Integrable anisotropic spin-ladder model

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We present an integrable spin-ladder model, which possesses a free parameter besides the rung coupling J. Wang's system based on the $SU(4)$ symmetry can be obtained as a special case. The model is exactly solvable by means of the Bethe ansatz method. We determine the dependence on the anisotropy parameter of the phase transition between gapped and gapless spin excitations and present the phase diagram. Finally, we show that the model is a special case of a more general Hamiltonian with three free parameters.

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With the discovery of high-temperature superconductivity in doped copper oxide (or cuprate) materials,[1] a tremendous effort has been made to understand the physics underlying this phenomenon. In the absence of doping these compounds are reasonably approximated by the two-dimensional Heisenberg model[2,3] or some suitable generalization describing spin-exchange-type interaction. It is well known that in one dimension the Heisenberg model is exactly solvable via Bethe ansatz methods and from this solution the spectrum of elementary spin excitation is gapless. On the other hand, the existence of the spin gap is critical for the observed phenomenon of superconductivity to occur under doping. To maximize the interaction between theory and experiment, much work is now focused on quasi-one-dimensional models known as ladders. The introduction of these ladder systems has brought about a significantly increased understanding of the physics of the cuprate compounds.

By introducing the concept of the ladder model the apparent contradiction in the excitation spectrum is resolved, since the ladder allows for the formation of singlet states along the rungs which are responsible for the formation of the spin gap. However, the usual Heisenberg ladder model cannot be solved. In order to gain some results in the theory of spin ladder systems, many authors have considered generalized models, which incorporate additional interaction terms that guarantee exact solvability. Remarkably, such generalized models still exhibit realistic physical properties such as the existence of a spin gap[4] and the magnetization plateaus at fractional values of the total magnetization.[5]

This approach has been used to derive quasi-one-dimensional systems using the well-established theories from the one-dimensional case.[4–15] In all cases cited above, no free parameters are present, other than the rung interaction coupling and applied magnetic field, due to the strict conditions of integrability. With the presence of free parameters it is reasonable to expect that the solution may provide better test models for describing the various behaviors associated with ladder systems.

The purpose of this paper is to present an integrable generalized spin ladder with one extra parameter, characterizing anisotropy, without violating integrability. This model is exactly solvable by the Bethe ansatz and it reduces to the model introduced by Wang[4] for a special limit of this extra parameter. The situation here is akin to the generalization of the XXX chain to the anisotropic XXZ version. The introduction of the additional free parameter in the present case allows for an example of a model with a critical line varying continuously with the anisotropy. More specifically, the size of the gap in the massive region depends explicitly on the anisotropy parameter, which in turn shows dependence of the anisotropy parameter for the points at which the gap closes that define the phase transition.

Let us begin by introducing the generalized spin-ladder model, whose Hamiltonian reads

$$H^{(1)}=\sum_{j=1}^{L}\left[h_{j,j+1}+\frac{1}{2}J(\vec{\sigma}_j\cdot\vec{\tau}_j-1)\right], \tag{1}$$

where

$$\begin{aligned}h_{j,j+1}=&\tfrac{1}{4}(1+\sigma_j^z\sigma_{j+1}^z)(1+\tau_j^z\tau_{j+1}^z)+(\sigma_j^+\sigma_{j+1}^-+\sigma_j^-\sigma_{j+1}^+)\\&\times(\tau_j^+\tau_{j+1}^-+\tau_j^-\tau_{j+1}^+)+\tfrac{1}{2}(1+\sigma_j^z\sigma_{j+1}^z)(t^{-1}\tau_j^+\tau_{j+1}^-\\&+t\tau_j^-\tau_{j+1}^+)+\tfrac{1}{2}(t^{-1}\sigma_j^+\sigma_{j+1}^-+t\sigma_j^-\sigma_{j+1}^+)\\&\times(1+\tau_j^z\tau_{j+1}^z).\end{aligned}$$

Above $\vec{\sigma}_j$ and $\vec{\tau}_j$ are Pauli matrices acting on site j of the upper and lower legs, respectively, J is the strength of the rung coupling (we will consider only the case $J>0$ in the subsequent analysis corresponding to *antiferromagnetic* coupling), and t is a free parameter representing an anisotropy in the legs and interchain interaction. Throughout, L is the number of rungs (equivalently, the length of the ladder) and periodic boundary conditions are imposed. By setting $t\to 1$ in Eq. (1), Wang's model based on the $SU(4)$ symmetry[4] can be recovered. (Strictly speaking, it is $SU(4)$ invariant in the absence of the rung interactions.) The Hamiltonian is invariant under interchange of the legs; i.e., $\vec{\sigma}_j\leftrightarrow\vec{\tau}_j$. Moreover, under spin inversion for both leg spaces the Hamiltonian is invariant with the interchange $t\leftrightarrow t^{-1}$. For this reason we see that the parameter t plays the role of spin anisotropy.

The energy eigenvalues of the Hamiltonian are given by

$$E=-\sum_{j=1}^{M_1}\left(\frac{1}{\lambda_j^2+1/4}-2J\right)+(1-2J)L, \tag{2}$$

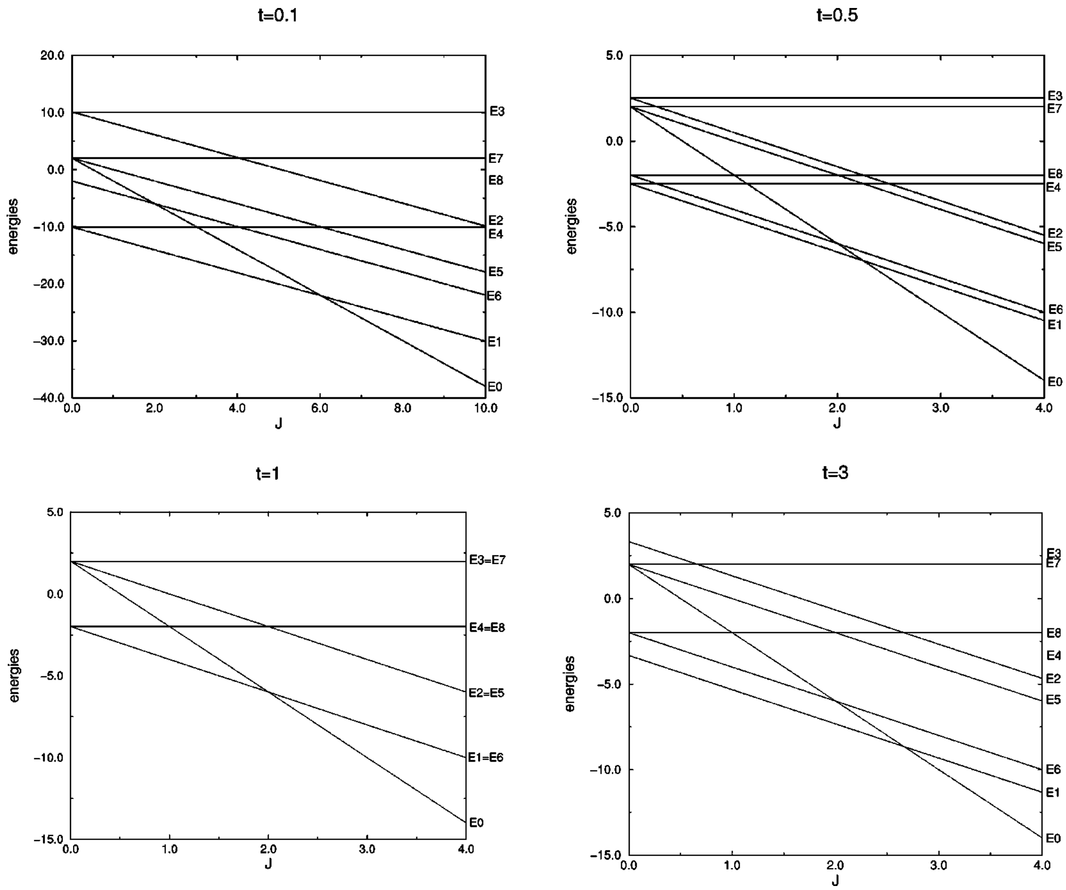

FIG. 1. Energies E_i $(i=0,\ldots,8)$ versus rung coupling J for different values of the anisotropy t. Notice that there is mainly competition between E_0 and E_1 to be the lowest-energy level of the model. In addition, the critical value of J above which E_0 is the ground-state energy varies with t reaching its minimum value when $t=1$.

where λ_j are solutions to the Bethe ansatz Eqs. (3) below. The Bethe ansatz equations arise from the exact solution of the model through the nested algebraic Bethe ansatz method and read

$$t^{(L-2M_3)}\left(\frac{\lambda_j-i/2}{\lambda_j+i/2}\right)^L=\prod_{l\neq j}^{M_1}\frac{\lambda_j-\lambda_l-i}{\lambda_j-\lambda_l+i}\prod_{\alpha=1}^{M_2}\frac{\lambda_j-\mu_\alpha+i/2}{\lambda_j-\mu_\alpha-i/2},$$

$$t^{(L-2M_3)}\prod_{\beta\neq\alpha}^{M_2}\frac{\mu_\alpha-\mu_\beta-i}{\mu_\alpha-\mu_\beta+i}=\prod_{j=1}^{M_1}\frac{\mu_\alpha-\lambda_j-i/2}{\mu_\alpha-\lambda_j+i/2}\times\prod_{\delta=1}^{M_3}\frac{\mu_\alpha-\nu_\delta-i/2}{\mu_\alpha-\nu_\delta+i/2}, \qquad (3)$$

$$t^{(L+2M_2-2M_1)}\prod_{\gamma\neq\delta}^{M_3}\frac{\nu_\delta-\nu_\gamma-i}{\nu_\delta-\nu_\gamma+i}=\prod_{\alpha=1}^{M_2}\frac{\nu_\delta-\mu_\alpha-i/2}{\nu_\delta-\mu_\alpha+i/2}.$$

We remark that although the anisotropy parameter t does not appear explicitly in the energy expression (2), the solutions λ_j for the Bethe ansatz equation do depend on t as will be illustrated later.

The exact diagonalization of the two-site Hamiltonian shows that for $J>1+\frac{1}{2}(t+1/t)$ the (unique) ground state assumes the form of the product of the singlets with energy $E_0=2-4J$ and the energies of the excitations are given by

$$E_1=-2J-(t+1/t), \quad E_2=-2J+(t+1/t),$$

$$E_3=(t+1/t), \quad E_4=-(t+1/t),$$

$$E_5=-2J+2, \quad E_6=-2J-2, \quad E_7=2, \quad E_8=-2.$$

A sample of these numerical results are presented in Fig. 1 above.

For L sites it follows that the ground state is still given by a product of rung singlets when $J>1+\frac{1}{2}(t+1/t)$ and the energy is $(1-2J)L$. This is in fact the reference state used in the Bethe ansatz calculation and corresponds to the case $M_1=M_2=M_3=0$ for the Bethe ansatz Eq. (3). To describe an elementary spin-1 excitation, we take $M_1=1$ and $M_2=M_3=0$ in Eq. (3), which leads to the imaginary solution for the variable λ (strictly, the lattice length L is assumed to be even),

$$\lambda=\frac{i}{2}\frac{t-1}{t+1}, \qquad (4)$$

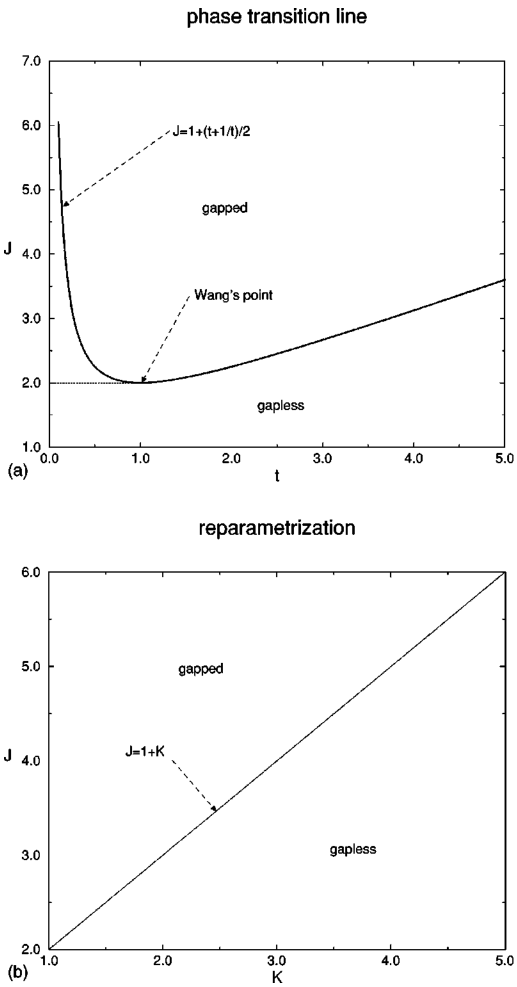

FIG. 2. (a) Rung coupling J versus anisotropy t. This graphic represents the phase diagram and the dotted line shows Wang's point. The curve $[J=1+(t+1/t)/2]$ divides the gapped and gapless phases. (b) Rung coupling J versus reparametrization parameter K. This graphic shows a reparametrization of the curve $J=1+(t+1/t)/2$ in terms of $K=(t+1/t)/2$. In this parameterization, the phase boundary is a straight line.

giving the minimal excited-state energy. The energy gap can easily be calculated using the exact Bethe ansatz solution and has the form

$$\Delta=2\left(J-1-\frac{1}{2}\left(t+\frac{1}{t}\right)\right). \tag{5}$$

By solving $\Delta=0$ for J we find the critical value $J^c=1+\frac{1}{2}(t+1/t)$, indicating the critical line at which the quantum phase transition from the dimerized phase to the gapless phase occurs.

The phase diagram in Fig. 2(a) assumes a simpler form after a suitable reparametrization. We introduce a new parameter K given by $K=(t+1/t)/2\geqslant 1$. In Fig. 2(b) the phase diagram is represented in terms of K and J. The phase boundary is now a straight line given by $J=1+K$. As a preliminary attempt to characterize the gapless phase we have studied by numerical diagonalization the energy spectra of Eq. (1) on ladders of sizes up to eight rungs and several values of J and K.

As mentioned above, if $J>1+K$ the spectra is gapped and the ground state a product of singlets on each rung. As we cross the phase transition line a state with finite magnetization becomes the ground state. By further decreasing J, with K fixed, the ground-state magnetization initially increases and then, if J is made small enough, drops to zero. This behavior resembles that of the one-dimensional anisotropic Heisenberg model, also called the XXZ chain, in the presence of a magnetic field.[16] In the XXZ chain with no magnetic field applied, the anisotropy can be tuned to bring the system into a massive antiferromagnetic (AF) phase, where the ground state is a Néel state. By tuning the magnetic field inside this AF phase we observe a behavior similiar to the one found here in terms of the parameter J. In particular, the Pokrovsky-Talapov[16,17] phase transition appearing in the phase diagram of the XXZ chain has many features in common with the phase transition found here, with J and K playing the roles of the magnetic field and the anisotropy, respectively. Based on this analogy, we conjecture that the ground state with null magnetization found in our model for small values of J suggests the presence of an AF gapped phase (Néel phase). Therefore, another phase-transition line is expected to exist below the one presented here.

The integrability of this model can be shown by the fact that it can be mapped [see Eq. (7) below] to the following Hamiltonian, which can be derived from an R matrix obeying the Yang-Baxter algebra for $J=0$, while for $J\neq 0$ the rung interactions take the form of a chemical-potential term.

$$\hat{H}^{(1)}=\sum_{j=1}^{L}\left[\hat{h}_{j,j+1}-2JX_j^{00}\right], \tag{6}$$

where

$$\begin{aligned}\hat{h}_{j,j+1}=\sum_{\alpha=0}^{3} & X_j^{\alpha\alpha}X_{j+1}^{\alpha\alpha}+X_j^{20}X_{j+1}^{02}+X_j^{02}X_{j+1}^{20}+X_j^{13}X_{j+1}^{31}\\ &+X_j^{31}X_{j+1}^{13}+t(X_j^{10}X_{j+1}^{01}+X_j^{12}X_{j+1}^{21}+X_j^{03}X_{j+1}^{30}\\ &+X_j^{23}X_{j+1}^{32})+t^{-1}(X_j^{01}X_{j+1}^{10}+X_j^{21}X_{j+1}^{12}+X_j^{30}X_{j+1}^{03}\\ &+X_j^{32}X_{j+1}^{23}).\end{aligned}$$

Above $X_j^{\alpha\beta}=|\alpha_j\rangle\langle\beta_j|$ are the Hubbard operators with $|\alpha_j\rangle$ the orthogonalized eigenstates of the local operator $\vec{\sigma}_j\cdot\vec{\tau}_j$, as in Wang's case.[4] The local Hamiltonians (1) and (6) are related through the following basis transformation:

$$\begin{aligned}&|\uparrow,\uparrow\rangle\rightarrow 1/\sqrt{2}(|\uparrow,\downarrow\rangle-|\downarrow,\uparrow\rangle),\quad |\uparrow,\downarrow\rangle\rightarrow|\uparrow,\uparrow\rangle,\\ &|\downarrow,\uparrow\rangle\rightarrow 1/\sqrt{2}(|\uparrow,\downarrow\rangle+|\downarrow,\uparrow\rangle),\quad |\downarrow,\downarrow\rangle\rightarrow|\downarrow,\downarrow\rangle.\end{aligned} \tag{7}$$

The following R matrix,

$$R=\left(\begin{array}{cccc|cccc|cccc|cccc}
a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & t^{-1}b & 0 & 0 & c & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & b & 0 & 0 & 0 & 0 & 0 & c & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & tb & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c & 0 & 0 & 0\\
\hline
0 & c & 0 & 0 & tb & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & tb & 0 & 0 & c & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & b & 0 & 0 & 0 & 0 & 0 & c & 0 & 0\\
\hline
0 & 0 & c & 0 & 0 & 0 & 0 & 0 & b & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & c & 0 & 0 & t^{-1}b & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & tb & 0 & 0 & c & 0\\
\hline
0 & 0 & 0 & c & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t^{-1}b & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & c & 0 & 0 & 0 & 0 & 0 & b & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c & 0 & 0 & t^{-1}b & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a
\end{array}\right), \qquad (8)$$

with

$$a=x+1, \quad b=x, \quad \text{and} \quad c=1$$

obeys the Yang-Baxter algebra

$$R_{12}(x-y)R_{13}(x)R_{23}(y)=R_{23}(y)R_{13}(x)R_{12}(x-y) \qquad (9)$$

and originates the Hamiltonian (6) for $J=0$ by the standard procedure[18]

$$\hat{h}_{j,j+1}=P\frac{d}{dx}R(x)|_{x=0},$$

where P is the permutation operator.

This model studied above represents one particular case of a more general Hamiltonian that has three free parameters and reads

$$H^{g}=\sum_{j=1}^{L}\left[h^{g}_{j,j+1}+\frac{1}{2}J(\vec{\sigma}_j\cdot\vec{\tau}_j-1)\right], \qquad (10)$$

where

$$h^{g}_{j,j+1}=\sigma_j^{+}\sigma_{j+1}^{-}\left[\frac{t_1^{-1}}{4}(1+\tau_j^z)(1+\tau_{j+1}^z)+\frac{t_2}{4}(1-\tau_j^z)(1-\tau_{j+1}^z)+t_3\tau_j^{+}\tau_{j+1}^{-}+\tau_j^{-}\tau_{j+1}^{+}\right]+\sigma_j^{-}\sigma_{j+1}^{+}$$
$$\times\left[\frac{t_1}{4}(1+\tau_j^z)(1+\tau_{j+1}^z)+\frac{t_2^{-1}}{4}(1-\tau_j^z)(1-\tau_{j+1}^z)+\tau_j^{+}\tau_{j+1}^{-}+t_3^{-1}\tau_j^{-}\tau_{j+1}^{+}\right]+\frac{1}{4}(1+\sigma_j^z)(1+\sigma_{j+1}^z)$$
$$\times[\tfrac{1}{2}(1+\tau_j^z\tau_{j+1}^z)+t_1^{-1}\tau_j^{+}\tau_{j+1}^{-}+t_1\tau_j^{-}\tau_{j+1}^{+}]+\tfrac{1}{4}(1-\sigma_j^z)(1-\sigma_{j+1}^z)[\tfrac{1}{2}(1+\tau_j^z\tau_{j+1}^z)+t_2\tau_j^{+}\tau_{j+1}^{-}+t_2^{-1}\tau_j^{-}\tau_{j+1}^{+}].$$

This Hamiltonian can be mapped to [see Eq. (7)]

$$\hat{H}^{g}=\sum_{j=1}^{L}\,[\hat{h}^{g}_{j,j+1}-2JX^{00}_{j}], \tag{11}$$

where

$$\begin{aligned}\hat{h}^{g}_{j,j+1}=&\sum_{\alpha=0}^{3}X^{\alpha\alpha}_{j}X^{\alpha\alpha}_{j+1}+X^{20}_{j}X^{02}_{j+1}+X^{02}_{j}X^{20}_{j+1}+t_1(X^{10}_{j}X^{01}_{j+1}\\&+X^{12}_{j}X^{21}_{j+1})+t_2(X^{30}_{j}X^{03}_{j+1}+X^{32}_{j}X^{23}_{j+1})+t_3X^{31}_{j}X^{13}_{j+1}\\&+t_1^{-1}(X^{01}_{j}X^{10}_{j+1}+X^{21}_{j}X^{12}_{j+1})+t_2^{-1}(X^{03}_{j}X^{30}_{j+1}\\&+X^{23}_{j}X^{32}_{j+1})+t_3^{-1}X^{13}_{j}X^{31}_{j+1}.\end{aligned}$$

Using the algebraic nested Bethe ansatz method, this model can be exactly solved. The energy eigenvalues of the Hamiltonian (10) are also given by Eq. (2) while the Bethe ansatz equations reads

$$t_1^{(L-M_3)}t_2^{M_3}t_3^{-M_3}\left(\frac{\lambda_j-i/2}{\lambda_j+i/2}\right)^{L}=\prod_{l\neq j}^{M_1}\frac{\lambda_j-\lambda_l-i}{\lambda_j-\lambda_l+i}\prod_{\alpha=1}^{M_2}\frac{\lambda_j-\mu_\alpha+i/2}{\lambda_j-\mu_\alpha-i/2},$$

$$t_1^{(L-M_3)}t_2^{M_3}t_3^{-M_3}\prod_{\beta\neq\alpha}^{M_2}\frac{\mu_\alpha-\mu_\beta-i}{\mu_\alpha-\mu_\beta+i}=\prod_{j=1}^{M_1}\frac{\mu_\alpha-\lambda_j-i/2}{\mu_\alpha-\lambda_j+i/2}\prod_{\delta=1}^{M_3}\frac{\mu_\alpha-\nu_\delta-i/2}{\mu_\alpha-\nu_\delta+i/2},$$

$$t_1^{(M_2-M_1)}t_2^{-(L-M_1+M_2)}t_3^{-(M_1-M_2)}\prod_{\gamma\neq\delta}^{M_3}\frac{\nu_\delta-\nu_\gamma-i}{\nu_\delta-\nu_\gamma+i}=\prod_{\alpha=1}^{M_2}\frac{\nu_\delta-\mu_\alpha-i/2}{\nu_\delta-\mu_\alpha+i/2}.$$

The physics of the integrable model presented here is expected to be much richer, since the presence of these extra parameters will certainly influence the phase diagram of the model. More details of this model are being studied and the results will be shown in a future work.

To summarize, we have introduced a generalization of Wang's spin-ladder model based on the $SU(4)$ symmetry. This was achieved by introducing one extra parameter into the system without violating integrability. The Bethe ansatz equations as well as the energy expression of the model are presented. We show also that the model has a gap that depends on the free parameter and the critical point, and the phase diagram was obtained. We note that our model with one free parameter is a special case of a more general integrable Hamiltonian that has three free parameters. A comprehensive analysis of the general model will be undertaken in future work.

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