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Nonabelian fluids and helicities

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Abstract: In analogy with the non-Abelian gauge helicities conserved in time for "null fields", that we have defined previously, in this paper we first define non-Abelian fluid helicities and then total non-Abelian helicities for combined non-Abelian fluid and gauge fields. For a U(N) group the helicities considered are for both gluonic-type fluids, composed of particles in the adjoint representation, and quark-type fluids, in the N-dimensional Cartan subalgebra. We write down various Lagrangian formulations for the uncoupled and coupled systems. In each case we determine the equations of motion, symmetries, and a Hamiltonian formulation. Taking the velocity of the fluid in the adjoint representation, we find that in the case of the gluonic fluid, we can write a non-Abelian gluonic fluid Euler-Yang-Mills equation that conserves the defined helicities, and comes from a Lagrangian.

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1 Introduction

Non-Abelian fluids exist in nature. It is well known that heavy ion collisions, like those done in RHIC and the LHC, create a strongly-coupled quark-gluon plasma which is a fluid. In fact it is very plausible that the very early universe before the deconfining/confining transition has a form of a ball of non-Abelian fluid.

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The dynamics of non-relativistic perfect fluids is described by the Euler and continuity equations. One can couple them to external electromagnetic fields, as in [2]. "Null gauge fields" admit topologicaly non-trivial knot solutions associated with the conservation of four helicities $\mathcal{H}_{mm}, \mathcal{H}_{me}, \mathcal{H}_{em}, \mathcal{H}_{ee}$(see [3, 4] and references therein). In [5], a map between these electromagnetic helicities and fluid helicities $\mathcal{H}_f$ was derived. Correspondingly Hopfion fluid solution and other knot solutions were written down. For the system of a fluid coupled to an external eletromagnetic fields, instead of the individual fluid helicity $\mathcal{H}_f$, the spatial CS form of the velocity, and the electromagnetic $\mathcal{H}_{mm}$ helicity, a total fluid-electromagnetic helicity $\mathcal{H}_{\text{tot}}$ is conserved [6].

In [7], we have defined helicities for Yang-Mills fields, generalizing the ones for electromagnetism. The notion of non-Abelian fluids was considered previously from various points of view, for instance in [1] and [8].

But in this paper, we propose to think about it from the point of view of extending the notion of non-Abelian helicities from Yang-Mills to total ones for a combined system of non-Abelian fluid coupled to Yang-Mills fields.

We consider fluids that are either "gluonic" in type, i.e., have an index a in the adjoint, thought of as coming from a strongly-coupled gluon plasma, or "quark" in type, i.e., have an index f that takes N values for $\mathrm{U}(N)$, specifically in the N-dimensional Cartan sub-algebra.

We will first define the helicities, and then construct Euler equations, Lagrangians and classical solutions that conserve the helicities.

In order to gain some insight, we review the non-relativistic, then the relativistic, Abelian fluids, from the point of view of a Lagrangian (with the corresponding symmetries) and Hamiltonian formulations. An essential point of the construction will be a Clebsch parametrization of the velocity, $u_\mu \propto a_\mu = \partial_\mu\theta + \alpha\partial_\mu\beta$. A natural generalization would be to write a non-Abelian generalization in terms of $g, h \in \mathcal{G}$, and $u^\mu \propto a_\mu = g^{-1}\partial_\mu g + \alpha h^{-1}\partial_\mu h$, but we will find that it does not lead to a reasonable Euler equation (one that is covariant, and written in terms of u^μ, not the Clebsch parameters), and so does not conserve our helicities.

Simple generalizations to the non-Abelian case of these Lagrangians are possible, and we find two of them, but one will give terms that depend only on the Clebsch parameters, and not on the full velocities, while the other gives only one equation (a "traced" equation, instead of a full matrix equation, i.e., N^2 components, as we want).

To continue, we will review the construction of [1], which does lead to an Euler-like equation coupled to Yang-Mills fields through a Lorentz-like force, just that a single equation, not N^2 equations, one for each a index. The main difference between the approach of [1] and our approach is that we take the fluid velocity in the adjoint representation, namely u^a_μ. In both formulations the current is non-Abelian but in [1] it is assumed that only the density carries an a index and not the velocity, whereas in our formulation both ρ and u are in the adjoint representation.[1]

We then construct a Euler equation for non-Abelian gluonic fluids coupled to Yang-Mills fields, that does conserve our constructed helicities, and then find a Lagrangian formulation for it. We will also consider solutions for it. Finally, we present an attempt at an Euler

[1]Note that in [9, 10] some problems were claimed to be identified with this construction, specifically with a non-Abelian chemical potential, but with normal fluid velocity u_μ.

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equation for a non-Abelian "quark"-like (fundamental) fluid coupled to Yang-Mills fields, but the result is somewhat trivial.

Fluid dynamics has a long history of studies.[2] The topic of non-Abelian fluids was analyzed in the past by many authors. A Hamiltonian description appears in [15], non-Abelian fluids were studied in [1], multi-fluids were discussed in [16], extensions to higher dimensions, internal symmetries and supersymmetry were explored in [17], various aspects of anomalies in fluid dynamics were studied in [18, 19]and [20].

The paper is organized as follows. In section 2 we review the Abelian helicities, and in section 3 we present our proposed non-Abelian helicities. In section 4 we review the Lagrangian formulations for the Abelian, non-relativistic and relativistic fluid dynamics, coupled to electromagnetism. In section 5 we present the non-Abelian fluid constructions. We start with the non-Abelian Clebsch parametrization formulations, then review the formulation of [1], after which we present the gluonic Euler equation coupled to Yang-Mills that conserves the corresponding helicities, with classical solutions, and end with the attempt at a "quark" (fundamental) fluid coupled to Yang-Mills.

2 Abelian helicities

In Maxwell's electromagnetism there is a sector of "null fields" $\vec{E}\cdot\vec{B}=0$, $E^2=B^2$, for which on top of the ordinary SO$(4,2)$ symmetry, there are four types of conserved helicities that play the role of topological charges for knot solutions. In analogy to that, one can also define a null fluid with $(\vec{v})^2=1$, for which the fluid dynamics admits a nontrivial conserved helicity [5, 21] (in the absence of the null condition, the fluid conserves the fluid helicity only if it has boundary conditions at infinity for fields vanishing fast enough). It turns out that a generalization of such a helicity is also conserved in the theory of a abelian charged fluid coupled to background electric and magnetic fields. In the following subsections we review these three types of helicities.

2.1 Abelian gauge helicities

In electromagnetism in vacuum, defining the vector potential $\vec{A}$ and its dual $\vec{C}$,

$$\vec{E}=\vec{\nabla}\times\vec{C};\quad \vec{B}=\vec{\nabla}\times\vec{A}\,, \tag{2.1}$$

one can define helicities as spatial integrals of spatial Chern-Simons (electric-electric and magnetic-magnetic) or BF forms (mixed electric-magnetic) made up from $\vec{A}$ and $\vec{C}$,

$$\begin{aligned}
\mathcal{H}_{ee} &= \int d^3x\vec{C}\cdot\vec{E}=\int d^3x\vec{C}\cdot\vec{\nabla}\times\vec{C}=\int d^3x\epsilon^{ijk}C_i\partial_jC_k\,,\\
\mathcal{H}_{mm} &= \int d^3x\vec{A}\cdot\vec{B}=\int d^3x\epsilon^{ijk}A_i\partial_jA_k\,,\\
\mathcal{H}_{em} &= \int d^3x\vec{C}\cdot\vec{B}=\int d^3x\epsilon^{ijk}C_i\partial_jA_k\,,\\
\mathcal{H}_{me} &= \int d^3x\vec{A}\cdot\vec{E}=\int d^3x\epsilon^{ijk}A_i\partial_kC_k\,,
\end{aligned} \tag{2.2}$$

[2]A standard physics text on the subject is [11]. A mathematical treatment is [12]. The relation between Langrange and Euler descriptions of a fluid is discussed by [13]; see also [14].

which are conserved, by use of the Maxwell equations, if $\vec{E}\cdot\vec{B}\propto\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}=0$ and $\vec{E}^2-\vec{B}^2\propto F_{\mu\nu}F^{\mu\nu}=0$ (for "null" configurations, with null Riemann-Silberstein vector $\vec{F}=\vec{E}+i\vec{B}$, $\vec{F}^2=0$), since (see for instance [5, 21])

$$\begin{aligned}\partial_t\mathcal{H}_{mm}&=\int d^3x(\partial_t\vec{A}\cdot\vec{B}+\vec{A}\cdot\partial_t\vec{B})=-\int d^3x(\vec{E}\cdot\vec{B}+\vec{A}\cdot(\vec{\nabla}\times\vec{E}))\\&=-2\int d^3x\vec{E}\cdot\vec{B}=-\int d^3x\epsilon^{ijk}\partial_i(E_jA_k)\,,\\\partial_t\mathcal{H}_{ee}&=\int d^3x(\partial_t\vec{C}\cdot\vec{E}+\vec{C}\cdot\partial_t\vec{E})=-\int d^3x(\vec{B}\cdot\vec{E}+\vec{C}\cdot(\vec{\nabla}\times\vec{B}))\\&=-2\int d^3x\vec{E}\cdot\vec{B}=-\int d^3x\epsilon^{ijk}\partial_i(E_jA_k)\,,\\\partial_t\mathcal{H}_{me}&=\int d^3x(\partial_t\vec{A}\cdot\vec{E}+\vec{A}\cdot\partial_t\vec{E})=\int d^3x(-\vec{E}\cdot\vec{E}+\vec{A}\cdot(\vec{\nabla}\times\vec{B}))\\&=-\int d^3x(\vec{E}^2-\vec{B}^2)\,,\\\partial_t\mathcal{H}_{em}&=\int d^3x(\partial_t\vec{C}\cdot\vec{B}+\vec{C}\cdot\partial_t\vec{B})=\int d^3x(-\vec{B}\cdot\vec{B}+\vec{C}\cdot(\vec{\nabla}\times\vec{E}))\\&=\int d^3x(\vec{E}^2-\vec{B}^2)\,,\end{aligned}\tag{2.3}$$

although, if one considers $\vec{A},\vec{C}\to0$ sufficiently fast at the boundary, such that the boundary terms above are zero in the absence of the null conditions, $\mathcal{H}_{ee}$ and $\mathcal{H}_{mm}$ are still conserved.

2.2 Fluid helicity

In fluid dynamics, one can define (as defined and later studied by Moffat [22–24]) the analog of H_{mm}, the fluid helicity defined as the spatial Chern-Simons form of the velocity, i.e.,

$$\mathcal{H}_f=\frac{1}{\Gamma^2}\int d^3x\vec{v}\cdot\vec{\nabla}\times\vec{v}=\frac{m^2}{h^2}\int d^3x\vec{v}\cdot\vec{\omega}\,,\tag{2.4}$$

where Γ is a normalization constant, here taken to be $=h/m$ in order for $\mathcal{H}$ to be integer valued (in superconductivity this arises from the conditions on the wave function, but it was recently shown that this can be proved in generality, for any fluid [18]).

Using the Euler equation, one finds that the fluid helicity is conserved (time independent) if the velocity goes to zero sufficiently fast at infinity (so that a boundary term involving the velocity vanishes).

2.3 The helicity of a charged fluid coupled to electromagnetic background

For the Euler equation coupled to external electromagnetism, Abanov and Wiegmann [2] considered a *total* fluid + electromagnetism helicity, defined in terms of the canonical momentum

$$\vec{\pi}=m\vec{v}+\vec{A},\tag{2.5}$$

as the spatial Chern-Simons form of $\vec{\pi}$, i.e.,

$$\begin{aligned}\mathcal{H}_{\text{tot}}&=\frac{1}{h^2}\int d^3x\vec{\pi}\cdot\vec{\nabla}\times\vec{\pi}=\frac{1}{h^2}\int d^3x\left[m^2\vec{v}\cdot\vec{\omega}+\vec{A}\cdot\vec{B}+2m\vec{v}\cdot\vec{B}\right]\\&=\mathcal{H}_f+\mathcal{H}_{mm}+2\mathcal{H}_{fm}\,,\end{aligned}\tag{2.6}$$

where by partial integration $\int d^3x \vec{v}\cdot\vec{B} = \int d^3x \vec{A}\cdot\vec{\omega}$, so the 2 cross-terms (or "cross-helicities") are equal, giving what we called the cross fluid-electromagnetic helicity, $\mathcal{H}_{fm}$.

Abanov and Wiegmann [2] (see also [6] where solutions of the Euler coupled to electromagnetism with this total helicity were found) found that this total helicity is now conserved (time independent),

$$\frac{d}{dt}\mathcal{H}_{\rm tot} = 0\,, \tag{2.7}$$

instead of the individual terms, a fact based on the Euler equation and the vanishing of v_i, A_i at infinity.

The proof was simple in a 4-dimensional formalism (even though the fluid theory was still non-relativistic) where one defined also a π_0 by

$$\pi_0 = \Phi + A_0\,, \quad -\Phi = \mu + \frac{m\vec{v}^2}{2}\,, \tag{2.8}$$

and with the usual 4-current $j^\mu = (\rho, \rho v^i)$, the Euler equation became

$$j^\mu \Omega_{\mu\nu} = 0\,, \;\; \Omega_{\mu\nu} \equiv \partial_\mu \pi_\nu - \partial_\nu \pi_\mu\,, \tag{2.9}$$

and the total helicity density 3-form was

$$h = \pi\wedge d\pi = \pi\wedge\Omega \Rightarrow dh = \Omega\wedge\Omega = 0. \tag{2.10}$$

3 Non-abelian helicities

A natural question is whether one can generalize the abelian helicities also to non-abelian helicities. In a similar manner to the abelian case, we will analyze first the issue for gauge fields and then for non-abelian fluids of different types.

3.1 Non-abelian gauge helicities

The electromagnetic helicities were generalized to non-Abelian (Yang-Mills) fields in [7] as follows.

The non-Abelian singlet helicity extending $\mathcal{H}_{mm}$ was defined as the spatial Chern-Simons form of the non-Abelian field A_i^a, in the $A_0^a = 0$ gauge,

$$\begin{aligned}\mathcal{H}^{NA}_{mm} &= \int d^3x\,{\rm Tr}\left[A\wedge dA + \frac{2}{3}A\wedge A\wedge A\right] = \int d^3x\,{\rm Tr}\left[A\wedge F - \frac{1}{3}A\wedge A\wedge A\right]\\ &= \int d^3x\left[A_i^a B_i^a - \frac{1}{3}\epsilon^{ijk} f^{abc} A_i^a A_j^b A_k^c\right],\end{aligned} \tag{3.1}$$

which is conserved by the use of the YM equations, integration by parts and the $A_0^a = 0$ gauge,

$$\partial_t \mathcal{H}^{NA}_{mm} = 2\int d^3x \vec{E}^a\cdot\vec{B}^a\,, \tag{3.2}$$

provided

$$\vec{E}^a\cdot\vec{B}^a = 0\,. \tag{3.3}$$

The dual $\mathcal{H}_{ee}$ singlet helicity, the CS form of the dual C_i^a field,

$$\begin{aligned} E_i^a &= F_{i0}^a = \frac{1}{2}\epsilon_i{}^{jk}(\partial_j C_k^a - \partial_k C_j^a + f^a{}_{bc}C_j^b C_k^c) \\ B_i^a &= \frac{1}{2}\epsilon_{ijk}F^{ajk} = -\partial_t C_i^a \,, \end{aligned} \tag{3.4}$$

is defined as

$$\begin{aligned} \mathcal{H}_{ee}^{NA} &= \int d^3x\, \mathrm{Tr}\left[C\wedge dC + \frac{2}{3}C\wedge C\wedge C\right] = \int d^3x\, \mathrm{Tr}\left[C\wedge \tilde{F} - \frac{1}{3}C\wedge C\wedge C\right] \\ &= \int d^3x \left[C_i^a E_i^a - \frac{1}{3}\epsilon^{ijk}f^{abc}C_i^a C_j^b C_k^c\right] , \end{aligned} \tag{3.5}$$

and is also conserved provided (3.3) is satisfied.

On the other hand, the mixed singlet helicities, defined as the BF terms,

$$\begin{aligned} \mathcal{H}_{em}^{NA} &= \int d^3x\, \mathrm{Tr}\left[C\wedge dA + \frac{1}{2}A\wedge A\wedge C\right] = \int d^3x\, \mathrm{Tr}\left[C\wedge F - \frac{1}{2}A\wedge A\wedge C\right] \\ &= \int d^3x \left[C_i^a B_i^a - \frac{1}{2}\epsilon^{ijk}f^{abc}A_i^a A_j^b C_k^c\right] \\ \mathcal{H}_{me}^{NA} &= \int d^3x\, \mathrm{Tr}\left[A\wedge dC + \frac{1}{2}C\wedge C\wedge A\right] = \int d^3x\, \mathrm{Tr}\left[A\wedge \tilde{F} - \frac{1}{2}C\wedge C\wedge A\right] \\ &= \int d^3x \left[A_i^a E_i^a - \frac{1}{2}\epsilon^{ijk}f^{abc}C_i^a C_j^b A_k^c\right] \end{aligned} \tag{3.6}$$

are not conserved,

$$\partial_t \mathcal{H}_{em}^{NA} = \int d^3x \left[\vec{E}^a\cdot\vec{E}^a - \vec{B}^a\cdot\vec{B}^a + \frac{1}{2}\epsilon^{ijk}f_{abc}(A_i^a A_j^b B_k^c - C_i^a C_j^b E_k^c)\right] . \tag{3.7}$$

Non-singlet non-Abelian helicities are defined similarly. The non-Abelian version of $\mathcal{H}_{mm}$ is again defined as the spatial Chern-Simons term of A_i^a, with T^a inserted and in the $A_0^a = 0$ gauge,

$$\begin{aligned} \mathcal{H}^{NA^a}{}_{mm} &= \int d^3x\, \mathrm{Tr}\left[T^a\left(A\wedge dA + \frac{2}{3}A\wedge A\wedge A\right)\right] \\ &= \int d^3x\, \mathrm{Tr}\left[T^a\left(A\wedge F - \frac{1}{3}A\wedge A\wedge A\right)\right] \\ &= \int d^3x\; d^a{}_{bc}\left[A_i^b B_i^c - \frac{1}{3}\epsilon^{ijk}f^{bde}A_i^c A_j^d A_k^e\right] , \end{aligned} \tag{3.8}$$

now conserved if $d^a{}_{bc}\vec{E}^b\cdot\vec{B}^c = 0$. The same condition gives the conservation of the dual $\mathcal{H}_{ee}$ helicity,

$$\begin{aligned} \mathcal{H}^{NA^a}{}_{ee} &= \int d^3x\, \mathrm{Tr}\left[T^a\left(C\wedge dC + \frac{2}{3}C\wedge C\wedge C\right)\right] \\ &= \int d^3x\, \mathrm{Tr}\left[T^a\left(C\wedge \tilde{F} - \frac{1}{3}C\wedge C\wedge C\right)\right] \\ &= \int d^3x\; d^a{}_{bc}\left[C_i^b E_i^c - \frac{1}{3}\epsilon^{ijk}f^{bde}C_i^c C_j^d C_k^e\right] . \end{aligned} \tag{3.9}$$

The mixed non-singlet helicities

$$\begin{aligned}
\mathcal{H}^{NA^a}{}_{em} &= \int d^3x \, \mathrm{Tr}\left[T^a\left(C\wedge dA + \frac{1}{2}A\wedge A\wedge C\right)\right] \\
&= \int d^3x \, \mathrm{Tr}\left[T^a\left(C\wedge F - \frac{1}{2}A\wedge A\wedge C\right)\right] \\
&= \int d^3x \; d^a{}_{bc}\left[C_i^b B_i^c - \frac{1}{2}\epsilon^{ijk} f^{bde} A_i^c A_j^d C_k^e\right] \\
\mathcal{H}^{NA^a}{}_{me} &= \int d^3x \, \mathrm{Tr}\left[T^a\left(A\wedge dC + \frac{1}{2}C\wedge C\wedge A\right)\right] \\
&= \int d^3x \, \mathrm{Tr}\left[T^a\left(A\wedge \tilde{F} - \frac{1}{2}C\wedge C\wedge A\right)\right] \\
&= \int d^3x \; d^a{}_{bc}\left[A_i^b E_i^c - \frac{1}{2}\epsilon^{ijk} f^{bde} C_i^c C_j^d A_k^e\right] ,
\end{aligned} \tag{3.10}$$

are not conserved, though the condition $d^a{}_{bc}\left(\vec{E}^b\cdot\vec{E}^c - \vec{B}^b\cdot\vec{B}^c\right) = 0$ comes close to conserving them (they would be conserved for solutions that are duality symmetric, exchanging $\vec{A}^a$ with $\vec{C}^a$).[3]

In the following, we will want to define a non-Abelian version of a fluid, and couple it to Yang-Mills fields, just like the regular fluid was coupled to electromagnetism.

3.2 Non-Abelian fluid helicities

The first step in attempting to generalize the concept of non-Abelian helicities to fluid dynamics is to define what is a non-Abelian fluid. There are several different ways to do it. The starting point is the non-Abelian fluid current that generates the non-Abelian global symmetry in the case that there are no non-Abelian gauge fields, and the local symmetry for the system coupled to non-Abelian gauge fields. Thus, the Abelian currents and their conservation law are uplifted as follows.

$$j_\mu \to j_\mu^a \qquad \partial^\mu j_\mu^a = 0 \;\; (global) \quad \text{or} \qquad D^\mu j_\mu^a \equiv \partial^\mu j_\mu^a + f^{abc} A^\mu{}_b j_{\mu_c} = 0 \;\; (local). \tag{3.11}$$

The Abelian fluid current is built from the density ρ and velocity

$$j^\mu = j u^\mu \qquad j = \sqrt{j^\mu j_\mu} = \rho\sqrt{1-v^2}\,, \tag{3.12}$$

so that for the non-relativistic case

$$j^\mu = \rho(1,\vec{v}). \tag{3.13}$$

For the non-Abelian current there are several options of how to build the non-relativistic current. Possibilities one can consider are

$$(1)\; j_\mu^a = \rho^a(1,\vec{v})\,, \qquad (2)\; j_\mu^a = \rho^a(1,\vec{v}^a)\,, \qquad (3)\; j_\mu^a = j_\mu Q^a = \rho(1,\vec{v})Q^a. \tag{3.14}$$

Here a natural option would be for a to be an adjoint index, but we will explore later also the possibility to have a fundamental index.

[3]This is related to the fact that $\vec{A}^a$ and the dual $\vec{C}^a$ cannot be *simultaneously* defined in YM theory.

In option (1) the flow velocity is the same for all the different non-Abelian components of the density ρ^a. This would be a natural one if one is interested in a fluid where all the components move at the same velocity, except in [1], the variant (3) was considered, as we will review later. In option (2) each component has different velocity. There is no summation over the indices a in (2).

In the relativistic case the three options of the non-Abelian currrents can be written as

$$(1)\ j^a_\mu = j^a u_\mu\,, \qquad (2)\ j^a_\mu = j^a u^a_\mu\,, \qquad (3)\ j^a_\mu = j_\mu Q^a\,, \tag{3.15}$$

where

$$j^a = \sqrt{j^a_\mu j^{\mu a}} = \rho^a\sqrt{1-v^2}\,, \tag{3.16}$$

namely, using the singlet velocity and with no summation over a.

Thus, to get fluid helicities which are the analogs of the singlet and non-singlet non-Abelian helicities we need to adopt option (2).

The singlet helicity reads

$$\begin{aligned}\mathcal{H}^{NAs}_{\rm fl} &= \int d^3x\, \mathrm{Tr}\left[\mathbf{V}\wedge d\mathbf{V} + \frac{2}{3}\mathbf{V}\wedge\mathbf{V}\wedge\mathbf{V}\right]\\ &= \int d^3x \left[V^a_i \epsilon^{ijk}\partial_j V^a_k - \frac{1}{3}\epsilon^{ijk} f^{abc} V^a_i V^b_j V^c_k\right].\end{aligned} \tag{3.17}$$

Here $\mathbf{V} = V^a_i T_a dx^i$ is a non-Abelian spatial one-form, and $\mathbf{W} = d\mathbf{V}$.

In a similar manner to the non-Abelian adjoint helicities for the gauge fields, also for the fluid case one has to insert a matrix T^a into the trace as follows.

$$\begin{aligned}\mathcal{H}^{NAa}{}^a_{fl} &= \int d^3x\, \mathrm{Tr}\left[T^a\left(\mathbf{V}\wedge d\mathbf{V} + \frac{2}{3}\mathbf{V}\wedge\mathbf{V}\wedge\mathbf{V}\right)\right]\\ &= \int d^3x\, \mathrm{Tr}\left[T^a\left(V\wedge\mathbf{W} - \frac{1}{3}\mathbf{V}\wedge\mathbf{V}\wedge\mathbf{V}\right)\right]\\ &= \int d^3x\; d^a{}_{bc}\left[V^b_i W^c_i - \frac{1}{3}\epsilon^{ijk} f^{bde} V^c_i V^d_j V^e_k\right].\end{aligned} \tag{3.18}$$

3.3 Helicities of a "gluonic" fluid coupled to gauge fields

Next we consider the non-Abelian helicities of the fluid coupled to gauge fields. We will treat two separate cases.

It is not clear what a non-Abelian fluid can correspond to, but in this subsection we are considering the possibility of a fluid made up of only gluons, so a "gluonic fluid", interacting with the Yang-Mills fields themselves.

In other words, in the case of a strongly coupled plasma like the one obtained in heavy ion collisions (at RHIC or Alice at the LHC), if we ignore the quarks, we should have a gluon plasma, and because of the strong coupling, this can be described as a fluid. And yet, there is still the Yang-Mills field around, and it should interact with the gluonic fluid, giving the model we are after.

There are many models one can have about the non-Abelian fluid. Instead of starting with a Lagrangian or an equation of motion, we find it more convenient to start with a definition of helicity, and look for models that conserve it in time.

If a is an index in the adjoint representation, define then the natural generalization of the Abelian case, namely option (2) for the non-coupled fluids, i.e., a velocity $\vec{v}_a$, fluid density ρ_a, such that

$$\vec{j}_a = \rho_a \vec{v}_a \,, \tag{3.19}$$

and mass m_a of the fluid particle, chemical potential μ_a of the same, and charge Q_a, so

$$\Pi_\mu = \Pi_\mu^a T_a \,, \;\; \vec{\Pi}_a = m_a \vec{v}_a + Q_a \vec{A}_a \,, \;\; \Pi_{0a} = -\mu_a - m_a \frac{\vec{v}_a^2}{2} + A_{0a} \,, \tag{3.20}$$

where there is no sum over a. Note that

$$\rho_a(\vec{x}, t) = \sum_n m_a \delta^3(\vec{x} - \vec{x}_{an}(t)) \,, \tag{3.21}$$

where $\vec{x}_{an}(t)$ is defined by some initial position $\vec{x}_{an}$ and some velocity $\vec{v}_{an}(t)$, so we can consider the case that all m_a are equal, $m_a = m$, as long as the $\vec{v}_a$ are different for different a (such that ρ_a are all different). We can also consider all Q_a to be equal, meaning all fluid particles interact the same way with the Yang-Mills fields, though we will leave it as it is for now.

In terms of them, define the forms

$$\Pi = \Pi^a T_a \,, \;\; \Omega = \Omega^a T_a \,, \;\; \Omega = d\Pi + \Pi \wedge \Pi \,, \tag{3.22}$$

and thus the non-Abelian helicity densities

$$h^a = \mathrm{Tr}\left[T^a \left(\Pi \wedge d\Pi + \frac{2}{3} \Pi \wedge \Pi \wedge \Pi \right) \right] \,, \tag{3.23}$$

as well as the singlet non-Abelian helicity density h, with T^a replaced by the identity, such that

$$dh^a = \mathrm{Tr}\left[T^a \left(\Omega \wedge \Omega \right) \right] \,, \tag{3.24}$$

and similarly for dh.

The total singlet non-Abelian helicity is then

$$\begin{aligned} \mathcal{H}^{NA}_{\text{total}} &= \int d^3x \, \mathrm{Tr}\left[\Pi \wedge d\Pi + \frac{2}{3} \Pi \wedge \Pi \wedge \Pi \right] = \int d^3x \, \mathrm{Tr}\left[\Pi \wedge \Omega - \frac{1}{3} \Pi \wedge \Pi \wedge \Pi \right] \\ &= \int d^3x \left[\Pi_i^a \epsilon^{ijk} \Omega_{jk}^a - \frac{1}{3} \epsilon^{ijk} f^{abc} \Pi_i^a \Pi_j^b \Pi_k^c \right] \\ &= \mathcal{H}^{NA}_{mm} + \mathcal{H}^{NA}_{fm} + \mathcal{H}^{NA}_{f} \,, \end{aligned} \tag{3.25}$$

where $\mathcal{H}^{NA}_{mm}$ is the magnetic Yang-Mills helicity, $\mathcal{H}^{NA}_{f}$ is a non-Abelian analog of the fluid helicity, and $\mathcal{H}^{NA}_{fm}$ is a mixed term. The total non-singlet non-Abelian helicity is

$$\begin{aligned} {\mathcal{H}^{NA}}^a{}_{mm} &= \int d^3x \, \mathrm{Tr}\left[T^a \left(\Pi \wedge d\Pi + \frac{2}{3} \Pi \wedge \Pi \wedge \Pi \right) \right] \\ &= \int d^3x \, \mathrm{Tr}\left[T^a \left(\Pi \wedge \Omega - \frac{1}{3} \Pi \wedge \Pi \wedge \Pi \right) \right] \\ &= \int d^3x \; d^a{}_{bc} \left[\Pi_i^b \Pi_i^c - \frac{1}{3} \epsilon^{ijk} f^{bde} \Pi_i^c \Pi_j^d \Pi_k^e \right] \\ &= {\mathcal{H}^{NA}}^a{}_{mm} + {\mathcal{H}^{NA}}^a{}_{fm} + {\mathcal{H}^{NA}}^a{}_{f} \,, \end{aligned} \tag{3.26}$$

with a similar definition for the 3 terms.

We will find that these helicities are conserved by a non-Abelian generalization of the Euler equation.

3.4 Fundamental (”quark fluid”) helicities

One could think perhaps of a “quark fluid” instead of the “gluon fluid”, i.e., that only the quarks interact strongly and form a fluid, and then this fluis interacts with the resulting Yang-Mills field of the gluons.

Since the quarks q are in the fundamental representation, they should have an index i in the fundamental representation. But, moreover, they should have another index, f, that labels the type of fluid particle, and the particle is associated with a T_f in the Cartan subalgebra of the Lie algebra. This is a situation similar to the one considered by [1], though the rest of the construction is different.

We consider then the canonical momentum (we have specialized to $m_f = m$, but by a similar argument as in the gluon case from the previous subsection, this doesn't change anything)

$$\vec{\Pi}_f = m\vec{v}_f \sum_a \delta^{fa} T_a + \sum_{\tilde{b}} Q_{f\tilde{b}} \vec{A}^{\tilde{b}} T_{\tilde{b}} \equiv m\vec{v}_f + \vec{A}_f\,, \quad \Pi_f = \vec{\Pi}_f d\vec{x}\,, \tag{3.27}$$

where

$$Q_{f\tilde{b}} \equiv \sum_{i,j} \bar{q}_{fi} (T_{\tilde{b}})_{ij} q_{fj} \tag{3.28}$$

is the charge coupling of a $\bar{q}q$ pair to Yang-Mills gluons.

T_f must be in the Cartan subalgebra (commuting elements), which is N-dimensional for U(N), and then $T_{\tilde{b}}$ is taken to be another subset of T_a.

We can define then the total helicity

$$\begin{aligned}\mathcal{H}'_{\text{tot}} &= \sum_f \int d^3x \,\text{Tr}\left[\Pi_f \wedge d\Pi_f + \frac{2}{3}\Pi_f \wedge \Pi_f \wedge \Pi_f\right] \\ &= m^2 \int d^3x \sum_f \vec{v}_f \cdot \vec{\omega}_f + \int d^3x \sum_f \text{Tr}\left[A_f \wedge dA_f + \frac{2}{3} A_f \wedge A_f \wedge A_f\right] + \text{more}.\end{aligned} \tag{3.29}$$

In order to not obtain extra terms (the “more” above) we would need to have the conditions

$$\text{Tr}[T_f \tilde{T}_{\tilde{a}}] = 0\,, \quad \text{Tr}[T_{f_1}[T_{f_2}, T_{f_3}]] = 0\,, \quad \text{Tr}[T_{f_1}[\tilde{T}_{\tilde{a}}, \tilde{T}_{\tilde{b}}]] = 0\,, \text{Tr}[[T_{f_1}, T_{f_2}]\tilde{T}_{\tilde{a}}] = 0. \tag{3.30}$$

If we choose $T_{\tilde{a}}$ to be in the Lie algebra, except the Cartan subalgebra, we only find the extra term

$$\int d^3x 2m \sum_f \text{Tr}[v_f \wedge A_f \wedge A_f]. \tag{3.31}$$

Another possibility is that we define the sum over f already in Π, so

$$\Pi = \sum_f \Pi_f\,, \quad A \equiv \sum_f Q_{f\tilde{b}} A^{\tilde{b}} T_{\tilde{b}}\,, \tag{3.32}$$

in which case we would define the total helicity as

$$\begin{aligned}\mathcal{H}''_{\text{tot}} &= \int d^3x\,\text{Tr}\left[\Pi\wedge d\Pi + \frac{2}{3}\Pi\wedge\Pi\wedge\Pi\right]\\ &= m^2\int d^3x\sum_f \vec{v}_f\cdot\vec{\omega}_f + \int d^3x\sum_f \text{Tr}\left[A\wedge dA + \frac{2}{3}A\wedge A\wedge A\right]\\ &\quad + 2m\int d^3x\,\text{Tr}[v\wedge A\wedge A]. \end{aligned}\tag{3.33}$$

But the question is: is either $\mathcal{H}'_{\text{tot}}$ or $\mathcal{H}''_{\text{tot}}$ conserved by a non-Abelian Euler equation with an index f? We will see that, unfortunately, while we can conserve $\mathcal{H}'_{\text{tot}}$, the answer is somewhat trivial, since the fluid types f are non-interacting.

4 Lagrangian formalism of Abelian fluid dynamics

At this point we would like to place the conserved helicities in the framework of Lagrangian formulations of fluid dynamics. We begin with the ordinary uncharged fluid first in the non-relativistic limit and then in relativistic regime. We then present a Lagrangian formulation for a charged fluid coupled to an Abelian gauge field again both in the nonrelativistic and relativistic domains.

4.1 The nonrelativistic fluid

First, let us review the Lagrangian formulation and the corresponding equations of motion of the nonrelativistic fluid that were considered in [1]. We then address the issues of the dimensions of the fields, the Hamiltonian formulation, and the symmetries. The equations of motion and the conservation laws are shown to be compatible with the continuity and Euler equations of the fluid.

4.1.1 The Lagrangian density

The nonrelativistic fluid degrees of freedom were characterized in [1] by the fluid density ρ, the flow velocity $\vec{v}$ and the auxiliary gauge field a_μ. The system was defined by the following Lagrangian density

$$\mathcal{L}(\rho,\vec{v},a_\mu) = -j^\mu a_\mu + \frac{1}{2}\rho\vec{v}^2 - V(\rho)\,, \tag{4.1}$$

where V is some potential giving a pressure (or force), j^μ is the 4-current defined as

$$j^\mu = (c\rho, \rho\vec{v})\,. \tag{4.2}$$

The variation with respect to a_μ leads to $j_\mu = 0$ clearly not describing a fluid. However, if one parametrizes the gauge field [1] in the Clebsch parametrization

$$a_\mu = \partial_\mu\theta + \alpha\partial_\mu\beta\,, \tag{4.3}$$

so that now in fact the Lagrangian density is $\mathcal{L}(\rho,\vec{v},\theta,\alpha,\beta)$, it takes the explicit form

$$\mathcal{L}(\rho,\vec{v},\theta,\alpha,\beta) = -\rho(\partial_t\theta + \alpha\partial_t\beta) - \rho\vec{v}\cdot(\vec{\nabla}\theta + \alpha\vec{\nabla}\beta) + \rho\frac{\vec{v}^2}{2} - V\,, \tag{4.4}$$

which can be rewritten as

$$\mathcal{L} = \rho \left[\frac{\vec{v}^2}{2} - \frac{V}{\rho} - \frac{D\theta}{Dt} - \alpha \frac{D\beta}{Dt} \right] , \tag{4.5}$$

where

$$\frac{D\theta}{Dt} = \partial_t \theta + \vec{v} \cdot \vec{\nabla} \theta \qquad \frac{D\beta}{Dt} = \partial_t \beta + \vec{v} \cdot \vec{\nabla} \beta. \tag{4.6}$$

A variant, though equivalent, action was proposed in [25],

$$\int d^4x \mathcal{L} = \int d^4x \left\{ \frac{\rho \vec{v}^2}{2} - \rho \partial_0 \tilde{\mu} + \phi \left(\frac{D\rho}{Dt} + \rho \vec{\nabla} \cdot \vec{v} \right) - \rho \alpha_a \frac{D\beta^a}{Dt} \right\}$$

$$= \int d^4x \left\{ \rho \left[\frac{\vec{v}^2}{2} - \partial_0 \tilde{\mu} - \frac{D\phi}{Dt} - \alpha^a \frac{D\beta^a}{Dt} \right] \right\} , \tag{4.7}$$

where the second form was obtained by partial integration, and it is then equivalent to (4.6), with $-\partial_0 \tilde{\mu}$ being the same term as $-V/\rho$.

4.1.2 Dimensions of the fields

One can assign dimensions to the various fields as

$$\begin{aligned} &[a_\mu] = 1, \qquad [j_\mu] = 3, \qquad [\rho] = 3, \qquad [v_\mu] = 0, \\ &\frac{1}{2}\rho v^2 \to \frac{1}{2} m \rho v^2 \qquad V(\rho) \to m^4 V\left(\frac{\rho}{m^3}\right). \end{aligned} \tag{4.8}$$

In this assignment ρ has the meaning of a number density or charge density. If however we want that the dimension of ρ to reflect the fact that it is mass density, namely $[\rho] = 4$, and that as before $[j_\mu] = 3$ and $[v_\mu = 0]$, we have to modify the relation between j_μ and v_μ as $j_\mu = \frac{\rho}{m} v_\mu$.

4.1.3 The equations of motion

- The equation of motion associated with the variation of $\vec{v}$ gives the velocity in terms of the Clebsch parametrization,

$$\vec{v} = \vec{\nabla} \theta + \alpha \vec{\nabla} \beta. \tag{4.9}$$

- The variation with respect to θ yields the equation of motion

$$\partial_\mu j^\mu = \partial_t \rho + \vec{\nabla} \cdot (\rho \vec{v}) = 0 , \tag{4.10}$$

  which is the continuity equation.

- The equations of motion that follow from the variation of β, α are

$$\begin{aligned} j^\mu \partial_\mu \alpha &= \rho(\dot{\alpha} + \vec{v} \cdot \vec{\nabla} \alpha) \equiv \rho \frac{D\alpha}{Dt} = 0 \\ j^\mu \partial_\mu \beta &= \rho(\dot{\beta} + \vec{v} \cdot \vec{\nabla} \beta) \equiv \rho \frac{D\beta}{Dt} = 0 , \end{aligned} \tag{4.11}$$

  where in the first equation we have used the continuity equation $\partial_\mu j^\mu = 0$.

- Varying ρ gives the Bernoulli-type equation,

$$\dot{\theta}+\alpha\dot{\beta}+\frac{\vec{v}^2}{2}=-\frac{\delta}{\delta\rho}\int d^d\vec{r}V\equiv -1-\frac{P}{\rho}\,, \tag{4.12}$$

  and the Euler equation then is obtained from a combination of these equations, namely from a derivative of the Bernoulli equation, using the α,β equations.

4.1.4 Hamiltonian formulation

The momentum conjugate to a given field ϕ_i is given by

$$\pi_i=\frac{\delta\mathcal{L}}{\delta\dot{\phi}_i}\,, \tag{4.13}$$

and the corresponding Hamiltonian density $\mathcal{H}$ is

$$\mathcal{H}=\sum_i\pi_i\dot{\phi}_i-\mathcal{L}. \tag{4.14}$$

For the Lagrangian density (4.1) the only nontrivial conjugate momenta are

$$\pi_\theta=-\rho,\qquad \pi_\beta=-\alpha\rho, \tag{4.15}$$

and all the other vanish,

$$\pi_\alpha=\pi_\rho=\pi_{\vec{v}}=0. \tag{4.16}$$

The associated Hamiltonian density is

$$\mathcal{H}=-\rho\dot{\theta}-\rho\alpha\dot{\beta}-\mathcal{L}=\rho\vec{v}\cdot(\vec{\nabla}\theta+\alpha\vec{\nabla}\beta)+\frac{1}{2}\rho\vec{v}^2+V(\rho). \tag{4.17}$$

This Hamiltonian density, not surprisingly, coincides with T_{00}, that will be derived as a Noether current in (4.19) and from the coupling to background metric, as we will do later, in (5.69).

From the Hamiltonian formulation one can proceed and canonically quantize the system. The fact that $\pi_\rho=\pi_{\vec{v}}=\pi_\alpha=0$ implies that these are constraints and the canonical quantization has to proceed using the Dirac brackets. We will leave this to a future research project.[4]

4.1.5 Symmetries

The symmetries of the action of the nonrelativstic fluid are:

- Invariance under space-time translations. The corresponding Noether current, the energy momentum tensor, is given by

$$T_{\mu\nu}=-\sum_{\phi^q=\theta,\beta,\rho,\vec{v},\alpha}\frac{\partial\mathcal{L}}{\partial\partial^\mu\phi^q}\partial_\nu\phi^q+\eta^{\mu\nu}\mathcal{L}\,, \tag{4.18}$$

[4]Although the system was only classically described, one could imagine, for instance, a quantum fluid, as in our motivation, the strongly-coupled quark-gluon plasma (sQGP) obtained at RHIC and ALICE.

so

$$
\begin{aligned}
T_{00} &= -\rho(\partial_t\theta + \alpha\partial_t\beta) - \mathcal{L} = \frac{1}{2}\rho v^2 + V \\
T_{0i} &= -\rho(\partial_i\theta + \alpha\partial_i\beta) = -\rho v_i \Rightarrow T^0{}_i = \rho v_i \\
T_{ij} &= +\rho(\partial_i\theta + \alpha\partial_i\beta)v_j + \delta_{ij}\mathcal{L} \\
&= +\rho v_i v_j - \delta_{ij}\left(-\rho\frac{\delta}{\delta\rho}\int V + V\right),
\end{aligned} \tag{4.19}
$$

where in $\mathcal{L}$ we have used the equation of motion for $\vec{v}$, $\vec{v} = \vec{\nabla}\theta + \alpha\vec{\nabla}\beta$, and the Bernoulli equation. Note that the bracket multiplying $-\delta_{ij}$ becomes in the nonrelativistic limit $-P + \dots$.

Since in the nonrelativistic limit

$$
V \simeq \rho + \dots, \tag{4.20}
$$

the conservation of the energy-momentum tensor reads

$$
\begin{aligned}
0 = \partial_t T_{00} - \partial_i T_{i0} &= \partial_t\left(\frac{\rho v^2}{2} + V\right) + \partial_i(\rho v_i) \\
&\simeq [\partial_t\rho + \partial_i(\rho v_i)] + \partial_t(\rho v^2) \\
0 = -\partial_t T_{0i} + \partial_j T_{ji} &= \rho[\partial_t v_i + v_j\partial_j v_i - f_i] + v_i[\partial_t\rho + \partial_j(\rho v_j)],
\end{aligned} \tag{4.21}
$$

the first being the continuity equation (plus higher order terms), and the second a linear combination of the continuity equation, and the Euler equation with force per particle $f_i = \partial_i\left(V - \rho\frac{\delta}{\delta\rho}\int V\right)$.

- Invariance under rotations. The angular momentum charge is given by

$$
J_{ij} = \int d^3x J_{0ij} = \int d^3x(T_{0i}x_j - T_{0j}x_i) = \int d^3x\rho(v_i x_j - v_j x_i). \tag{4.22}
$$

  Needless to say that the nonrelativistic action is not invariant under boosts.

- The action is invariant under the time reversal symmetry

$$
t \to -t \qquad \theta(-t) = -\theta(t) \qquad \beta(-t) \to -\beta(t), \tag{4.23}
$$

  where obviously the flip of sign of β can be interchanged by a flit of signs of α, these latter two implying $\vec{v} \to -\vec{v}$, as they should.

- In a similar manner, the parity transformations

$$
\vec{x} \to -\vec{x} \qquad \theta(-\vec{x}) = -\theta(\vec{x}) \qquad \beta(-\vec{x}) \to -\beta(\vec{x}) \tag{4.24}
$$

  also keep the action invariant.

- A shift symmetry of θ,
$$\theta \to \theta + a. \tag{4.25}$$
The associated conserved current is the fluid current, given in (4.2),
$$j_{(\theta)\mu} = \rho v_\mu = (\rho, \rho\vec{v})\,, \tag{4.26}$$
whose conservation we saw that corresponds to the $\vec{v}$ equation of motion.

- A shift symmetry of β,
$$\beta \to \beta + b. \tag{4.27}$$
The associated current is
$$j_{(\beta)\mu} = \alpha\rho v_\mu = \alpha(\rho, \rho\vec{v}). \tag{4.28}$$
The conservation of this current is the equation of motion
$$\partial^\mu j_{\mu\beta} = \partial^\mu(\alpha\rho v_\mu) = \rho\frac{D\alpha}{Dt} = 0. \tag{4.29}$$

4.1.6 Nonrelativistic fluid coupled to electromagnetism

The coupling to external electromagnetism was obtained in [1] by replacing $a_\mu \to a_\mu + A_\mu$,
$$a_\mu \to a_\mu + A_\mu = \partial_\mu\theta + \alpha\partial_\mu\beta + A_\mu\,, \tag{4.30}$$
where A_μ is the electromagnetic field.

The physical meaning of this coupling is that the fluid current j^μ takes the role of the electromagnetic current j^μ_{EM}, and in particular the fluid density ρ is now the electric charge density.

The equations of motion of the coupled system follow again from the variations of $\rho, \vec{v}, \theta, \alpha$ and β:

- A variation with respect to A_μ is relevant only when the gauge fields are dynamical, namely by adding the $-\frac{1}{4}F_\mu F^{\mu\nu}$ term to the Lagrangian density $\mathcal{L}$. In that case one gets the familiar Maxwell's equations $\partial^\mu F_{\mu\nu} = j_\nu$. Here we consider a coupling to an electromagnetic background with no backreaction, so we should not vary the Lagrangian density with respect to A_μ.

- The variation with respect to θ, α and β yield the same equations as for the uncoupled system.

- From the variation with respect to $\vec{v}$ one finds
$$\vec{v} = \vec{\nabla}\theta + \alpha\vec{\nabla}\beta + \vec{A}. \tag{4.31}$$

- The variation with respect to ρ yields the same equation as (4.12).

  Again if we take the ∂^i derivative and use the other equations of motion we get the electromagnetic-coupled Euler equation (with $q = 1$, $m = 1$),
$$\begin{aligned}\partial_t v^i + (\vec{v}\cdot\vec{\nabla})v^i &= E^i + [\vec{v}\times\vec{B}]^i - \partial^i\frac{\delta}{\delta\rho}\int drV\\ \frac{\delta}{\delta\rho}\int drV &= 1 + \frac{P}{\rho}.\end{aligned} \tag{4.32}$$

The symmetries of the coupled system are the same as those of the uncoupled one. A realization of gauge invariance requires turning the gauge fields into dynamical fields and incorporating the backreaction of the fluid on the electric and magnetic fields.

4.2 The relativistic fluid

The relativistic fluid is by definition described by a Lorentz invariant action. The Lagrangian density is a functional of $a^\mu = \partial^\mu\theta + \alpha\partial^\mu\beta$. The fluid degrees of freedom can be expressed either in terms of j^μ or in terms of ρ and u^μ. These two options yield different equations of motion. It is convenient to define

$$j \equiv \sqrt{j^\mu j_\mu/c^2} = \rho\sqrt{1-\vec{v}^2/c^2} = \frac{\rho}{\gamma}\,, \qquad j_\mu = j u_\mu. \tag{4.33}$$

Using this definition, [1] proposed the Lagrangian density

$$\mathcal{L} = -j^\mu a_\mu - f(j)\,, \tag{4.34}$$

where f is an arbitrary function of the relativistic number density j. For the free case, corresponding to the nonrelativistic Lagrangian with $V=0$, f takes the form of

$$f(j) = f_0(j) = jc^2\,; \qquad \mathcal{L} = -j^\mu a_\mu - jc^2. \tag{4.35}$$

In the presence of a potential V, the non-relativistic limit of this action has

$$f(j) = jc^2 + V(j)\,, \quad j \simeq \rho - \frac{1}{2c^2}\rho\vec{v}^2\,, \quad u^\mu \simeq (c,\vec{v}). \tag{4.36}$$

The dimension assignment in this relativistic case is:

- Assuming the dimension of the auxiliary gauge field is the canonical $[a_\mu]=1$, then again $[j_\mu]=3$, so that the $f(j)$ term has to be of the form

$$f(j) \to m^4 f\left(\frac{j}{m^3}\right). \tag{4.37}$$

4.2.1 Equations of motion

- The variation with respect to θ is the same as in (4.10). This yields, as before, the continuity equation. The variations with respect to α and β are given in (4.11).

- At this point there are two options for what are the fields with respect to which one varies the Lagrangian density: (i) variation with respect to j^μ; (ii) variation with respect to ρ and u^μ. The former is manifestly relativistic invariant and the latter is not, since ρ is a zero component of a vector and not a scalar. These two options are different, since in the first option the components j^i are composite of two fields, ρ and v^i of the second one, and a variation with respect to products of fields is different than the separate variations of them.

- The variation with respect to j^μ yields

$$a_\mu = -\frac{u_\mu}{c^2} f'(j) = -\frac{j_\mu}{jc^2} f'(j). \tag{4.38}$$

It was shown in [1] how to get from this equation of motion the relativistic Euler equation. One first takes the curl of the last equation

$$\partial_\mu a_\nu - \partial_\nu a_\mu = \frac{1}{c^2}(\partial_\nu(u_\mu f'(j)) - \partial_\mu(u_\nu f'(j))). \tag{4.39}$$

The left hand side is equal to $\partial_\mu \alpha \partial_\nu \beta - \partial_\nu \alpha \partial_\mu \beta$, which vanishes when projected on u^μ due to (4.11). In this way we get the relativistic Euler equation

$$u^\mu \partial_\mu \left[u_\nu f'(j)\right] - \partial_\nu f'(j) = 0. \tag{4.40}$$

- Next we check the separate variation with respect to ρ and u^μ. The variation of ρ is

$$-\frac{u^\mu a_\mu}{\gamma} - \frac{f'(j)}{\gamma} = 0. \tag{4.41}$$

This equation is the same as the equation (4.38), upon multiplying it with u^μ.

- The variation with respect to $\vec{v}$ of the free action yields

$$-j\vec{a} + \frac{\rho\vec{v}}{\sqrt{1-\frac{v^2}{c^2}}} = 0 \rightarrow \qquad \vec{a} = \frac{\vec{v}}{\left(1-\frac{v^2}{c^2}\right)}. \tag{4.42}$$

In the non-relativistic limit this coincides with (4.9).

4.2.2 Hamiltonian formulation

As in the nonrelativistic case, here also only θ and β have nontrivial conjugate momenta, $\pi_\theta = -\rho$ and $\pi_\beta = -\rho\alpha$, so that the corresponding Hamiltonian density is

$$\mathcal{H} = -\rho\dot{\theta} - \rho\alpha\dot{\beta} - \mathcal{L} = \rho\vec{v}^2 + f(j)\,, \tag{4.43}$$

which indeed in the nonrelativistic limit goes into (4.17).

4.2.3 Symmetries

- The symmetries for this system are the same as those of the non-relativistic case, apart from the fact that now the action is also invariant under boosts.
- The energy-momentum tensor that follows from the Noether procedure reads

$$\begin{aligned} T_{\mu\nu} &= -\sum_{\phi_q=\theta,\beta} \frac{\partial\mathcal{L}}{\partial(\partial^\mu\phi_q)}\partial_\nu\phi_q + \eta_{\mu\nu}\mathcal{L} \\ &= -\frac{u_\mu u_\nu}{c^2} j f'(j) + \eta_{\mu\nu}(jf'(j) - f(j))\,, \end{aligned} \tag{4.44}$$

where we have used (4.38). As was shown in [1], the conservation of the energy-momentum tensor translates to the continuity and relativistic Euler equations as follows:

$$\partial^\mu T_{\mu\nu} = -\frac{1}{c^2}\left[\partial^\mu(ju_\mu)u_\nu f'(j) + j\left\{u_\mu\partial^\mu(u_\nu)f'(j) - c^2\partial_\nu f'(j)\right\}\right]. \tag{4.45}$$

It is easy to see that the vanishing of the first term corresponds to the continuity equation, and the vanishing of the second term is the relativistic Euler equation.

4.3 The coupling of the relativistic fluid to electromagnetism

The coupling of a charged relativistic fluid to electromagnetic fields is done in exactly the same way as for the nonrelativistic fluid, namely introducing the replacement (4.30) into the action (4.34), so

$$\mathcal{L} = -j^\mu(a_\mu + A_\mu) - f(j)\,. \tag{4.46}$$

The procedures of writing the equations of motion, the Hamiltonian formulation, the symmetries and the conservation laws follow the same line as before.

The right-hand side of the Euler equation (4.40) becomes just $c^2 j^\mu F_{\mu\nu}$, the correct Lorentz force coupling to electromagnetism.

5 The Lagrangian formalism for non-Abelian fluid dynamics

5.1 Preliminary discussion

Upon uplifting an ordinary fluid and a fluid that couples to Abelian gauge fields to non-Abelian fluid and a fluid that couples to non-Abelian gauge fields, one has to devise an uplift to the basic fields of the fluid theory namely: ρ, the fluid density, u^μ, the flow velocity, j^μ, the fluid current and θ, α and β, the Clebsch parametrization of the auxiliary gauge fields a_μ. In this section we propose two prescriptions of performing the uplift for non-Abelian fluids, first without coupling to non-Abelian gauge fields and then with such coupling.

- Obviously, the current and the auxiliary gauge field have to be mapped into elements of the algebra of the non-Abelian group, namely

$$j_\mu \to j_\mu = j^a_\mu T^a \qquad a_\mu \to a_\mu = a^a_\mu T^a. \tag{5.1}$$

- It is thus clear how to uplift j:

$$j \equiv 2\sqrt{\mathrm{Tr}[j^\mu j_\mu]} = \sqrt{j^{\mu a} j^a_\mu}. \tag{5.2}$$

- The uplift of the auxiliary gauge fields a_μ implies a prescription of the uplift to the Clebsh fields θ, α and β. The prescription should be such that when one goes back from a non-Abelian group $\mathcal{G}$ to the Abelian group U(1), the ordinary Clebsch parameters are retrieved. We propose two methods to achieve this goal:

  (1) We make use to group elements

$$a_\mu = \partial_\mu\theta + \alpha\partial_\mu\beta \to -i(g^{-1}\partial_\mu g + \alpha h^{-1}\partial_\mu h)\,, \tag{5.3}$$

  where $g, h \in \mathcal{G}$. It is obvious, as expected, that for a $G = \mathrm{U}(1)$ group the parameters are the Clebsch ones.

  (2) We use auxiliary fields that are in the algebra of the non-Abelian group

$$a_\mu = \partial_\mu\theta + \alpha\partial_\mu\beta \to a^a_\mu = \partial_\mu\theta^a + d^a_{bc}\alpha^b\partial_\mu\beta^c\,. \tag{5.4}$$

In the following subsection we adopt these uplifts of the Clebsch coefficients and develop the corresponding Lagrangian formulations, equations of motion, Hamiltonian formulations, symmetries and the coupling to non-Abelian gauge fields.

Then, in the subsection after that, we summarize the approach of [1] including the generalization to flavored fluids.

After that, we consider another formulation for a gluonic fluid coupled to Yang-Mills fields, that conserves our defined non-Abelian gluon helicities, with a Lagrangian formulation and the conservation of the energy-momentum tensor, related to the Euler equation. Some classical solutions in this set-up are then considered. Finally, we mention an attempt at an Euler equation for a "quark" (non-Abelian in the fundamental) fluid, but we find that it is somewhat trivial.

5.2 The Lagrangian formulation

Using the uplifts discussed above, the Lagrangian densities for the models of non-Abelian relativistic fluid read

$$\mathcal{L}_1 = -\operatorname{Tr}[j^\mu(g^{-1}\partial_\mu g + \alpha h^{-1}\partial_\mu h)] - f(j)\,. \tag{5.5}$$

$$\mathcal{L}_2 = -j^\mu_a(\partial_\mu\theta^a + d^a{}_{bc}\alpha^b\partial_\mu\beta^c) - f(j)\,. \tag{5.6}$$

Notice that these Lagrangian densities describe a non-Abelian fluid even in the absence of a coupling to non-Abelian gauge fields, in a similar way to what we have done above in the Abelian case. This is different than the approach of [1], where the non-Abelian fluid was defined only when the fluid couples to the non-Abelian gauge fields.

5.2.1 The dimensions of the fields

The passage to non-abelian fluid does not change the discussion of the dimensions of the fields a_μ, j_μ, ρ and $\vec{v}$. The only new ingredients in (1) are the group elements g and h, which obviously have zero dimensions, $[g] = [h] = 0$. In (2) θ^a, α^a and β^a also have zero dimensions.

5.2.2 The equations of motion for the formulation (1)

- The variation with respect to $g \in \mathcal{G}$ reads

$$\begin{aligned}&0 = \operatorname{Tr}[(-g^{-1}\delta g g^{-1}\partial_\mu g + g^{-1}\partial_\mu\delta g)j^\mu] \to \\ &\operatorname{Tr}\left[-\left\{g^{-1}\delta g\left(\partial_\mu j^\mu + [g^{-1}\partial_\mu g, j^\mu]\right)\right\}\right] = 0\,,\end{aligned} \tag{5.7}$$

  which implies that

$$D_{g_\mu} j^\mu \equiv \partial_\mu j^\mu + [A^g_\mu, j^\mu] \equiv \partial_\mu j^\mu + [g^{-1}\partial_\mu g, j^\mu] = 0. \tag{5.8}$$

  This is a non-Abelian generalization of the continuity equation. As it stands it depends on the auxiliary field A^g_μ.

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- The equation of motion associated with the variation of α is

$$\mathrm{Tr}[h^{-1}\partial_\mu h j^\mu] = 0\,, \tag{5.9}$$

which is a generalization of (4.11) of the form

$$\mathrm{Tr}[\rho^a T^a(h^{-1}\partial_t h + \vec{v}\cdot h^{-1}\vec{\nabla}h)] = 0. \tag{5.10}$$

- In a similar manner to the variation of g, the variation of h yields

$$\alpha D_{h\mu} j^\mu + j^\mu \partial_\mu \alpha \equiv \alpha\left(\partial_\mu j^\mu + [A^h_\mu, j^\mu]\right) + j^\mu\partial_\mu\alpha = 0. \tag{5.11}$$

- Variation with respect to j^μ yields

$$a_\mu = g^{-1}\partial_\mu g + \alpha h^{-1}\partial_\mu h = -\frac{u_\mu}{c^2}f'(j) = -\frac{j_\mu}{jc^2}f'(j). \tag{5.12}$$

- To get the "non-Abelian Euler equation" we follow similar steps as those used to derive (4.38). However, now we want this equation to be covariant, or at least invariant, under $\mathcal{G}$. Indeed, we would like the equation to describe, ideally, N^2 degrees of freedom, coupled to Yang-Mills fields. And we know that the coupling to Yang-Mills will be covariant, or at least invariant: it should be the "Lorentz" force $j^\mu F_{\mu\nu}$ (or, at least, its trace). We thus again take the curl of the last equation, but now we also add the commutator, to get the field strength $F_{(a)\mu\nu}$. From the left-hand side of the equation (5.12), we get

$$\begin{aligned}
\partial_\mu a_\nu - \partial_\nu a_\mu &= [A^g_\nu, A^g_\mu] + \alpha[A^h_\nu, A^h_\mu] - (\partial_\nu\alpha)A^h_\mu + (\partial_\mu\alpha)A^h_\nu\\
[a_\mu, a_\nu] &= [A^g_\mu, A^g_\nu] + \alpha^2[A^h_\mu, A^h_\nu] + \alpha\left([A^g_\mu, A^h_\nu] - [A^g_\nu, A^h_\mu]\right) \Rightarrow\\
F_{(a)\mu\nu} &= \alpha(\alpha-1)[A^h_\mu, A^h_\nu] + \alpha\left([A^g_\mu, A^h_\nu] - [A^g_\nu, A^h_\mu]\right) + (\partial_\mu\alpha)A^h_\nu - (\partial_\nu\alpha)A^h_\mu.
\end{aligned} \tag{5.13}$$

We now multiply this expression with j^μ and use the relations (5.8), (5.9) and (5.11), to eliminate $(\partial_\mu\alpha)$, obtaining

$$\begin{aligned}
j^\mu F_{(a)\mu\nu} &= \alpha(\alpha-1)j^\mu[A^h_\mu, A^h_\nu] + \alpha j^\mu\left([A^g_\mu, A^h_\nu] - [A^g_\nu, A^h_\mu]\right)\\
&\quad + \alpha[A^g_\mu - A^h_\mu, j^\mu]A^h_\nu - (\partial_\nu\alpha)j^\mu A^h_\mu.
\end{aligned} \tag{5.14}$$

Unfortunately, there is no way to make this vanish, as it did in the Abelian case (we can, in fact, check that the above vanishes in the Abelian case, since commutators vanish and $j^\mu A^h_\mu \to j^\mu\partial_\mu\beta = 0$ by the α equation of motion). That wouldn't be a problem, a priori, except for the fact that we cannot rewrite this in terms of the 4-velocity u^μ and j, it is only a function of the "Clebsch parametrization" A^g_μ, A^h_μ, α. So that does not look like extra terms in a non-Abelian Euler equation. Moreover, since it is written only in terms of the Clebsch parameters, but not in terms of the velocity and density, it is clear that the equation will not preserve the helicities we introduced (which *are* written only in terms of the velocity and density).

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We might think that taking the trace of the above could help (then we would, at least, get one equation right), but it does not. We can now eliminate the term with $\partial_\nu \alpha$, since $\mathrm{Tr}[j^\mu A^h_\mu] = 0$ by the α equation of motion, and after some manipulations inside the trace, the rest of the terms become

$$\mathrm{Tr}[j^\mu F_{(a)\mu\nu}] = -\alpha\, \mathrm{Tr}\left([A^h_\nu, j^\mu](A^g_\mu + A^h_\mu)\right) + \alpha(\alpha-1)\, \mathrm{Tr}\left(j^\mu [A^h_\mu, A^h_\nu]\right) , \tag{5.15}$$

which is still nonzero, and still not expressible in terms of u^μ and j^μ only. Only the term $A^g_\mu + A^h_\mu = a_\mu = -u^\mu f'(j)/c^2$ is, the rest are not.

Finally, the curl plus commutator of the right-hand side of the equation (5.12), multiplied by j^μ, which is what we hoped would be the true non-Abelian Euler equation (without extra terms), is again

$$\frac{j}{c^2}\left\{u^\mu \partial_\mu [u_\nu f'(j)] - \partial_\nu f'(j)\right\} + \frac{j}{c^4}[f'(j)]^2 u^\mu [u_\mu, u_\nu]\,, \tag{5.16}$$

except, of course, this is now a non-Abelian formula.

5.2.3 The equations of motion for the formulation (2)

- The variation with respect to θ^a yields, like in the abelian case, a conservation of the current with ordinary, not a covariant, derivative,

$$\partial^\mu j^a_\mu = 0. \tag{5.17}$$

- From the variation with respect to α^a we get

$$d^a{}_{bc}(\partial^\mu \beta^b) j^c_\mu = 0. \tag{5.18}$$

- In a similar manner, the variation with respect to β^a yields

$$d^a{}_{bc}\partial^\mu(\alpha^b j^c_\mu) = 0. \tag{5.19}$$

- From the variation with respect to the current j^μ_a, one finds

$$a^a_\mu = \partial_\mu \theta^a + d^a{}_{bc}\alpha^b \partial_\mu \beta^c = -\frac{j^a_\mu}{jc^2} f'(j). \tag{5.20}$$

- As before we take now the curl of the last equation,

$$\partial_\mu a^a_\nu - \partial_\nu a^a_\mu = d^a{}_{bc}(\partial_\mu \alpha^b \partial_\nu \beta^c - \partial_\nu \alpha^b \partial_\mu \beta^c) = \frac{1}{c^2}(\partial_\mu(u^a_\nu f'(j)) - \partial_\nu(u^a_\mu f'(j))). \tag{5.21}$$

  Upon using the equations of motion derived above, associated with the variations of θ^a, α^a and β^a, we have that

$$d^a{}_{bc}(\partial_\mu \alpha^b \partial_\nu \beta^c - \partial_\nu \alpha^b \partial_\mu \beta^c) j^\mu_a = 0\,, \tag{5.22}$$

  where we see that the index a is summed over (as is, of course, the Lorentz index μ), which means that this is a single equation (not a matrix equation, i.e., N^2 equations).

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Thus, in order to be able to use it, we must also multiply the equation (5.21) with j_a^μ and sum over a.

Moreover, as in the case of the formulation (1), we must consider a covariant equation, also because later we will want to couple to Yang-Mills fields, and the coupling in the Euler equation must be in terms of the covariant $F^a_{\mu\nu}$. Therefore, we must again add the commutator $[a_\mu, a_\nu]$, also multiplied with j_a^μ. But in the commutator, we can replace the same equation (5.20), and so ignore it.

Thus, we get the non-Abelian Euler equation that reads

$$j_a^\mu \partial_\mu \left[j_\nu^a \frac{f'(j)}{j} \right] - \frac{1}{2} \partial_\nu (j^2) \frac{f'(j)}{j} - j^2 \partial_\nu \left[\frac{f'(j)}{j} \right] = 0. \tag{5.23}$$

In this case, we did obtain a non-Abelian Euler equation, but it is not a matrix equation (it is a single equation, not N^2 equations, like we want).

5.2.4 Hamiltonian formulations

In analogy to the Abelian case, for the case (1), only g and h (but not α or j^a_μ) have nontrivial conjugate momenta, $\pi_g^a = -(j^0 g^{-1})^a$ and $\pi_h^a = -\alpha(j^0 h^{-1})^a$, so the Hamiltonian density is

$$\mathcal{H} = -\operatorname{Tr}[j^0(g^{-1}\partial_0 g + \alpha h^{-1}\partial_0 h)] - \mathcal{L} = \operatorname{Tr}\left[\vec{j} \cdot \left(g^{-1}\vec{\nabla} g + \alpha h^{-1}\vec{\nabla} h\right)\right] + f(j). \tag{5.24}$$

For the Lagrangian density (2), θ^a and β^a have conjugate momenta

$$\pi_{\theta^a} = -j_0^a = -\rho^a \qquad \pi_{\beta^a} = -d^a{}_{bc}\alpha^b j_0^c = -d^a{}_{bc}\alpha^b \rho^c. \tag{5.25}$$

The corresponding Hamiltonian density is

$$\mathcal{H} = -\rho^a \dot{\theta}_a - d^a{}_{bc}\alpha^b \rho^c \dot{\beta}_a - \mathcal{L} = \vec{j}^a(\vec{\nabla}\theta_a + d_{abc}\alpha^b \vec{\nabla}\beta^c) + f(j). \tag{5.26}$$

5.2.5 The symmetries

- As in the Abelian case, we have Lorentz symmetry, parity and time-reversal symmetry, where in the case (1) we have $g(-t) = -g(t)$, $h(-t) = -h(t)$ and $g(-\vec{x}) = -g(\vec{x})$, $h(-\vec{x}) = -h(\vec{x})$, and in the case (2) we have the obvious non-Abelian generalization of the Abelian case.

- In the case (1), we also have $\mathcal{G}$ symmetry invariance, separately for g and for h, so we have really $\mathcal{G} \times \mathcal{G}$ invariance.

- Translational invariance, giving the Noether current = energy-momentum tensor. In the case (1),

$$\begin{aligned} T_{\mu\nu} &= -\sum_{\phi_q = g,h} \frac{\partial \mathcal{L}}{\partial \partial^\mu \phi_q} \partial_\nu \phi_q + \eta_{\mu\nu}\mathcal{L} \\ &= \operatorname{Tr}\left[j_\mu \left(g^{-1}\partial_\nu g + \alpha h^{-1}\partial_\nu h\right)\right] - \eta_{\mu\nu}\left\{\operatorname{Tr}\left[j^\rho\left(g^{-1}\partial_\rho g + \alpha h^{-1}\partial_\rho h\right)\right] + f(j)\right\}, \end{aligned} \tag{5.27}$$

whereas in the case (2), we have

$$\begin{aligned} T_{\mu\nu} &= -\sum_{\phi_q=\theta,\beta} \frac{\partial \mathcal{L}}{\partial \partial^\mu \phi_q} \partial_\nu \phi_q + \eta_{\mu\nu}\mathcal{L} \\ &= \left[j^a_\mu \left(\partial_\nu \theta^a + d_{abc}\alpha^b \partial_\nu \beta^c \right) \right] - \eta_{\mu\nu} \left\{ j^\rho_a \left(\partial_\rho \theta^a + d_{abc}\alpha^b \partial_\rho \beta^c \right) + f(j) \right\} , \end{aligned} \tag{5.28}$$

5.2.6 The fluid coupled to non-Abelian gauge fields

So far we have discussed a non-Abelian fluid, but without coupling it to non-Abelian gauge fields. This was the analog of ordinary Abelian fluid not coupled to external electromagnetic fields. Turning on a coupling to a background of non-Abelian gauge fields, one has to add to the auxiliary non-Abelian gauge field a_μ a genuine gauge field A_μ. It is important to emphasize that similarly to the coupling to the electromagnetic fields, here also we do not include the backreaction of the fluid on the non-Abelian gauge fields. The incorporation of the interaction is thus done by the replacement $a_\mu \to a_\mu + A_\mu$, which in the case (1) becomes

$$a_\mu \to a_\mu + A_\mu = -i(g^{-1}\partial_\mu g + \alpha h^{-1}\partial_\mu h) + A^a_\mu T^a \,, \tag{5.29}$$

so that the Lagrangian density reads

$$\mathcal{L} = -\operatorname{Tr}\left[j^\mu (g^{-1}\partial_\mu g + \alpha h^{-1}\partial_\mu h + A_\mu) \right] - f(j) \,, \tag{5.30}$$

and in the case (2) becomes

$$a^a_\mu \to a^a_\mu + A^a_\mu = \partial_\mu \theta^a + d^a{}_{bc}\alpha^b \partial_\mu \beta^c + A^a_\mu \,, \tag{5.31}$$

and the Lagrangian becomes

$$\mathcal{L} = -j^\mu_a (\partial_\mu \theta^a + d^a{}_{bc}\alpha^b \partial_\mu \beta^c + A^a_\mu) - f(j). \tag{5.32}$$

The "Euler" equation will now have the source

$$j^\mu F_{\mu\nu} \,, \tag{5.33}$$

as a matrix source in the case (1), which is the correct "Lorentz" force coupling to Yang-Mills fields. Except, of course, we saw that the resulting Euler equation is not very useful. And the source $j^\mu_a F^a_{\mu\nu}$ in the case (2), which is good, but it gives only one equation, as we said.

5.3 The approach of [1] for a fluid coupled to non-Abelian gauge fields

A different uplift of the Clebsh coefficients that leads to a different Lagrangian formulation was proposed in [1]. This paper considers a non-Abelian current $J^\mu_a(t, \vec{r})$ that, however, splits into a standard Abelian current $j^\mu(t, \vec{r}) = \rho(t, \vec{r}) u^\mu(t, \vec{r})$, with an Abelian 4-velocity u^μ, together with a non-Abelian charge $Q_a(t, \vec{r})$, obeying a Wong equation along the fluid particle worldlines,

$$\frac{d\mathcal{Q}_a(\tau)}{d\tau} + f_{abc} \frac{dX^\mu(\tau)}{d\tau} A^b_\mu (X^\nu(\tau)) \mathcal{Q}_c(\tau) = 0 \,, \tag{5.34}$$

which can be expressed as the covariant conservation

$$j^\mu (D_\mu Q)_a = 0\,, \tag{5.35}$$

and is the equivalent of the α_a, β^a equations of motion in the Abelian case. If it is satisfied, this, together with the conservation $Q\partial^\mu j_\mu = 0$, gives

$$D_\mu (j^\mu Q) = D_\mu J^\mu = 0\,. \tag{5.36}$$

The nonrelativistic non-Abelian Lagrangian is written in terms of a an arbitrary group element $g \in \mathcal{G}$, taking the role of the parameters of the Clebsch parametrization (θ, α, β), and a fixed Lie algebra element T_0 (so $g = \exp[i\alpha^a T_a]$), as[5]

$$\mathcal{L} = j^\mu 2\,\mathrm{Tr}[T_0 g^{-1} D_\mu g] - f(j) + \mathcal{L}(A_\mu)\,, \quad D_\mu = \partial_\mu + A_\mu. \tag{5.37}$$

Note the normalization is $\mathrm{Tr}[T^a T_b] = -(1/2)\delta^a_b$. This is invariant under the gauge group $g \in \mathcal{G}$ as follows: left action on g, $g \to U \cdot g$, and adjoint action on A_μ, $A_\mu \to \mathrm{U}(A_\mu + \partial_\mu)U^{-1}$.

If we ignore the A_μ Lagrangian $\mathcal{L}(A_\mu)$, we get coupling with *external* (non-dynamical) electromagnetism, just like in the cases studied in [2, 6].

Then:

- The equation of motion of g gives the current conservation, (5.36).

- Varying with respect to j^μ gives a sort of Bernoulli equation, and then taking ∂_ν on it and antisymmetrizing in (μ, ν), manipulating the result, one obtains the non-Abelian Euler-type equation

$$\frac{ju^\mu}{c^2}\partial_\mu (u_\nu f'(j)) - j\partial_\nu f'(j) = 2\,\mathrm{Tr}[J^\mu F_{\mu\nu}] = -j^\mu Q_a F^a_{\mu\nu}\,, \tag{5.38}$$

  with the right-hand side describing the non-Abelian Lorentz force term, the generalization of the Abelian term $j^\mu F_{\mu\nu}$, which for $\nu = i$ gives

$$j^\mu F_{\mu i} = \rho \left(\vec{E} + \frac{\vec{v}}{c} \times \vec{B}\right)_i\,, \tag{5.39}$$

  and the left-hand side the Euler equation multiplied by ρ.

  Indeed, the non-relativistic limit is now

$$\partial_t \vec{v} + \vec{v} \cdot \vec{\nabla}\vec{v} + \frac{\vec{\nabla} P}{\rho} = Q^a \left[\vec{E}_a + \frac{\vec{v}}{c} \times \vec{B}_a\right]\,, \tag{5.40}$$

  and $E^i_a \equiv cF^a_{0i}$, $B^i_a \equiv -\frac{c}{2}\epsilon^{ijk}F^a_{jk}$.

[5] Note that in [26] a related version of a non-Abelian Clebsch parametrization was considered, and Chern-Simons terms were also analyzed, but in this case still we had a fixed Lie algebra element projection, and the velocity u^μ and current j^μ were still Abelian. Further, in [27], various such Chern-Simons terms were added to the action as topological terms, but were not considered as non-Abelian helicities, as we do.

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5.4 Flavored colored fluid

One can consider the generalization corresponding to several types of fluid velocity, which is by choosing several directions in the Lie algebra, T_f, such that the T_f's commute among each other, i.e., they belong to the Cartan subalgebra, and equal in number the rank of the group (so N for U(N)). This is then somewhat similar to the case of the quark fluid that we considered in the previous section. Each component f has an associated velocity, density and current, so u^μ_f, j_f, j^μ_f, and one has

$$\begin{aligned}J^\mu &= \sum_{f=1}^{N} Q_f j^\mu_f = \sum_{f=1}^{N}\sum_{a=1}^{N^2} Q_{af}T^a j^\mu_f \\ Q_f &= Q_{af}T^a = gT_f g^{-1}.\end{aligned} \tag{5.41}$$

The Lagrangian is a sum over f, specifically

$$\mathcal{L} = \sum_{f=1}^{N} j^\mu_f 2\,\mathrm{Tr}[T_f g^{-1}D_\mu g] - f(\{j_f\}) + \mathcal{L}(A^a_\mu)\,, \tag{5.42}$$

with the Wong equation

$$\sum_f j^\mu_f D_\mu Q_f = 0. \tag{5.43}$$

One can write would-be Euler equations for each f channel, so

$$\frac{j_f u^\mu_f}{c^2}\partial_\mu\left(u^\nu_f\frac{\partial f}{\partial j_f}\right) - j_f\partial^\nu\frac{\partial f}{\partial j_f} = 2\,\mathrm{Tr}\left[j^\mu_f(D_\mu Q_f)(D^\nu g)g^{-1}\right] + 2\,\mathrm{Tr}[j_{\mu f}Q_f F^{\mu\nu}]. \tag{5.44}$$

However, as we can easily see, *only summing over f do we get something like the Euler equation coupling on the right-hand side*, obtaining

$$\sum_f\left\{\frac{j_f u^\mu_f}{c^2}\partial_\mu\left(u^\nu_f\frac{\partial f}{\partial j_f}\right) - j_f\partial^\nu\frac{\partial f}{\partial j_f}\right\} = -\sum_f j_{\mu f}Q_{af}F^{a\mu\nu}\,, \tag{5.45}$$

and the right-hand side becomes, for $\nu = i$, the Lorentz force

$$\sum_f \rho_f Q_{af}\left(\vec{E}_a + \frac{\vec{v}_f}{c}\times\vec{B}_a\right)_i. \tag{5.46}$$

So, this summed over Euler equation in the nonrelativistic limit becomes (with $f = \sum_f j_f c^2 - V$)

$$\sum_f \rho_f\left[\partial_t\vec{v}_f + \vec{v}_f\cdot\vec{\nabla}\vec{v}_f\right] + \vec{\nabla}P = \sum_f \rho_f Q_{af}\left(\vec{E}_a + \frac{\vec{v}_f}{c}\times\vec{B}_a\right)\,, \tag{5.47}$$

but **the non-summed Euler equation doesn't have a simple answer on the right-hand side, so cannot be understood as a true Euler equation**.

In conclusion, this case only gives *one* sensible Euler equation, and not one for each fluid component (and in any case the fluid components were quark-like, not gluon-like).

5.5 Nonabelian gluonic fluid coupled to Yang-Mills fields

Different non-Abelian fluid Euler equations were considered, for instance, in [1] and [8]. However, as we said, our goal is to write non-Abelian generalizations of the Euler equation that can conserve the helicities we defined, and perhaps even be obtained from a Lagrangian. Also, as opposed to the case in [1], we will obtain Euler equations for each adjoint index a, and not only a single one.

For the (effective) gluonic fluid, interacting with Yang-Mills fields, we then propose the generalized Euler equation

$$\left[\partial_t \vec{v}_a + \vec{v}_a \cdot \vec{\nabla} \vec{v}_a\right] + \vec{\nabla}\mu_a = Q_a \left(\vec{E}_a + \frac{\vec{v}_a}{c} \times \vec{B}_a\right) , \tag{5.48}$$

with no sum over a, and with (again, without sum over a, unless specifically written)

$$\Pi_\mu = \sum_a \Pi^a_\mu T_a\,, \quad \vec{\Pi}_a = m_a \vec{v}_a + Q_a \vec{A}_a\,, \quad \Pi_{0a} = -\mu_a - m_a \frac{\vec{v}_a^2}{2} + A_{0a}. \tag{5.49}$$

Using the 4-dimensional notation (even though we have a nonrelativistic theory), as done in [2] in the Abelian case, for Π_μ and the 4-current

$$j^a_\mu = \rho^a u^a_\mu\,, \tag{5.50}$$

the gluonic Euler equation (5.48) implies (with no sum over a)

$$j^{a\mu}\Omega^a_{\mu\nu} = 0\,. \tag{5.51}$$

Note that $\Omega^a_{0i} = E^a_i, \Omega^a_{jk} = \epsilon_{ijk} B^a_i$.

This 4-dimensional form of the non-Abelian Euler equation can be used, like in the Abelian case in [2], to prove the conservation of the helicities we have defined.

Indeed, defining

$$\Pi = \sum_a \Pi^a T_a\,, \quad \Omega = \sum_a \Omega^a T_a\,, \quad \Omega = d\Pi + \Pi \wedge \Pi\,, \tag{5.52}$$

the helicity densities

$$h^a = \mathrm{Tr}\left[T^a \left(\Pi \wedge d\Pi + \frac{2}{3}\Pi \wedge \Pi \wedge \Pi\right)\right] , \tag{5.53}$$

obey

$$dh^a = \mathrm{Tr}\left[T^a \left(\Omega \wedge \Omega\right)\right]. \tag{5.54}$$

Then, by a generalization of the argument in the Abelian case, we have $\Omega^a \wedge \Omega^b = 0$, since $\Omega^a \wedge \Omega^b$ is a 4-form, thus proportional to the volume form. But, if we contract with $j^{a\mu}$, so (with no sum over a) $j^{a\mu}\Omega^a_{[\mu\nu}\Omega^b_{\rho\sigma]} = 0$, we get zero because of the Euler equation above. But that is only possible if the original proportionality constant was zero, so $\Omega^a \wedge \Omega^b$ was zero to begin with. Explicitly, with indices, we have the identity

$$\epsilon^{\alpha\nu\lambda\rho} 4\Omega^a_{[\mu\nu}\Omega^b_{\lambda\rho]} = \delta^\alpha_\mu \epsilon^{\tau\nu\lambda\rho}\Omega^a_{\tau\nu}\Omega^b_{\lambda\rho}\,, \tag{5.55}$$

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which is correct, since both α and μ must be different than $\nu\lambda\rho$, hence must be equal, and the rest follows.

But then, multiplying with $j^{a\mu}$ gives zero because of the gluonic Euler equation, while on the right-hand side we get $j^{a\alpha}(\Omega^a \wedge \Omega^b)$, so $\Omega^a \wedge \Omega^b = 0$. But then

$$dh^a = \sum_{b,c} \mathrm{Tr}[T^a T^b T^c]\Omega^b \wedge \Omega^c = 0\,, \tag{5.56}$$

so all the N^2 $\mathcal{H}^{NA\,a}_{mm}$ (as well as the singlet one $\mathcal{H}^{NA}_{\text{total}}$, by the same argument) helicities are conserved!

5.6 Lagrangian formulation and dynamical Yang-Mills fields

Now we want to obtain the equations (5.48), together with the non-Abelian current conservation equation, from a Lagrangian.

Unlike the case in [1], we consider $j^\mu = j^{\mu a}T_a$ a fully nonabelian current, without a split into j^μ and Q^a, so that we have a chance to obtain the Euler equation with an index a.

The action we try is then (note that there is no $\alpha h^{-1}\partial_\mu h$ term, and $g^{-1}\partial_\mu g + A_\mu$ is replaced by $g^{-1}D_\mu g$, with respect to the previous attempt)

$$S = \int d^4x \left\{ \mathrm{Tr}[j^\mu g^{-1} D_\mu g] - \sum_a \sqrt{-j^{\mu a} j^a_\mu} - V\left(\sum_a \sqrt{-j^{\mu a} j^a_\mu}\right) + \mathcal{L}(A^a_\mu) \right\}, \tag{5.57}$$

where V is a potential, depending on the nonrelativistic non-Abelian generalization of the $V(j)$ in the Abelian case,

$$\begin{aligned} D_\mu g &= \partial_\mu g + A_\mu g \\ j^{\mu a} &= \rho^a_0 u^{\mu a} = \rho^a \sqrt{1 - \vec{v}_a^2}\,\frac{dx^{\mu a}}{d\tau}\,, \end{aligned} \tag{5.58}$$

and $u^{\mu a}$ is the 4-velocity of the fluid particle of a type.

Note that

$$-j^{a\mu} j^a_\mu = \rho^a_0 \Rightarrow \frac{d}{dj^{\mu a}}\sqrt{-j^{\mu a} j^a_\mu} = u^a_\mu\,. \tag{5.59}$$

Then we obtain:

- the j^a_μ equation of motion as a simpler non-Abelian generalization of the Clebsch parametrization of the velocity (4.9),

$$u^a_\mu(1 + V') = (g^{-1}D_\mu g)^a. \tag{5.60}$$

- the continuity equation is obtained from the variation of the action with respect to g, *after which we fix the gauge* $g = 1$, which gives

$$(D_\mu j^\mu)^a = 0. \tag{5.61}$$

- the A^a_μ equation of motion, if we consider the gauge fields as dynamical, is

$$(D_\nu F^{\nu\mu})^a = j^{\mu a}. \tag{5.62}$$

If we ignore $\mathcal{L}(A^a_\mu)$, the gauge fields are treated as external.

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Taking ∂_μ (5.60)$_\nu$ $-(\mu \leftrightarrow \nu)$ gives

$$\partial_\mu(g^{-1}D_\nu g)^a - \partial_\nu(g^{-1}D_\mu g)^a = (\partial_\mu u^a_\nu - \partial_\nu u^a_\mu)(1+V') + u^a_\mu \partial_\nu V' - u^a_\nu \partial_\mu V'. \tag{5.63}$$

Multiplying by $\rho_0^a u^{\mu a}$, ignoring the V' in $(1+V')$ and $u^{\mu a}\partial_\mu V'$ as being subleading in the nonrelativistic limit (otherwise we keep it there), and since $u^{\mu a}\partial_\nu u^a_\mu = \partial_\nu(u^{\mu a}u^a_\mu) = \partial_\nu(-1) = 0$, the right-hand side gives approximately

$$\rho_0^a u^{\mu a}\partial_\mu u^a_\nu - \rho_0^a \partial_\nu V', \tag{5.64}$$

which for $\nu = i$ and in the nonrelativistic limit gives the left-hand side of the Euler equation (with an index a).

The left-hand side of (5.63), contracted with T_a and taking the trace, gives

$$\begin{aligned}\partial_\mu \operatorname{Tr}[g^{-1}D_\nu g] - \partial_\nu \operatorname{Tr}[g^{-1}D_\mu g] &= \operatorname{Tr}[D_\mu(g^{-1}D_\nu g) - D_\nu(g^{-1}D_\mu g)]\\ &= \operatorname{Tr}\left\{g^{-1}[D_\mu, D_\nu]g - [g^{-1}D_\mu g, g^{-1}D_\nu g]\right\}\\ &= \operatorname{Tr}\left\{g^{-1}F_{\mu\nu}g - [g^{-1}D_\mu g, g^{-1}D_\nu g]\right\}.\end{aligned} \tag{5.65}$$

But now we peel off the T_a with the trace (so the $(T_a)_{ji}$), by multiplication with $(T^a)^{kl}$ and completeness of the T_a matrices, we remember that we have (5.60), so we get

$$(g^{-1}F_{\mu\nu}g)^a - \sum_{b,c} f^a{}_{bc} u^b_\mu u^c_\nu, \tag{5.66}$$

and this was contracted with $\rho_0^a u^{\mu a}$, in order to (hope to) give the right-hand side of the Euler equation.

But we saw that the gauge $g = 1$ is needed (it was needed to obtain the correct continuity equation (5.61)), and then, putting both sides together and $g = 1$, the Euler equation *in the nonrelativistic limit* is now

$$\rho_0^a u^{\mu a}\partial_\mu u^a_\nu - \rho_0^a \partial_\nu V' = \rho_0^a F^a_{\mu\nu} u^{\mu a} - \rho_0^a \sum_{b,c} f_{abc} u^{\mu a} u^b_\mu u^c_\nu = \rho_0^a F^a_{\mu\nu} u^{\mu a}, \tag{5.67}$$

since the last term vanishes ($u^{\mu a}u^b_\mu$ is symmetric in (ab), but f_{abc} is antisymmetric).

This is indeed just the Euler equation we wanted.

5.7 Belinfante energy-momentum tensor and the Euler equation

The standard Euler equation is understood as the nonrelativistic limit of the conservation equation for the energy-momentum tensor of a fluid, $\partial^\nu T_{\mu\nu} = 0$, where

$$T^{\mu\nu} = \rho u^\mu u^\nu + P(\eta^{\mu\nu} + u^\mu u^\nu). \tag{5.68}$$

So it is natural to ask: can the same be true in the non-Abelian case presented here?

But first, we want to understand how this works in the Abelian case, not from the $T_{\mu\nu}$ of a fluid, as we did in the previous fluid cases, but from the *Belinfante* energy-momentum tensor (that should therefore give the same result as the Noether current calculation for the Abelian cases in section 4) associated with the nonrelativistic Abelian Lagrangian for the fluid,

$$\mathcal{L} = -j^\mu a_\mu - \sqrt{-j^\mu j_\mu} - V\,. \tag{5.69}$$

Coupling it to gravity, we get

$$\frac{\mathcal{L}}{\sqrt{-g}} = -j^\mu a_\mu - \sqrt{-j^\mu j^\nu g_{\mu\nu}} - V\left(\sqrt{-j^\mu j^\nu g_{\mu\nu}}\right)\,, \tag{5.70}$$

since A_μ and j^μ are the fundamental fields (note that for a particle $j^\mu = q\frac{dx^\mu}{d\tau}\delta^4(x - x(\tau))$), and where

$$\begin{aligned} j^\mu &= \rho\sqrt{1-v^2}u^\mu = \rho_0\frac{dx^\mu}{d\tau} \\ u^\mu &= \frac{dx^\mu}{d\tau} = \frac{1}{\sqrt{1-v^2}}(1,\vec{v}) = \frac{\tilde{u}^\mu}{\sqrt{1-v^2}}\,,\quad \tilde{u}^\mu = \frac{dx^\mu}{dt}. \end{aligned} \tag{5.71}$$

Then the Belinfante energy-momentum tensor is obtained as

$$T^{\mu\nu} = -\frac{j^\mu j^\nu}{\sqrt{-j^\rho j_\rho}}(1+V') + \eta^{\mu\nu}\mathcal{L} = \rho\tilde{u}^\mu\tilde{u}^\nu(1+V') + \eta^{\mu\nu}\mathcal{L}\,, \tag{5.72}$$

and the on-shell Lagrangian becomes (since $a_\mu = u_\mu(1+V')$) $\mathcal{L} \simeq \rho V' - V \simeq \rho V'$ (assuming we can neglect V), meaning we get

$$T^{\mu\nu} = \rho\tilde{u}^\mu\tilde{u}^\nu + (\eta^{\mu\nu} + \tilde{u}^\mu\tilde{u}^\nu)\,(\rho V')\,, \tag{5.73}$$

which exactly matches the energy-momentum tensor of the fluid, (5.68), which means that it will lead to the correct Euler equation ($\partial^\mu T_{\mu 0} = 0$ gives the continuity equation, and $\partial^\mu T_{\mu i} = 0$ gives a combination of the continuity equation and the Euler equation).

In the non-Abelian case then, we again couple the action (for simplicity, without the kinetic term for A_μ^a, though that can be introduced as well) to gravity, obtaining

$$S = \int d^4x \left\{ \mathrm{Tr}[j^\mu g^{-1} D_\mu g] - \sum_a \sqrt{-j^{\mu a} j^{\nu a} g_{\mu\nu}} - V\left(\sum_a \sqrt{-j^{\mu a} j^{\nu a} g_{\mu\nu}}\right)\right\}\,, \tag{5.74}$$

leading to the Belinfante energy-momentum tensor

$$T^{\mu\nu} = -\sum_a \frac{j^{\mu a} j^{\nu a}}{\sqrt{-j^{\rho a} j^a_\rho}} + \eta^{\mu\nu}\mathcal{L} = \sum_a \rho^a \tilde{u}^{a\mu}\tilde{u}^{a\nu} + \eta^{\mu\nu}\mathcal{L}\,, \tag{5.75}$$

and on-shell (at $A_\mu = 0$, in the $g = 1$ gauge) $\mathcal{L} \simeq \sum_a \rho^a V'$, so we get

$$T^{\mu\nu} \simeq \rho\sum_a \tilde{u}^{a\mu}\tilde{u}^{a\nu} + \sum_a (\eta^{\mu\nu} + \tilde{u}^{a\mu}\tilde{u}^{a\nu})\,\rho^a V' \equiv \sum_a T^{\mu\nu a}. \tag{5.76}$$

We see then that we could *define* a $T^a_{\mu\nu}$ for each a index, and then the conservation of each $T^a_{\mu\nu}$ will give the left-hand sides of the Euler equation with index a.

We could also keep the Yang-Mills fields A^a_μ, both the coupling and the dynamical terms, and the result would generalize as well.

5.8 Null fields for classical solutions

Here we review a formalism that has been used to find classical solutions of electromagnetism, e.g. in [4], fluids [5, 21] and electromagnetism coupled to fluids [6].

The electromagnetic fields $\vec{E}$ and $\vec{B}$ can be combined into a complex vector, the so-called Riemann-Silberstein vector,

$$\vec{F} = \vec{E} + i\vec{B}\,. \tag{5.77}$$

The Maxwell's equations of motion in vacuum, in terms of $\vec{F}$, are

$$\vec{\nabla}\cdot\vec{F} = 0\,;\ \partial_t\vec{F} + i\vec{\nabla}\times\vec{F} = 0. \tag{5.78}$$

One can then define null fields by

$$F^2 = 0 \qquad \leftrightarrow \qquad \vec{E}\cdot\vec{B} = 0 \qquad E^2 - B^2 = 0. \tag{5.79}$$

A useful parametrization of the null fields is in terms of the Bateman ansatz [28],

$$\vec{F} = \vec{\nabla}\alpha\times\vec{\nabla}\beta\,, \tag{5.80}$$

in terms of the two complex scalar fields $\alpha, \beta \in \mathbb{C}$.

In terms of electric and magnetic fields, the ansatz is

$$\begin{aligned} E^i &= \epsilon^{ijk}\left(\partial_j\alpha_R\partial_k\beta_R - \partial_j\alpha_I\partial_k\beta_I\right) \\ B^i &= \epsilon^{ikj}\left(\partial_j\alpha_R\partial_k\beta_I + \partial_j\alpha_I\partial_k\beta_R\right)\,, \end{aligned} \tag{5.81}$$

where the indices I and R refer to the imaginary and real parts, respectively.

The first Maxwell equation in (5.78) is trivially satisfied by the Bateman ansatz (5.80) and the second Maxwell equation takes the form

$$i\nabla\times(\partial_t\alpha\nabla\beta - \partial_t\beta\nabla\alpha) = \nabla\times\vec{F}\,, \tag{5.82}$$

which is satisfied if one has

$$i(\partial_t\alpha\nabla\beta - \partial_t\beta\nabla\alpha) = \vec{F}. \tag{5.83}$$

In fact in a more general solution the right-hand side could be $\vec{F} + \vec{G}$, where $\nabla\times\vec{G} = 0$, but $\vec{G}$ can be time dependent.

We note that *on-shell*, we can write

$$A_\mu = \frac{1}{2}\mathrm{Im}[\alpha\partial_\mu\beta - \beta\partial_\mu\alpha]\,, \tag{5.84}$$

and upon denoting $\alpha = \alpha_R + i\alpha_I$ and $\beta = \beta_R + i\beta_I$, the gauge field takes the form

$$A_\mu = \frac{1}{2}\mathrm{Im}\left[\alpha_R\partial_\mu\beta_I + \alpha_I\partial_\mu\beta_R - [(\partial_\mu\alpha_R)\beta_I + \partial_\mu\alpha_I)\beta_R]\right]. \tag{5.85}$$

Solutions of (5.82) have necessarily a zero norm, since

$$F^2 = (\partial_t\alpha\nabla\beta - \partial_t\beta\nabla\alpha)(\vec{\nabla}\alpha\times\vec{\nabla}\beta) = 0\,. \tag{5.86}$$

The simplest null electromagnetic field is made up of perpendicular constant electric and magnetic fields, namely

$$\vec{E} = (E, 0, 0)\,, \qquad \vec{B} = (0, B, 0)\,, \qquad |E| = |B|. \tag{5.87}$$

In terms of α and β, for $E = -B = -4$, this solution takes the form

$$\alpha = 2i(t + z) - 1\,, \qquad \beta = 2(x - iy). \tag{5.88}$$

Another class of well-known null electromagnetic solutions are the plane waves

$$\vec{F} = (\hat{x} + i\hat{y})e^{i(z-t)}\,, \tag{5.89}$$

which in terms of the Bateman variables read

$$\alpha = e^{i(z-t)}\,, \qquad \beta = x + iy. \tag{5.90}$$

The basic knotted solution, the Hopfion solution is given by

$$\alpha = \frac{A - iz}{A + it}\,, \;\; \beta = \frac{x - iy}{A + it}\,, \;\; A = \frac{1}{2}(x^2 + y^2 + z^2 - t^2 + 1). \tag{5.91}$$

The Hopfion is characterized by non-trivial helicity $\mathcal{H}_{mm} = \frac{1}{4}$.

This solution can be obtained by applying the special conformal transformation (SCT)

$$x^\mu \to \frac{x^\mu - b^\mu x_\sigma x^\sigma}{1 - 2b_\sigma x^\sigma + b_\rho b^\rho x_\sigma x^\sigma} \tag{5.92}$$

on (5.88), with $b^\mu = (i, 0, 0, 0)$.

From the Maxwell electromagnetism, we can map to null fluids, with $\vec{v}^2 = 1$, so $u^\mu u_\mu = 0$, via the map

$$\rho \leftrightarrow \frac{1}{2}(\vec{E}^2 + \vec{B}^2); \quad v_i \leftrightarrow \frac{[\vec{E} \times \vec{B}]_i}{\frac{1}{2}(\vec{E}^2 + \vec{B}^2)}\,, \tag{5.93}$$

valid for null electromagnetic fields, $\vec{F}^2 = 0$.

Then one finds the basic fluid Hopfion (knotted) solution of the Euler equation and continuity equations,

$$\begin{aligned}
v_x &= \frac{2(y + x(t - z))}{1 + x^2 + y^2 + (t - z)^2}\,, \qquad v_y = \frac{-2(x - y(t - z))}{1 + x^2 + y^2 + (t - z)^2}\,, \\
v_z &= \pm\sqrt{1 - v_x^2 - v_y^2} = \pm\frac{1 - x^2 - y^2 + (t - z)^2}{1 + x^2 + y^2 + (t - z)^2}\,, \\
\rho &= \frac{16(1 + x^2 + y^2 + (t - z)^2)^2}{(t^4 - 2t^2(x^2 + y^2 + z^2 - 1) + (1 + x^2 + y^2 + z^2)^2)^3}\,.
\end{aligned} \tag{5.94}$$

5.9 Classical solutions

In order to find classical solutions of the non-Abelian Euler fluid coupled to Yang-Mills equations, we do the same as in the Abelian case, which was considered in [6].

We can then find two types of solutions:

1) We can use the null fluid-electromagnetism map from [5, 21] to define the null velocity field for each index a,

$$v_a^i = \frac{\epsilon^{ijk} E_a^j B_a^k}{\vec{B}_a^2} \Rightarrow -(\vec{v}_a \times \vec{B}_a)_i = -\epsilon_{ijk} v_a^j B_a^k = -\frac{\epsilon^{ijk}\epsilon^{jlm} E_l^a B_m^a B_k^a}{\vec{B}_a^2} = E_i^a\,, \tag{5.95}$$

with no sum over a.

That means that the source of the non-Abelian Euler equation (its right-hand side) vanishes, so if the fluid v_a^i is a solution of our generalized Euler equation (with index a), derived from the solution of the Yang-Mills equation, then it is a solution of the combined system.

2) We can, alternatively, consider the YM fields as just *external* fields, which only need to satisfy the Bianchi identities $D_{[\mu}F_{\nu\rho]} = 0$, saying that $F^a_{\mu\nu}$ is derived from an A^a_μ.

But then,

$$v_a^i = \frac{\epsilon^{ijk} E_a^j B_a^k}{\frac{1}{2}(\vec{E}_a^2 + \vec{B}^2)} \tag{5.96}$$

satisfies

$$u_a^\mu F^a_{\mu\nu} = 0 : \vec{v}_a \cdot \vec{E}_a = 0 \quad \& \quad \vec{E}_a + \vec{v}_a \times \vec{B}_a = 0. \tag{5.97}$$

Then we replace $F^a_{\mu\nu}$ with

$$\Omega^a_{\mu\nu} = F^a_{\mu\nu} + \mathcal{V}^a_{\mu\nu}\,, \tag{5.98}$$

where

$$\mathcal{V}^a_{0i} = m(\partial_0 v_i^a - v_j^a \partial_i v_a^j)\,, \quad \mathcal{V}^a_{ij} = m(\partial_i v_j^a - \partial_j v_i^a)\,, \tag{5.99}$$

in the YM solution $F^a_{\mu\nu} = \ldots$, which means we now have, instead,

$$u_a^\mu \Omega^a_{\mu\nu} = 0\,, \tag{5.100}$$

which is our generalized Euler equation (provided $n^a = n$ is the same).

But by construction $\partial_{[\mu}\mathcal{V}^a_{\nu\rho]} = 0$, and we also have (since it satisfies the YM equations) $D_{[\mu}\Omega^a_{\nu\rho]} = 0$, which means that the Bianchi identities are satisfied,

$$D_{[\mu}F^a_{\nu\rho]} = D_{[\mu}\Omega^a_{\nu\rho]} - D_{[\mu}\mathcal{V}^a_{\nu\rho]} = D_{[\mu}\Omega_{\nu\rho]} - \partial_{[\mu}\mathcal{V}^a_{\nu\rho]} = 0\,, \tag{5.101}$$

as well, as we wanted.

Of course, the most general case is when the we have a solution in the presence of dynamical Yang-Mills fields, which means we would need to satisfy, besides the non-Abelian Euler equation and Bianchi identities, also the equation of motion

$$(D_\mu F^{\mu\nu})^a = j^{\nu a}. \tag{5.102}$$

But that is beyond what we can do at this time.

5.10 Attempt at an Euler equation for non-Abelian fluid in the fundamental ("quark")

For completeness, we describe here also an attempt to construct Euler equations for the non-Abelian fluid in the fundamental representation (for a "quark" fluid), though we will see that while we can preserve the helicities $\mathcal{H}'_f$ defined in (3.29), there is no interaction between the fluid types, so it is not clear if these Euler equations are useful.

We could, indeed, *define* the generalized non-Abelian Euler equation for fluid velocity with an index f in the Cartan subalgebra,

$$\left[\partial_t \vec{v}_f + \vec{v}_f \cdot \vec{\nabla} \vec{v}_f\right] + \vec{\nabla}\mu_f = \sum_a Q_{af}\left(\vec{E}_a + \frac{\vec{v}_f}{c} \times \vec{B}_a\right), \tag{5.103}$$

with no sum over f. We could also consider the case that the sum over a runs only over an index $\tilde{b}$ that excludes the Cartan subalgebra.

The Euler equation (5.103) preserves the helicity $\mathcal{H}'_{\rm tot}$ defined in (3.29), which is seen as follows. The Euler equation (5.103) can be rewritten as (no sum over f)

$$j^\mu_f \Omega^f_{\mu\nu} = 0\,, \quad \Omega^f = d\Pi_f\,, \tag{5.104}$$

where $j^\mu_f = \rho_f v^\mu_f$, and by the same arguments as in the Abelian case in [2], the helicity $\mathcal{H}'_{\rm tot}$ is conserved:

$$dh_f = {\rm Tr}[\Omega_f \wedge \Omega_f]\,, \tag{5.105}$$

but then by $j^\mu_f \Omega^f_{\mu\nu} = 0$ (no sum over f), we have $\Omega_f \wedge \Omega_f = 0$, which implies that $dh_f = 0$, so $\mathcal{H}'_f = \int h_f$ is conserved. But *there is no interaction between the various f's, so this is somewhat trivial.*

6 Conclusions

In this paper we have constructed non-Abelian helicities for non-Abelian fluids, first by themselves, and then coupled to Yang-Mills fields. One type of fluid was the "gluonic" fluid, in the adjoint, with densities ρ^a and velocities $u^{\mu a}$, for which we have defined the total helicities $\mathcal{H}^{NA}{}^a_{mm}$ and $\mathcal{H}^{NA}_{\rm tot}$, and the other was the "quark" fluid, with densities ρ_f and velocities u^μ_f, with f in the Cartan subalgebra of $\mathrm{U}(N)$, for which we have defined the helicities $\mathcal{H}'_{\rm tot}$ and $\mathcal{H}''_{\rm tot}$.

We then have constructed non-Abelian Euler equations that conserve the above helicities. In the gluonic case, we have obtained Euler equations with an index a, in which the current j^a_μ does not factorize into j_μ and Q^a, as in previous attempts. We have also found a Lagrangian that gives these equations, and constructed classical solutions. In the quark case, the non-Abelian Euler equation we have found conserves the corresponding helicity, but the fluid types f do not interact.

In our construction, we have reviewed the Abelian Lagrangian construction for non-relativistic and relativistic fluids, using a Clebsch parametrization, coupled to electromagnetism, and written also the symmetries, and Hamiltonian descriptions, which were perhaps not previously done. Following that, we have constructed two direct non-Abelian analogs of

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the Abelian Clebsch construction, but we have found that for one, the resulting Euler-like equation involved also the Clebsch fields, and did not conserve the helicities, so did not serve our purposes, while for the other we had a single equation (not a matrix, or N^2 components, equation). But perhaps these can be thought of as a good mathematical physics problems, that can have other applications.

There are many things left for further work, like for instance finding classical solutions in the presence of dynamical Yang-Mills fields, which we have not done. Also, the generalization to higher knottedness of the solutions is still left for the future.

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A The fluid Hopfion solution

In [5, 21], by mapping the electromagnetic Hopfion solution, the fluid Hopfion was derived. The velocity components take the form

$$\begin{aligned} v_x &= \frac{2(y + x(t-z))}{1 + x^2 + y^2 + (t-z)^2}, \\ v_y &= \frac{-2(x - y(t-z))}{1 + x^2 + y^2 + (t-z)^2}, \\ v_z^2 &= 1 - v_x^2 - v_y^2. \end{aligned} \tag{A.1}$$

The vorticity components are given by

$$\omega_x = \frac{2\left(t^2 y - 2z(ty + x) + 2tx + x^2 y + y^3 + yz^2 + y\right)}{((t-z)^2 + x^2 + y^2 + 1)}, \tag{A.2}$$

$$\omega_y = -\frac{2\left(x\left((t-z)^2 + y^2 + 1\right) + 2y(z-t) + x^3\right)}{\left((t-z)^2 + x^2 + y^2 + 1\right)^2}, \tag{A.3}$$

$$\omega_z = -\frac{4\left((t-z)^2 + 1\right)}{\left((t-z)^2 + x^2 + y^2 + 1\right)^2}. \tag{A.4}$$

It is easy to check that $\vec{\nabla} \cdot \vec{\omega} = 0$. The scalar product of $\vec{v}$ and $\vec{\omega}$ is given by

$$\vec{v} \cdot \vec{\omega} = \frac{4\left((t-z)^2 + x^2 + y^2 - 1\right)}{\left((t-z)^2 + x^2 + y^2 + 1\right)^2}. \tag{A.5}$$

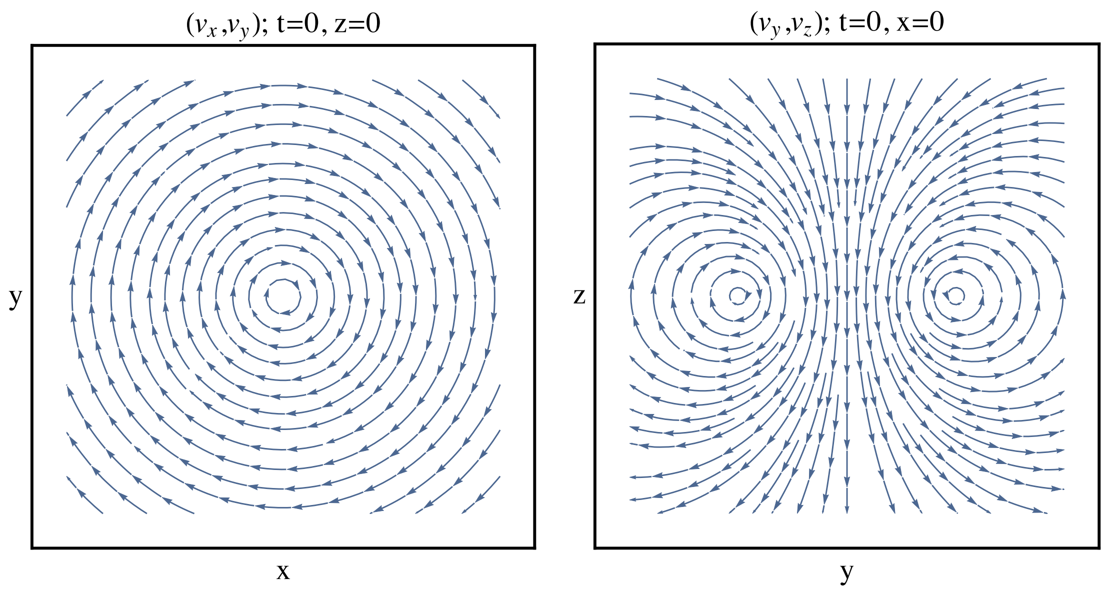

Figure 1. Orthogonal sections of the velocity field for the Hopfion solution, on the (x, y) plane (left) and (y, z) plane (right). Using rotational symmetry in the (x, y) directions the linked torus structure is apparent.

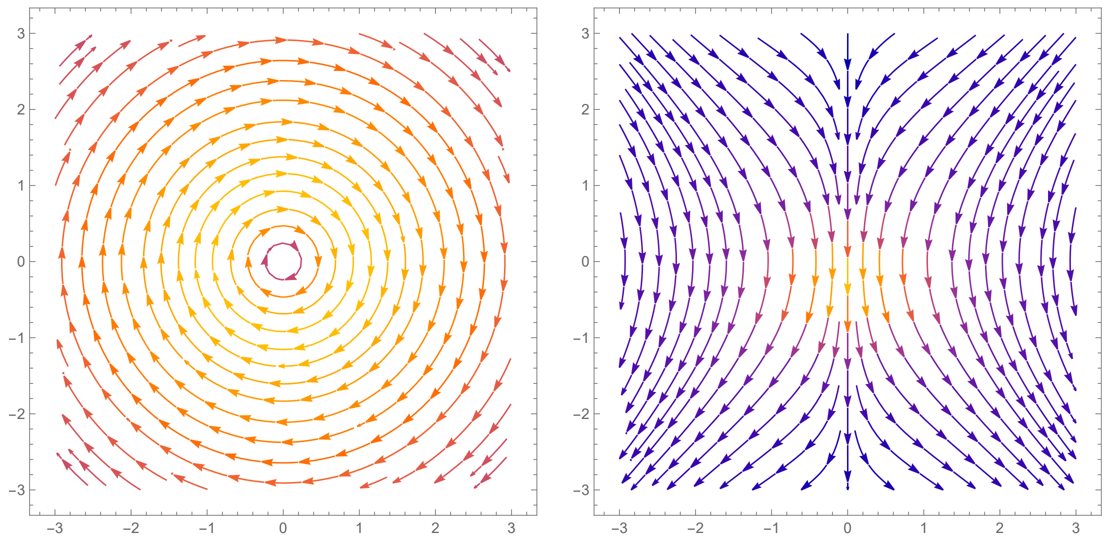

Figure 2. Orthogonal sections of the vorticity field for the Hopfion solution, (ω_x, ω_y) on the (x, y) plane (left) and $(\omega_y, \omega_z)(y, z)$ plane (right).

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Code Availability Statement. This article has no associated code or the code will not be deposited.

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