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The Dirac equation in a non-Riemannian manifold. I. An analysis using the complex algebra

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The Dirac wave equation is obtained in the non-Riemannian manifold of the Einstein–Schrödinger nonsymmetric theory. A new internal connection is determined in terms of complex vierbeins, which shows the coupling of the electromagnetic potential with gravity in the presence of a spin-½ field.

I. INTRODUCTION

The Einstein–Schrödinger (ES) nonsymmetric theory[1] was an attempt to geometrize, in an unitary way, the gravitational and the electromagnetic fields. However, structure problems in the theory have not permitted a coherent interpretation of the field equations. On the other hand, the technique of geometrizing fields introduced by Einstein continues to be a useful tool in classical field theory. Actually there has been some work[2] where the electromagnetic field is made explicit; this is accomplished by first taking the skew-symmetric part of the metric as being proportional to the electromagnetic tensor and, then, adding a sourcelike term to the Lagrangian of the theory. This permits one to reobtain the Einstein–Maxwell equations through a correspondence principle. When we face the problem of developing a Dirac theory using the space-time manifold of the ES nonsymmetric theory, we use then complex vierbeins. This is equivalent to introducing an internal **C** space in the manifold of the general relativity theory. However, we have the problem of how to obtain a coherent mechanism that fits the correct rules for the transformation group. We can see, for example, that the way the problem was developed by Marques and Oliveira,[4] in a study of the geometrical properties of **C**, **Q**, and **O** tangent spaces, does not permit the corresponding Dirac field equations to exist (we will consider the complex case presently). The reason is that the internal **C** connection was ignored when they introduced complex vierbeins. Instead, they generalize the tangent real connection (the one originating on the local real-tangent space) to a complex one. This induced them to generalize both the Lorentz group (to a pseudo-unitary group) and its representation U(L), for the generalized Dirac field theory. However, in spite of it being possible to show that, on the local tangent space, the trace of the symmetric part of the tangent connection should correspond to the electromagnetic field, there is no way to obtain the desirable correspondence to an (actual **R**) Dirac field theory. The problem can be solved by considering besides the tangent connection, the connection corresponding to an internal **C** space. This forces us to maintain the Lorentz group as being that of the (local) space-time transformations on the (local) tangent space. On the other hand, we must also have an "internal" transformation, corresponding to the internal **C** space.

In the space-time of general relativity it is possible to generalize the Dirac field equation by doing the transition

$$\gamma^\mu \to \gamma^\mu(x)\,, \quad \psi_{,\mu} \to \psi_{\|\mu} = \psi_{,\mu} + \Delta_\mu \psi\,,$$

where $\psi(x)$ is now the electron wave function in the curved space-time, and Δ_μ is the geometrical connection with relation to the internal space generated by the constant γ matrices $\{\Gamma_i\}$. It is easy to show that Δ_μ is given by

$$\Delta_\mu = \tfrac{1}{8}([\gamma^\nu,\gamma_{\nu,\mu}] - \{^{\rho}_{\mu\nu}\}[\gamma^\nu,\gamma_\rho])\,, \tag{1.1}$$

where $\{^{\rho}_{\mu\nu}\}$ are the Christoffel symbols for the space-time connection. In terms of real vierbeins $h_a^{\ \nu}$ (and their inverses $h^a_{\ \nu}$), (1.1) is written as

$$\Delta_\mu = (-1/4i)(h^{\nu b} h^{\ a}_{\nu,\mu} - \{^{\rho}_{\mu\nu}\} h^{\ a}_{\rho} h^{\nu b})\sigma_{ab}\,. \tag{1.2}$$

The function $\psi^i(x)$, $i = 1,...,4$, above, satisfies a Dirac equation defined on the curved space-time manifold of general relativity, which now has the form

$$\gamma^\mu(x)(\psi_{,\mu} + \Delta_\mu \psi) - \mu\psi = 0\,, \tag{1.3}$$

where μ is a mass coefficient. Thus the gravitational field is present in this equation through the connection Δ_μ.

The nonsymmetric manifold of the ES theory with the locally defined complex vierbeins referred to above will be used in this work. These vierbeins define new Fock–Ivanenko coefficients which permit the construction of the corresponding Dirac equations related to the non-Riemannian manifold of the ES theory. In Sec. II we will present briefly the properties of the complex tangent space as well as the corresponding field equations obtained in the ES nonsymmetric theory. In Sec. III the generalized Fock–Ivanenko coefficients will be determined, as well as the new Dirac equation. In Sec. IV we will proceed to their analysis. Throughout, we will use the $\mu,\nu,...$ indices as those on the non-Riemannian manifold; the $a,b,...$ indices will be those on the complex tangent space.

II. THE COMPLEX TANGENT SPACE

According to the correspondence principle there exists, at each point of the curved space-time of general relativity, a local tangent space[5] with the structure of a flat space-time, with the metric given by the Minkowski tensor η_{ab}. Therefore, we must have the line element

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$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu = \eta_{ab}\, dx^a\, dx^b$, locally, where, $g_{\mu\nu} = g_{\nu\mu}$.

In the ES nonsymmetric theory, the metric of curved space-time has the symmetry property $g^*_{\mu\nu} = g_{\nu\mu}$, $g_{\mu\nu} = g_{\underset{\vee}{\mu\nu}} + ik_{\underset{\vee}{\mu\nu}}$. Defining complex vierbeins e^a_μ (and their inverse e^μ_a), we have that[6]

$$g_{\mu\nu} = e^{*a}_{\ \nu} e^b_\mu \eta_{ab}, \tag{2.1}$$

$$g^{\mu\nu} = e^{*\mu}_{\ a} e^\nu_b \eta^{ab}, \tag{2.2}$$

where η_{ab} (and its inverse η^{ab}) is the metric of the tangent space which we take here as the Minkowski tensor. The metric $g_{\mu\nu}$ and its inverse $g^{\mu\nu}$ are such that $g^{\mu\nu} g_{\sigma\nu} = \delta^\mu_\sigma$, where the order of indices are significant. From there, we obtain the orthogonality conditions for the complex vierbeins:

$$e^{*a}_{\ \mu} e^\mu_b = e^a_\mu e^{*\mu}_{\ b} = \delta^a_b, \tag{2.3}$$

$$e^{*a}_{\ \mu} e^\nu_a = e^a_\mu e^{*\nu}_{\ a} = \delta^\nu_\mu. \tag{2.4}$$

As is well known, the transformation law for vectors in the complex tangent space, local to a curved space-time, is defined by

$$e'^a_\mu(x) = L^a{}_b(x) e^b_\mu(x), \tag{2.5}$$

where $L^a{}_b$ are the Lorentzian rotation matrices, which have the property

$$L^T \eta L = \eta \tag{2.6}$$

and, as $e^a_\mu(x)$ is a complex function, $e^a_\mu = e^a_{\mu_R} + i e^a_{\mu_I}$. Then

$$\bar{e}^a_\mu(x) = e^a_{\mu_R} - i e^a_{\mu_I} \tag{2.7}$$

is the conjugate of e^a_μ. This means that we have attached to the Minkowskian tangent space, an "internal space," the **C** space. The "internal" transformation law of an object of the **C** space, K, is

$$K' = U(1) K, \tag{2.8}$$

where $U(1)$ stands for a unitary 1×1 (local) transformation matrix, $U(1) = e^{i\phi(x)}$, and

$$\bar{K}' = \bar{U}(1) \bar{K}, \tag{2.9}$$

where $\bar{U}(1) = U^{-1}(1) = e^{-i\phi(x)}$. A more general transformation law for the complex vierbeins now should be

$$e'^a_\mu(x) = U(1) L^a{}_b(x) e^b_\mu(x). \tag{2.10}$$

The covariant derivative of the vierbeins e^a_μ and $e^{*a}_{\ \mu}$ on this tangent space are now given by

$$e^a_{\mu\|\nu} = e^a_{\mu,\nu} + \Lambda_\nu{}^a{}_b e^a_\mu + C_\nu e^b_\mu, \tag{2.11}$$

$$e^{*a}_{\ \mu\|\nu} = e^{*a}_{\ \mu,\nu} + \Lambda_\nu{}^a{}_b e^{*b}_{\ \nu} - C_\nu e^{*a}_{\ \mu}, \tag{2.12}$$

where $\Lambda_\nu{}^a{}_b$ is the tangent connection related to the Minkowskian space and C_ν is the "internal connection." Their transformation laws are, respectively,

$$\Lambda'_\nu = L \Lambda L^{-1} - L_{,\nu} L^{-1} \quad \text{(space-time transformations)}, \tag{2.13}$$

$$C'_\nu = U(1) C_\nu U^{-1}(1) - U_{,\nu}(1) U^{-1}(1) \quad \text{(internal transformations)}. \tag{2.14}$$

Here C_ν transforms as a vector under space-time transformations. Considering the particular case where we have only the internal transformations represented by the matrices $U(1) = 1 + i\phi$, the internal connection C_ν transforms in first order, as

$$C'_\nu = C_\nu + i\phi_{,\nu}, \tag{2.15}$$

which is the same as a gauge transformation law for an electromagnetic potential. We also have the relation

$$R^\rho{}_{\mu\nu\gamma} e^a_\rho - S_{\nu\gamma}{}^a{}_b e^b_\mu = 0, \tag{2.16}$$

where $R^\rho_{\ \mu\nu\gamma}$ is the curvature in the non-Riemannian space-time written in terms of the nonsymmetic affinity, and $S_{\nu\gamma}$ is the curvature over the complex tangent space,

$$S_{\nu\gamma} = \Lambda_{\nu,\gamma} - \Lambda_{\gamma,\nu} - [\Lambda_\nu, \Lambda_\gamma], \tag{2.17}$$

which is skew symmetric with respect to the curved-space indices and anti-Hermitian in the tangent space indices. The internal curvature can be also obtained:

$$P_{\nu\gamma} = C_{\nu,\gamma} - C_{\gamma,\nu}. \tag{2.18}$$

Then, in the particular case of (2.15), the internal curvature can be considered to correspond to the Maxwell electromagnetic tensor.

One of the field equations of the ES nonsymmetric theory, obtained through a variational principle, is $g_{\underset{+}{\mu}\underset{-}{\nu};\alpha} = 0$, where the symbol (;) means that the connection used in this equation is the Schrödinger connection,[1] $\theta^\rho{}_{\mu\nu}$, $\theta_\mu = \theta^\rho{}_{\underset{\vee}{\mu\rho}} = 0$. (The notation used in this work is the same used by Marques and Oliveira in Ref. 4. It has been kept the tranditional use of the "$+$ $-$" covariant derivative used in the ES nonsymmetric theory, Ref. 1. See also, M. A. Tonellat, Ref. 9.) (In general, $\theta^\rho{}_{\mu\alpha}$ is a nonsymmetric connection such that $\theta^{*\rho}{}_{\mu\alpha} = \theta^\rho{}_{\alpha\mu}$.) This equation corresponds to the following vierbein equations:

$$e^a_{\underset{+}{\mu}|\alpha} = (e^{*a}_{\ \underset{-}{\mu}|\alpha})^* = e^a_{\mu,\alpha} - \theta^\rho{}_{\mu\alpha} e^a_\rho + \Lambda_\alpha{}^a{}_b e^b_\mu + C_\alpha e^a_\mu = 0, \tag{2.19}$$

where (2.1) was used. Taking the inverse equation: $g^{\underset{+}{\mu}\underset{-}{\nu}}{}_{;\alpha} = 0$, and (2.2), we have the corresponding equations for the inverse vierbeins

$$e^{*\overset{\mu}{+}}_{\ a|\alpha} = (e^{\overset{\mu}{-}}_{a|\alpha})^* = e^{*\mu}_{\ a,\alpha} + \theta^\mu{}_{\rho\alpha} e^{*\rho}_{\ a} - \Lambda_\alpha{}^b{}_a e^{*\mu}_{\ b} - C_\alpha e^{*\mu}_{\ b} = 0. \tag{2.20}$$

We can rewrite Eqs. (2.13) and (2.14) as

$$e^a_{\underset{+}{\mu}|\alpha} = e^a_{\mu,\alpha} - \theta^\mu{}_{\rho\alpha} e^\rho_a - \Lambda_\alpha{}^b{}_a e^\mu_b = 0, \tag{2.21}$$

$$e^{*\overset{\mu}{+}}_{\ a|\alpha} = e^{*\mu}_{\ a,\alpha} + \theta^\mu{}_{\rho\alpha} e^{*\rho}_{\ a} + \Lambda_\alpha{}^b{}_a e^{*\mu}_{\ b} = 0, \tag{2.22}$$

where

$$\Lambda_\alpha{}^a{}_b = \Lambda_\alpha{}^a{}_b + \delta^a_b C_\alpha, \quad \Lambda^*{}_\alpha{}^a{}_b = \Lambda_\alpha{}^a{}_b - \delta^a_b C_\alpha. \tag{2.23}$$

From (2.19) and (2.20) we obtain the relation

$$\Lambda_\alpha{}^a{}_b = e^a_\mu e^{*\overset{\mu}{+}}_{\ b;\alpha} = -e^a_{\underset{+}{\mu};\alpha} e^{*\mu}_{\ b}, \tag{2.24}$$

and from (2.23),

$$\Lambda_\alpha = \mathrm{Re}\left[e^a_\mu e^{*\overset{\mu}{+}}_{\ b;\alpha}\right] = \mathrm{Re}\left[-e^a_{\underset{+}{\mu};\alpha} e^{*\mu}_{\ b}\right], \tag{2.25}$$

$$C_\alpha = i(\mathrm{Im}\left[e^a_\mu e^{*\overset{\mu}{+}}_{\ b;\alpha}\right]) = i(\mathrm{Im}\left[-e^a_{\underset{+}{\mu};\alpha} e^{*\mu}_{\ b}\right]). \tag{2.26}$$

Taking (2.25), we can expand it in terms of real and imaginary parts. We then obtain for Λ_α

$$\Lambda_\alpha = \mathrm{Re}\left[e^a_\mu e^{*\underset{+}{\mu}}_{b;\alpha} \right]$$
$$= (e^a_{\alpha_R} e^\mu_{b_{R,\alpha}} + e^a_{\mu_I} e^\mu_{b_{I,\alpha}} + e^a_{\mu_R} \theta^\mu_{\underline{\rho\alpha}} e^\rho_{b_R} + e^a_{\mu_I} \theta^\mu_{\underline{\rho\alpha}} e^\rho_{b_I}$$
$$- e^a_{\mu_I} \theta^\mu_{\underset{\vee}{\rho\alpha}} e^\rho_{b_R} + e^a_{\mu_R} \theta^\mu_{\underset{\vee}{\rho\alpha}} e^\rho_{b_I}) \tag{2.27}$$

or

$$\Lambda_\alpha = \mathrm{Re}\left[- e^a_{\underset{+}{\mu;\alpha}} e^{*\mu}_b \right]$$
$$= (- e^a_{\mu_{R,\alpha}} e^\mu_{b_R} - e^a_{\mu_{I,\alpha}} e^\mu_{b_I} + \theta^\rho_{\underline{\mu\alpha}} e^a_{\rho_R} e^\mu_{b_R} + \theta^\rho_{\underline{\mu\alpha}} e^a_{\rho_I} e^\mu_{b_I}$$
$$+ \theta^\rho_{\underset{\vee}{\mu\alpha}} e^a_{\rho_R} e^\mu_{b_I} - \theta^\rho_{\underset{\vee}{\mu\alpha}} e^a_{\rho_I} e^\mu_{b_R}) . \tag{2.28}$$

Analogously, from (2.26), we obtain for C_α

$$C_\alpha = i(\mathrm{Im}\left[e^a_\mu e^{*\underset{+}{\mu}}_{a;\alpha} \right])$$
$$= i(e^a_{\mu_I} e^\mu_{b_{R,\alpha}} - e^a_{\mu_R} e^\mu_{b_{I,\alpha}} + e^a_{\mu_R} \theta^\mu_{\underset{\vee}{\rho\alpha}} e^\rho_{b_R} + e^a_{\mu_I} \theta^\mu_{\underset{\vee}{\rho\alpha}} e^\rho_{b_I}$$
$$+ e^a_{\mu_I} \theta^\mu_{\underline{\rho\alpha}} e^\rho_{b_R} - e^a_{\mu_R} \theta^\mu_{\underline{\rho\alpha}} e^\rho_{b_I}) \tag{2.29}$$

or

$$C_\alpha = i(\mathrm{Im}\left[- e^a_{\underset{+}{\mu;\alpha}} e^{*\mu}_b \right])$$
$$= i(e^a_{\mu_{R,\alpha}} e^\mu_{b_I} - e^a_{\mu_{I,\alpha}} e^\mu_{b_R} - \theta^\rho_{\underline{\mu\alpha}} e^a_{\rho_R} e^\mu_{b_I} + \theta^\rho_{\underline{\mu\alpha}} e^a_{\rho_I} e^\mu_{b_R}$$
$$+ \theta^\rho_{\underset{\vee}{\mu\alpha}} e^a_{\rho_R} e^\mu_{b_R} + \theta^\rho_{\underset{\vee}{\mu\alpha}} e^a_{\rho_I} e^\mu_{b_I}) . \tag{2.30}$$

Suppose we have a theory where the antisymmetrical part of the space-time connection is zero, $\theta^\rho{}_{\underset{\vee}{\mu\nu}} = 0$, but still with complex vierbeins, i.e., a theory where we have a complex, antiysmmetrical part for the metric. We now obtain for the tangent and internal connections,

$$\Lambda_\nu = (e^a_{\mu_R} e^\mu_{b_{R,\nu}} + e^a_{\mu_I} e^\mu_{b_{I,\nu}} + e^a_{\mu_R} \Gamma^\mu_{\rho\nu} e^\rho_{b_R} + e^a_{\mu_I} \Gamma^\mu_{\rho\nu} e^\rho_{b_I}) \quad .$$
$$= (- e^a_{\mu_{R,\nu}} e^\mu_{b_R} - e^a_{\mu_{I,\nu}} e^\mu_{b_I} + e^a_{\rho_R} \Gamma^\rho_{\mu\nu} e^\mu_{b_R} + e^a_{\rho_I} \Gamma^\rho_{\mu\nu} e^\mu_{b_I}) \tag{2.31}$$

and

$$C_\nu = i(e^a_{\mu_I} e^\mu_{b_{R,\nu}} - e^a_{\mu_R} e^\mu_{b_{I,\nu}} + e^a_{\mu_I} \Gamma^\mu_{\rho\nu} e^\rho_{b_R} - e^a_{\mu_R} \Gamma^\mu_{\rho\nu} e^\rho_{b_I})$$
$$= i(e^a_{\mu_{R,\nu}} e^\mu_{b_I} - e^a_{\mu_{I,\nu}} e^\mu_{b_R} - e^a_{\rho_R} \Gamma^\rho_{\mu\nu} e^\mu_{b_I} + e^a_{\rho_I} \Gamma^\rho_{\mu\nu} e^\mu_{b_R}) , \tag{2.32}$$

where we used the notation $\Gamma^\rho_{\mu\nu}$ for the symmetrical connection.

We can see that the relation of the (complex) metric, with the new complex vierbeins, adds new extra terms to the tangent connection. Also, the internal connection has a relation with the vierbeins, which would not exist if the vierbeins are real. It is noticeable, from (2.31) and (2.32), that the same happens in a "complex theory" without a complex torsion term. It easy to conclude, as in Ref. 2, that the Einstein–Maxwell theory is reached in a convenient limit such as to eliminate the complex part of the metric and therefore the corresponding complex ones for the vierbeins. However, some years ago, this fact was criticized by theoretical analysis,[3] which does not change the power of a geometrical analysis. Thinking from a geometrical point of view, we will go forward, obtaining the (Dirac) field equations and see what we can get in this "complex theory."

III. THE GENERALIZATION OF THE FOCK–IVANENKO COEFFICIENTS

The Dirac constant γ matrices satisfy the anticommutation relations

$$\{\gamma_a, \gamma_b\} = 2\eta_{ab} \mathbf{1}_4 , \tag{3.1}$$

$$\{\gamma^a, \gamma^b\} = 2\eta^{ab} \mathbf{1}_4 , \tag{3.2}$$

where η_{ab} (and its inverse η^{ab}) is the Minkowski tensor with signature $+2$, and to the relation $\gamma^a{}_{,b} = 0$. The set formed with combinations of γ matrices,

$$\{\Gamma_i\} = \{\mathbf{1}_4, \gamma_a, \sigma_{ab} = (i/2)[\gamma_a, \gamma_b],$$
$$\gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3 , \gamma_5\gamma_a\} ,$$

composes a linearly independent set in the internal space of the Dirac wavefunctions ψ.

Now, multiplying (3.1) by e^{*a}_ν and e^b_μ, and using (2.1), we obtain

$$\{\gamma_\mu, \dot\gamma_\nu\} = 2g_{\mu\nu} \mathbf{1}_4 , \tag{3.3}$$

where $g_{\mu\nu}$ is now the ES nonsymmetric metric. In (3.4) we have defined γ_μ and $\dot\gamma_\mu$ by

$$e^a_\mu \gamma_a = \gamma_\mu , \quad e^{*a}_\mu \gamma_a = \dot\gamma_\mu . \tag{3.4}$$

Analogously, multiplying (3.2) by $e^{*\mu}_a$ and e^ν_b, we obtain

$$\{\dot\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \mathbf{1}_4 , \tag{3.5}$$

where

$$e^\mu_a \gamma^a = \gamma^\mu , \quad e^{*\mu}_a \gamma^a = \dot\gamma^\mu , \tag{3.6}$$

and the relation (2.2) was used. The covariant derivative of $\gamma^\mu(x)$ over the non-Riemannian manifold of nonsymmetric theory, is given by

$$\gamma_{\underset{+}{\mu}|\nu} = \gamma_{\mu,\nu} - \Omega^\rho{}_{\mu\nu}\gamma_\rho + [\Delta_\nu, \gamma_\mu] , \tag{3.7}$$

where Δ_μ is the internal connection, corresponding to the space of generalized γ matrices (or, also, of Dirac wave functions space), and $\Omega^\rho{}_{\mu\nu}$ is a more general space-time affinity (that, at least in principle, includes the internal connection C_μ). Taking the identity (3.5) and the Eq. (2.19), we have that

$$\gamma_{\underset{+}{\mu}|\nu} = (e^a_{\underset{+}{\mu}} \gamma_a)_{|\nu} = (e^a_{\underset{+}{\mu}|\nu})\gamma_a = 0 , \tag{3.8}$$

since γ_a is a constant matrix. In the same way, we obtain

$$\dot\gamma_{\underset{-}{\mu}|\nu} = (e^{*a}_{\underset{-}{\mu}|\nu})\gamma_a = 0 . \tag{3.9}$$

Expanding (3.8) and (3.9) we have

$$\gamma_{\underset{+}{\mu}|\nu} = \gamma_{\mu,\nu} - \theta^\rho{}_{\mu\nu}\gamma_\rho + [\Delta_\nu, \gamma_\mu] + C_\nu\gamma_\mu = 0 , \tag{3.10}$$

$$\dot\gamma_{\underset{-}{\mu}|\nu} = \dot\gamma_{\mu,\nu} - \theta^\rho{}_{\nu\mu}\dot\gamma_\rho + [\Delta_\nu, \dot\gamma_\mu] - C_\nu\dot\gamma_\mu = 0 . \tag{3.11}$$

We can observe then, from (3.10) and (3.11), that we obtain a relation similar of that of general relativity, i.e.,

$$\Delta_\nu = (1/4i)\Lambda_\nu^{\overset{ab}{\vee}}\sigma_{ab} = (1/4i)\mathrm{Re}\left[e^a_\mu e^{*\overset{\mu b}{+}}{}_{;\nu} \right]\sigma_{ab} , \tag{3.12}$$

or

$$\Delta_\nu = (1/4i)\mathrm{Re}\left[- e^a_{\underset{+}{\mu;\nu}} e^{*\mu b} \right]\sigma_{ab} , \tag{3.13}$$

where it was used (2.21) for Λ_ν.

If we now consider $\psi(x)$ as the wave function of a spin-$\frac{1}{2}$ particle of mass m, placed in a non-Riemmanian manifold of ES nonsymmetric theory, $\bar\psi(x) = \psi^\dagger\gamma_0$ will be the wave function of its antiparticle, and the corresponding Dirac wave equations are, respectively,

$$\gamma^\mu(\vec{\partial}_\mu + \Delta_\mu + C_\mu)\psi - \mu\psi = 0\,, \tag{3.14}$$

$$-\bar{\psi}(\overleftarrow{\partial}_\mu + \Delta_\mu - C_\mu)\dot{\gamma}^\mu - \mu\bar{\psi} = 0\,, \tag{3.15}$$

where $\mu = mc/\hbar$.

The new operator $(\partial_\mu + \Delta_\mu + C_\mu)$ comes from the covariant derivation of the function $\psi(x)$, which, besides being an object that transforms, locally, under the representation of Lorentz group $[\mathrm{U}(L)]$, also transforms under the (internal) U(1) group. Equations (3.14) and (3.15) describe particles placed in a curved non-Riemannian space-time of the ES nonsymmetric theory, since the connections Δ_μ and C_μ are now related to complex vierbeins, as well as the complex space-time connection.

Another way to obtain Eqs. (3.14) and (3.15) is through a minimal action principle. In this case the action is

$$A = \int \mathscr{L}\, d^4x\,,$$

where the Lagrangian is given by

$$\mathscr{L} = \sqrt{-g}\,[\bar{\psi}\gamma^\mu(\psi_{,\mu} + \Delta_\mu\psi + C_\mu\psi) + (\bar{\psi}_{,\mu} + \bar{\psi}\Delta_\mu - \bar{\psi}C_\mu)\dot{\gamma}^\mu\psi - \mu\bar{\psi}\psi]\,. \tag{3.16}$$

From (3.15), the wave equation for the charge conjugate function, ψ^c, is

$$\dot{\gamma}^\mu(\psi^c_{,\mu} + \Delta_\mu\psi^c - C_\mu\psi^c) - \mu\psi^c = 0\,, \tag{3.17}$$

where $\psi^c = C\bar{\psi}^T$, and C is the charge conjugate matrix. Therefore, if the wave equation of a particle is constructed with the set γ^μ and $(\Delta_\mu + C_\mu)$, the wave equation for its "charge conjugate" will be constructed with the set $\dot{\gamma}^\mu$ and $(\Delta_\mu - C_\mu)$.

Let us write now the internal connection C_μ as

$$C_\mu = ileA_\mu(x)\,. \tag{3.18}$$

Then, after (2.15), we can interpret e as the electric charge for the electron, $A_\mu(x)$ as the electromagnetic potential, and l will be a constant such that it balances units. Equations (3.14) and (3.17) can be written now as

$$\gamma^\mu(\partial_\mu + \Delta_\mu + ielA_\mu)\psi - \mu\psi = 0\,, \tag{3.19}$$

$$\dot{\gamma}^\mu(\partial_\mu + \Delta_\mu - ielA_\mu)\psi^c - \mu\psi^c = 0\,. \tag{3.20}$$

IV. CONCLUSION

We have learned that complexifying the space-time manifold of general relativity is equivalent to attaching it an internal **C** space. The new metric is no longer symmetric and its antisymmetric part should be proportional to the electromagnetic tensor [see Einstein, Ref. 1, Eqs. (11)–(17) in Sec. III]. Through complex vierbeins, it is possible to obtain a (complex) tangent space local to the nonsymmetric curved space of the ES type. Using these concepts we obtained here Dirac field equations for a spin-$\frac{1}{2}$ particle. The internal complex connection corresponds to the electromagnetic potential.

We observe that it is possible to define the complex vierbeins as

$$e^a_\mu = e^a_{\mu_R} + i\kappa\lambda n^a_\mu\,, \tag{4.1}$$

where κ is considered now as a parameter, and λ is a constant. We use here, as in Ref. 2,

$$\lambda \propto \frac{e}{L^2} \sim \frac{c^3 e}{\hbar G} = 1.82\times 10^{56}\,\frac{\text{statvolt}}{\text{cm}}\,.$$

In a limit where the parameter $\kappa\to 0$, we obtain from (2.27) and (2.28) the real connection Λ_ν from general relativity. Using the above expression for the vierbeins, we can display the interesting behavior of the new Dirac equations that appears when we split up $\gamma^\mu(x)$ in terms of its real and imaginary parts, and also suppose a complex mass term: $\mu = \mu_R + i\mu_I$, where we again can take $\mu_I = \kappa\lambda m$. Then, from (3.19),

$$[e^\mu_{a_R}\gamma^a(\partial_\mu + \Delta_\mu + ielA_\mu)\psi + \nu\mu_R\psi] + ik\lambda\,[n^\mu_a\gamma^a(\partial_\mu + \Delta_\mu + ielA_\mu)\psi + m\psi] = 0\,. \tag{4.2}$$

In the limit of the parameter $\kappa\to 0$, we should get the normal Dirac equations in the presence of gravitational and electromagnetic fields. Therefore, it means that we can get another identical set of Dirac equations if we take $n^\mu_a \equiv e^\mu_{a_R} \sim h^\mu_a$, and $m\equiv\mu_R$, where h^μ_a and μ_R will be vierbeins and the mass term of general relativity theory.

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APPENDIX: COMMENTS ON A MORE GENERAL TRANSFORMATION LAW IN A TANGENT SPACE ASSOCIATED TO A COMPLEX INTERNAL SPACE

Let us consider, instead of (2.10), a more general transformation law for objects in the complex tangent space. Considering, for instance, the vectors e^a_μ, it can be defined as

$$e'^a_\mu(x) = L^a{}_b(x)e^b_\mu(x)\,, \tag{A1}$$

$$e'^{*a}_\mu(x) = L^{*a}{}_b e^{*b}_\mu(x)\,. \tag{A2}$$

The complex matrix $\mathbf{L}^a{}_b$ now is a kind of pseudo-Lorentz matrix that follows the relation

$$\mathbf{L}^\dagger\eta\mathbf{L} = \eta\,. \tag{A3}$$

The covariant derivative of e^a_μ on this complex tangent space is then defined as

$$e^a_{\mu|\nu} = e^a_{\mu,\nu} + \Lambda_\nu{}^a{}_b e^b_\mu\,, \tag{A4}$$

$$e^{*a}_{\mu|\nu} = e^{*a}_{\mu,\nu} + \Lambda^*{}_\nu{}^a{}_b e^{*b}_\mu\,, \tag{A5}$$

where the affinity is complex. Its transformation law is

$$\Lambda'_\mu = \mathbf{L}\Lambda_\mu\mathbf{L}^{-1} - \mathbf{L}_{,\mu}\mathbf{L}^{-1},$$

$$\Lambda'^*_\mu = L^*\Lambda^*\mathbf{L}^{*-1} - \mathbf{L}^*_{,\mu}\mathbf{L}^{*-1}\,. \tag{A6}$$

It is directly shown (see Ref. 4 that, through the Einstein field equations for the nonsymmetric theory (a complex theory), $g_{\underset{+}{\mu}\underset{-}{\nu};\alpha} = 0$, we obtain the same corresponding field equations for the vierbeins described in Eqs. (2.21) and

(2.22). However, we must also have $\eta_{\underset{+}{a}\underset{-}{b}|\mu} = 0$, where the "minus" sign corresponds to the complex conjugate of the affinity Λ_μ:

$$\eta_{\underset{+}{a}\underset{-}{b}|\mu} = \eta_{ab,\mu} - \Lambda_\mu{}^c{}_a \eta_{cb} - \Lambda^*{}_\mu{}^c{}_b \eta_{ac} = 0 . \tag{A7}$$

As η_{ab} lowers indices, we have that Λ_μ is anti-Hermitian with respect to the index of the tangent space. Then, we have

$$\Lambda_{\mu ab} = \Lambda_{\mu \underset{\vee}{ab}} + i\Lambda_{\mu \underline{ab}} . \tag{A8}$$

The expansion of L in first order is, from (A.1)–(A.3),

$$\mathbf{L} \cong \mathbf{1} + \epsilon + i\mu, \quad \mathbf{L}^{-1} \cong \mathbf{1} - \epsilon - i\mu , \tag{A9}$$

where $\epsilon = \epsilon(x)$ are infinitesimal rotation matrices as before and $\mu = \mu(x)$ are symmetric infinitesimal matrices. We can write the latter as

$$\mu_{ab} = (a + \mathbf{1}\tfrac{1}{4}\,\mathrm{Tr}\,\mu)_{ab} , \tag{A10}$$

where a is a symmetric trace-free matrix. Considering then, a particular transformation such that

$$\mathbf{L} \cong \mathbf{1} + \tfrac{1}{4}K , \quad K = \mathrm{Tr}\,\mu , \tag{A11}$$

the affinity Λ_α of this complex theory transforms as

$$\Lambda'_\alpha = \Lambda_\alpha - (i/4)K_{,\alpha} , \quad \mathrm{Tr}\,\Lambda'_\alpha = \mathrm{Tr}\,\Lambda_\alpha - iK_\alpha , \tag{A12}$$

which is similar to the gauge transformation of an electromagnetic potential. (In the same way, we can show that the complex part of a nonsymmetric tangent curvature obtained with the above Λ_α will be related to the Maxwell electromagnetic tensor.)

Now, from (A7), (3.8), and (3.9), we can easily obtain a relation between Λ_α and the connection Δ_α:

$$\begin{aligned} \Lambda_\alpha{}^{ab} e_{\mu b} \gamma_a &= \eta^{ac} e_{\mu c} [\Delta_\alpha^{(1)}, \gamma_a] , \\ \Lambda^*{}_\alpha{}^{ab} e^*_{\mu b} \gamma_a &= \eta^{ab} e^*_{\mu c} [\Delta_\alpha^{(2)}, \gamma_a] , \end{aligned} \tag{A13}$$

where $\Delta_\alpha^{(1)}$ and $\Delta_\alpha^{(2)}$ are now (general) Dirac connections corresponding to the Fock–Ivanenko coefficients. However, expanding $\Delta_\alpha^{(1)}$ and $\Delta_\alpha^{(2)}$ in terms of the set $\{\Gamma_i\}$, we can see that the real part of Λ_α is of the same form as in general relativity, but that there is no way to relate the complex symmetrical part of Λ_α in terms of that expansion, since the only symmetrical term there, which is proportional to the unit element of the set (the unit 4×4 matrix), is eliminated through the commutator in (A13). This shows that this is not the correct choice for the transformation matrix **L**. As we saw the correct one is the product expressed in (2.10).

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