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On the synchronization of two metronomes and their related dynamics

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Abstract. Synchronization was first reported by Christiaan Huygens in 1665 when he observed anti-phase synchronization achieved by two pendulum clocks hanging on a common base. Since then researchers have tried to understand the results reported by Huygens using their own ways to reproduce his experiment and applying several methods of analysis. Each researcher has reported different results, even compared with those reported by Huygens. In this paper a simple model is proposed to study in-phase and anti-phase synchronization of two metronomes based on a normal mode analysis using van der Pol oscillators. The instantaneous frequency of the responses from both simulations and experimental data is used in the analysis. Unlike previous studies, measurements are made using videos and the time domain responses of the metronomes extracted by means of tracking software. Plots showing how the initial conditions lead to both synchronization states are also presented.

1. Introduction

Synchronization means occurring in the same time [1]. It comes from the interaction between two or more bodies. Since the first known report of synchronization in 1665 from Christiaan Huygens, who showed that two pendulum clocks hanging from a wooden movable bar always synchronize so that their pendulums were in anti-phase, many researchers have tried to reproduce and understand his results. Bleckhman [2] studied a laboratory reproduction of Huygens's coupled clock system, comprised of two pendulums connected to a rigid frame, which were constrained to oscillate in one dimension. He modeled the system as a three-degree-of-freedom system and used van der Pol damping [3-4] to represent the escapement mechanism of the clocks. Bleckhman predicted that in-phase and anti-phase synchronization states are achievable and are stable under the same set of circumstances. Bennett et al. [5] studied a system composed of two pendulums hanging from a common movable frame free to move in one dimension. The pendulums were driven by an escapement mechanism. They found that the clocks synchronize in anti-phase when the system mass ratio is comparable with that reported by Huygens and conclude from numerical simulations that in-phase synchronization is an unstable state, while anti-phase synchronization and oscillation-death are stable synchronization states. Shortly after, Pantaleone [6] studied a system composed of two metronomes resting on a light wooden board resting on two empty soda cans. He modelled the metronomes using van der Pol oscillators and the coupling between them from an undamped motion of a mass-like base. In contrast with Bennett et al [5] he concluded that the in-phase synchronization state is a stable state and suggested that anti-phase synchronization is unstable, he stated that anti-



phase synchronization can be produced in a metronome system by either adding large damping to the base motion, or by having very large oscillation frequencies. More recently additional work, including simulations and experiments, have been conducted by several researchers, for example [7-10] who have proposed systems intended to reproduce Huygens's original experience, and different models to analyze those systems. Most of them appear to be complex, leading to results and predictions also difficult to be understood and to compare their relationship with the previous research.

In this paper a simple normal-mode model is proposed to study the in-phase and the antiphase synchronization of two oscillating metronomes positioned on top of a movable wooden base. van der Pol damping is used to model the escapement mechanism in the metronomes. From numerical simulations, the relationship between the final synchronization state, the relevant parameters and the initial conditions (IC) is explored. Videos together with some software [11] were used to collect data from experiments. A method based on the Instantaneous Frequency (*IF*) [12] concept is used to determine whether synchronization occurs, and to observe what happens to the frequency of the oscillators before and after synchronization.

2. The model for in-phase and antiphase Synchronization

The in-phase and the anti-phase synchronization of the two metronomes on top of a mobile wooden base, as shown in figure 1, is of interest. The metronomes are self-excited oscillators that use potential energy stored in a spring to create oscillating motion of their pendulums. Their dynamics are coupled through the wooden base. After an initial transient they maintain a stable oscillation, with a constant period – a limit cycle.

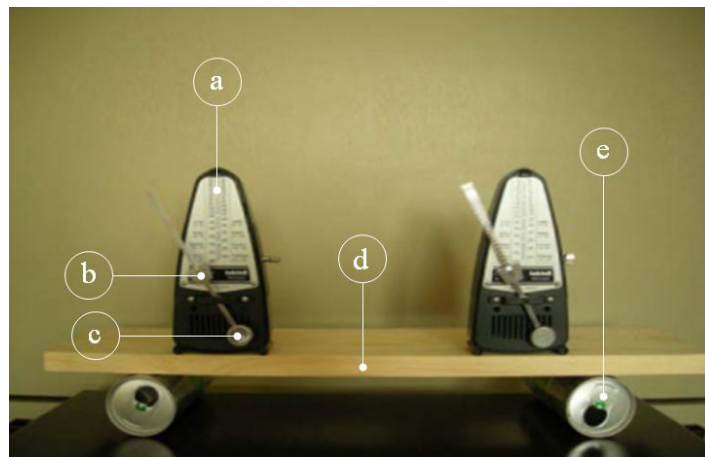


Figure 1. Two oscillating metronomes supported by a mobile wooden base resting on empty soda cans to permit the base motion in the horizontal direction [13]. (a) Metronome, (b) mass in the metronome to set up the oscillation frequency, (c) pendulum bob, (d) wooden base and (e) empty soda can.

This system can be represented crudely by a simple model if just the horizontal motion of the pendulum's bobs is taken into account and the masses of the metronomes and the base are considered as concentrated masses coupled by mean of linear springs. Such as system is shown in figure 2, where x_1 , x_2 and x_M represent the displacements of the masses of the metronome pendulum bobs m_1 , m_2 , and the displacement of the wooden base of mass M , respectively. In the analysis conducted here it is assumed that $m = m_1 = m_2$ and $k = k_1 = k_2$, which means that in the steady-state both the metronomes oscillate at the same frequency.

Initially neglecting the van der Pol damping terms c_{vPol} , the system of equations for this model is given by

$$\begin{bmatrix} m & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & m \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_M \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{bmatrix} \begin{Bmatrix} x_1 \\ x_M \\ x_2 \end{Bmatrix} = 0 \quad (1)$$

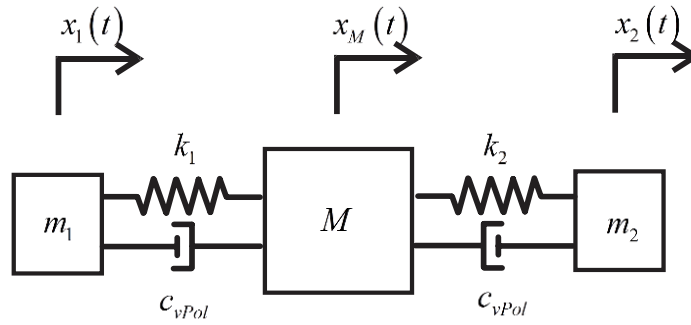


Figure 2. 3DOF system representing the masses of two metronomes coupled to a base in the centre by mean of a linear springs and van der Pol dampers.

Assuming harmonic motion of the form $x = X e^{j\omega t}$ in which $j = \sqrt{-1}$, and introducing the non-dimensional parameters $\Omega = \omega / \omega_n$ in which $\omega_n = (k/m)^{1/2}$, and $\sigma = m/M$ equation (1) can be written as

$$\begin{bmatrix} 1 - \Omega^2 & -\Omega^2 & 0 \\ -\Omega^2 & 2 - \Omega^2 / \sigma & -\Omega^2 \\ 0 & -\Omega^2 & 1 - \Omega^2 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_M \\ X_2 \end{Bmatrix} = \mathbf{0} \quad (2)$$

From equation (2) the undamped natural frequencies and their associated mode-shapes are given by

$$\Omega_1 = 0, \quad \Omega_2 = 1, \quad \Omega_3 = \sqrt{1 + 2\sigma}$$

$$\mathbf{x}_1 = \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}, \quad \mathbf{x}_2 = \begin{Bmatrix} 1 \\ 0 \\ -1 \end{Bmatrix}, \quad \mathbf{x}_3 = \begin{Bmatrix} 1 \\ -2\sigma \\ 1 \end{Bmatrix} \quad (3a,b,c)$$

The first mode corresponds to rigid-body motion and is not of interest here. The second mode corresponds to anti-phase motion of the metronome masses in which there is no motion of the base and the third mode corresponds to in-phase motion of the metronome masses with a (small) anti-phase motion of the base, the amplitude of which is dependent upon the mass ratio σ . The second mode corresponds to the case when there is anti-phase synchronisation and the third mode corresponds to in-phase synchronisation. Equation (3b) suggests that the frequency of oscillation of the system is higher when the system is synchronized in-phase than when it is synchronized in antiphase by a factor which depends on the mass ratio σ .

To obtain in-phase and anti-phase synchronization from this model (without the van der Pol dampers) it is necessary for the ICs to correspond exactly to the normal modes equations (3b) and (3c). When the ICs do not correspond to the normal modes the system never achieves synchronization. Instead it exhibits a beating behaviour.

The escapement mechanism which inputs energy into the metronome pendulum is crudely modelled by van der Pol damping $c_{vPol} = c(1 - (\Delta x)^2)$ in which c is a linear viscous damping

coefficient and Δx is the relative displacement across the damper, as in [6]. This type of damper supplies energy to the system when the relative displacement is small and dissipates energy when the relative displacement is large. The van der Pol damping term can be written in non-dimensional form as $-\mu(1-(\Delta\hat{x})^2)$ in which $\mu = c/(m\omega_n)$ and where $\hat{x} = x/x_0$ is the dimensionless displacement in which x_0 is the displacement where the damping force equals zero. Note that for the metronome model $\mu \ll 1$ so that their coupled motions are almost sinusoidal in the steady-state [6]. To illustrate the behavior of a single van der Pol oscillator from an initial position of rest, figure 3 shows its non-dimensional displacement as a function of non-dimensional time. It can be seen that after an initial transient the van der Pol oscillator reaches a stable oscillation with quasi-harmonic oscillations, which is similar to the behaviour of a metronome. Note also that the maximum non-dimensional displacement has a maximum amplitude of 2 in the steady-state [4].

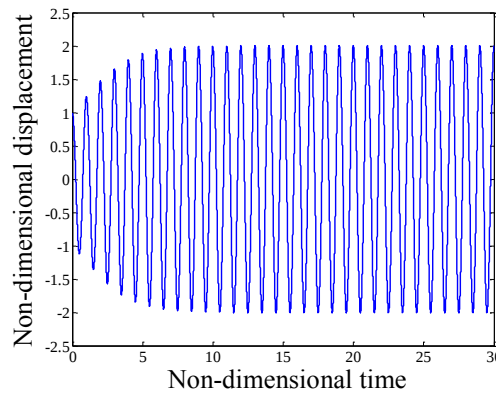


Figure 3. Time-domain behaviour for a single van der Pol oscillator with $\mu = 0.1$.

Including the van der Pol dampers into the model given by equation (1) results in the following equation which is written in non-dimensional form, in which $\hat{x}' = d\hat{x}/d\tau$ and $\hat{x}'' = d\hat{x}'/d\tau$ are the non-dimensional velocity and acceleration

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\sigma & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \hat{x}_1'' \\ \hat{x}_2'' \\ \hat{x}_3'' \end{Bmatrix} - \mu \begin{bmatrix} 1-(\hat{x}_1-\hat{x}_2)^2 & -\{1-(\hat{x}_1-\hat{x}_2)^2\} & 0 \\ -\{1-(\hat{x}_2-\hat{x}_1)^2\} & 2-(\hat{x}_2-\hat{x}_1)^2-(\hat{x}_2-\hat{x}_3)^2 & -\{1-(\hat{x}_2-\hat{x}_3)^2\} \\ 0 & -\{1-(\hat{x}_3-\hat{x}_2)^2\} & 1-(\hat{x}_3-\hat{x}_2)^2 \end{bmatrix} \begin{Bmatrix} \hat{x}_1' \\ \hat{x}_2' \\ \hat{x}_3' \end{Bmatrix} + \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \end{Bmatrix} = \mathbf{0} \quad (4)$$

Equation (4) can be solved numerically for different values of the parameters μ and σ and for a large range of ICs. Figures 4a and b illustrate the time domain displacements for each of the masses in system shown in figure 2 when $\mu = 0.1$ and $\sigma = 0.1$. Figure 4(a) corresponds to in-phase synchronisation and figure 4(b) corresponds to antiphase synchronization. Zoom plots for these graphs can also be seen in these figures. The final synchronized state of the system depends on the combination of ICs and the values of the parameters μ and σ . In Section 4 the relationship between the final synchronization state, the ICs and some different parameters is further investigated.

3. Instantaneous Frequency and Synchronization

Synchronization of two or more coupled oscillators is defined from the similarity of its periods, or frequencies. In this work the IF is calculated from the displacement time histories of the masses of the oscillators. In this section it is illustrated on the responses of the model shown in figure 2, and in the

next section the method is applied to experimental measurements from two metronomes configured as shown in figure 1.

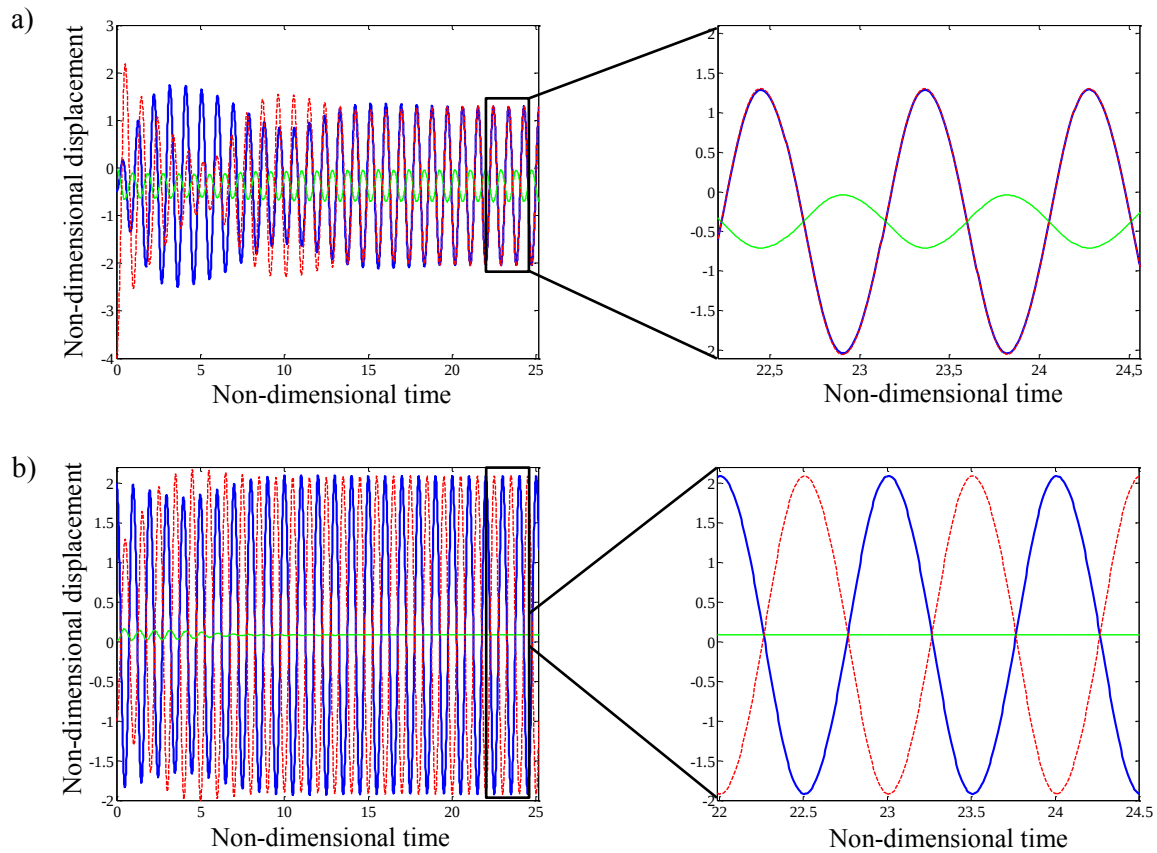


Figure 4. In-phase and antiphase synchronization of the 3 DOF system with linear springs and van der Pol damping when $\mu=0.1$ and $\sigma=0.1$. (a) In-phase synchronization and zoom of its steady state respectively for ICs of $x_1 = -0.5$, $x_2 = -4$, $x_M = 0$ and $\dot{x}_i = 0$. (b) antiphase synchronization and zoom of its steady state respectively for ICs of $x_1 = 2$, $x_2 = -1$, $x_M = 0$ and $\dot{x}_i = 0$. Solid blue line — m_1 , dashed red line — m_2 and solid thin green line — M .

The IF of a signal is calculated in a process which uses the Hilbert Transform (HT). The HT enables the analytic signal to be calculated which has real and imaginary parts in the time domain [11]. The analytic signal $w(t)$ is given by

$$w(t) = x(t) + j\tilde{x}(t) \quad (5)$$

where $x(t)$ is the original signal and $\tilde{x}(t)$ is the HT of $x(t)$. The instantaneous phase is first obtained by $\varphi(t) = \arctan(\tilde{x}(t)/x(t))$. The IF is related to this by $IF = d\varphi(t)/dt$, which can be calculated approximately from measured (or simulated) discrete time data at time t_i by

$$IF(t_i) = \frac{\varphi_{i+1}(t) - \varphi_i(t)}{t_{i+1} - t_i} \quad (6)$$

If the IF is calculated for a sinusoidal function, it will be constant with a value corresponding to the frequency of the sine wave. If the IF is calculated for a periodic function comprising two or more harmonics the IF will oscillate about the value of the fundamental frequency. For example if the signal

has a fundamental frequency of ω with an amplitude A , and a third harmonic with a frequency of 3ω with an amplitude of B , the IF is given by

$$IF = \omega + \alpha \cos 2\omega t \quad (7)$$

where $\alpha = 2\omega B/A$. In the case of synchronization of metronomes the relevant frequency is the fundamental frequency, which is also the frequency of synchronization and the oscillation about the fundamental frequency will be due to higher harmonics. In general equation (8) can be written as

$$IF = \omega + \alpha_n \cos(n-1)\omega t \quad (8)$$

where $\alpha_n = (n-1)\omega B/A$, in which n is the number of the harmonic being added to the fundamental frequency.

In-phase and anti-phase synchronization are obtained for the system in figure 2 by choosing appropriate ICs. For the synchronized system (both in-phase and anti-phase states) with parameters set at $\mu = 0.1$ and $\sigma = 0.1$, the IF is calculated from the displacements of the masses m_1 , m_2 and M , and is shown in figures 5(a) and (b) respectively together with zoomed graphs. An IF of 1 corresponds to the undamped natural frequency of the anti-phase motion oscillator. It can be seen in figure 5(a) that the IF oscillates about the mean value $IF = 1.1$, which corresponds to the undamped natural frequency of the in-phase mode. The oscillation of the IF indicates the presence of small higher harmonics in the displacement signals. Figure 5b shows the IF of the displacement of mass m_1 and m_2 when the system has synchronised so that it is in anti-phase. The IF of M is zero as it has no motion. It can be seen that the IF oscillates about 1.

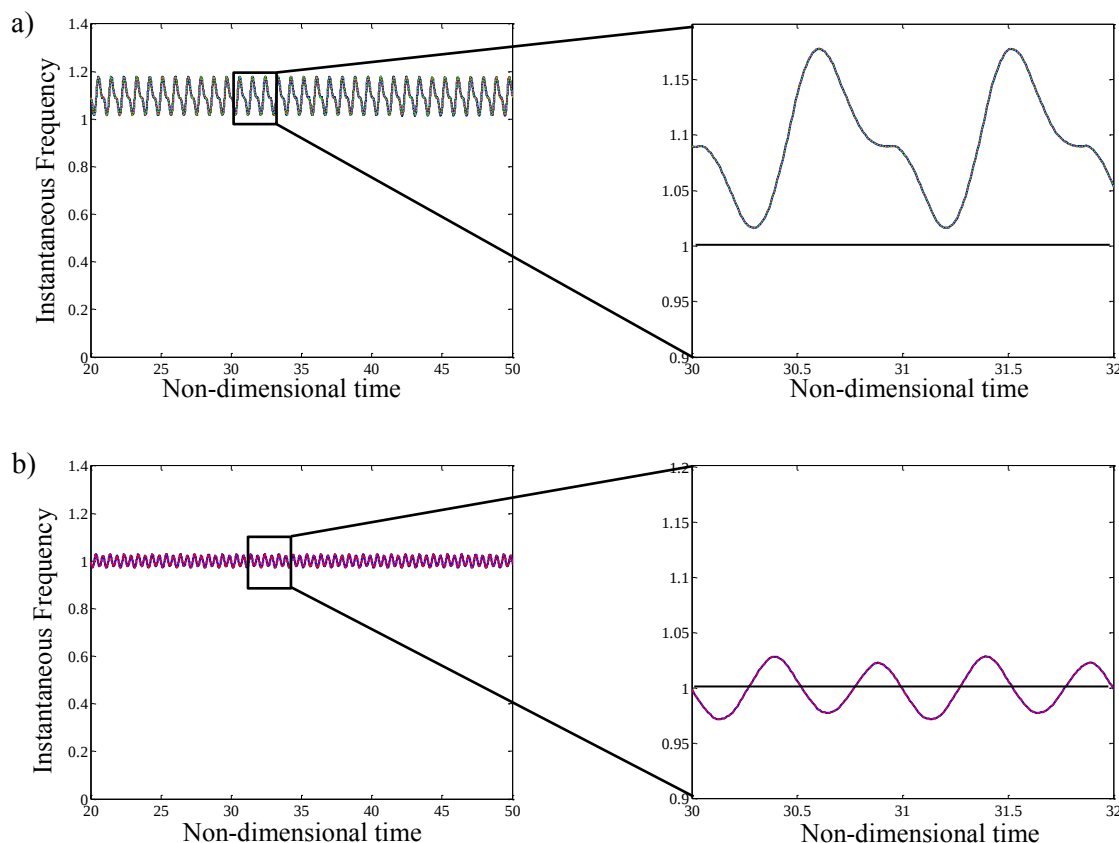


Figure 5. Instantaneous frequency for the displacements of m_1 , m_2 and M when the system is in-phase and anti-phase synchronization with $\mu = 0.1$ and $\sigma = 0.1$. (a) IF for m_1 , m_2 and M when the system has synchronised in-phase. (b) IF for the displacements of m_1 and m_2 when the system has synchronised in anti-phase the motion of $M = 0$. Solid blue line — IF of m_1 , dashed red line — IF of m_2 and solid thin green line — IF for M .

4. Experiments

In this subsection the *IF* approach is applied to experimental data. Experiments were made with two metronomes on top of a wooden base resting on two soda cans, as shown in figure 1. The metronomes are Wittner Takell-Piccolo (Serie 830); the mass of each metronome is 146 g with a pendulum bob mass of 30 g. The mass of the wooden base is 118.3 g, giving a mass ratio between the pendulum bob and the equivalent base mass (base mass and the difference between the metronome's mass and the pendulum bob mass) of $\sigma = 0.12$.

Videos of the experiments were taken with a digital Panasonic DMC-FS12 camera with 848×480 pixels and 30frames per second. From the videos the horizontal displacement data were extracted using Tracker[®] software [11]. The software permits easy extraction of the displacement, velocity, acceleration and time data from videos of the experiment. The experiments were carried out in two steps, first videos taken of the motion of each metronome alone positioned on a rigid base, both oscillating at 1.67 Hz (200 bpm). The metronomes were then positioned on the wooden base and initial conditions set to obtain in-phase synchronization. The motion of the system was then captured by video. These experiments were then repeated with the metronomes set to a frequency of 1.6 Hz (192 bpm) and with different IC to obtain antiphase synchronization.

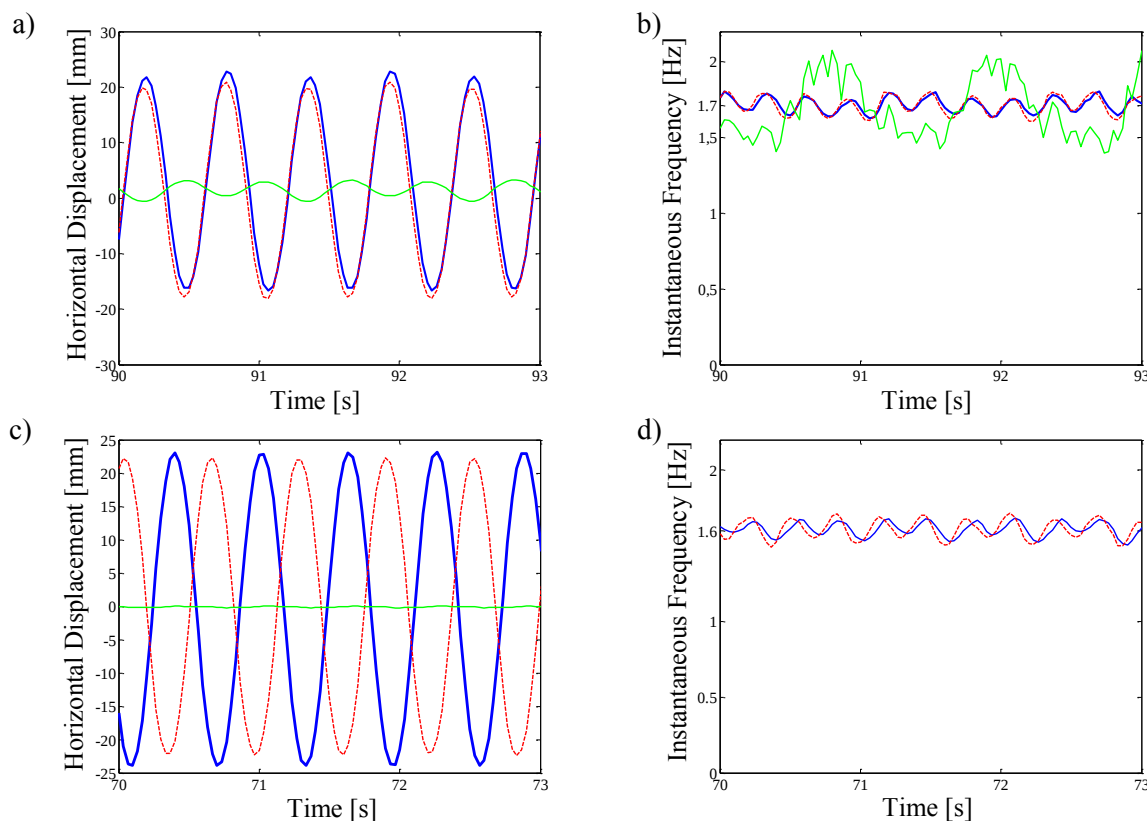


Figure 6. Experimental results showing the horizontal displacements of the metronomes on the wooden base and their *IF*s. (a) metronomes set to oscillate at 1.67 Hz and exhibiting in-phase synchronization with IC of $x_1 = -18.15$, $x_2 = -13.83$, $x_M = -1.83$ [mm] and $\dot{x}_i = 0$. (b) *IF* from (a). (c) metronomes set to oscillate at 1.6 Hz and exhibiting anti-phase synchronization with IC of $x_1 = 15.83$, $x_2 = 15.24$, $x_M = 0$ [mm] and $\dot{x}_i = 0$. (d) *IF* from (c). Solid blue line $-m_1$, dashed red line $-m_2$ and solid tiny green line $-M$.

The results for the coupled motion of the metronomes are shown in figure 6. A zoom of the time history of the displacements of each metronome oscillating at an initial set up frequency of 1.67 Hz is shown in figure 6(a) and the corresponding *IF* is shown in figure 6 (b). It can be seen that for the metronomes oscillating in-phase there is the presence of small amplitude higher harmonics in the

responses. Examining figure 6b, it can be seen that the synchronization frequency is slightly higher than the initial set up frequency. A zoom of the time history of the displacements of each metronome oscillating at an initial set up frequency of 1.6 Hz is shown in figure 6(c) and the corresponding *IF* is shown in figure 6 (d). It can be seen that now the metronomes are oscillating in anti-phase. As in the previous case there is also the presence of small amplitude higher harmonics. Note that to obtain both kinds of synchronization two different set up frequencies and two different IC were used. In the next section the relationship between the parameters, ICs and the final synchronization state is studied.

5. Basins of attraction for synchronization

It has been shown above that the model in figure 2 can capture the qualitative behaviour of two couple metronomes. The model is now used to investigate how the initial conditions affect whether the system synchronizes so that it is in-phase or anti-phase. For a given set of ICs and for a given set of system parameters it is possible to determine the final synchronization state of the model shown in figure 2 by numerically integrating equation (5). There are six ICs, three for the initial displacements of m_1 , m_2 and M , and their corresponding initial velocities. If all the velocities are considered to be zero and the initial displacement of M is set to zero, only the initial displacements for m_1 and m_2 need to be considered.

Two different cases are considered. In the first case five basins of attraction are shown for five different values for μ , while the mass ratio is set so that $\sigma = 0.1$. In the second case, five basins of attraction are shown for five different values for the mass ratio σ while the damping ratio is set so that $\mu = 0.1$. Note that the oscillators are the same, which represents the case when the natural frequency of each metronome are equal.

Figure 7 shows the basins of attraction for a range of non-dimensional initial displacements for m_1 and m_2 when the damping parameter increases from $\mu = 0.1$ up to $\mu = 0.5$. There is a coloured pixel for each combination of initial conditions. A Light green pixel represents in-phase synchronization for the specific combination, and a dark blue pixel represents anti-phase synchronization. The results in figure 7(a) show that the final states of synchronization are almost symmetric, suggesting that, when $\sigma = 0.1$, and the system has small damping the final synchronization state generally depends on the sign of the initial displacements. It can be seen that ICs with the same sign generally lead to in-phase synchronization, while ICs with a different sign generally lead to anti-phase synchronization.

An increase of the damping in the system causes that the system to bias so that it synchronizes more often in anti-phase than in-phase, as shown for example in figure 7b. By further increasing the damping in the system, this process is reversed and for the same range of ICs the system reaches anti-phase synchronization more often than in-phase synchronization, as shown in figures 7(c-e). This suggest that in-phase synchronization is easier to obtain for large damping values, when $\sigma = 0.1$. Note also the increasing complexity of the basins of attraction as the damping values is increased.

In figure 8 are shown the basins of attraction for five different values of the mass ratio σ while $\mu = 0.1$ is constant. These values for σ are equivalent to a base mass M equal to 50, 20, 10, 5 and 1 times larger than m . Figure 8(a) shows that for this level of damping and for a large base mass compared to the metronomes, anti-phase synchronization is achieved more often than in-phase synchronization. There is also an interesting feature in the basin of attraction shown in figure 8(a) for the set of ICs near the origin. For a base mass of 20 or 10 times larger than m , the in-phase and anti-phase synchronized states occur almost equally often as shown in figures 8(b) and (c) When M is 5 times m again the anti-phase state start to become more common, and is even more dominant when $M = m$. This simulation suggests that for light damping the anti-phase synchronization state is more common than the in-phase state. The lighter the base the more dominant the anti-phase state becomes.

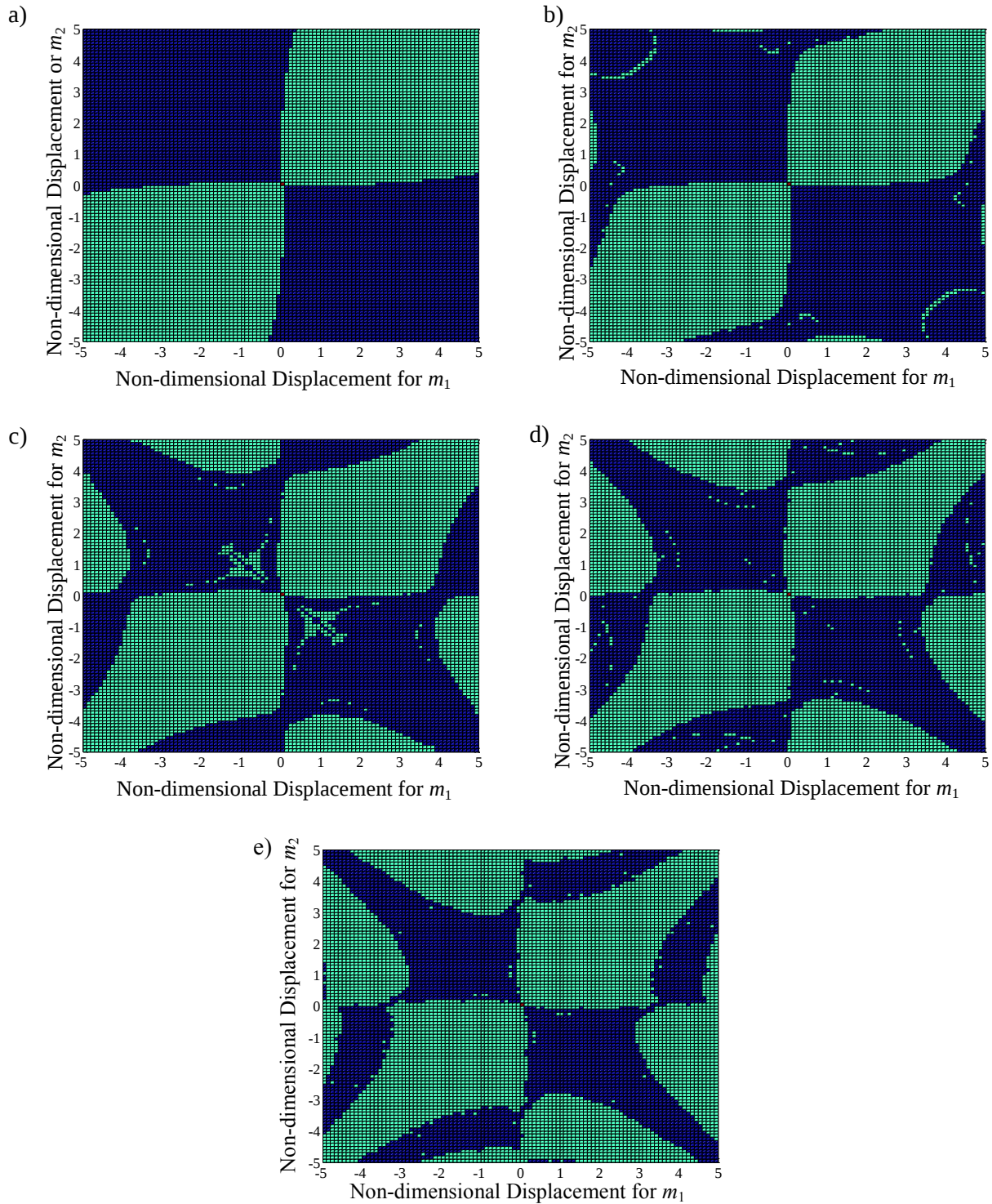


Figure 7. Basins of attraction showing when the system is attracted to either in-phase or antiphase synchronization state for five different damping values μ and for a constant mass ratio $\sigma = 0.1$. The horizontal axis are initial positions for m_1 and the vertical axis are initial positions for m_2 , when $x_M = 0$ and $\dot{x}_i = 0$. (a) $\mu = 0.1$, (b) $\mu = 0.2$, (c) $\mu = 0.3$, (d) $\mu = 0.4$, (e) $\mu = 0.5$. Green (light) dots – in-phase synchronization, blue (dark) dots – antiphase synchronization.

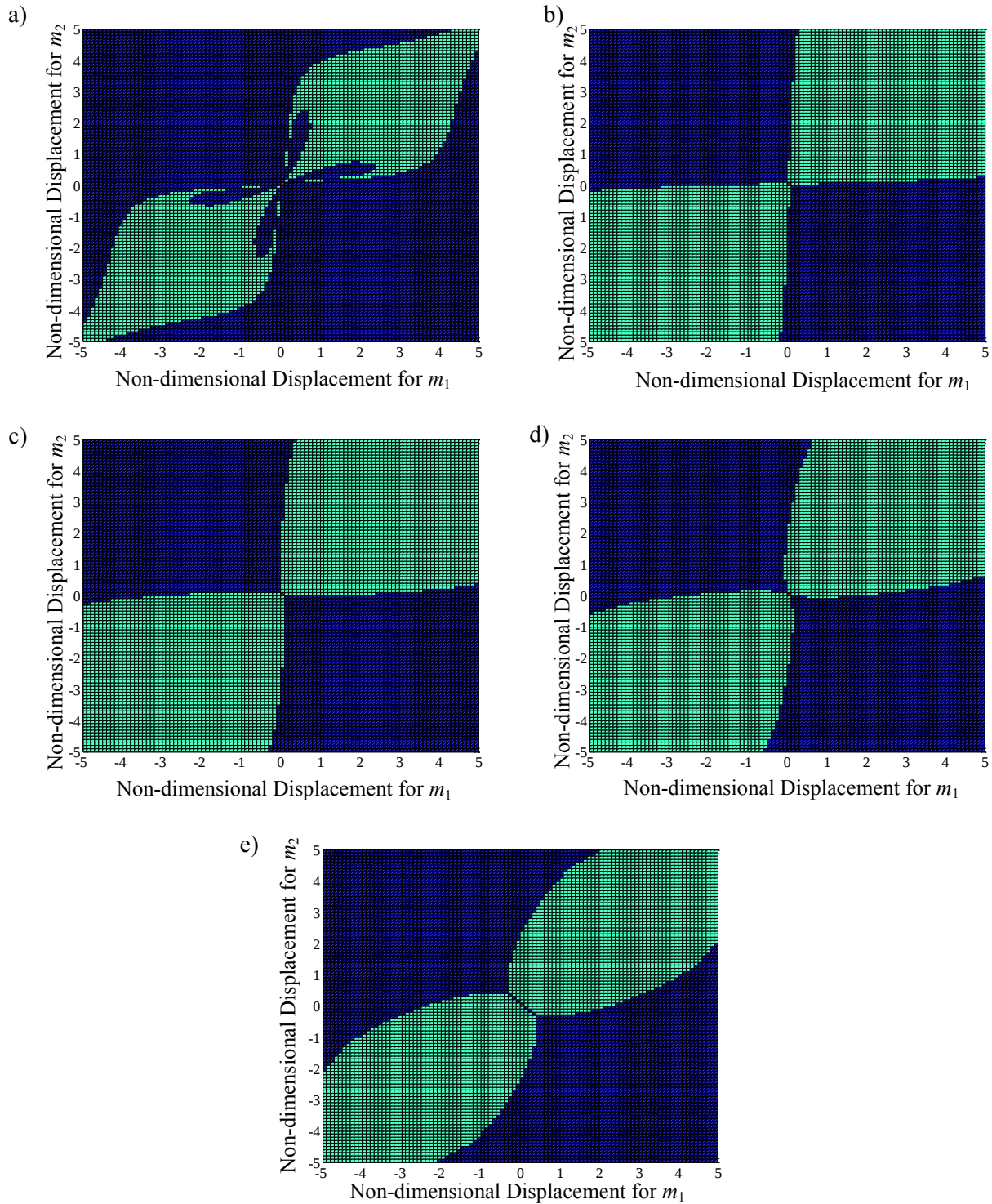


Figure 8. Basins of attraction showing when the system is attracted to either in-phase or antiphase synchronization state for five different mass ratio values σ and for a constant damping $\mu=0.1$. The horizontal axis are initial positions for m_1 and the vertical axis are initial positions for m_2 , when $x_M = 0$ and $\dot{x}_i = 0$. (a) $\sigma=0.02$, (b) $\sigma=0.05$, (c) $\sigma=0.1$, (d) $\sigma=0.2$, $\sigma=1$. Green (light) dots – in-phase synchronization, blue (dark) dots – antiphase synchronization.

6. Conclusions

In this paper a simple normal modes model with van der Pol damping has been proposed to study the in-phase and antiphase synchronization of two metronomes. Numerical simulations have shown that from this model it is possible to obtain both of the synchronization states. Experiments were also conducted using two couple metronomes positioned on a wooden base. A camera was used to capture videos of the motion of the system, and time histories of the displacements of the metronomes and the base were extracted from the videos using Tracker[®] software. By comparing the experimental results with those from the model, it was shown the model can capture the qualitative behaviour of the coupled metronomes. The model was then used to investigate the way in which the initial conditions affect the final synchronized state of the system – whether the metronomes oscillate in-phase or in anti-phase. It was found that for a mass ratio of $\sigma = 0.1$ and considering similar oscillators, that ICs with the same sign generally lead to in-phase synchronization, while ICs with a different sign generally lead to anti-phase synchronization. Increasing the damping but maintain the same mass ratio results in a bias so that the system synchronizes more often in anti-phase than in-phase. By further increasing the damping in the system, this process is reversed and for the same range of ICs the system reaches anti-phase synchronization more often than in-phase synchronization.

It was also found that for a damping value $\mu = 0.1$ and for a base mass ranging from 1 to 50 times the metronome masses results in the situation where anti-phase synchronization state is more common than in-phase synchronisation.

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