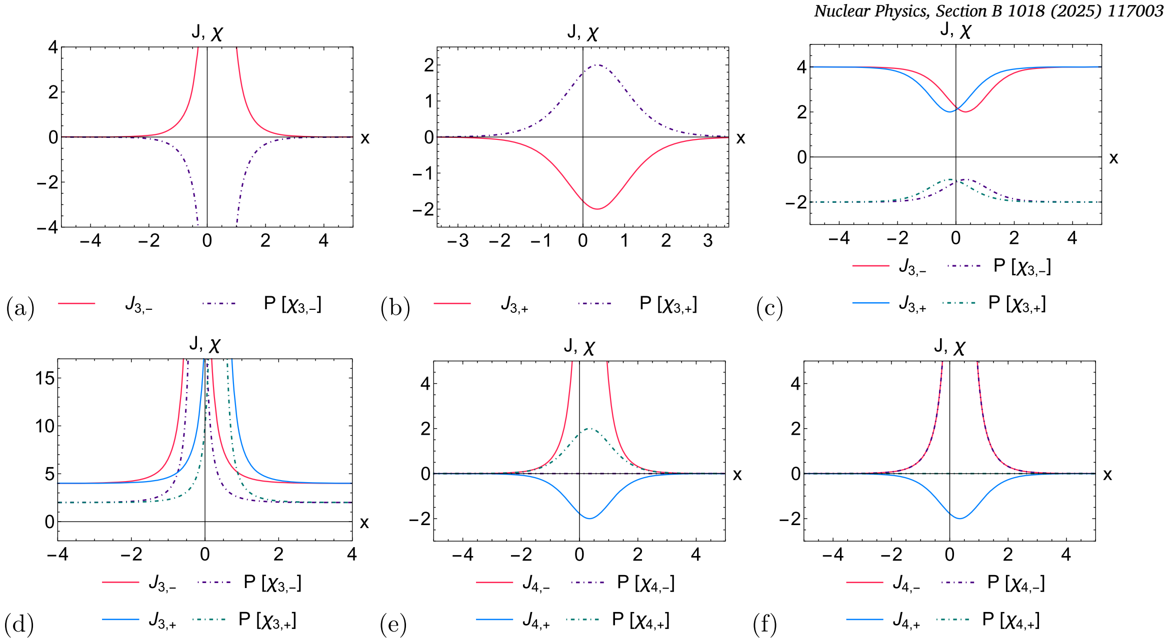

Fig. 2. Plots of the KdV-I and KdV-II solutions. We use both the positive and negative super Miura transformation for the bosonic field (equations (3.6) and (3.7)) and only the positive one for the fermionic field. **(a-b)** Using the zero vacuum solution $v_5, \bar\psi_5$, we obtain two different solutions for sKdV. We obtain a discontinuous solution by using the first Miura, and by changing the super Miura leads to soliton/dark-soliton profile. **(c)** Now using the non-zero positive vacuum $v_0 = 2$, we obtain dark-solitons for the bosonic field and solitons for the fermionic projection for sKdV-II solutions. In such case, changing the super Miura only leads to a phase shift. **(d)** Setting a negative smKdV-II vacuum $v_0 = -2$ leads to discontinuous solutions, although now the quadratic nature of the super Miura transformation leads to a positive vacuum for the sKdV-II field. **(e-f)** The c_1 dependent solution $v_4, \bar\psi_4$ is quite interesting for the sKdV case. As the bosonic field does not depend on c_1, we have in both cases a discontinuous solution for the first super Miura, and dark-soliton for the second one. For the fermionic projection, in **(e)** the choice $c_1 = -1$, leads to a null solution for the first Miura, and a dark-soliton for the second one. In turn, in **(f)** with the choice $c_1 = 1$, the first super Miura leads to a discontinuous solution, while the second one gives a solution that is always zero.

$$J^{\pm}_{3,+} = v_0^2 - 2\partial_x^2 \ln\left(1 + \frac{(\bar k_1 - v_0)}{2\bar k_1}\rho_N\right), \qquad \bar\chi^{\pm}_{3,+} = \mp v_0\bar\psi_0 \pm \frac{2\bar\psi_0}{v_0}\partial_x^2 \ln\left(1 + \frac{(\bar k_1 - v_0)}{2\bar k_1}\rho_N\right). \tag{7.9}$$

In addition, for the mixed vacuum configuration $(v_0, \bar\psi_0) = (0, \bar\psi_0)$, we have

$$J^{\pm}_{1,-} = J^{\pm}_{1,+} = 0, \qquad \bar\chi^{\pm}_{1,-} = \bar\chi^{\pm}_{1,+} = \pm 4k_1^2 c_1 \bar\psi_0 \rho_N, \tag{7.10}$$

and

$$J^{\pm}_{4,-} = -2\partial_x^2 \ln\left(1 - \frac{1}{2}\rho_N\right), \qquad \bar\chi^{\pm}_{4,-} = \pm\frac{(c_1+1)\,\bar\psi_0}{k_1}\,\partial_x^2 \ln\left(1 - \frac{1}{2}\rho_N\right), \tag{7.11}$$

$$J^{\pm}_{4,+} = -2\partial_x^2 \ln\left(1 + \frac{1}{2}\rho_N\right), \qquad \bar\chi^{\pm}_{4,-} = \pm\frac{(c_1-1)\,\bar\psi_0}{k_1}\,\partial_x^2 \ln\left(1 + \frac{1}{2}\rho_N\right). \tag{7.12}$$

An interesting feature about this solution is the possibility of having a vanishing fermionic solution for $\bar\chi_{4,-}$ if $c_1 = -1$, or for $\bar\chi_{4,+}$ if $c_1 = 1$. In Fig. 2, we have plotted the obtained solutions for skdV, for $t = 0$, where the upper index is always considered to be positive. Again, several interesting solutions appear, such as solitons, dark-solitons and discontinuous solutions.

8. Discussion and further developments

In this paper we have extended the algebraic construction of positive flows to integrate the negative grade sector for both smKdV and sKdV hierarchies. These follow the ideas developed for the pure bosonic case [49] with the inclusion of fermionic fields.

The vacuum orbit plays a crucial role in classifying the hierarchies, splitting both smKdV and sKdV into two different types: the type-I hierarchy, which allows only a zero bosonic and fermionic vacuum, and a type-II hierarchy, which only has a non-zero bosonic vacuum. For the smKdV, the type I and II share the same positive flows but have different negative flows, while for the sKdV they share the same equations of motion for both the positive and negative part.

Soliton solutions for the smKdV were constructed by applying the dressing method. This approach generates different vacuum operators that form an abelian subalgebra, which is essential for the involution of the flows: $\{\mathcal{E}^{(2n+1)}\}$ for the smKdV-I, and $\{\Omega_{2m+1}, \Gamma_{2m+1}\}$ for smKdV-II. Five different vertex operators were obtained in such context, leading to two different types of one-soliton solutions for smKdV-I, and five solutions for smKdV-II.

Concerning the super Miura map, we have extended this transformation to all integer flows, whether they are positive or negative. For the negative flows, it was necessary to introduce an additional condition on the time derivatives of the sKdV field, the so-called temporal super Miura. These temporal relations are extremely important to show consistency of the gauge transformation for both super Miura and the dressing transformation, but also to prove the supersymmetry of $sKdV(-1)$. Another interesting result is the coalescence of two subsequent negative flows of smKdV hierarchy into a single flow of the sKdV. The soliton solutions for the sKdV hierarchy was obtained by using the super Miura transformation. Due to the existence of four possible super Miura transformation, we find that each smKdV solution lead to multiple solutions for sKdV.

On the other hand, regarding the vacuum structure, note that $\mathcal{E}^{(2n+1)}$ has degree $2n+1$ according to the grading operator $\hat{Q} \equiv 2\hat{d} + \frac{1}{2}M_1^{(0)}$, whereas Ω_{2m+1} and Γ_{2m+1} seem to have different grades in each part of its components.[11] This issue can be solved if we associate degree 1 to v_0 and $\frac{1}{2}$ to $\bar{\psi}_0$, by redefining the grading operator as follows,

$$\widetilde{Q} \equiv \hat{Q} + \widetilde{d} \quad \text{where} \quad \widetilde{d} \equiv v_0 \frac{\partial}{\partial v_0} + \frac{1}{2}\bar{\psi}_0 \frac{\partial}{\partial \bar{\psi}_0}, \tag{8.1}$$

in such a way that both Ω_{2m+1} and Γ_{2m+1} have degree $2n+1$, and the vacuums v_0, $\bar{\psi}_0$ could be interpreted as the new spectral parameters. Thus, each term in $\mathcal{L}^{\text{smKdV-II}}_{2n+1,\text{vac}}$ and $\mathcal{L}^{\text{smKdV-II}}_{-2n,\text{vac}}$ has now grade $2n+1$ and $2n$, as would be expected. This idea was previously discussed in [49], where v_0 only was assumed to be a new spectral parameter, leading to define a two-loop algebra [55] associated with it.

In the supersymmetric case is necessary to introduce two new spectral parameters, making unclear whether this concept can be extended to a three-loop algebra. We intend to investigate this further.

In addition, it would be also interesting to explore different solutions with non-zero vacuum, such as multi-soliton solutions and quasi-periodic solutions in future investigations. The quasi-periodic solutions have already been obtained for supersymmetric systems by using different methods [56,57], but we would like to extend the dressing method to obtain it more systematically. We also would like to implement the Bäcklund transformations [58] via a regular ansatz [28]. This will allow us to obtain the defect matrix for systems such as sKdV(-1) and smKdV(-2), and to use it for performing the scattering of the solutions. Finally, it would be also interesting to explore what would happen to the ZCC if we use a semi-integer ansatz for the Lax Pair, i.e. $N \in \mathbb{Z} + \frac{1}{2}$. Some of these issues are currently under investigations, and we hope to report it in the future.

CRediT authorship contribution statement

Y.F. Adans: Writing – review & editing, Writing – original draft, Project administration, Methodology, Investigation, Formal analysis, Conceptualization. **A.R. Aguirre:** Writing – review & editing, Supervision, Investigation, Conceptualization. **J.F. Gomes:** Writing – review & editing, Supervision, Methodology, Conceptualization. **G.V. Lobo:** Writing – review & editing, Methodology, Investigation, Formal analysis, Conceptualization. **A.H. Zimerman:** Supervision.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A. The $sl(2,1)$ superalgebra

In this section, we present the algebraic structure used to construct the smKdV and sKdV integrable hierarchies. Let us consider super Kac-Moody algebra $\mathcal{G} = sl(2,1)$ generated by

$$\begin{aligned} L_0 &= \left\{ h_1^{(m)} = \lambda^m h_1,\ h_2^{(m)} = \lambda^m h_2,\ E_{\pm\alpha_1}^{(m)} = \lambda^m E_{\pm\alpha_1}, \hat{\kappa} \right\}, \\ L_1 &= \left\{ E_{\pm\alpha_2}^{(m)} = \lambda^m E_{\pm\alpha_2},\ E_{\pm(\alpha_1+\alpha_2)}^{(m)} = \lambda^m E_{\pm(\alpha_1+\alpha_2)} \right\}, \end{aligned} \tag{A.1}$$

where $\lambda \in \mathbb{C}$, $m \in \mathbb{N}$, $\hat{\kappa}$ is central extension term and

[11] The grades would be $2n$, $2n+\frac{1}{2}$, $2n+1$ and $2n+1$, $2n+\frac{3}{2}$ respectively.

$$h_1=\begin{pmatrix}1&0&0\\0&-1&0\\0&0&0\end{pmatrix},\quad h_2=\begin{pmatrix}0&0&0\\0&1&0\\0&0&1\end{pmatrix},\quad E_{\alpha_1}=\begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix},\quad E_{-\alpha_1}=E^{\dagger}_{\alpha_1},$$
$$E_{\alpha_2}=\begin{pmatrix}0&0&0\\0&0&1\\0&0&0\end{pmatrix},\quad E_{\alpha_1+\alpha_2}=\begin{pmatrix}0&0&1\\0&0&0\\0&0&0\end{pmatrix},\quad E_{-\alpha_2}=E^{\dagger}_{\alpha_2},\quad E_{-(\alpha_1+\alpha_2)}=E^{\dagger}_{(\alpha_1+\alpha_2)}. \tag{A.2}$$

The L_0 and L_1 are called the bosonic and fermionic parts of algebra, respectively, satisfying the following relations

$$\left[L_0,L_0\right]\subset L_0,\qquad \left[L_0,L_1\right]\subset L_1,\qquad \left[L_1,L_1\right]\subset L_0.$$

The *principal grading operator* is defined by

$$Q=2\hat{d}+\frac{1}{2}h_1, \tag{A.3}$$

where $\hat{d}$ is a derivation operator,

$$\left[\hat{d},T_a^{(m)}\right]=m\,T_a^{(m)},\qquad T_a^{(m)}\in\mathcal{G}.$$

The *principal grading operator* decomposes the algebra in graded subspaces, $\mathcal{G}=\bigoplus_a\mathcal{G}_a$, where

$$\left[Q,\mathcal{G}_a\right]=a\,\mathcal{G}_a,\qquad \left[\mathcal{G}_a,\mathcal{G}_b\right]\in\mathcal{G}_{a+b},$$

for $a,b\in\mathbb{Z}$. For our purposes, the subspaces to consider are:

$$\begin{aligned}
\mathcal{G}_{2m}&=\left\{h_1^{(m)}\right\},\\
\mathcal{G}_{2m+\frac{1}{2}}&=\left\{E_{\alpha_2}^{(m+\frac{1}{2})},\,E_{\alpha_1+\alpha_2}^{(m)},\,E_{-\alpha_2}^{(m)},\,E_{-\alpha_1-\alpha_2}^{(m+\frac{1}{2})}\right\},\\
\mathcal{G}_{2m+1}&=\left\{E_{\alpha_1}^{(m)},\,E_{-\alpha_1}^{(m+1)},\,h_2^{(m+\frac{1}{2})}\right\},\\
\mathcal{G}_{2m+\frac{3}{2}}&=\left\{E_{\alpha_2}^{(m+1)},\,E_{\alpha_1+\alpha_2}^{(m+\frac{1}{2})},\,E_{-\alpha_2}^{(m+\frac{1}{2})},\,E_{-\alpha_1-\alpha_2}^{(m+1)}\right\}.
\end{aligned} \tag{A.4}$$

Another key ingredient in constructing our models is the grade one constant element, defined as

$$\mathcal{E}^{(1)}=E_{\alpha_1}^{(0)}+E_{-\alpha_1}^{(1)}+h_1^{(\frac{1}{2})}+2h_2^{(\frac{1}{2})}=\begin{pmatrix}\sqrt{\lambda}&1&0\\\lambda&\sqrt{\lambda}&0\\0&0&2\sqrt{\lambda}\end{pmatrix}, \tag{A.5}$$

which decomposes the algebra into $\mathcal{G}=\mathcal{K}\bigoplus\mathcal{M}$, where $\mathcal{K}_{\mathcal{E}}=\{x\in\mathcal{G}\mid[\mathcal{E},x]=0\}$ is the kernel of $\mathcal{E}$, and $\mathcal{M}_{\mathcal{E}}$ its complement,

$$\begin{aligned}
&\mathcal{K}_{\text{Bose}}=\mathcal{K}_{\mathcal{E}}\cap L_0=\left\{K_1^{(2m+1)},\,K_2^{(2m+1)}\right\},\qquad \mathcal{M}_{\text{Bose}}=\mathcal{M}_{\mathcal{E}}\cap L_0=\left\{M_1^{(2m)},M_2^{(2m+1)}\right\},\\
&\mathcal{K}_{\text{Fermi}}=\mathcal{K}_{\mathcal{E}}\cap L_1=\left\{F_1^{(2m+\frac{1}{2})},\,F_2^{(2m+3/2)}\right\},\quad \mathcal{M}_{\text{Fermi}}=\mathcal{M}_{\mathcal{E}}\cap L_1=\left\{G_1^{(2m+3/2)},\,G_2^{(2m+\frac{1}{2})}\right\}.
\end{aligned} \tag{A.6}$$

The bosonic generators are defined as

$$\begin{aligned}
K_1^{(2m+1)}&=E_{\alpha_1}^{(m)}+E_{-\alpha_1}^{(m+1)}, & M_1^{(2m)}&=h_1^{(m)},\\
K_2^{(2m+1)}&=h_1^{(m+\frac{1}{2})}+2h_2^{(m+\frac{1}{2})}, & M_2^{(2m+1)}&=E_{\alpha_1}^{(m)}-E_{-\alpha_1}^{(m+1)},
\end{aligned}$$

and the fermionic generators are

$$\begin{aligned}
F_1^{(2m+\frac{1}{2})}&=\left(E_{\alpha_2}^{(m+\frac{1}{2})}+E_{\alpha_1+\alpha_2}^{(m)}\right)+\left(E_{-\alpha_2}^{(m)}+E_{-(\alpha_1+\alpha_2)}^{(m+\frac{1}{2})}\right),\\
F_2^{(2m+\frac{3}{2})}&=\left(E_{\alpha_2}^{(m+1)}+E_{\alpha_1+\alpha_2}^{(m+\frac{1}{2})}\right)-\left(E_{-\alpha_2}^{(m+\frac{1}{2})}+E_{-(\alpha_1+\alpha_2)}^{(m+1)}\right),\\
G_1^{(2m+\frac{3}{2})}&=\left(E_{\alpha_2}^{(m+1)}-E_{\alpha_1+\alpha_2}^{(m+\frac{1}{2})}\right)+\left(E_{-\alpha_2}^{(m+\frac{1}{2})}-E_{-(\alpha_1+\alpha_2)}^{(m+1)}\right),\\
G_2^{(2m+\frac{1}{2})}&=\left(E_{\alpha_2}^{(m+\frac{1}{2})}-E_{\alpha_1+\alpha_2}^{(m)}\right)-\left(E_{-\alpha_2}^{(m)}-E_{-(\alpha_1+\alpha_2)}^{(m+\frac{1}{2})}\right).
\end{aligned}$$

From (A.3) and (A.5), we can reorganize the graded subspaces (A.4) in terms of the kernel-image decomposition in the following way,

$$
\begin{aligned}
\mathcal{G}_{2m} &= \left\{ M_1^{(2m)} \right\}, \\
\mathcal{G}_{2m+\frac{1}{2}} &= \left\{ F_1^{(2m+\frac{1}{2})},\, G_2^{(2m+\frac{1}{2})} \right\}, \\
\mathcal{G}_{2m+1} &= \left\{ K_1^{(2m+1)},\, K_2^{(2m+1)},\, M_2^{(2m+1)} \right\}, \\
\mathcal{G}_{2m+\frac{3}{2}} &= \left\{ F_2^{(2m+\frac{3}{2})},\, G_1^{(2m+\frac{3}{2})} \right\}.
\end{aligned}
\tag{A.8}
$$

The commutation relations of the algebra are then given by

$$
\begin{aligned}
&\left[K_1^{(2m+1)}, K_1^{(2n+1)}\right] = \hat{\kappa}(m-n)\delta_{n+m+1,0}, &&\left[K_1^{(2m+1)}, G_2^{(2n+1/2)}\right] = -G_1^{(2(m+n)+3/2)}, \\
&\left[K_1^{(2m+1)}, K_2^{(2n+1)}\right] = 0, &&\left[K_2^{(2m+1)}, F_1^{(2n+1/2)}\right] = -F_2^{(2(m+n)+3/2)}, \\
&\left[K_1^{(2m+1)}, M_1^{(2n)}\right] = -2M_2^{(2m+2n+1)}, &&\left[K_2^{(2m+1)}, F_2^{(2n+3/2)}\right] = -F_1^{(2(m+n+1)+1/2)}, \\
&\left[K_1^{(2m+1)}, M_2^{(2n+1)}\right] = -2M_1^{(2(m+n+1))} - \hat{\kappa}(m+n)\delta_{n+m+1,0}, &&\left[K_2^{(2m+1)}, G_1^{(2n+3/2)}\right] = -G_2^{(2(m+n+1)+1/2)}, \\
&\left[K_2^{(2m+1)}, K_2^{(2n+1)}\right] = -\hat{\kappa}(2m+1)\delta_{n+m+1,0}, &&\left[K_2^{(2m+1)}, G_2^{(2n+1/2)}\right] = -G_1^{(2(m+n)+3/2)}, \\
&\left[K_2^{(2m+1)}, M_1^{(2n)}\right] = 0, &&\left[M_1^{(2m)}, F_1^{(2n+1/2)}\right] = -G_2^{(2(m+2)+1/2)}, \\
&\left[K_2^{(2m+1)}, M_2^{(2n+1)}\right] = 0, &&\left[M_1^{(2m)}, F_2^{(2n+3/2)}\right] = -G_1^{(2(m+2)+3/2)}, \\
&\left[M_1^{(2m)}, M_1^{(2n)}\right] = 2\hat{\kappa}m\delta_{m+n,0}, &&\left[M_1^{(2m)}, G_1^{(2n+3/2)}\right] = -F_2^{(2(m+2)+3/2)}, \\
&\left[M_1^{(2m)}, M_2^{(2n+1)}\right] = 2K_1^{(2m+2n+1)}, &&\left[M_1^{(2m+1)}, G_2^{(2n+1/2)}\right] = -F_1^{(2(m+2)+1/2)}, \\
&\left[M_2^{(2m+1)}, M_2^{(2n+1)}\right] = -\hat{\kappa}(m-n)\delta_{n+m+1,0}, &&\left[M_2^{(2m+1)}, F_1^{(2n+1/2)}\right] = -G_1^{(2(m+n)+3/2)}, \\
&\left[K_1^{(2m+1)}, F_1^{(2n+1/2)}\right] = F_2^{(2(m+n)+3/2)}, &&\left[M_2^{(2m+1)}, F_2^{(2n+3/2)}\right] = -G_2^{(2(m+n+1)+1/2)}, \\
&\left[K_1^{(2m+1)}, F_2^{(2n+3/2)}\right] = F_1^{(2(m+n+1)+1/2)}, &&\left[M_2^{(2m+1)}, G_1^{(2n+3/2)}\right] = F_1^{(2(m+n+1)+1/2)}, \\
&\left[K_1^{(2m+1)}, G_1^{(2n+3/2)}\right] = -G_2^{(2(m+n+1)+1/2)}, &&\left[M_2^{(2m+1)}, G_2^{(2n+1/2)}\right] = F_2^{(2(m+n)+3/2)},
\end{aligned}
$$

and the anti-commutations relations, given by

$$
\begin{aligned}
&\left\{F_1^{(2m+1/2)}, F_1^{(2n+1/2)}\right\} = 2(K_1^{(2m+2n+1)} + K_2^{(2m+2n+1)}) - \hat{\kappa}(4m+1)\delta_{n+m+1,0}, \\
&\left\{F_1^{(2m+1/2)}, F_2^{(2n+3/2)}\right\} = 0, \\
&\left\{F_1^{(2m+1/2)}, G_1^{(2n+3/2)}\right\} = -2M_1^{(2(m+n+1))} + \hat{\kappa}\delta_{n+m+1,0}, \\
&\left\{F_1^{(2m+1/2)}, G_2^{(2n+1/2)}\right\} = -2M_2^{(2m+2n+1)}, \\
&\left\{F_2^{(2m+3/2)}, F_2^{(2n+3/2)}\right\} = -(K_1^{(2(m+n+1)+1)} + K_2^{(2(m+n+1)+1)}), \\
&\left\{F_2^{(2m+3/2)}, G_1^{(2n+3/2)}\right\} = 2M_2^{(2(m+n+1)+1)}, \\
&\left\{F_2^{(2m+3/2)}, G_2^{(2n+1/2)}\right\} = 2M_1^{(2(m+n+1))} - \hat{\kappa}\delta_{n+m+1,0}, \\
&\left\{G_1^{(2m+3/2)}, G_1^{(2n+3/2)}\right\} = -2(K_1^{(2(m+n+1)+1)} - K_2^{(2(m+n+1)+1)}), \\
&\left\{G_1^{(2m+3/2)}, G_2^{(2n+1/2)}\right\} = \hat{\kappa}\,(2m-2n-1)\,\delta_{n+m+1,0}, \\
&\left\{G_2^{(2m+1/2)}, G_2^{(2n+1/2)}\right\} = 2(K_1^{(2m+2n+1)} + K_2^{(2m+2n+1)}).
\end{aligned}
$$

Appendix B. Grade decomposition of zero curvature condition

In this appendix, we present explicitly the grade by grade decomposition proposed for each ansatz in section 2.

- The smKdV positive flows

$$\left[\partial_x + A_x,\ \partial_{t_N} + D_N^{(N)} + D_N^{(N-\frac{1}{2})} + \cdots + D_N^{(0)}\right] = 0, \tag{B.1}$$

decomposes as follows

$$\left[\mathcal{E}^{(1)}, D_N^{(N)}\right] = 0, \tag{B.2a}$$

$$\left[\mathcal{E}^{(1)}, D_N^{(N-\frac{1}{2})}\right] + \left[A_{\frac{1}{2}}, D_N^{(N)}\right] = 0, \tag{B.2b}$$

$$\left[\mathcal{E}^{(1)}, D_N^{(N-1)}\right] + \left[A_{\frac{1}{2}}, D_N^{(N-\frac{1}{2})}\right] + \left[A_0, D_N^{(N)}\right] + \partial_x D_N^{(N)} = 0, \tag{B.2c}$$

$$\vdots$$

$$\left[A_{\frac{1}{2}}, D_N^{(0)}\right] + \left[A_0, D_N^{(\frac{1}{2})}\right] + \partial_x D_N^{(\frac{1}{2})} - \partial_{t_N} A_{\frac{1}{2}} = 0, \tag{B.2d}$$

$$\left[A_0, D_N^{(0)}\right] + \partial_x D_N^{(0)} - \partial_{t_N} A_0 = 0. \tag{B.2e}$$

- For sKdV positive flows, we have

$$\left[\partial_x + A_x,\ \partial_{t_N} + \mathcal{D}_N^{(N)} + \mathcal{D}_N^{(N-\frac{1}{2})} + \cdots + \mathcal{D}_N^{(-1)}\right] = 0, \tag{B.3}$$

$$\left[\mathcal{E}^{(1)}, \mathcal{D}_N^{(N)}\right] = 0, \tag{B.4a}$$

$$\left[\mathcal{E}^{(1)}, \mathcal{D}_N^{(N-\frac{1}{2})}\right] = 0, \tag{B.4b}$$

$$\partial_x \mathcal{D}_N^{(N)} + \left[\mathcal{E}^{(1)}, \mathcal{D}_N^{(N-1)}\right] = 0, \tag{B.4c}$$

$$\vdots$$

$$\partial_x \mathcal{D}_N^{(-\frac{1}{2})} + \left[A_{-1}, \mathcal{D}_N^{(\frac{1}{2})}\right] + \left[A_{-\frac{1}{2}}, \mathcal{D}_N^{(0)}\right] - \partial_{t_N} A_{-\frac{1}{2}} = 0, \tag{B.4d}$$

$$\partial_x \mathcal{D}_N^{(-1)} + \left[A_{-1}, \mathcal{D}_N^{(0)}\right] + \left[A_{-\frac{1}{2}}, \mathcal{D}_N^{(-\frac{1}{2})}\right] - \partial_{t_N} A_{-1} = 0. \tag{B.4e}$$

- The smKdV negative flows

$$\left[\partial_x + A_x,\ \partial_{t_{-N}} + D_{-N}^{(-\frac{1}{2})} + D_N^{(-1)} + \cdots + D_{-N}^{(-N)}\right] = 0, \tag{B.5}$$

decomposes as follows

$$\left[A_0, D_{-N}^{(-N)}\right] + \partial_x D_{-N}^{(-N)} = 0, \tag{B.6a}$$

$$\left[A_{\frac{1}{2}}, D_{-N}^{(-N)}\right] + \left[A_0, D_{-N}^{(-N+\frac{1}{2})}\right] + \partial_x D_{-N}^{(-N+\frac{1}{2})} = 0, \tag{B.6b}$$

$$\vdots$$

$$\left[\mathcal{E}^{(1)}, D_{-N}^{(-1)}\right] + \left[A_0, D_{-N}^{(-\frac{1}{2})}\right] - \partial_{t_{-N}} A_0 = 0, \tag{B.6c}$$

$$\left[\mathcal{E}^{(1)}, D_{-N}^{(-\frac{1}{2})}\right] - \partial_{t_{-N}} A_{\frac{1}{2}} = 0. \tag{B.6d}$$

- The sKdV negative flows

$$\left[\partial_x + A_x,\ \partial_{t_{-N}} + \mathcal{D}_{-N}^{(-\frac{1}{2})} + \mathcal{D}_N^{(-1)} + \cdots + \mathcal{D}_{-N}^{(-N-2)}\right] = 0, \tag{B.7}$$

decomposes in the following way

$$\left[A_{-1}, \mathcal{D}_{-N}^{(-N-2)}\right] = 0, \tag{B.8a}$$

$$\left[A_{-\frac{1}{2}}, \mathcal{D}_{-N}^{(-N-2)}\right] + \left[A_{-1}, \mathcal{D}_{-N}^{(-N-\frac{3}{2})}\right] = 0, \tag{B.8b}$$

$$\partial_x \mathcal{D}_{-N}^{(-N-2)} + \left[A_{-1}, \mathcal{D}_{-N}^{(-N-1)}\right] + \left[A_{-\frac{1}{2}}, \mathcal{D}_{-N}^{(-N-\frac{3}{2})}\right] = 0, \tag{B.8c}$$

$$\vdots$$

$$\partial_x \mathcal{D}_{-N}^{(-1)} + \left[A_{-\frac{1}{2}}, \mathcal{D}_{-N}^{(-\frac{1}{2})}\right] + \left[\mathcal{E}^{(1)}, \mathcal{D}_{-N}^{(-2)}\right] - \partial_{t_{-N}} A_{-1} = 0, \tag{B.8d}$$

$$\partial_x \mathcal{D}_{-N}^{(-\frac{1}{2})} + \left[\mathcal{E}^{(1)}, \mathcal{D}_{-N}^{(-\frac{3}{2})}\right] - \partial_{t_{-N}} A_{-\frac{1}{2}} = 0, \tag{B.8e}$$

$$\left[\mathcal{E}^{(1)}, \mathcal{D}_{-N}^{(-1)}\right] = 0, \tag{B.8f}$$

$$\left[\mathcal{E}^{(1)}, \mathcal{D}_{-N}^{(-\frac{1}{2})}\right] = 0. \tag{B.8g}$$

Appendix C. Lax pairs for smKdV and sKdV

Here we present the explicit form of the Lax pairs for some specific flows of both sKdV and smKdV hierarchies.

C.1. The smKdV hierarchy

- $N = 3$

$$\begin{aligned} A_{t_3}^{\mathrm{smKdV}} = \;& K_1^{(3)} + K_2^{(3)} + \bar{\psi}\, G_2^{(\frac{5}{2})} + \upsilon\, M_1^{(2)} + \frac{1}{2}\upsilon\,\bar{\psi}\, F_2^{(\frac{3}{2})} + \frac{1}{2}\partial_x\bar{\psi}\, G_1^{(\frac{3}{2})} \\ & - \frac{1}{2}\left\{\upsilon^2 + \bar{\psi}\partial_x\bar{\psi}\right\} K_1^{(1)} + \frac{1}{2}\bar{\psi}\partial_x\bar{\psi}\, K_2^{(1)} + \frac{1}{2}\partial_x\upsilon\, M_2^{(1)} \\ & + \frac{1}{4}\left\{\upsilon\partial_x\bar{\psi} - \partial_x\upsilon\bar{\psi}\right\}\, F_1^{(\frac{1}{2})} + \left\{\frac{1}{4}\partial_x^2\bar{\psi} - \frac{1}{2}\upsilon^2\bar{\psi}\right\}\, G_2^{\frac{1}{2}} \\ & + \left\{\frac{1}{4}\partial_x^2\upsilon - \frac{1}{2}\upsilon^3 - \frac{3}{4}\upsilon\bar{\psi}\partial_x\bar{\psi}\right\}\, M_1^{(0)}. \end{aligned} \tag{C.1}$$

- $N = 5$

$$A_{t_5}^{\mathrm{smKdV}} = D_5^{(5)} + D_5^{(\frac{9}{2})} + \cdots + D_5^{(0)}, \tag{C.2}$$

with

$$\begin{aligned} D_5^{(5)} &= K_1^{(5)} + K_2^{(5)} = \mathcal{E}^{(5)}, \\ D_5^{(\frac{9}{2})} &= \bar{\psi}\, G_2^{(9/2)}, \\ D_5^{(4)} &= \upsilon\, M_1^{(4)}, \\ D_5^{(\frac{7}{2})} &= \frac{1}{2}\upsilon\bar{\psi}\, F_2^{(7/2)} + \frac{1}{2}\partial_x\bar{\psi}\, G_1^{(7/2)}, \\ D_5^{(3)} &= -\frac{1}{2}\left(\bar{\psi}\partial_x\bar{\psi} + \upsilon^2\right)\, K_1^{(3)} + \frac{1}{2}\bar{\psi}\partial_x\bar{\psi}\, K_2^{(3)} + \frac{1}{2}\partial_x\upsilon\, M_2^{(3)}, \\ D_5^{(\frac{5}{2})} &= \frac{1}{4}\left(\upsilon\partial_x\bar{-}\partial_x\upsilon\bar{\psi}\right) F_1^{(5/2)} + \frac{1}{4}\left(-2\upsilon^2\bar{\psi} + \partial_x^2\bar{\psi}\right)\, G_2^{(5/2)}, \\ D_5^{(2)} &= \frac{1}{4}\left(-3\upsilon\bar{\psi}\partial_x\bar{\psi} - 2\upsilon^3 + \partial_x^2\upsilon\right)\, M_1^{(2)}, \\ D_5^{(\frac{3}{2})} &= \frac{1}{8}\left(\upsilon\partial_x^2\bar{\psi} + \bar{\psi}\partial_x^2\upsilon - \upsilon^3\bar{\psi} - \partial_x\upsilon\partial_x\bar{\psi}\right)\, F_2^{(3/2)} + \frac{1}{8}\left(\partial_x^3\bar{\psi} - 3\upsilon^2\partial_x\bar{\psi} - 3\upsilon\partial_x\upsilon\bar{\psi}\right)\, G_1^{(3/2)}, \\ D_5^{(1)} &= \frac{1}{8}\left(\bar{\psi}\partial_x^3\bar{\psi} - \partial_x\bar{\psi}\partial_x^2\bar{\psi} - 8\upsilon^2\bar{\psi}\partial_x\bar{\psi} - (\partial_x\upsilon)^2 + 2\upsilon\partial_x^2\upsilon - 3\upsilon^4\right)\, K_1^{(1)} \\ &\quad + \frac{1}{8}\left(\bar{\psi}\partial_x^3\bar{\psi} - \partial_x\bar{\psi}\partial_x^2\bar{\psi} - 4\upsilon^2\bar{\psi}\partial_x\bar{\psi}\right)\, K_2^{(1)} + \frac{1}{8}\left(\partial_x^3\upsilon - 6\upsilon^2\partial_x\upsilon - 4\partial_x\upsilon\bar{\psi}\partial_x\bar{\psi} - 2\upsilon\bar{\psi}\partial_x^2\bar{\psi}\right)\, M_2^{(1)}, \\ D_5^{(\frac{1}{2})} &= \frac{1}{16}\left(4\upsilon^2\partial_x\upsilon\bar{\psi} - 4\upsilon^3\partial_x\bar{\psi} - \partial_x\upsilon\partial_x^2\bar{\psi} + \partial_x^2\upsilon\partial_x\bar{\psi} + \upsilon\partial_x^3\bar{\psi} - \partial_x^3\upsilon\bar{\psi}\right)\, F_1^{(1/2)} \\ &\quad + \frac{1}{16}\left(\partial_x^4\bar{\psi} - 4\upsilon^2\partial_x^2\bar{\psi} - 8\upsilon\partial_x\upsilon\partial_x\bar{\psi} - 2(\partial_x\upsilon)^2\bar{\psi} - 6\upsilon\partial_x^2\upsilon\bar{\psi} + 6\upsilon^4\bar{\psi}\right)\, G_2^{(1/2)}, \\ D_5^{(0)} &= \frac{1}{16}\left(\partial_x^4\upsilon - 10\upsilon^2\partial_x^2\upsilon - 10\upsilon(\partial_x\upsilon)^2 + 6\upsilon^5 + 20\upsilon^3\bar{\psi}\partial_x\bar{\psi} - 5\partial_x(\partial_x\upsilon\bar{\psi}\partial_x\bar{\psi}) - 5\upsilon\bar{\psi}\partial_x^3\bar{\psi}\right) M_1^{(2)}. \end{aligned}$$

- $N = \frac{1}{2}$

$$A^{\text{smKdV}}_{t_{\frac{1}{2}}} = \xi F_1^{(\frac{1}{2})} + \xi \bar{\psi} M_1^{(0)}. \tag{C.4}$$

- $N = -1$

$$A^{\text{smKdV}}_{t_{-1}} = \cosh 2\phi\, K_1^{(-1)} + K_2^{(-1)} - \sinh 2\phi\, M_2^{(-1)} - \psi \sinh\phi\, F_2^{(-\frac{1}{2})} - \psi \cosh\phi\, G_1^{(-\frac{1}{2})}, \tag{C.5}$$

where $v(x, t_{-1}) = \partial_x \phi(x, t_{-1})$.

- $N = -2$

$$\begin{aligned} A^{\text{smKdV}}_{t_{-2}} &= M_1^{(-2)} - \frac{e^{\phi}\psi_-}{2}\left(F_1^{(-\frac{3}{2})} + G_2^{(-\frac{3}{2})}\right) - \frac{e^{-\phi}\psi_+}{2}\left(F_1^{(-\frac{3}{2})} - G_2^{(-\frac{3}{2})}\right) \\ &\quad + a_-\left(K_1^{(-1)} + M_2^{(-1)}\right) - a_+\left(K_1^{(-1)} - M_2^{(-1)}\right) + \left(1 + \psi_-\psi_+\right) K_2^{(-1)} \\ &\quad + \Omega_+\left(F_2^{(-\frac{1}{2})} + G_1^{(-\frac{1}{2})}\right) + \Omega_-\left(F_2^{(-\frac{1}{2})} - G_1^{(-\frac{1}{2})}\right), \end{aligned} \tag{C.6}$$

where

$$v = \partial_x \phi, \qquad \psi_\pm = \partial_x^{-1}\left(e^{\pm\phi}\bar{\psi}\right), \qquad a_\pm = e^{\pm 2\phi}\partial_x^{-1}\left[e^{\mp 2\phi}\left(1 + \psi_\mp \partial_x \psi_\pm\right)\right],$$

and

$$\Omega_\pm = \frac{e^{\pm\phi}}{2}\partial_x^{-1}\left[e^{\mp 2\phi}\psi_\pm - \psi_\mp \mp \partial_x \psi_\mp \left(1 + \psi_-\psi_+ \mp 2a_\pm\right)\right].$$

C.2. The sKdV hierarchy

- $N = 3$

$$\begin{aligned} A^{\text{sKdV}}_{t_3} = &\; K_1^{(3)} + K_2^{(3)} + \frac{\bar{\chi}}{2} G_1^{\left(\frac{3}{2}\right)} - \frac{J}{2} M_2^{(1)} + \frac{\partial_x \bar{\chi}}{4} G_2^{\left(\frac{1}{2}\right)} \\ &- \frac{\partial_x J}{4} M_1^{(0)} + \left(\frac{\partial_x^2 \chi}{8} - \frac{J\bar{\chi}}{4}\right)\left(G_1^{\left(\frac{2}{2}\right)} + F_2^{\left(-\frac{1}{2}\right)}\right) \\ &+ \left(\frac{\partial_x^2 J}{8} - \frac{J^2}{4} - \frac{\bar{\chi}\partial_x \bar{\chi}}{8}\right)\left(K_1^{(-1)} - M_2^{(-1)}\right). \end{aligned} \tag{C.7}$$

- $N = 5$

$$A^{\text{sKdV}}_{t_5} = \mathcal{D}_5^{(5)} + \mathcal{D}_5^{(\frac{9}{2})} + \cdots + \mathcal{D}_5^{(-1)}, \tag{C.8}$$

with

$$\begin{aligned} \mathcal{D}_5^{(5)} &= K_1^{(5)} + K_2^{(5)} = \mathcal{E}^{(5)}, \\ \mathcal{D}_5^{(\frac{9}{2})} &= 0, \\ \mathcal{D}_5^{(4)} &= 0, \\ \mathcal{D}_5^{(\frac{7}{2})} &= \frac{\bar{\chi}}{2} G_1^{(\frac{7}{2})}, \\ \mathcal{D}_5^{(3)} &= -\frac{J}{2} M_2^{(3)}, \\ \mathcal{D}_5^{(\frac{5}{2})} &= \frac{\partial_x \bar{\chi}}{4} G_2^{(\frac{5}{2})}, \\ \mathcal{D}_5^{(2)} &= -\frac{\partial_x J}{4} M_1^{(2)}, \\ \mathcal{D}_5^{(\frac{3}{2})} &= \frac{J\bar{\chi}}{8} F_2^{(\frac{3}{2})} + \frac{1}{8}\left(\partial_x^2 \bar{\chi} - 2J\bar{\chi}\right) G_1^{(\frac{3}{2})}, \\ \mathcal{D}_5^{(1)} &= \frac{1}{8}\left(J^2 + \bar{\chi}\partial_x \bar{\chi}\right) K_1^{(1)} - \frac{1}{8}\bar{\chi}\partial_x \bar{\chi} K_2^{(1)} + \frac{1}{8}\left(-\partial_x^2 J + 2J^2 + \bar{\chi}\partial_x \bar{\chi}\right) M_2^{(1)}, \\ \mathcal{D}_5^{(\frac{1}{2})} &= \frac{1}{16}\left(-\partial_x J\bar{\chi} + J\partial_x \bar{\chi}\right) F_1^{(\frac{1}{2})} + \frac{1}{16}\left(-\partial_x^3 \bar{\chi} - 3\partial_x (J\bar{\chi})\right) G_2^{(\frac{1}{2})}, \end{aligned}$$

$$\mathcal{D}_5^{(0)} = \frac{1}{16}\left(-\partial_x^3 J + 3\partial_x J^2 + 2\bar{\chi}\partial_x^2\bar{\chi}\right) M_1^{(0)},$$

$$\mathcal{D}_5^{(-\frac{1}{2})} = \frac{1}{32}\left(\partial_x^4\bar{\chi} - 3\partial_x^2(J\bar{\chi}) + 6J^2\bar{\chi} - J\partial_x^2\bar{\chi} - \partial_x^2 J\bar{\chi}\right)\left(F_2^{(-\frac{1}{2})} + G_1^{(-\frac{1}{2})}\right),$$

$$\mathcal{D}_5^{(-1)} = \frac{1}{32}\left(\partial_x^4 J - 6(\partial_x J)^2 - 8J\partial_x^2 J + 6J^3 - 4\bar{\chi}\partial_x^3\bar{\chi} + 8J\bar{\chi}\partial_x\bar{\chi}\right)\left(K_1^{(-1)} - M_2^{(-1)}\right).$$

- $N = \frac{1}{2}$

$$A^{\mathrm{sKdV}}_{t_{\frac{1}{2}}} = \xi F_1^{(\frac{1}{2})} - \xi\bar{\chi}\left(K_1^{(-1)} - M_2^{(-1)}\right). \tag{C.10}$$

- $N = -1$

$$A^{\mathrm{sKdV}}_{t_{-1}} = \mathcal{D}_{-1}^{(-3)} + \mathcal{D}_{-1}^{(-\frac{5}{2})} + \cdots + \mathcal{D}_{-1}^{(-\frac{1}{2})}, \tag{C.11}$$

with

$$\mathcal{D}_{-1}^{(-3)} = -\frac{1}{8}\Big\{\partial_x(\partial_{t_{-1}}\partial_x\eta + \partial_x\gamma) - 2\partial_x\eta\left(\partial_{t_{-1}}\eta + \gamma\right)$$
$$+\partial_x\bar{\eta}\left(\bar{\nu}_- - \bar{\nu}_+ + 2\bar{\gamma} - \partial_{t_{-1}}\partial_x\bar{\gamma}\right)\Big\}\left(K_1^{(-3)} - M_2^{(-3)}\right),$$

$$\mathcal{D}_{-1}^{(-\frac{5}{2})} = \frac{1}{8}\left(\partial_x(\bar{\nu}_+ + \bar{\nu}_-) - \partial_{t_{-1}}\partial_x^2\bar{\gamma} + 2\partial_x\bar{\gamma}\,\gamma + 2\partial_x\bar{\gamma}\right)\left(F_2^{(-\frac{5}{2})} + G_1^{(-\frac{5}{2})}\right),$$

$$\mathcal{D}_{-1}^{(-2)} = \frac{1}{4}\left(\partial_{t_{-1}}\partial_x\eta + \partial_x\gamma\right) M_1^{(-2)},$$

$$\mathcal{D}_{-1}^{(-\frac{3}{2})} = \frac{1}{4}\left(\bar{\nu}_+ - \bar{\nu}_- - 2\bar{\gamma}\right) F_1^{(-\frac{3}{2})} - \frac{1}{4}\partial_{t_{-1}}\partial_x\bar{\gamma}\, G_2^{(-\frac{3}{2})},$$

$$\mathcal{D}_{-1}^{(-1)} = \frac{1}{2}\gamma\left(K_1^{(-1)} + K_2^{(-1)}\right) + \frac{1}{2}\partial_{t_{-1}}\eta\, K_1^{(-1)} + K_2^{(-1)},$$

$$\mathcal{D}_{-1}^{(-\frac{1}{2})} = \frac{1}{2}\partial_{t_{-1}}\bar{\gamma}\, F_2^{(-\frac{1}{2})}.$$

Appendix D. Supersymmetry transformation for smKdV(−2)

Let us consider the field equations for smKdV(−2),

$$\partial_{t_{-2}}\partial_x\phi = -2\left(a_- + a_+\right) + 2\bar{\psi}\left(\Omega_- + \Omega_+\right), \tag{D.1a}$$

$$\partial_{t_{-2}}\bar{\psi} = -2\left(\Omega_+ - \Omega_-\right), \tag{D.1b}$$

with

$$\psi_\pm = \partial_x^{-1}\left(\mathrm{e}^{\pm\phi}\bar{\psi}\right), \tag{D.2a}$$

$$a_\pm = \mathrm{e}^{\pm 2\phi}\partial_x^{-1}\left[\mathrm{e}^{\mp 2\phi}\left(1 + \psi_\mp\partial_x\psi_\pm\right)\right], \tag{D.2b}$$

and

$$\Omega_\pm = \frac{\mathrm{e}^{\pm\phi}}{2}\partial_x^{-1}\left[\mathrm{e}^{\mp 2\phi}\psi_\pm - \psi_\mp \mp \left(\partial_x\psi_\mp\right)\left(1 + \psi_-\psi_+ \mp 2a_\pm\right)\right], \tag{D.3}$$

where the anti-derivative operator is defined by $\partial_x^{-1} f = \int^x f(y)\mathrm{d}y$. By applying the supersymmetry transformation

$$\delta v = \xi\,\partial_x\bar{\psi} \quad \rightarrow \quad \delta\phi = \xi\bar{\psi}, \tag{D.4a}$$

$$\delta\bar{\psi} = -\xi\, v = -\xi\partial_x\phi, \tag{D.4b}$$

on the fields $\psi_\pm$, $a_\pm$ and $\Omega_\pm$, we have:

$$\partial_x\delta\psi_+ = \delta\left(\mathrm{e}^{\phi}\bar{\psi}\right) = -\xi\partial_x\left(\mathrm{e}^{\phi}\right), \tag{D.5}$$

and then

$$\delta\psi_+ = -\xi\mathrm{e}^{\phi}. \tag{D.6}$$

Similarly, for ψ_- we get

$$\delta\psi_- = \xi\mathrm{e}^{-\phi}. \tag{D.7}$$

Now consider for a_+,

$$\begin{aligned}\delta a_+ &= \delta\left\{e^{2\phi}\partial_x^{-1}\left[e^{-2\phi}\left(1+\psi_-\partial_x\psi_+\right)\right]\right\}\\ &= 2\delta\phi a_+ + e^{2\phi}\partial_x^{-1}\left\{\delta\left[e^{-2\phi}\left(1+\psi_-\partial_x\psi_+\right)\right]\right\}.\end{aligned} \tag{D.8}$$

Now, let us evaluate the second term

$$\begin{aligned}\delta\left[e^{-2\phi}\left(1+\psi_-\partial_x\psi_+\right)\right] &= -2\delta\phi e^{-2\phi}\left(1+\psi_-\partial_x\psi_+\right) + e^{-2\phi}\delta\left(\psi_-\partial_x\psi_+\right)\\ &= -2\xi\bar{\psi}e^{-2\phi} + e^{-2\phi}\xi e^{-\phi}e^{\phi}\bar{\psi} - e^{-2\phi}\psi_-\xi\partial_x\phi e^{\phi}\\ &= -\xi\partial_x\left(\psi_- e^{-\phi}\right),\end{aligned}$$

leading to

$$\begin{aligned}\delta a_+ &= 2\delta\phi a_+ + e^{2\phi}\partial_x^{-1}\left\{\delta\left[e^{-2\phi}\left(1+\psi_-\partial_x\psi_+\right)\right]\right\}\\ &= 2\xi\bar{\psi}a_+ - \xi e^{2\phi}\partial_x^{-1}\left\{\partial_x\left(\psi_- e^{-\phi}\right)\right\}\\ &= 2\xi\bar{\psi}a_+ - \xi e^{\phi}\psi_-.\end{aligned} \tag{D.9}$$

For a_-, the calculation is rather similar and gives us the following result

$$\delta a_- = -2\xi\bar{\psi}a_- + \xi e^{-\phi}\psi_+. \tag{D.10}$$

Finally for Ω_+, we have

$$\delta\Omega_+ = \delta\phi\Omega_+ + \frac{e^{\phi}}{2}\partial_x^{-1}\left\{\delta\left[e^{-2\phi}\psi_+ - \psi_- - \left(\partial_x\psi_-\right)\left(-2a_+ + \psi_-\psi_+ + 1\right)\right]\right\}. \tag{D.11}$$

Calculating the second term, we find

$$\begin{aligned}&\delta\left[e^{-2\phi}\psi_+ - \psi_- - \left(\partial_x\psi_-\right)\left(-2a_+ + \psi_-\psi_+ + 1\right)\right] =\\ &\xi e^{-2\phi}\psi_+\bar{\psi} - 2\xi e^{-\phi} + \partial_x\phi e^{-\phi}\left(-2a_+ + \psi_-\psi_+ + 1\right) + 3\xi\bar{\psi}\psi_-.\end{aligned} \tag{D.12}$$

By noting that the following relation holds,

$$\begin{aligned}&\partial_x\left(e^{-\phi}\left(-2a_+ - 1 - \psi_-\psi_+\right)\right) =\\ &\partial_x\phi e^{-\phi}\left(-2a_+ + 1 + \psi_-\psi_+\right) - 2\xi e^{-\phi} + \xi e^{-2\phi}\psi_+\bar{\psi} + 3\xi\bar{\psi}\psi_-,\end{aligned} \tag{D.13}$$

we can write

$$\delta\left[e^{-2\phi}\psi_+ - \psi_- - \left(\partial_x\psi_-\right)\left(-2a_+ + \psi_-\psi_+ + 1\right)\right] = \xi\partial_x\left(e^{-\phi}\left(-2a_+ - 1 - \psi_-\psi_+\right)\right), \tag{D.14}$$

and finally, we get

$$\begin{aligned}\delta\Omega_+ &= \xi\bar{\psi}\Omega_+ + \frac{\xi e^{\phi}}{2}\partial_x^{-1}\left\{\partial_x\left(e^{-\phi}\left(-2a_+ - 1 - \psi_-\psi_+\right)\right)\right\},\\ \delta\Omega_+ &= \xi\bar{\psi}\Omega_+ - \xi a_+ - \xi\frac{\psi_-\psi_+ + 1}{2}.\end{aligned} \tag{D.15}$$

Now, one can calculate for Ω_- and obtain

$$\delta\Omega_- = -\xi\bar{\psi}\Omega_- + \xi a_- - \xi\frac{\psi_-\psi_+ + 1}{2}. \tag{D.16}$$

By using (D.6), (D.7), (D.9), (D.15) and (D.16), we can finally evaluate the supersymmetry of the field equation, namely

$$\begin{aligned}\delta\partial_{t_{-2}}\bar{\psi} &= -2\left(\delta\Omega_+ - \delta\Omega_-\right)\\ &= -2\left(\xi\bar{\psi}\Omega_+ - \xi a_+ - \xi\frac{\psi_-\psi_+ + 1}{2} + \xi\bar{\psi}\Omega_- - \xi a_- + \xi\frac{\psi_-\psi_+ + 1}{2}\right)\\ &= -\xi\left[2\bar{\psi}\left(\Omega_+ + \Omega_-\right) - 2\left(a_+ + a_-\right)\right] = -\xi\partial_{t_{-2}}\partial_x\phi,\end{aligned} \tag{D.17}$$

implying that

$$\delta\partial_{t_{-2}}\bar{\psi} = -\xi\partial_{t_{-2}}\partial_x\phi. \tag{D.18}$$

If we check the other way around, we have

$$\begin{aligned}\delta\partial_x\partial_{t_{-2}}\phi &= -2\left(\delta a_- + \delta a_+\right) + 2\delta\bar{\psi}\left(\Omega_- + \Omega_+\right) + 2\bar{\psi}\left(\delta\Omega_- + \delta\Omega_+\right)\\ &= \xi\left[2\bar{\psi}\left(a_- - a_+ + 1 + \psi_-\psi_+\right) + 2\psi_- e^{\phi} - 2\psi_+ e^{-\phi} - 2\partial_x\phi\left(\Omega_+ + \Omega_-\right)\right],\end{aligned} \tag{D.19}$$

and then we can verify that the fermionic equation satisfies

$$\begin{aligned}\partial_{t_{-2}}\partial_x\bar{\psi} &= -2\left(\partial_x\Omega_+ - \partial_x\Omega_-\right)\\ &= 2\bar{\psi}\left(a_- - a_+ + 1 + \psi_-\psi_+\right) + 2\psi_-\mathrm{e}^{\phi} - 2\psi_+\mathrm{e}^{-\phi} - 2\partial_x\phi\left(\Omega_+ + \Omega_-\right),\end{aligned} \tag{D.20}$$

from which we conclude that this pair is supersymmetric, namely

$$\delta\partial_x\partial_{t_{-2}}\phi = \xi\partial_{t_{-2}}\partial_x\bar{\psi}. \tag{D.21}$$

Appendix E. Determining $\theta_{-\frac{1}{2}}$

In order to completely determine the element $\theta_{-\frac{1}{2}}$ in (4.28), we need to solve the following ordinary differential equation

$$(\partial_x\phi)\bar{\psi} = \partial_x\Xi - \bar{\psi}_0 v_0 + \frac{3\Xi}{2}\bar{\psi}_0\bar{\psi}. \tag{E.1}$$

Therefore, let us analyze it for different vacuum configurations:

1. For $(v_0, \bar{\psi}_0) = (0, 0)$ and $(v_0, \bar{\psi}_0) = (v_0, 0)$, the equation (E.1) simplifies as follows

$$(\partial_x\phi)\bar{\psi} = \partial_x\Xi, \tag{E.2}$$

   and then we get the following solution

$$\Xi = \partial_x^{-1}\left(\phi_x\bar{\psi}\right). \tag{E.3}$$

2. For $(v_0, \bar{\psi}_0) = (v_0, \bar{\psi}_0)$, the equation (E.1) simplifies as follows

$$\left(\partial_x\phi + v_0\right)\bar{\psi} = \partial_x\Xi + \frac{3\Xi}{2}\bar{\psi}_0\bar{\psi}. \tag{E.4}$$

   In this case, the solution is slightly more complex, and can be written as follows

$$\Xi = e^{-\partial_x^{-1}\left(\frac{3\bar{\psi}_0\bar{\psi}}{2}\right)}\partial_x^{-1}\left(e^{\partial_x^{-1}\left(\frac{3\bar{\psi}_0\bar{\psi}}{2}\right)}\left(\phi_x\bar{\psi} + v_0\bar{\psi}_0\right)\right), \tag{E.5}$$

   where is possible to take the limit $\bar{\psi}_0 \to 0$.

Appendix F. Vacuum projection and Heisenberg subalgebra for sKdV hierarchy

In section 7, we have obtained the solutions for sKdV hierarchy from the smKdV solutions by applying the super Miura transformation. However, we have not exhibited explicitly the sKdV vacuum operators. They are presented in this appendix.

Let us start from the vacuum projection of the spatial sKdV Lax

$$A_{x,\mathrm{vac}}^{\mathrm{sKdV}} = \Lambda = \mathcal{E}^{(1)} + \frac{J_0}{2}\left(K_1^{(-1)} - M_2^{(-1)}\right) + \frac{\chi_0}{2}\left(F_2^{-\frac{1}{2}} + G_1^{-\frac{1}{2}}\right). \tag{F.1}$$

The Kernel of Λ is given by

$$\mathcal{K}_\Lambda = \left\{\Lambda_{2m+1}, \Pi_{2m+1}\right\} \quad \text{such} \quad [X, \Lambda] = 0, \quad X \in \mathcal{K}_\Lambda, \tag{F.2}$$

where

$$\begin{aligned}\Lambda_{(2m+1)} &= \mathcal{E}^{(2m+1)} + \frac{J_0}{2}\left(K_1^{(2m-1)} - M_2^{(2m-1)}\right) + \frac{\bar{\chi}_0}{2}\left(G_1^{(2m-\frac{1}{2})} + F_2^{(2m-\frac{1}{2})}\right),\\ \Pi_{(2m+1)} &= K_2^{(2m+1)} - \frac{\bar{\chi}_0}{J_0}F_2^{(2m+\frac{3}{2})}.\end{aligned} \tag{F.3}$$

We notice that any linear combination of these operators will satisfy the ZCC in the vacuum.

Firstly, in the case of positive flows, we set $(J, \bar{\chi}) = (J_0, \chi_0)$, and obtain the following vacuum projection

$$\begin{aligned}
A^{\mathrm{sKdV}}_{t_1,\mathrm{vac}} &= A^{\mathrm{sKdV}}_{x,\mathrm{vac}} = \Lambda_1, \\
A^{\mathrm{sKdV}}_{t_3,\mathrm{vac}} &= \Lambda_3 + \frac{J_0}{2}\left(\Pi_1 - \Lambda_1\right), \\
A^{\mathrm{sKdV}}_{t_5,\mathrm{vac}} &= \Lambda_5 + \frac{J_0}{2}\left(\Pi_3 - \Lambda_3\right) + \frac{3J_0^2}{8}\left(\Lambda_1 - \Pi_1\right), \\
A^{\mathrm{sKdV}}_{t_{2m+1},\mathrm{vac}} &= \Lambda_{2m+1} + \sum_{i=0}^{m-1} v_0^{2(m-i)}\left(a_i\Lambda_{2i+1} + b_i\Pi_{2i+1}\right).
\end{aligned} \tag{F.4}$$

Clearly, the zero vacuum limit is given by

$$A^{\mathrm{sKdV}}_{t_{2m+1},\mathrm{vac}} = \mathcal{E}^{(2m+1)}. \tag{F.5}$$

Then, the zero curvature condition projected on the vacuum, i.e.

$$\left[A^{\mathrm{KdV}}_{x,\,\mathrm{vac}}, A^{\mathrm{KdV}}_{t_N,\,\mathrm{vac}}\right] = 0, \tag{F.6}$$

is satisfied as long as we can express the vacuum potential in terms of $\Lambda_{(2m+1)}$ and $\Pi_{(2m+1)}$.

For the negative sKdV flows, the vacuum projection is a little more subtle. We notice that the Lax pair given by equations (2.25) and (2.26a), is highly dependent of $\partial_{t_{-1}}\eta$ and $\partial_{t_{-1}}\bar{\gamma}$. Therefore, besides specifying the vacuum, $(J, \bar{\chi}) = (\partial_x\eta, \partial_x\bar{\gamma}) = (J_0, \chi_0)$, we must also specify the time derivatives. To do that, we must use the temporal and spatial Miura relations for a specified vacuum. For a zero vacuum solution, we can project (3.16a) and (3.16b) to obtain:

$$\eta_x = \bar{\gamma}_x = \bar{\gamma}_t = 0 \quad \text{and} \quad \eta_t = 2, \tag{F.7}$$

leading to the following vacuum operator

$$A^{\mathrm{sKdV}}_{t_{-1},\mathrm{vac}} = K_1^{(-1)} + K_2^{(-1)} = \mathcal{E}^{(-1)}, \tag{F.8}$$

which only commutes with other zero vacuum flows. However, we can also use the relations (3.18a) and (3.18b), which also holds for non-zero vacuum flows, and obtain

$$\eta_x = v_0^2, \quad \bar{\gamma}_x = -v_0\bar{\psi}_0, \quad \bar{\gamma}_t = \frac{\bar{\psi}_0(v_0 - 1)}{v_0^2} \quad \text{and} \quad \eta_t = \frac{2}{v_0}, \tag{F.9}$$

and then the vacuum operator is given by

$$A^{\mathrm{sKdV}}_{t_{-1},\mathrm{vac}} = \frac{1}{v_0}\left(\Lambda_{-1} - \Pi_{-1}\right) + \Pi_{-1} = J_0^{-\frac{1}{2}}\left(\Lambda_{-1} - \Pi_{-1}\right) + \Pi_{-1}, \tag{F.10}$$

which only commutes with non-zero vacuum operators. For lower flows, the procedure follows the same approach: to identify the negative vacuum operator, we must first establish the vacuum configuration. This will then lead to:

$$A^{\mathrm{sKdV}}_{t_{-2m+1},\mathrm{vac}} = \mathcal{E}^{(-2m+1)}, \tag{F.11}$$

for a zero vacuum, or to

$$A^{\mathrm{sKdV}}_{t_{-2m+1},\mathrm{vac}} = \sum_{i=1}^{m} J_0^{-(m-i)-\frac{1}{2}}\left(c_i\Lambda_{-2i+1} + d_i\Pi_{-2i+1}\right), \tag{F.12}$$

for a non-zero vacuum.

Data availability

No data was used for the research described in the article.

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