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Probing the infrared behavior of the ghost-gluon vertex in Quantum Chromodynamics

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Abstract

The present work concerns the ghost-gluon vertex of Quantum Chromodynamics, which, according to the Taylor identity, has no perturbative corrections to any order, in the Landau gauge and for a specific momentum configuration. We study this vertex for a momentum configuration for which there is no proof of such a result, which is the one with a zero gluon momentum.

The framework we adopt for it is the "Dynamical Perturbation Theory" approach, which consists of inserting some nonperturbative information of the theory into its perturbative expansion. It is a phenomenological attempt only, intended to probe infrared properties of the theory by means of loop calculations.

We have made use of two nonperturbative informations: First, a finite gluon mass, since there are even more indications that the gluon presents a mass, though it is a dynamical one – it intrinsically changes from finite in the infrared, to zero in the ultraviolet. Second, a recent result on the effective charge of Quantum Chromodynamics, which itself considers a dynamical gluon mass.

We calculate the 1-loop correction to the ghost-gluon vertex, aiming at verifying how close to 1 the ghost-gluon vertex renormalization function ($\tilde{Z}_1$) is. The result obtained was positive in this direction: $\tilde{Z}_1$ does not differ much from unity, as shown in Chap.5. Moreover, our result fits better the lattice data when we consider the mentioned effective charge, than when we set the coupling constant as a fit parameter.

Therefore, our somewhat phenomenological approach based on a dynamical gluon mass is, at least, consistent, and supports the approximation $\tilde{Z}_1 \equiv 1$, usually performed in the study of the Schwinger-Dyson equations of Quantum Chromodynamics.

Keywords: Quantum Chromodynamics. Dynamical gluon mass. Schwinger-Dyson equations. Dynamical Perturbation Theory.

Resumo

O presente trabalho diz respeito ao vértice ghost-gluon da Cromodinâmica Quântica, o qual, de acordo com a identidade de Taylor, não possui correções perturbativas no calibre de Landau para uma determinada configuração de momentos. Estudamos este vértice numa configuração para a qual não há provas de um tal resultado, que é para o momento do gluon igual a zero.

Para tanto, adotamos a abordagem da "Teoria Dinâmica de Perturbação", que consiste em inserir características não perturbativas da teoria em sua expansão perturbativa. Trata-se de uma tentativa de caráter fenomenológico apenas, que objetiva explorar propriedades da teoria no domínio infravermelho por meio de cálculos de loop.

Utilizamos duas informações não perturbativas: Primeiramente, uma massa finita do gluon, visto que há consideráveis indicações de que ele apresente uma massa, embora esta seja o que se chama de dinâmica – ela, inerentemente, varia de um valor finito no infravermelho para zero no ultravioleta. Em segundo, um resultado recente acerca da carga efetiva da Cromodinâmica Quântica, na qual é considerada uma massa dinâmica do gluon.

Calculamos então a correção, a 1 loop, do vértice ghost-gluon, com o fim de verificar o quão próxima a função de renormalização ($\tilde{Z}_1$) desse vértice é de 1. O resultado obtido foi positivo neste sentido: $\tilde{Z}_1$ difere pouco de 1, como mostrado no Cap.5. O resultado, ainda, é melhor ajustado aos dados da rede quando consideramos a referida carga efetiva, do que quando usamos a constante de acoplamento como um parâmetro de ajuste.

Portanto, nossa abordagem um tanto fenomenológica, baseada numa massa dinâmica do gluon, é ao menos consistente e dá suporte à aproximação $\tilde{Z}_1 \equiv 1$, comumente efetuada no estudo das equações de Schwinger-Dyson da Cromodinâmica Quântica.

Palavras-chave: Cromodinâmica Quântica. Massa dinâmica do gluon. Equações de Schwinger-Dyson. Teoria Dinâmica de Perturbação.

Chapter 1

Introduction

The main subject of this work is Quantum Chromodynamics (QCD). In order to understand it, a good first step is asking: What is it intended to be about?

First, it is a theory of matter, that constitutes part of nature, whose description is the purpose of Physics – which actually began to address this question with the birth of Quantum Mechanics (QM) in the late 19th century. QM succeeded in characterizing the physics of microscopic scales, being able to account quite well for the atomic spectrum – i.e. the periodic table, and also molecules and their observed properties. All atoms are made of electrons and a nucleus, which in turn was seen to be made of more fundamental entities (called particles), the protons and neutrons.

However, even the protons and neutrons, such as other particles that were known, seemed to have a substructure, leading physicists to suppose that they would be composed of even more fundamental particles. Up to the present, no experiment has shown us these particles. But there has been more and more indirect evidence that these entities would indeed be. They are named quarks and gluons. In fact, quarks and gluons are the basic dynamic constituents of QCD, which is a quantum field theory: a gauge – or, even more specifically, a Yang-Mills theory.

So, the purpose of QCD is to describe the very being of protons, neutrons and all particles alike, and also their behavior – which, as we shall explain, is summarized by the term *strong interactions*. Then, in order to present QCD we shall first give an overview of observed facts and the inferences they led to: Why quarks, and why gluons? Then, why Quantum Field Theory (QFT) seems to be an appropriate framework for this description. Finally, why gauge theories also seem suited to it.

After showing how QCD meets some features of strong interactions, we display more formal details on QFT and QCD themselves. The processes described by the theory are related to its correlation (or Green) functions. The perturbative method expresses, at least in principle, any correlation function as being constructed by more fundamental ones, namely the propagators and vertices of the theory.

QCD contains quarks and gluons propagators, and the vertices among them. However, the quantization of gauge theories requires gauge fixing, and for some gauges it corresponds to introducing a new field, the ghost one. Then, besides the basic correlation functions cited above, QCD has the ghost propagator and the ghost-gluon vertex also as fundamental quantities. As we show in Chap.3, these Green functions, and the others of the theory, are all coupled to each other in a precise way. This is the content of the Schwinger-Dyson equations (SDE), which are an object of study in order to explore the theory.

One purpose of the present work is to deal with some class of results of SDE, concerning the gluon propagator. It is well known in QFT that the propagators are quantities related to the mass of the particles which are the fundamental excitations of the given field. In the mainstream, the gluon is widely considered to be a massless particle, since it is so to any order in perturbation. However, the perturbative method fails, for QCD, in the infrared (IR) domain – that is, at low energies. On the other hand, SDE studies of the gluon propagator led to the conclusion that it could display a massive character. This gluon mass is called dynamical, and it is energy-dependent: it goes to zero in the ultraviolet (UV) – i.e. at high energies, *and* assume a finite value as the energy is decreased.

Now, there comes a technical problem. The perturbative method is a powerful and successful tool in extracting, from the Green functions, physical observables, such as cross sections, and decay rates of particles. But it is so only in its domain of validity – which for QCD excludes the IR region, that is surely an important one, since it would be responsible for confinement (i.e. the fact that we do not observe quarks and gluons alone, but only bound into certain composite particles).

Among the proposals to deal with this issue, one underlies our work: it is the "Dynamical Perturbation Theory" (DPT). It consists in the idea of introducing nonperturbative information into the perturbative series. That is, for example: consider the above result for the gluon propagator – that it would have no mass in the UV, and be massive in the IR. Now take the perturbative expansion of a given correlation function, up to certain number of loops containing gluon propagators. So, one

DPT-like procedure would be to rewrite this perturbative expansion with that complete propagator in place of the original one (called bare) inside the loops. A more rigorous one would be to do the same for all other propagators and vertices within the loop expansion.

However, the former case is what we have done, except for one detail. Instead of considering a gluon mass that depends on the momentum (being finite in the IR and zero in the UV), we consider a gluon propagator with a constant finite mass. This could of course be less accurate. But, like DPT itself, this procedure we have employed is of phenomenological character. It is not proved to be valid, and it is no more than an attempt to explore the theory, trying to obtain some useful information and see its consequences.

So, our methods and limitations having been clarified, let us be more specific. This whole procedure was applied to the ghost-gluon vertex. It is affirmed by the Taylor identity[3] that its complete expression equals its bare one – in other words, as we shall explain in the text, its renormalization function, $\tilde{Z}_1$, is identically equal to 1. Moreover, this identification is an useful simplification for SDE calculations. On the other hand, the Taylor identity has been proven to hold to each order of perturbation, so its validity in the IR domain is unknown.

Then, our work aims at investigating whether the renormalization function of the ghost-gluon vertex is identically equal to 1, within the framework and proposals that we have just described.

Finally, before we start, we should set our conventions: we take $c = \hbar = 1$. That is, speeds are measured in units of the speed of light (in vacuum), and actions are measured in units of the Planck constant (over 2π), so that quantities may be written equivalently as in powers of units of mass, energy, momentum or length.

Chapter 2

Strong interactions and Quantum Chromodynamics

In the early 20th century, it was still lacking in Physics an understanding of matter[4],[5]. At that time QM arose as the theory aiming at that, and it succeeded quite well in many applications in the atomic scale[6],[7]. In a short time, Dirac improved QM to be compatible with Special Relativity[8], then predicting the existence of particles and antiparticles, which was soon confirmed by experiment[9].

Furthermore, Dirac accomplished the quantization of the electromagnetic field, providing great advances in what can be considered one main foundation of QFT. We will discuss QFT in much more detail later, but for now we anticipate that it is a quantum theory that can account for the appearance and disappearance of particles, as in processes characterized by transformation of some particles into others, such as those involved in nuclei decays, for instance.

So, we begin with a brief narrative about processes involving particles, and discoveries that led to the theoretical development that we will focus on.

2.1 The place of strong interactions

2.1.1 Hadrons and their composition

As the endeavor to understand matter established some results, such as the atomic model and the nucleus composition, even more empirical discoveries came up. For their history and more detailed presentations we refer to [9] and [10]. In few words, the observation of particles, and of each process they took part in, led to a classification of both interactions and particles. Besides gravitation, the interactions were regarded as occurring in three kinds: the electromagnetic, the weak, and the strong one[9].

The particles' classification, on the other hand, contains hierarchies. The first two classes are *hadrons* and *leptons,* which differ in the very subject of this chapter: the hadrons are the particles that participate in the strong interactions, while the leptons are those which do not so. The latter are[9]: the electron, the muon, the tau, and their respective neutrinos, and are, up to now, understood to be elementary particles – unlike the hadrons, which are divided up in two subclasses: the mesons and the baryons, according to the substructure they seem to have.

The suggestion that the hadrons would not be elementary came from the observation, during the 1950's and 1960's, of a considerable variety of them, which was put schematically in 1961, by Gell-Mann and, similarly and independently, by Ne'eman, too. This organization is called the *eightfold way*, and is represented below:

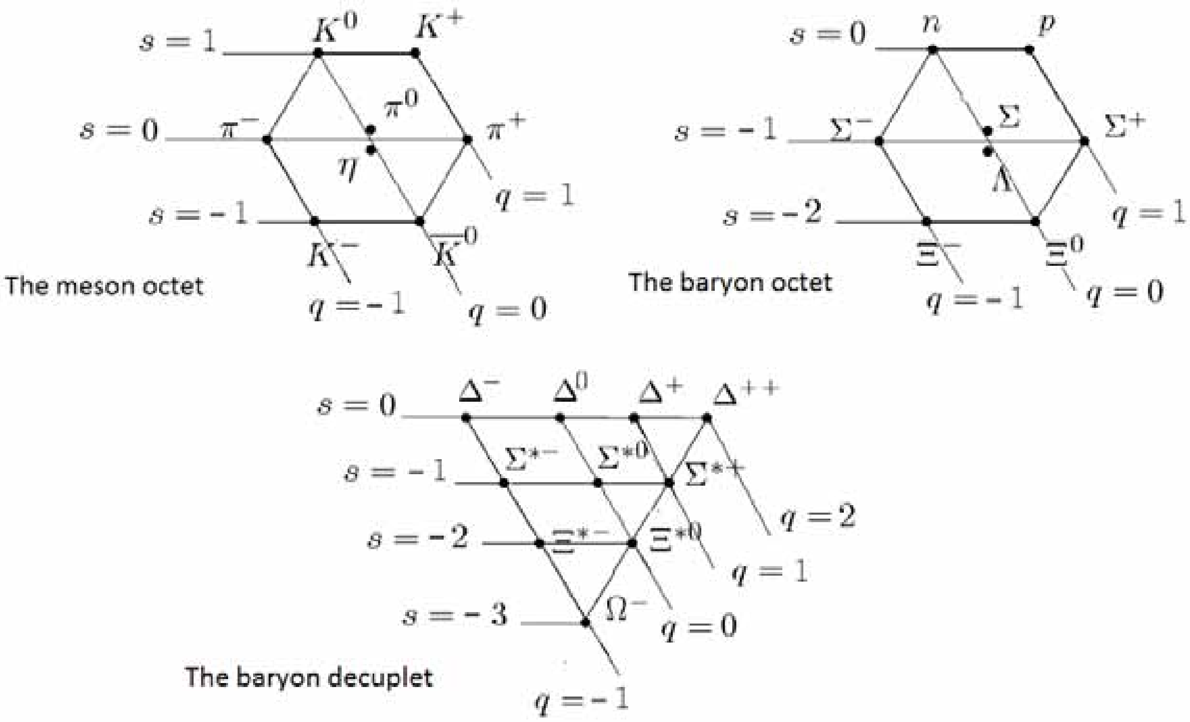

Figure 2.1: The eightfold way, showing the quantum numbers involved in this classification: the electric charge, Q, and the strangeness, S.

With this whole fauna of particles, each one presenting some quantum numbers and lifetime, there comes a natural question: could this diversity be actually explained by some substructure of the hadrons? So, in 1964, Gell-Mann and (also independently) Zweig proposed that there would indeed be such a substructure: all hadrons would be constituted of more basic elements, particles which were coined by the name *quarks*. The original quark model contained three kinds of quarks, then called flavors: *up* (u), *down* (d), and *strange* (s), following the pattern below.

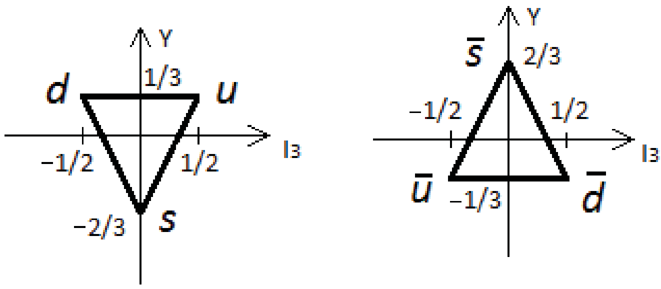

Figure 2.2: The quark model with the flavors, up, down, and strange, showing their respective antiquarks, and their quantum numbers I_3 (the azimuthal component of the isospin I) and hypercharge $Y = 2(Q - I_3)$.

In a short time the three-flavor quark model was extended to contain a fourth flavor, the *charm* (c), which was empirically evidenced in 1974. Later, two more quarks were proposed in the model, the *bottom* (b) and the *top* (t). The former was evidenced in 1977, but the top only in 1995, due to its large mass.

These are the supposedly elementary particles that would constitute the mesons and the baryons. Until the quark model was proposed, the evidence for them was just the great diversity of particles and the reactions they participated in. However, along with particles' discoveries, another strong evidence came in the late 1960's: the appearance of some dynamics inside the hadrons. We now see how this came up.

2.1.2 More indication for quarks – the parton model

The evidence for an internal dynamics within hadrons arose in deep inelastic scattering (DIS) experiments, in which a hadron-hadron, or a lepton-hadron scattering produces a final state with a distinct particle content – for example, the electron-proton inelastic scattering, represented below in its one-photon exchange approximation:

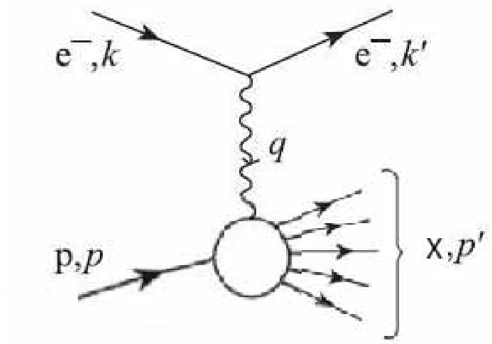

Figure 2.3: Electron-proton inelastic scattering, in the 1-photon exchange approximation.

The X stands for the sum over all possible hadrons in the final state. The cross-section for this process is given by:

$$\frac{\mathrm{d}^2\sigma}{\mathrm{d}Q^2\mathrm{d}\nu} = \frac{\pi\alpha^2}{4k^2\sin^4(\theta/2)}\frac{1}{k_0k_0'}\left[W_2\cos^2(\theta/2) + 2W_1\sin^2(\theta/2)\right] ,$$

where, neglecting the electron mass,

$$\begin{cases} Q^2 = -(k-k')^2 = 2k_0k_0'(1-\cos\theta) , \\ p^2 = M^2 , \qquad (p')^2 = W^2 , \\ p.(k-k') = M\nu , \quad 2M\nu = Q^2 + W^2 - M^2 . \end{cases}$$

The unknown details of the process are carried by the structure functions, W_1 and W_2, dependent on both Q^2 and ν. In 1969, Bjorken[11] obtained the result that these functions, in the deep inelastic region of high Q^2 and high ν, would scale – that is, would depend only on one variable, $Q^2/M\nu$:

$$\begin{cases} Q^2 \to \infty \\ \nu \to \infty \end{cases} , \text{ with } Q^2/\nu \text{ finite} \implies \begin{cases} MW_1(Q^2,\nu) = F_1(Q^2/M\nu) \\ \nu W_2(Q^2,\nu) = F_2(Q^2/M\nu) \end{cases} , \text{ both finite.}$$

This behavior was indeed observed to agree quite well to experiment[12],p.93, calling attention for the lack of understanding of this scaling. Feynman offered an intuitive interpretation introducing the so-called partons: basic constituents of the hadrons, that would individually elastically scatter with the intermediate photon which, at higher energies, would probe shorter distance scales. This is depicted in Fig.2.4:

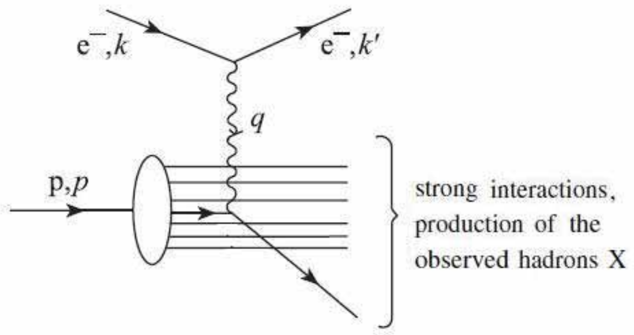

Figure 2.4: Representation of a parton scattering within the proton.

Putting together the parton and the quark model, the scaling functions F are related to probability distributions for each quark, leading to sum rules which consist in constraints among these distributions. For more details we again refer to [12]. It is of our interest to mention one of them:

$$\int_0^1 \mathrm{d}x\, x \left[u\left(x\right) + \bar{u}\left(x\right) + d\left(x\right) + \bar{d}\left(x\right) + s\left(x\right) + \bar{s}\left(x\right)\right] = 1 - \varepsilon \ ,$$

where ($\bar{u}$) u , ($\bar{d}$) d , and ($\bar{s}$) s denote the probability distributions for (anti-)up, (anti-)down, and (anti-)strange quarks, respectively, and ε stands for a participation of more partons than these three quarks ad their antiquarks. Ref. [12] mentions data for which ε is close to $1/2$ for the proton, thus supporting the possibility that it might be composed by more particles, charged or not.

Therefore, the quark parton model served as a dynamic evidence of a substructure of hadrons, not only with quarks, but also suggesting more possibilities of particles that could count as partons – the QCD's gluons, for instance. These are related to the subject of the next subsection, the last of our chosen topics on pre-QCD phenomenology.

2.1.3 The concept of color: phenomenology

We have shown that there was plenty of evidence of a substructure of hadrons, from both their variety and dynamics. Now we mention indicatives for a third aspect of hadrons substructure: an internal degree of freedom.

First, we consider the Δ baryons, discovered in the 1950's. In particular, Δ^{++}, which is composed by three u quarks and has azimuthal spin $3/2$. By its fermionic character, its wave function should be anti-symmetric. Therefore, in order to be so, if the ground state is spatially symmetric there must be another characteristic for the quarks, so that the Δ^{++}'s wave function may be like $\psi_{\Delta^{++}} = \psi_{\text{space}}\psi_{\text{flavor}}\psi_{\text{spin}}\psi_{\text{else}}$, with ψ_{else} anti-symmetric.

Moreover, the ratio between cross-sections

$$R = \frac{\sigma\left(e^- e^+ \to \text{hadrons}\right)}{\sigma\left(e^- e^+ \to \mu^- \mu^+\right)}$$

is found to be proportional to $\sum_{\text{f}} {Q_{\text{f}}}^2$, where the sum is over the quark flavors, each with charge Q_{f}. The experimental results are shown below (figure taken from [13]):

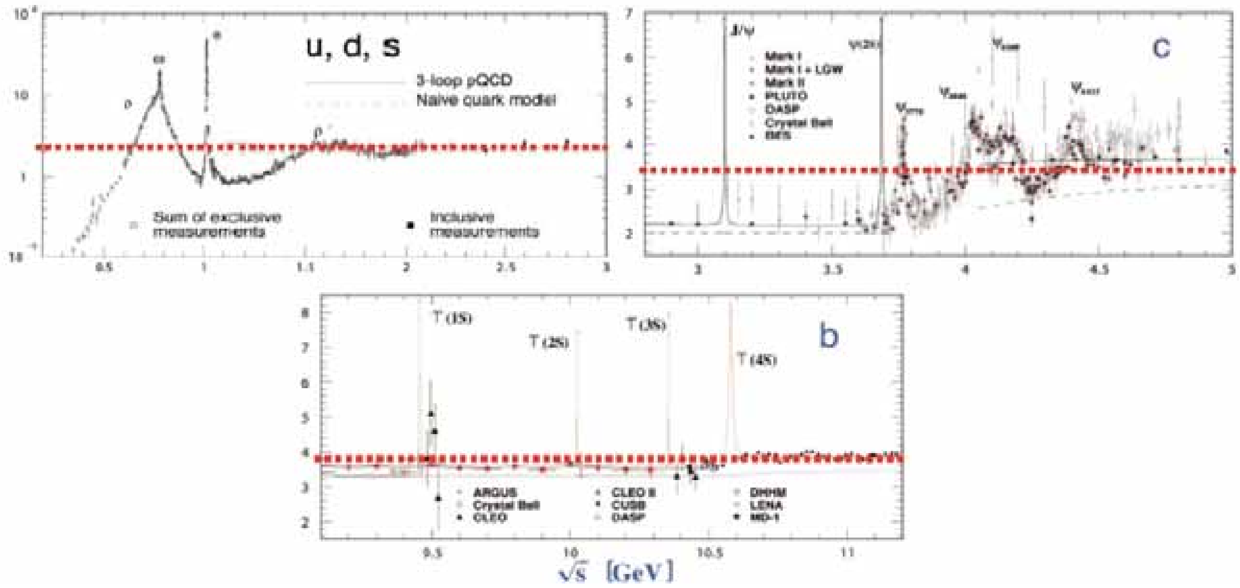

Figure 2.5: The ratio R for increasing energy, with red lines at the thresholds for the flavors in the hadronic final states.

It was found that good agreement with these data is obtained with a multiplicative factor near 3. Similar considerations for the pion decay into two photons, and for anomaly cancellation in the Standard Model[13], are also successful with a factor 3.

Then, if this factor is due to an internal degree of freedom, it seems to be equally common to all flavors, and the data indicates it to assume three possible configurations. These are precisely the QCD colors. We will see in Sect.2.3 that their role in QCD is actually distinct, upgraded, one may say. But up to now the quarks present, therefore, a color symmetry, which is formalized by the group $SU(3)$, so that hadrons consist in the *singlet* representations of this symmetry, in order for the color wave function to always be completely anti-symmetric, consistently with observations that led to the conception just described.

That all hadrons are color singlets is considered to be one condition for confinement – expression that characterizes the empirical fact that quarks are not observed alone, but only combined into mesons and baryons. This recalls an important feature of QFT, namely infrared slavery, to which we now turn.

2.2 Phenomenology meets QFT

In the beginning of this chapter we have pointed out the place that phenomena involving particles take in theoretical physics, which is in the domain of relativistic quantum theory. Since QFT is the ruling paradigm for the study of strong interactions, the first point to be addressed is its suitability as the formalism for particle physics. As introduced in [14], QFT "arose out of our need to describe the ephemeral nature of life." Basically speaking, field theory deals with dynamic variables distributed throughout some space. For this reason, QFT, despite its emergence from the conciliation between QM and Special Relativity, is employed both in many-body and in particle physics. For the latter, QFT meets quite well the dynamics of creation and annihilation of particles.

Furthermore, the also mentioned quantization of the electromagnetic field led to the development of Quantum Electrodynamics (QED)[6],[15], which became a framework for the development of QFT, conducting some advances, such as regularization and renormalization procedures[15], and giving a great success to the perturbative method, since QED predictions agree impressively with experimental data to a great precision[16].

Besides the advances of QFT that were accomplished in QED, and among other ones (such as, for example, spontaneous, and dynamical symmetry breaking[17],[18]), another major advance was accomplished in the context of condensed matter physics: the concept of the renormalization group. In the remaining of this section we shall briefly discuss the benefits it gives to QCD.

2.2.1 Renormalization and running charges

We begin our discussion of renormalization, which is mainly based on [14], Chap.VI.8 (to which we refer for further details), with the following considerations. A QFT is, as will be shown in the next chapter, described by a Lagrangian, that contains the dynamic information of the theory and is a functional of its fields (i.e. degrees of freedom) and parameters (such as masses and coupling constants). It also starts with the hypothesis that these degrees of freedom are, within the whole range of their possible configurations, well described by the local fields present in the Lagrangian.

Renormalization may be related to a restriction in the validity of the local field concept to describe the physics in the distance (in the space where the fields are defined) scales under consideration. As for a quantum theory, to a distance scale it corresponds an energy scale, and it was in fact seen, in the development of QFT, that for an interacting theory the momentum scale in which a given process takes place seems to be a significant quantity.

The standard approach to this subject in particle physics texts begins with the necessity of introducing regularization parameters in order to have well-defined calculations. Then, these parameters are put, as detailed in Sect.3.2, into extra terms – called counterterms, that are set apart in the Lagrangian. When these counterterms have the same form as the terms they were originated from in the former Lagrangian, the theory is said to be renormalizable. Renormalizability is in fact a fundamental requirement for a QFT to be considered successful, or even valid.

This division with counterterms amounts to separating[[14],p.344] the field configurations that would be relevant to the physics for some scale. There is, indeed, always a scale μ with dimension of mass, no matter which regularization procedure is employed. However, the Lagrangian itself must be independent of changes in this scale, so the relevant quantities derived from it (such as the correlation functions, as we shall see in Chap.3). That is,

$$\frac{\mathsf{d}}{\mathsf{d}\mu}(\text{correlation function}) = 0\ ,$$

thus allowing the theory to relate, in a precise way, its own description for different energy scales, and implying a differential equation that contains the masses and couplings, and their dependence on the renormalization scale μ. Our interest focus on the latter.

The dependence of each coupling g_i ($i = 1, \ldots, n$) with the scale is given by the so-called β function:

$$\beta_i\left(g_1, \ldots, g_n\right) = \frac{\mathsf{d}g_i}{\mathsf{d}t}\ ,$$

where $t = \log\left(\mu/\mu_0\right)$, for a specific scale μ_0. So, the β functions describe the couplings (g_i) as quantities depending on t, analogously to a phase space flow given by (β_i). This flow accounts for the behavior of the interactions of the theory with the energy scale.

Before we proceed to discuss two possible behaviors of $g_i\left(\mu\right)$, we note that one content of the renormalization concept is that there *is* a diversity of processes, and this diversity is related to distance – or energy, scales. Intuitively, it may be said that interactions lead to quantum fluctuations, which may affect properties of the interactions themselves. Again citing [14]: "the quantum vacuum is just as much a dielectric as a lump of actual material".

We now explain how this variety of behaviors suits so well what is observed in strong phenomena.

2.2.2 Infrared slavery (or not), and asymptotic freedom

As we have seen, strong interactions, however they are described, *must* present two features: first, confinement, i.e. the fact that no partons (quarks and what else would constitute hadrons) were ever observed, but only their bound states are. Second, scaling, that is, at high energies the partons interact electromagnetically behaving as free particles. So, this is very well met by the diversity we spoke of: for some processes the partons would behave as free from each other, while for other they would be very strongly bound, and there appears to be some significance of energy scales.

Furthermore, these specific behaviors are fulfilled by solutions g for a β function[19]. Although it is still an open question, it is generally supposed that confinement might be described by possible solutions that give increasing g for some momentum range. One, called IR slavery, correspond to a coupling that increases indefinitely, becoming infinite at some value of momentum. Another one is when a negative β goes to 0 for a nonzero g, so that g increases to a stable finite value in the IR.

On the other hand, there are solutions with a high energy behavior that indeed accounts for scaling. This behavior is called asymptotic freedom, and corresponds to a decreasing g that goes to zero as the momentum scale goes to infinity, so that the higher the energy, the weaker the interaction among the fields of the theory – which would be the partons.

Then, we see that a QFT is capable of describing both an increasing (with distance) bind among its basic constituents, and a decreasing (with energy) one until they are almost free from each other. In particular, there is a class of QFT that, besides being renormalizable, is one of the few[19] that present the property of asymptotic freedom: it consists of the non-Abelian Yang-Mills theories (with possible addition of up to, in the $SU(3)$ case, 16 fermions). They are a type of gauge theories, which also meet quite well some features that strong interactions seem to possess, as we show next.

2.3 Phenomenology meets gauge theories

We have seen that quarks seem to have three possible internal configurations, associated to the $SU(3)$ color symmetry. It is widely known the statement of Noether first theorem that continuous global symmetries lead to conservation laws[20]. On the other hand, local symmetries present a distinct consequence which provided an important paradigm to Physics: the gauge principle, according to which the symmetries dictate the dynamics, as we shall see below.

2.3.1 Gauge field theories

A gauge theory starts from imposing a local symmetry to a given system. We consider the general case of a multiplet of fields, which transforms, under the action of a Lie group (usually

$SU(N)$[21]), as:

$$\varphi_i(x) \mapsto \varphi'_i(x) = U_{ij}(x)\,\varphi_j(x)\ , i, j = 1, \ldots, N\ , \tag{2.1}$$

with

$$U(x) = U(\theta^a(x)) = e^{-i\theta^a(x)T^a}$$

an unitary operator of the group's representation, whose Lie algebra generators are T^a, $a = 1, \ldots, d$. The algebra has dimension d, and is defined by the operation and the structure constants $\left[T^a, T^b\right] = if^{cab}T^c$. The local operator $U(x)$ is parametrized by $\theta^a(x)$, and x belongs to the Minkowski space. For infinitesimal transformations, the variation of each field is

$$\delta\varphi = i\theta^a(x)\,T^a\varphi(x) + \mathcal{O}\left(\theta^2\right)\ , \tag{2.2}$$

where, from now on, we omit the multiplet, group, or Lorentz indices, when understood and more convenient.

For a Lagrangian $\mathcal{L}$ that depends on the field φ and on its first derivatives only, (2.2) implies that

$$\delta\mathcal{L} = -i\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}T^a\varphi\,(\partial_\mu\theta^a) + \mathcal{O}\left(\theta^2\right)\ ,$$

so it is clear that the variation of the Lagrangian under the local symmetry comes from the term $\delta(\partial_\mu\varphi)$, that is, from the expected fact that the derivatives of the fields do not transform in the same way as the fields themselves. So, in order for the action to be invariant, i.e. for the dynamics of the system to present this local symmetry, one proposes to change the ordinary derivative, ∂, to the so-called covariant derivative[1], D, that must act on φ in a way such that $D\varphi$ transforms in the same way as φ:

$$D_\mu\varphi \mapsto U(D_\mu\varphi)\ \Leftarrow\ D \mapsto UDU^\dagger\ . \tag{2.3}$$

It is then attempted to add a linear combination of the generators T to ∂:

$$D_\mu = \partial_\mu - igA^a_\mu(x)\,T^a\ , \tag{2.4}$$

therefore introducing a new, called gauge field, $A_\mu(x) \equiv A^a_\mu(x)\,T^a$, which is then coupled to φ

[1] More details on the subject of this subsection, in particular the covariant derivative, are presented in [23],Chap.13.

with intensity given by g. Then, $\delta D = \delta A$ under the gauge group, and (2.3) implies that

$$A_\mu \mapsto U\left(A_\mu\right) U^\dagger + \frac{i}{g} U\left(\partial_\mu U^\dagger\right) \,, \tag{2.5}$$

which for infinitesimal transformations assumes the form

$$\delta A_\mu = -\frac{1}{g}\partial_\mu \theta^a T^a - i\theta^a \left[T^a, A_\mu\right] + \mathcal{O}\left(\theta^2\right) \,, \tag{2.6}$$

meaning that the gauge field belongs to the adjoint representation of the group's Lie algebra[22].

Thus, by $\partial \mapsto D$ it is obtained a Lagrangian that is invariant under the local transformations (2.1) and (2.5). Since the requirement of local invariance introduces the gauge field, one may explore its dynamics, by introducing kinetic terms, and possibly other interaction terms, still preserving invariance under (2.5). Besides gauge invariance, there are other requirements, such as renormalizability[19], which restrict the possible terms for a Lagrangian to be viable for a quantum theory. One is

$$\mathcal{L}\left(\varphi, \partial\varphi, A, \partial A\right) = -\frac{1}{4} F^a_{\mu\nu} F^{a\,\mu\nu} + \mathcal{L}_\varphi \,, \tag{2.7}$$

where $-F^2/4$ is the Yang-Mills[2] term, F is the field strength given by

$$F_{\mu\nu} := \frac{i}{g}\left[D_\mu, D_\nu\right] = \partial_\mu A_\nu - \partial_\nu A_\mu - ig\left[A_\mu, A_\nu\right] \,, \tag{2.8}$$

and $\mathcal{L}_\varphi$ is the original Lagrangian for φ, with the covariant derivative in place of the usual one.

This replacement of ∂ by D is called the gauge principle. It allows the promotion of a global symmetry to be local, adding new fields that interact with the original ones in a manner specified by (2.4), which clearly depends on which group representation the fields φ_i belong to.

In fact, the gauge principle is one main paradigm in Particle Physics, and accounts for the descriptions in vogue for both the strong, and the electroweak interactions. QED, for instance, is a quantized Yang-Mills theory with the Abelian group $U(1)$, and the gauge field is precisely the electromagnetic, whose quantum is the photon. Now we direct ourselves to QCD.

2.3.2 Quantum Chromodynamics

QCD is based on the promotion of $SU(3)$ color symmetry to a local one. It is intended to

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2 in reference to the pioneering work [25].

described the dynamics of the quarks that are supposed to constitute the hadrons. So, the QCD version of (2.7) is

$$\mathcal{L} = -\frac{1}{4}F^2 + \bar{\psi}^j \left(iD - m_j\right)\psi^j \ ,$$

where each quark of flavor j has mass m_j and belongs to the fundamental representation[24] of $SU(3)$, F is given by (2.8), and the quanta of the field A are called the gluons. So, this dynamics of quarks and gluons is expected to be able to describe the strong interaction and all its features.

Nevertheless, quantization is required for a theory that aims at phenomena at such small scales. However, the quantization of gauge field theories is a complex, and still open matter. This issue is addressed in Appendix B, in which we show the Faddeev-Popov method[3]. It leads to the presence, at the theoretical (linguistic, actually) level only, of the so-called ghost fields, that are both Grassmann variables and Lorentz scalars, denoted by $\bar{\chi}^a, \chi^a$. They, as the gauge fields, also belong to the adjoint representation of the group. Then, QCD is described by the effective Lagrangian

$$\mathcal{L} = -\frac{1}{4}F^2 + \bar{\psi}^j \left(iD - m_j\right)\psi^j + \left(\partial^\mu \bar{\chi}^a\right)\left(D_\mu \chi^a\right) - \frac{1}{2\eta}\left(\partial_\mu A^{a\mu}\right)\left(\partial_\nu A^{a\nu}\right) \ , \qquad (2.9)$$

where the covariant derivative in the adjoint representation is

$$D_\mu \chi = \partial_\mu \chi - ig\left[A_\mu, \chi\right] \ ,$$

and the last term is a gauge-fixing to the class of covariant gauges.

Moreover, as for a quantum field theory, this Lagrangian must be renormalized, but for convenience and clarity we hold it over to the next chapter. For now, we write more explicitly the terms in (2.9):

$$\begin{aligned}\mathcal{L} \ = \ & -\frac{1}{4}\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) + \frac{ig}{2}\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)\left[A^\mu, A^\nu\right] + \frac{g^2}{4}\left[A_\mu, A_\nu\right]\left[A^\mu, A^\nu\right] \\ & + \bar{\psi}^j \left(i\partial\!\!\!/ - m_j\right)\psi^j + g\bar{\psi}^j A_\mu \psi^j + \left(\partial^\mu \bar{\chi}\right)\left(\partial_\mu \chi\right) - ig\left(\partial^\mu \bar{\chi}\right)\left[A_\mu, \chi\right] - \frac{1}{2\eta}\left(\partial . A\right)^2 \ . \qquad (2.10)\end{aligned}$$

So we note that the gluons couple not only to the quarks, but also to themselves (as explained in Appendix B, the ghost-gluon interaction terms actually consist in gluon self-interactions). This

[3] As the quantization of gauge theories stands on its own a whole matter of study, the discussion of it could not be but incomplete, so we shall restrict ourselves to briefly mentioning problems on the Faddeev-Popov quantization procedure.

gives the theory a much complex dynamics, which will be made somewhat precise at the end of the next chapter when we set the SDEs for QCD.

2.4 A not so close meeting

The just mentioned complexity of QCD dynamics might be precisely what would enrich the theory in such a way that it would present all properties required for a theory to describe strong phenomena. However, this is simply unknown yet: the perturbative method, whether as an expansion on the coupling or on the Planck constant, has intrinsic limitations and is unable to reach the IR region, which is essential for the study of confinement. Moreover, there is no solution for QCD: even semi-analytical methods seem to be far from giving complete success to the theory, and possibly the furthest indication of success comes from lattice QFT, which computes transition amplitudes and correlation functions in an approximated space-time lattice[26].

Besides the fact that QCD is not yet completely solved, there is another important problem. Although it seems appropriate to speak of quarks and gluons, there is an essential open gap between QCD and the object it aims to describe: the physics of hadrons. Empirically, one observes amplitudes of hadronic processes, while with QCD one works with quarks and gluons correlation functions, and, as we shall see next, begins with asymptotic states of quarks and gluons. Thus far, field theory is unable to build this bridge, and models are needed to describe the hadronization processes[27].

Clearly, these two problems are related to each other. But if they might in fact be solved unitedly is still a great open question, and object of plenty of work in the present. Most of them work, in a variety of approaches, within QCD, that is in fact the ruling paradigm for the study of strong interactions. Therefore, within this paradigm, there is a certain need for the study of the IR region of QCD, for which one must employ non-perturbative methods – in the sense of being not only an expansion in powers of any coupling constant. We will concern ourselves with two of these methods: lattice QFT, and the Schwinger-Dyson equations (SDEs), the latter being our focus.

Chapter 3

Quantum field theory and Schwinger-Dyson equations

As shall be described, the SDE are an infinite set of integral equations which couple the various Green functions of the theory. So we first discuss Green functions, starting with the reason why they are relevant: their direct relation to transition amplitudes. For simplicity, we will discuss the case of a scalar field.

3.1 Reduction (LSZ) formula

In a quantum theory one basic object to study the dynamics of some system is the S-matrix, which is determined by the system's Hamiltonian and is given by the transition amplitudes

$$\langle p_1, \ldots, p_n \text{ out}|\, q_1, \ldots, q_l \text{ in}\rangle \equiv \langle p_1, \ldots, p_n \text{ in}|\, S \,|q_1, \ldots, q_l \text{ in}\rangle\,.$$

In the standard approach, one starts with the hypothesis that the above states are generated, from an unique vacuum state $|\Omega\rangle$, by the Heisenberg field $\varphi(x)$ which satisfy the asymptotic (adiabatic) condition

$$\lim_{x_0\to\mp\infty} \langle\alpha|\,\varphi(x)\,|\beta\rangle = Z^{1/2}\,\langle\alpha|\,\varphi_{\substack{\text{in}\\ \text{out}}}(x)\,|\beta\rangle \tag{3.1}$$

for all $|\alpha\rangle\,, |\beta\rangle$ eigenstates of the free field $\varphi_{\text{in}}(x) = S\varphi_{\text{out}}(x)\,S^{-1}$. The multiplicative factor $\sqrt{Z}$ is just the component of $\varphi(x)\,|\Omega\rangle$ along $\varphi_{\substack{\text{in}\\ \text{out}}}(x)\,|\Omega\rangle$.

As a consequence of this *hypothesis*, any transition amplitude between on-shell momentum eigenstates is related to the correlation functions of the field $\varphi(x)$,

$$G^{(n)}(x_1, \ldots, x_n) := \langle\Omega|\,\mathrm{T}\left[\varphi(x_1)\cdots\varphi(x_n)\right]|\Omega\rangle \tag{3.2}$$

where T is the time-ordering operator. This relation is the so-called reduction[4] formula:

$$\begin{aligned}\langle p_1, \ldots, p_n \text{ out} | \, q_1, \ldots, q_l \text{ in} \rangle &= \left(\frac{i}{\sqrt{Z}}\right)^{n+l} \int \mathsf{d}^D x_1 \cdots \mathsf{d}^D y_n \, e^{i\left(\sum_{j=1}^{n} p_j \cdot y_j - \sum_{k=1}^{l} q_k \cdot x_k\right)} \times \\ &\times \left(\Box_{x_1} + m^2\right) \cdots \left(\Box_{y_n} + m^2\right) G^{(n)}\left(y_1, \ldots, y_n, x_1, \ldots, x_l\right) \\ &+ \text{disconnected terms,} \end{aligned} \tag{3.3}$$

where $m^2 = p_j^2 = q_k^2 \; \forall j, k$, and the disconnected terms are those in which at least one particle in the process is not affected by it, that is, its initial and final states are the same.

By means of the reduction formula, the calculation of transition amplitudes amounts to determining certain Green functions of the interacting theory. We therefore proceed to a discussion of the relevant types of Green functions in the context of this work.

3.2 Generating functionals and Green functions

In the quantum theory path integral formalism in D dimensions, the vacuum expectation value (3.2) is related to the following functional integral over the dynamic variables

$$\mathcal{Z}[J] = \int \mathcal{D}\varphi \exp i \int \mathsf{d}^D x \; \{\mathcal{L}[\varphi(x)] + J(x)\varphi(x)\} \tag{3.4a}$$

$$= \mathcal{Z}[0] \sum_{n=0}^{\infty} \frac{i^n}{n!} \int \mathsf{d}^D x_1 \cdots \mathsf{d}^D x_n \, J(x_1) \cdots J(x_n) \, G^{(n)}(x_1, \ldots, x_n) \tag{3.4b}$$

which is named the generating functional for the Green functions, that are then explicitly given by:

$$G^{(n)}(x_1, \ldots, x_n) = \frac{1}{\mathcal{Z}[0]} \left. \frac{\delta^n \mathcal{Z}[J]}{i\delta J(x_1) \cdots i\delta J(x_n)} \right|_{J=0} . \tag{3.5}$$

Their Fourier transforms are the Green functions in momentum space, denoted as:

$$\begin{aligned}\tilde{G}^{(n)}(p_1, \ldots, p_n) &= \int \mathsf{d}^D x_1 \cdots \mathsf{d}^D x_n \, e^{-i \sum_{j=1}^{n} p_j \cdot x_j} G^{(n)}(x_1, \ldots, x_n) \\ &= (2\pi)^D \delta^D \left(\textstyle\sum_{j=1}^{n} p_j\right) G^{(n)}(p_1, \ldots, p_n) . \end{aligned} \tag{3.6}$$

In particular, $G^{(2)}(p, -p)$ is called the complete propagator, and near the mass shell $p^2 \sim m^2$ it approaches $iZ/\left(p^2 - m^2\right)$.

[4] or LSZ, for H. Lehmann, K. Symanzik, and W. Zimmermann.

Another type of Green functions consists of the connected ones, corresponding to Feynman diagrams which are not separable in two or more disjoint diagrams. That is, the connected Green functions $G_c^{(n)}$ are related to processes in which no subsets of particles interact independently from each other. They are generated by the functional $\mathcal{W}[J]$ defined by

$$\mathcal{Z}[J] = \exp \mathcal{W}[J] , \tag{3.7}$$

so that

$$G_c^{(n)}(x_1, \ldots, x_n) := \left. \frac{\delta^n \mathcal{W}[J]}{i\delta J(x_1) \cdots i\delta J(x_n)} \right|_{J=0} . \tag{3.8}$$

More details on this definition, and examples of some low-order connected Green functions are given in [15],Sect.5-1-5, and [28],pp.398-400, respectively.

Now, dividing any Green function by the two-point functions of each one of its external lines defines the amputated Green functions:

$$\begin{cases} G_a^{(n)}(p_1, \ldots, p_n) := \left(\prod_{j=1}^n \left[G^{(2)}(p_j, -p_j) \right]^{-1} \right) G^{(n)}(p_1, \ldots, p_n) \ , \ n > 2 \\ G_a^{(2)}(p, -p) := \dfrac{1}{G^{(2)}(p, -p)} . \end{cases} \tag{3.9}$$

Recalling that near the mass shell $G^{(2)}(p, -p) = iZ/(p^2 - m^2)$, we see that the amputated functions give a compact expression for the reduction formula:

$$\begin{aligned} \langle p_1, \ldots, p_n \text{ out} | \, q_1, \ldots, q_l \text{ in} \rangle &= \left(iZ^{-1/2} \right)^{n+l} \left(-q_1^2 + m^2 \right) \cdots \left(-(-p_n)^2 + m^2 \right) \times \\ &\quad \times G^{(n+l)}(-p_1, \ldots, -p_n, q_1, \ldots, q_l) \\ &= Z^{(n+l)/2} G_a^{(n+l)}(-p_1, \ldots, -p_n, q_1, \ldots, q_l) \ . \end{aligned} \tag{3.10}$$

For Grassmann fields[5], we define the even order Green functions as

$$G_c^{(2n)}(x_1 \ldots, x_n, y_1, \ldots, y_n) := \left. \left(\frac{\delta}{i\delta\bar{\eta}(x_1)} \cdots \frac{\delta}{i\delta\bar{\eta}(x_n)} \frac{i\delta}{\delta\eta(y_1)} \cdots \frac{i\delta}{\delta\eta(y_n)} \mathcal{W}[\eta, \bar{\eta}] \right) \right|_{\eta, \bar{\eta}=0} . \tag{3.11}$$

Moreover, we consider the Legendre transform[6] of the generating functional for the connected

5 Grassmann derivatives anticommute.

6 It consists in $f(x) \longrightarrow g(f'(x)) := f - xf'$, then satisfying $\mathrm{d}g = -x\mathrm{d}f'$, while $\mathrm{d}f = f'\mathrm{d}x$ [17].

Green functions $\mathcal{W}[J]$. Given

$$\phi(x,J) := \frac{\delta\mathcal{W}[J]}{i\delta J(x)} , \tag{3.12}$$

we have

$$i\Gamma[\phi] := \left[\mathcal{W}[J] - i\int \mathsf{d}^D x\ J(x)\,\phi(x)\right]\Bigg|_{J(x)=J(x,\phi)} , \tag{3.13}$$

where the inverse $J(x,\phi)$ is well defined as long as $\left.\frac{\delta\mathcal{W}[J]}{\delta J(x)}\right|_{J=0} = i\phi(x,0) \equiv 0$, and consequently

$$i\frac{\delta\Gamma[\phi]}{\delta\phi(y)} = \int \mathsf{d}^D x\ \left(\frac{\delta\mathcal{W}[J]}{i\delta J(x)} - i\phi(x)\right)\frac{\delta J(x)}{\delta\phi(y)} - iJ(y,\phi)\ ,$$

$$\Longrightarrow\quad J(x,\phi) \equiv -\frac{\delta\Gamma[\phi]}{\delta\phi(x)} . \tag{3.14}$$

The functional $\Gamma[\phi]$ is traditionally called the effective action, and generates another set of Green functions, the proper (or 1PI, i.e. 1-particle irreducible) ones, by:

$$G_p^{(n)}(x_1,\ldots,x_n) := \left.\frac{\delta^n\Gamma[\phi]}{\delta\phi(x_1)\cdots\delta\phi(x_n)}\right|_{\phi=0} , \tag{3.15}$$

while for Grassmann fields

$$G_p^{(2n)}(x_1,\ldots,x_n,y_1,\ldots,y_n) := \left[\left(-\frac{\delta}{\delta\bar\phi(x_1)}\right)\cdots\left(-\frac{\delta}{\delta\bar\phi(x_n)}\right)\frac{\delta}{\delta\phi(y_1)}\cdots\frac{\delta}{\delta\phi(y_n)}\Gamma\left[\phi,\bar\phi\right]\right]\Bigg|_{\phi=0} . \tag{3.16}$$

From the very definition of the field ϕ,

$$\frac{\delta}{\delta\phi(y)}\frac{\delta\mathcal{W}[J]}{\delta J(x)} = i\delta^D(x-y)\ ,$$

$$\Rightarrow\quad \delta^D(x-y) = -i\int \mathsf{d}^D z\ \frac{\delta J(z)}{\delta\phi(y)}\frac{\delta^2\mathcal{W}[J]}{\delta J(z)\,\delta J(x)} = i\int \mathsf{d}^D z\ \frac{\delta^2\Gamma[\phi]}{\delta\phi(y)\,\delta\phi(z)}\frac{\delta^2\mathcal{W}[J]}{\delta J(z)\,\delta J(x)}\ , \tag{3.17}$$

that is, $G_p^{(2)}(y,z)$ is the inverse (by convolution) of the propagator $-iG^{(2)}(z,x)$. On the other hand, for Grassmann fields the corresponding relation is given by

$$\delta_{\alpha\beta}\delta^D(x-y) = \frac{\delta}{\delta\phi_\alpha(x)}\phi_\beta(y) = \frac{\delta}{\delta\phi_\alpha(x)}\frac{\delta\mathcal{W}}{i\delta\bar\eta_\beta(y)}$$

$$\Rightarrow \ \delta_{\alpha\beta}\delta^D\left(x-y\right) = \int \mathsf{d}^D z \left(-\frac{\delta^2\Gamma}{\delta\phi_\alpha\left(x\right)\delta\bar{\phi}_\gamma\left(z\right)}\frac{\delta}{\delta\eta_\gamma\left(z\right)} + \frac{\delta^2\Gamma}{\delta\phi_\alpha\left(x\right)\delta\phi_\gamma\left(z\right)}\frac{\delta}{\delta\bar{\eta}_\gamma\left(z\right)}\right)\frac{\delta\mathcal{W}}{i\delta\bar{\eta}_\beta\left(y\right)}$$

$$= i\int \mathsf{d}^D z \, \frac{\delta^2\Gamma}{\delta\phi_\alpha\left(x\right)\delta\bar{\phi}_\gamma\left(z\right)}\frac{\delta^2\mathcal{W}}{\delta\eta_\gamma\left(z\right)\delta\bar{\eta}_\beta\left(y\right)} \ . \tag{18}$$

In either case, this leads to the following relation in momentum space:

$$G_p^{(2)}\left(p,-p\right) = \frac{i}{G_c^{(2)}\left(p,-p\right)} \ . \tag{3.19}$$

More examples of proper Green functions are shown in [28],Sect.12.7 and [15],Sect.6-2-2.

Now we have a whole package of concepts to work with. Before moving on to discuss the SDE, we briefly introduce the idea of renormalized Lagrangians.

The factor Z in (3.1) actually corresponds to the renormalization of the Green functions of the theory, which is also expressed in (3.3) and (3.10). An alternative procedure is to renormalize them from the beginning, renormalizing the source J,

$$J\left(x\right) \longrightarrow Z^{1/2}J\left(x\right) \ ,$$

in the derivatives, and then obtaining the renormalized Green functions

$$\begin{aligned} G_R^{(n)}\left(x_1,\ldots,x_n\right) &= \frac{\left(-i\right)^n}{\mathcal{Z}\left[0\right]}\left.\frac{\delta^n\mathcal{Z}\left[\sqrt{Z}J\right]}{\delta\left(\sqrt{Z}J\left(x_1\right)\right)\cdots\delta\left(\sqrt{Z}J\left(x_n\right)\right)}\right|_{J=0} \\ &= \frac{\left(-i\right)^n}{\mathcal{Z}\left[0\right]}\frac{1}{\sqrt{Z}^n}\left.\frac{\delta^n\mathcal{Z}\left[J\right]}{\delta J\left(x_1\right)\cdots\delta J\left(x_n\right)}\right|_{J=0} \\ &= Z^{-n/2}G^{(n)}\left(x_1,\ldots,x_n\right) \ , \end{aligned} \tag{3.20}$$

while

$$G_{c\ R}^{(n)} = Z^{-n/2}G_c^{(n)} \ , \tag{3.21}$$

$$G_{a\ R}^{(n)} = Z^{n/2}G_a^{(n)} \ , \tag{3.22}$$

$$G_{p\ R}^{(n)} = Z^{n/2}G_p^{(n)} \ , \tag{3.23}$$

which, therefore, already takes the Z factors of the reduction formula (3.10) into account.

In what concerns the generating functionals $\mathcal{Z}\left[J\right]$ and $\mathcal{W}\left[J\right]$, the above redefinition of J is equivalent to the redefinition of the field φ,

$$\varphi(x) \longrightarrow Z^{-1/2}\varphi(x) \ ,$$

and, for the effective action $\Gamma[\phi]$, to $\phi(x) \longrightarrow Z^{-1/2}\phi(x)$. The Lagrangian may then be rewritten in terms of this redefined field, leading to an expression which contains the counterterms mentioned in Sect.2.2. To exemplify it, we consider the scalar φ^4 theory, with the Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\varphi)(\partial^\mu\varphi) - \frac{1}{2}m_0^2\varphi^2 - \lambda_0\varphi^4 \ ,$$

which as a function of the renormalized field reads

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}Z(\partial_\mu\varphi)(\partial^\mu\varphi) - \frac{1}{2}m_0^2 Z\varphi^2 - \lambda_0 Z^2\varphi^4 \\ &= \frac{1}{2}(\partial_\mu\varphi)(\partial^\mu\varphi) - \frac{1}{2}m^{\,2}\varphi^2 - \lambda\varphi^4 + \mathcal{L}_c \ , \end{aligned}$$

where

$$\mathcal{L}_c = \frac{1}{2}(Z-1)(\partial_\mu\varphi)(\partial^\mu\varphi) - \frac{1}{2}\delta_m\varphi^2 - \delta_\lambda\varphi^4 \ ,$$

with $\delta_m = m_0^2 Z - m^2$ and $\delta_\lambda = \lambda_0 Z^2 - \lambda$.

Following this procedure, the whole renormalized theory may be constructed with the Lagrangian which now contains the counterterms in $\mathcal{L}_c$, that are usually treated as additional interactions of the theory.

3.3 The Schwinger-Dyson equations

With the whole machinery at hands, we first state, for a scalar field, a general differential equation which is the starting point for the derivation of the SDE. It all starts with the *hypothesis* that the identity

$$\int \mathcal{D}\varphi \frac{\delta}{\delta\varphi} \equiv 0 \qquad (3.24)$$

holds in the theory under consideration,

$$\Longrightarrow \quad -i\int \mathcal{D}\varphi \frac{\delta}{\delta\varphi} \exp i\int \mathsf{d}^D x \ (\mathcal{L} + J\varphi) = \int \mathcal{D}\varphi \left[\frac{\delta\mathcal{L}}{\delta\varphi}(\varphi) + J\right] \exp i\int \mathsf{d}^D x \ (\mathcal{L} + J\varphi) = 0 \ .$$

Now, we consider $\dfrac{\delta\mathcal{L}}{\delta\varphi}$ not as a functional of $\varphi(x)$, but of the functional derivative $\dfrac{\delta}{i\delta J(x)}$, which enables us to write the above equation as:

$$\begin{aligned}\left[\frac{\delta\mathcal{L}}{\delta\varphi}\left(\frac{\delta}{i\delta J(x)}\right)+J(x)\right]\mathcal{Z}[J] &= \int\mathcal{D}\varphi\left[\frac{\delta\mathcal{L}}{\delta\varphi}\left(\frac{\delta}{i\delta J(x)}\right)+J(x)\right]\exp i\int \mathsf{d}^D x\ (\mathcal{L}+J\varphi)\\ &= \int\mathcal{D}\varphi\left[\frac{\delta\mathcal{L}}{\delta\varphi}(\varphi)+J\right]\exp i\int \mathsf{d}^D x\ (\mathcal{L}+J\varphi)\ ,\end{aligned}$$

$$\therefore\quad \left[\frac{\delta\mathcal{L}}{\delta\varphi}\left(\frac{\delta}{i\delta J(x)}\right)+J(x)\right]\mathcal{Z}[J]=0\ . \tag{3.25}$$

which in the sequence we apply to QCD.

3.3.1 Quantum Chromodynamics

The first step in order is to write down the renormalized Lagrangian for QCD. From (2.10), making explicit all indices, we have:

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4}Z_3\left(\partial_\mu A^a_\nu-\partial_\nu A^a_\mu\right)\left(\partial^\mu A^{a\nu}-\partial^\nu A^{a\mu}\right)-\frac{1}{2\eta}\left(\partial.A^a\right)\left(\partial.A^a\right)+\tilde{Z}_3\left(\partial^\mu\bar{\chi}^a\right)\left(\partial_\mu\chi^a\right)\\ &\quad +Z_2\bar{\psi}^j\left(i\partial\!\!\!/-Z_{m_j}m_j\right)\psi^j-\tilde{Z}_1 g f^{abc}\left(\partial^\mu\bar{\chi}^a\right)\chi^b A^c_\mu+Z_{1F}g\bar{\psi}^j T^a_{jk}A\!\!\!/^a\psi^k\\ &\quad -Z_1\frac{g}{2}f^{abc}\left(\partial_\mu A^a_\nu-\partial_\nu A^a_\mu\right)A^{b\mu}A^{c\nu}-Z_4\frac{g^2}{4}f^{abe}f^{cde}A^a_\mu A^b_\nu A^{c\mu}A^{d\nu}\ ,\end{aligned} \tag{3.26}$$

where

$$\tilde{Z}_1:=Z_g\tilde{Z}_3 {Z_3}^{1/2}\ ,\ Z_{1F}:=Z_g Z_2 {Z_3}^{1/2}\ ,\ Z_1:=Z_g {Z_3}^{3/2}\ ,\ Z_4:={Z_g}^2 {Z_3}^2\ , \tag{3.27}$$

which are relations that will be discussed in the next section. We denote the corresponding sources as

$$\begin{cases} J^{a\mu}(x)=-\dfrac{\delta\Gamma}{\delta a^a_\mu(x)}\ , \quad \bar{\eta}^j(x)=\dfrac{\delta\Gamma}{\delta\phi^j(x)}\ , \quad \eta^j(x)=-\dfrac{\delta\Gamma}{\delta\bar{\phi}^j(x)}\ ,\\ \bar{\lambda}^a(x)=\dfrac{\delta\Gamma}{\delta c^a(x)}\ , \quad \lambda^a(x)=-\dfrac{\delta\Gamma}{\delta\bar{c}^a(x)}\ ,\end{cases} \tag{3.28}$$

with the fields given by

$$\begin{cases} a^{a\mu}(x)=\dfrac{\delta\mathcal{W}}{i\delta J^a_\mu(x)}\ , \quad \phi^j(x)=\dfrac{\delta\mathcal{W}}{i\delta\bar{\eta}^j(x)}\ , \quad \bar{\phi}^j(x)=\dfrac{i\delta\mathcal{W}}{\delta\eta^j(x)}\ ,\\ c^a(x)=\dfrac{\delta\mathcal{W}}{i\delta\bar{\lambda}^a(x)}\ , \quad \bar{c}^a(x)=\dfrac{i\delta\mathcal{W}}{\delta\lambda^a(x)}\ .\end{cases} \tag{3.29}$$

A detailed diagrammatic approach to obtaining SDEs is given in [29] for QED. We have formally derived the SDE for the ghost propagator, this calculation is shown in Appendix C. However, it is sufficient for our purposes to only sketch the result, and then show diagrammatically some SDEs of QCD. The general procedure, considering (3.25), is to start with the specific version of

$$\frac{\delta}{\delta J(y)}\left\{\left[\frac{\delta\mathcal{L}}{\delta\varphi}\left(\frac{\delta}{i\delta J(x)}\right)+J(x)\right]\mathcal{Z}[J]\right\}=0\,, \tag{3.30}$$

possibly applying the derivative more times, and manipulate it until it results in an integral equation such as

$$\begin{aligned} 0 \;=\; & -\tilde{Z}_3\partial^2\delta^{af}\delta^D(x-z')+\Bigg[\frac{\delta^2\Gamma}{\delta\bar{c}^f(z')\,\delta c^a(x)}+\tilde{Z}_1 g f^{abc}\partial^\mu\int \mathrm{d}^D z\,\mathrm{d}^D y \\ & \frac{\delta^2\mathcal{W}}{\delta J^{c\mu}(x)\,\delta J^d_\nu(z)}\frac{\delta^3\Gamma}{\delta a^{d\nu}(z)\,\delta\bar{c}^f(z')\,\delta c^{e'}(y)}\frac{\delta^2\Gamma}{\delta\bar{c}^{e'}(y)\,\delta c^b(x)}^{-1}\Bigg]\Bigg|_{\lambda,\bar{\lambda},c,\bar{c},J,a=0}\,, \end{aligned} \tag{3.31}$$

which in momentum space is written as

$$\begin{aligned} G_p^{(2)af}(p,-p) \;=\; & -\tilde{Z}_3\delta^{af}p^2-\tilde{Z}_1 g f^{abc}\int\frac{\mathrm{d}^D q}{(2\pi)^D}p^\mu G^{(2)cd}_{c\,\mu\nu}(p+q,-p-q)\times \\ & \times G_p^{(3)def\,\nu}(p+q,-p,-q)\,G_c^{(2)be}(q,-q)\;, \end{aligned} \tag{3.32}$$

and is represented by the diagram below, where connected Green functions are designed as blank circles, and the proper ones as shaded circles.

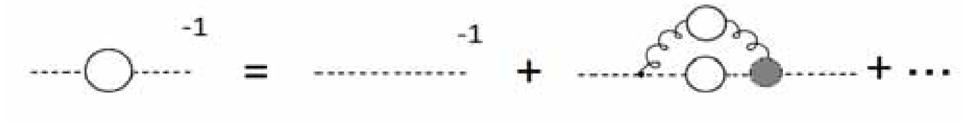

Figure 3.1: Representation of the SDE for the ghost propagator.

In this figure, and in the other ones to follow, we show only the 1PI contributions, and the further terms (2PI and so on) are represented by "$+\cdots$".

So, the full (or dressed) ghost propagator is related to both the full gluon propagator and the full ghost-gluon vertex. For the gluon, the SDE corresponds to:

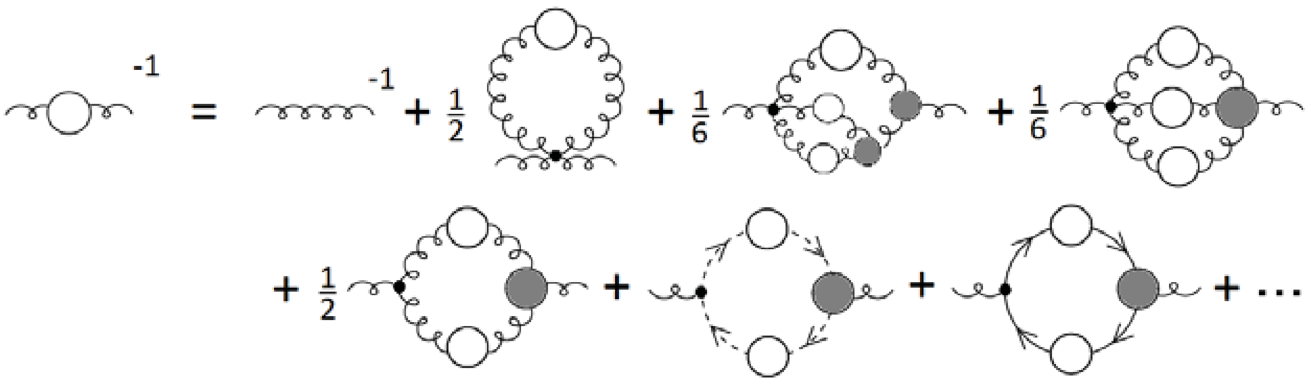

Figure 3.2: Diagrams for the SDE of the gluon propagator.

where the fractions are symmetrization factors from the Feynman rules in each diagram. Next, we show the SDEs for the quark propagator and for the quark-gluon vertex.

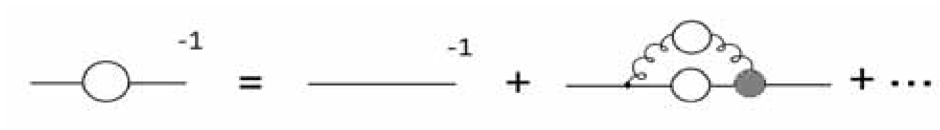

Figure 3.3: Diagrams for the quark SDE.

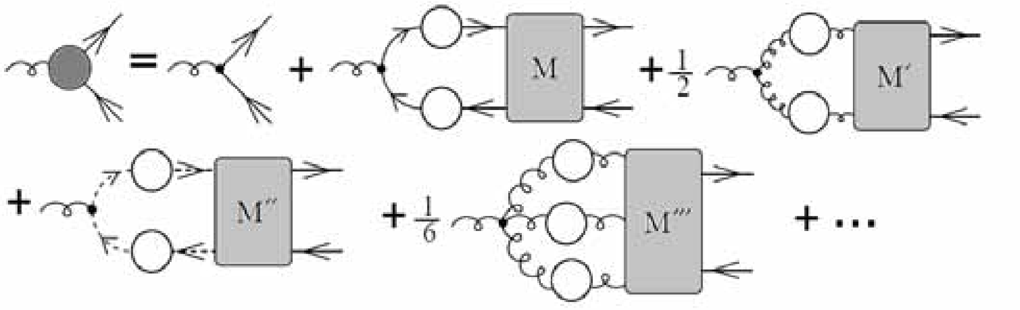

Figure 3.4: Representation of the SDE for the quark-gluon vertex.

where the amplitudes M, M', M'' and M''' are all proper, each one involving a series of diagrams[30]. Finally, the SDE for the ghost-gluon vertex is:

Figure 3.5: Representation of the SDE for the ghost-gluon vertex.

which is the main object of study in this work.

From (3.23) and (3.26), the full (dressed, complete) ghost-gluon vertex is related to the bare (tree-level) one by

$$(\text{full vertex}) = \tilde{Z}_1 \, (\text{bare vertex}) \ . \tag{3.33}$$

Therefore, any knowledge of $\tilde{Z}_1$ may be welcome for the solving of the ghost and the gluon SDEs (Figs. 3.1 and 3.2), which will be the subject of the next chapter. In principle, $\tilde{Z}_1$ is a nontrivial function of the momenta of the three legs of the vertex. Nevertheless, there is the possibility of simplifications due to the gauge symmetry – which is the case we deal with.

3.4 Slavnov-Taylor identities

It is a well known fact that, although the quantization procedure requires gauge fixing, the symmetries of a given theory may be carried to the quantum level, thus leading, for instance, to the Ward identities in QED[17]. For QCD there are, correspondingly, the Slavnov-Taylor identities (STI) which, by imposing invariance on the generating functional $\mathcal{Z}$, imply functional differential equations for $\mathcal{Z}$, $\mathcal{W}$, and Γ [17],Sect.7.6 that, furthermore, imply identities for Green functions and also relations among some of them.

It turns out, however, that non-Abelian theories are distinct, since the Faddeev-Popov Lagrangian indeed presents a symmetry that includes the ghost fields[17],Sect.7.5. This is the BRST[31] symmetry, and is given by the particular gauge transformation when the gauge parameter is $\theta = -\lambda g \chi_2$, where λ is a Grassmann number, and $\chi_2 = (\chi - \bar{\chi}) / i\sqrt{2}$. Then, under an infinitesimal BRST transformation:

$$\begin{cases} \delta A_\mu = \lambda D_\mu \chi_2 \ , \\ \delta\psi = \lambda i g T \chi_2 \psi \ , \\ \delta\chi_1 = \lambda \dfrac{i}{\eta} \partial . A \ , \\ \delta\chi_2 = \lambda \dfrac{i}{2} g \left[\chi_2, \chi_2\right] \ , \end{cases} \tag{3.34}$$

where the term $[\chi_2, \chi_2]$ in the last line is the Lie algebra bracket. We show now some consequences of BRST invariance for QCD correlation functions[32],Sect.2.5.

One result concerns the gluon propagator, whose expansion in terms of the self-energy $\Pi_{\mu\nu}$ is given by

$$D^{ab}_{\mu\nu}(k) = D^{ab}_{(0)\ \mu\nu}(k) + D^{ac}_{(0)\ \mu\rho}(k)\, \Pi^{cd\ \rho\sigma}(k)\, D^{db}_{(0)\ \sigma\nu}(k) + \cdots \ ,$$

where $D^{ab}_{(0)\mu\nu}$ is the bare propagator. It states that the gluon self-energy $\Pi_{\mu\nu}$ is transversal to all orders perturbatively, i.e. for all momenta k:

$$\frac{1}{\eta} k^\mu k^\nu D^{ab}_{\mu\nu}(k) = \delta^{ab} \ ,$$

so that the self-energy may be written as

$$\Pi^{ab}_{\mu\nu}(k) = \delta^{ab}\left(k_\mu k_\nu - k^2 g_{\mu\nu}\right)\Pi\left(k^2\right) \ ,$$

and the full propagator in covariant gauges as

$$D^{ab}_{\mu\nu}(k) = \frac{\delta^{ab}}{k^2}\left(\frac{g_{\mu\nu} - k_\mu k_\nu/k^2}{1 + \Pi(k^2)} + \frac{1}{\eta}\frac{k_\mu k_\nu}{k^2}\right) \ .$$

In other words, the longitudinal term of the gluon propagator is not renormalized.

Moreover, the gauge symmetry imposes constraints among the renormalization functions in (3.26), implying that, to all orders,

$$\frac{Z_1}{Z_3} = \frac{\tilde{Z}_1}{\tilde{Z}_3} = \frac{Z_{1F}}{Z_2} = \frac{Z_4}{Z_1} \ ,$$

which, from (3.27), is equivalent to

$$\frac{Z_g {Z_3}^{3/2}}{Z_3} = \frac{Z_g \tilde{Z}_3 {Z_3}^{1/2}}{\tilde{Z}_3} = \frac{Z_g Z_2 {Z_3}^{1/2}}{Z_2} = \frac{{Z_g}^2 {Z_3}^2}{Z_g {Z_3}^{3/2}} \ .$$

That is, the renormalized coupling does not depend on which vertex is renormalized.

Finally, we consider the ghost-gluon vertex renormalization function $\tilde{Z}_1$, which presents the following simple feature. In the limit of the incoming ghost momentum going to zero:

$$\tilde{Z}_1 = 1 \text{ , for } \eta = 0 \quad \text{(Landau gauge)} . \tag{3.35}$$

This identity was shown, by Taylor[3], to be true to all orders perturbatively, and it means that this vertex receives no perturbative corrections.

Furthermore, there is a conjecture[33] stating that, also for the Landau gauge, this is true in the limit of any momentum going to zero. We shall explore the zero gluon momentum configuration within an improved perturbative correction, which will be discussed in the next two chapters.

3.5 Remarks on SDEs

The diagrammatic representations given above for the SDEs show that these constitute an infinite set of equations coupling the Green functions of various (all, in fact) orders for the theory under consideration. However, an important point to make is that the conceptual complexity is not only the infinitude of the system of equations, but their very coupling.

In order to grasp their significance, consider the gluon SDE. It depends on all vertices, each of which satisfies its own SDE, that also depend on all 2-point functions of the theory, and so on. So, the SDEs express the fact that there is no independent correlation function in QFT.

Hence, if the interactions are ultimately related to the correlation functions, the way quarks interact with each other, for instance, is intrinsically related to the behavior of *all* fields of the theory. At last, QFT holds a whole distinct picture of interactions: it does not speak of interactions *between* systems, but *among* them. It is worthwhile to note that this picture is brought just when describing strong phenomena via QFT.

On the other hand, we come back to the fact that this complexity is carried from the conceptual to the formal place, where it must be dealt with, and turn to the technical problem of finding solutions to SDEs. This amounts to a whole body of works[30],[34], some of which[35],[36] led to one basic motivation for the present study: a massive-like solution for the gluon propagator, to which we direct, then.

Chapter 4

The dynamical gluon mass

From the discussion at the end of the previous chapter, it is clear that the whole set of SDEs is intractable in practice: rigorously, solving exactly one SDE of a theory amounts to solving all of them. Therefore, unless some kind of convergent limiting procedure is known, truncation is unavoidable in order to reduce the set of equations to a finite one.

4.1 IR finite gluon propagator

Among SDE calculations, one important for the context of the present study was accomplished by Mandelstam in his 1979 work [37]. In order to obtain a closed equation for the gluon 2-point function in the Landau gauge, he made some assumptions and approximations that are still influential for more recent works, whether in their support or refutation. They were: (a) working with pure QCD, that is, the quarks are treated as nondynamic in what concerns the dynamics of the gluons themselves; (b) neglecting ghosts contributions, which leads to a quite simple form of the STI which relates the full 3-gluon vertex to the bare one and to the gluon propagator[34]; (c) leaving out the contribution of all other higher-order Green functions.

The result of Mandelstam, and also a slight variation of it made by Brown and Pennington[38],[36], for the gluon propagator were verified to be compatible with an infrared (IR) behavior like $(k^2)^{-2}$. This specific IR enhanced (that is, more singular than $1/k^2$ as $k^2 \to 0$) behavior is phenomenologically appealing, as the following two aspects exemplify. First, a propagator like $(k^2)^{-2}$ may be related[36] to a linearly rising potential, accounting for a Schrödinger-like description of a bound state of quarks. Secondly, it satisfies the Wilson area law, which serves as a criterion for confinement in pure QCD[39]. Both of these aspects strongly lie in (a), an hypothesis that invokes the idea of the gluons being responsible for the interaction between quarks which compose the observed hadrons. Although this is, as noted in the discussion of Chap.3, not characteristic of QFT, its status is summarized in [39] (emphasis are ours):

> "the idea of a static potential breaks down when dynamical quarks are included, and the Wilson criterion becomes inapplicable. One *hopes*, however, that if a pure-gauge theory is confining according to Wilson criterion, then it is *an indication* that dynamical quarks will be confined when they are introduced."

Nevertheless, two distinct approaches oppose this IR enhanced result. One of them is the lattice QCD, which, although is not sufficiently precise in the extreme IR, gives results that indicate the plausibility of an IR finite gluon propagator[40]. Another one started with the conclusion, by Cornwall[35], that the gluons would present an effective mass[41].

Cornwall introduced the so-called pinch-technique, which consists in a different construction of the SDEs. Rearranging Feynman graphs combined in a gauge-invariant manner, and defining effective Green functions that are compatible with multiplicative renormalization, he obtained a non-trivial, gauge-invariant self-energy for his new gluon propagator, so that the gluon has a dynamical mass.

It is important to say that this dynamical mass does not mean that there is a massive on-shell condition for the gluons – if there is any sense at all to speak of an on-shell condition for the QCD fields. The massive solution consists in the presence of a running mass, which goes to zero in the UV region, so that perturbative QCD results may be recovered. We mention, for our own use in the next chapter, that Cornwall also obtained an IR finite coupling constant, that behaves as

$$\alpha_{\mathrm{Cnw}} = \frac{1}{4\pi b \log\left[\left(4M^2(k^2) + k^2\right)/\Lambda^2_{\mathrm{QCD}}\right]} \,, \tag{4.1}$$

in Euclidean space, where $b = (33 - 2n_f)/48\pi^2$, and n_f is the number of fermions[7].

Furthermore, QCD phenomenology[42] seems to prefer Cornwall's massive-like solutions, and more recent numerical solutions[36],[41] for the gluon SDE also indicate an IR finite gluon propagator. The solution of [36], besides presenting a dynamical mass which preserves gauge-invariance[8],

[7] The inclusion of fermions is somehow *ad hoc* since, up to now, no SDE aproach has included them.

[8] It is proven in [36],Sect.4.5 that their solution indeed satisfies the STI, hence does not violate gauge-invariance.

regains the UV logarithmic perturbative behavior and can be fitted by

$$D\left(k^2\right) = \frac{1}{k^2 + M^2\left(k^2\right)}, \quad \text{with} \quad M^2\left(k^2\right) = \frac{m_0^4}{k^2 + m_0^2}\ , \tag{4.2}$$

from which were obtained values for the ratio m_0/Λ_{QCD} in a quite narrow interval around 2.

It is also calculated in [36], within the same approximations of [38], a solution of the gluon SDE with ghosts considered. To solve numerically the coupled equations for the ghost and the gluon, it was assumed that $\tilde{Z}_1 = 1$, $Z_1 = 1$, and the result was again an IR finite gluon propagator, which recovers the UV $\log$ behavior, too. The SDE approach in [36] and in [44] also derives, from this dynamical mass, an effective running charge that is in accordance with lattice QCD results[44].

Even more recent papers[43], both within lattice QCD and SDE, continue to support an IR finite gluon propagator. Now we turn to discuss some way to make a more phenomenological use of these results.

4.2 Dynamical Perturbation Theory

Also in 1979, it was introduced[45] the idea of DPT: a scheme to improve the usual perturbative series of QCD, introducing into it nonperturbative informations. If this could correspond to some kind of resummation[46] of the usual series in the bare coupling remains as a speculative matter only. Furthermore, any formal demonstration in support of DPT still lacks. Yet one way to explore this idea is taking the nonperturbative information to be the Green functions calculated via SDE, and inserting, into a perturbative calculation, expressions for propagator(s) and/or coupling in the whole range of momenta – that is, dressing them in the perturbative expansion, and verifying the resulting behavior.

One may take, for instance, Cornwall's expression (4.1), together with the massive fit for the gluon propagator, (4.2), and insert them in place of the corresponding bare quantities when appropriate. This was performed in some phenomenological studies[42], such as the pion form factor, a modelling of hadronic cross-sections, and the Bjorken sum rule.

In general, as shown in [42], a variety of applications involving very different energy scales fits reasonably well the experiments, and they all lead to the same, and quite narrow, range for the gluon mass at $q^2 = 0$,

$$m_0 \approx \mathcal{O}\left(2\Lambda_{\text{QCD}}\right) \ . \tag{4.3}$$

One possibility in particular is to use, in the usual perturbative series, the full propagators when occurring in loops. It would be an option whether or not to use the full vertices, and/or the running coupling, too. In any case, it could be a device to probe infrared properties by means of loop calculations, which in the usual perturbative approach is clearly misleading.

4.3 Our proposal

We propose to investigate, within a DPT scheme, the renormalization function for the ghost-gluon vertex, $\tilde{Z}_1$. We calculate the 1-loop correction to this vertex, using all bare Green functions except for the gluon propagator, which we consider to be, in Minkowski space,

$$\Delta_{\mu\nu}\left(k\right) = \frac{1}{k^2 - M^2}\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) \ , \tag{4.4}$$

where we took the Landau gauge ($\eta = 0$) for the following reason.

It is our intention to probe, within this DPT scheme, the approximation $\tilde{Z}_1 = 1$ that is made in the procedures, described in Sect.4.1, of solving the gluon SDE. We employ our calculations for the particular kinematical configuration of the gluon momentum equal to zero, so that we calculate the sum of diagrams below:

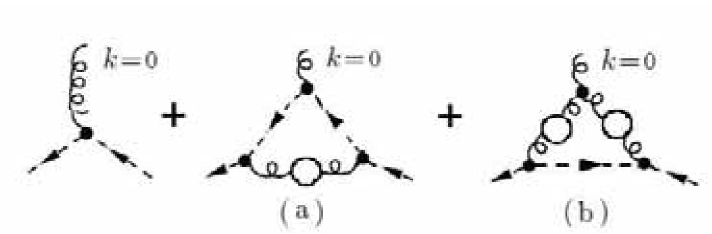

Figure 4.1: The ghost-gluon vertex in the zero gluon momentum configuration: the bare vertex, plus the 1-loop corrections (a) and (b).

In the MOM scheme, (3.33) is:

$$\Lambda_\nu^{abc}\left(\mu^2, q^2\right) \equiv -igf^{abc}q_\nu\Lambda\left(\mu^2, q^2\right) = \tilde{Z}_1\left(\mu^2, q^2\right)\ \Lambda_{\nu\ \text{bare}}^{abc}\left(q\right) \ ,$$

where $\Lambda^{abc}_{\nu\ \text{bare}}(q) = -igf^{abc}q_\nu$ corresponds to the bare vertex, and the renormalized vertex factor $\Lambda(\mu^2, q^2)$ is given by the renormalization condition imposed at the energy scale μ, fixing the value of $\Lambda(\mu^2, \mu^2)$. Then, if $\Lambda_{\text{unr.}}(q^2)$ is the unrenormalized expression obtained perturbatively, then:

$$\tilde{Z}_1\left(\mu^2, q^2\right) = \Lambda\left(\mu^2, q^2\right) = \Lambda\left(\mu^2, \mu^2\right)\frac{\Lambda_{\text{unr.}}\left(q^2\right)}{\Lambda_{\text{unr.}}\left(\mu^2\right)}\,. \tag{4.5}$$

Now we have all that is necessary to analyse our results, so we proceed to them.

Chapter 5

Results and Conclusions

Our calculations of diagrams (a) and (b) of Fig.4.1 are shown in Appendix D. The unrenormalized result is:

$$\xi^\mu = igf^{abc}q^\mu\Lambda_{\text{unr.}}\left(q^2\right) = igf^{abc}q^\mu\frac{N\alpha}{16\pi}\zeta\left(q^2\right) \ ,$$

where

$$\zeta\left(q^2\right) = \frac{7}{12}+\frac{1}{2}\frac{m^2}{q^2}-\frac{1}{6}\frac{q^2}{m^2}-\frac{m^2}{q^2}\left(\frac{5}{6}+\frac{1}{2}\frac{m^2}{q^2}\right)\log\left(1+\frac{q^2}{m^2}\right)+\frac{1}{6}\frac{q^2}{m^2}\left(3+\frac{q^2}{m^2}\right)\log\left(1+\frac{m^2}{q^2}\right) \ ,$$

which, as also shown in Appendix D, goes to unity as m goes to zero, thus recovering the massless result $\tilde{Z}_1 = 1$, in accordance with Taylor identity (3.35) and with the conjecture mentioned in Sect.3.4.

We choose to renormalize the vertex to unity: $\Lambda\left(\mu^2,\mu^2\right) = 1$ – that is, at $q^2 = \mu^2$ the full vertex equals its tree-level expression. Therefore, from (4.5), our renormalized result is:

$$\Lambda\left(\mu^2,q^2\right) = \frac{1-\dfrac{N\alpha}{16\pi}\zeta\left(q^2\right)}{1-\dfrac{N\alpha}{16\pi}\zeta\left(\mu^2\right)} \ , \tag{5.1}$$

$$\Rightarrow \ \tilde{Z}_1{}^{-1}\left(\mu^2,q^2\right) = \frac{1-\dfrac{N\alpha}{16\pi}\zeta\left(\mu^2\right)}{1-\dfrac{N\alpha}{16\pi}\zeta\left(q^2\right)} \ . \tag{5.2}$$

It is standard to expand this expression to

$$\tilde{Z}_1{}^{-1}\left(\mu^2,q^2\right) \cong 1+\frac{N\alpha}{16\pi}\left[\zeta\left(q^2\right)-\zeta\left(\mu^2\right)\right]+\mathcal{O}\left(\alpha^2\right) \ , \tag{5.3}$$

which is indeed consistent with the usual perturbative expansion, since it considers contributions up to order α *only*. However, we work within a DPT proposition, i.e. a perturbative approach that is supposed to contain some non-perturbative character, and not to be solely an expansion in orders of α. Therefore, we consider that it is worthwhile to use *both* (5.2) and (5.3) to investigate the result of our calculations.

5.1 General behavior of the solution

The first point we check is whether there is any significant difference between the results of (5.2) and of (5.3). Furthermore, both expressions depend on three parameters: μ, α, and m, so we shall study how sensitive our result is to each one of these. At first, we have analysed plots within the intervals $\mu \in [1, 10]$GeV, $\alpha \in [0.1, 6.5]$, and $m \in [1, 700]$MeV, and in every case the distinction between (5.2) and (5.3) was seen to be insignificant.

Although the result is practically the same, (5.2) involves one less approximation, so we choose to use it in order to study the behavior of our solution for $\tilde{Z}_1{}^{-1}$, since our proposal is an attempt of perturbative expansion that is not restricted to powers of α. We sketch the dependence with each one of the three parameters by fixing two of them while plotting $\tilde{Z}_1{}^{-1}(q^2)$ for some values of the third.

The variation with μ is the expected one for multiplicative renormalization, as shown in Fig.5.1:

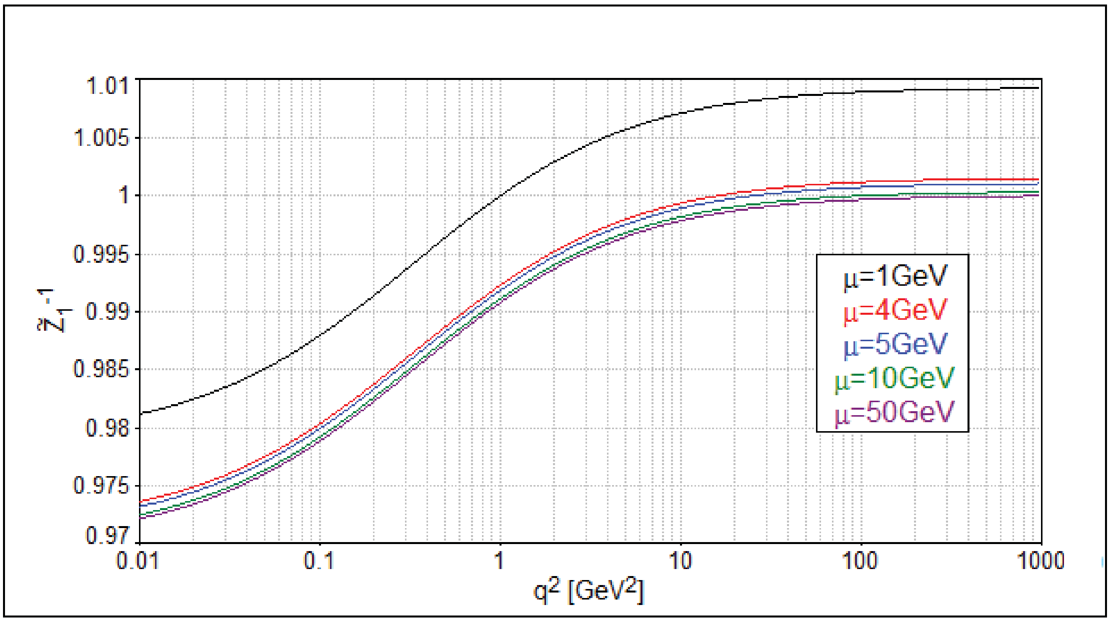

Figure 5.1: The ghost-gluon vertex renormalization function for various renormalization scales, for $m = 250$MeV and $\alpha = 0.5$.

We see that $\tilde{Z}_1{}^{-1}(q^2)$ does not assume values much different than 1. The same occurs for the following interval of m:

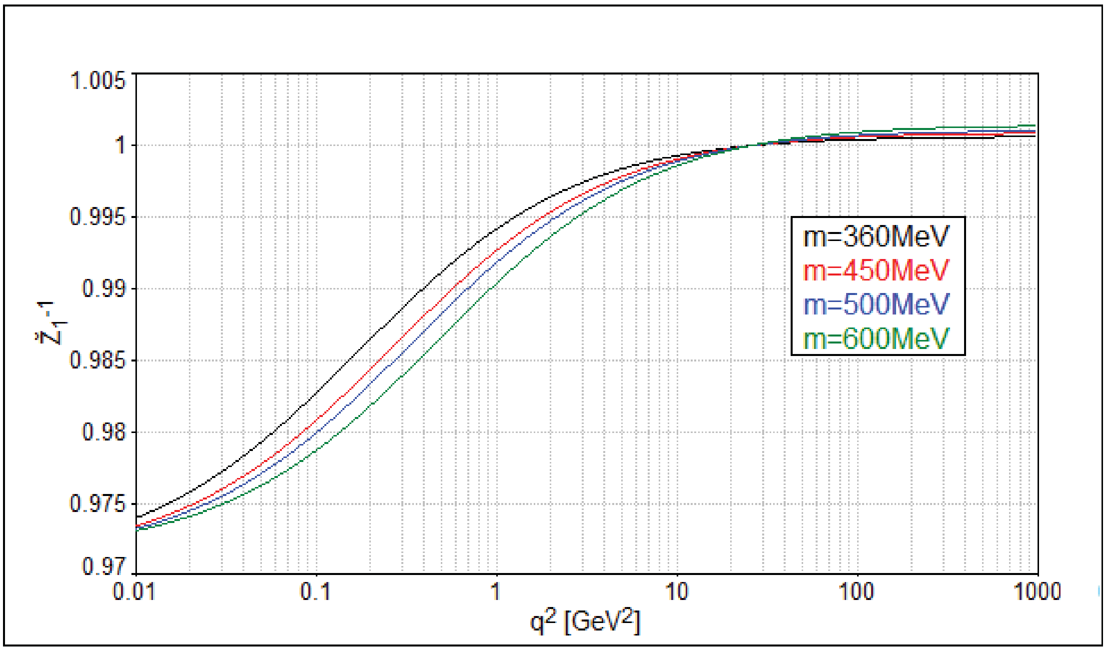

Figure 5.2: The vertex renormalization function for masses between $1.2\Lambda_{\mathrm{QCD}}$ and $2\Lambda_{\mathrm{QCD}}$ (with $\Lambda_{\mathrm{QCD}} = 300$MeV), with $\alpha = 0.5$ and renormalized at $\mu = 5$GeV.

Paying attention to the scale, the greatest variation of $\tilde{Z}_1{}^{-1}$ for this interval of masses is ~ 0.005, i.e. 0.5% of the value 1 it assumes for high q^2. So this renormalization function seems to be very little sensitive to m, then it can hardly be useful for indicating some range of finite values for the gluon mass. Still, could it at least point towards a finite rather than a null mass? The answer is: hardly, too. In our first observation within $m \in [1, 700]$MeV, it was noted that, as m gets close to 1MeV, $\tilde{Z}_1{}^{-1}$ turns to be almost identically equal to 1, the massless limit. Therefore, this result could suggest, whether an infinitesimal mass or $m \sim \Lambda_{\mathrm{QCD}}$, only if compared with some reference with a precision great enough to distinguish 1 from 0.975.

However, the only reference we have is from lattice calculations, [1] and [2]. Both contain error bars which are ≥ 0.5, and consequently we can already anticipate that, whichever conclusions our result may lead to, it certainly will not give any information on the mass term for the gluon, within our DPT scheme.

Nevertheless, one more parameter remains to be sketched. It is actually the most influential one, since a greater variation from unity can be obtained only with higher values of α, as seen below:

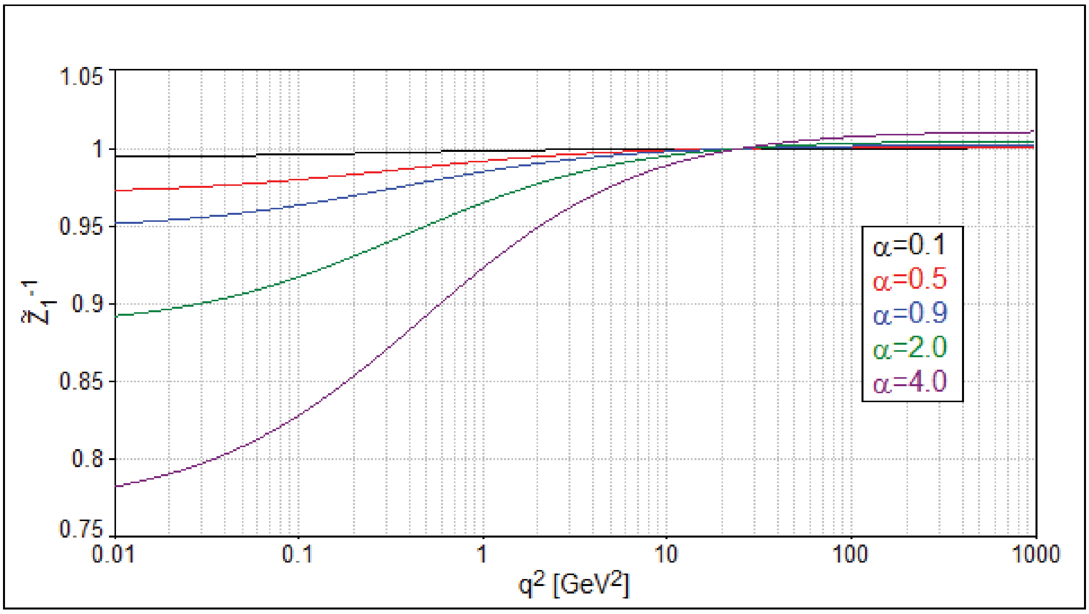

Figure 5.3: $\tilde{Z}_1{}^{-1}(q^2)$ for various values of the coupling, for $m = 250$MeV and renormalized at $\mu = 5$GeV.

All plots in Fig.5.3 above are for constant values of α in (5.2). On the other hand, it may be considered in the following possible ways: (1) as a fit parameter, as usual in comparisons with lattice data[47]; (2) related to the lattice's bare coupling $\beta = 2N/4\pi\alpha$ [26]; (3) as the running coupling obtained from SDE results that fit lattice data, given in [44]; (4) a constant equal to this running result evaluated at the renormalization scale; (5) or the Cornwall's expression (4.1).

Now, also because of our proposal of an improved perturbative expansion, we choose the options (3) and (5), which are both running couplings supposed to contain nonperturbative information of the theory. A more consistent approach would be to integrate these running couplings in the 1-loop calculations of Appendix 4. However, we choose the simpler analysis of just replacing the α in

(5.2) for each running charge, obtaining

$$\tilde{Z}_1^{\ -1}\left(\mu^2, q^2\right) = \frac{1 - \dfrac{N\alpha\left(\mu^2\right)}{16\pi}\zeta\left(\mu^2\right)}{1 - \dfrac{N\alpha\left(q^2\right)}{16\pi}\zeta\left(q^2\right)}\ , \tag{5.4}$$

for $\alpha = \alpha_{\text{ABP}}$ or α_{Cnw}, option (3) or (5), respectively.

Reference [44] presents an effective charge α_{ABP} for gluon masses equal (in the Landau gauge) to 500 and to 600MeV, both with the same qualitative behavior. We show here the latter, fitted by a simple expression that we shall use ahead[9].

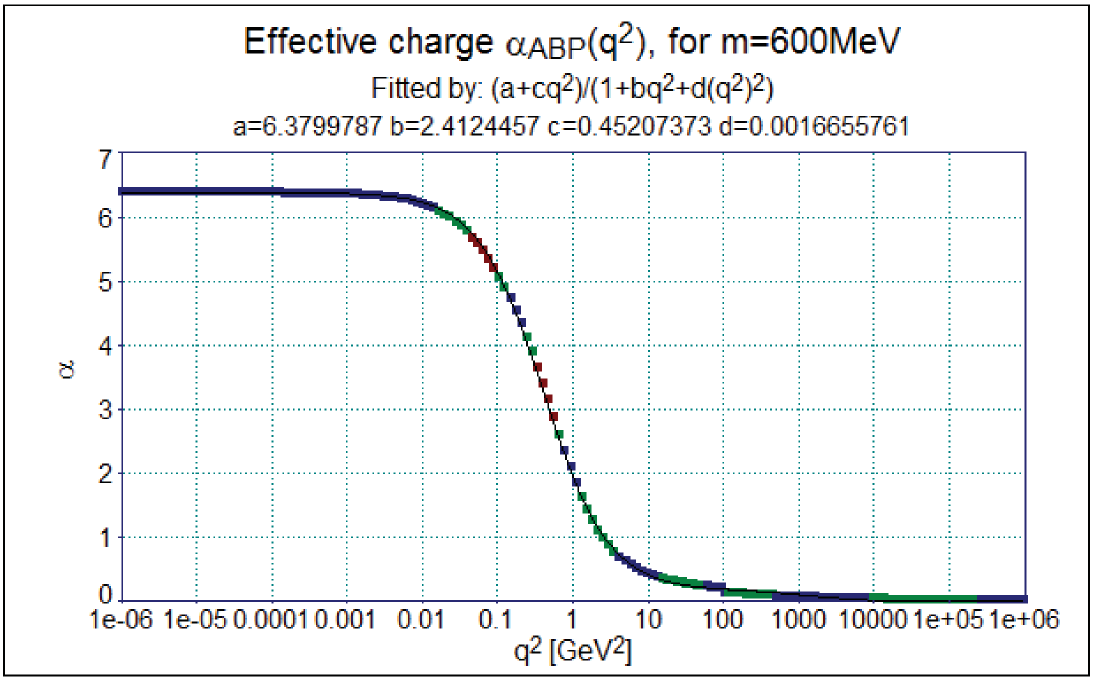

Figure 5.4: Effective charge α_{ABP}, for $m = 600\text{MeV}$, fitted by the rational equation shown in the graphic, which accounts for the qualitative behavior of both results of [44].

We note, also, that this fit is made only to be used in (5.4), so our choice of function was for simplicity rather than consistency with the behavior of the QCD charge to any order of perturbation. In fact, all we require from this function $\left(a + cq^2\right) / \left(1 + bq^2 + d(q^2)\right)$ is that it reproduces accurately

9 In this figure, and in other ones to come, blue points are those with less than 1 standard deviation (SD), the green ones with less than 2SD, and the red ones with more than 2SD.

the points from the numerical SDE calculation of [44], specially in the interval $q^2 \in (0.01, 100)$, where stand the lattice data we shall compare our results with, in the next section.

Before that, we, in order to compare the resulting behavior with the one for a constant α, show (5.4) for options (3), (4) and (5), $\alpha(q^2) = \alpha_{\text{ABP}}(q^2)$, $\alpha_{\text{ABP}}(\mu^2)$ and $\alpha_{\text{Cnw}}(q^2)$ respectively, for $\Lambda_{\text{QCD}} = 300\text{MeV}$ and $m = 2\Lambda_{\text{QCD}}$:

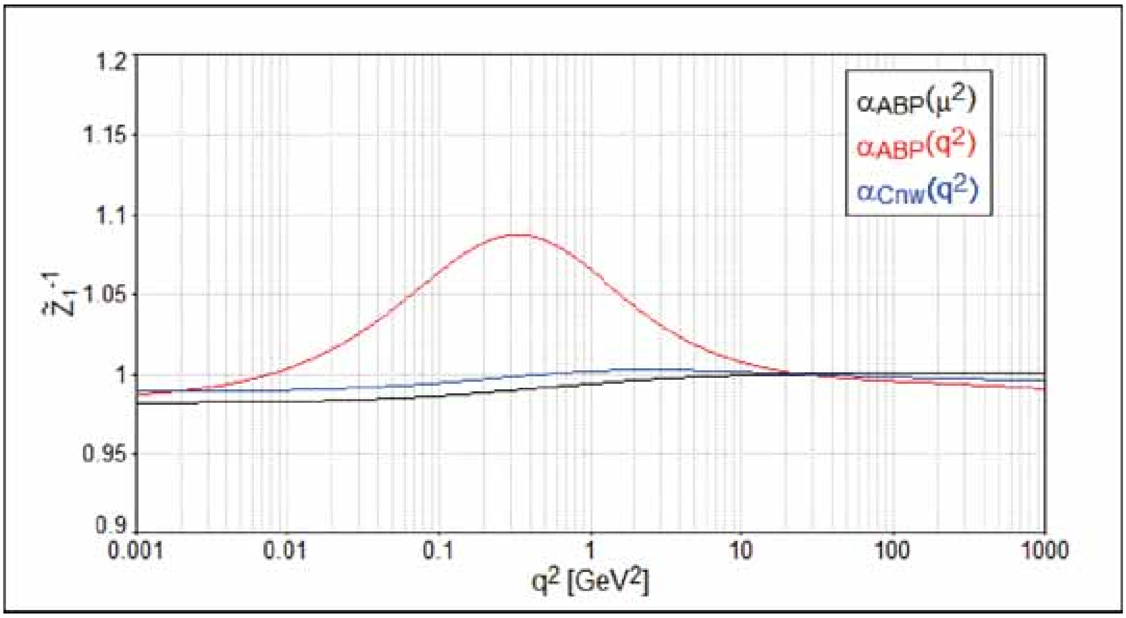

Figure 5.5: $\tilde{Z}_1{}^{-1}(q^2)$ for options (3), (4), and (5) as the coupling α.

So, option (3) seems to be the only one to give values higher than unity, and, as we shall see in the next section, this choice fits better the results obtained in lattice computations[1],[2].

5.2 Fitting lattice data

We now compare our results with lattice data taken from [1] and [2]. The former was calculated for $SU(3)$, while the latter for $SU(2)$. None of them can be fitted to our result if we consider option (1), i.e. a constant α. They were fitted only for running charges, as we show now.

5.2.1 SU(2) data

We were able to fit the data from [2], renormalized at $\mu^2 = 10.5\text{GeV}^2$, with expressions corresponding to options (3) and (5), giving the two following graphics[10]:

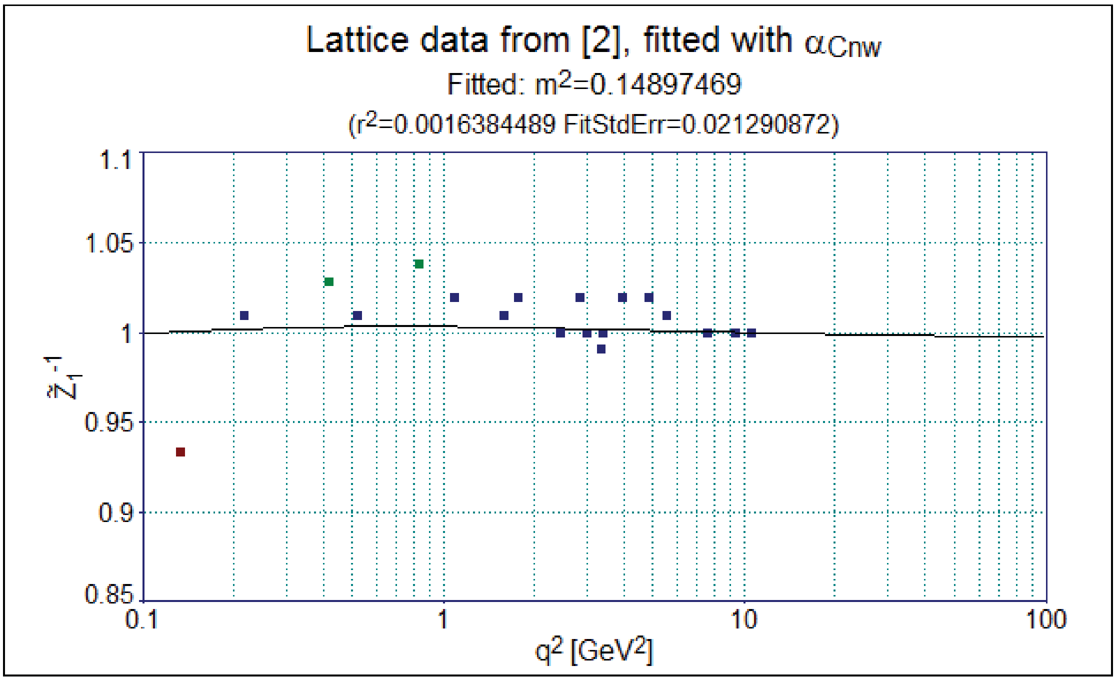

Figure 5.6: $\tilde{Z}_1{}^{-1}(q^2)$ data from [2], fitted by (5.4), with Cornwall's expression (4.1) for α.

10 The title of each graphic contains the standard error, FitStdErr, and coefficient of determination, r^2, of the fit. The better the fit, the closer FitStdErr is to 0 and r^2 is to 1.

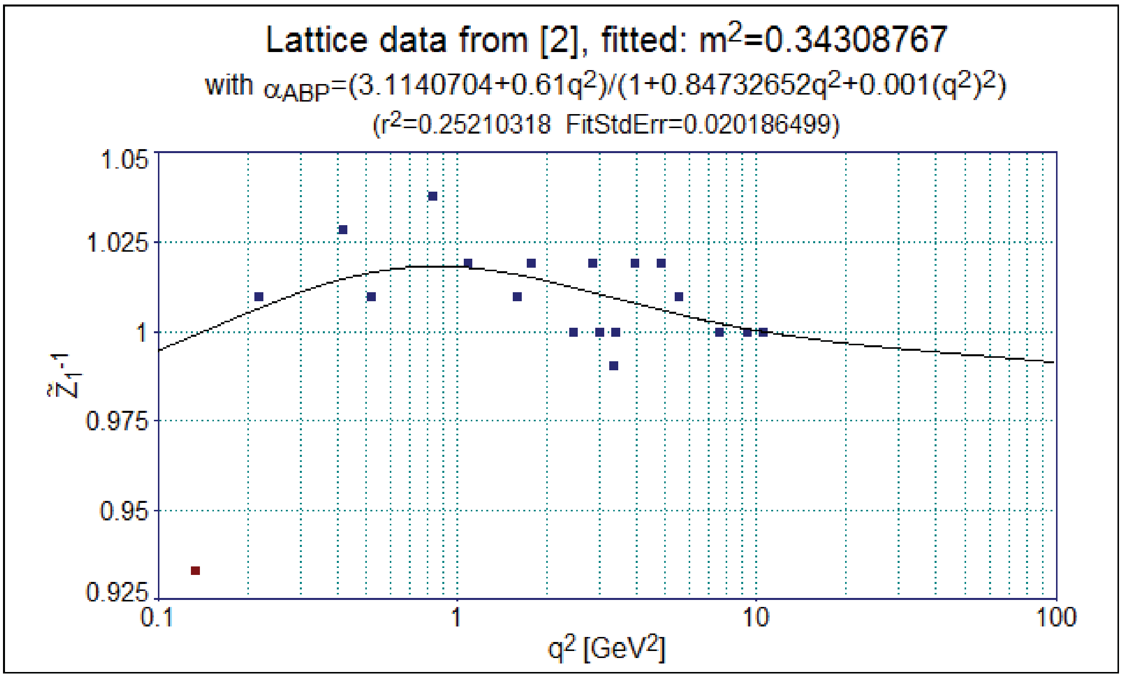

Figure 5.7: $\tilde{Z}_1^{\ -1}(q^2)$ data from [2], fitted by (5.4), also with a fit $\alpha = (a + cq^2)\,/\,(1 + bq^2 + 0.001(q^2))$ for $\alpha_{\text{ABP}}(q^2)$.

It is clear that a charge of the form obtained in [44] fits better this lattice data, and the fit residuals are, except for the lowest q^2 point, within the range of the data error bars, that have width 0.05. Within these errors, however, the data could fit $\tilde{Z}_1^{\ -1}(q^2) \equiv 1$ as well. The same can occur for the data from [1], as we see now.

5.2.2 SU(3) data

We start by pointing out that all errors of the data from [1] lie between 0.05 and 0.12. This restricts our conclusions in the same way as before: within the errors, the data (which we renormalized at $\mu^2 = 65.727\text{GeV}^2$) might be close to 1 enough to be fitted by $\tilde{Z}_1^{\ -1}(q^2) \equiv 1$. We fitted them anyway, but now we have succeeded in fitting only when using an expression for $\alpha_{\text{ABP}}(q^2)$, as shown below.

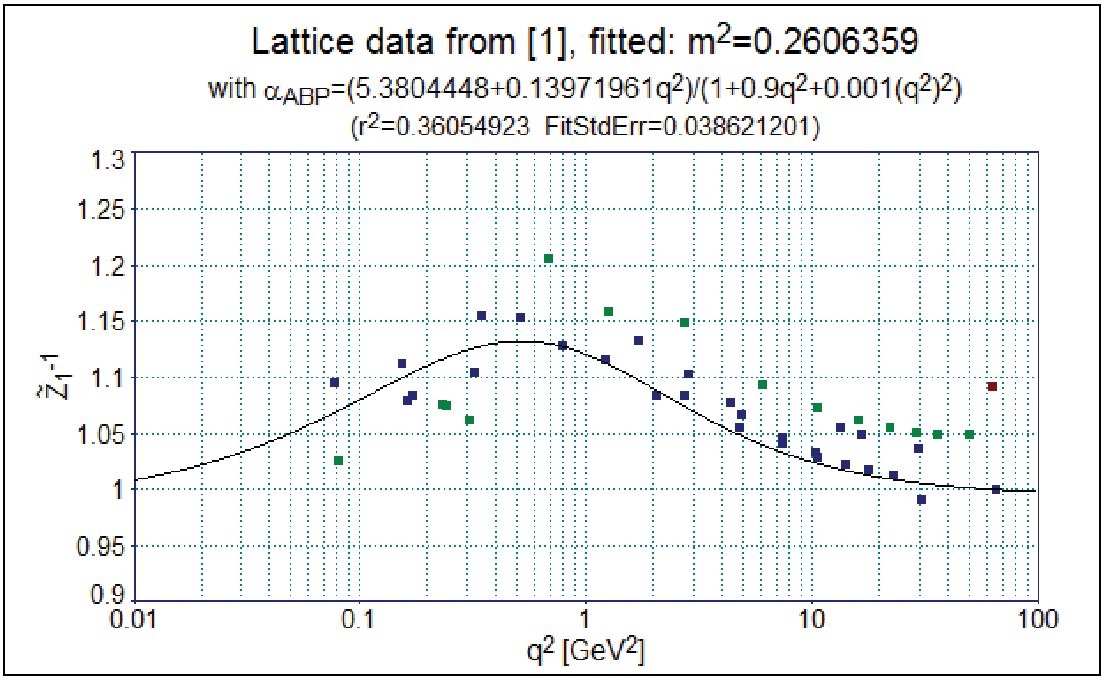

Figure 5.8: $\tilde{Z}_1^{\ -1}(q^2)$ data from [1], fitted by (5.4), with a fit $\alpha = (a + cq^2) \,/\, (1 + bq^2 + 0.001(q^2))$ for $\alpha_{\text{ABP}}(q^2)$.

Once again, most of the fit residuals are within the errors of the lattice data. Therefore, the best expression to fit them is indeed (5.4) with an effective coupling that has the same behavior as $\alpha_{\text{ABP}}(q^2)$ obtained in [44]. However, as we have noted before, the data contain errors wide enough to make this conclusion not much more reliable than saying that the data would be fitted by our expression with a constant α, or even by the massless result $\tilde{Z}_1^{\ -1}(q^2) \equiv 1$.

So, the least we can say is: (I) Our DPT-like results are well fitted to lattice data, giving a *possible* predilection for $\alpha_{\text{ABP}}(q^2)$, within a mass range $m \in (490, 590)$ consistent with the previous estimatives for the gluon mass discussed in Chap.4; (II) In general, the results do not differ much from unity, sustaining, or at least not denying, the approximation of $\tilde{Z}_1^{\ -1}$ to be identically equal to 1, as usually considered in SDE calculations, without significant nonperturbative IR contributions.

Chapter 6

Final remarks

We have employed a specific calculation based on some propositions and results obtained for QCD: indications, from both lattice and SDE computations, for an IR finite gluon propagator; the idea of DPT; the Taylor identity valid perturbatively; and conjectures or suppositions also concerning the ghost-gluon vertex.

First, we calculated an effective 1-loop correction to this vertex in the Landau gauge and for one momenta configuration only: zero gluon momentum. Other and more general configurations could be a matter of further investigation, such as a gauge-dependence of the results – that might achieve conclusions further than the Taylor identity and extend the possibilities of using simple expressions for this vertex in SDE calculations in other gauges.

However, it must be stressed that our work made use of two approaches: dressing the gluon propagators in the loop calculations and using, in the expression for the renormalization function, (i) an effective running charge $\alpha\left(q^2\right)$; and (ii) a constant value α. Anyhow, they are attempts of introducing a nonperturbative quality of the theory into its perturbative expansion. There is no proof of DPT's validity or efficacy, and it is only supported by the *phenomenological* attempt itself.

Despite these limitations, our DPT-like approach gave a somewhat satisfactory result. It was quite successful in sustaining the validity of (at least approximately) $\tilde{Z}_1 \cong 1$, and on indicating a possible predilection for the case (i) above, specially for the effective charges with behavior like the ones from [44]. These charges were obtained via SDE computations that were fitted to lattice data and considered the gluon having a dynamical mass, indicating, then, some consistency of the result.

Finally, there is one related work in progress. Recently[47], a calculation similar to ours was employed for the gluon and the ghost propagators with an effective massive Yang-Mills Lagrangian. Besides renormalizing the gluon propagator like in a massive on-shell condition, and using the coupling α as a fit parameter, the results presented in [47] were obtained by rescaling their original result, i.e. renormalizing them at arbitrary convenient scales so that they would fit closely the lattice data. We shall modify their result in two aspects: first, by imposing a renormalization condition compatible with the ultraviolet massless behavior of the gluon; and then studying the changes implied by the use of an effective running charge. This could be a further step in the phenomenological direction taken in the present work.

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Appendix A Perturbative Method

In this appendix we show the general expression from which Green functions may be expanded in orders of the coupling(s) of the theory under consideration. From the generator functional integral expression (3.4a), the separation of the Lagrangian into the sum of a free and an interacting terms, $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_i$ respectively, leads to:

$$\begin{aligned} \mathcal{Z}[J] &= \int \mathcal{D}\varphi \left(\exp i \int \mathsf{d}^D x\, \mathcal{L}_i[\varphi] \right) \exp i \int \mathsf{d}^D x \left\{ \mathcal{L}_0[\varphi] + J\varphi \right\} \\ &= \left(\exp i \int \mathsf{d}^D x\, \mathcal{L}_i \left[\frac{\delta}{i\delta J} \right] \right) \int \mathcal{D}\varphi \exp i \int \mathsf{d}^D x \left\{ \mathcal{L}_0[\varphi] + J\varphi \right\} \ , \end{aligned} \tag{A.1}$$

where $\mathcal{L}_0$ is specified by a differential operator K, which is supposed to be invertible and is given by

$$\int \mathsf{d}^D x\, \mathcal{L}_0 = -\frac{1}{2} \int \mathsf{d}^D x\, \varphi(x)\, K(x,x)\, \varphi(x) + \text{surface terms} .$$

Now, with the change of variables

$$\varphi(x) \mapsto \varphi(x) - \textstyle\int \mathsf{d}^D y\, K^{-1}(x,y)\, J(y) \ , \tag{A.2}$$

$$\begin{aligned} \int \mathsf{d}^D x \left\{ \mathcal{L}_0[\varphi] + J\varphi \right\} \ \mapsto \ \int \mathsf{d}^D x \Big\{ & -\frac{1}{2}\varphi(x)\, K(x,x)\, \varphi(x) + J(x)\, \varphi(x) \\ & -\frac{1}{2} \int \mathsf{d}^D y\, J(y)\, K^{-1}(y,x)\, J(x) \Big\} \end{aligned}$$

up to surface terms,

$$\Rightarrow \int \mathcal{D}\varphi \exp i \int \mathsf{d}^D x \left\{ \mathcal{L}_0 + J\varphi \right\} \propto \exp \frac{-i}{2} \int \mathsf{d}^D x \mathsf{d}^D y\, J(y)\, K^{-1}(y,x)\, J(x) \ .$$

Since $\mathcal{Z}[J]$ is defined up to any multiplicative factor (see eq.3.5), (A.1) becomes

$$\mathcal{Z}[J] = \exp \left(i \int \mathsf{d}^D x\, \mathcal{L}_i \left[\frac{\delta}{i\delta J} \right] \right) \exp \left(\frac{-i}{2} \int \mathsf{d}^D x \mathsf{d}^D y\, J(y)\, K^{-1}(y,x)\, J(x) \right) \ , \tag{A.3}$$

and then, by expanding the exponential series of $\mathcal{L}_i[\delta/i\delta J]$, one obtains, for any Green function, a perturbative expansion in the coupling constant(s) contained in $\mathcal{L}_i$. Clearly, the reliability of this perturbative series is limited at least by the magnitudes of these couplings, a point that can be addressed from renormalization group considerations.

One final remark must be done: we have chosen, for brevity, to show the expansion of the Green functions generated by $\mathcal{Z}$ in terms of the interactions of the theory. However, a distinct, and perhaps more conceptually fruitful expansion is obtained for the effective action Γ, the generator of proper correlation functions, as a formal series[11] in powers of the Planck constant $\hbar$. It may be shown that this in fact consists in a loopwise expansion of the quantum fluctuations of the field theory. For this subject we refer to [15],Sect.9-2-2.

[11] That is, a power series in an unknown z, but with no assignment of values to it, and no concerns with convergence.

Appendix B Faddeev-Popov quantization

In this appendix we present the Faddeev-Popov procedure, which is a method for the quantization of gauge fields within the functional integral formalism of QFT. The essential principle is that the integration is *over the dynamic variables*, which is not given by $\int \mathcal{D}A$, since this would redundantly integrate over gauge orbits[12]. The Faddeev-Popov procedure is an *ansatz* that amounts to taking a different (and supposed non-redundant[13]) measure of integration for the gauge fields.

This is accomplished by factoring out of the generating functional the volume of the gauge group[[14],Chap.III.4,[17],Sect.7.2], as follows: Let $S\left(A, \partial A\right)$ be the action for the gauge fields, and

$$\Delta_{FP}\left(A\right)^{-1} := \int \mathcal{D}U \, \delta\left[f\left(A_U\right)\right] \ , \tag{B.1}$$

where $\mathcal{D}U$ is the invariant[[32],[24]] integration measure of the gauge group, A_U is mapped from A by U, and f is some function of the gauge fields, so that the delta functional corresponds to some gauge fixing. Δ_{FP} is the so-called Faddeev-Popov determinant, and is straightfowardly shown to be gauge-invariant, as also are the measure $\mathcal{D}A$ and the action S for the gauge fields. Under the above conditions, we may write the generating functional as[14]

$$\begin{aligned} \mathcal{Z}\left[0\right] &= \int \mathcal{D}A \, e^{iS} \, \Delta_{FP}\left(A\right) \int \mathcal{D}U \, \delta\left[f\left(A_U\right)\right] \\ &= \int \mathcal{D}U \int \mathcal{D}A \, \Delta_{FP}\left(A\right) \, \delta\left[f\left(A\right)\right] \, e^{iS} \ . \end{aligned}$$

If one considers gauges of the form

$$f\left(A_U\right) \equiv G_\mu A_U^{a\mu} - B^a \ ,$$

with G and B to be chosen, and suppose f to vanish for one specific U[15], one obtains the following

[12] A gauge orbit is the set of field configurations related to each other by some gauge transformation. That is, $\int \mathcal{D}A$ would overcount the degrees of freedom.

[13] It is actually not assured that every Faddeev-Popov measure takes one and only one representative of each gauge orbit. This issue is relevant at the nonperturbative level and is referred to as the Gribov problem (or copies, or ambiguity), and is outlined in [15] and [48].

[14] Although we shall not discuss this mathematical subtlety, we note that the step just below involves interchanging the order of integrals at least one of which (since U is a function of coordinate space) is divergent.

[15] This supposition is an avoidance of the Gribov problem (see footnote number 13).

expression for Δ_{FP} :

$$\Delta_{FP}(A) = \det \left. \frac{\delta\left(G_\mu A_U^{a\mu}(x)\right)}{\delta\theta^b(y)} \right|_{f=0} , \tag{B.2}$$

where θ is related to U by $U = e^{i\theta^a(y)T^a}$, and T^a are the generators of the adjoint representation of the Lie algebra of the gauge group[49]. Taking the clever step of averaging $\mathcal{Z}$ over B as

$$\mathcal{Z} \to \int \mathcal{D}B\, \mathcal{Z} \exp \frac{-i}{2\eta} \int \mathrm{d}^D x \, \left(B^a(x)^2\right) ,$$

exponentiating the $G_\mu A_U^{a\mu}$ term, then leads to the usual gauge fixing term added to the Lagrangian:

$$\mathcal{Z}[J] = \int \mathcal{D}A\, \Delta_{FP}(A) \, \exp i \int \mathrm{d}^D x \, \left(\mathcal{L} - \frac{1}{2\eta}\left(G_\mu A^{a\mu}\right)^2 + J_\mu A^\mu\right) . \tag{B.3}$$

It is clear that $\mathcal{Z}$ depends on the choice of gauge, or class of gauges, which determines B.2, which, on the other hand, may depend or not on the gauge fields[32],p.51. If not, Δ_{FP} is simply a constant that can be discarded from B.3. But if Δ_{FP} is a function of the gauge fields, it is of great usefulness the following functional integral expression for an operator's determinant:

$$\det M = \int \mathcal{D}\chi \mathcal{D}\bar{\chi} \, \exp \int \mathrm{d}^D x \mathrm{d}^D y \bar{\chi}^a(x) M^{ab}(x,y) \chi^b(y) ,$$

where χ and $\bar{\chi}$ are Lorentz scalar, and Grassmann quantities, then $\Delta_{FP}(A)$ is (up to irrelevant multiplicative i factors) written as:

$$\Delta_{FP}(A) = -i \int \mathcal{D}\chi \mathcal{D}\bar{\chi} \, \exp \int \mathrm{d}^D x \mathrm{d}^D y \bar{\chi}^a(x) \left[\frac{\delta\left(G_\mu A_U^{a\mu}(x)\right)}{\delta\theta^b(y)}\right] \chi^b(y) , \tag{B.4}$$

with the Grassmann fields χ and $\bar{\chi}$ belonging to the adjoint representation of the gauge group. These are the so-called Faddeev-Popov ghosts (or "ghosts" only). In covariant gauges, $G = \partial$ and

$$\frac{\delta\left(G_\mu A_U^{a\mu}(x)\right)}{\delta\theta^b(y)} = \left(\delta^{ab}\Box - g f^{abc}\partial^\mu A_\mu^c\right)\delta^D(x-y) ,$$

which in B.4, and integrating by parts, gives

$$\Delta_{FP}(A) = i \int \mathcal{D}\chi \mathcal{D}\bar{\chi} \, \exp \int \mathrm{d}^D x \left[\partial^\mu \bar{\chi}^a(x)\right] D_\mu^{ab} \chi^b(x) , \tag{B.5}$$

where $D^{bc} = \delta^{bc}\partial - ig\left(T^a\right)^{bc} A^a$ is the covariant derivative[49], $D_\mu^{ab} = \delta^{ab}\partial_\mu - g f^{abc} A_\mu^c$ in the adjoint representation. Thus, the Faddeev-Popov determinant is exponentiated, leading B.4 to

$$\mathcal{Z}[J] = \int \mathcal{D}A \mathcal{D}\chi \mathcal{D}\bar{\chi} \, \exp i \int \mathrm{d}^D x \, \left(\mathcal{L} - \frac{1}{2\eta}\left(G_\mu A^{a\mu}\right)^2 + \left(\partial^\mu \bar{\chi}^a\right) D_\mu^{ab}\chi^b + J_\mu A^\mu\right) ,$$

so that the dynamics of the gauge fields is studied with the aid of the ghost fields, whose dynamics may also be considered (solely due to its functional integration above). To be more precise, the ghost fields are a feature of language only – they are purely formal, and their coupling to the gauge fields is a device to describe the actual gauge fields' self-interaction.

Therefore, one can finally consider the gauge theory as being described by an effective Lagrangian, given by

$$\mathcal{L}_{\text{eff}} = \mathcal{L} - \frac{1}{2\eta} \left(G_\mu A^{a\mu}\right)^2 + \left(\partial^\mu \bar{\chi}^a\right) D^{ab}_\mu \chi^b \,, \tag{B.6}$$

and study it through the generating functional

$$\mathcal{Z}\left[J, \lambda, \bar{\lambda}\right] = \int \mathcal{D}A \mathcal{D}\chi \mathcal{D}\bar{\chi} \, \exp i \int \mathsf{d}^D x \, \left(\mathcal{L}_{\text{eff}} + J_\mu A^\mu + \bar{\lambda}^a \chi^a + \bar{\chi}^a \lambda^a\right) \,.$$

Appendix C Calculation of the ghost SDE

In order to briefly show the process of calculation of SDEs, we now derive the SDE for the ghost propagator. The procedure shown below is standard to obtain each SDE, for which it is enough to apply more functional derivatives corresponding to the Green function of interest.

Since the functional derivative is

$$\frac{\delta \mathcal{L}}{\delta \bar{\chi}^a} = -\partial_\nu \frac{\partial \mathcal{L}}{\partial\left(\partial_\nu \bar{\chi}^a\right)} = -\tilde{Z}_3\left(\partial^2 \chi^a\right) + Z_g \tilde{Z}_3 {Z_3}^{1/2} g f^{abc} \partial_\nu\left(\chi^b A^{c\nu}\right) ,$$

then (3.25) takes the form:

$$\begin{aligned}
0 &= \left[-\tilde{Z}_3 \partial^2 \frac{\delta}{i\delta\bar{\lambda}^a(x)} + \tilde{Z}_1 g f^{abc} \partial^\mu \frac{\delta}{i\delta\bar{\lambda}^b(x)} \frac{\delta}{i\delta J^{c\mu}(x)} + \lambda^a(x)\right] \exp \mathcal{W}\left[J, \eta, \bar{\eta}.\lambda, \bar{\lambda}\right] \\
&= \left[-\tilde{Z}_3 \partial^2 \tfrac{\delta \mathcal{W}}{i\delta\bar{\lambda}^a(x)} + \tilde{Z}_1 g f^{abc} \partial^\mu \left(\tfrac{\delta \mathcal{W}}{i\delta\bar{\lambda}^b(x)} \tfrac{\delta \mathcal{W}}{i\delta J^{c\mu}(x)} + \tfrac{\delta^2 \mathcal{W}}{i\delta\bar{\lambda}^b(x) i\delta J^{c\mu}(x)}\right) + \lambda^a(x)\right] e^{\mathcal{W}}
\end{aligned}$$

$$\Rightarrow \quad 0 = \frac{\delta}{\delta\lambda^e(y)} \left[-\tilde{Z}_3 \partial^2 \tfrac{\delta \mathcal{W}}{i\delta\bar{\lambda}^a(x)} + \tilde{Z}_1 g f^{abc} \partial^\mu \left(\tfrac{\delta \mathcal{W}}{i\delta\bar{\lambda}^b(x)} \tfrac{\delta \mathcal{W}}{i\delta J^{c\mu}(x)} + \tfrac{\delta^2 \mathcal{W}}{i\delta\bar{\lambda}^b(x) i\delta J^{c\mu}(x)}\right) + \lambda^a(x)\right] e^{\mathcal{W}}\Big|_{\lambda,\bar{\lambda},c,\bar{c},J,a=0}$$

$$= \left[-\tilde{Z}_3 \partial^2 \frac{\delta^2 \mathcal{W}}{i\delta\lambda^e(y)\,\delta\bar{\lambda}^a(x)} + \tilde{Z}_1 g f^{abc} \partial^\mu \left(\frac{\delta^3 \mathcal{W}}{i\delta\lambda^e(y)\,\delta\bar{\lambda}^b(x)\, i\delta J^{c\mu}(x)}\right) + \delta^{ae}\delta^D(x-y)\right] e^{\mathcal{W}}\Big|_{\lambda,\bar{\lambda},c,\bar{c},J,a=0}$$

$$\Rightarrow \quad 0 = \left[-i\tilde{Z}_3 \partial^2 \frac{\delta^2 \mathcal{W}}{\delta\bar{\lambda}^a(x)\,\delta\lambda^e(y)} + \tilde{Z}_1 g f^{abc} \partial^\mu \frac{\delta}{\delta J^{c\mu}(x)} \left(\frac{\delta^2 \mathcal{W}}{\delta\bar{\lambda}^b(x)\,\delta\lambda^e(y)}\right) + \delta^{ae}\delta^D(x-y)\right]\Bigg|_{\lambda,\bar{\lambda},c,\bar{c},J,a=0}$$

$$\begin{aligned}
= \; & -\tilde{Z}_3 \partial^2 \left.\frac{\delta^2 \Gamma}{\delta\bar{c}^e(y)\,\delta c^a(x)}^{-1}\right|_{\lambda,\bar{\lambda},c,\bar{c},J,a=0} + \delta^{ae}\delta^D(x-y) \\
& -i\tilde{Z}_1 g f^{abc} \partial^\mu \left[\int \mathsf{d}^D z \, \frac{\delta a^{d\nu}(z)}{\delta J^{c\mu}(x)} \frac{\delta}{\delta a^{d\nu}(z)} \frac{\delta^2 \Gamma}{\delta\bar{c}^e(y)\,\delta c^b(x)}^{-1}\right]\Bigg|_{\lambda,\bar{\lambda},c,\bar{c},J,a=0} \\
= \; & -\tilde{Z}_3 \partial^2 \left.\frac{\delta^2 \Gamma}{\delta\bar{c}^e(y)\,\delta c^a(x)}^{-1}\right|_{\lambda,\bar{\lambda},c,\bar{c},J,a=0} + \delta^{ae}\delta^D(x-y) + \tilde{Z}_1 g f^{abc} \partial^\mu \left[\int \mathsf{d}^D z \mathsf{d}^D z_1 \mathsf{d}^D z_2 \, \times \right. \\
& \left. \times \; \frac{\delta^2 \mathcal{W}}{\delta J^{c\mu}(x)\,\delta J^d_\nu(z)} \frac{\delta^2 \Gamma}{\delta\bar{c}^e(y)\,\delta c^{b'}(z_1)}^{-1} \frac{\delta^3 \Gamma}{\delta a^{d\nu}(z)\,\delta\bar{c}^{b'}(z_1)\,\delta c^{e'}(z_2)} \frac{\delta^2 \Gamma}{\delta\bar{c}^{e'}(z_2)\,\delta c^b(x)}^{-1}\right]_{\lambda,\bar{\lambda},c,\bar{c},J,a=0} ,
\end{aligned}$$

where we made use of $\frac{\partial}{\partial X} M^{-1} = -M^{-1} \frac{\partial M}{\partial X} M^{-1}$. Finally, applying $\int \mathsf{d}^D y \, \frac{\delta^2 \Gamma}{\delta\bar{c}^f(z')\,\delta c^e(y)}$ (with implicit sum over all repeated indices) we obtain:

$$\begin{aligned} 0 &= -\tilde{Z}_3\partial^2\delta^{af}\delta^D(x-z') + \left.\frac{\delta^2\Gamma}{\delta\bar{c}^f(z')\,\delta c^a(x)}\right|_{\lambda,\bar{\lambda},c,\bar{c},J,a=0} \\ &\quad + \tilde{Z}_1 g f^{abc}\partial^\mu \left.\left[\int \mathsf{d}^D z \mathsf{d}^D y \, \frac{\delta^2\mathcal{W}}{\delta J^{c\mu}(x)\,\delta J^d_\nu(z)} \frac{\delta^3\Gamma}{\delta a^{d\nu}(z)\,\delta\bar{c}^f(z')\,\delta c^{e'}(y)} \frac{\delta^2\Gamma}{\delta\bar{c}^{e'}(y)\,\delta c^b(x)}^{-1}\right]\right|_{\lambda,\bar{\lambda},c,\bar{c},J,a=0} \end{aligned}$$

$$\begin{aligned} \Rightarrow \quad 0 &= G_p^{(2)fa}(x,z') - \tilde{Z}_3\partial^2\delta^{af}\delta^D(x-z') \\ &\quad - i\tilde{Z}_1 g f^{abc}\partial^\mu \int \mathsf{d}^D z \mathsf{d}^D y \, G_{c\ \mu}^{(2)cd\nu}(x,z)\, G_{p\ \nu}^{(3)dfe}(z,z',y)\, G_c^{(2)eb}(y,x) \ . \end{aligned}$$

Taking the corresponding Fourier transforms,

$$\begin{aligned} 0 &= \int \frac{\mathsf{d}^D p}{(2\pi)^D} e^{ip\cdot(x-z')} G_p^{(2)af}(p,-p) - \tilde{Z}_3\partial^2\delta^{af}\int \frac{\mathsf{d}^D p}{(2\pi)^D} e^{ip\cdot(x-z')} \\ &\quad - i\tilde{Z}_1 g f^{abc}\partial^\mu \int \mathsf{d}^D z \mathsf{d}^D y \frac{\mathsf{d}^D p_1}{(2\pi)^D}\cdots\frac{\mathsf{d}^D p_4}{(2\pi)^D} e^{i[p_1\cdot(x-z)+p_3\cdot z+p_4\cdot z'-(p_3+p_4)\cdot y+p_2\cdot(y-x)]} \times \\ &\quad \times G_{c\,\mu\nu}^{(2)cd}(p_1,-p_1)\, G_p^{(3)def\,\nu}(p_3,p_4,-p_3-p_4)\, G_c^{(2)be}(p_2,-p_2) \end{aligned}$$

$$\begin{aligned} \Rightarrow \quad 0 &= \int \frac{\mathsf{d}^D p}{(2\pi)^D} e^{ip\cdot(x-z')} \Big[G_p^{(2)af}(p,-p) - \tilde{Z}_3\delta^{af} i^2 p^2 \\ &\quad - i\tilde{Z}_1 g f^{abc} \int \frac{\mathsf{d}^D q}{(2\pi)^D} i p^\mu G_{c\,\mu\nu}^{(2)cd}(p+q,-p-q)\, G_p^{(3)def\,\nu}(p+q,-p,-q)\, G_c^{(2)be}(q,-q) \Big] \end{aligned}$$

$$\therefore \qquad \delta^{af}[D(p)]^{-1} = \tilde{Z}_3\delta^{af}\left[D_{(0)}(p)\right]^{-1} - \tilde{Z}_1 g^2 f^{abc} f^{cbf} \int \frac{\mathsf{d}^D q}{(2\pi)^D} \Lambda^\mu_{(0)}(p,q)\, D_{\mu\nu}(p+q)\, \Lambda^\nu(q,p)\, D(q) \ ,$$

Adopting the conventions below

$$\left\{ \begin{array}{c} G_{p\,\mu\nu}^{(2)ab}(p,-p) = \left[-iG_{c\,\mu\nu}^{(2)ab}(p,-p)\right]^{-1} = \left[D_{\mu\nu}^{ab}(p)\right]^{-1} = \delta^{ab}\left[D_{\mu\nu}^{ab}(p)\right]^{-1} \ , \\ G_p^{(2)ab}(p,-p) = \left[-iG_c^{(2)ab}(p,-p)\right]^{-1} = \left[D^{ab}(p)\right]^{-1} = \delta^{ab}[D(p)]^{-1} \ , \\ D^{ab}_{(0)}(p) = \delta^{ab} D_{(0)}(p) = -\frac{1}{p^2}\delta^{ab} \ , \\ -igf^{abc}\Lambda^\mu(p,q) = -iG_p^{(3)abc\,\mu}(q-p,p,-q) = -igf^{abc}\Lambda^\mu(-q,-p) \ , \\ -igf^{abc}\Lambda^\mu_{(0)}(p,q) = -igf^{abc}p^\mu \ . \end{array} \right. \tag{C.1}$$

and using the $SU(N)$ relation $f^{abc}f^{cbf} = f^{abc}f^{fcb} = -f^{abc}f^{fbc} = -N\delta^{af}$, we finally obtain:

$$[D(p)]^{-1} = \tilde{Z}_3\left[D_{(0)}(p)\right]^{-1} + \tilde{Z}_1 g^2 N \int \frac{\mathsf{d}^D q}{(2\pi)^D} \Lambda^\mu_{(0)}(p,q)\, D_{\mu\nu}(p+q)\, \Lambda^\nu(q,p)\, D(q) \ .$$

Appendix D The 1-loop calculations

In this appendix we show an outline of the 1-loop calculations we have made, as described in Sect.4.3. The loop integrals were calculated in D dimensions, and then dimensionally regularized[50].

D.1 Ghost-ghost-gluon loop

The diagram on Fig.4.1(a) contributes with

$$\xi^{\mu}_{(1)} = -\frac{g^3}{\mu^{D-4}}\int \frac{\mathrm{d}^D p}{(2\pi)^D} f^{d'f'c} r^{\rho} \frac{\delta^{ff'}}{(p+k)^2} f^{ae'f}(p+k)^{\mu} \frac{\delta^{ee'}}{p^2} f^{dbe} p^{\nu} \delta^{d'd}\left[\Delta^{(0)}_{\nu\rho}(q-p)\right] ,$$

where $\Delta_{\mu\nu}$ is given by (4.4). Then, in the Landau gauge

$$\xi^{\mu}_{(1)} = \frac{Ng^3}{2\mu^{D-4}} f^{abc}\int \frac{\mathrm{d}^D p}{(2\pi)^D} \frac{(p+k)^{\mu} p^{\nu} r^{\rho}}{p^2 (p+k)^2\left[(q-p)^2 - m^2\right]}\left(g_{\nu\rho} - \frac{(q-p)_{\nu}(q-p)_{\rho}}{(q-p)^2}\right) ,$$

since $f^{aef} f^{dbe} f^{dfc} = -(N/2) f^{abc}$. For $k=0$, $q=r$, and so:

$$\begin{aligned}
\xi^{\mu}_{(1)} &= \frac{Ng^3}{2\mu^{D-4}} f^{abc}\int \frac{\mathrm{d}^D p}{(2\pi)^D} \frac{p^{\mu}}{(p^2)^2\left[(q-p)^2 - m^2\right]}\left(p^2 - \frac{\left[p.(q-p)\right]^2}{(q-p)^2}\right) \\
&= \frac{Ng^3}{2\mu^{D-4}} f^{abc}\int \frac{\mathrm{d}^D p}{(2\pi)^D}\left\{\frac{p^{\mu}}{p^2\left[(q-p)^2 - m^2\right]} - \frac{1}{4}\frac{p^{\mu}\left[q^2 - p^2 - (q-p)^2\right]\left[q^2 - p^2 - (q-p)^2\right]}{(p^2)^2\left[(q-p)^2 - m^2\right](q-p)^2}\right\}
\end{aligned}$$

writing $p.(q-p) = \frac{1}{2}\left[q^2 - p^2 - (q-p)^2\right]$.

$$\begin{aligned}
\therefore\quad \xi^{\mu}_{(1)} &= \frac{Ng^3}{2\mu^{D-4}} f^{abc}\int \frac{\mathrm{d}^D p}{(2\pi)^D}\left\{\frac{p^{\mu}}{p^2\left[(q-p)^2 - m^2\right]} - \frac{1}{4}\frac{p^{\mu}(q^2-p^2)^2}{(p^2)^2\left[(q-p)^2 - m^2\right](q-p)^2}\right. \\
&\quad \left. + \frac{1}{2}\frac{p^{\mu}(q^2-p^2)}{(p^2)^2\left[(q-p)^2 - m^2\right]} - \frac{1}{4}\frac{p^{\mu}\left[(q-p)^2 - m^2 + m^2\right]}{(p^2)^2\left[(q-p)^2 - m^2\right]}\right\} \\
&= \frac{Ng^3}{2\mu^{D-4}} f^{abc}\int \frac{\mathrm{d}^D p}{(2\pi)^D}\left\{\frac{1}{2}\frac{p^{\mu}}{p^2\left[(p-q)^2 - m^2\right]} - \frac{1}{4}\frac{(p+q)^{\mu}}{(p^2-m^2)p^2}\right. \\
&\quad \left. + \frac{1}{4}\frac{(2q^2 - m^2)\ p^{\mu}}{(p^2)^2\left[(p-q)^2 - m^2\right]} + \frac{1}{2}\frac{q^2\ (p+q)^{\mu}}{(p+q)^2(p^2-m^2)p^2} - \frac{1}{4}\frac{(q^2)^2\ (p+q)^{\mu}}{\left((p+q)^2\right)^2(p^2-m^2)p^2}\right\} ,
\end{aligned}$$

where we have shifted the integration variable from p to $p-q$ when convenient in order to possibly obtain simpler integrands when Feynman-parametrizing:

$$
\begin{aligned}
\xi^{\mu}_{(1)} &= \frac{Ng^3}{2\mu^{D-4}} f^{abc} \int \frac{\mathsf{d}^D p}{(2\pi)^D} \int_0^1 \mathsf{d}x \left\{ \frac{\Gamma(2)}{2} \frac{p^\mu}{\left[\left((p-q)^2 - m^2\right)x + p^2(1-x)\right]^2} \right. \\
&\quad - \frac{\Gamma(2)}{4} \frac{(p+q)^\mu}{\left[(p^2-m^2)x + p^2(1-x)\right]^2} + \frac{\Gamma(3)}{4\Gamma(2)} \frac{(2q^2 - m^2)\ (1-x)\, p^\mu}{\left[\left((p-q)^2 - m^2\right)x + p^2(1-x)\right]^3} \\
&\quad + \int_0^x \mathsf{d}y \left[\frac{\Gamma(3)}{2} \frac{q^2\ (p+q)^\mu}{\left[(p^2-m^2)y + p^2(x-y) + (p+q)^2(1-x)\right]^3} \right. \\
&\qquad \left.\left. - \frac{\Gamma(4)}{4\Gamma(2)} \frac{(q^2)^2\ (1-x)(p+q)^\mu}{\left[(p^2-m^2)y + p^2(x-y) + (p+q)^2(1-x)\right]^4} \right] \right\} \\
&= \frac{Ng^3}{2\mu^{D-4}} f^{abc} \int \frac{\mathsf{d}^D p}{(2\pi)^D} \int_0^1 \mathsf{d}x \left\{ \frac{\Gamma(2)}{2} \frac{(p+xq)^\mu}{\left[p^2 + q^2 x(1-x) - m^2 x\right]^2} \right. \\
&\quad - \frac{\Gamma(2)}{4} \frac{(p+q)^\mu}{(p^2 - m^2 x)^2} + \frac{\Gamma(3)}{4\Gamma(2)} \frac{(2q^2 - m^2)\ (1-x)(p+xq)^\mu}{\left[p^2 + q^2 x(1-x) - m^2 x\right]^3} \\
&\quad \left. + \int_0^x \mathsf{d}y \left[\frac{\Gamma(3)}{2} \frac{q^2\ (p+xq)^\mu}{\left[p^2 + q^2 x(1-x) - m^2 y\right]^3} - \frac{\Gamma(4)}{4\Gamma(2)} \frac{(q^2)^2\ (1-x)(p+xq)^\mu}{\left[p^2 + q^2 x(1-x) - m^2 y\right]^4} \right] \right\} .
\end{aligned}
$$

Making use of well-known results for the momentum integrals[50], we obtain:

$$
\begin{aligned}
\xi^{\mu}_{(1)} &= \frac{iNg^3}{2(4\pi)^2} f^{abc} q^\mu \int_0^1 \mathsf{d}x \left\{ \frac{\Gamma(\epsilon/2)}{2} x \left(\frac{m^2 x - q^2 x(1-x)}{4\pi\mu^2} \right)^{-\epsilon/2} - \frac{\Gamma(\epsilon/2)}{4} \left(\frac{m^2 x}{4\pi\mu^2} \right)^{-\epsilon/2} \right. \\
&\quad + \frac{\Gamma(1+\epsilon/2)\ (2q^2 - m^2)\, x(1-x)}{4\Gamma(2)\ \left[q^2 x(1-x) - m^2 x\right]} \left(\frac{m^2 x - q^2 x(1-x)}{4\pi\mu^2} \right)^{-\epsilon/2} \\
&\quad \left. + \int_0^x \mathsf{d}y \left[\frac{\Gamma(1+\epsilon/2)\ q^2 x}{2\left[q^2 x(1-x) - m^2 y\right]} - \frac{\Gamma(2+\epsilon/2)\ (q^2)^2\, x(1-x)}{4\Gamma(2)\left[q^2 x(1-x) - m^2 y\right]^2} \right] \left(\frac{m^2 y - q^2 x(1-x)}{4\pi\mu^2} \right)^{-\epsilon/2} \right\}
\end{aligned}
$$

$$= \frac{iNg^3}{2\,(4\pi)^2} f^{abc} q^\mu \int_0^1 \mathsf{d}x \left\{ \frac{1}{2}\left(\frac{2}{\epsilon}-\gamma\right) x \left[1-\frac{\epsilon}{2}\log\left(\frac{-q^2x}{4\pi\mu^2}\right)-\frac{\epsilon}{2}\log\left(1-x-m^2/q^2\right)\right] \right.$$
$$-\frac{1}{4}\left(\frac{2}{\epsilon}-\gamma\right)\left[1-\frac{\epsilon}{2}\log\left(\frac{m^2x}{4\pi\mu^2}\right)\right]+\frac{1}{4}\left(2q^2-m^2\right)\frac{x\,(1-x)}{[q^2x\,(1-x)-m^2x]}$$
$$\left. +\int_0^x \mathsf{d}y \left[\frac{q^2x}{2\,[q^2x\,(1-x)-m^2y]}-\frac{(q^2)^2\,x\,(1-x)}{4\,[q^2x\,(1-x)-m^2y]^2}\right]\right\}+\mathcal{O}(\epsilon)\ ,$$

where, as conventional, we do not consider all positive powers of ϵ. The integrals over the Feynman parameters are calculated using the results in [51].

$$\Rightarrow\quad \xi^\mu_{(1)} = \frac{iNg^3}{2\,(4\pi)^2} f^{abc} q^\mu \int_0^1 \mathsf{d}x \left\{ \frac{1}{2}\left(\frac{2}{\epsilon}-\gamma\right)x-\frac{1}{4}\left(\frac{2}{\epsilon}-\gamma\right)+\frac{1}{4}\log\left(\frac{m^2x}{4\pi\mu^2}\right)\right.$$
$$-\frac{1}{2}x\log\left(\frac{-q^2x}{4\pi\mu^2}\right)-\frac{1}{2}x\log\left(1-x-m^2/q^2\right)+\frac{1}{4}\left(2-\frac{m^2}{q^2}\right)\frac{x\,(1-x)}{[x\,(1-x)-xm^2/q^2]}$$
$$\left.+\frac{1}{2}x\left[-\frac{q^2}{m^2}\log\left(x\,(1-x)-ym^2/q^2\right)\right]_0^x-\frac{1}{4}x\,(1-x)\left[\frac{q^2}{m^2}\frac{1}{x\,(1-x)-ym^2/q^2}\right]_0^x\right\}$$

$$= \frac{iNg^3}{4\,(4\pi)^2} f^{abc} q^\mu \int_0^1 \mathsf{d}x \left\{\frac{1}{2}\log\left(\frac{m^2x}{4\pi\mu^2}\right)-x\log\left(\frac{-q^2x}{4\pi\mu^2}\right)+\frac{q^2}{m^2}x\log(1-x)+\frac{1}{2}\frac{q^2}{m^2}\right.$$
$$\left.-\left(1+\frac{q^2}{m^2}\right)x\log\left(1-x-m^2/q^2\right)+\frac{1}{2}\left(2-\frac{m^2}{q^2}-\frac{q^2}{m^2}\right)\frac{1-x}{1-x-m^2/q^2}\right\}$$

$$= \frac{iNg^3}{8\,(4\pi)^2} f^{abc} q^\mu \left\{\frac{q^2}{m^2}+\left[x\log\left(\frac{m^2x}{4\pi\mu^2}\right)-x\right]_0^1-\left[x^2\log\left(\frac{-q^2x}{4\pi\mu^2}\right)-\frac{x^2}{2}\right]_0^1\right.$$
$$+\frac{q^2}{m^2}\left[\left(x^2-1\right)\log(1-x)-\left(\frac{x^2}{2}+x\right)\right]_0^1$$
$$-\left(1+\frac{q^2}{m^2}\right)\left[\left(x^2-\left(1-m^2/q^2\right)^2\right)\log\left(1-x-m^2/q^2\right)-\left(\frac{x^2}{2}+\left(1-m^2/q^2\right)x\right)\right]_0^1$$
$$\left.+\left(2-\frac{m^2}{q^2}-\frac{q^2}{m^2}\right)\left[x-m^2/q^2\log\left(1-x-m^2/q^2\right)\right]_0^1\right\}$$

$$
\begin{aligned}
= & \frac{iNg^3}{8\left(4\pi\right)^2} f^{abc} q^\mu \left\{ \log\left(\frac{m^2}{4\pi\mu^2}\right) - \log\left(\frac{-q^2}{4\pi\mu^2}\right) \right. \\
& + \left[\left(2 - \frac{m^2}{q^2} - \frac{q^2}{m^2}\right)\left(1 - \frac{m^2}{q^2}\right) - \left(1 + \frac{q^2}{m^2}\right)\frac{m^2}{q^2}\left(2 - \frac{m^2}{q^2}\right) - \left(2 - \frac{m^2}{q^2} - \frac{q^2}{m^2}\right)\right] \log\left(-\frac{m^2}{q^2}\right) \\
& + \left[\left(2 - \frac{m^2}{q^2} - \frac{q^2}{m^2}\right) - \left(1 + \frac{q^2}{m^2}\right)\left(1 - \frac{m^2}{q^2}\right)^2 - \left(2 - \frac{m^2}{q^2} - \frac{q^2}{m^2}\right)\left(1 - \frac{m^2}{q^2}\right)\right] \log\left(1 - \frac{m^2}{q^2}\right) \\
& + \left[-\frac{3}{2}\frac{q^2}{m^2} + \frac{q^2}{m^2} + \left(1 + \frac{q^2}{m^2}\right)\left(\frac{3}{2} - \frac{m^2}{q^2}\right) + \left(2 - \frac{m^2}{q^2} - \frac{q^2}{m^2}\right) - 1 + \frac{1}{2}\right] \\
= & \frac{iNg^3}{8\left(4\pi\right)^2} f^{abc} q^\mu \left[\frac{m^2}{q^2}\left(3 - 2\frac{m^2}{q^2}\right)\log\left(1 - \frac{q^2}{m^2}\right) - \frac{q^2}{m^2}\log\left(1 - \frac{m^2}{q^2}\right) + \left(1 - \frac{m^2}{q^2}\right)\right] ,
\end{aligned}
$$

after some algebraic manipulations.

D.2 Ghost-gluon-gluon loop

The contribution of the diagram on Fig.4.1(b) is

$$
\xi^\mu_{(2)} = -\frac{g^3}{\mu^{D-4}} \int \frac{\mathrm{d}^D p}{\left(2\pi\right)^D} f^{fe'c} r^\sigma \frac{\delta^{ee'}}{p^2} \delta^{ff'} \left[\Delta_{\sigma\tau}\left(p - r\right)\right] \left[V^{f'ad',\tau\mu\rho}\left(p - r, k, q - p\right)\right] \delta^{dd'} \left[\Delta_{\rho\nu}\left(q - p\right)\right] f^{dbe} p^\nu ,
$$

where $V^{abc}_{\mu\nu\rho}\left(p, q, r\right) = f^{abc}\left[g_{\mu\nu}\left(p - q\right)_\rho + g_{\nu\rho}\left(q - r\right)_\mu + g_{\rho\mu}\left(r - p\right)_\nu\right]$ and $\Delta_{\mu\nu}$ is given by (4.4). Then, since $f^{fad} f^{dbe} f^{fec} = -\left(N/2\right) f^{abc}$, for the configuration $k = r - q = 0$, and in the Landau gauge $\eta = 0$:

$$
\xi^\mu_{(2)} = \frac{Ng^3}{2\mu^{D-4}} f^{abc} \int \frac{\mathrm{d}^D p}{\left(2\pi\right)^D} \frac{\left[g^{\tau\mu} p^\rho + g^{\mu\rho} p^\tau - 2g^{\rho\tau} p^\mu\right]}{\left(p + q\right)^2 \left(p^2 - m^2\right)^2} \left(p + q\right)^\nu q^\sigma \left(g_{\sigma\tau} - \frac{p_\sigma p_\tau}{p^2}\right)\left(g_{\rho\nu} - \frac{p_\rho p_\nu}{p^2}\right) ,
$$

where we have shifted the integration variable in order to simplify most terms of the expression.

$$
\begin{aligned}
\therefore \quad \xi^\mu_{(2)} = & \frac{Ng^3}{2\mu^{D-4}} f^{abc} \int \frac{\mathrm{d}^D p}{\left(2\pi\right)^D} \frac{\left[g^{\tau\mu} p^\rho + g^{\mu\rho} p^\tau - 2g^{\rho\tau} p^\mu\right]}{\left(p + q\right)^2 \left(p^2 - m^2\right)^2} \left(q_\tau - p_\tau \frac{p.q}{p^2}\right)\left(q_\rho - p_\rho \frac{p.q}{p^2}\right) \\
= & \frac{Ng^3}{2\mu^{D-4}} f^{abc} \int \frac{\mathrm{d}^D p}{\left(2\pi\right)^D} \frac{1}{\left(p + q\right)^2 \left(p^2 - m^2\right)^2} \left[q^\mu p^\rho + \left(\frac{p.q}{p^2} p^\rho - 2q^\rho\right) p^\mu\right]\left(q_\rho - p_\rho \frac{p.q}{p^2}\right) \\
= & \frac{Ng^3}{2\mu^{D-4}} f^{abc} \int \frac{\mathrm{d}^D p}{\left(2\pi\right)^D} \frac{1}{\left(p + q\right)^2 \left(p^2 - m^2\right)^2} 2\left(-q^2 + \frac{\left(p.q\right)^2}{p^2}\right) p^\mu \\
= & \frac{Ng^3}{2\mu^{D-4}} f^{abc} \int \frac{\mathrm{d}^D p}{\left(2\pi\right)^D} \left[\frac{p.q\, p^\mu}{p^2\left(p^2 - m^2\right)^2} - q^2 \frac{p.q\, p^\mu}{\left(p + q\right)^2 \left(p^2 - m^2\right)^2 p^2} - \frac{\left(2q + p\right).q\, p^\mu}{\left(p + q\right)^2 \left(p^2 - m^2\right)^2}\right]
\end{aligned}
$$

$$= \frac{Ng^3}{2\mu^{D-4}} f^{abc} \int \frac{\mathsf{d}^D p}{(2\pi)^D} \int_0^1 \mathsf{d}x \left\{ -q^2 \frac{\Gamma(4)}{\Gamma(2)} \int_0^x \mathsf{d}y\, y \frac{[(p-(1-x)q).q]\, p^\mu}{\left[p^2 - q^2(1-x)^2 + q^2(1-x) - m^2 y\right]^4} \right.$$

$$\left. + \frac{\Gamma(3)}{\Gamma(2)} x \left[q_\nu \frac{p^\mu p^\nu}{(p^2 - m^2 x)^3} - \frac{[(2q + p - (1-x)q).q]\, p^\mu}{\left[p^2 - q^2(1-x)^2 + q^2(1-x) - m^2 x\right]^3} \right] \right\}$$

$$= \frac{Ng^3}{2\mu^{D-4}} f^{abc} \frac{q^\mu}{D} \int \frac{\mathsf{d}^D p}{(2\pi)^D} \int_0^1 \mathsf{d}x \left\{ -q^2 \frac{\Gamma(4)}{\Gamma(2)} \int_0^x \mathsf{d}y\, y \frac{p^2}{[p^2 + q^2 x(1-x) - m^2 y]^4} \right.$$

$$\left. + \frac{\Gamma(3)}{\Gamma(2)} x \left[\frac{p^2}{(p^2 - m^2 x)^3} - \frac{p^2}{[p^2 + q^2 x(1-x) - m^2 x]^3} \right] \right\}$$

$$= \frac{iNg^3}{2(4\pi)^2} f^{abc} q^\mu \frac{1}{4-\epsilon} \int_0^1 \mathsf{d}x \left\{ \frac{\Gamma(3-\epsilon/2)\,\Gamma(\epsilon/2)}{\Gamma(2)\,\Gamma(2-\epsilon/2)}\, x \left[\left(\frac{m^2 x}{4\pi\mu^2} \right)^{-\epsilon/2} - \left(\frac{-q^2 x(1-x) + m^2 x}{4\pi\mu^2} \right)^{-\epsilon/2} \right] \right.$$

$$\left. - q^2 \frac{\Gamma(3-\epsilon/2)\,\Gamma(1+\epsilon/2)}{\Gamma(2)\,\Gamma(2-\epsilon/2)} \int_0^x \mathsf{d}y\, \frac{y}{q^2 x(1-x) - m^2 y} \left(\frac{-q^2 x(1-x) + m^2 y}{4\pi\mu^2} \right)^{-\epsilon/2} \right\}$$

$$= \frac{iNg^3}{4(4\pi)^2} f^{abc} q^\mu \int_0^1 \mathsf{d}x \left\{ x \left[\log\left(\frac{-q^2 x}{4\pi\mu^2} \right) - \log\left(\frac{m^2 x}{4\pi\mu^2} \right) + \log\left(1 - x - m^2/q^2\right) \right] \right.$$

$$\left. + \frac{q^2}{m^2} \left[y + \frac{q^2}{m^2} x(1-x) \log\left(x(1-x) - y m^2/q^2 \right) \right]_0^x \right\} + \mathcal{O}(\epsilon)$$

$$= \frac{iNg^3}{4(4\pi)^2} f^{abc} q^\mu \int_0^1 \mathsf{d}x \left\{ x \log\left(-\frac{q^2}{m^2} \right) + x \log\left(1 - x - m^2/q^2\right) \right.$$

$$\left. + \frac{q^2}{m^2} x + \left(\frac{q^2}{m^2} \right)^2 x(1-x) \left[\log\left(1 - x - m^2/q^2\right) - \log(1-x) \right] \right\}$$

$$= \frac{iNg^3}{4(4\pi)^2} f^{abc} q^\mu \left\{ \frac{1}{6} \frac{q^2}{m^2} - \frac{5}{12} + \frac{1}{2} \frac{m^2}{q^2} + \frac{1}{2} \log\left(-\frac{q^2}{m^2} \right) \right.$$

$$+ \left(-\frac{2}{3} \frac{m^2}{q^2} + \frac{1}{2} \frac{m^2}{q^2} \frac{m^2}{q^2} + \frac{1}{6} \frac{q^2}{m^2} \frac{q^2}{m^2} \right) \log\left(1 - \frac{m^2}{q^2} \right)$$

$$\left. + \left(\frac{1}{2} + \frac{2}{3} \frac{m^2}{q^2} - \frac{1}{2} \frac{m^2}{q^2} \frac{m^2}{q^2} \right) \log\left(-\frac{m^2}{q^2} \right) \right\}$$

$$= \frac{iNg^3}{4\,(4\pi)^2} f^{abc} q^\mu \left\{ -\frac{5}{12} + \frac{1}{6}\frac{q^2}{m^2}\left[1 + \frac{q^2}{m^2}\log\left(1 - \frac{m^2}{q^2}\right)\right] + \frac{m^2}{q^2}\left[\frac{1}{2} + \left(\frac{1}{2}\frac{m^2}{q^2} - \frac{2}{3}\right)\log\left(1 - \frac{q^2}{m^2}\right)\right]\right\}$$

D.3 Their sum and massless limit

We have obtained:

$$\xi^\mu_{(1)} = \frac{iNg^3}{4\,(4\pi)^2} f^{abc} q^\mu \zeta_1\,,$$

where

$$\zeta_1 = 1 - \frac{m^2}{q^2} + \frac{1}{2}\frac{m^2}{q^2}\left(3 - 2\frac{m^2}{q^2}\right)\log\left(1 - \frac{q^2}{m^2}\right) - \frac{1}{2}\frac{q^2}{m^2}\log\left(1 - \frac{m^2}{q^2}\right)\,,$$

whose limit as $m \to 0$ is:

$$\zeta_1 \xrightarrow{m\to 0} 1 - 0 + \frac{1}{2}(0) - \frac{1}{2}(-1) = \frac{3}{2}\,.$$

The second diagram contributes with

$$\xi^\mu_{(2)} = \frac{iNg^3}{4\,(4\pi)^2} f^{abc} q^\mu \zeta_2\,,$$

where

$$\zeta_2 = -\frac{5}{12} + \frac{1}{6}\frac{q^2}{m^2}\left[1 + \frac{q^2}{m^2}\log\left(1 - \frac{m^2}{q^2}\right)\right] + \frac{m^2}{q^2}\left[\frac{1}{2} + \left(\frac{1}{2}\frac{m^2}{q^2} - \frac{2}{3}\right)\log\left(1 - \frac{q^2}{m^2}\right)\right]\,,$$

whose limit as $m \to 0$ is, then:

$$\zeta_2 \xrightarrow{m\to 0} -\frac{5}{12} + \frac{1}{6}\left(-\frac{1}{2}\right) + 0 = -\frac{1}{2}\,.$$

Therefore, we obtain the expected limit of $\tilde{Z}_1 = 1$ for massless gluon propagators:

$$\xi^\mu = igf^{abc}q^\mu \frac{N\alpha}{16\pi}(\zeta_1 + \zeta_2) \xrightarrow{m\to 0} igf^{abc}q^\mu \frac{N\alpha}{16\pi}\,,$$

where we write explicitly the correction to the tree-level expression $-igf^{abc}q^\mu$. Finally, in order to show the result based on our proposal (Sect.4.3), we write the factors ζ_1 and ζ_2 above in terms of $-q^2$, since our analysis in Chap.5 considers spacelike q^2, and our calculation also considered $q^2 < 0$[16]:

$$\xi^\mu = igf^{abc}q^\mu \frac{N\alpha}{16\pi}\zeta\left(q^2\right)\,,$$

[16] This assumption does not affect our results. It only keeps the integrands regular, discarding some harmless boundary terms of the Feynman integrals which would, anyhow, be gone in the renormalization process.

$$\zeta = \frac{7}{12} + \frac{1}{2}\frac{m^2}{q^2} - \frac{1}{6}\frac{q^2}{m^2} \; - \frac{m^2}{q^2}\left(\frac{5}{6} + \frac{1}{2}\frac{m^2}{q^2}\right)\log\left(1 + \frac{q^2}{m^2}\right) + \frac{1}{6}\frac{q^2}{m^2}\left(3 + \frac{q^2}{m^2}\right)\log\left(1 + \frac{m^2}{q^2}\right) \; ,$$

where, as said above, q^2 denotes $|q^2|$.