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Zeros of Jacobi functions of second kind

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Abstract

The number of zeros in $(-1, 1)$ of the Jacobi function of second kind $Q_n^{(\alpha,\beta)}(x)$, $\alpha, \beta > -1$, i.e. the second solution of the differential equation

$$(1-x^2)y''(x) + (\beta - \alpha - (\alpha+\beta+2)x)y'(x) + n(n+\alpha+\beta+1)y(x) = 0,$$

is determined for every $n \in \mathbb{N}$ and for all values of the parameters $\alpha > -1$ and $\beta > -1$. It turns out that this number depends essentially on α and β as well as on the specific normalization of the function $Q_n^{(\alpha,\beta)}(x)$. Interlacing properties of the zeros are also obtained. As a consequence of the main result, we determine the number of zeros of Laguerre's and Hermite's functions of second kind.

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1. Introduction

Consider the Jacobi differential equation

$$(1-x^2)y''(x)+(\beta-\alpha-(\alpha+\beta+2)x)y'(x)+n(n+\alpha+\beta+1)y(x)=0, \tag{1}$$

where $n\in\mathbb{N}$, $\alpha,\beta>-1$. It is classically known that it has two linearly independent solutions, one of them being the Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$. The behaviour of the n zeros of this polynomial has been thoroughly investigated, mainly because of their important role as nodes of Gaussian quadrature and because of their nice electrostatic interpretation. Surprisingly enough, very little is known about the zeros of the non-polynomial solution $Q_n^{(\alpha,\beta)}(x)$ of (1).

There are few results in the literature concerning the zeros of $Q_n^{(\alpha,\beta)}(x)$. Sturm's theorem guarantees that the zeros of $P_n^{(\alpha,\beta)}(x)$ and of $Q_n^{(\alpha,\beta)}(x)$ interlace. Stieltjes [9] (see also [10, Theorem 6.9.2]) proved that the Legendre function of second kind $Q_n^{(0,0)}(x)$ has exactly $n+1$ zeros in $(-1,1)$. Moreover, he established the inequalities $\nu\pi/(n+\frac{1}{2})<\theta<(\nu+\frac{1}{2})\pi/(n+\frac{1}{2})$, $\nu=0,1,2,\ldots,n$, for the zeros of $Q_n^{(0,0)}(\cos\theta)$. Recently, Peherstorfer and Schmuckenschläger [6] calculated the number of zeros of the function $Q_0^{(\alpha,\beta)}(x)$ and proved that this number depends on the parameters α and β.

The reason for the lack of results might be the simple fact that, if $Q_n^{(\alpha,\beta)}(x)$ is a "second" solution, then, for any value of the real constant C, the function

$$\widetilde{Q}_n^{(\alpha,\beta)}(x)=Q_n^{(\alpha,\beta)}(x)+C\,P_n^{(\alpha,\beta)}(x), \tag{2}$$

is also a solution of (1) and $\widetilde{Q}_n^{(\alpha,\beta)}(x)$ may be called a "second" solution, too. Roughly speaking, the problem is in the specific normalization which should be adopted. While the number and location of zeros of the polynomial solution does not depend on the normalization, in the case of the second solution both the number and location of the zeros of $\widetilde{Q}_n^{(\alpha,\beta)}(x)$ depend essentially on the value of the constant C. In this paper, we adopt a specific normalization of the second solution and determine the number of its zeros in $(-1,1)$. Grosjean [4] proved that

$$Q_n^{(\alpha,\beta)}(x)=\frac{\Gamma(\alpha+\beta+2)(1-x)^{-\alpha}\,(1+x)^{-\beta}}{2^{\alpha+\beta+1}\,\Gamma(\alpha+1)\,\Gamma(\beta+1)}\mathscr{P}\int_{-1}^{1}\frac{P_n^{(\alpha,\beta)}(s)(1-s)^{\alpha}(1+s)^{\beta}}{x-s}\,\mathrm{d}s, \tag{3}$$

where $\mathscr{P}$ denotes the Cauchy principal value and $P_n^{(\alpha,\beta)}(x)$ is the nth Jacobi polynomial normalized by $P_n^{(\alpha,\beta)}(1)=(\alpha+1)_n/n!$, is a solution of (1) that is linearly independent to $P_n^{(\alpha,\beta)}(x)$. Let $x_{n,k}(\alpha,\beta)$ be the zeros of $P_n^{(\alpha,\beta)}(x)$, arranged in decreasing order,

$$-1<x_{n,n}(\alpha,\beta)<\cdots<x_{n,2}(\alpha,\beta)<x_{n,1}(\alpha,\beta)<1.$$

We shall denote by $q_{n,i}(\alpha,\beta)$ the zeros of $Q_n^{(\alpha,\beta)}(x)$ in $(-1,1)$, arranged also in decreasing order. According to Sturm's theorem about interlacing of zeros of linearly independent solutions of differential equations, there is exactly one simple zero of $Q_n^{(\alpha,\beta)}(x)$ in each of the intervals surrounded by two consecutive zeros of $P_n^{(\alpha,\beta)}(x)$. In other words, the inequalities $x_{n,k+1}(\alpha,\beta)<q_{n,k}(\alpha,\beta)<x_{n,k}(\alpha,\beta)$ hold for $k=1,\ldots,n-1$, and $Q_n^{(\alpha,\beta)}(x)$ has at least $n-1$ zeros in $(-1,1)$. In what follows we shall omit the parameters α and β in the denotations of these zeros using the more succinct forms $x_{n,k}$ and $q_{n,k}$. We shall prove the following result.

Table 1
Number of zeros of $Q_n^{(\alpha,\beta)}(x)$ in $(-1, 1)$

$\alpha\backslash\beta$	$-1<\beta\leqslant -\frac{1}{2}$	$-\frac{1}{2}<\beta$
$-1<\alpha\leqslant -\frac{1}{2}$	$n-1$	n_ℓ
$-\frac{1}{2}<\alpha$	n_r	$n+1$

Theorem 1. *Let $n \in \mathbb{N}$. The number of zeros of the Jacobi function of the second kind $Q_n^{(\alpha,\beta)}(x)$, defined by* (3), *in the interval* $(-1, 1)$, *is*

(a) *$n-1$ when $-1<\alpha,\ \beta\leqslant -\frac{1}{2}$;*
(b) *n when either $-1<\alpha\leqslant -\frac{1}{2},\ \beta> -\frac{1}{2}$, or $\alpha> -\frac{1}{2}, -1<\beta\leqslant -\frac{1}{2}$;*
(c) *$n+1$ when $\alpha, \beta> -\frac{1}{2}$.*

Moreover, the following interlacing properties hold:

(A) *The $n-1$ zeros of $Q_n^{(\alpha,\beta)}(x)$ satisfy $x_{n,k+1}(\alpha,\beta)<q_{n,k}(\alpha,\beta)<x_{n,k}(\alpha,\beta)$ for $k=1,\ldots,n-1$.*
(B′) *When $-1<\alpha\leqslant -\frac{1}{2},\ \beta> -\frac{1}{2}$, except for the zeros, described in* (A), *$Q_n^{(\alpha,\beta)}(x)$ has one additional zero $q_{n,n}\in(-1, x_{n,n})$.*
(B″) *When $\alpha> -\frac{1}{2}, -1<\beta\leqslant -\frac{1}{2}$, except for the zeros, described in* (A), *$Q_n^{(\alpha,\beta)}(x)$ has one additional zero $q_{n,0}\in(x_{n,1}, 1)$.*
(C) *When $\alpha, \beta>-\frac{1}{2}$ the function $Q_n^{(\alpha,\beta)}(x)$ has $n+1$ zeros satisfying the interlacing properties described in* (A), (B′) *and* (B″).

The results of the theorem are summarized in Table 1. There n_ℓ means that the additional zero of $Q_n^{(\alpha,\beta)}(x)$ appears to the left of the smallest zero $x_{n,n}$ of $P_n^{(\alpha,\beta)}(x)$, and n_r means that the extra zero of $Q_n^{(\alpha,\beta)}(x)$ appears to the right of the largest zero $x_{n,1}$ of $P_n^{(\alpha,\beta)}(x)$.

The cases when either $\alpha=-\frac{1}{2}$ or $\beta=-\frac{1}{2}$ are of particular interest. Our analysis yields the following:

Proposition 2. *The equalities* $\lim_{x\uparrow 1} Q_n^{(-1/2,\beta)}(x)=0$ *and* $\lim_{x\downarrow -1} Q_n^{(\alpha,-1/2)}(x)=0$ *hold for any α, $\beta> -1$.*

Moreover, when $\alpha=\beta=-\frac{1}{2}$, for the second kind Chebyshev function is given by

$$Q_n^{(-1/2,-1/2)}(x)=\sqrt{1-x^2}\,U_{n-1}(x),$$

where $U_n(x)$ is the Chebyshev polynomial of the second kind (see [4]).

The above-mentioned version of Sturm's theorem implies that $Q_n^{(\alpha,\beta)}(x)$ may possess eventually additional zeros, but at most one in each of the intervals $(-1, x_{n,n})$ and $(x_{n,1}, 1)$. Despite that this analysis depends only on the behaviour of the Jacobi function of the second kind close to the end-points of $(-1, 1)$, Theorem 1 provides the first complete and correct characterization of the number of zeros of $Q_n^{(\alpha,\beta)}(x)$. It might be worth mentioning that Lemma 2 in [11] erroneously states that $Q_n^{(\alpha,\beta)}(x)$ always has $n+1$ zeros, no matter what the values of α and β are, while Theorem 1 in the present paper shows that it happens if and only if $\alpha > -\frac{1}{2}$ and $\beta > -\frac{1}{2}$. We thank Professors Wong and Zhang for a discussion on this topic.

The paper is organized as follows. In Section 2 we recall some results obtained in [4,6]. Theorem 1 is proved in Section 3. In Section 4 we investigate the positions of zeros of $Q_{n+1}^{(\alpha,\beta)}(x)$, relative to those of $(\mathrm{d}/\mathrm{d}x)Q_n^{(\alpha,\beta)}(x)$, $Q_{n-1}^{(\alpha,\beta)}(x)$ and $Q_n^{(\alpha,\beta)}(x)$. In the last section we determine the number of zeros of the Laguerre and Hermite functions of the second kind in $(0, +\infty)$ and $(-\infty, +\infty)$, respectively.

2. Preliminaries

Let $\alpha > -1$, $\beta > -1$, and $n \geqslant 0$. As it was already mentioned, except for the case $n = 0$, $\alpha + \beta + 1 = 0$, which was considered in [10, Theorem 4.61.1], in what follows we consider the solution $y(x) = Q_n^{(\alpha,\beta)}(x)$ of (1) as given in (3). Grosjean [4, Eq. (41)] proved that, for every $x \in (-1, 1)$,

$$Q_n^{(\alpha,\beta)}(x) = P_n^{(\alpha,\beta)}(x)\left[(\alpha+\beta+1)\int_0^x \frac{\mathrm{d}t}{(1-t)^{\alpha+1}(1+t)^{\beta+1}} + \frac{\Gamma(\alpha+\beta+2)}{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)}\Lambda_{\alpha\beta}\right] - \frac{W_{n-1}^{(\alpha,\beta)}(x)}{(1-x)^{\alpha}(1+x)^{\beta}}, \tag{4}$$

where [4, p. 268, Eq. (32)]

$$\Lambda_{\alpha\beta} = (\alpha-\beta)\sum_{n=1}^{\infty}\frac{1}{(n+\alpha)(n+\beta)} + \frac{\alpha}{\beta+1}\,{}_3F_2\left(\left.\begin{matrix}1-\alpha, 1, 1\\ \beta+2, 2\end{matrix}\,\right|\,-1\right) - \frac{\beta}{\alpha+1}\,{}_3F_2\left(\left.\begin{matrix}1-\beta, 1, 1\\ \alpha+2, 2\end{matrix}\,\right|\,-1\right),$$

and, for $n \in \mathbb{N}$,

$$W_{n-1}^{(\alpha,\beta)}(x) = \frac{\Gamma(\alpha+\beta+2)}{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)}\int_{-1}^{1}\frac{P_n^{(\alpha,\beta)}(x) - P_n^{(\alpha,\beta)}(s)}{x-s}(1-s)^{\alpha}(1+s)^{\beta}\,\mathrm{d}s,$$

are the first associated polynomials [4,5,10].

Note that, for $n \in \mathbb{N}$,

$$\frac{\mathrm{d}}{\mathrm{d}x}Q_n^{(\alpha,\beta)}(x) = \frac{2(1+\alpha)(1+\beta)(1+\alpha+\beta+n)}{(2+\alpha+\beta)(3+\alpha+\beta)}Q_{n-1}^{(\alpha+1,\beta+1)}(x). \tag{5}$$

Also, in Grosjean [4, p. 286, Eq. (87)], we have

$$W_n^{(\alpha,\beta)}(x) = \frac{(\alpha+\beta+1)(n+\alpha+\beta+2)}{2}\sum_{k=0}^{n}\frac{P_k^{(-\alpha,-\beta)}(x)P_{n-k}^{(\alpha,\beta)}(x)}{(k+1)(n-k+\alpha+\beta+1)}.$$

On using the normalization

$$P_n^{(\alpha,\beta)}(1) = \frac{(\alpha+1)_n}{n!}, \tag{6}$$

we obtain

$$W_n^{(0,\beta)}(1) = \frac{(1+\beta)(2+\beta+n)}{2} \sum_{k=0}^{n} \frac{1}{(k+1)(n-k+\beta+1)}, \tag{7}$$

when $\alpha = 0$ and

$$W_n^{(\alpha,\beta)}(1) = \frac{1}{2\alpha} \left(\frac{(1+\alpha+\beta)\Gamma(n+\alpha+2)}{\Gamma(\alpha+1)\Gamma(n+2)} - \frac{\Gamma(\alpha+\beta+2)\Gamma(n+\beta+2)}{\Gamma(\beta+1)\Gamma(n+\alpha+\beta+2)} \right), \tag{8}$$

otherwise. Obviously, we have $W_n^{(\alpha,\beta)}(1) = (-1)^n W_n^{(\beta,\alpha)}(-1)$ and

$$Q_n^{(\alpha,\beta)}(-x) = (-1)^{n-1} Q_n^{(\beta,\alpha)}(x), \quad \text{for every } n \in \mathbb{N}. \tag{9}$$

It follows from [4, (39)] that, for $\alpha, \beta > -1$, and $x \in (-1, 1)$, we have

$$\begin{aligned} Q_0^{(\alpha,\beta)}(x) &= (\alpha+\beta+1) \int_0^x \frac{\mathrm{d}t}{(1-t)^{\alpha+1}(1+t)^{\beta+1}} + \frac{\Gamma(\alpha+\beta+2)}{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)} \Lambda_{\alpha\beta} \\ &= \frac{\Gamma(\alpha+\beta+2)}{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)(1-x)^{\alpha}(1+x)^{\beta}} \mathscr{P} \int_{-1}^{1} \frac{(1-t)^{\alpha}(1+t)^{\beta}}{x-t}\, \mathrm{d}t. \end{aligned} \tag{10}$$

It was proved in [6] that

$$\lim_{x \uparrow 1} Q_0^{(\alpha,\beta)}(x) = \begin{cases} -\dfrac{\pi\Gamma(\alpha+\beta+2)}{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)} \cot(\alpha\pi), & \alpha < 0, \\ +\infty, & \alpha \geqslant 0, \end{cases} \tag{11}$$

provided $\alpha+\beta+1 \neq 0$.

We shall make use of the reflection formula for the Gamma function [1, p. 256, Eq. (6.1.17)]

$$\Gamma(\alpha+1)\Gamma(-\alpha) = -\frac{\pi}{\sin(\alpha\pi)}, \quad -1 < \alpha < 0, \tag{12}$$

and of the sum

$$\sum_{k=1}^{\infty} \frac{1}{(k+\alpha)(k-\alpha-1)} = -\frac{\pi \cot(\alpha\pi)}{2\alpha+1}, \quad \alpha \notin \mathbb{Z}, \quad \alpha \neq -\frac{1}{2}. \tag{13}$$

3. Number of zeros of $Q_n^{(\alpha,\beta)}(x)$ in $(-1, 1)$

Proof of Theorem 1. From (4), if $x_{n,k}(\alpha,\beta)$ is a zero of the nth degree Jacobi polynomial, then

$$Q_n^{(\alpha,\beta)}(x_{n,k}(\alpha,\beta)) = -\frac{W_{n-1}^{(\alpha,\beta)}(x_{n,k}(\alpha,\beta))}{(1-x_{n,k}(\alpha,\beta))^{\alpha}(1+x_{n,k}(\alpha,\beta))^{\beta}}.$$

Since the zeros of $W_{n-1}^{(\alpha,\beta)}(x)$ and $P_n^{(\alpha,\beta)}(x)$ interlace [3, p. 86], then

$$\operatorname{sign}(Q_n^{(\alpha,\beta)}(x_{n,k}(\alpha,\beta))) = \operatorname{sign}\left(-\frac{W_{n-1}^{(\alpha,\beta)}(x_{n,k}(\alpha,\beta))}{(1-x_{n,k}(\alpha,\beta))^{\alpha}(1+x_{n,k}(\alpha,\beta))^{\beta}}\right) = (-1)^k. \tag{14}$$

The behaviour of $Q_n^{(\alpha,\beta)}(x)$ close to the end points ± 1 remains to be investigated. The identity

$$Q_n^{(\alpha,\beta)}(x) = P_n^{(\alpha,\beta)}(x)Q_0^{(\alpha,\beta)}(x) - \frac{W_{n-1}^{(\alpha,\beta)}(x)}{(1-x)^{\alpha}(1+x)^{\beta}}, \quad x \in (-1,1), \tag{15}$$

follows from (4) and (10). We study the behaviour of $Q_n^{(\alpha,\beta)}(x)$ as $x \uparrow 1$, taking the limit, as $x \uparrow 1$, in the above expression. Let us begin with the case $\alpha+\beta+1=0$, $(\alpha,\beta) \neq (-\frac{1}{2},-\frac{1}{2})$. Since in this case $\alpha<0$, then either by using (4) and the explicit expression for $\Lambda_{\alpha,-1-\alpha}$, or by employing (15) and (10), we obtain

$$\lim_{x\uparrow 1} Q_n^{(\alpha,-1-\alpha)}(x) = P_n^{(\alpha,-1-\alpha)}(1)\frac{2\alpha+1}{\Gamma(\alpha+1)\Gamma(-\alpha)}\sum_{k=1}^{\infty}\frac{1}{(k+\alpha)(k-\alpha-1)}.$$

Then Eqs. (12) and (13) immediately yield

$$\lim_{x\uparrow 1} Q_n^{(\alpha,-1-\alpha)}(x) = \frac{(\alpha+1)_n\cos(\alpha\pi)}{n!}.$$

Therefore the sign of this limit is negative for $\alpha \in (-1,-\frac{1}{2})$ and positive for $\alpha \in (-\frac{1}{2},0)$.

Now we proceed the study of $\lim_{x\uparrow 1} Q_n^{(\alpha,\beta)}(x)$ for the general situation when $\alpha+\beta+1 \neq 0$. We shall distinguish the cases: $\alpha<0$, $\alpha=0$ and $\alpha>0$.

(1) $\alpha<0$

By using (11), (6) and (8) we have

$$\begin{aligned}\lim_{x\uparrow 1} Q_n^{(\alpha,\beta)}(x) &= \lim_{x\uparrow 1}\left\{P_n^{(\alpha,\beta)}(x)Q_0^{(\alpha,\beta)}(x) - \frac{W_{n-1}^{(\alpha,\beta)}(x)}{(1-x)^{\alpha}(1+x)^{\beta}}\right\}\\ &= P_n^{(\alpha,\beta)}(1)\lim_{x\uparrow 1} Q_0^{(\alpha,\beta)}(x) = -\frac{(\alpha+1)_n}{n!}\frac{\pi\Gamma(\alpha+\beta+2)}{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)}\cot(\alpha\pi).\end{aligned}$$

Obviously the right-hand side of the latter expression depends on α:

(1.1) $-1<\alpha<-\frac{1}{2}$

$$\lim_{x\uparrow 1} Q_n^{(\alpha,\beta)}(x) < 0.$$

Therefore $Q_n^{(\alpha,\beta)}(x)$ does not vanish in $(x_{n,1}(\alpha,\beta),1)$.

(1.2) $-\frac{1}{2}<\alpha<0$

$$\lim_{x\uparrow 1} Q_n^{(\alpha,\beta)}(x) > 0.$$

Hence, $Q_n^{(\alpha,\beta)}(x)$ has exactly one zero in the interval $(x_{n,1}(\alpha,\beta),1)$.

(1.3) $\alpha = -\frac{1}{2}$

In this case $\lim_{x\uparrow 1} Q_n^{(-1/2,\beta)}(x) = 0$.

(2) $\alpha = 0$

It follows from (11), (6) and (7) that

$$\lim_{x\uparrow 1} Q_n^{(0,\beta)}(x) = \lim_{x\uparrow 1}\left\{P_n^{(0,\beta)}(x)Q_0^{(0,\beta)}(x) - \frac{W_{n-1}^{(0,\beta)}(x)}{(1+x)^\beta}\right\}$$
$$= \lim_{x\uparrow 1} Q_0^{(0,\beta)}(x) - \frac{(1+\beta)(2+\beta+n)}{2^{\beta+1}}\sum_{k=0}^{n}\frac{1}{(k+1)(n-k+\beta+1)} = +\infty.$$

Therefore, $Q_n^{(0,\beta)}(x)$ has exactly one zero $q_{n,0}(0,\beta)$ in the interval $(x_{n,1}(0,\beta),1)$. This result generalizes [10, Theorem 6.9.2], which provides the number of zeros of the Legendre functions of the second kind in $(-1,1)$.

(3) $\alpha > 0$

Multiplying (4) by $(1-x)^\alpha$, we obtain

$$Q_n^{(\alpha,\beta)}(x)(1-x)^\alpha = P_n^{(\alpha,\beta)}(x)\left[(\alpha+\beta+1)(1-x)^\alpha\int_0^x \frac{\mathrm{d}t}{(1-t)^{\alpha+1}(1+t)^{\beta+1}}\right.$$
$$\left. +(1-x)^\alpha \frac{\Gamma(\alpha+\beta+2)}{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)}\Lambda_{\alpha\beta}\right] - \frac{W_{n-1}^{(\alpha,\beta)}(x)}{(1+x)^\beta}.$$

Since

$$\lim_{x\uparrow 1}(1-x)^\alpha\int_0^x \frac{\mathrm{d}t}{(1-t)^{\alpha+1}(1+t)^{\beta+1}} = \frac{1}{\alpha 2^{\beta+1}},$$

then the sign of $Q_n^{(\alpha,\beta)}(x)$ at the points from $(-1,1)$ that are close to 1 is the same as the sign of

$$\frac{(\alpha+1)_n}{n!}\frac{\alpha+\beta+1}{\alpha 2^{\beta+1}} - \frac{W_{n-1}^{(\alpha,\beta)}(1)}{2^\beta} = \frac{2^{-1-\beta}\Gamma(2+\alpha+\beta)\Gamma(1+\beta+n)}{\alpha\Gamma(1+\beta)\Gamma(1+\alpha+\beta+n)},$$

which is always positive. Thus, $Q_n^{(\alpha,\beta)}(x)$ has a zero $q_{n,0}(\alpha,\beta)$ in $(x_{n,1}(\alpha,\beta),1)$.

Thus, we proved that $Q_n^{(\alpha,\beta)}(x)$ has one zero, $q_{n,0}(\alpha,\beta)$, in the interval $(x_{n,1}(\alpha,\beta),1)$ if and only if $\alpha \in (-\frac{1}{2},+\infty)$. Then the symmetry relation (9) implies that $Q_n^{(\alpha,\beta)}(x)$ has one zero, $q_{n,n}(\alpha,\beta)$, in $(-1,x_{n,n}(\alpha,\beta))$ if and only if $\beta \in (-\frac{1}{2},+\infty)$. This completes the proof of the theorem. Moreover, we established Proposition 2. □

At the end of this section we return to the comments we made in the introduction about the dependence of the number of zeros of a specific solution of (1) on the normalization adopted. It is clear from the proof of Theorem 1 that, if we choose the constant C in the representation (2) of $\widetilde{Q}_n^{(\alpha,\beta)}(x)$, to be sufficiently large, then $\widetilde{Q}_n^{(\alpha,\beta)}(x)$ would have $n+1$ zeros, independently on the values of α and β. Indeed, it certainly happens if

$$C > \frac{\pi \Gamma(\alpha+\beta+2)}{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)} \cot(\alpha\pi),$$

because in this case we have $\lim_{x\uparrow 1} \widetilde{Q}_n^{(\alpha,\beta)}(x) > 0$ for all $n \in \mathbb{N}$, $\alpha, \beta > -1$.

4. Relative position of zeros of $Q_{n+1}^{(\alpha,\beta)}(x)$, $(\mathrm{d}/\mathrm{d}x)\, Q_n^{(\alpha,\beta)}(x)$, $Q_{n-1}^{(\alpha,\beta)}(x)$ between two consecutive zeros of $Q_n^{(\alpha,\beta)}(x)$

We begin this section with the following interlacing property of zeros of the Jacobi functions of second kind.

Theorem 3. *Let $n \in \mathbb{N}$. Then there is exactly one zero $q_{n+1,i+1}(\alpha,\beta)$ of $Q_{n+1}^{(\alpha,\beta)}(x)$ and exactly one zero $q_{n-1,i}(\alpha,\beta)$ of $Q_{n-1}^{(\alpha,\beta)}(x)$ in each interval $(q_{n,i+1}(\alpha,\beta), q_{n,i}(\alpha,\beta))$ between two consecutive zeros of $Q_n^{(\alpha,\beta)}(x)$.*

This fact can be established in various ways. One is to use Sturm's comparison theorem (see [10, Theorem 1.82.1]). Indeed, since the function

$$u(x) = (1-x)^{(\alpha+1)/2}\,(1+x)^{(\beta+1)/2} Q_n^{(\alpha,\beta)}(x)$$

is a solution of (see [10, Eq. (4.24.1)])

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}u(x) + f(x,n)u(x) = 0, \tag{16}$$

where

$$f(x,n) = \frac{1}{4}\left\{\frac{1-\alpha^2}{(1-x)^2} + \frac{1-\beta^2}{(1+x)^2} + 4\frac{n(n+\alpha+\beta+1)+(\alpha+1)(\beta+1)/2}{1-x^2}\right\},$$

and obviously $f(x,n) < f(x,n+1)$, then the interlacing property follows.

It can be justified also by the recurrence relation (see [4, (16)])

$$Q_{-1}^{(\alpha,\beta)}(z) = 1,$$

$$z Q_n^{(\alpha,\beta)}(z) = A_n Q_{n+1}^{(\alpha,\beta)}(z) - B_n Q_n^{(\alpha,\beta)}(z) + C_n Q_{n-1}^{(\alpha,\beta)}(z), \quad n \geqslant 0, \tag{17}$$

where $Q_0^{(\alpha,\beta)}(z)$ is given by (10), $C_0 := 1$, and

$$A_n = \frac{2(n+1)(n+\alpha+\beta+1)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)},$$

$$B_n = \frac{\alpha^2-\beta^2}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)},$$

$$C_n = \frac{2(n+\alpha)(n+\beta)}{(2n+\alpha+\beta)(2n+\alpha+\beta+1)}.$$

Hence the Jacobi functions of second kind satisfy a Christoffel–Darboux formula which yields the interlacing property [3].

In order to study the location of the zeros of the derivatives, we shall use the following structure relations. From [5, Eq. (1.2.13)] and [2] we have

$$\sigma(x)\frac{\mathrm{d}}{\mathrm{d}x}Q_n^{(\alpha,\beta)}(x)=\Psi_n^{(\alpha,\beta)}(x)Q_n^{(\alpha,\beta)}(x)-\frac{2(n+\alpha+\beta+1)(n+1)}{2n+\alpha+\beta+2}Q_{n+1}^{(\alpha,\beta)}(x), \tag{18}$$

where $\sigma(x)=1-x^2$ and

$$\Psi_n^{(\alpha,\beta)}(x)=-\frac{n+\alpha+\beta+1}{2n+\alpha+\beta+2}\left(\beta-\alpha-(2n+\alpha+\beta+2)x\right).$$

Then, from (18) and (17), we obtain

$$\sigma(x)\frac{\mathrm{d}}{\mathrm{d}x}Q_n^{(\alpha,\beta)}(x)=\Phi_n^{(\alpha,\beta)}(x)Q_n^{(\alpha,\beta)}(x)+\frac{2(\alpha+n)(\beta+n)}{2n+\alpha+\beta}Q_{n-1}^{(\alpha,\beta)}(x),\quad n\in\mathbb{N}, \tag{19}$$

where

$$\Phi_n^{(\alpha,\beta)}(x)=n\left(\frac{\alpha-\beta}{2n+\alpha+\beta}-x\right).$$

By the previous result, in every interval $(q_{n,i+1}(\alpha,\beta),q_{n,i}(\alpha,\beta))$ between two consecutive zeros of $Q_n^{(\alpha,\beta)}(x)$ there is exactly one zero of $Q_{n+1}^{(\alpha,\beta)}(x)$ and of $Q_{n-1}^{(\alpha,\beta)}(x)$. By Rolle's theorem, and by Theorem 1, in that interval there is exactly one zero $t_{n,i}$ of $(\mathrm{d}/\mathrm{d}x)Q_n^{(\alpha,\beta)}(x)$. Now we study the mutual location of $q_{n+1,i+1}(\alpha,\beta)$, $q_{n-1,i}(\alpha,\beta)$, and $t_{n,i}$.

Theorem 4. *For any fixed $n\in\mathbb{N}$ and $\alpha,\beta>-1$, let*

$$c^*=\frac{\beta-\alpha}{2n+\alpha+\beta+2}$$

and

$$d^*:=\frac{\alpha-\beta}{2n+\alpha+\beta}.$$

(i) *If, for some index i, we have $q_{n+1,i+1}(\alpha,\beta)<c^*$, then $q_{n+1,i+1}(\alpha,\beta)>t_{n,i}$. Otherwise, if $q_{n+1,i+1}(\alpha,\beta)>c^*$, then $q_{n+1,i+1}(\alpha,\beta)<t_{n,i}$.*

(ii) *If, for some index j, we have $q_{n-1,j}(\alpha,\beta)<d^*$, then $q_{n-1,j}(\alpha,\beta)<t_{n,j}$. Otherwise, if $q_{n-1,j}(\alpha,\beta)>d^*$, then $q_{n-1,j}(\alpha,\beta)>t_{n,j}$.*

Proof. First we shall establish the statement (i). Let us restrict our considerations only to $x\in I$, where $I:=(q_{n,i+1}(\alpha,\beta),q_{n,i}(\alpha,\beta))$. Since the zeros of $Q_n^{(\alpha,\beta)}(x)$ and of its derivative are simple, and $\sigma(x)=1-x^2$ is positive in $(-1,+1)$, then

$$\sigma(x)\frac{(\mathrm{d}/\mathrm{d}x)Q_n^{(\alpha,\beta)}(x)}{Q_n^{(\alpha,\beta)}(x)}>0\quad\text{for } x\in(q_{n,i+1}(\alpha,\beta),t_{n,i})$$

and

$$\frac{(\mathrm{d}/\mathrm{d}x)\,Q_n^{(\alpha,\beta)}(x)}{Q_n^{(\alpha,\beta)}(x)}<0 \quad \text{for } x\in(t_{n,i},q_{n,i}(\alpha,\beta)).$$

Then relation (18) implies immediately that

$$\Psi_n^{(\alpha,\beta)}(q_{n+1,i+1}(\alpha,\beta))>0 \quad \text{if and only if } q_{n+1,i+1}(\alpha,\beta)\in(q_{n,i+1}(\alpha,\beta),t_{n,i}),$$

and

$$\Psi_n^{(\alpha,\beta)}(q_{n+1,i+1}(\alpha,\beta))<0 \quad \text{if and only if } q_{n+1,i+1}(\alpha,\beta)\in(t_{n,i},q_{n,i}(\alpha,\beta)).$$

The latter statement is equivalent to (i). The proof of (ii) is similar. It uses the relation (19). □

It is worth mentioning that for $\alpha=\beta$ we have $c^*=d^*=0$ for all $n\in\mathbb{N}$, and then the inequalities

$$q_{n+1,i+1}(\alpha,\alpha)<t_{n,i}<q_{n-1,i}(\alpha,\alpha)$$

hold for the positive zeros of $Q_{n+1}^{(\alpha,\alpha)}(x)$ and of $Q_{n-1}^{(\alpha,\alpha)}(x)$. In particular, in the Legendre case these coincide with results given in [7,10]. The corresponding result for the zeros of Jacobi polynomials and of their zeros was obtained in [8].

5. Laguerre and Hermite cases

In this section, we obtain the number of zeros of Laguerre and Hermite functions of second kind in the intervals where the corresponding polynomials are orthogonal.

5.1. Zeros of Laguerre functions of the second kind

The change of variables

$$x=1-\frac{2s}{\beta}, \qquad \beta>0,$$

in (1) yield the following hypergeometric differential equation:

$$s\left(1-\frac{s}{\beta}\right)y''(s)+\left(\alpha+1-s\left(1+\frac{\alpha+2}{\beta}\right)\right)y'(s)+n\left(1+\frac{n+\alpha+1}{\beta}\right)y(s)=0.$$

When β tends to infinity, we obtain the Laguerre differential equation

$$sy''(s)+(\alpha+1-s)y'(s)+ny(s)=0,$$

whose polynomial solution is the Laguerre polynomial $L_n^{(\alpha)}(x)$, and denoted by $Q_n^{(\alpha)}(s)$ is its second solution which is obtained by the limit relation (see [2])

$$\lim_{\beta\to+\infty}\left(\frac{-\beta}{2}\right)^{n-2}Q_n^{(\alpha,\beta)}\left(1-\frac{2s}{\beta}\right)=Q_n^{(\alpha)}(s).$$

The linear change of variables

$$x(s) = 1 - \frac{2s}{\beta}$$

transforms the zeros $x_{n,k}(\alpha, \beta)$ of $P_n^{(\alpha,\beta)}(x)$ to the zeros $x_{n,k}(\alpha)$ of $L_n^{(\alpha)}(x)$, and the zeros $q_{n,k}(\alpha, \beta)$ of $Q_n^{(\alpha,\beta)}(x)$ are mapped to the zeros $q_{n,k}(\alpha)$ of the second kind Laguerre functions $Q_n^{(\alpha)}(x)$. Moreover, the mapping reverses the order of the zeros and it also maintains the interlacing properties of the zeros.

Now Table 1 immediately implies the following facts about the number of zeros of $Q_n^{(\alpha)}(s)$:

(1) If $\alpha > -\frac{1}{2}$, then $Q_n^{(\alpha)}(x)$ has $n+1$ zeros in $(0, +\infty)$.
(2) If $-1 < \alpha \leqslant -\frac{1}{2}$, then $Q_n^{(\alpha)}(x)$ has n zeros in $(0, +\infty)$.

5.2. Zeros of Hermite functions of the second kind

By using limit relations between Jacobi and Hermite functions of the second kind, obtained in [2], we conclude that the Hermite function of second kind $Q_n(x)$ has exactly $n+1$ zeros in $(-\infty, +\infty)$.

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