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**Dynamic Behavior of Finite Mono-Coupled Periodic Structures
with Variable Cross-Section and Coupled Resonators**

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**Dynamic Behavior of Finite Mono-Coupled Periodic Structures
with Variable Cross-Section and Coupled Resonators**

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In memory of my Mother, Mafalda Albertin Xavier da Silva.

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“If you want to find the secrets of the universe, think in terms of energy, frequency, and vibration”.

Nikola Tesla

ABSTRACT

The interest in periodic structures has grown significantly in recent years, driven by their ability to efficiently manipulate waves, making them a promising solution for various applications in modern engineering systems. These structures are notable for their capacity to reduce vibrations within specific frequency ranges, with widespread applications in acoustic insulation in the aerospace sector and vibration control in civil structures. Recently, the metastructures, which consist of the integration of vibration absorbers into periodic structures, have been investigated as a means to expand the frequency ranges where vibration attenuation occurs. This study focuses on analyzing the dynamic behavior of finite mono-coupled periodic structures composed of rods with catenoidal, exponential, and conical shapes. To model these rods, their receptance and dynamic stiffness matrices are determined. The transfer matrix method is employed to derive the global properties of the structure based on the transmissibility of a single unit cell. The influence of the structure's geometry is analyzed by considering symmetric and asymmetric cells. It was observed that when the cells are composed of rods of the same length, asymmetric configurations exhibit greater wave attenuation, albeit limited to higher frequency ranges. The introduction of dynamic vibration absorbers into symmetric cells was investigated to create super-attenuation bands and multiple frequency attenuation bands. To validate the proposed methodology, experimental tests were conducted on a structure composed of conical rods, both in the absence and presence of dynamic vibration absorbers. The experimental results showed good agreement with theoretical predictions, reinforcing the effectiveness of the developed approach.

Keywords: Periodic Structures; Symmetrical / Asymmetrical Structure; Super -attenuation / Multi-frequency bands; Dynamic Vibration Absorbers; Metastructure.

RESUMO

O interesse por estruturas periódicas tem crescido significativamente nos últimos anos, impulsionado por sua capacidade de manipular ondas de maneira eficiente, o que as torna uma solução promissora para diversas aplicações em sistemas de engenharia modernos. Essas estruturas destacam-se por sua habilidade de reduzir vibrações em faixas de frequência específicas, sendo amplamente aplicadas em isolamento acústico no setor aeroespacial e no controle de vibrações em estruturas civis. Recentemente, as metaestruturas, que consistem na integração de absorvedores de vibração em estruturas periódicas, têm sido investigadas como uma forma de ampliar as faixas de frequência onde ocorre a atenuação vibracional. Este trabalho tem como foco a análise do comportamento dinâmico de estruturas periódicas monoacopladas finitas, compostas por barras com geometrias catenoidal, exponencial e cônica. Para modelar essas barras, são determinadas suas matrizes de receptância e rigidez dinâmica. O método da matriz de transferência é empregado para derivar as propriedades globais da estrutura a partir da transmissibilidade de uma única célula unitária. A influência da geometria da estrutura é analisada considerando células simétricas e assimétricas. Observou-se que, quando as células são formadas por barras de mesmo comprimento, as configurações assimétricas apresentam maior atenuação de ondas, embora restrita a faixas de frequência mais elevadas. A introdução de absorvedores dinâmicos de vibração em células simétricas foi investigada, visando à criação de bandas de superatenuação e a formação de múltiplas bandas de frequência atenuadas. Para validar a metodologia proposta, foram realizados ensaios experimentais em uma estrutura composta por barras cônicas, tanto na ausência quanto na presença de absorvedores dinâmicos de vibração. Os resultados experimentais demonstraram boa concordância com as previsões teóricas, reforçando a eficácia da abordagem desenvolvida.

Palavras-chave: Estruturas Periódicas; Estrutura Simétrica / Assimétrica, Super banda de atenuação / Bandas multifrequenciais; Absorvedores Dinâmicos de Vibração; Metaestruturas.

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LIST OF ABBREVIATIONS AND ACRONYMS

AutoMac	Modal Assurance Criterion
BE	Bragg Scattering Effect
CMIF	Complex Mode Indicator Function
DSM	Dynamic Stiffness Method
DOF	Degree of Freedom
DVA	Dynamic Vibration Absorber
EMA	Experimental Modal Analysis
FEM	Finite Element Method
FRFs	Frequency Response Functions
LRE	Local Resonance Effect
SVD	Singular Value Decomposition

LIST OF SYMBOLS

Designation	Explanation
Latin symbols	
A	Constant cross-section area
A_L	Cross-section area at left-side end
A_R	Cross-section area at right-side end
$A(x)$	Variable cross-section area as a function of the position x
a_1, a_2	Constants for the uniform rod general solution
b_1, b_2	Constants for the catenoidal shaped rod general solution
c_0	Wave velocity for a rod with uniform cross-section area
c_1, c_2	Constants for the exponential shaped rod general solution
D_{att}	Dynamic stiffness of an attached vibration absorber
$D_{11}, D_{12}, D_{21}, D_{22}$	Elements of the dynamic stiffness matrix
d_1, d_2	Constants for the conical shaped rod general solution
E	Young's modulus of the rod
E_a, E_c	Young's modulus of the of the dynamic vibration absorber used in the experimental procedure
e_1, e_2	Constants for the Cayley-Hamilton Theorem
F_L	Amplitude of the excitation force at left end side
F_R	Amplitude excitation force at right end side
J_q	Bessel functions of the first kind of order q
k	Wavenumber for catenoidal shaped rods
$\hat{k}$	Wavenumber for exponential shaped rods
k_c	Wavenumber of the single cell
k_a	Longitudinal wavenumber of the rod calculated for the undamped natural frequency of the vibration absorber
k_0	Wavenumber of a rod with a uniform cross-section
l	Length of the rod

l_{cell}	Length of the single cell
m	Mass of the cell
m_a	Mass of the dynamic vibration absorber
m_n	Mass of the nut which composed the dynamic vibration absorber in the experimental procedure
m_s	Mass of the screw which composed the dynamic vibration absorber in the experimental procedure
m_1	Mass of the primary absorber
N	Number of cells
n_a	Number of absorbers
p	Constant of Bessel's equation
p_1, p_2, p_3	Peaks in the transmissibility response
q	Order of the Bessel functions
r_c	Radius of one of the parts that composes the dynamic vibration absorber in the experimental procedure
r_L	Radius of the left side end of structure
$R_{11}, R_{12}, R_{21}, R_{22}$	Elements of the receptance matrix
s_a	Stiffness of dynamic vibration absorber
s_1	Stiffness of the primary absorber
U_L	Amplitude of the longitudinal displacement at left end side
U_R	Amplitude of the longitudinal displacement at right end side
u	Amplitude of the longitudinal displacement along the axial direction
$\mathbf{T}, \mathbf{T}_{cell}$	Transfer matrix of a single cell
$\mathbf{T}_{att}$	Transfer matrix of a vibration absorber
T_1	Trough of the transmissibility response
$T_{11}, T_{12}, T_{21}, T_{22}$	Elements of the receptance matrix
t_a	Thickness of one of the parts that composes the dynamic vibration absorber in the experimental procedure
t	Time

x_L	Parameter that defines the shape of the cross-sectional for conical shaped rods
w_a	Width of one of the parts that composes the dynamic vibration absorber in the experimental procedure
Y_q	Bessel functions of the second kind of order q

Designation	Explanation
Greek symbols	
α	Parameter that defines the shape of the cross-sectional for catenoidal shaped rods
β	Parameter that defines the shape of the cross-sectional for exponential shaped rods
δ	Ratio of the cross-sectional area of the stepped rod
γ	Frequency ratio
Δ	Constant mass variation
ε	Longitudinal strain along the axial direction
η	Loss factor of the rod
η_a	Loss factor of the dynamic vibration absorber used in the experimental procedure
Λ	Eigenvalue of the transfer matrix
λ_0	Wavelength of rod with uniform cross-section
μ	Floquet exponent
ρ	Mass density of the rod
ρ_a	Mass density of the dynamic vibration absorber used in the experimental procedure
σ	Longitudinal stress along the axial direction
υ	Mass ratio
φ	Constant of Bessel's equation
ω	Circular frequency
ω_1	Natural frequency of the primary absorber

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1 INTRODUCTION

This chapter provides a brief motivation and a review of the literature on periodic structures, which form the foundation of this work. Additionally, the problem statement and objectives are outlined. Finally, the structure of the thesis report is presented.

1.1 Motivation

For several reasons, vibration can be considered an unwanted phenomenon in industrial systems and processes. Excessive vibration in vehicles, machines, or equipment can cause physical and mental discomfort to occupants or operators, and cause damage and premature failure to structures, machines, and devices (HUANG; GRIFFIN, 2014; LI; HUANG, 2018). When intense and continuous, vibration can lead the machine or equipment to material fatigue, and consequently to mechanical failure (LU *et al.*, 2023; WATBAN KHALID FAHMI; REZA KASHYZADEH; GHORBANI, 2022). These situations make vibration attenuation have an important role in the mechanical design stage, in which it becomes essential to apply vibration control techniques to resolve the presence of unwanted vibration, ensuring structural integrity and mechanical performance.

In recent decades, different vibration control techniques have been studied, known as passive, semi-active, or active. Among these techniques, the use of passive metamaterials has been highlighted. Passive metamaterials are periodic structures that exhibit periodicity based on geometric, material discontinuities, or the addition of resonators distributed in the structure. Due to this periodicity, these structures have the particular characteristic of forming frequency bands where the vibrational wave does not propagate, known as bandgaps (LIU; HUSSEIN, 2012). In this way, passive metamaterials become a useful alternative to suppress vibrations without the necessity for active vibration control (SANTOS *et al.*, 2023).

Several studies have been dedicated to reducing the propagation of vibration within a specific frequency range using such periodic systems. Recent research seeks to observe the behavior of the frequency band in relation to the geometry of periodic structures, as in the case of mono-coupled structures composed of stepped rod cells, which present satisfactory results in the attenuation of longitudinal waves mentioned in the use of stepped rods is the sudden change in the cross-sectional area, which can cause stress concentration at discontinuities (CARNEIRO *et al.*, 2021; SANTOS *et al.*, 2023). One of the methods to

reduce stress concentration is avoid corner, using smooth geometries, for instance, as explored in the work of Taylor *et al.* (2011). Periodic structures composed of rods with smooth and variable cross-sectional areas could potentially solve the problem of stress concentrations and also guarantee the Bragg stop-band, due to the variation in the cross-sectional area.

Some advances still seen in the design of periodic structures to reduce the transmission of vibrations explore the use of vibration absorbers to broaden the frequency ranges in which frequency bands occur or to create multi-frequency bandgaps in lower frequency ranges. Cleante *et al.* (2022) explored the local resonance of the dynamic vibration absorber in periodic structures in order to obtain an ultra-wide frequency band, known as the super stop-band. Hao *et al.* (2019) concluded that the positions of local resonance and Bragg bandgaps can be adjusted by changing the natural frequency of the vibration absorbers. Mao, Zhang and Fan (2023) analyzed the vibration bandgaps in a non-uniform beam with periodically continuous variable cross-sections with coupled absorbers and concluded that disordered absorbers can broaden the bandgaps and generate various local resonance bandgaps.

Given this context, this research aims to investigate the dynamic behavior of finite mono-coupled periodic structures with variable cross-section and coupled resonators. The objective is to analyze how the variation in cross-sectional geometry and the inclusion of resonators influence the formation and characteristics of vibration attenuation bands. The study involves analytical modeling, numerical simulations, and experimental validations to optimize the performance of these periodic structures. By integrating variable geometries and coupled resonators, this research contributes to the advancement of vibration control strategies, providing effective solutions for engineering applications

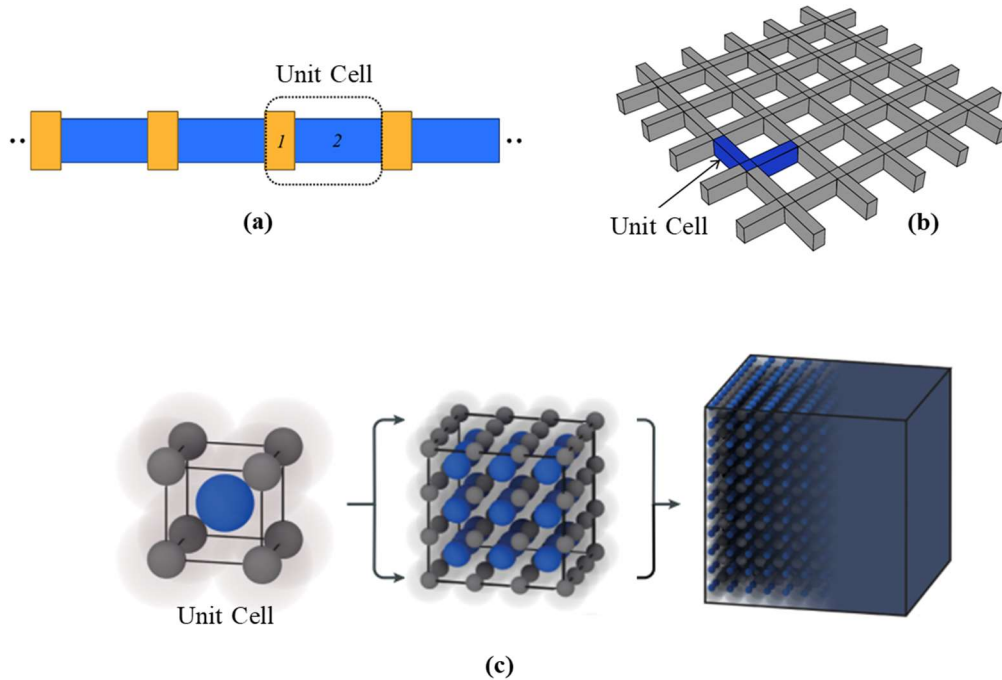
1.2 Literature Review

1.2.1 Periodic Structures – Definition

Periodic structures are arrangements of repeating structural units, commonly referred to as cells, which are uniformly interconnected. These structures can exhibit periodicity in one, two, or three dimensions, as illustrated in Figure 1. In mono-coupled configurations, wave transmission occurs predominantly in one direction, while multi-coupled structures exhibit multiple degrees of freedom between cells, allowing wave propagation in multiple

directions.

Figure 1 – Illustrations of periodic structures: (a) unidimensional; (b) two-dimensional; (c) three-dimensional.



Source: Adapted from Cenedese, Belloni and Braghini (2021), Yang *et al.* (2022) and Kadic *et al.* (2019).

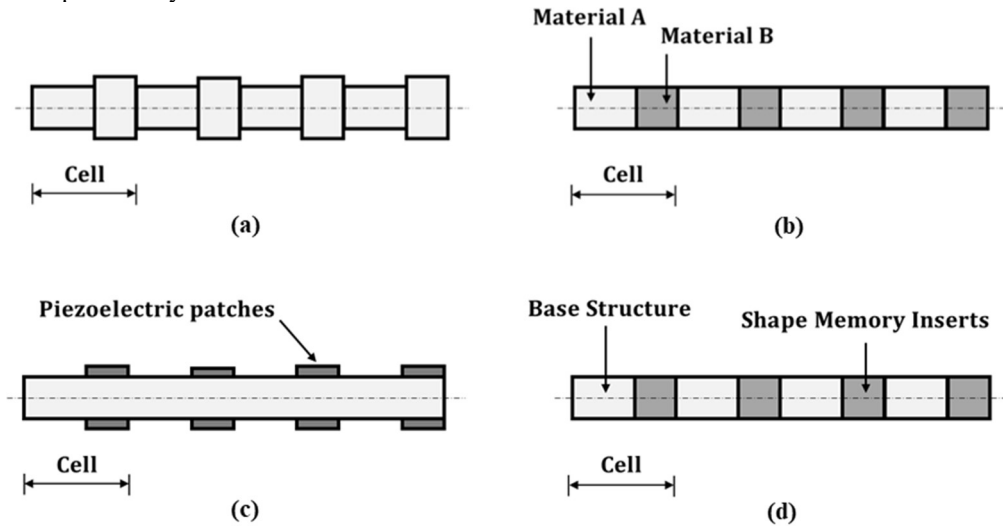
Periodic structures are broadly classified as passive, active, and semiactive. Passive structures rely on inherent properties like geometry or material discontinuities, while active structures incorporate components such as piezoelectric patches to adapt their dynamic response by directly injecting energy into the system. Semiactive structures, on the other hand, utilize controllable elements—such as shape memory alloys (SMAs) used to modify stiffness or damping—which adjust parameters without adding mechanical energy to the system. Figure 2 illustrates examples of all three classifications.

Due to periodicity shown in Figure 2, periodic structures exhibit unique dynamic properties. The Bragg scattering effect is one of the phenomena presents in such systems. This phenomenon occurs when the wavelength becomes comparable to the length of the structural cells, resulting in frequency regions where waves do not propagate. Such regions are known as “bandgaps” (BRILLOUIN, 1946).

Therefore, periodic structures are often called mechanical filters to the propagation of waves. The frequency bands where waves propagate along periodic structures are called “pass bands” and the specific frequency bands where waves do not propagate (for infinite

periodic structure) are called “bandgaps”, or “stop-bands” (for finite periodic structure), where the waves are attenuated. In passive periodic structures, the spectral width and location of these bands are fixed, whereas in active periodic structures the bands are adjustable in relation to the vibration response (BAZ, 2019). It is important to distinguish between the concepts of bandgaps and attenuation bands. Bandgaps refer to frequency ranges in infinite structures where waves do not propagate freely within a waveguide, while attenuation bands are associated with frequency ranges in finite periodic structures where waves are attenuated (DA SILVA *et al.*, 2025). According to the investigation carried out by Carneiro *et al.* (2021), in finite mono-coupled periodic structures, when the cells are symmetric, the attenuation band occurs in the same frequency range as the bandgap in an infinite structure. In the case of asymmetric cells, the attenuation band tends to be wider for a smaller number of cells, but decreases until it becomes approximately equivalent to the bandgap of an infinite asymmetric structure (DOMADIYA *et al.*, 2016; HVATOV; SOROKIN, 2015).

Figure 2 – Examples of passive, active and semiactive periodic structures: (a) geometric discontinuity; (b) material discontinuity; (c) active piezoelectric patches; (d) active shape memory inserts.



Source: Adapted from Baz (2019).

1.2.2 Periodic Structures - Historical Background

The study of periodic structures began with the work of Brillouin (1946), who investigated the theoretical foundations of wave propagation in periodic systems. In this work, Brillouin (1946) approached how the periodicity of a structure could influence wave transmission, establishing the basis for the concept of bandgaps, which characterize

frequency regions where waves do not propagate. This idea aroused significant interest within the scientific community, particularly due to the potential for controlling vibrations and waves in mechanical structures.

Since then, the interest in periodic structures has grown exponentially, especially because of their ability to manipulate mechanical and vibrational waves. One of the main phenomena associated with these structures is Bragg scattering, which plays a crucial role in the formation of bandgaps. This phenomenon occurs when incident waves encounter periodic interfaces, where the geometric or material periodicity of the structure causes destructive interference in propagating waves, blocking them in specific frequency ranges (BRILLOUIN, 1946).

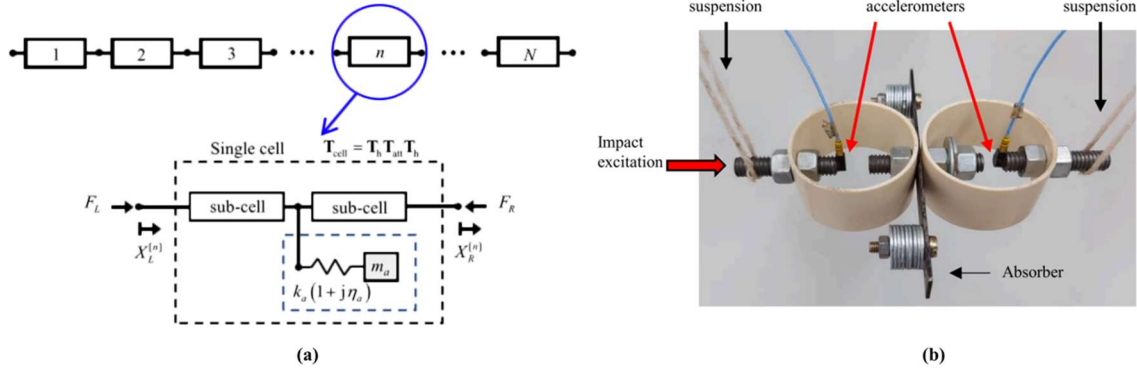
The concept of bandgaps, introduced by Brillouin (1946), was expanded upon by subsequent research investigating how structural characteristics, such as cell size and geometry, affect the position and width of these bands. Mead (1970 and 1971) was one of the pioneers in relating structural periodicity to the dynamic behavior of waves, providing a theoretical foundation for developing structures capable of attenuating vibrations in specific frequency ranges. Simultaneously, Cremer and Heckl (1973) contributed to the characterization of vibrational behavior in periodic systems with various structural configurations.

In more recent years, advancements in analytical and numerical methods, such as the transfer matrix method and Cayley-Hamilton theorem, have enabled more detailed modeling of Bragg scattering in finite mono-coupled structures. These methods have been applied to accurately predict the dynamic behavior of structures, particularly in analyzing the displacement transmissibility at the structure's ends, as extensively explored by Gonçalves, Brennan and Cleante (2021). Additionally, the use of Floquet-Bloch theorem is widely employed in the analysis of infinite periodic structures, allowing for accurate prediction of frequency bandgaps (HVATOV; SOROKIN, 2015).

The interaction between Bragg scattering and local resonators has become a significant topic in recent research. Studies such as those by Cenedese, Belloni and Braghin (2021) demonstrated that combining adjustable local resonators with periodic structures could result in broader and more effective bandgaps, enabling vibration control over more diverse frequency ranges. Raghavan and Phani (2013) investigated periodic resonators and coupled masses on a finite structural beam using receptance coupling methods and Bloch's theorem. Cleante *et al.* (2022) focused on studying the dynamic behavior of mono-coupled

periodic structures with the addition of vibration absorbers, Figure 3 providing a detailed study on the influence of absorber parameters in achieving super-stop band.

Figure 3 – Periodic structure with a coupled resonator: (a) schematic representation of the single cell (b) experimental test of the single cell.



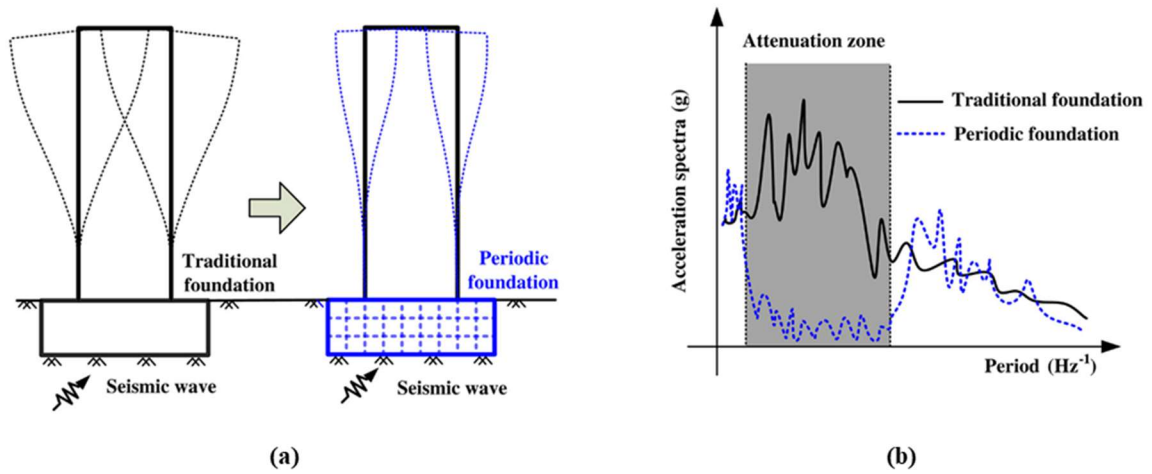
Source: Adapted from Cleante *et al.* (2022).

This integration paved the way for developing adjustable systems, such as adding smart materials to structures, including shape memory alloys and piezoelectrics, which opened new possibilities for designing active structures (LIU *et al.*, 2024; SONG; SHEN, 2019; KUMBHAR *et al.*, 2018).

In engineering applications, periodic structures have demonstrated great versatility in a wide range of uses due to their ability to manipulate mechanical waves and control vibrations. In civil engineering, the use periodic foundation are used for the control of seismic events and to serve as a vibration isolator, as shown in Figure 4 (CHENG *et al.*, 2020; CHENG; SHI, 2017; XIANG *et al.*, 2012; YAN *et al.*, 2015).

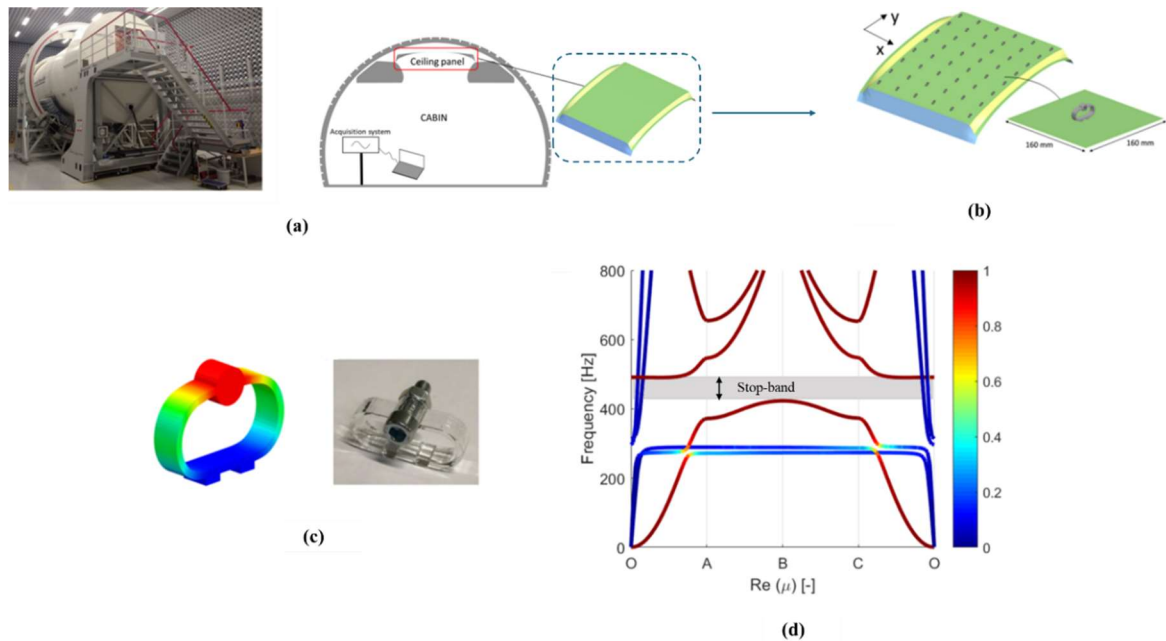
In aeronautical engineering, there are studies on the application of locally resonant metamaterials in the aircraft panel to improve sound insulation, according to Figure 5 (ALVES PIRES *et al.*, 2022). There are also applications in aerospace engineering, where there is the development of metamaterials for applications in satellites (RAMACHANDRAN; FARUQUE; ISLAM, 2020).

Figure 4 – Illustration of the use of an isolation mechanism using a periodic foundation: (a) representation of the replacement of the traditional solid foundation by a periodic foundation; (b) Highlighting of the attenuation zone in the seismic response of the structure with the periodic foundation.



Source: Adapted from Cheng and Shi (2017).

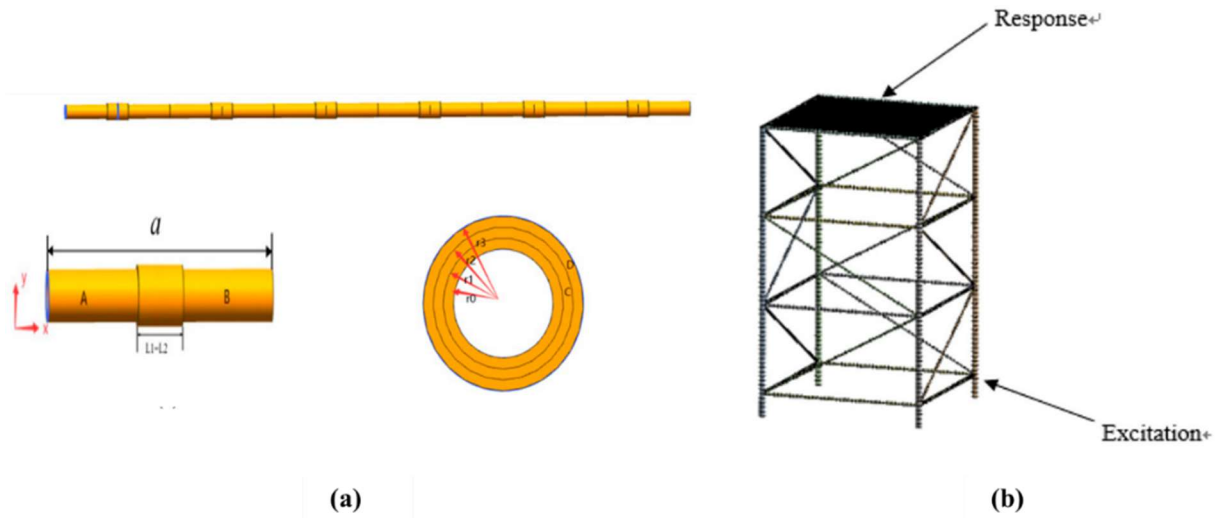
Figure 5 – Schematic of the application of locally resonant metamaterials in an aircraft panel to enhance sound transmission loss: (a) Acoustic fuselage demonstrator at Airbus and schematic interior; (b) Metamaterial configuration with an example of a unit cell; (c) Representation and photograph of the resonator used in the metamaterial; (d) Dispersion diagram highlighting the stop band.



Source: Adapted from Alves Pires *et al.* (2022) .

In naval engineering, many studies are focused on the application of periodic structures on offshore platforms to realize the vibration control of the truss structure, as shown in Figure 6 (AZEVEDO VASCONCELOS *et al.*, 2024; LAO *et al.*, 2022; ZHAOWANG *et al.*, 2020).

Figure 6 – Schematic of the application of periodic structures in offshore platforms: (a) the periodic pipeline structure (b) offshore platform truss structure model.



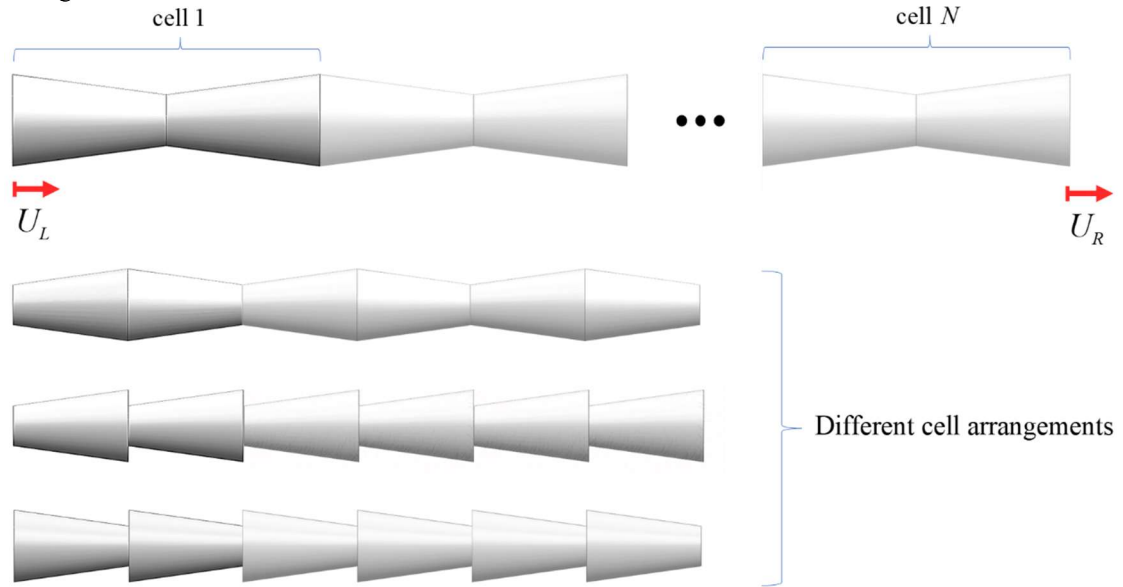
Source: Adapted from Lao *et al.* (2022) .

This overview demonstrates the ongoing impact of studying periodic structures, both in theory and practical applications, and their relevance in contemporary science and engineering.

1.3 Problem Statement

Since the use of periodic structures is a useful alternative to control the propagation of longitudinal vibration, the work aims to explore the use of finite mono-coupled periodic structures to obtain vibrational wave attenuation, through the evaluation of the displacement transmissibility response of the ends of the structure. Figure 7 exemplifies a periodic structure with geometric variation of the cross-sectional area, composed of N cells and exemplifies different arrangements of cells of which this structure can be composed.

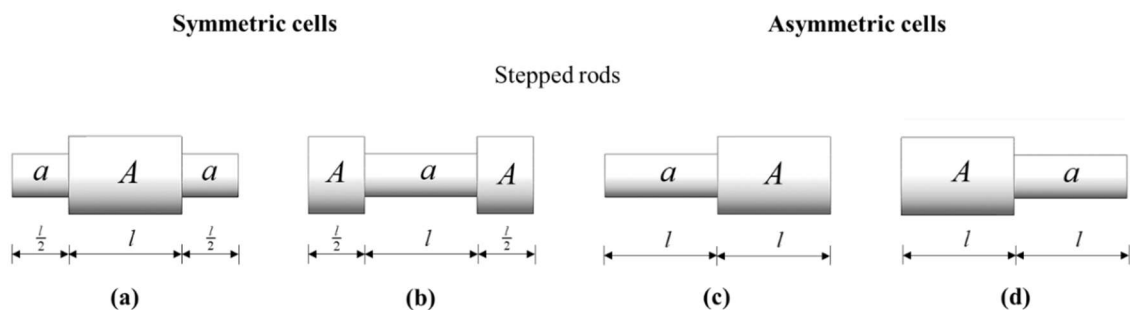
Figure 7 – Finite mono-coupled periodic structure with different cell arrangements. U_L represents the amplitude of displacement on the left-hand side, and U_R represents the amplitude of the displacement on the right-hand side.



Source: Author own elaboration.

As mentioned previously, the sets of cells formed by stepped rods, shown in Figure 8, may present problems related to stress concentration at discontinuities, and a solution that can potentially solve this problem is the change to cells composed of rods with areas smooth and variable cross-sections. Given this, the work proposes a finite monocoupled periodic structure with new arrangements of periodic cells composed of rods that have variable cross-sectional areas, to guarantee the attenuation of longitudinal waves in the periodic structure through the formation of the Bragg stop-band.

Figure 8 – Sets of cells formed by stepped rods: (a) and (b) symmetric cells; (c) and (d) asymmetric cells. All cells have length $2l$.

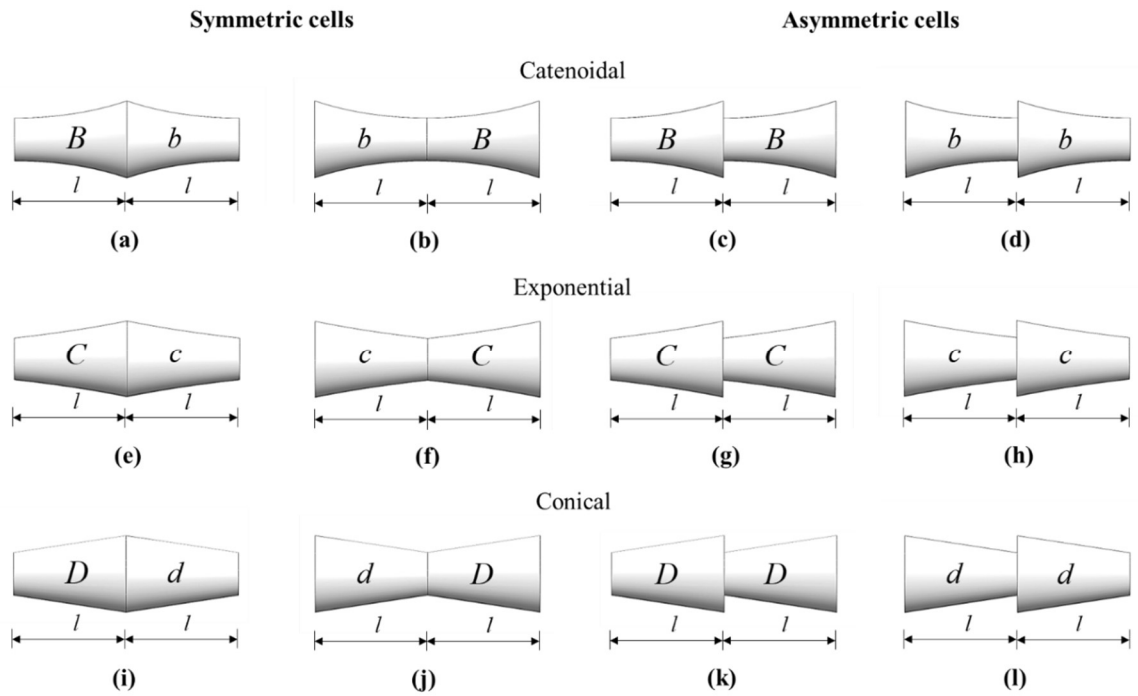


Source: Author own elaboration.

The sets of cells formed by rods with variable cross-sectional areas are shown in Figure 9. The cells are composed of two rods of the same length that have catenoidal,

exponential, or conical shapes, and are divided into subsets of symmetric or asymmetric cells.

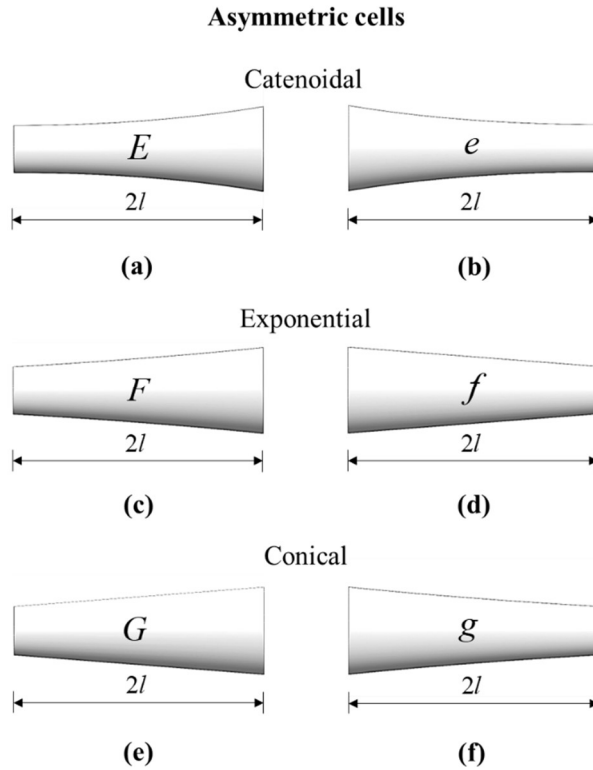
Figure 9 – Different configurations of a single cell for: (a) and (b) catenoidal symmetric cells; (c) and (d) catenoidal asymmetric cells; (e) and (f) exponential symmetric cells; (g) and (h) exponential asymmetric cells; (i) and (j) conical symmetric cells; (k) and (l) conical asymmetric cells. All cells have length $2l$ and each segment of the cell has length l .



Source: Author own elaboration.

Since the asymmetric cell arrangements in Figure 9 present a sudden change in cross-section, a significant influence on the occurrence of attenuation band is expected, which may occur in different frequency ranges of the symmetric cells. Thus, the work addresses a second subset of asymmetric cells, which has the same cell length, but without a sudden change in the cross-sectional area of the cell.

Figure 10 – Second subsets of asymmetric cells: (a) and (b) catenoidal asymmetric cells; (c) and (d) exponential asymmetric cells; (e) and (f) conical asymmetric cells. All cells have length $2l$.



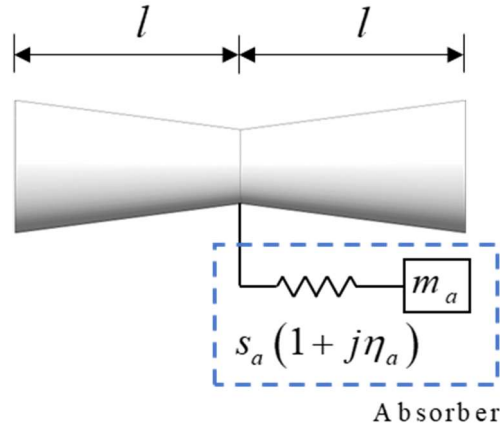
Source: Author own elaboration.

To analyze the propagation and attenuation of the longitudinal wave in the periodic structure, the work focuses on obtaining the response of the displacement transmissibility between the ends of the periodic structure (U_R/U_L). The displacement transmissibility allows visualization regions in which the frequency bands for wave attenuation occurred. Therefore, based on analytical equations of displacement transmissibility, the work aims to reduce vibration in specific frequency bands. To achieve this, the work is based on the method described by Gonçalves, Brennan and Cleante (2021), in which the displacement transmissibility of a single element can be used to estimate the dynamic response of a finite structure composed of N periodic cells. Therefore, to apply the proposed methodology, it is first necessary to model the system, for this purpose, the dynamic stiffness and receptance matrices of the rods that form the periodic structures are obtained. Next, the transfer matrix method is applied, and consequently, the analytical transmissibility equations of the cells that form the periodic structure are obtained.

To improve vibration attenuation, the work explores the addition of dynamic vibration absorbers in the symmetric cell, as shown in Figure 11, to obtain wider frequency

ranges of attenuation, through the combination of the local resonance of the absorber and the Bragg stop-band or obtain multi-frequency attenuation bands from disordered local resonances.

Figure 11 – Representation of symmetric cell with length $2l$ with addition of dynamic vibration absorber attached at its center. Parameters of absorber: stiffness s_a , mass m_a and loss factor η_a .



Source: Author own elaboration.

Experimental tests must be carried out to highlight and validate the results obtained in the proposed methodologies, emphasizing the effects obtained in attenuation bands.

1.4 Objectives

The main objective of this thesis is to analyze the dynamic behavior of finite mono-coupled periodic structures composed of rods with variable cross-sectional areas, focusing on the formation of stop-bands to achieve attenuation of longitudinal vibration waves. By incorporating dynamic vibration absorbers into the structure, the aim is to broaden the vibration attenuation bands or create attenuation bands at lower frequencies. The specific objectives are as follows:

1. To develop dynamic stiffness and receptance matrices for the rods with variable cross-sectional areas that constitute the periodic structure.
2. To derive analytical equations defining the transmissibility of a single cell.
3. To analyze the geometric effects regarding the composition of the structure by symmetric or asymmetric cells.

4. To analyze the dynamic effects of adding dynamic vibration absorbers to symmetric cells and compare these results with those of cells without absorbers.
5. To validate the results through experimental tests.

1.5 Contributions to Knowledge

This thesis makes several significant contributions to the field of vibration control:

1. Analytical Framework:

This work develops a detailed analytical framework for modeling the dynamic behavior of finite mono-coupled periodic structures composed of rods with non-uniform cross-sectional areas. It provides analytical expressions for the transmissibility of individual cells, enabling efficient and accurate predictions of the dynamic response.

2. Geometrical Insights:

The study advances the understanding of how different cell geometries (symmetric and asymmetric) affect vibration attenuation. It demonstrates that smoothly varying geometries, such as catenoidal, exponential, and conical bars, reduce stress concentrations while maintaining or enhancing the effectiveness of attenuation bands compared to stepped rods.

3. Coupling of Dynamic Vibration Absorbers:

The thesis explores the coupling of dynamic absorbers in symmetric cells, showcasing their ability to create super-attenuation bands and multi-frequency attenuation bands. Rainbow-shaped absorbers are highlighted for their capacity to broaden the local resonance band, expanding the operational frequency range of periodic structures.

4. Experimental Validation:

The methodology undergoes experimental validation, ensuring that theoretical predictions align with observed behavior. A series of controlled experiments confirm the accuracy of the analytical and numerical models, demonstrating the practical feasibility of the proposed structures and their effectiveness in vibration attenuation.

1.6 Thesis Structure

This thesis is divided into seven chapters. Chapter 1 introduces the study, providing motivation, a literature review on periodic structures, the problem statement, and the objectives of the work. Chapter 2 presents the modeling of the rods used to compose the

periodic structures under study, including uniform rods and rods with catenoidal, exponential, and conical shapes. Chapter 3 describes the methodology for obtaining the analytical displacement responses of cell assemblies. Chapter 4 investigates the geometric parameters of the cells, focusing on symmetric and asymmetric configurations, based on the resulting stop-bands. Chapter 5 explores the addition of dynamic vibration absorbers to symmetric cells. Chapter 6 details the experimental tests, comparing the results with the analytical expressions derived in Chapters 4 and 5. Finally, Chapter 7 summarizes the conclusions of the thesis.

2 LONGITUDINAL VIBRATIONS IN RODS

This chapter outlines the methodology employed for the mathematical modeling of rods with uniform and variable cross-sectional areas. The Dynamic Stiffness Method (DSM), based on classical theory, is employed as an exact analytical approach (LIU *et al.*, 2022). The rods are modeled using dynamic stiffness and receptance matrices. The theory of longitudinal wave propagation in rods with variable cross-sections is well-established in the literature, as discussed by Graff (1991).

This chapter considers rods with uniform cross-sectional areas, as well as those with catenoidal, exponential, and conical shapes. The modeling of conical-shaped rods necessitates the use of Bessel functions, which are comprehensively discussed in Bowman (1958).

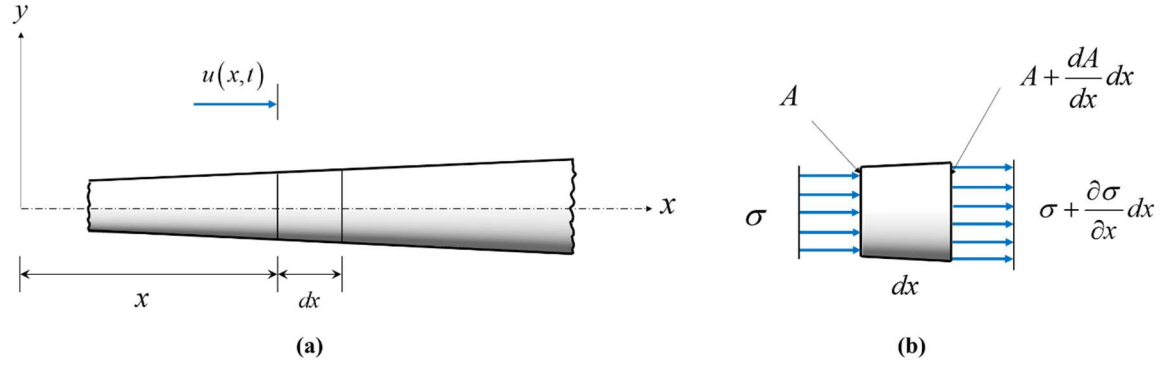
2.1 Governing equations for axial deformation in rods

Axial deformation refers to the elongation or compression of a rod along its axis due to an applied axial force. This phenomenon is fundamental to understanding the dynamic behavior of rods, as it forms the basis for analyzing longitudinal vibrations. The analysis is based on classical mechanics and assumes linear elasticity, where the stress-strain relationship is governed by Hooke's Law (TIMOSHENKO; GOODIER, 1951).

In this section, the governing equations for axial deformation are derived based on the following key assumptions: the axis of the rod remains straight during deformation; cross-sections remain planar and perpendicular to the rod's axis; the material is homogeneous and linearly elastic; material properties, such as Young's modulus E , mass density ρ , are constant for uniform rods but may vary with position for rods with variable cross-sections (CRAIG; ANDREW KURDILA, 2006).

Figure 12 shows a portion of a rod with variable cross-section area along with its respective free-body diagram. Let $u(x, t)$ represent the displacement of the cross-section along the axial direction. It is assumed that the rod is under a dynamically varying stress field $\sigma(x, t)$, so that adjacent sections are subjected to varying stresses. The cross-sectional area is represented by parameter $A = A(x)$.

Figure 12 – Illustration of a portion of a rod: (a) undergoing axial deformation; (b) free-body diagram.



Source: Adapted from Craig and Kurdila (2006); Graff (1991).

Applying Newton's Second law to the free-body diagram in Figure 12(b):

$$-\sigma A + \left(\sigma + \frac{\partial \sigma}{\partial x} dx \right) \left(A + \frac{dA}{dx} dx \right) = \frac{1}{2} \rho \left[A + \left(A + \frac{dA}{dx} dx \right) \right] dx \frac{\partial^2 u}{\partial t^2} \quad (1)$$

The mass is calculated on the basis of the average area, so the equation (1) reduces to

$$A \frac{\partial \sigma}{\partial x} + \sigma \frac{dA}{dx} = \rho A \frac{\partial^2 u}{\partial t^2} \quad (2)$$

where terms of order dx^2 have been neglected.

The equation (2) can be rewritten as

$$\frac{1}{A} \frac{\partial}{\partial x} (\sigma A) = \rho \frac{\partial^2 u}{\partial t^2} \quad (3)$$

According to Hooke's Law for linearly elastic materials, the normal stress σ is related to the extensional strain ε through the constitutive relationship

$$\sigma = E \varepsilon \quad (4)$$

The deformation can be described using the strain-displacement relationship, where the strain $\varepsilon(x, t)$ is defined as the spatial derivative of the axial displacement $u(x, t)$

$$\varepsilon = \frac{F}{EA} = \frac{\partial u}{\partial x}(x, t) \quad (5)$$

where the $F(x, t)$ is externally applied axial force per unit length with resultant lying along the centroidal axis of the member.

Substituting equations (5) and (4) into equation (3), the equation of motion is obtained (GRAFF, 1991):

$$\frac{\partial^2 u}{\partial x^2} + \frac{1}{A} \frac{\partial A}{\partial x} \frac{\partial u}{\partial x} = \frac{1}{c_0^2} \frac{\partial^2 u}{\partial t^2} \quad (6)$$

or it can be written as

$$u''(x, t) + \frac{A'(x)}{A(x)} u'(x, t) - \frac{1}{c_0^2} \ddot{u}(x, t) = 0 \quad (7)$$

where the quotation mark $(\cdot)'$ corresponds to $\partial(\cdot)/\partial x$, and the point on top $(\dot{\cdot})$ corresponds to $\partial(\cdot)/\partial t$. The parameter $c_0 = \sqrt{E/\rho}$ is the velocity of a wave propagation of rod with uniform cross-section area.

If $A(x) = A$ is constant, $A'(x) = 0$, and the equation (7) simplifies to the equation of motion for a rod with a uniform cross-section:

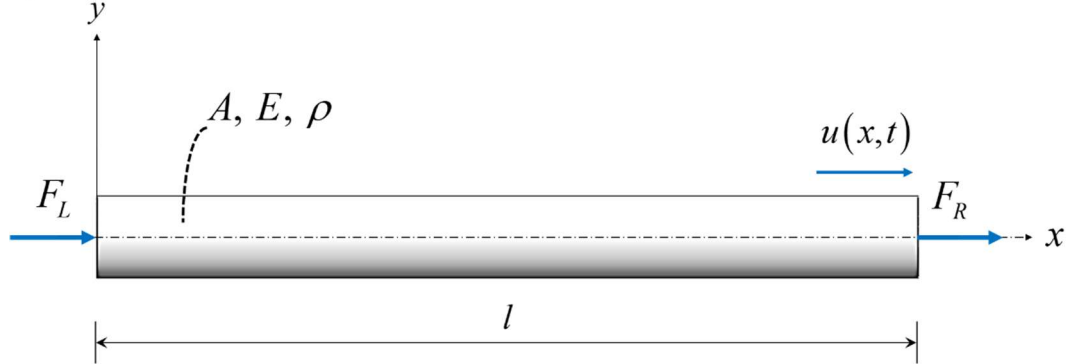
$$u''(x, t) - \frac{1}{c_0^2} \ddot{u}(x, t) = 0 \quad (8)$$

2.2 Rod with uniform cross-section

The rod with a uniform cross-sectional area is illustrated in Figure 13. The rod has a uniform cross-sectional area A , Young's modulus E , mass density ρ and length l . A

force with complex amplitude F_L can be applied to the left-hand side and a force with complex amplitude F_R can be applied at the right end side.

Figure 13 – Illustration of a rod with a uniform cross-sectional area.



Source: Author own elaboration.

Considering the equation of motion for a rod with a uniform cross-section (equation (8)) and applying the method of separation of variables, the displacement solution can be expressed as

$$u(x, t) = U(x)T(t) \quad (9)$$

where $U(x)$ is a displacement dependent function and $T(t)$ is a time dependent function.

Considering harmonic motion, such that the $T(t) = \cos(\omega t)$, equation (8) can be rewritten in the space domain as

$$U''(x) + k_0^2 U(x) = 0 \quad (10)$$

where $k_0 = \omega \sqrt{\rho/E}$ known as the wavenumber of a rod with uniform cross-section, and ω is the circular frequency.

The solution to the differential equation (10) is given by Liu *et al.* (2022):

$$U(x) = a_1 \cos(k_0 x) + a_2 \sin(k_0 x) \quad (11)$$

where a_1 and a_2 are two constants that depend on the boundary conditions.

To obtain the receptance and dynamic stiffness matrices, the four boundary conditions are considered.

$$EAU'(0) = -F_L|_{F_R=0}, \quad U(0) = U_L|_{U_R=0} \quad (12)$$

$$EAU'(l) = F_R|_{F_L=0}, \quad U(l) = U_R|_{U_L=0} \quad (13)$$

To determine the receptance matrix $\mathbf{R} = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix}$, the boundary conditions from equation (12) are applied to equation (11) to calculate the parameters R_{11} and R_{21} . Subsequently, the second boundary conditions, given in equation (13), are applied to obtain the parameters R_{12} and R_{22} . Since the system is reciprocal, $R_{12} = R_{21}$, ensuring the symmetry of the receptance matrix. Therefore, the receptance matrix is found and is shown in equation (14) (LIU *et al.*, 2022).

$$\begin{Bmatrix} U_L^{[A]} \\ U_R^{[A]} \end{Bmatrix} = -\frac{l}{EA} \frac{1}{k_0 l} \begin{bmatrix} \cot(k_0 l) & \csc(k_0 l) \\ \csc(k_0 l) & \cot(k_0 l) \end{bmatrix} \begin{Bmatrix} F_L^{[A]} \\ F_R^{[A]} \end{Bmatrix} \quad (14)$$

where the superscript $^{[A]}$ corresponds to the rod 'A' presented in Figure 8.

The dynamic stiffness matrix $\mathbf{D} = \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix}$ can be found from the inverse of the receptance matrix $\mathbf{D} = \mathbf{R}^{-1}$, so the dynamic stiffness matrix is, according to Liu *et al.* (2022):

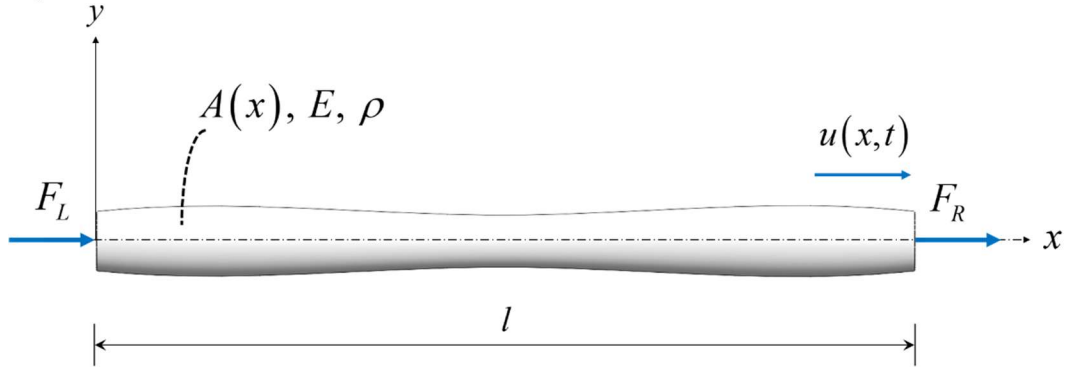
$$\begin{Bmatrix} F_L^{[A]} \\ F_R^{[A]} \end{Bmatrix} = \frac{EA}{l} k_0 l \begin{bmatrix} \cot(k_0 l) & -\csc(k_0 l) \\ -\csc(k_0 l) & \cot(k_0 l) \end{bmatrix} \begin{Bmatrix} U_L^{[A]} \\ U_R^{[A]} \end{Bmatrix} \quad (15)$$

Note that in equation (14), the parameter $\frac{EA}{l}$ is the axial stiffness of the rod with a uniform cross-sectional area.

2.3 Rods of variable cross-section

Figure 14 illustrates an example of a rod with a variable cross-sectional area. The distinguishing property of this rod is its varying cross-sectional area $A(x)$, while the other parameters Young's Modulus E , mass density ρ and length l are the same as those of a rod with a uniform cross-sectional area.

Figure 14 - Illustration of a rod of variable cross-sectional area.



Source: Author own elaboration

Considering the equation of motion for a rod with a variable cross-section (equation (7)) and applying the method of separation of variables, the equation of motion in the space domain $U(x)$ is given by , according to Graff (1991):

$$U''(x) + \frac{A'(x)}{A(x)} U'(x) + k_0^2 U(x) = 0 \quad (16)$$

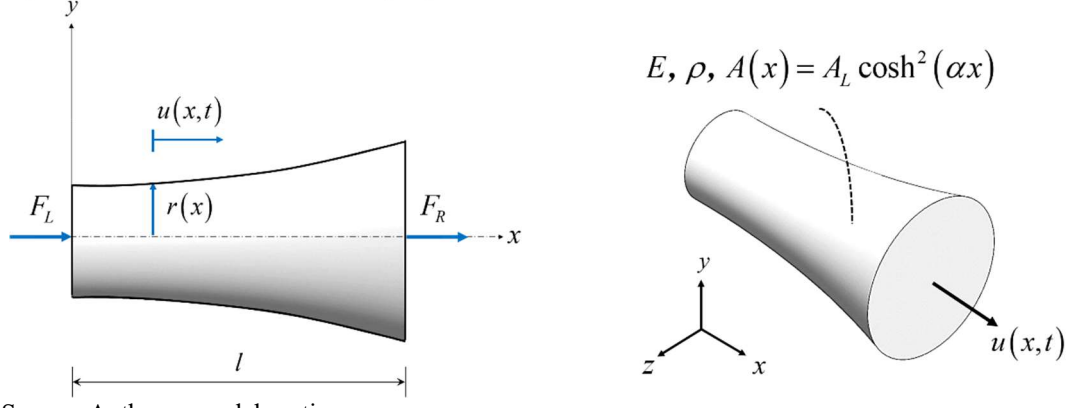
where $k_0 = \omega \sqrt{\rho/E}$ is the wavenumber of a rod with a uniform cross-section, as defined in the previous section.

2.3.1 Rod with catenoidal varying cross-sectional area

The system shown in Figure 15 is a rod with catenoidal varying cross-sectional area. The cross-section area changes according to a catenoid function. On the left-hand side of the rod, the cross-section is defined by $A_L = \pi r_L^2$. At position x , the cross-section is given by $A(x) = A_L \cosh^2(\alpha x)$. The constant α governs the size of the final diameter as x

increases. Rod properties are considered: Young's Modulus E , mass density ρ and length l .

Figure 15 – Rod with catenoidal varying cross-sectional area.



Source: Author own elaboration.

Using the equation (16), and substituting the area equation that governs the rod profile $A(x) = A_L \cosh^2(\alpha x)$, the equation of motion in the space domain can be written as

$$U''(x) + 2\alpha \tanh(\alpha x) U'(x) + k_0^2 U(x) = 0 \quad (17)$$

The solution to differential equation (17) is given by, according to Graff (1991):

$$U(x) = [b_1 \cosh(kx) + b_2 \sinh(kx)] \operatorname{sech}(\alpha x) \quad (18)$$

where $k = \sqrt{k_0^2 - \alpha^2}$, b_1 and b_2 are two constants that depend on the boundary conditions.

To obtain the receptance and dynamic stiffness, four boundary conditions are evaluated according to

$$EA_L U'(0) = -F_L|_{F_R=0}, \quad U(0) = U_L|_{U_R=0} \quad (19)$$

$$EA_L \cosh^2(\alpha l) U'(l) = F_R|_{F_L=0}, \quad U(l) = U_R|_{U_L=0} \quad (20)$$

Therefore, the receptance is obtained

$$\begin{Bmatrix} U_L^{[B]} \\ U_R^{[B]} \end{Bmatrix} = \frac{l}{EA_L} \frac{1}{kl} \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \begin{Bmatrix} F_L^{[B]} \\ F_R^{[B]} \end{Bmatrix} \quad (21)$$

where the superscript $^{[B]}$ corresponds to the rod ‘ B ’ presented in Figure 9. The terms R_{11} , R_{12} , R_{21} and R_{22} are given by

$$R_{11} = \frac{\alpha l \sinh(\alpha l) \sinh(kl) - kl \cosh(\alpha l) \cosh(kl)}{\alpha l \sinh(\alpha l) \cosh(kl) - kl \sinh(kl) \cosh(\alpha l)} \quad (22)$$

$$R_{12} = R_{21} = -\frac{kl}{\alpha l \sinh(\alpha l) \cosh(kl) - kl \sinh(kl) \cosh(\alpha l)} \quad (23)$$

$$R_{22} = \frac{kl \cosh(kl) \operatorname{sech}(\alpha l)}{\alpha l \sinh(\alpha l) \cosh(kl) - kl \sinh(kl) \cosh(\alpha l)} \quad (24)$$

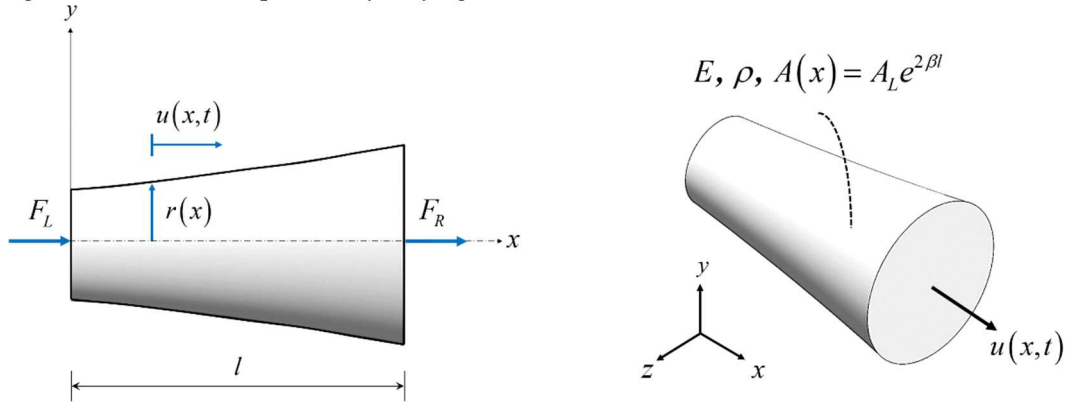
The dynamic stiffness matrix is given by

$$\begin{Bmatrix} F_L^{[B]} \\ F_R^{[B]} \end{Bmatrix} = \frac{EA_L}{l} kl \begin{bmatrix} \frac{\cosh(kl)}{\sinh(kl)} & \frac{\cosh(\alpha l)}{\sinh(kl)} \\ \frac{\cosh(\alpha l)}{\sinh(kl)} & \frac{\cosh(\alpha l)}{\sinh(kl)} \end{bmatrix} \cdot \left[\cosh(\alpha l) \cosh(kl) - \frac{\alpha l}{kl} \sinh(\alpha l) \sinh(kl) \right] \begin{Bmatrix} U_L^{[B]} \\ U_R^{[B]} \end{Bmatrix} \quad (25)$$

2.3.2 Rod with exponentially varying cross-sectional area

A rod with exponentially varying cross-sectional area is shown in Figure 16. The cross-section area of the rod changes according to an exponential function, in which at position x , the cross-section is given by $A(x) = A_L e^{2\beta x}$. The constant β governs the size of the final diameter as x increases, and A_L is the cross-section area in the left-hand side of rod and is defined by $A_L = \pi r_L^2$. The other rods properties are the Young’s Modulus E , mass density ρ and the length l .

Figure 16 – Rod with exponentially varying cross-sectional area.



Source: Author own elaboration.

Substituting the equation that governs the cross-section area, $A(x) = A_L e^{2\beta x}$ on the equation (16), the equation of motion in the space domain is obtained:

$$U''(x) + 2\beta U'(x) + k_0^2 U(x) = 0 \quad (26)$$

The solution of equation (26) can be written, according to Kalkowski, Muggleton and Rustighi (2016):

$$U(x) = e^{-\beta x} \left[c_1 \cos(\hat{k}x) + c_2 \sin(\hat{k}x) \right] \quad (27)$$

where $\hat{k} = \sqrt{k_0^2 - \beta^2}$, c_1 and c_2 are constants that depends on the boundary conditions.

Four boundary conditions are evaluated according to

$$EA_L U'(0) = -F_L \Big|_{F_R=0}, \quad U(0) = U_L \Big|_{U_R=0} \quad (28)$$

$$EA_L e^{2\beta l} U'(l) = F_R \Big|_{F_L=0}, \quad U(l) = U_R \Big|_{U_L=0} \quad (29)$$

The receptance and the dynamic stiffness matrices are given by

$$\begin{Bmatrix} U_L^{[C]} \\ U_R^{[C]} \end{Bmatrix} = \frac{l}{EA_L} \frac{\hat{k}l}{(k_0 l)^2} \begin{bmatrix} \frac{\beta l}{\hat{k}l} - \cot(\hat{k}l) & -e^{-\beta l} \csc(\hat{k}l) \\ -e^{-\beta l} \csc(\hat{k}l) & e^{-2\beta l} \left(-\frac{\beta l}{\hat{k}l} - \cot(\hat{k}l) \right) \end{bmatrix} \begin{Bmatrix} F_L^{[C]} \\ F_R^{[C]} \end{Bmatrix} \quad (30)$$

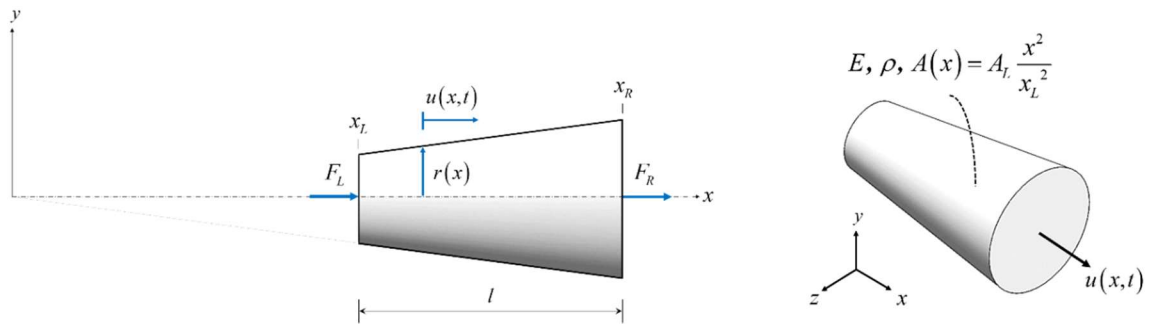
$$\begin{Bmatrix} F_L^{[C]} \\ F_R^{[C]} \end{Bmatrix} = \frac{EA_L}{l} \hat{k}l \begin{bmatrix} \frac{\beta l}{\hat{k}l} + \cot(\hat{k}l) & -e^{\beta l} \csc(\hat{k}l) \\ -e^{\beta l} \csc(\hat{k}l) & e^{2\beta l} \left(-\frac{\beta l}{\hat{k}l} + \cot(\hat{k}l) \right) \end{bmatrix} \begin{Bmatrix} U_L^{[C]} \\ U_R^{[C]} \end{Bmatrix} \quad (31)$$

where the superscript ^[C] corresponds to the rod ‘C’ presented in Figure 9.

2.2.3 Rod with conically varying cross-sectional area

The system shown in Figure 17 is rod with conically varying cross-section area. In this case, the cross-section governed by equation $A(x) = A_L \frac{x^2}{x_L^2}$, where $A_L = \pi r_L^2$ at position x_L . The properties of the rod are considered: Young’s Modulus E , mass density ρ and the length l .

Figure 17 – Rod with conically varying cross-sectional area.



Source: Author own elaboration.

The equation of motion in space domain is obtain using the equation (16), and is given by, according to Graff (1991):

$$U''(x) + \frac{2}{x}U'(x) + k_0^2 U(x) = 0 \quad (32)$$

To solve the equation (32), it is necessary to use Bessel's equations. Appendix A and B present the mathematical procedure to find the half-order Bessel equation that governs the solution of the differential equation. Therefore, it can be assumed that the solution is given by

$$U(x) = \frac{1}{\sqrt{x}} \left\{ d_1 J_{\frac{1}{2}}(k_0 x) + d_2 J_{-\frac{1}{2}}(k_0 x) \right\} \quad (33)$$

where the constants d_1 and d_2 depend on the boundary conditions and according to the

properties of the Bessel function $J_{\frac{1}{2}}(k_0 x) = \sqrt{\frac{2}{\pi k_0 x}} \sin(k_0 x)$ and

$J_{-\frac{1}{2}}(k_0 x) = \sqrt{\frac{2}{\pi k_0 x}} \cos(k_0 x)$. Substituting these equations in equation (33), the solution of equation (32) is given by

$$U(x) = \frac{1}{k_0 x} \left[d_1 \sin(k_0 x) + d_2 \cos(k_0 x) \right] \quad (34)$$

Assuming the four boundary conditions as

$$EA_L U'(x_L) = -F_L|_{F_R=0}, \quad U(x_L) = U_L|_{U_R=0} \quad (35)$$

$$EA_L \frac{x_R^2}{x_L^2} U'(x_R) = F_R|_{F_L=0}, \quad U(x_R) = U_R|_{U_L=0} \quad (36)$$

where $x_R = x_L + l$.

The receptance and dynamic stiffness matrices are given by

$$\begin{Bmatrix} U_L^{[D]} \\ U_R^{[D]} \end{Bmatrix} = \frac{l}{EA_L} \hat{x}_L \begin{bmatrix} \frac{\hat{x}_R k_0 l \cos(k_0 l) - \sin(k_0 l)}{k_0 l \cos(k_0 l) - \sin(k_0 l) (1 + \hat{x}_L \hat{x}_R (k_0 l)^2)} & \frac{\hat{x}_L k_0 l}{k_0 l \cos(k_0 l) - \sin(k_0 l) (1 + \hat{x}_L \hat{x}_R (k_0 l)^2)} \\ \frac{\hat{x}_L k_0 l}{k_0 l \cos(k_0 l) - \sin(k_0 l) (1 + \hat{x}_L \hat{x}_R (k_0 l)^2)} & \frac{\hat{x}_L}{\hat{x}_R} \left(\frac{\hat{x}_L k_0 l \cos(k_0 l) + \sin(k_0 l)}{k_0 l \cos(k_0 l) - \sin(k_0 l) (1 + \hat{x}_L \hat{x}_R (k_0 l)^2)} \right) \end{bmatrix} \begin{Bmatrix} F_L^{[D]} \\ F_R^{[D]} \end{Bmatrix} \quad (37)$$

$$\begin{Bmatrix} F_L^{[D]} \\ F_R^{[D]} \end{Bmatrix} = \frac{EA_L}{l} \frac{1}{\hat{x}_L} \begin{bmatrix} \frac{\hat{x}_L k_0 l \cos(k_0 l) + \sin(k_0 l)}{\sin(k_0 l)} & -\frac{\hat{x}_R k_0 l}{\sin(k_0 l)} \\ -\frac{\hat{x}_R k_0 l}{\sin(k_0 l)} & \frac{\hat{x}_R}{\hat{x}_L} \cdot \frac{\hat{x}_R k_0 l \cos(k_0 l) + \sin(k_0 l)}{\sin(k_0 l)} \end{bmatrix} \begin{Bmatrix} U_L^{[D]} \\ U_R^{[D]} \end{Bmatrix} \quad (38)$$

where $\hat{x}_L = x_L / l$ and $\hat{x}_R = 1 + \hat{x}_L$.

2.4 Displacement transmissibility of the rods with variable cross-sectional areas

The displacement transmissibility of rods with variable cross-sectional areas is evaluated. In this analysis, the rods are excited at the left-hand end (i.e. $F_R = 0$), and the displacement transmissibility is defined as the ratio between the displacement at the right-hand end and the displacement at the left-hand end U_R / U_L .

The displacement transmissibility can be determined using the elements of the receptance matrices presented in the previous sections, where $U_R / U_L = R_{21} / R_{11}$.

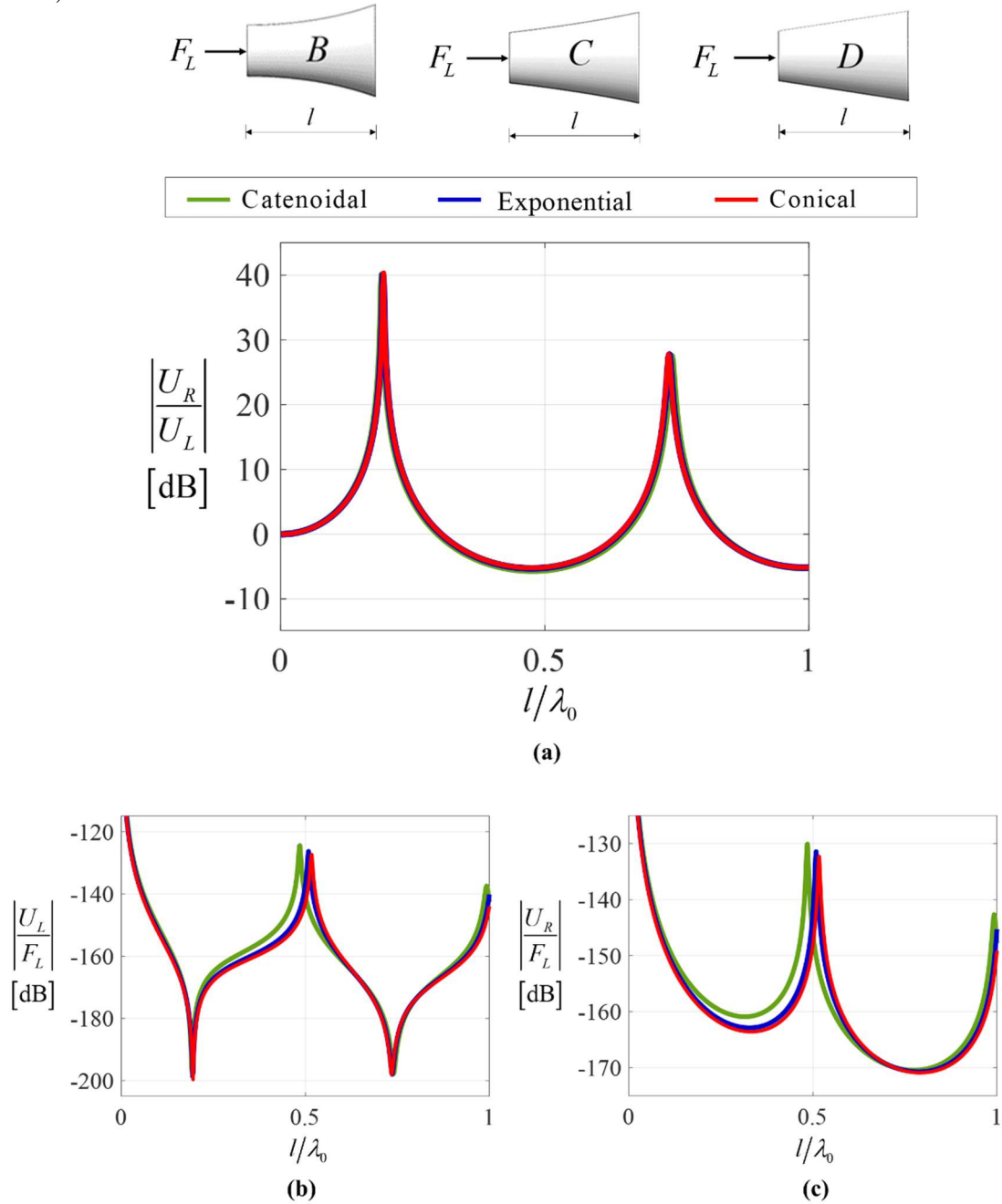
To analyze the displacement transmissibility, Figure 18(a) presents the displacement transmissibility responses of the rods discussed in the previous section: the rods with variable cross-sections, including the catenoidal-shaped rod, the exponential-shaped rod, and the conical-shaped rod. Figures 18(b) and 18(c) present the point receptance and transfer receptance, which are defined as the ratio between the displacement at the first or last nodes and the applied force. For better comparison, the transmissibility and receptance responses are plotted using the same cross-sectional areas at the left-hand and right-hand sides.

To facilitate interpretation, the responses are plotted using a non-dimensional wavenumber, l / λ_0 , where $\lambda_0 = 2\pi / k_0$ represents the wavelength of a rod with a uniform cross-section and l is the length of the rod that composed the cell. The parameters used in Figure 18 to obtain the results are $E = 3 \text{ GPa}$, $\rho = 1130 \text{ kg/m}^3$, $l = 0.15 \text{ m}$, $r_L = 0.025 \text{ m}$,

$\alpha = 7.95 \text{ m}^{-1}$, $\beta = 3.91 \text{ m}^{-1}$ ($\beta l = 0.5865$) and $x_L = 0.1875 \text{ m}$. A loss factor $\eta = 0.01$ is assumed that is incorporated in Young's modulus in complex form. These parameters were specifically selected to reflect the physical system discussed in Chapter 6, which focuses on experimental validation, ensuring consistency and comparability between the numerical and experimental analyses.

In Figure 18, it can be observed that rods with catenoidal, exponential and conical profiles present similar dynamic responses with slight differences in their resonance frequencies. These slight differences are due the different volumes that rods have.

Figure 18 – Comparison of transmissibilities responses of the rods with variable cross section: (a) Displacement transmissibility; (b) Point receptance (dB ref. 1m/N); (c) Transfer receptance (dB ref. 1m/N).



Source: Author own elaboration.

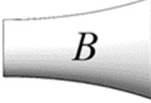
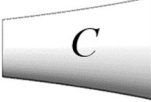
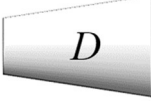
2.5 Chapter Remarks

This chapter presented the mathematical development required for obtaining the receptance and dynamic stiffness matrices of rods with uniform cross-sectional areas as well as rods with variable cross-sectional areas, such as catenoidal, exponential, and conical profiles.

The mathematical model was developed based on classical theory, exclusively considering axial deformation along the rod, focusing only on longitudinal behavior. The receptance matrices (Table 1) and dynamic stiffness matrices (Table 2) of the rods with variable cross-sectional area were derived and will be applied to the modeling of periodic structures through their corresponding transfer matrix. Analytical equations for the transmissibility of the cells will be derived, enabling a detailed analysis of the dynamic behavior of these periodic structures under different configurations and excitation frequencies.

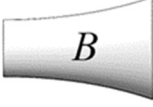
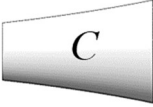
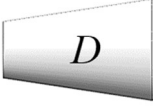
Additionally, it was observed that the displacement transmissibility of the catenoidal, conical, and exponential rods exhibited similar responses. This indicates that, despite their different geometric profiles, the general transmissibility characteristics are comparable.

Table 1 – Receptance matrices of the rods with variable cross-sectional area.

Rod	Receptance matrix $\mathbf{R} = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix}$
	$\mathbf{R}^{[B]} = \frac{l}{EA_L} \frac{1}{kl} \begin{bmatrix} \frac{\alpha l \sinh(\alpha l) \sinh(kl) - kl \cosh(\alpha l) \cosh(kl)}{\alpha l \sinh(\alpha l) \cosh(kl) - kl \sinh(kl) \cosh(\alpha l)} & -\frac{kl}{\alpha l \sinh(\alpha l) \cosh(kl) - kl \sinh(kl) \cosh(\alpha l)} \\ \frac{kl}{\alpha l \sinh(\alpha l) \cosh(kl) - kl \sinh(kl) \cosh(\alpha l)} & \frac{kl \cosh(kl) \operatorname{sech}(\alpha l)}{\alpha l \sinh(\alpha l) \cosh(kl) - kl \sinh(kl) \cosh(\alpha l)} \end{bmatrix}$
	$\mathbf{R}^{[C]} = \frac{l}{EA_L} \frac{\hat{k}l}{(k_0l)^2} \begin{bmatrix} \frac{\beta l}{\hat{k}l} - \cot(\hat{k}l) & -e^{-\beta l} \csc(\hat{k}l) \\ -e^{-\beta l} \csc(\hat{k}l) & e^{-2\beta l} \left(-\frac{\beta l}{\hat{k}l} - \cot(\hat{k}l) \right) \end{bmatrix}$
	$\mathbf{R}^{[D]} = \frac{l}{EA_L} \hat{x}_L \begin{bmatrix} \frac{\hat{x}_R k_0 l \cos(k_0 l) - \sin(k_0 l)}{k_0 l \cos(k_0 l) - \sin(k_0 l) (1 + \hat{x}_L \hat{x}_R (k_0 l)^2)} & \frac{\hat{x}_L k_0 l}{k_0 l \cos(k_0 l) - \sin(k_0 l) (1 + \hat{x}_L \hat{x}_R (k_0 l)^2)} \\ \frac{\hat{x}_L k_0 l}{k_0 l \cos(k_0 l) - \sin(k_0 l) (1 + \hat{x}_L \hat{x}_R (k_0 l)^2)} & \frac{\hat{x}_L}{\hat{x}_R} \left(\frac{\hat{x}_L k_0 l \cos(k_0 l) + \sin(k_0 l)}{k_0 l \cos(k_0 l) - \sin(k_0 l) (1 + \hat{x}_L \hat{x}_R (k_0 l)^2)} \right) \end{bmatrix}$

Source: Author own elaboration.

Table 2 – Dynamic stiffness matrices of the rods with variable cross-sectional area.

Rod	Dynamic stiffness matrix $\mathbf{D} = \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix}$
	$\mathbf{D}^{[B]} = \frac{EA_L}{l} \frac{1}{\hat{x}_L} \begin{bmatrix} \frac{\hat{x}_L k_0 l \cos(k_0 l) + \sin(k_0 l)}{\sin(k_0 l)} & -\frac{\hat{x}_R k_0 l}{\sin(k_0 l)} \\ -\frac{\hat{x}_R k_0 l}{\sin(k_0 l)} & \frac{\hat{x}_R}{\hat{x}_L} \cdot \frac{\hat{x}_R k_0 l \cos(k_0 l) + \sin(k_0 l)}{\sin(k_0 l)} \end{bmatrix}$
	$\mathbf{D}^{[C]} = \frac{EA_L}{l} \hat{k} l \begin{bmatrix} \frac{\beta l}{\hat{k} l} + \cot(\hat{k} l) & -e^{\beta l} \csc(\hat{k} l) \\ -e^{\beta l} \csc(\hat{k} l) & e^{2\beta l} \left(-\frac{\beta l}{\hat{k} l} + \cot(\hat{k} l) \right) \end{bmatrix}$
	$\mathbf{D}^{[D]} = \frac{EA_L}{l} \frac{1}{\hat{x}_L} \begin{bmatrix} \frac{\hat{x}_L k_0 l \cos(k_0 l) + \sin(k_0 l)}{\sin(k_0 l)} & -\frac{\hat{x}_R k_0 l}{\sin(k_0 l)} \\ -\frac{\hat{x}_R k_0 l}{\sin(k_0 l)} & \frac{\hat{x}_R}{\hat{x}_L} \cdot \frac{\hat{x}_R k_0 l \cos(k_0 l) + \sin(k_0 l)}{\sin(k_0 l)} \end{bmatrix}$

Source: Author own elaboration.

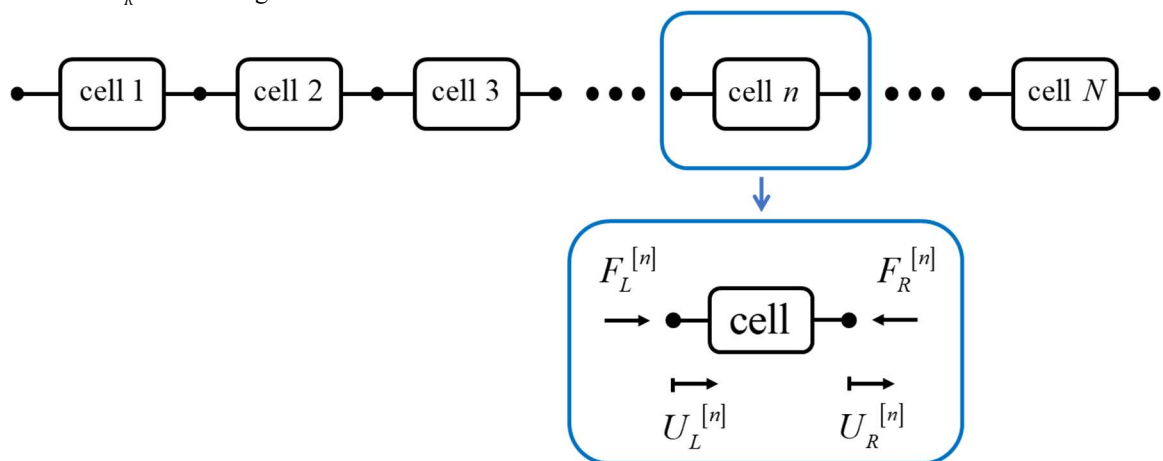
3 MODELING A PERIODIC STRUCTURE

This chapter introduces the mathematical procedure for modeling periodic structures, building upon the framework established by Gonçalves, Brennan and Cleante (2021). The transfer matrix method is employed to determine the dynamic behavior of the entire structure based on the transmissibility characteristics of a single cell. Additionally, the Cayley-Hamilton theorem is introduced to facilitate the derivation of simplified analytical expressions for the displacement transmissibility of periodic structures. The dynamic stiffness and receptance matrices will also be considered in the modeling process. Consequently, this chapter provides analytical expressions for each cell to be studied in the scope of this work.

3.1 The transfer matrix method

This section summaries the transfer matrix method, which is used to estimate the dynamic response of unidimensional periodic structures composed by N cells, from the transmissibility of a single cell. Consider a mono-coupled periodic structure composed of N identical periodic cells, as shown in Figure 19.

Figure 19 – General mono-coupled structure comprising N periodic cells, highlighting a single cell n . The displacement amplitudes within the single cell are represented by $U_L^{[n]}$ on the left-hand side and $U_R^{[n]}$ on the right-hand side. The excitation force amplitudes in the single cell are denoted as $F_L^{[n]}$ on the left-hand side and $F_R^{[n]}$ on the right-hand side.



Source: Author own elaboration.

The relationship between the force and displacement at the right end of the structure in Figure 19 to the force and displacement at the left end is given by, according to Mead (1996):

$$q_R^{[N]} = \mathbf{T}^N q_L^{[1]} \quad (39)$$

where $q_R = \{F_R, U_R\}^T$, $q_L = \{F_L, U_L\}^T$, $\mathbf{T} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}$ is the transfer matrix of a single cell, and the superscript T denotes the transpose of the vector.

From the work of Rubin (1967) it is possible to obtain the transfer matrix from the transformation of the receptance matrix. Consider the receptance matrix as:

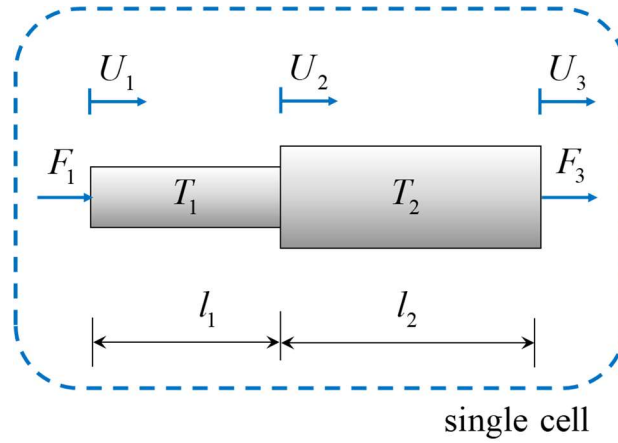
$$\begin{Bmatrix} U_L \\ U_R \end{Bmatrix} = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \begin{Bmatrix} F_L \\ F_R \end{Bmatrix} \quad (40)$$

Applying the transformation proposed by Rubin (1967), the transfer matrix is given by

$$\begin{Bmatrix} F_R \\ U_R \end{Bmatrix} = \begin{bmatrix} R_{12}^{-1} R_{11} & -R_{12}^{-1} \\ -R_{22} R_{12}^{-1} R_{11} + R_{12} & R_{22} R_{12}^{-1} \end{bmatrix} \begin{Bmatrix} F_L \\ U_L \end{Bmatrix} \rightarrow \begin{Bmatrix} F_R \\ U_R \end{Bmatrix} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \begin{Bmatrix} F_L \\ U_L \end{Bmatrix} \quad (41)$$

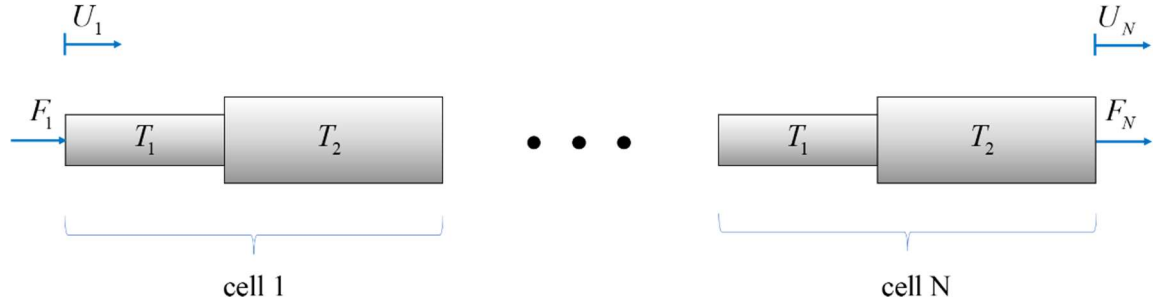
The transfer matrix of the cell is obtained by multiplication of the transfer matrices of the cell elements. For example, Figure 20 shows a single cell composed of two elements.

Figure 20 – A single cell composed of two elements.



Source: Author own elaboration.

The transfer matrix of the cell is obtained by $\mathbf{T} = \mathbf{T}_2 \mathbf{T}_1$. Therefore, for a periodic structure composed of N cells, according to Figure 21, the transfer matrix is given by $\mathbf{T}^{[N]} = (\mathbf{T}_2 \mathbf{T}_1)^N = \mathbf{T}^N$. Here, $\mathbf{T}^{[N]}$ is the transfer matrix of the entire periodic structure, which results from raising the transfer matrix of a single cell $\mathbf{T}$, to the power N .

Figure 21 – Periodic structure composed N cells where the cell is composed of two elements.

Source: Author own elaboration.

3.1.1 Transfer matrices of the rod elements

The analytical expressions of the transfer matrices for the rod elements shown in Chapter 2 are determined and presented in Table 3. The remaining rod elements a , b and c which form the sets of cells of Figure 8 in Chapter 1 are also calculated and shown in Table 3. The receptance matrix of these rods to obtain the transfer matrices is determined in a similar manner shown in Chapter 2.

Table 3 - Transfer matrices of the rod elements. All rods have length l .

(to be continued)

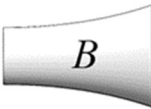
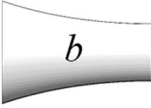
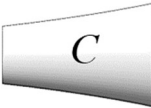
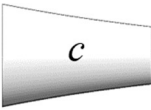
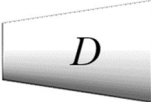
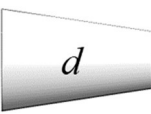
Rod	Transfer Matrix
	$\mathbf{T}_A = \begin{bmatrix} \cos(k_0 l) & \frac{EA}{l} k_0 l \sin(k_0 l) \\ -\frac{l}{EA} \frac{\sin(k_0 l)}{k_0 l} & \cos(k_0 l) \end{bmatrix}$
where A is the constant cross-sectional area.	
	$\mathbf{T}_B = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix},$ where $T_{11} = \cosh(\alpha l) \cosh(kl) - \frac{\alpha l}{kl} \sinh(\alpha l) \sinh(kl)$, $T_{12} = \frac{EA_L}{l} \frac{kl}{\tanh(kl)} \left[\frac{\cosh(\alpha l)}{\cosh(kl)} - \cosh(\alpha l) \cosh(kl) + \frac{\alpha l}{kl} \sinh(\alpha l) \sinh(kl) \right],$ $T_{21} = -\frac{l}{EA_L} \frac{\sinh(kl)}{kl \cosh(\alpha l)} \quad \text{and} \quad T_{22} = \frac{\cosh(kl)}{\cosh(\alpha l)}.$
	$\mathbf{T}_b = \begin{bmatrix} \frac{\cosh(kl)}{\cosh(\alpha l)} & \frac{EA_L}{l} \frac{kl}{\tanh(kl)} \left[\frac{\cosh(\alpha l)}{\cosh(kl)} - \cosh(\alpha l) \cosh(kl) + \frac{\alpha l}{kl} \sinh(\alpha l) \sinh(kl) \right] \\ -\frac{l}{EA_L} \frac{\sinh(kl)}{kl \cosh(\alpha l)} & \cosh(\alpha l) \cosh(kl) - \frac{\alpha l}{kl} \sinh(\alpha l) \sinh(kl) \end{bmatrix}$
	$\mathbf{T}_C = \begin{bmatrix} e^{\beta l} \left(\cos(\hat{k}l) - \frac{\beta l}{\hat{k}l} \sin(\hat{k}l) \right) & \frac{EA_L}{l} \frac{e^{\beta l} \left((\beta l)^2 + (\hat{k}l)^2 \right) \sin(\hat{k}l)}{\hat{k}l} \\ -\frac{l}{EA_L} \frac{e^{-\beta l} \sin(\hat{k}l)}{\hat{k}l} & e^{-\beta l} \left(\cos(\hat{k}l) + \frac{\beta l}{\hat{k}l} \sin(\hat{k}l) \right) \end{bmatrix}$
	$\mathbf{T}_c = \begin{bmatrix} e^{-\beta l} \left(\frac{\beta l}{\hat{k}l} \sin(\hat{k}l) + \cos(\hat{k}l) \right) & \frac{EA_L}{l} \frac{e^{\beta l} \left((\beta l)^2 + (\hat{k}l)^2 \right) \sin(\hat{k}l)}{\hat{k}l} \\ -\frac{l}{EA_L} \frac{e^{-\beta l} \sin(\hat{k}l)}{\hat{k}l} & e^{\beta l} \left(-\frac{\beta l}{\hat{k}l} \sin(\hat{k}l) + \cos(\hat{k}l) \right) \end{bmatrix}$

Table 3 - Transfer matrices of the rod elements. All rods have length l .

(conclusion)

Rod	Transfer Matrix
	$\mathbf{T}_D = \begin{bmatrix} \frac{\hat{x}_R k_0 l \cos(k_0 l) - \sin(k_0 l)}{\hat{x}_L k_0 l} & \frac{EA_L}{l} \frac{\left(\left(\hat{x}_R \hat{x}_L (k_0 l)^2 + 1 \right) \sin(k_0 l) - k_0 l \cos(k_0 l) \right)}{\hat{x}_L^2 k_0 l} \\ -\frac{l}{EA_L} \frac{\hat{x}_L \sin(k_0 l)}{\hat{x}_R k_0 l} & \frac{\hat{x}_L k_0 l \cos(k_0 l) + \sin(k_0 l)}{\hat{x}_R k_0 l} \end{bmatrix}$
	$\mathbf{T}_d = \begin{bmatrix} \frac{\hat{x}_L k_0 l \cos(k_0 l) + \sin(k_0 l)}{\hat{x}_R k_0 l} & \frac{EA_L}{l} \frac{\left(\left(\hat{x}_R \hat{x}_L (k_0 l)^2 + 1 \right) \sin(k_0 l) - k_0 l \cos(k_0 l) \right)}{\hat{x}_L^2 k_0 l} \\ -\frac{l}{EA_L} \frac{\hat{x}_L \sin(k_0 l)}{\hat{x}_R k_0 l} & \frac{\hat{x}_R k_0 l \cos(k_0 l) - \sin(k_0 l)}{\hat{x}_L k_0 l} \end{bmatrix}$

Source: Author own elaboration.

3.2 The Cayley-Hamilton theorem

The Cayley-Hamilton theorem is presented as a complement to obtain expression for displacement transmissibility $\frac{U_R^{[N]}}{U_L^{[1]}}$ of periodic structure with consists of N mono-coupled identical cells. The foundational expressions were established by Faulkner and Hong (1985), who derived analytical equations for the transfer matrix of mono-coupled periodic systems by applying the Cayley-Hamilton theorem.

The matrix $\mathbf{T}^N$ of equation (39) can be written with combination linear of eigenvalues to N - power $\Lambda_{1,2}^N$, where $\Lambda_1 = \Lambda$ and $\Lambda_2 = \frac{1}{\Lambda}$, thus can be written as, according to Ayres Jr (1962):

$$\Lambda_1^N = \Lambda^N = \mathbf{e}_1 \left(\Lambda + \frac{1}{\Lambda} \right) + \mathbf{e}_2 \left(\Lambda - \frac{1}{\Lambda} \right) \quad (42)$$

$$\Lambda_2^N = \frac{1}{\Lambda^N} = \mathbf{e}_1 \left(\Lambda + \frac{1}{\Lambda} \right) - \mathbf{e}_2 \left(\Lambda - \frac{1}{\Lambda} \right) \quad (43)$$

where e_1 and e_2 are constants. Note that $\Lambda = e^\mu$, so it is possible to assume that $\Lambda + \frac{1}{\Lambda} = 2 \cosh(\mu)$ and $\Lambda - \frac{1}{\Lambda} = 2 \sinh(\mu)$. Substituting these affirmations in equations (42) and (43), the constants are found, then

$$e_1 = \frac{\cosh(N\mu)}{2 \cosh(\mu)}, \quad e_2 = \frac{\sinh(N\mu)}{2 \sinh(\mu)} \quad (44)$$

Note that $\mu = e_1 \cosh(\Theta)$, where $\Theta = \frac{T_{11} + T_{22}}{2}$ is related with eigenvalues of transfer matrix $\mathbf{T}$ by $\Lambda_{1,2} = \Theta \pm \sqrt{\Theta^2 - 1}$, so

$$\mathbf{T}^N = e_1 [\mathbf{T} + \mathbf{T}^{-1}] + e_2 [\mathbf{T} - \mathbf{T}^{-1}] \quad (45)$$

where $\mathbf{T} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}$ is the transfer matrix of the single cell and $\mathbf{T}^{-1} = \begin{bmatrix} T_{22} & -T_{12} \\ -T_{21} & T_{11} \end{bmatrix}$ is the inverse matrix. Therefore, the equation (45) can be written as

$$\mathbf{T}^{[N]} = \begin{bmatrix} \cosh(N\mu) + (T_{11} - T_{12}) \frac{\sinh(N\mu)}{2 \sinh(\mu)} & T_{12} \frac{\sinh(N\mu)}{\sinh(\mu)} \\ T_{21} \frac{\sinh(N\mu)}{\sinh(\mu)} & \cosh(N\mu) + (T_{11} - T_{12}) \frac{\sinh(N\mu)}{2 \sinh(\mu)} \end{bmatrix} \quad (46)$$

Rubin (1967) proposes the transformation from transfer matrix to receptance matrix, so

$$\begin{Bmatrix} U_L \\ U_R \end{Bmatrix} = \begin{bmatrix} -T_{12}^{-1} T_{11} & -T_{12}^{-1} \\ -(T_{12}^{-1})^T & -T_{22} T_{12}^{-1} \end{bmatrix} \begin{Bmatrix} F_L \\ F_R \end{Bmatrix} \rightarrow \begin{Bmatrix} U_L \\ U_R \end{Bmatrix} = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \begin{Bmatrix} F_L \\ F_R \end{Bmatrix} \quad (47)$$

Therefore, applying the transformation of Rubin (1967) in matrix of equation (46), the receptance matrix of periodic structure, is given by

$$\mathbf{R}^{[N]} = -\frac{\sinh(\mu)}{T_{12} \sinh(N\mu)} \begin{bmatrix} \cosh(N\mu) + (T_{11} - T_{12}) \frac{\sinh(N\mu)}{2\sinh(\mu)} & 1 \\ 1 & \cosh(N\mu) + (T_{11} - T_{12}) \frac{\sinh(N\mu)}{2\sinh(\mu)} \end{bmatrix} \quad (48)$$

Considering $F_R = 0$, the relationship between displacement $U_R^{[N]}$ and $U_L^{[1]}$ is given by, according to Gonçalves, Brennan and Cleante (2021)

$$\frac{U_R^{[N]}}{U_L^{[1]}} = \frac{2\sinh(\mu)}{2\sinh(\mu)\cosh(N\mu) + (T_{11} - T_{22})\sinh(N\mu)} \quad (49)$$

For symmetric structures, the elements of the transfer matrix $T_{11} = T_{22}$. So, the equation (49) for periodic structures composed by symmetric cells, reduces to, according to Gonçalves, Brennan and Cleante (2021):

$$\frac{U_R^{[N]}}{U_L^{[1]}} = \frac{1}{\cosh(N\mu)} \quad (50)$$

3.2.1 Displacement transmissibility of the cells

The analytical expressions of displacement transmissibility of cells are obtained and are shown in Table 4. To do this, using the transfer matrices in Table 3 for each rod configuration, the transfer matrices for cell configuration shown in Figures 8 and 9 of Chapter 1 can be calculated by multiplying the transfer matrices of the rod elements, as mentioned in Section 3.1. Once this is done, with the cell transfer matrix obtained, the displacement transmissibility can be calculated using equation (49) or equation (50) for symmetric cell.

Table 4 - Equation of transmissibility $T_r = U_R^{[2]} / U_L^{[1]}$, for a single cell. All cells have length $2l$.

(to be continued)

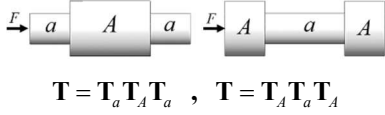
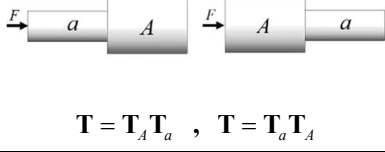
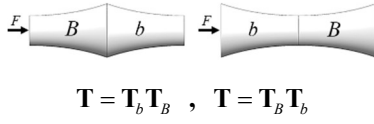
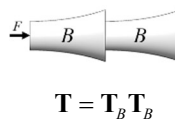
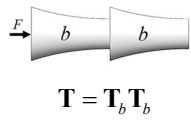
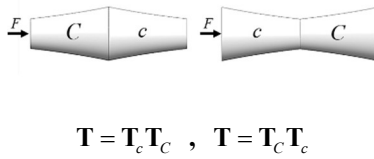
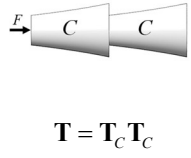
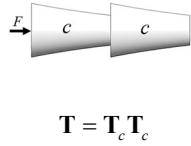
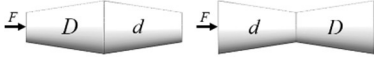
Sets of cells	Displacement Transmissibility
Stepped Rod	
 $T_r = \frac{4\delta}{(\delta-1)^2 - (\delta+1)^2 \cos(2k_0 l)}, \quad \delta = \frac{\text{cross-sectional area of rod } A}{\text{cross-sectional area of rod } a}$ $\mathbf{T} = \mathbf{T}_a \mathbf{T}_A \mathbf{T}_a, \quad \mathbf{T} = \mathbf{T}_A \mathbf{T}_a \mathbf{T}_A$	
 $T_r = \frac{2}{(\delta+1) \cos(2k_0 l) - (\delta-1)}$ $\mathbf{T} = \mathbf{T}_A \mathbf{T}_a, \quad \mathbf{T} = \mathbf{T}_a \mathbf{T}_A$	
Catenoidal	
 $T_r = -\frac{kl \cosh(\alpha l)}{\alpha l \sinh(\alpha l) \sinh(2kl) + kl \cosh(\alpha l) \cosh(2kl)}$ $\mathbf{T} = \mathbf{T}_b \mathbf{T}_B, \quad \mathbf{T} = \mathbf{T}_B \mathbf{T}_b$	
 $T_r = -\frac{(kl)^2 \cosh(\alpha l)}{\cosh(\alpha l) f_1 - \alpha l kl \sinh(\alpha l) f_2}, \text{ where}$ $f_1 = (\alpha l)^2 \sinh^2(\alpha l) \sinh^2(kl) + (kl)^2 \sinh^2(kl) + (kl)^2 \cosh^2(\alpha l) \cosh^2(kl)$ $f_2 = \sinh(2kl) + \sinh^2(\alpha l) \sinh(2kl) + \sinh(kl) \cosh(kl)$ $\mathbf{T} = \mathbf{T}_B \mathbf{T}_B$	
 $T_r = -\frac{4kl \cosh^2(\alpha l)}{\alpha l \sinh(2\alpha l) \sinh(2kl) - 4kl \sinh(kl) \cosh^2(\alpha l) - 4kl \cosh^2(kl)}$ $\mathbf{T} = \mathbf{T}_b \mathbf{T}_b$	
Exponential	
 $T_r = \frac{-(\hat{kl})^2}{2\left[(\hat{kl})^2 + (\beta l)^2\right] \sin^2(\hat{kl}) - (\hat{kl})^2}$ $\mathbf{T} = \mathbf{T}_c \mathbf{T}_C, \quad \mathbf{T} = \mathbf{T}_C \mathbf{T}_c$	
 $T_r = \frac{-(\hat{kl})^2}{\left[(\hat{kl})^2 + (\beta l)^2\right] \sin^2(\hat{kl}) - e^{2\beta l} (\hat{kl} \cos(\hat{kl}) - \beta l \sin(\hat{kl}))^2}$ $\mathbf{T} = \mathbf{T}_C \mathbf{T}_C$	
 $T_r = \frac{-(\hat{kl})^2 e^{2\beta l}}{\left[(\hat{kl})^2 + (\beta l)^2\right] e^{2\beta l} \sin^2(\hat{kl}) - (\hat{kl} \cos(\hat{kl}) + \beta l \sin(\hat{kl}))^2}$ $\mathbf{T} = \mathbf{T}_c \mathbf{T}_c$	

Table 4 - Equation of transmissibility $T_r = U_R^{[2]} / U_L^{[1]}$, for a single cell. All cells have length $2l$.

Sets of cells	Displacement Transmissibility
Conical	
 $\mathbf{T} = \mathbf{T}_d \mathbf{T}_D \quad , \quad \mathbf{T} = \mathbf{T}_D \mathbf{T}_d$	$T_r = \frac{\hat{x}_L \hat{x}_R (k_0 l)^2}{\left[\hat{x}_L \hat{x}_R (k_0 l)^2 + 1 \right] \cos(2k_0 l) + k_0 l \sin(2k_0 l) - 1}$
 $\mathbf{T} = \mathbf{T}_D \mathbf{T}_D$	$T_r = \frac{2\hat{x}_L^2 \hat{x}_R (k_0 l)^2}{2\hat{x}_R^3 (k_0 l)^2 \cos^2(k_0 l) - 2\left[\hat{x}_L^2 \hat{x}_R (k_0 l)^2 - 1 \right] \sin^2(k_0 l) - (2\hat{x}_R^2 + \hat{x}_L)k_0 l \sin(2k_0 l)}$
 $\mathbf{T} = \mathbf{T}_d \mathbf{T}_d$	$T_r = \frac{2\hat{x}_L \hat{x}_R^2 (k_0 l)^2}{2\hat{x}_L^3 (k_0 l)^2 \cos^2(k_0 l) - 2\left[\hat{x}_L \hat{x}_R^2 (k_0 l)^2 + 1 \right] \sin^2(k_0 l) + (2\hat{x}_L^2 + \hat{x}_R)k_0 l \sin(2k_0 l)}$

Source: Author own elaboration.

The analytical expressions of displacement transmissibility of the sets of cells in Figure 10 of Chapter 1 are also calculated. The cell transfer matrices are given from the transfer matrices in Table 3 since the cells are a rod element. Thus, using equation (49), the expressions are obtained and presented in Table 5.

Table 5 - Equation of transmissibility $T_r = U_R^{[2]} / U_L^{[1]}$, for a single cell. All cells have length $2l$.

Sets of cells	Displacement Transmissibility
Catenoidal	
	$T_r = \frac{kl}{kl \cosh(2\alpha l) \cosh(2kl) - \alpha l \sinh(2\alpha l) \sinh(2kl)}$
	$T_r = \frac{\cosh(2\alpha l)}{\cosh(2kl)}$
Exponential	
	$T_r = \frac{\hat{k}l e^{-2\beta l}}{(\hat{k}l \cot(2kl) - \beta l) \sin(2\hat{k}l)}$
	$T_r = \frac{\hat{k}l e^{2\beta l}}{(\beta l + \hat{k}l \cot(2kl)) \sin(2\hat{k}l)}$
Conical	
	$T_r = \frac{\hat{x}_L k_0 l}{\hat{x}_R k_0 l \cos(2k_0 l) - \sin(2k_0 l)}$
	$T_r = \frac{\hat{x}_R k_0 l}{\hat{x}_L k_0 l \cos(2k_0 l) + \sin(2k_0 l)}$

Source: Author own elaboration.

3.3 Chapter Remarks

This chapter presented the transfer matrix theory and the Cayley-Hamilton theorem as fundamental methodology for modeling periodic structures. Through these theories, it was initially possible to derive the analytical equations of the transfer matrices for each rod composing the cell sets to be investigated in the study, and subsequently, to develop analytical equations describing the dynamic behavior of these sets.

The transmissibility equation for a single cell, whose response will be analyzed in the next chapter, allows the prediction of the dynamic behavior of the entire periodic structure. This approach enables the identification of attenuation band regions (stop-bands) and provides an understanding of the factors influencing the structure's dynamic behavior, such as the symmetry or asymmetry of the cells.

4 NUMERICAL SIMULATIONS FOR A FINITE PERIODIC STRUCTURE

This chapter presents simulations for finite periodic structures composed of exponential-shaped rods to investigate their dynamic behavior through the displacement transmissibility response. For these simulations, the analytical transmissibility equations derived in Chapter 3 were used. The Cayley-Hamilton theorem is employed to determine the responses of the periodic structure.

Different cell configurations are analyzed to understand the factors influencing the formation of attenuation bands (stop-bands). An analysis is conducted regarding the parameter βl to understand its influence on the stop-band characteristics.

Furthermore, a comparison with stepped rods is presented to highlight the differences in dynamic behavior between the two configurations.

4.1 Dynamic behavior for a finite periodic structure composed of rods with variation cross-section areas

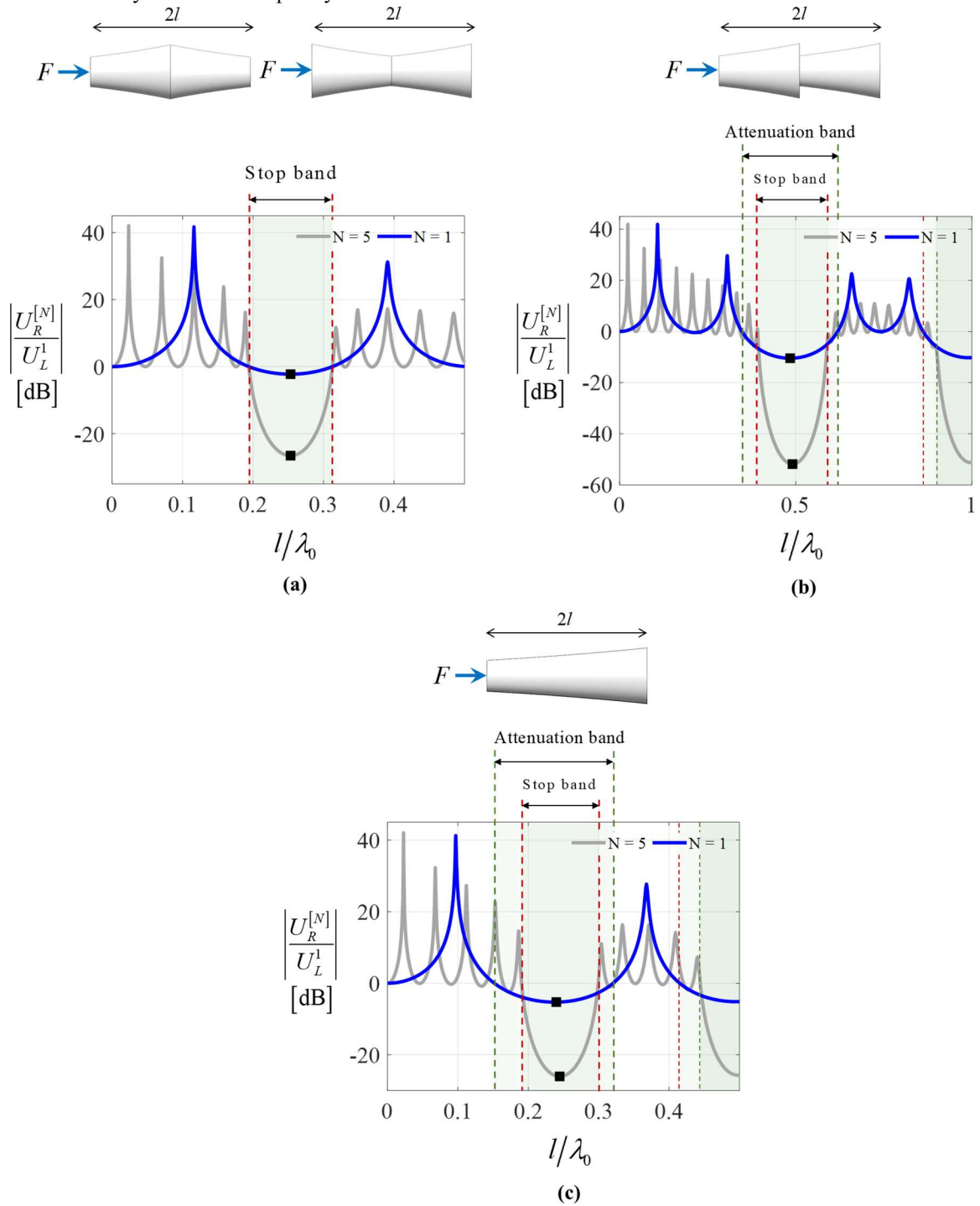
As seen previously in Figure 18, the catenoidal, the exponential and the conical-shaped rods present similar transmissibility responses. Because of this, the study in this chapter is concentrated only on the exponential shaped rods, as these cells present more simplified mathematical transmissibility equations, as seen in Table 4, when compared to catenoidal and conical rods.

Using equation (49) (for asymmetric cells) and equation (50) (for symmetric cells), the displacement transmissibility of a single cell and for a periodic structure with five cells are plotted in Figure 22. The parameters used in Figure 22 were selected as they represent the physical system discussed in Chapter 6, focusing on experimental validation. The parameters are $E = 3 \text{ GPa}$, $\rho = 1130 \text{ kg/m}^3$, $\eta = 0.01$, $r_L = 0.025 \text{ m}$. In Figures 22(a) and 17(b), the length of rod is $l = 0.15 \text{ m}$ and $\beta = 3.91 \text{ m}^{-1}$ ($\beta l = 0.5865$). Figure 22(c), the length of the rod is $2l = 0.3 \text{ m}$, and $\beta = 1.96 \text{ m}^{-1}$ ($\beta l = 0.294$).

In Figures 22(a) and 22(b), the dynamic behavior is observed in different cell configurations, formed by two rods of the same length l . It can be seen in Figure 22(a), the structure presents symmetric cells with repeated elements of length $2l$, while Figure 22(b), asymmetric cells have repeated elements of length l . This results in different frequency ranges where the stop/attenuation bands occur. For the attenuation band of the asymmetric

cell to occur in the same frequency range as the symmetric cell, it must have the same periodicity length $2l$, like the cell shown in Figure 22(c), which has the same periodicity length, in which the frequency range contrasts with that of symmetric cells.

Figure 22 - Displacement transmissibility of a periodic structure with: (a) symmetric cells; (b) asymmetric cells with periodicity l ; (c) asymmetric cells with periodicity $2l$. The black squares indicated the minimum transmissibility within the frequency band.

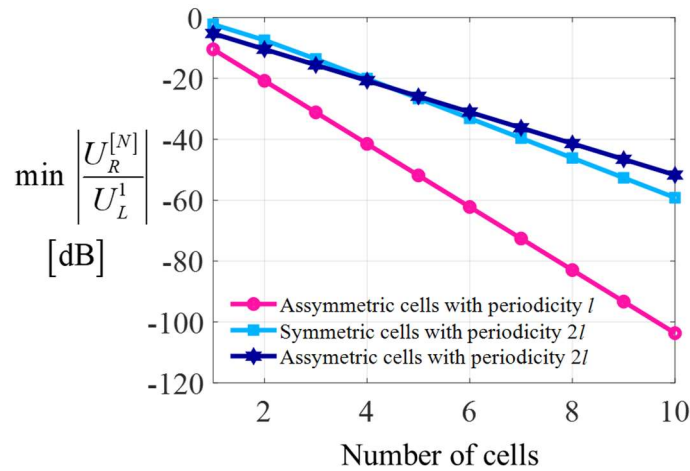


Source: Author own elaboration.

It is observed that in the symmetric cell structure, Figure 22(a), the transmissibility response is the same for the two symmetric cell configurations. It is also observed that with the increase in the number of cells, the stop-band, delimited by the red dashed lines in the figure, does not change the lower and upper limits of the stop-band. For this configuration, the limits found using a root finding algorithm (*fsolve* function in Matlab®) are $l/\lambda_{0-\text{lower}} = 0.1941$ and $l/\lambda_{0-\text{upper}} = 0.3125$. While in the asymmetric cell configuration, Figures 22(b) and 22(c) when the number of cells increases, the bandwidth of the attenuation band decreases until reaching the band limits of an infinite structure (Appendix C). Similarly, the stops band limits for five cells were found and are equal to $l/\lambda_{0-\text{lower}} = 0.3472$ and $l/\lambda_{0-\text{upper}} = 0.6189$ (Figure 22(b)), $l/\lambda_{0-\text{lower}} = 0.1904$ and $l/\lambda_{0-\text{upper}} = 0.3203$ (Figure 22(c)). It is observed that the limits of Figure 22(c) contrast with the limits found in the symmetric cell configuration, whereas the limits of Figure 22(b) are approximately twice as large.

The minimum transmissibility for one and five cells is highlighted in Figure 22 by the black squares. It is noted that as the number of cells increases, the minimum transmissibility within the frequency band increases. This behavior is shown in Figure 23, in which there is a linear increase in the minimum transmissibility in symmetric and asymmetric cells as the number of cells increases.

Figure 23 – Minimum of transmissibility within the lowest attenuation frequency band as a function of number of cells for symmetric and asymmetric cells.



Source: Author own elaboration.

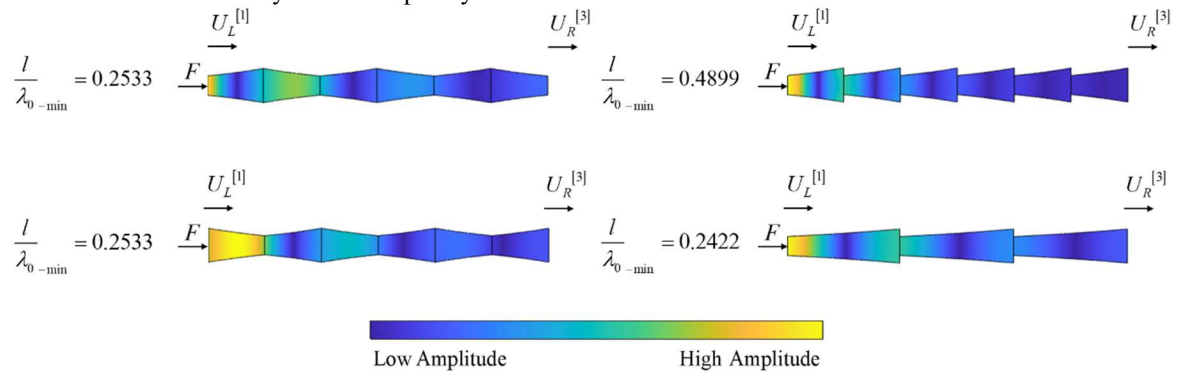
It is noted that when symmetric and asymmetric cells have the same periodicity, the minimum transmissibility responses present similar vibration attenuation. It is observed

that in a range of between one and five cells, the asymmetric cells present a greater vibration attenuation, however, this behavior is reversed when the structure starts to have more than five cells. In the structure with asymmetric cells with length periodicity l , the response presents a greater vibration attenuation when compared to that of symmetric cells, however this attenuation occurs at a higher frequency. This fact is due to these cells having twice as many repeated elements.

To visualize the vibration attenuation at minimum transmissibility, the deflections for the corresponding frequency of the minimum transmissibility of the structures discussed in Figure 24, consisting of 3 cells, are calculated. In the figure, it is possible to observe the vibrational attenuation through the gradual reduction of the displacement amplitude from left to right.

The calculation of the deflection shapes begins with the construction of the global dynamic stiffness matrix of the structure, which incorporates the properties of the rods with exponential geometry and the corresponding boundary conditions. Once the global nodal displacements are obtained, they are applied to the equation of the exponential-shaped rod (equation (27)), using the constants c_1 and c_2 from equation (27) to determine the displacements along each rod in the structure, including the displacements at the left and right ends. Finally, the calculated displacements are normalized with respect to the maximum displacement, enabling a relative analysis of the deflection shapes.

Figure 24 – Deflections shapes for symmetrical and asymmetrical structures excited at the frequency of the minimum transmissibility within frequency band.

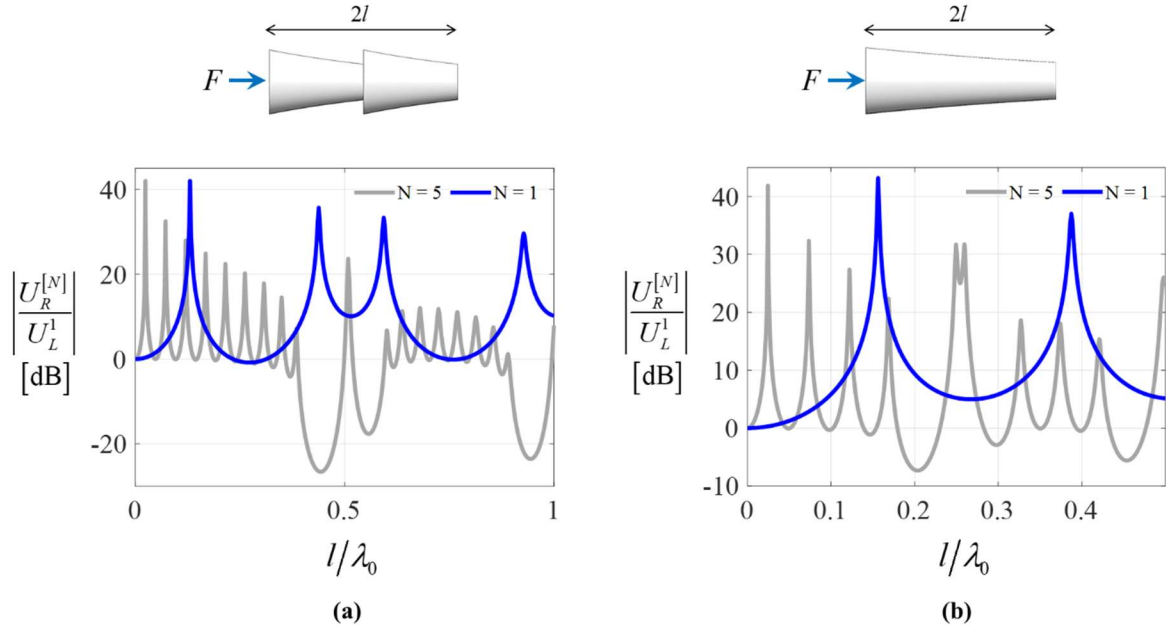


Source: Author own elaboration.

The second configuration of asymmetric cells is shown in Figure 25. It is noted that for a single cell, there is no presence of an attenuation band, but rather an amplification of the vibrational displacement. However, as the number of cells increases, there is the presence of attenuation bands, but with the presence of a peak in transmissibility. As the

excitation is carried out on the thicker side, the thinner side will present a large displacement, thus causing a transmissibility peak in the response. This highlights the importance of correctly orienting structures made up of asymmetric cells to achieve the desired effect of vibration attenuation.

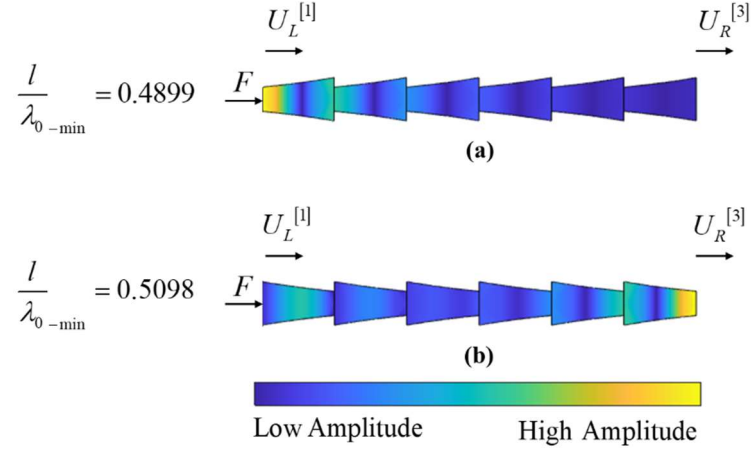
Figure 25 - Displacement transmissibility of periodic structures with asymmetric cells: (a) periodicity l ; (b) periodicity $2l$.



Source: Author own elaboration.

To better illustrate the importance of correctly orienting the asymmetric cells, Figure 26 presents the response of the deflections calculated for the frequency corresponding to the minimum transmissibility in structures with asymmetric cells with excitation on the thinner side (Figure 26(a)), and for frequency corresponding to the transmissibility peak for structures with asymmetric cells with excitation on the thicker side (Figure 26(b)). The calculation of the deflection shapes shown in this figure follows the same process mentioned previously.

Figure 26- Deflections shapes for an asymmetric structure with different orientations: (a) Excitation force at thinner left end; (b) Excitation force at the thicker left end.



Source: Author own elaboration.

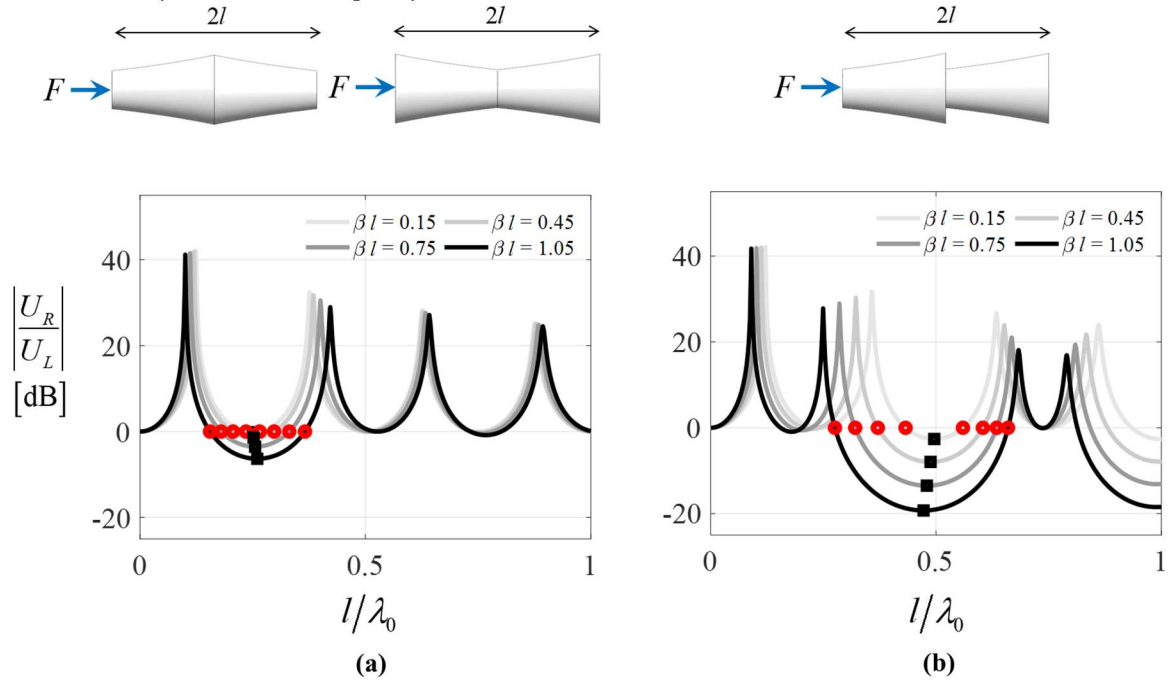
4.1.1 The effect of parameter βl on transmissibility

The effects of the parameter βl are investigated for structures with symmetric and asymmetric cells.

The parameter βl is defined as the product of the geometric parameter β and the characteristic length l of the rod. The parameter β characterizes the rate of variation of the cross-sectional area along the length of an exponential rod, being intrinsic to the geometry of the structure. Meanwhile, l represents the length of a single exponential rod, which, together with another rod, forms a complete cell. In this case, the length of the cell is given by $2l$, encompassing both rods.

The combined parameter βl is an essential parameter in analyzing the dynamic behavior of the structure, as it governs the interaction between the rod's variable geometric properties and the waves propagating along it. This parameter directly influences the formation of stop-bands, determining the frequencies at which vibration attenuation occurs. These analyses were investigated by Da Silva *et al.* (2025). Figure 27 shows the displacement transmissibility for these structures for a single cell for $\beta l = 0.15, 0.45, 0.75, 1.05$.

Figure 27 - Displacement transmissibility of a periodic cell with different values of βl : (a) symmetric cell; (b) asymmetric cell; the red circles give the attenuation band limits and the black square gives the minimum transmissibility within these frequency bands.



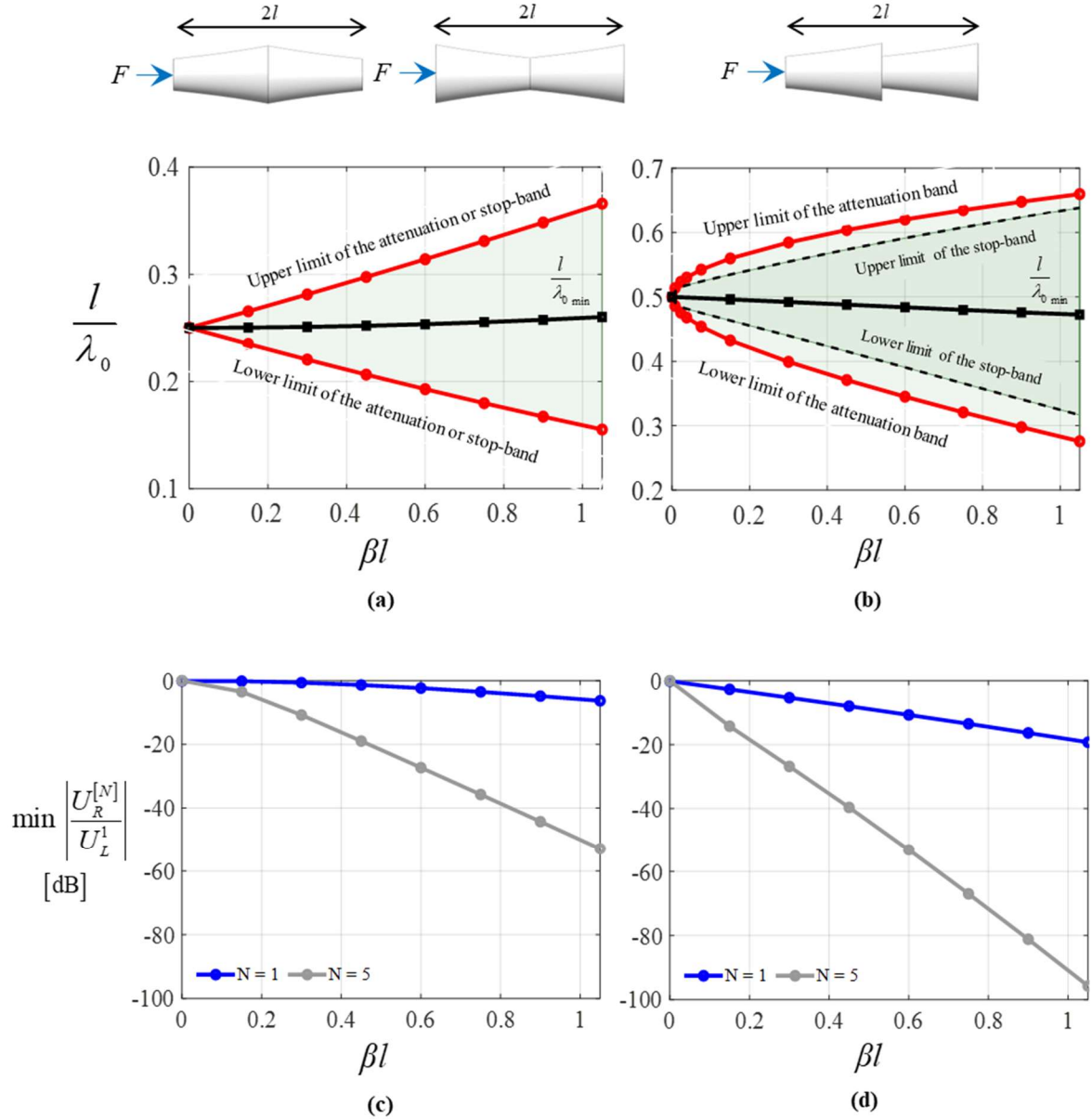
Source: Author own elaboration.

Figure 27 shows that the symmetric cell exhibits only one stop-band across the entire frequency range, while the asymmetric cells exhibit multiple attenuation bands. This phenomenon occurs because the asymmetric cells undergo an abrupt change in the cross-sectional area, which does not happen in symmetric cells (DA SILVA *et al.*, 2025).

Also, in Figure 27, the behavior of the stop-band is analyzed with respect to the parameter βl . It is observed that as the parameter βl increases, both the bandwidth and the minimum transmissibility increase. However, in structures with symmetric cells, the frequency corresponding to the minimum transmissibility increases slightly as the parameter increases, whereas for asymmetric cells, this frequency decreases slightly.

To better illustrate the behavior of the stop/attenuation band as a function of the parameter βl , the bandwidths are plotted in Figures 28(a) and 28(b), delimited by the lower and upper limits of the stop/attenuation bands. Additionally, Figures 28(c) and 28(d), present the minimum transmissibility as a function of the parameter βl .

Figure 28 - Stop/attenuation band as a function of the parameter βl : (a) and (b) Frequency limits of the attenuation/stop-band and the frequency defining the minimum attenuation for different βl values; (c) and (d) Minimum of transmissibility as a function of the βl .



Source: Author own elaboration.

In Figure 28(a) it is observed that the limits of the stop-band increase linearly, as do the frequencies corresponding to the minimum transmissibilities. For asymmetric cells, however, the stop-band limits increase quadratically, while the frequencies corresponding to the minimum transmissibilities decrease linearly. In Figures 28(c) and 28(d) it is noted that the attenuations increase linearly as the parameter βl increases. As mentioned earlier, asymmetric cells exhibit twice the attenuation due have twice as many repeated elements.

4.2 Comparison with Stepped Rods

The use of stepped rods for vibration control is well-discussed in the literature (CARNEIRO *et al.*, 2021; NIELSEN; SOROKIN, 2015; OH; YOON, 2021; SANTOS *et al.*, 2023). Stepped rod, due to geometric discontinuities, exhibit favorable results in vibration attenuation through the formation of frequency bands where vibrational waves are attenuated, attributed to Bragg scattering. However, it is expected that such discontinuities potentiate stress concentrations at these points due to the abrupt change in cross-sectional area. Exponential shaped rods potentially reduce stress concentrations at the discontinuities by featuring smooth changes in cross-sectional area, while still exhibiting Bragg band formation, as discussed in the previous section.

In this context, the section presents an analysis regarding the response of displacement transmissibility, regarding the formation of the stop/attenuation band and maximum vibration attenuation, in cells formed by stepped rods and exponential shaped rods. Figure 29 presents a comparison of the displacement transmissibility response and minimum transmissibility of symmetric cells formed by exponential shaped and stepped rods. For the simulation, the parameters used in the previous section were considered, and for the comparison, it was considered that the cells had the same cross-sectional areas at the ends and the same periodicity $2l$, so that the stopband/attenuation occurs in similar frequency ranges.

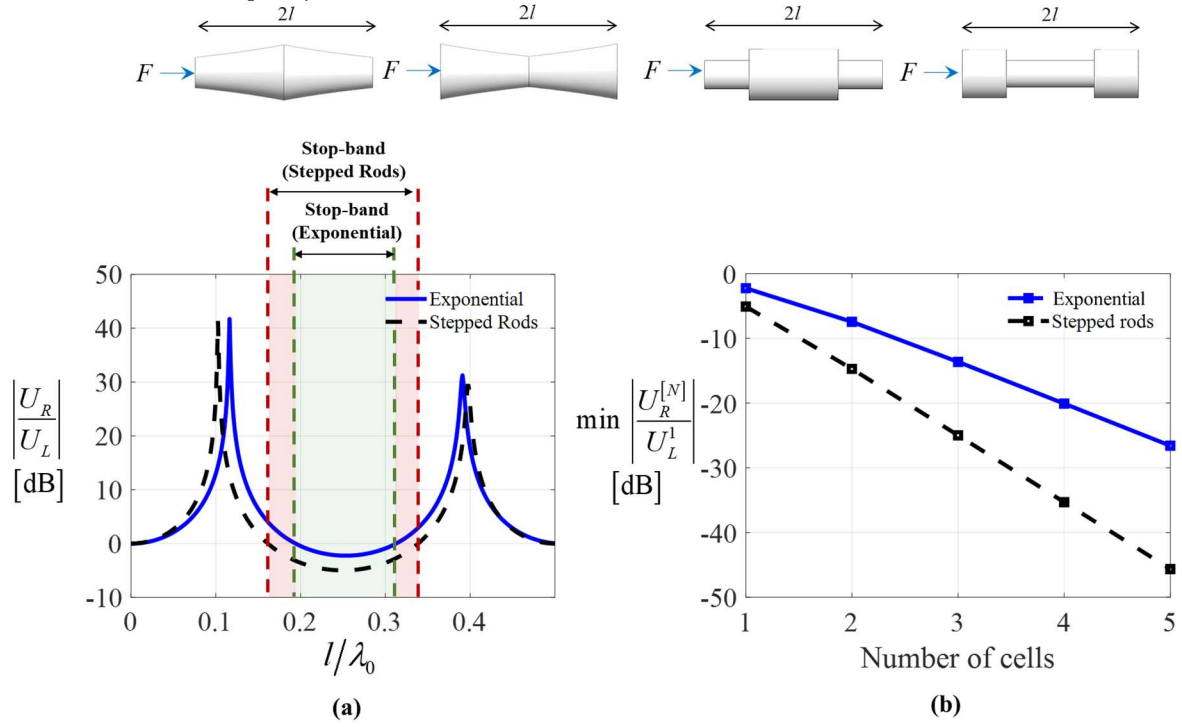
As expected, cells formed by stepped rods have a wider bandwidth, as seen in Figure 29(a). These values can also be seen in Table 6, which presents the bandwidth values using the *fsolve* function of the Matlab®. This effect is due to the difference in volume that the cells present. A similar effect was discussed in the previous section based on the variation of the parameter βl in cells formed by exponential shaped rods.

Still in Figure 29(a), it can be noted that the frequency corresponding to the minimum transmissibility of cells formed by stepped rods is slightly lower than cells formed by exponential shaped rods. A similar behavior was discussed in the previous section with variation of the parameter βl , which as the parameter increases, the frequency that governs the minimum transmissibility increases. So in comparison with the cells formed by stepped rods, it can be said that as the symmetric cell decreases in volume, the frequency corresponding to the minimum transmissibility decreases.

In Figure 29(b), it is noted that the cells formed by stepped rods present a slightly greater attenuation than the exponential shaped rods as the number of cells increases, and

the attenuation increases linearly, just like the exponential shaped rods, a fact evidenced in Table 6.

Figure 29 – Comparison of the exponential and stepped rod transmissibilities (a) Displacement transmissibility of a periodic structure with symmetric cells; (b) Minimum of transmissibility within the lowest attenuation frequency band as a function of number of cells.

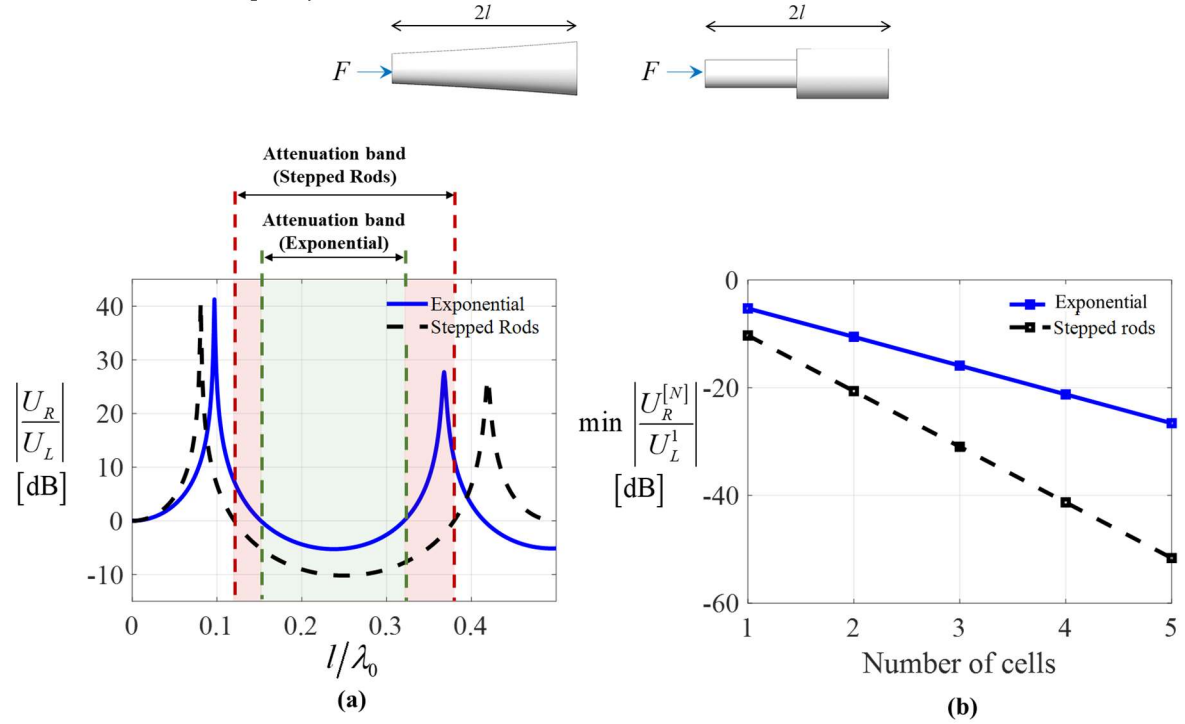


Source: Author own elaboration.

Figure 30 presents a comparison of the displacement transmissibility response and minimum transmissibility of asymmetric cells formed by exponential shaped and stepped rods with excitation from the thinner side of the cell.

The behavior seen in Figures 30(a) and 30(b) is similar to that discussed for symmetric cells. However, in asymmetric cells the frequency corresponding to minimum transmissibility increases when the cell volume decreases. Table 6 presents the corresponding values for bandwidth and minimum transmissibility, for a range of one to five cells. In the table it can be seen that the cells formed by the stepped rods have a greater bandwidth and greater attenuation as the number of cells increases.

Figure 30 - Comparison of the exponential and stepped rod transmissibilities (a) Displacement transmissibility of a periodic structure with asymmetric cells; (b) Minimum of transmissibility within the lowest attenuation frequency band as a function of number of cells.



Source: Author own elaboration.

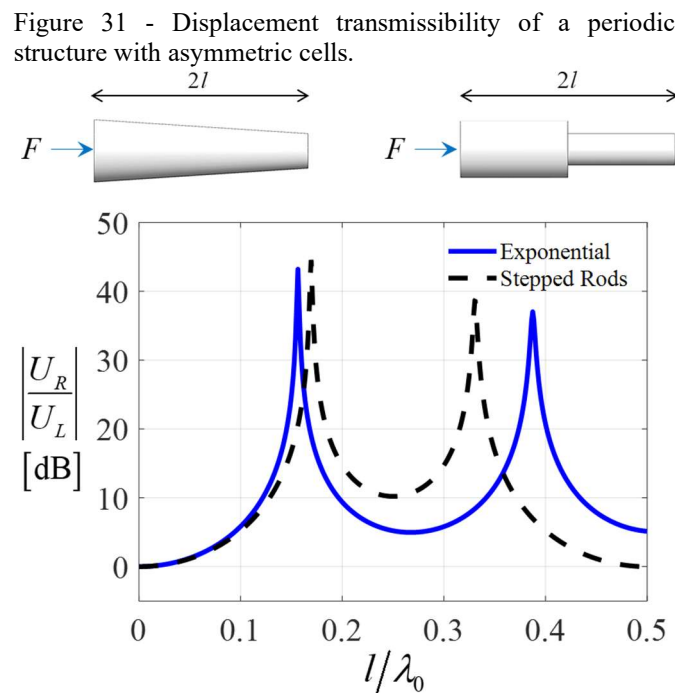
Table 6 - Values of the attenuation bandwidth for symmetric and asymmetric cells and the minimum transmissibility.

Exponential			Stepped Rods	
Bandwidth				
	Symmetric	Asymmetric	Symmetric	Asymmetric
Bandwidth $\Delta = \frac{l}{\lambda_{0-\text{upper}}} - \frac{l}{\lambda_{0-\text{lower}}}$		0.1695 (1 cell)	0.1767 independent of the number of cells	0.2615 (1 cell)
	0.1184	0.1356 (2 cells)		0.2194 (2 cells)
	independent of the number of cells	0.1212 (3 cells)		0.1957 (3 cells)
		0.1143 (4 cells)		0.1954 (4 cells)
		0.1087 (5 cells)		0.1911 (5 cells)
Maximum Attenuation within the Attenuation/Stop-band				
	Symmetric	Asymmetric	Symmetric	Asymmetric
Minimum Transmissibility $\min \left \frac{U_R^{[N]}}{U_L^1} \right \text{ [dB]}$	-2.3 (1 cell)	-5.27 (1 cell)	-5.08 (1 cell)	-10.29 (1 cell)
	-7.4 (2 cells)	-10.55 (2 cells)	-14.71 (2 cells)	-20.62 (2 cells)
	-13.6 (3 cells)	-15.89 (3 cells)	-24.98 (3 cells)	-30.95 (3 cells)
	-20.1 (4 cells)	-21.23 (4 cells)	-35.31 (4 cells)	-41.29 (4 cells)
	-26.6 (5 cells)	-26.58 (5 cells)	-45.64 (5 cells)	-51.62 (5 cells)

Source: Author own elaboration.

Although the values presented in Table 6, cells formed by stepped rods present greater efficiency in terms of greater bandwidth and greater attenuation, cells formed by exponential shaped rods also present satisfactory results and with the advantage of reducing the stress concentration at discontinuities. Therefore, both symmetric and asymmetric cells (excited by the thinner side) formed by stepped and exponential shaped rods are effective for attenuating the longitudinal vibrational wave.

Figure 31 presents the responses for asymmetric cells formed by exponential shaped and stepped rods with excitation from the thicker side of the cell.



Source: Author own elaboration.

It is noted that the cells formed by stepped rods also present a peak in the transmissibility response. The frequency range where the frequency peak occurs is smaller in cells formed by stepped rods, however the magnitude is greater.

4.3 Chapter Remarks

This chapter was dedicated to performing numerical simulations for the analysis of periodic structures composed of different configurations of cell arrangements formed by exponential-shaped rods. Exponential-shaped rods were chosen due to their greater mathematical simplification in the equations, as well as their displacement transmissibility

response being similar to that of catenoidal and conical rods. The cells analyzed varied between symmetric and asymmetric configurations.

For asymmetric cells, it was observed that when they are composed of rods with the same length as those in symmetric cells, the frequency bands are located at higher frequencies compared to symmetric cells. Additionally, asymmetric cells were found to exhibit attenuation bands distributed throughout the entire frequency range. They showed greater attenuation due to their periodicity being twice that of symmetric cells. When this periodicity is comparable to that of symmetric cells, the dynamic responses of the two configurations become similar.

For asymmetric cells to display attenuation bands effectively, it is essential that they are correctly oriented, with the thinner end of the rod facing the point of force application.

The influence of the parameter βl was also analyzed, showing that the greater the value of this parameter, the wider the bandwidth and the higher the maximum attenuation observed. Finally, a comparison was made with stepped rods. As expected, the stepped rods exhibited superior bandwidth and maximum attenuation. However, cells formed by exponential-shaped rods also showed good results in the formation of attenuation bands. Additionally, exponential rods had the advantage of reducing stress concentrations compared to stepped rods, which can be a significant benefit in practical applications.

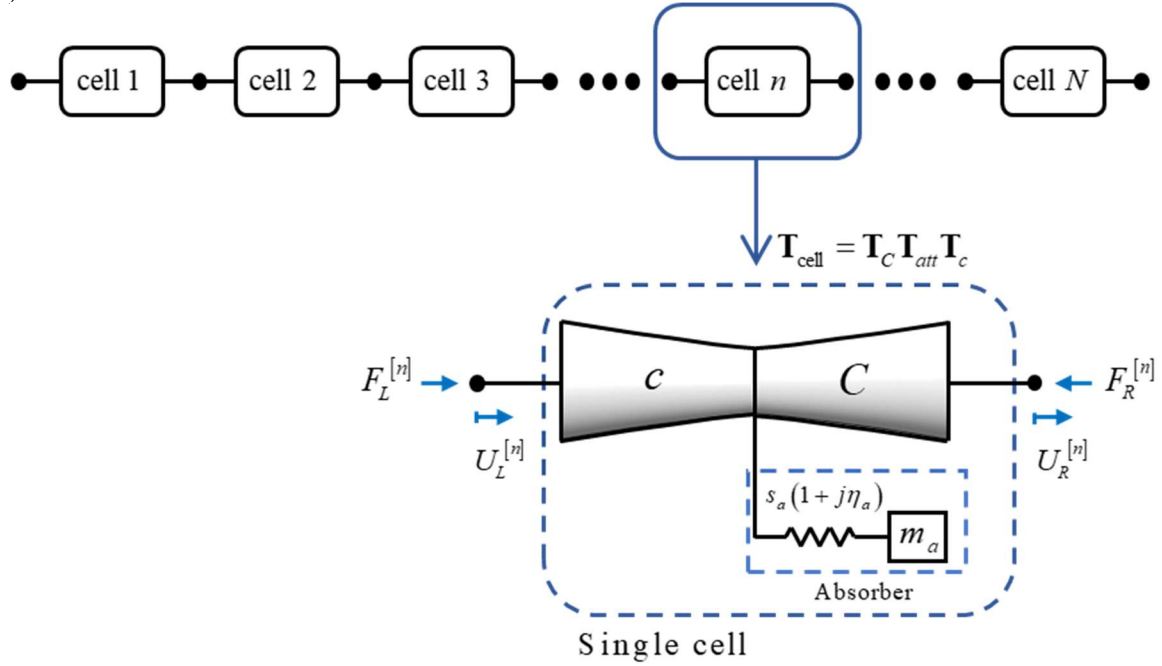
5 SIMULATION FOR SYMMETRIC CELLS WITH THE ADDITION OF THE DYNAMIC VIBRATION ABSORBER

Periodic structures composed of exponential-shaped rods exhibit stop-bands associated with the Bragg scattering effect. One of the challenges of these structures is achieving such attenuation bands at lower frequencies. To address this limitation, the addition of locally resonant absorbers presents an effective alternative to achieve attenuation bands at lower frequencies in these structures. In this context, this chapter investigates the conditions for creating a super-attenuation band and multi-frequency bands. To achieve this, the chapter describes the method used to obtain the displacement transmissibility of a finite mono-coupled periodic structure with the addition of a dynamic vibration absorber (DVA). The theory employed is based on recent works by Cleante *et al.* (2022) and Gonçalves, Brennan, Cleante (2021), which explore the effects of adding absorbers to a single cell of the periodic structure. As previously discussed in Chapter 3, the displacement transmissibility of a single cell can be used to estimate the dynamic response of a periodic structure composed of N cells.

5.1 Displacement transmissibility for cells with the addition of the dynamic vibration absorber

The finite periodic structure of interest is shown in Figure 32, with the n -th cell of the structure highlighted. The structure consists of N symmetrical and identical mono-coupled periodic cells. Each cell is composed of rods with exponential shapes and a dynamic vibration absorber coupled at the central point of the cell. The left and right ends of the cell are represented by the subscripts L and R, respectively, in the inputs and outputs of the n -th cell, which are denoted by the forces $F_{L,R}^{[n]}$ and displacements $U_{L,R}^{[n]}$.

Figure 32 – General mono-coupled structure comprising N periodic cells, highlighting a single cell n composed of two exponential-shaped rods with an additional dynamic vibration absorber attached at its center. The parameters of the absorber are illustrated: stiffness (s_a), mass (m_a) and loss factor (η_a). The displacement amplitudes within the single cell are represented by $U_L^{[n]}$ on the left-hand side and $U_R^{[n]}$ on the right-hand side. The excitation force amplitudes in the single cell are denoted as $F_L^{[n]}$ on the left-hand side and $F_R^{[n]}$ on the right-hand side. The matrix $\mathbf{T}_{cell}$ represents the transfer matrix of the cell, which is composed of the matrices for rods C (matrix $\mathbf{T}_C$) and c (matrix $\mathbf{T}_c$) as well as the absorber (matrix $\mathbf{T}_{att}$).



Source: Author own elaboration.

To calculate the displacement transmissibility of the periodic structure, the transfer matrix method is used, as previously mentioned in Chapter 3. For the cell configuration of Figure 32, the transfer matrix of the cell is given by $\mathbf{T}_{cell} = \mathbf{T}_C \mathbf{T}_{att} \mathbf{T}_c$, in which $\mathbf{T}_C$ and $\mathbf{T}_c$ are the transfer matrices for the rods that compose the cell, and are shown in Table 3. The matrix $\mathbf{T}_{att}$ is the transfer matrix of absorber attached in the central point of the cell, and can be obtained by

$$\mathbf{T}_{att} = \begin{bmatrix} 1 & D_{att} \\ 0 & 1 \end{bmatrix} \quad (51)$$

where D_{att} is the dynamic stiffness of the vibration absorber attached in the lumped structure, and is given by, according to Mead (1999):

$$D_{att} = \frac{m_a \omega^2 [s_a + j \omega m_a \omega_a \eta_a]}{s_a - m_a \omega^2 + j \omega m_a \omega_a \eta_a} \quad (52)$$

where the parameters of absorber are given by the stiffness s_a , the structural damping with a material loss factor η_a and the mass of dynamic vibration absorber m_a . The frequency of absorber can be calculated by $\omega_a = \sqrt{s_a/m_a}$. The equation (52) can be rewritten in terms of wavenumber

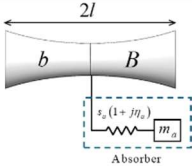
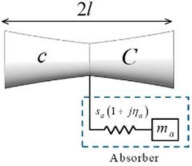
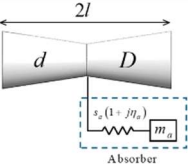
$$D_{att} = \frac{\nu EV}{l^2} (k_0 l)^2 \frac{\left[1 + j \eta_a \left(\frac{k_0 l}{k_a l} \right) \right]}{1 - \left[\frac{(k_0 l)^2}{(k_a l)^2} \right] + j \eta_a \left(\frac{k_0 l}{k_a l} \right)} \quad (53)$$

where $\nu = \frac{m_a}{m}$ is the mass ratio, where m is the total mass of the two rods that compose the cell and m_a is mass of dynamic vibration absorber. The Young's Modulus E , the length of the rod l , and the total volume of the rods V are the parameters of the host cell composed of two rods with variable cross-sectional areas. As previously defined, the parameter $k_0 = \omega \sqrt{\rho/E}$ is the wavenumber of a rod with a uniform cross-section and the ω is the circular frequency. The parameter k_a is a constant that corresponds to the longitudinal wavenumber of the rod at the undamped natural frequency of the vibration absorber, so that $k_a = \omega_a \sqrt{\rho/E}$.

Using the theory mentioned in Chapter 3, the displacement transmissibility equations for the single cells composed of catenoidal, exponential and conical-shaped rods with a dynamic vibration absorber (DVA) coupled at the central point of the cell, as depicted in Figure 32, are obtained and outlined in Table 7. For a periodic structure composed of N cells, the displacement transmissibility can be calculated using equation

(50) shown in Chapter 3, considering the transfer matrix of the cell as $\mathbf{T}_{cell} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}$.

Table 7 - Equation of transmissibility $T_r = U_R^{[2]} / U_L^{[1]}$, for a single cells with a dynamic vibration absorber (DVA) coupled at the central point of the cell. All cells have length $2l$.

Single cells	Displacement Transmissibility
 $\mathbf{T} = \mathbf{T}_B \mathbf{T}_{att} \mathbf{T}_b$	$T_r = -\frac{2A_L E k l \cosh^2(\alpha l)}{A_L E \left[\alpha l \sinh(2\alpha l) \sinh(2kl) + 2kl \cosh^2(\alpha l) (1 - 2 \cosh^2(kl)) \right] + D_{att} l \sinh(2kl)}$ The volume of the cell is given by $V = A_L l \left(1 + \frac{\sinh(2\alpha l)}{2\alpha l} \right)$.
 $\mathbf{T} = \mathbf{T}_C \mathbf{T}_{att} \mathbf{T}_c$	$T_r = \frac{2A_L E (\hat{k}l)^2}{2A_L E \left[(\hat{k}l)^2 \cos(2\hat{k}l) + (\beta l)^2 \cos(2\hat{k}l) - (\beta l)^2 \right] - D_{att} l \left[kl \sin(2\hat{k}l) + \beta l \cos(2\hat{k}l) - \beta l \right]}$ The volume of the cell is given by $V = A_L l \frac{(e^{2\beta l} - 1)}{\beta l}$.
 $\mathbf{T} = \mathbf{T}_D \mathbf{T}_{att} \mathbf{T}_d$	$T_r = \frac{2A_L E \hat{x}_L \hat{x}_R (k_0 l)^2}{A_L E \left[(\hat{x}_L \hat{x}_R (k_0 l)^2 + 1) 2 \cos(2k_0 l) + (\hat{x}_R - \hat{x}_L + 1) k_0 l \sin(2k_0 l) - 2 \right] + D_{att} \hat{x}_L l (1 - k_0 l \sin(2k_0 l) \hat{x}_R - \cos(2k_0 l))}$ The volume of the cell is given by $V = \frac{2}{3} A_L l \left(1 + \frac{\hat{x}_R^2}{\hat{x}_L^2} + \frac{\hat{x}_R}{\hat{x}_L} \right)$.

Source: Author own elaboration.

5.2 Formation of the super-attenuation band

The formation of the super-attenuation band is achieved by adding the dynamic vibration absorber into the symmetric cell. The purpose of the addition of the dynamic vibration absorber is to obtain the expansion of the stop-band in order to reach wider frequency ranges and consequently increase the attenuation of longitudinal vibration waves.

The formation of a super-attenuation band using a dynamic vibration absorber is obtained with the combination of the Bragg stop-band due to the periodicity of the structure, with the local resonance stop-band due to antiresonance formed by the dynamic vibration absorber. Cleante *et al.* (2022) investigated that the formation of the super attenuation band occurs in the symmetric periodic structure when the dynamic vibration absorber is tuned to the first natural frequency of the single cell.

Using the equation of the exponential-shaped rod from Table 7, the displacement transmissibilities are plotted in Figure 33. Point receptances and transfer receptances are

also plotted. All responses are presented as a function of a non-dimensional wavenumber l/λ_0 . In Figure 33, the parameters of the rods were considered as $E = 3$ GPa, $\rho = 1130$ kg/m³, $l = 0.150$ m, $r_L = 0.025$ m, $x_L = 0.1875$ m ($\hat{x}_L = 1.25$) while the parameters of the absorber are presented in Table 8.

Table 8 - Parameters of the dynamic vibration absorber.

Stiffness	Mass	Mass ratio	Frequency ratio
s_a [kN/m]	m_a [kg]	$\nu = \frac{m_a}{m}$	$\gamma = \frac{\omega_a}{\omega_n}$
66649	0.5936	0.4687	0.8
66649	0.3799	0.3000	1
66649	0.2638	0.2083	1.2

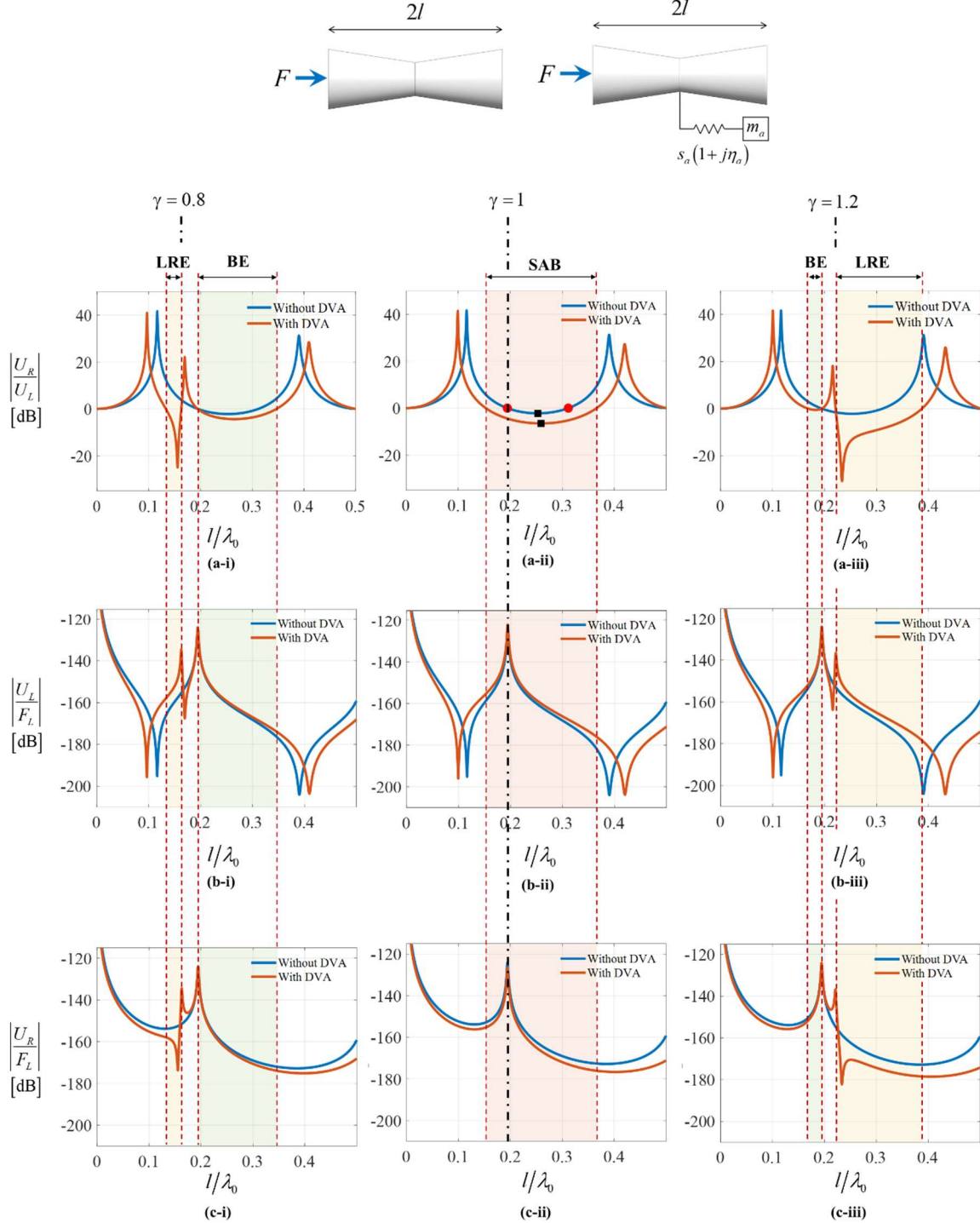
Source: Author own elaboration.

In Figure 33, the influence of the absorber on the displacement transmissibility response can be observed through the behavior of the local resonance and the stop band with changes in the absorber tuning frequency. When the tuning frequency ω_a is lower than the natural frequency of the structure ω_n , as in the case of $\gamma = 0.8$, the band arising from the local resonance of the absorber has a smaller bandwidth compared to the stop-band originating from the Bragg scattering effect, but it occurs in regions of lower frequencies.

In the case of $\gamma = 1$, where the absorber is tuned to the natural frequency, it has a considerable effect on the displacement transmissibility. Here, the super-attenuation band is caused by the combination of the local resonance of the absorber and the stop-band due to Bragg scattering.

At $\gamma = 1.2$, the local resonance bandwidth of the absorber is larger than the Bragg stop-band, since the absorber frequency is at a higher value relative to the natural frequency of the structure, in which the natural frequency influences the Bragg stop-band limit. For $\gamma = 0.8$, the lower limit of the Bragg stop-band $\frac{l}{\lambda_{0-\text{lower}}}$ corresponds to the natural frequency of the structure, while for $\gamma = 1.2$ the natural frequency corresponds to the upper limit of the Bragg band $\frac{l}{\lambda_{0-\text{upper}}}$.

Figure 33 – Numerical simulation responses with the addition of the dynamic vibration absorber to the symmetric cell: (a) Displacement transmissibilities; (b) Point receptances; (c) Transfer receptances; the orange area provides the range of super attenuation band (SAB), the green area provides the stop band due to the Bragg scattering effect (BE) and the yellow area provides the range of local resonance effect (LRE); the red circles provide the stop-band limits of the structure without dynamic vibration absorber; black squares give the minimum transmissibilities within the frequency bands in figure (a-ii).

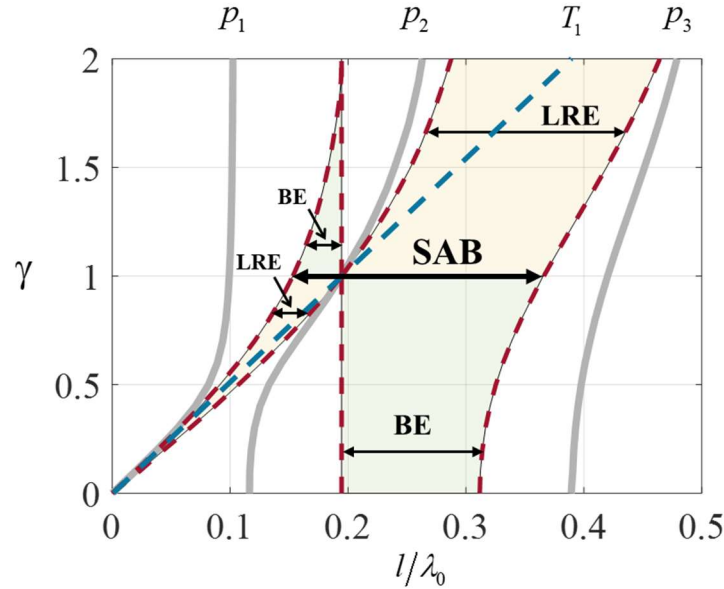


Source: Author own elaboration.

The effect of the absorber's tuning frequency on the formation of the frequency bands is also illustrated in Figure 34.

Figure 34 illustrates the dynamic evolution of the frequency bands formed by the Bragg scattering effect (denoted as BE in the figure) and the local resonance of the absorber (denoted as LRE in the figure) as the frequency ratio γ changes. It is observed that a super-attenuation band forms when the frequency ratio $\gamma = 1$ satisfies a specific condition. At this frequency ratio, the upper boundary frequency of the local resonance band coincides with the lower boundary frequency of the Bragg band, aligning with the trough of the transmissibility curve T_1 . Consequently, the second transmissibility peak p_2 does not appear, leading to the formation of the super-attenuation band, where the local resonance band merges seamlessly with the Bragg band. However, if there is a misalignment in the absorber's tuning, the second transmissibility peak will emerge, emphasizing the critical importance of precise absorber tuning to achieve the super-attenuation band.

Figure 34 - Frequency bands formed by the local resonance effect (LRE) and the Bragg effect (BE) are shown as γ is varied. The super-attenuation band is highlighted in $\gamma = 1$. The shaded areas represent the BE and LRE regions, while the red dashed lines indicate the boundaries of the frequency bands. p_1 , p_2 and p_3 correspond to the peaks in the transmissibility response, while T_1 refers to the trough of the transmissibility response, which occurs at the natural frequency of the absorber.

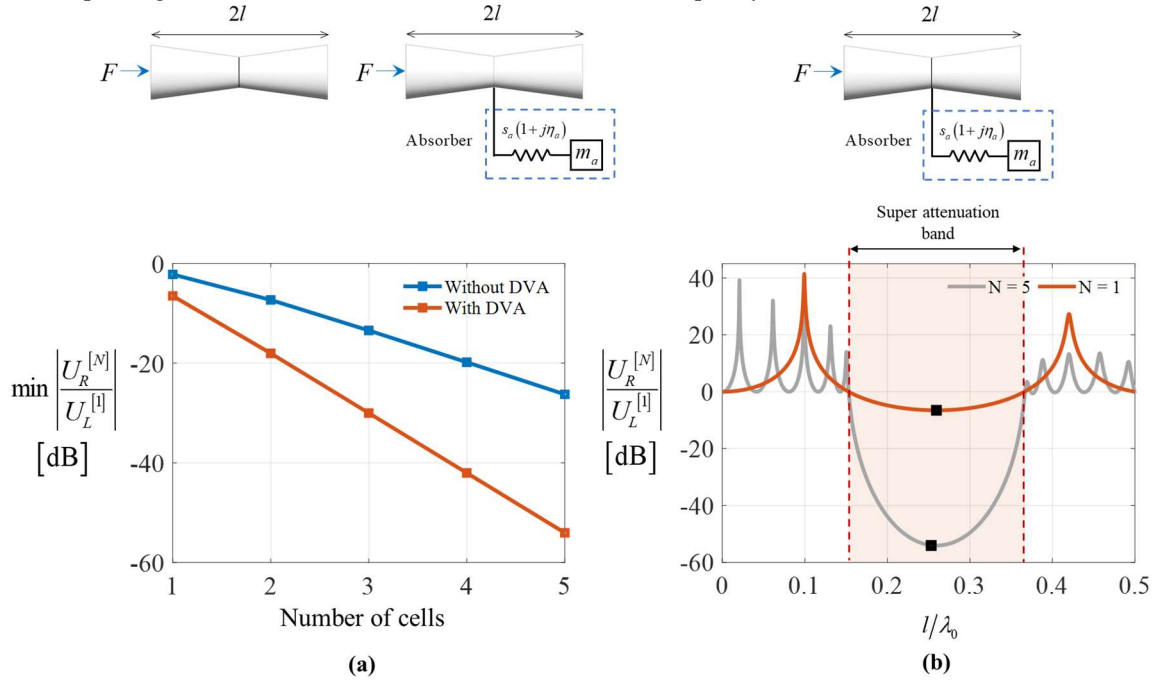


Source: Author own elaboration.

In response to the transmissibility of the super-attenuation band, Figure 33(a-ii) shows an increase in maximum attenuation, denoted by the black squares. As discussed in Chapter 4, for a mono-coupled finite periodic structure composed of symmetric cells, the

maximum attenuation increases with the number of cells, while the stop-band limits remain unchanged, resulting in the same bandwidth regardless of the value of N . For a structure with the addition of a DVA, this behavior is illustrated in Figure 35.

Figure 35 - Numerical simulation responses with the addition of the dynamic vibration absorber to the periodic structure: (a) Comparison of maximum attenuation within frequency bands about the number of cells; blue line without dynamic vibration absorber and orange line with dynamic vibration absorber; (b) Displacement transmissibility of a periodic structure of symmetric cells with dynamic vibration absorber; orange line $N = 1$ cell; gray line $N = 5$ cells; the red dashed lines provide the super attenuation band limits; black squares give the minimum transmissibilities within the frequency band.



Source: Author own elaboration.

In Figure 35(a), the maximum attenuation increases linearly with the number of cells, and the structure with an absorber exhibits greater attenuation as the number of cells increases. As expected, Figure 35(b) shows that the lower and upper limits of the super-attenuation band remain unchanged as the number of cells increases. The only observed effect is the increase in maximum attenuation, as illustrated in Figure 35(a).

It is worth highlighting that the addition of an absorber, tuned to a specific frequency ratio $\gamma = 1$, not only results in a wider attenuation frequency range but also achieves greater maximum attenuation within this range compared to a structure without the absorber. This observation is supported by the analysis presented in Table 9.

Table 9 - Values of the attenuation bandwidth and the minimum transmissibility for symmetric cells without DVA and with DVA.

Bandwidth		
	Symmetric cell without DVA	Symmetric cell with DVA
		$\gamma = 1$
Bandwidth	0.1184	0.2125
$\delta = \frac{l}{\lambda_{0\text{-upper}}} - \frac{l}{\lambda_{0\text{-lower}}}$	independent of the number of cells	independent of the number of cells
Maximum Attenuation within the Attenuation/Stop-band		
	-2.3 (1 cell)	-6.52 (1 cell)
Minimum Transmissibility	-7.4 (2 cells)	-18.04 (2 cells)
$\min \left \frac{U_R^{[N]}}{U_L^1} \right [\text{dB}]$	-13.6 (3 cells)	-30.01 (3 cells)
	-20.1 (4 cells)	-42.02 (4 cells)
	-26.6 (5 cells)	-54.04 (5 cells)

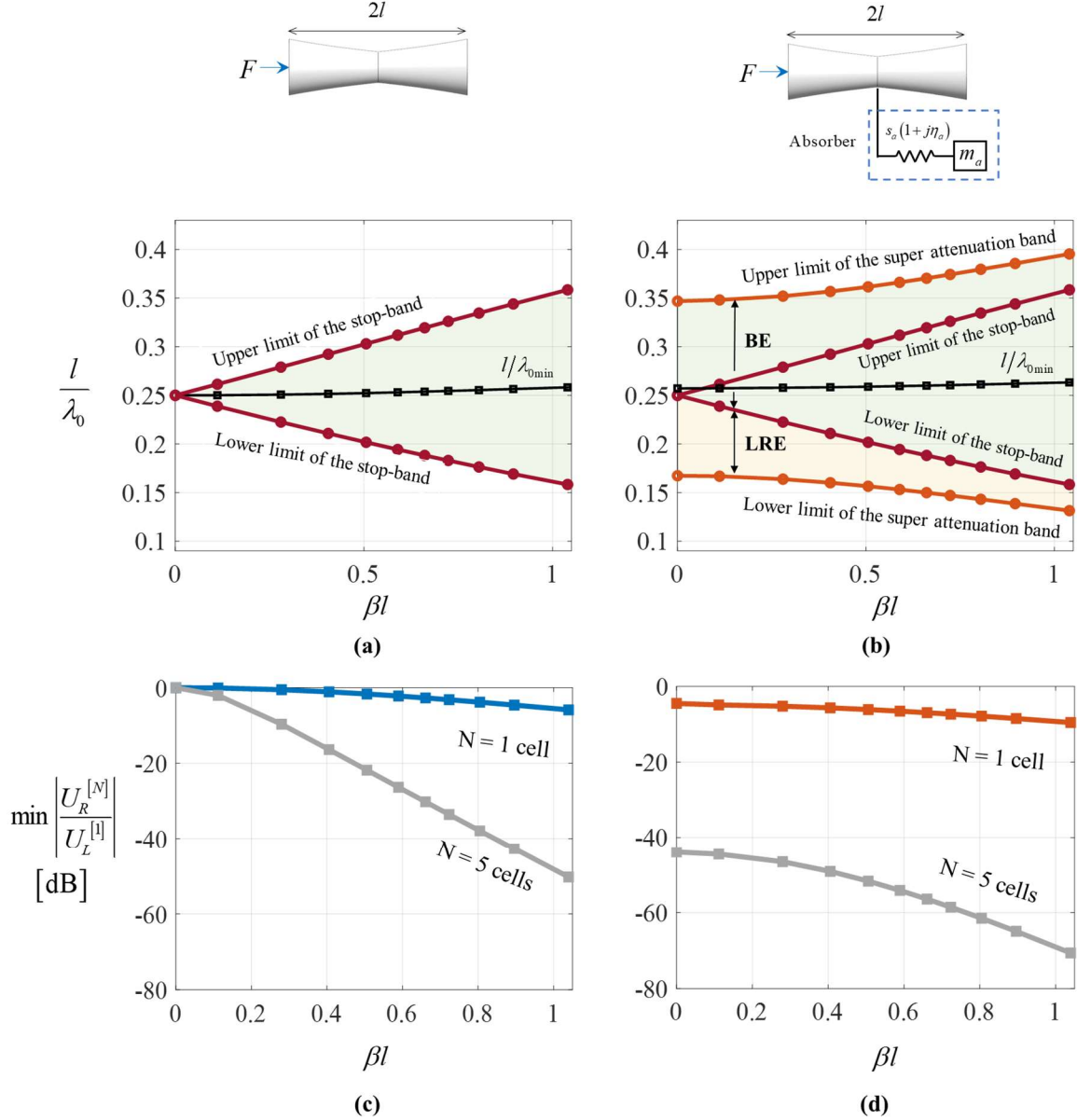
Source: Author own elaboration.

The bandwidth is analyzed in relation to the geometry of the exponential-shaped rod. In Figure 36, this analysis is performed by varying the parameter βl within the range of 0 to 1.25.

It can be observed in Figures 36(a) and 36(b) that the addition of the absorber effectively broadens the frequency band across the range of the parameter βl . For structures with smaller values of βl , the DVA significantly widens the local resonance band, as the first natural frequency of the structure corresponds to the lower limit of the stop-band, occurring at higher frequencies. Consequently, the larger local resonance bandwidth when the system is tuned to higher frequencies can be attributed to the more pronounced wave dispersion and increased energy dissipation characteristic of these frequencies.

Furthermore, the presence of the absorber introduces discontinuities of stiffness and mass at the central point of the cell. Consequently, adding the absorber also contributes to an increase in the width of the band resulting from Bragg scattering. Figures 36(c) and 36(d) illustrate that the addition of the absorber enhances the maximum attenuation within the frequency band. This effect becomes more pronounced with an increase in the number of cells, as evidenced by the comparison with the gray lines in the figures.

Figure 36 - Frequency bands as a function of the parameter βl for symmetric cell (a) without the DVA; (b) with the DVA ($\nu = 0.3$). Minimum transmissibility as a function of the parameter βl for symmetric cell (c) without the DVA; (d) with the DVA.



Source: Author own elaboration.

5.3 Formation of multi-frequency bands

This section explores the formation of multi-frequency bands through the addition of dynamic vibration absorbers in a symmetrical cell.

The resonant frequency band, achieved by tuning the vibration absorbers, can be formed at frequencies significantly lower than the Bragg band. According to Hao *et al.* 2019, the location of the local resonant frequency band is determined by the natural frequency of the oscillator. However, a limitation of using these absorbers is that the local

resonance band is generally much narrower than the Bragg band. To address this limitation and expand the frequency range of the resonant band, one potential approach is to use an ordered set of absorbers. Meng *et al.* (2020) demonstrated that metamaterials with resonators designed in a "rainbow-shaped" configuration could broaden the frequency band resulting from local resonance.

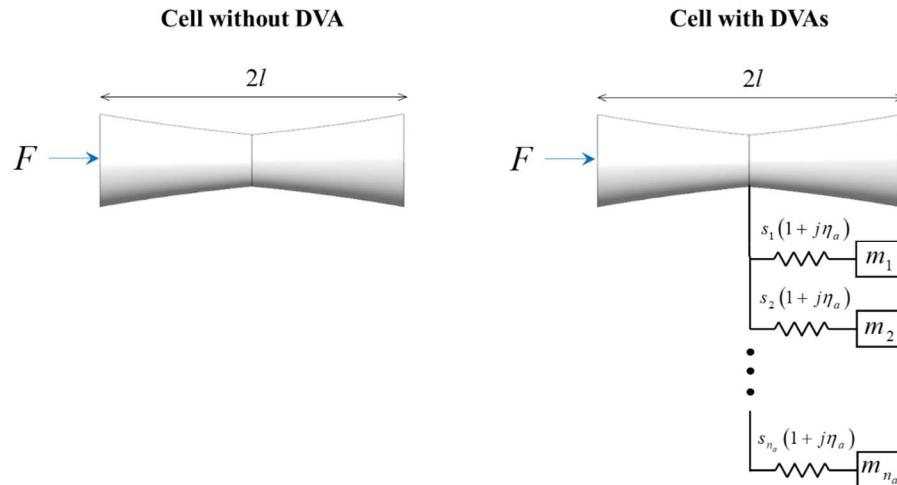
To enhance the local resonance band of symmetrical cells formed by exponential-shaped rods, rainbow-shaped dynamic absorbers were designed. Initially, eight absorbers were developed with a constant progressive mass variation, ensuring that the total mass of the absorbers amounted to 30% of the cell's mass. The total mass is expressed by the equation for the sum of an arithmetic progression:

$$m_a = \frac{n_a [2m_1 + (n_a - 1)\Delta m_1]}{2} \quad (54)$$

where n_a represents the number of absorbers, m_1 corresponds to the mass of the primary absorber, which defines the main frequency of the absorber set. The variable Δ indicates the constant mass variation of subsequent absorbers. Therefore, the mass of the next absorber is given by $m_{n_a} = m_1 [1 + (n_a - 1)\Delta]$ and so on, in a progressive manner.

Figure 37 presents the symmetrical unit cells formed by exponential-shaped rods, without absorbers and with rainbow-shaped absorbers composed n_a absorbers.

Figure 37 - Single symmetric cells composed by exponential-shaped rods (a) without DVA (b) with n_a rainbow-shaped dynamic vibration absorbers.



Source: Author own elaboration.

The displacement transmissibilities for both cases are plotted in Figure 38. The displacement transmissibility of the structure with rainbow-shaped dynamic vibration absorbers is obtained by summing the contributions of each absorber $D_{att} = D_{att[1]} + D_{att[2]} + D_{att[3]} + D_{att[4]} + D_{att[5]} + D_{att[6]} + D_{att[7]} + D_{att[8]}$. Then, the transmissibility equation from Table 7 is applied.

The displacement transmissibility of the structure with rainbow-shaped dynamic vibration absorbers is obtained by summing the contributions of each absorber. Then, the transmissibility equation from Table 7 is applied.

The parameters of the rod are identical to those used in the previous section, while the parameters of the rainbow-shaped dynamic vibration absorbers are listed in Table 10. Variations in mass were considered as follows: a 3% decrease ($\Delta = -0.03$), no variation in mass ($\Delta = 0$) and a 3% increase ($\Delta = 0.03$).

Table 10 - Parameters of the rainbow-shaped absorber.

Constant mass variation					
$\Delta = -0.03$		$\Delta = 0$		$\Delta = 0.03$	
m_1 [kg]	0.0531	all DVA have the same m_{n_a} [kg] 0.0475		m_1 [kg]	0.0430
m_2 [kg]	0.0515			m_2 [kg]	0.0443
m_3 [kg]	0.0499			m_3 [kg]	0.0456
m_4 [kg]	0.0483			m_4 [kg]	0.0468
m_5 [kg]	0.0467			m_5 [kg]	0.0481
m_6 [kg]	0.0451			m_6 [kg]	0.0494
m_7 [kg]	0.0435			m_7 [kg]	0.0507
m_8 [kg]	0.0419			m_8 [kg]	0.0520
all DVA have the same s_{n_a} [kN/m]	3325.9	all DVA have the same s_{n_a} [kN/m]	2976.7	all DVA have the same s_{n_a} [kN/m]	2693.9

Source: Author own elaboration.

The main frequency of the absorbers ($\omega_1 = \sqrt{\frac{s_1}{m_1}}$) presented in Figure 37 is tuned to the peak transmissibility of the response of a cell without a DVA. As discussed earlier, when the absorbers are tuned to frequencies lower than the lower limit of the Bragg band,

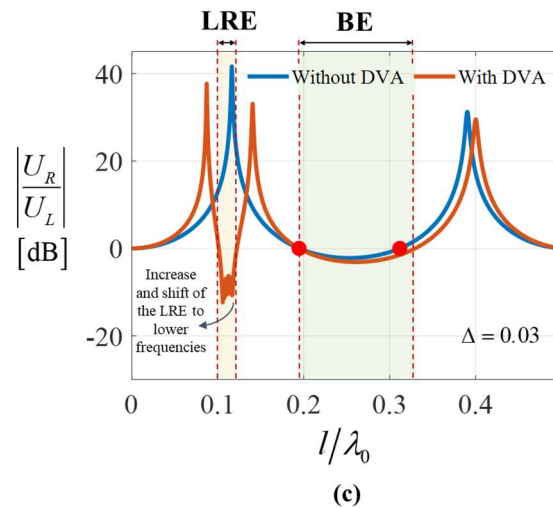
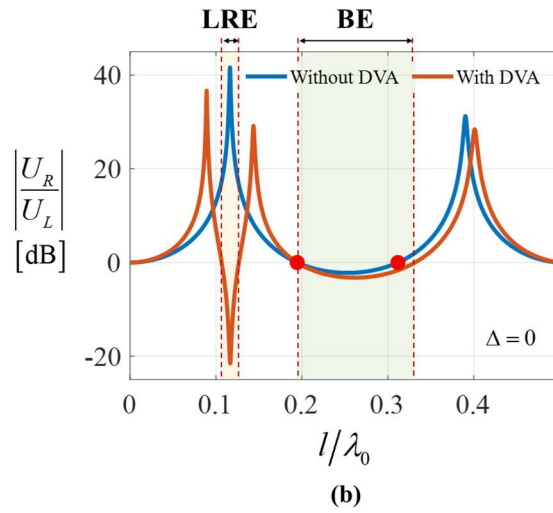
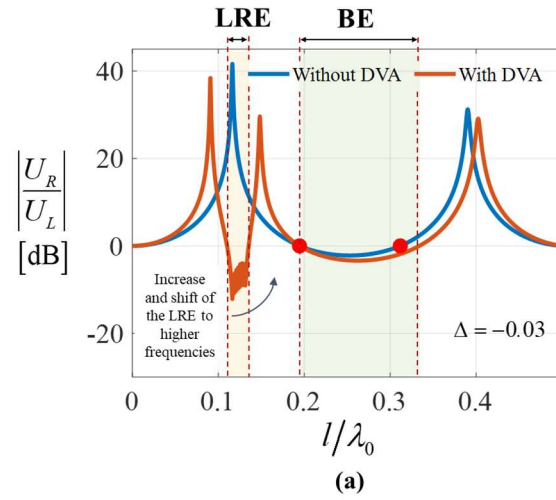
an attenuation band emerges due to the anti-resonance of the absorber, as illustrated in Figure 33(a-i).

Regarding the absorbers designed in a rainbow-shaped configuration, as described in Table 10, when the absorbers exhibit a variation in mass characterized by a decreasing trend ($\Delta = -0.03$), the subsequent absorbers exhibit higher natural frequencies relative to the primary frequency ω_1 . The small difference between the tuning frequencies prevents the emergence of transmissibility peaks within the tuning range, thereby increasing the bandwidth for higher frequencies. This behavior is evident in Figure 38(a), where the lower and upper limits of the attenuation band are $\frac{l}{\lambda_{0-\text{lower}}} = 0.1099$ and $\frac{l}{\lambda_{0-\text{upper}}} = 0.1351$, respectively. The maximum attenuation within the band is approximately -12.13 dB.

When the rainbow-shaped absorbers are configured with an increase in mass between the absorbers relative to the main absorber, the subsequent absorbers exhibit lower natural frequencies compared to the primary frequency ω_1 . In Figure 38(c), it can be observed that the lower and upper limits of the attenuation band are $\frac{l}{\lambda_{0-\text{lower}}} = 0.1011$ and $\frac{l}{\lambda_{0-\text{upper}}} = 0.1222$, respectively. This configuration results in an expanded band and a shift towards lower frequencies due to the reduced natural frequencies of the absorbers, as compared to the response shown in Figure 38(b), where the lower and upper limits are $\frac{l}{\lambda_{0-\text{lower}}} = 0.1067$ and $\frac{l}{\lambda_{0-\text{upper}}} = 0.1266$. The maximum attenuation of the band is approximately -11.42 dB, which is attributed to the increased mass of the rainbow-shaped absorber; the larger the mass, the greater the attenuation within the band.

When the absorbers are tuned to the same frequency, $\Delta = 0$, there is no variation in mass between the absorbers, as shown in Figure 38(b). This configuration results in a concentration of mass and stiffness at the main frequency, leading to an increase in maximum attenuation, which is approximately -25.48 dB.

Figure 38 - Displacement transmissibility of the single symmetric cells without DVA (blue line) and with rainbow-shaped absorbers (orange line): (a) with a decrease in mass $\Delta = -0.03$ (b) with no variation in mass $\Delta = 0$ (c) with an increase in mass $\Delta = 0.03$. The red dashed lines provide the frequency band limits.



Source: Author own elaboration.

5.4 Chapter Remarks

In this chapter, the analytical equations for displacement transmissibility were presented for the cells (catenoidal, exponential, and conical) with the absorber coupled at the center of the rods.

Given the limitation of Bragg stop-bands, which are located at higher frequencies, the formation of the super-attenuation band and multi-frequency bands was explored. It was observed that, to achieve the super-attenuation band, the DVA must be precisely tuned to the natural frequency of the structure. This condition promotes the merging of the local resonance band with the Bragg stop-band, resulting in an ultra-wide attenuation band.

Regarding the multi-frequency bands, it was found that the width of the local resonance band is significantly narrower than the width of the Bragg stop-bands. To expand this range, rainbow-shaped absorbers were designed. These absorbers proved effective in expanding the width of the local resonance band, albeit with a reduction in the maximum attenuation within the band.

Finally, the numerical simulations allowed a detailed understanding of the behavior of the local resonance bands and the Bragg stop-band concerning the tuning frequency.

6 EXPERIMENTAL VALIDATIONS

This chapter presents the laboratory experiments to validate the analyses described in Chapter 4 and 5. For this, rods with conically varying cross-section areas were used, because of ease of manufacture. The material chosen for the rod was Nylon 6.0, which had a Young's modulus of $E = 2.9 \text{ GPa}$ and a density of $\rho = 1140 \text{ kg/m}^3$. This ensured that the attenuation bands would be at frequencies of a few kHz rather than more than 10 kHz, for a rod of length $l = 0.15 \text{ m}$, $r_L = 0.025 \text{ m}$ and $r_R = 0.045 \text{ m}$ ($\hat{x}_L = 1.25$). The equivalent rod with exponential varying cross section would have a value of $\beta l = 0.5865$.

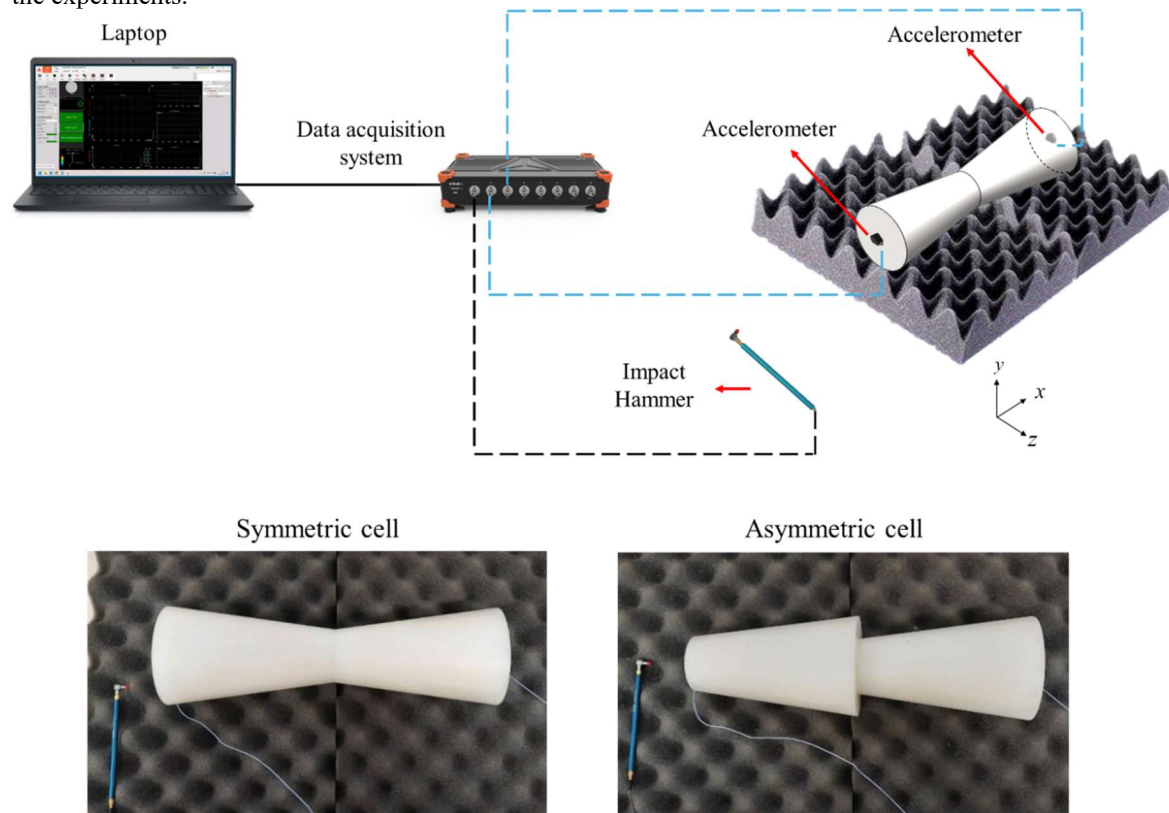
6.1 Symmetric and asymmetric cells

To validate the analysis of the symmetric and asymmetric cells described in Section 4, experimental tests were conducted using the cells shown in Figure 39. The conical-shaped rods were assembled using a small screw, and cyanoacrylate glue was applied to ensure a secure connection between the rods. To maintain free-free boundary conditions at each end of the cell, the structures were placed on a foam pad.

The experimental procedure used two accelerometers (model PCB 352A24) with a nominal sensitivity of 100 mV/g , positioned at the center of the left and right ends of each structure, as illustrated in Figure 34. An impact hammer (PCB 086E80) with a nominal sensitivity of 22.5 mV/N was used to excite the central point at the end of the structures. The signals were acquired with a sampling rate of 20 kHz and processed using a Dewesoft-Sirius data acquisition system connected to a laptop.

Figure 40 shows the results of experimental procedure, using 10 averages for each frequency response. The theoretical predictions, described in Section 3, are plotted together for comparison. The green shaded area indicates the stop/attenuation band of the theoretical prediction, where $|U_R / U_L| < 1$. In order for the theoretical response to correspond with the peaks of the experimental responses, a loss factor $\eta = 0.05$ in the Young's modulus was considered.

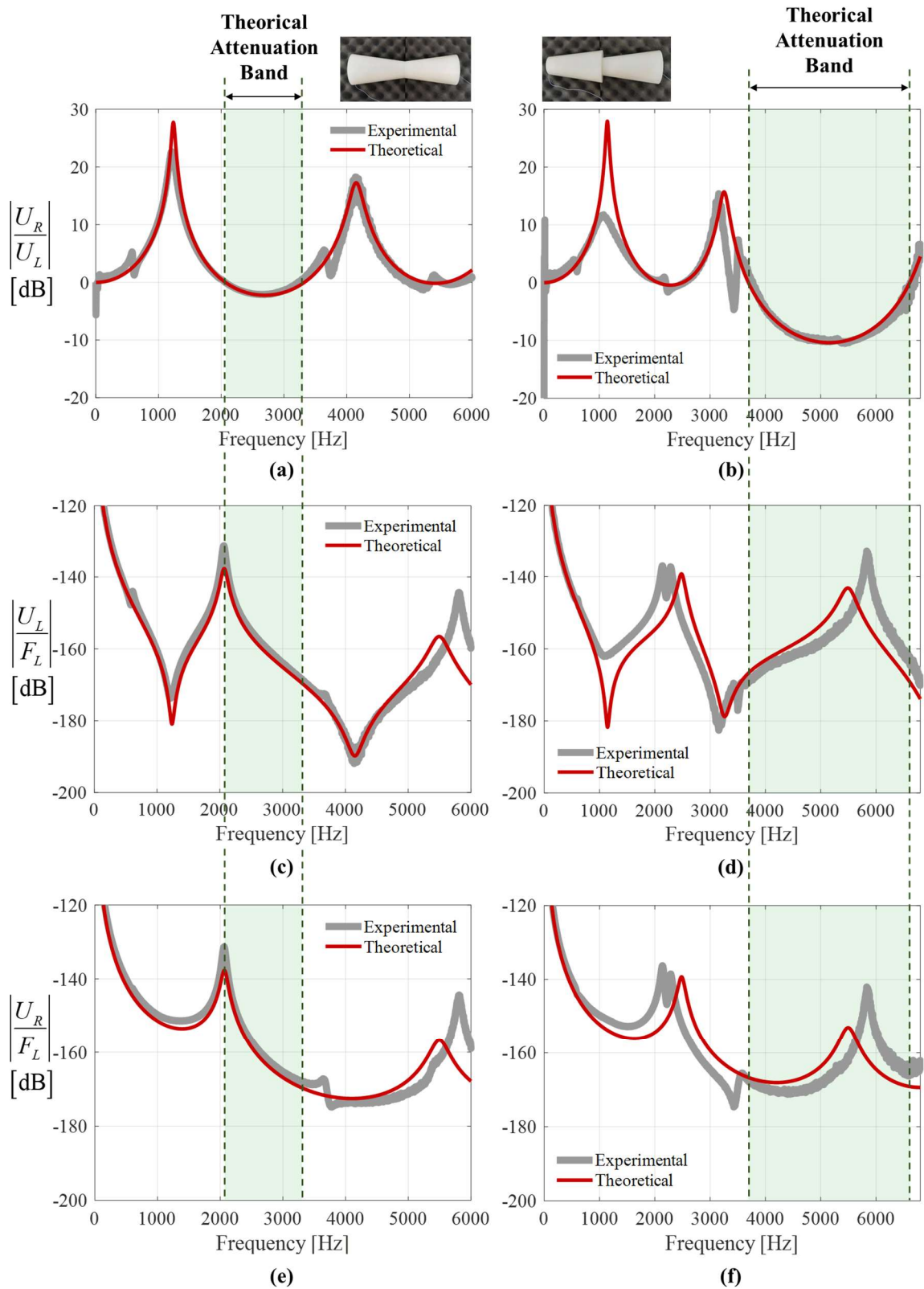
Figure 39 - Schematic representation of the experimental setup, highlighting the data acquisition system, the placement of accelerometers at the ends of a cell, and the laptop used for recording and processing the experimental data. Additionally, two photographs showcasing the symmetric and asymmetric cells utilized in the experiments.



Source: Author own elaboration.

The figures show that in both experimental responses, the frequency range where the stop/attenuation band occurs aligns with the theoretical predictions. The symmetric cell exhibits a minimum transmissibility of approximately -2.3 dB, while the asymmetric cell achieves approximately -10.4 dB, albeit at a higher frequency. Additionally, in Figures 40(a) and 40(b), some unexpected resonance peaks are evident in the experimental responses, which are also observable in Figures 40(c–f). These peaks are attributed to other vibration modes not accounted for in the analytical model, such as bending and longitudinal shear, as discussed in Appendix D.

Figure 40 - Comparison of experimental results with theoretical predictions using the model described in Section 4. (a) Transmissibility of a single symmetric cell, (b) Transmissibility of a single asymmetric cell (periodicity l). (c) Point receptance of a single symmetric cell, (d) Point receptance of a single asymmetric cell, (e) Transfer receptance of a single symmetric cell, (f) Transfer receptance of a single asymmetric cell.



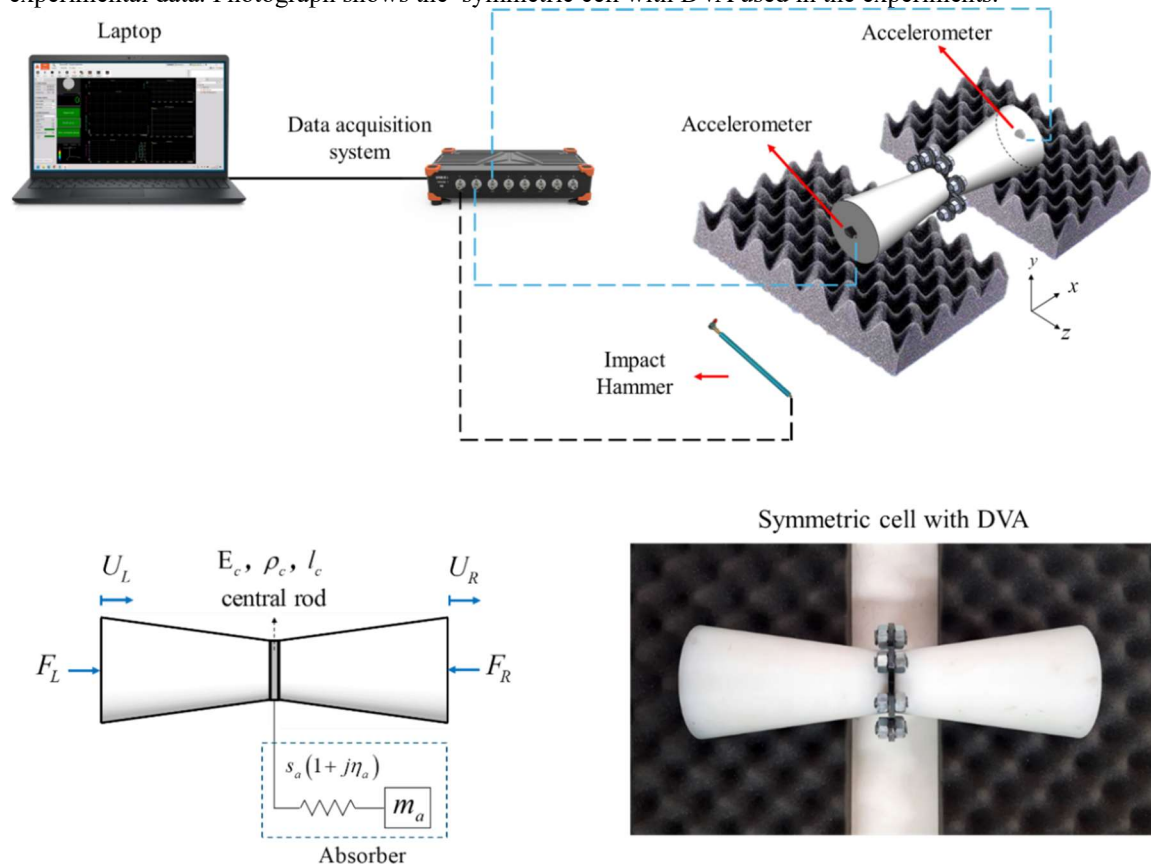
Source: Author own elaboration.

6.2 Addition of the dynamic vibration absorber

This section presents the experimental procedure used to validate the theory described in Chapter 5, focusing on obtaining the super-attenuation band and the multi-frequency band through the addition of a dynamic vibration absorber to a symmetric cell.

Figure 41 illustrates the conical-shaped rods with the dynamic vibration absorber in a symmetric configuration. The absorber is positioned at the center of the junction between the rods. The rods used in this experimental procedure are identical to those described in the previous section. The rods and the absorber were assembled using a small bolt and secured with cyanoacrylate glue. The schematic of the experimental setup is also the same as that employed in the previous section.

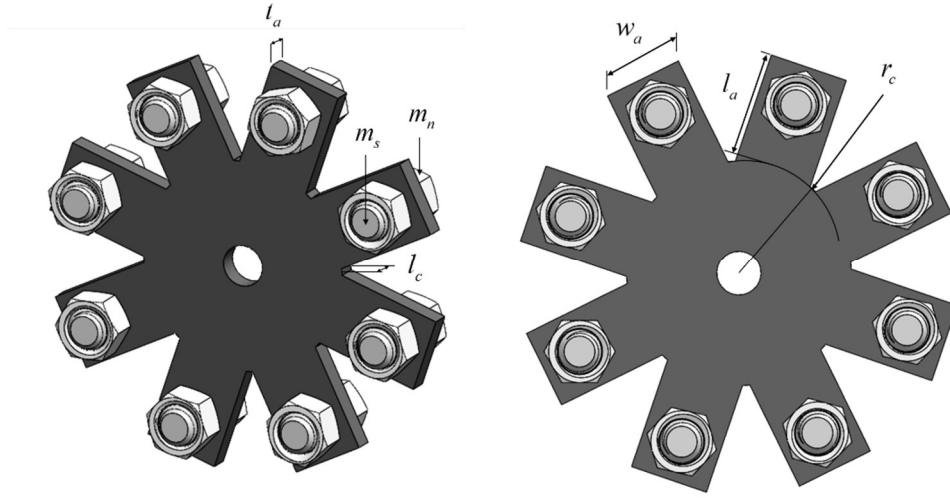
Figure 41 - Schematic of the experimental set-up. Highlighting the acquisition system, position of the accelerometers at the ends of a cell, the impact hammer and the laptop used to record and process the experimental data. Photograph shows the symmetric cell with DVA used in the experiments.



Source: Author own elaboration.

The dynamic vibration absorber consists of eight beams with masses attached to their ends, distributed in a disk-shaped configuration. The absorber was fabricated using 1020 steel, as shown in Figure 42.

Figure 42 - Illustration of dynamic vibration absorber. The screws and nuts act as the mass of the absorber.



Source: Author own elaboration.

In this stage, the experimental procedure consisted of two parts, the first part consisted of the formation of the super-attenuation band and the second part consisted of the formation of the multifrequency bands.

6.2.1 Super - attenuation band

As discussed in Section 5.2, to achieve the super attenuation band, the tuning frequency of the absorber must coincide with the lower limit of the Bragg band, which corresponds to the natural frequency of the structure. Therefore, the absorber was designed so that its tuning frequency matched exactly the natural frequency of the structure. To achieve this tuning frequency, nuts and screws were used as masses at the eight ends of the absorber, all with identical values. The parameters of the dynamic vibration absorber are shown in Table 11. The parameter s_a corresponds to the total stiffness of the absorber, while the parameter m_a represents its total mass, determined by the equation for a cantilever beam with a point mass at the free end $m_a = 8 \cdot [0.23(m_{beam}) + (m_s + 2m_n)]$ (RAO, 2010). The term m_s corresponds to the screw mass, m_n to the nut mass, and, to the beam mass. Since the disk has eight rods of equal mass, the total mass is obtained by multiplying the individual mass of each rod by the total number of rods. The beam mass m_{beam} is determined based on its geometric parameters (Table 11) and material density, calculated

as the product of volume and density. Once the total mass m_a its stiffness is determined through graphical analysis by identifying the absorber's tuning frequency, where

$$\omega_a = \sqrt{\frac{s_a}{m_a}}.$$

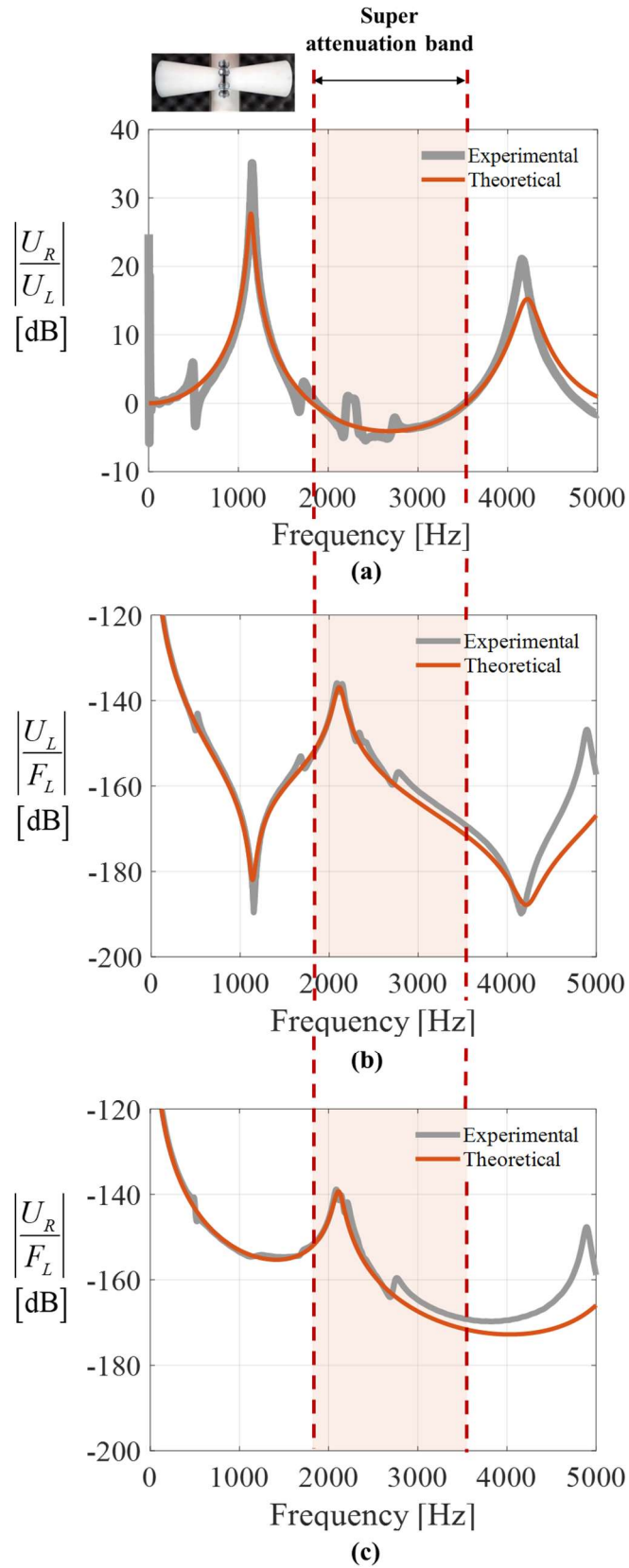
Table 11 - Parameters of the dynamic vibration absorber.

Disk	Young's modulus [GPa]	E_a, E_c	210
	Density [kg/m ³]	ρ_a, ρ_c	7870
	Thickness [m]	t_a, l_c	0.00475
	Center radius [m]	r_c	0.025
	Width of beam [m]	w_a	0.018
	Length of beam [m]	l_a	0.022
	Loss factor	η_a	0.05
	Stiffness [kN/m]	s_a	34535.8
Masses		m_a	0.1958
	Mass [kg]	m_s	0.0086
		m_n	0.0068

Source: Author own elaboration.

Figure 43 shows the results of the experimental procedure alongside the theoretical predictions described in Section 5.2. In the experimental procedure, 10 averages were considered for each frequency response. The super attenuation band predicted theoretically is indicated by the orange shaded area in Figure 43.

Figure 43 - Comparison of experimental results with theoretical predictions using the model described in Section 5.2 (a) Displacement transmissibility; (b) Point receptance (c) Transfer receptance.

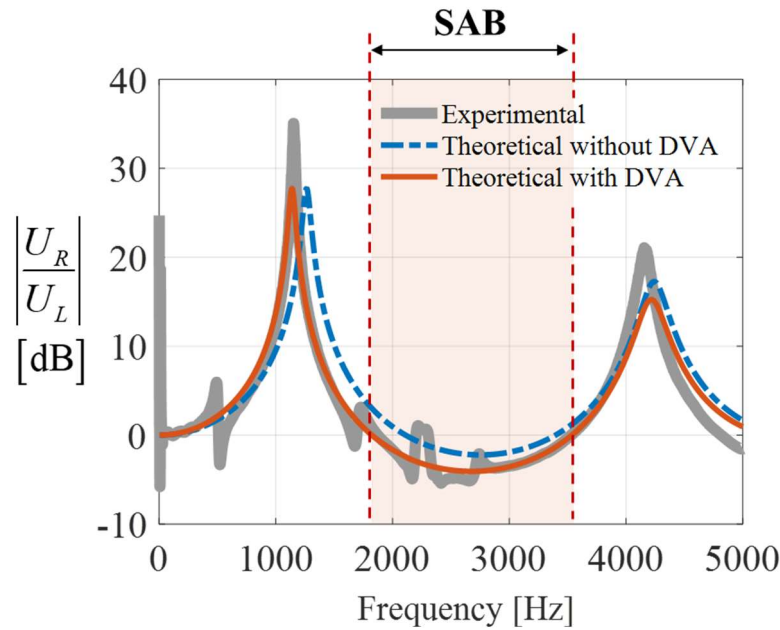


Source: Author own elaboration.

Figure 43(a) shows the experimental and theoretical results for the displacement transmissibility. It can be observed that the experimental response closely matches the theoretical prediction. The super attenuation band, in both responses, exhibits a minimum transmissibility of approximately -3.8 dB, whereas the cell without the absorber, discussed in the previous section, displayed -2.3 dB. In Figures 43(a–c), some unexpected resonance peaks can be observed in the experimental response, which, as mentioned in the previous section, originate from other vibration modes.

For a better analysis of the super attenuation band, Figure 44 presents a comparison between the results obtained and the theoretical prediction of the cell without the absorber.

Figure 44 - Comparison of experimental results with theoretical predictions with dynamic vibration absorber and without dynamic vibration absorber.



Source: Author own elaboration.

Figure 44 shows that, with the addition of an absorber, there is an increase in both the stop-band width and the minimum transmissibility. With the absorber, the lower and upper limits of the super attenuation band correspond to frequencies of 1816 Hz and 3490 Hz, respectively, while without the absorber, these limits were 2107 Hz and 3368 Hz. Thus, the results demonstrate that the addition of the absorbers is effective in achieving a wider frequency band, with the lower limit increasing by approximately 291 Hz and the upper limit by about 122 Hz.

6.2.2 Multi-frequential bands

As discussed in Section 5.3, when the absorber is tuned to frequencies below the lower limit of the Bragg band, it generates bands at lower frequencies. Rainbow-shaped absorbers were also discussed to promote the opening of these resulting bands from the absorber.

The experimental procedure was conducted in two steps: first, the absorber was tuned to a frequency below the lower limit of the Bragg band, specifically at the peak of transmissibility. In the second step, the masses of the absorber were varied in a rainbow-shaped configuration to enhance the band associated with the absorber's local resonance.

For the first step, the stiffness and mass parameters of the absorber are shown in Table 12. The parameter s_a corresponds to the total stiffness of the absorber and m_a is the total mass of absorber.

Table 12 - Parameters of the dynamic vibration absorber used in the experimental procedure.

Stiffness [kN/m]	s_a	34535.8
Mass [kg]	m_a	0.69
Frequency [rad/s]	ω_a	7074.7

Source: Author own elaboration.

For the second step, the stiffness and masses parameters of the absorber are presented in Table 13.

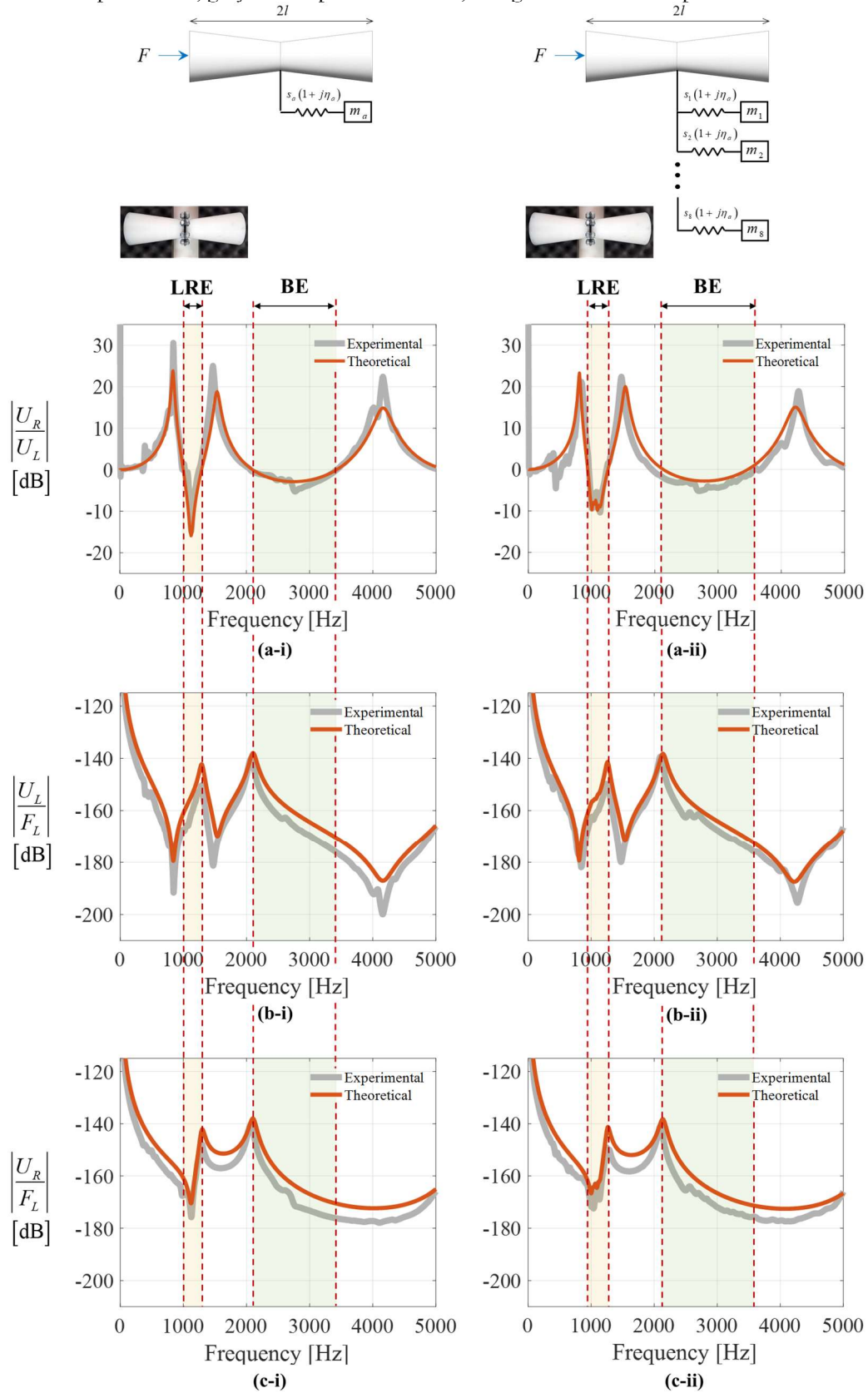
Table 13 - Parameters of the rainbow-shaped absorber used in the experimental procedure.

Stiffness [kN/m]	$s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8$	4316.9
	m_1, m_5	0.0852
	m_2, m_6	0.0919
Mass [kg]	m_3, m_7	0.1001
	m_4, m_8	0.1078
	ω_1, ω_5	7118.1
Frequency [rad/s]	ω_2, ω_6	6853.7
	ω_3, ω_7	6567.0
	ω_4, ω_8	6328.1

Source: Author own elaboration.

Figure 45 presents the results of the experimental procedure for both steps, alongside the theoretical predictions described in Section 5.3. In the experimental procedure, 10 averages were taken for each frequency response measurement.

Figure 45 - Comparison of experimental results with theoretical predictions using the model described in Section 5.3: (ai-ii) Displacement transmissibility (bi-ii) Point receptances; (c-ii) Transfer receptances; the yellow area provides the range of local resonance effect (LRE) of theoretical predictions; the green area provides the range of Bragg scattering effect (BE) of theoretical predictions; grey line: experiment results, orange line: theoretical prediction.



Source: Author own elaboration.

The transmissibility response is observed to align well with the theoretical prediction. When the absorbers are tuned to the same frequency, they exhibit more significant attenuation, as expected, reaching approximately -15.68 dB, with lower and upper limits at 1002 Hz and 1292 Hz, respectively. However, when the absorbers are arranged in a rainbow-shaped configuration, the attenuation decreases to -9.87 dB, with lower and upper limits at 937 Hz and 1267 Hz. An expansion of approximately 40 Hz is noted in the local resonance band, with this expansion being more pronounced at the lower limit.

6.3 Chapter Remarks

This chapter was dedicated to the experimental validations of the methodologies presented in this thesis. Initially, the experimental validation of cells was performed by conical-shaped rods with symmetric and asymmetric geometries. As theoretically predicted, the experiments showed that symmetric cells, when composed of rods of the same length as asymmetric cells, with different periodicities, exhibit stop-bands at lower frequencies. Although some bending modes appeared in the experimental responses, they did not significantly interfere with the formation of the frequency bands.

Subsequently, an absorber was added to the symmetric structure. The first objective of this experiment was to create a super attenuation band, and the results were consistent with theoretical predictions. In the second stage, an absorber in the "rainbow" format was introduced, tuned to operate at lower frequencies, specifically at the transmissibility peak of the cell. The results demonstrated, as theoretically predicted, that it was possible to open the local resonance band due to the use of the rainbow-shaped absorber. However, a reduction in attenuation within this band was observed.

7 CONCLUSIONS

This thesis investigated the propagation of longitudinal waves in rods composed of cells with catenoidal, conical, and exponentially varying cross-sectional areas. In Chapter 2, the structures were modeled in terms of receptance and dynamic stiffness matrices. Analytical equations for the displacement transmissibility of single cells were derived in Chapter 3, using the transfer matrix method in combination with the Cayley-Hamilton theorem. These analytical equations provided the foundation for analyzing the dynamic behavior of finite mono-coupled periodic structures.

In Chapter 4, the dynamic behavior of periodic structures was investigated in terms of single-cell dynamics and periodicity effects. It was found that structures composed of rod elements with catenoidal, conical and exponentially varying cross-sectional areas present similar responses for displacement transmissibility. Regarding the periodicity of the cell, it was verified that asymmetric cells that present the same periodicity as symmetric cells, present results similar to symmetric cells. However, when rods of the same length are arranged in symmetric and asymmetric cells, both do not have the same periodicity length. It was found that for such a configuration, the correctly oriented asymmetric cells (the thinner end of the rod oriented towards the excitation source) have an attenuation band at higher frequencies compared to the symmetric structure. Additionally, the use of exponential-shaped rods was shown to provide satisfactory results for vibration attenuation while reducing stress concentrations compared to stepped rods.

Chapter 5 explored the integration of dynamic absorbers into periodic structures. The inclusion of these absorbers demonstrated the ability to create super-attenuation bands (ultra-wide attenuation bands) when tuned to the natural frequency of the structure. However, such attenuation bands occur at higher frequencies. As an alternative to overcome this challenge, the absorbers were tuned to frequencies lower than the natural frequency, resulting in the creation of multifrequency bands: one associated with the local resonance of the absorber and another related to the Bragg scattering effect. Nevertheless, the attenuation bands associated with local resonance are narrower compared to the Bragg bandwidth. To increase the bandwidth of the local resonance, a rainbow configuration of the absorbers was proposed, a strategy that proved effective in expanding the local resonance bandwidth, significantly enhancing the structure's capability for vibration control at lower frequency ranges.

The experimental validation was conducted in Chapter 6 to compare theoretical predictions with experimental results. The initial tests evaluated the responses of simple symmetric and asymmetric cells composed of rods with conical transverse variations. Although the rods exhibited bending modes, as discussed in Appendix E, these modes did not interfere with the formation of the frequency band.

Subsequently, dynamic absorbers were incorporated into the symmetric cells to achieve super-attenuation bands or attenuation bands at lower frequencies. Initially, a super-attenuation band was obtained, whose responses showed good agreement with theoretical predictions. Later, the absorber was adjusted to operate at lower frequencies, where it presented a high attenuation value, albeit within a narrow frequency range. To overcome this limitation, expand the local resonance bandwidth, the configuration of rainbow-shaped absorbers was experimentally implemented. The results demonstrated that rainbow-shaped absorbers are capable of broadening the bandwidth, although with a reduction in maximum attenuation. Thus, it is concluded that the experimental results corroborated theoretical predictions, confirming the effectiveness of the proposed methodology.

7.1 Scientific contribution

Parts of this research have been published in the literature, which can be cited:

1. DA SILVA, CAMILA ALBERTIN XAVIER; SOROKIN, VLADISLAV ; BRENNAN, MICHAEL JOHN ; GONÇALVES, PAULO JOSÉ PAUPITZ . The dynamic behaviour of a finite periodic structure comprising either symmetric or asymmetric exponential- and conical-shaped rods. JOURNAL OF SOUND AND VIBRATION, v. 595, p. 118741, 2025. <https://doi.org/10.1016/j.jsv.2024.118741>
2. DA SILVA, CAMILA ALBERTIN XAVIER; GONCALVES, P. J. P. . Periodic structures with exponentially varying cross-section areas. In: CILAMCE-PANACM-2021, 2021, Rio de Janeiro. Proceedings of the Ibero-Latin-American Congress on Computational Methods in Engineering. Belo Horizonte/MG: Associação Brasileira de Métodos Computacionais em Engenharia (ABMEC), 2021.

7.2 Future work suggestions

Based on the findings of this study, the following directions are suggested for future research:

- Modeling shear modes with the Rayleigh-Love theory:

Extend the current modeling approach to include shear deformation effects by applying the Rayleigh-Love theory. This would provide a more comprehensive understanding of the dynamic behavior of rods, especially for higher-frequency applications or cases where shear modes significantly influence the response.

- Exploration of bending effects:

Investigate the bending dynamics in the structural behavior, including bending vibrations and their interactions with other vibration modes. This study could enhance the understanding of the overall dynamic response and improve the design of structures for applications requiring precise vibration control or wave manipulation.

- Integration of Acoustic Black Holes into structures with exponential-shaped rods:

Explore the potential of exponential rods in the creation of Acoustic Black Holes leveraging geometric profiles to concentrate and dissipate vibrational energy. The inclusion of viscoelastic layers can enhance wave absorption, significantly reducing reflection and increasing vibration attenuation.

- Study of periodic structures with mixed periodicity (geometric and material):

Investigate the dynamic behavior of periodic structures that combine both geometric and material periodicity. This approach could lead to novel metamaterial designs with enhanced tunability and adaptive characteristics, broadening the range of applications in vibration control and wave manipulation.

- Development of more complex structures:

Based on the geometry studied in the thesis, to explore the construction of more complex structures using the modeled rods. This may involve assembling the rods into multidimensional frameworks or integrating them with additional structural elements in periodic systems. These structures can be tailored for specific dynamic behaviors,

including enhanced vibration control and wave propagation management, for practical engineering applications.

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APPENDIX A – Solution to Bessel differential equation

This appendix presents the solution to Bessel differential equation. The Bessel differential equation is the linear second-order ordinary differential equation given by equation (A.1).

$$x^2 U'' + xU' + (x^2 - m^2)U(x) = 0 \quad (\text{A.1})$$

where m is a real constant, called the order of the Bessel equation.

Bowman (1958) suggests a transformation of the Bessel Equation that satisfies $U(x) = x^p J_q(\chi x^\varphi)$, then

$$U''(x) - \frac{2p-1}{x}U' + \left(\chi^2 \varphi^2 x^{2\varphi-2} + \frac{p^2 - q^2 \varphi^2}{x^2} \right) U(x) = 0 \quad (\text{A.2})$$

Therefore, the solution is given by equation (A.3).

$$U(x) = \begin{cases} x^p \left[d_1 J_q(\chi x^\varphi) + d_2 Y_q(\chi x^\varphi) \right] & \text{for integer, } q \\ x^p \left[d_1 J_q(\chi x^\varphi) + d_2 J_{-q}(\chi x^\varphi) \right] & \text{for nointeger, } q \end{cases} \quad (\text{A.3})$$

where $J_q(x)$ and $Y_q(x)$ are the Bessel functions of the first and second kind, and d_1 and d_2 are arbitrary constants.

APPENDIX B – Solution to the equation of motion in space domain of the rod with conically varying cross-section areas

In this appendix are considered the rod with conically varying cross-section areas, its equation of motion, given by equation (32), and the solution to Bessel differential equations, according to Appendix A.

The equation of motion in the space domain of rod with conically varying cross-section areas, according to Graff (1991) is given by the Bessel differential equation, according to equation (32). From the transformation proposed by Bowman (1958), according to equation (A.2), the values of p , φ , χ and q are respectively

$$p = -\frac{1}{2}; \quad \varphi = 1; \quad \chi = k_0; \quad q = \frac{1}{2}$$

Therefore, the solution for equation (32) is the Bessel equation of half-order, according to equation (B.1)

$$U(x) = \frac{1}{\sqrt{x}} \left[d_1 J_{\frac{1}{2}}(k_0 x) + d_2 J_{-\frac{1}{2}}(k_0 x) \right] \quad (\text{B.1})$$

According to the properties of the Bessel function, $J_{\frac{1}{2}}(x)$ and $J_{-\frac{1}{2}}(x)$ are obtained by expansions in power series, then

$$J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = \sqrt{\frac{2}{\pi x}} \sin(x) \quad (\text{B.2})$$

$$J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = \sqrt{\frac{2}{\pi x}} \cos(x) \quad (\text{B.3})$$

Substituting the equations (B.2) and (B.3) in equation (B.1), the solution of equation (32) is given by

$$U(x) = \frac{1}{x} \left[d_1 \sin(k_0 x) + d_2 \cos(k_0 x) \right] \quad (\text{B.4})$$

APPENDIX C – Floquet-Bloch theorem and the definition of stop-bands

In this appendix, the application of the Floquet-Bloch theorem in finite periodic systems and its relationship with the formation of stop-bands is discussed. The theorem is widely applied to infinite structures to calculate the dispersion properties of periodic structures and to estimate their wave modes and velocities (DARCHE; LOPEZ-CABALLERO; TIE, 2024; GUPTA *et al.*, 2023; PU; SHI, 2018).

With the transfer matrix of the unit cell well defined, the Floquet-Bloch theorem can be applied to relate the state vectors of the right and left faces of the periodic structure, where, $q_R^{[N]} = \Lambda q_L^{[1]}$, where Λ is the eigenvalue of the transfer matrix, given by.

$$\Lambda = e^{\mu} = e^{-i k_c l_{cell}} \quad (C.1)$$

where $i = \sqrt{-1}$, l_{cell} is the length of cell and k_c is the wavenumber of the cell, given by

$$k_c(\omega) = \frac{\ln(\Lambda(\omega))}{i l_{cell}} \quad (C.2)$$

where ω is the circular frequency.

In the context of periodic structures, the theorem allows defining frequency regions where wave propagation occurs, known as pass-bands. These occur when $|\Lambda| = 1$, where the wavenumber k_c is purely real, indicating that the wave propagates through the structure without attenuation. Frequency regions where waves do not propagate freely are known as bandgaps. In these regions, $|\Lambda| \neq 1$, with k_c has an imaginary part, indicating that the wave is evanescent and decays exponentially (BRILLOUIN, 1946).

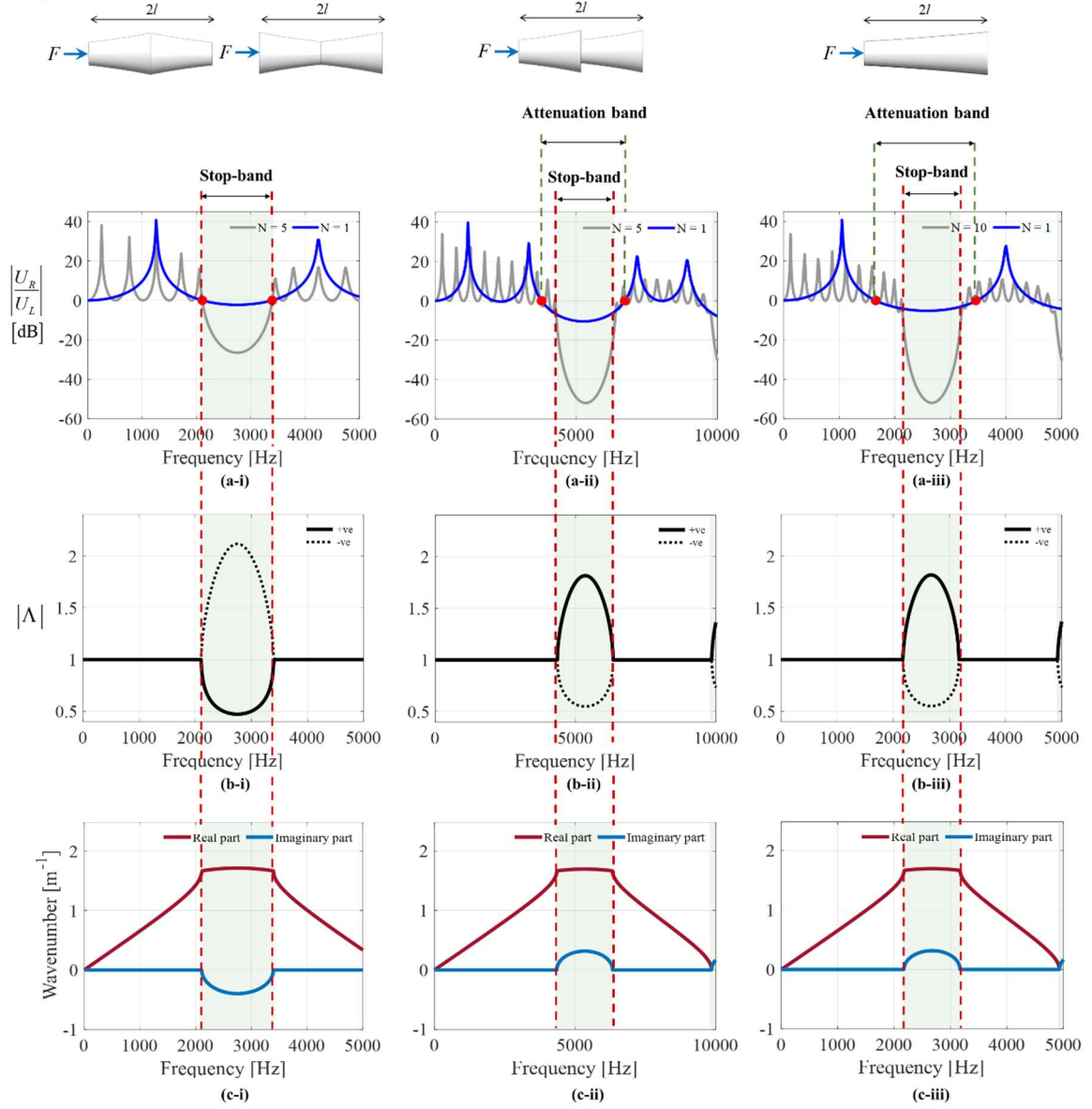
As mentioned, the theorem is widely used for infinite periodic structures, allowing the prediction of the frequencies at which stop-band will occur based on the dispersion curves. However, as discussed in Chapter 4, in structures with symmetric cells, the lower and upper limits of the stop-band do not change with an increasing number of cells. Thus, the lower and upper limits of the stop-band of a finite structure composed of symmetric cells are the same as those of the stop-band of an infinite structure. Therefore, for symmetric structures, the dispersion curves can be analyzed using the eigenvalues of the

unit cell transfer matrix, and it is also possible to perform an analysis based on the wavenumber (NIELSEN; SOROKIN, 2015).

However, it was discussed in Chapter 4 that in asymmetric structures, as the number of cells increases, the attenuation band narrows until it reaches the lower and upper limits of the stop-band of an infinite structure. Thus, for asymmetric structures with a reduced number of cells, wavenumber dispersion curves would not be an option to predict the attenuation bands accurately.

Figure C.1 presents the analysis of the relationship between displacement transmissibility and the wave and wavenumber dispersion curves to determine the stop-bands. For cells composed of exponential-shaped rods, $k_c = \hat{k}$, and $l_{cell} = 2l$.

Figure C.1 – Comparison of the stop-bands of finite structures with the bandgaps of infinite structures (a) displacement transmissibility; (b) wave dispersion (real part of Λ); (c) wavenumber; red circles provide the stop-band/attenuation band limits.



Source: Author own elaboration.

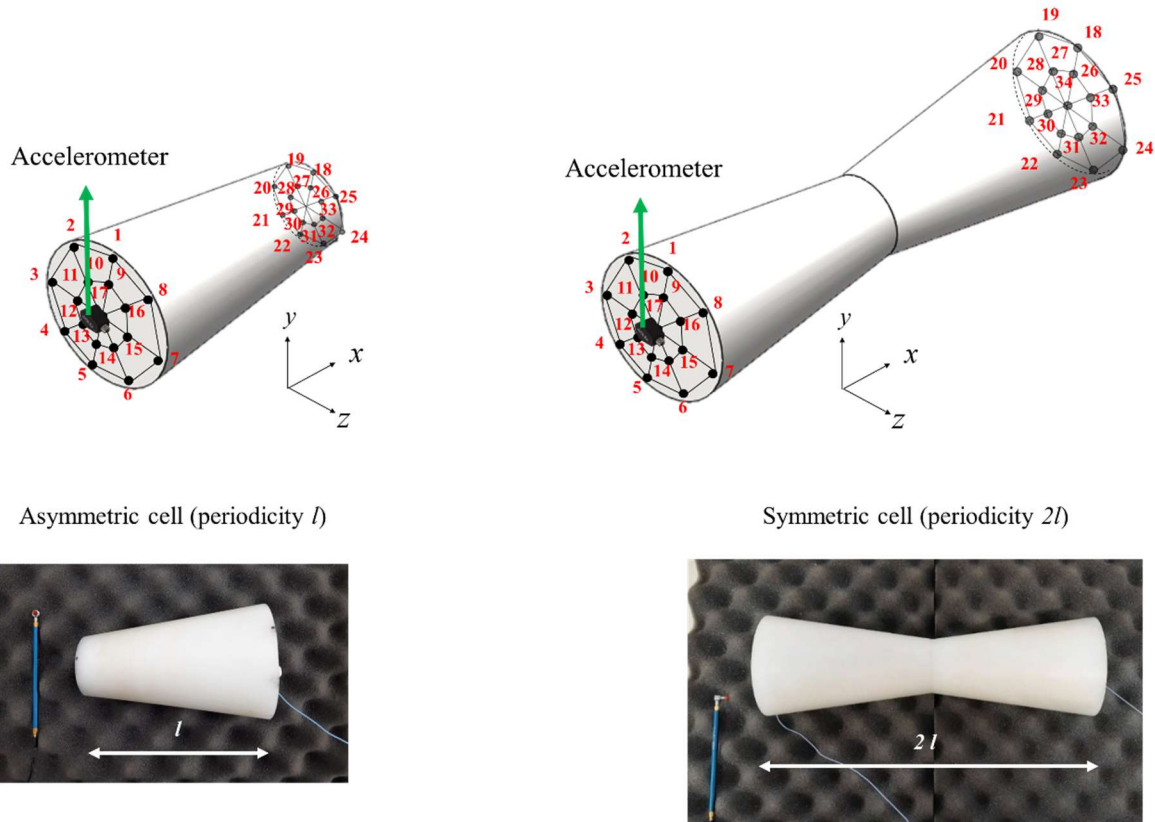
As can be observed in Figures C.1(ai–ci), the dispersion curve for symmetric structures corresponds to the stop-band predicted by the displacement transmissibility curve. However, in the case of asymmetric structures, shown in Figure C.1 (aii–ciii), a certain number of cells is required to reach the lower and upper limits of the stop-band of the infinite structure. This is because the greater the number of cells, the closer the structure will behave to an infinite one.

APPENDIX D – Experimental Modal Analysis (EMA)

This appendix describes the Experimental Modal Analysis (EMA) performed in symmetric and asymmetric cells. The experimental analysis was conducted as a complementary approach to investigating the dynamic behavior of the cells.

In the experimental procedure, the conical-shaped rods previously used in experimental setups were utilized in both symmetric and asymmetric configurations. The roving excitation method was employed. In this approach, the accelerometer (model PCB 352A24) with a nominal sensitivity of 100 mV/g was fixed at the center of the left end (point 17), as illustrated in Figure 42, while the impact hammer (model PCB 086E80) with a nominal sensitivity of 22.5 mV/N was moved to excite the structure at all the nodes indicated in Figure D.1. For each excitation point, an average of five frequency response measurements were performed. The signals were acquired and processed using a Dewesoft-Sirius system connected to a laptop.

Figure D.1 - Schematic representation of the roving excitation method, highlighting the nodes excited in the structure and the accelerometer positioned at the left end of a cell (point 17). Additionally, two photographs are provided showcasing the symmetric and asymmetric cells used in the experiments.



Source: Author own elaboration.

Signal processing was performed to extract and analyze the dynamic characteristics of the cells. The Complex Mode Indicator Function (CMIF) was computed as part of this process. CMIF is a method that provides a function for each reference degree of freedom (DOF), included in the analysis, using a poly-reference approach. This technique helps identify closely coupled modes and repeated roots by applying Singular Value Decomposition (SVD) to all measured Frequency Response Functions (FRFs). Peaks in the CMIF functions correspond to resonances, indicating the poles of the system under test (ALLEMANG; BROWN, 2006).

During data acquisition, the CMIF overlaid on the stabilization diagram to provide a comprehensive overview of the detected modes. Once the modes were identified, the autoMAC (Modal Assurance Criterion) was used to validate the distinction and accuracy of the estimated modes. The autoMAC evaluates the correlation between pairs of modes to ensure they are distinguishable based on the measured DOFs and FRFs. It produces values ranging from 0 to 1, where a value of 1 indicates perfect correlation (KHOO *et al.*, 2020).

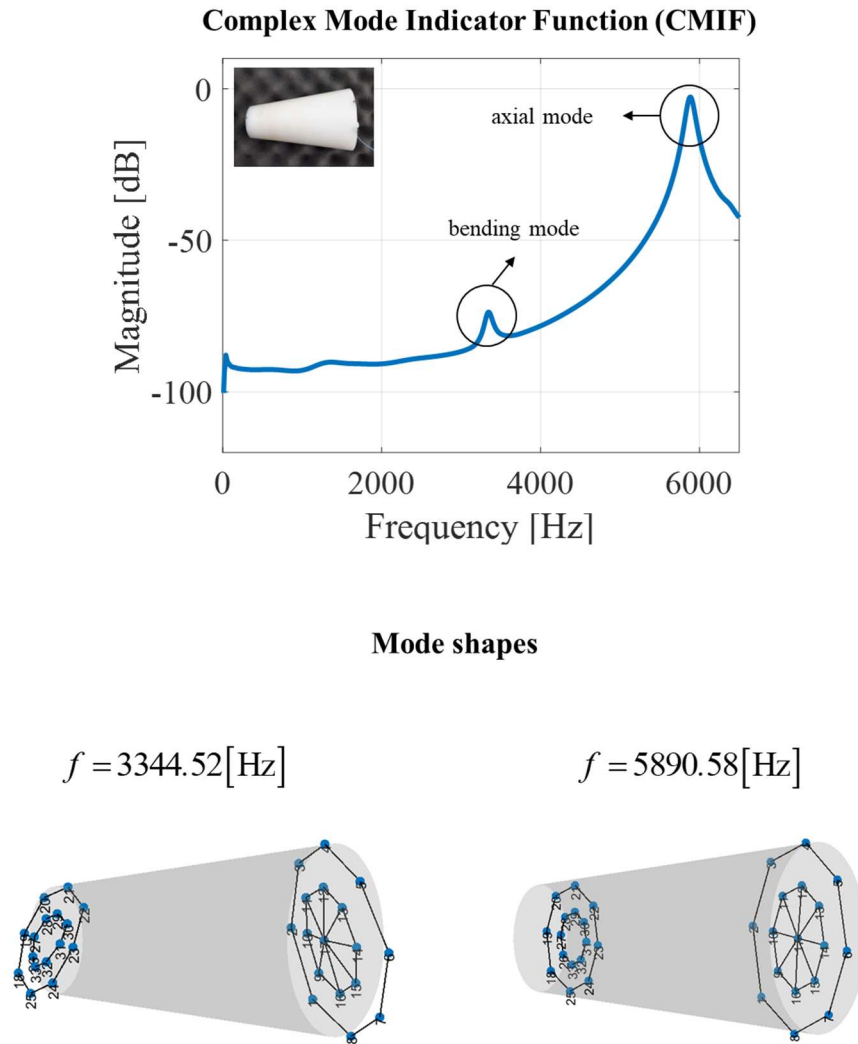
Figure D.2 presents the CMIF and mode shapes obtained from the experimental modal analysis for the asymmetric cell. Table D.1 shows the AutoMAC values for the correlation of each mode with itself.

Table D.1 - AutoMAC between modes to itself
(asymmetric cell).

Frequency [Hz]	3344.52	5890.58
3344.52	1	0.0522
5890.58	0.0523	1

Source: Author own elaboration.

Figure D.2 - Complex Mode Indicator Function (CMIF) and mode shapes of the asymmetric cell.



Source: Author own elaboration.

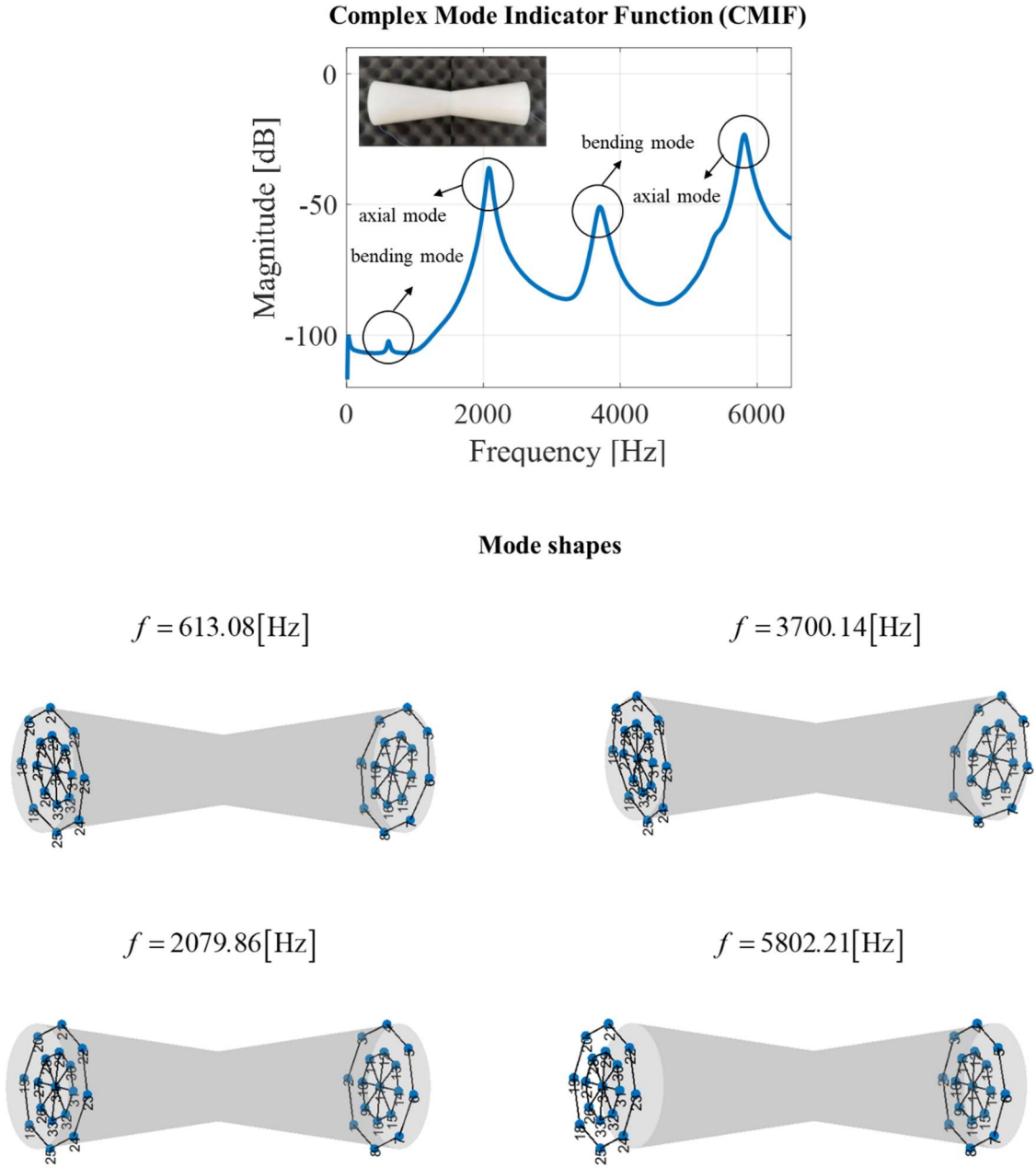
Figure D.3 presents the CMIF and mode shapes obtained from the experimental modal analysis for the symmetric cell and the Table D.2 shows the AutoMAC values for the correlation of each mode with itself.

Table D.2 - AutoMAC between modes to itself (symmetric cell).

Frequency [Hz]	613.08	2079.86	3700.14	5802.21
613.08	1	0.9110	0.1175	0.0191
2079.86	0.9110	1	0.0180	0.0062
3700.14	0.1175	0.018	1	0.0019
5802.21	0.0191	0.0062	0.0019	1

Source: Author own elaboration.

Figure D.3 - Complex Mode Indicator Function (CMIF) and mode shapes of the symmetric cell.



Source: Author own elaboration.

It can be observed that the modes obtained for both cells include not only the axial modes predicted by the theoretical predictions described in this thesis but also modes associated with flexural vibrations. The stop/attenuation band phenomenon discussed in this thesis is based solely on longitudinal displacement, excluding transverse displacement caused by Poisson's ratio effects (BANERJEE; ANANTHAPUVIRAJAH; PAPKOV, 2020) and flexural vibrations. These effects, which generally become significant at higher frequencies, can be analyzed using the Rayleigh-Love model (HAN *et al.*, 2014).

While these effects may have been present in the measurements discussed in Chapter 6 , their influence did not significantly affect the predicted attenuation bands or the minimum transmissibility. The additional effects observed in the experimental responses are primarily related to peaks associated with flexural vibrations.