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Complex Fermion Mass Poles in the Linear Sigma Model at Finite Temperature

Alisson Matheus Araujo Silva

Advisor

Gastão Inácio Krein

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Complex Fermion Mass Poles in the Linear Sigma Model at Finite Temperature

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Comissão Examinadora:

Prof. Dr. GASTÃO INÁCIO KREIN (Orientador)
Instituto de Física Teórica/UNESP

Prof. Dr. MARCUS BENGHI PINTO
Universidade Federal de Santa Catarina

Prof. Dra. ANA JULIA SILVEIRA MIZHER
Instituto de Física Teórica/UNESP

Conceito: Aprovado

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*Dedico esta tese a
minha mãe, que sempre me apoiou.*

Agradecimentos

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*"The best way to explain it,
is to do it."*

Lewis Carroll, *Alice in Wonderland*

Resumo

Investigamos o aparecimento de polos de massa complexa não físicos no propagador dos férmions no modelo sigma linear (LSM) a temperatura finita. O LSM é uma teoria de campos efetiva para a quebra espontânea da simetria quiral (SCSB) na cromodinâmica quântica (QCD). Os graus de liberdade efetivos são descritos por campos de mésons escalares e pseudoescalares e campos de férmions de spin-1/2. Quando os férmions são núcleons, o LSM realiza SCSB na fase confinada da QCD e quando os férmions são quarks, ele realiza SCSB na fase desconfiada da QCD. A presença de pólos complexos nos propagadores leva a resultados não físicos em observáveis, como na equação de estado termodinâmica e coeficientes de transporte. Especificamente, quantificamos ao nível de um loop as contribuições térmicas para a função espectral da autoenergia do férmion e os polos complexos. Encontramos que embora a temperatura altere as propriedades espectrais dos propagadores, ela não elimina os polos, mas próximo à temperatura de restauração quiral há mudanças qualitativas significativas nessas quantidades. Concluimos com algumas considerações sobre as implicações de nossos resultados para a interpretação do aparecimento de polos de massa complexa em propagadores da QCD como sinalização de confinamento de quark-glúon.

Palavras Chaves: Cromodinâmica Quântica; Simetria Quiral; Quebra Espontânea de Simetria, Teorias Efetivas; Modelo Sigma-Linear .

Áreas do conhecimento: Física de Partículas; Física Nuclear; Teoria Quântica de Campos

Abstract

We investigate the appearance of unphysical complex-mass poles in the fermion propagator in the linear sigma model (LSM) at finite temperature. The LSM is an effective field theory for spontaneous chiral symmetry breaking (SCSB) in quantum chromodynamics (QCD). The effective degrees of freedom are described by scalar and pseudoscalar meson fields and spin-1/2 fermion fields. When the fermions are nucleons, the LSM model realizes SCSB in the confined phase of QCD and when the fermions are quarks, it realizes SCSB in the QCD deconfined phase. The presence of complex poles in the propagators leads to unphysical results in observables such as in the thermodynamic equation of state and transport coefficients. Specifically, we quantify at the one-loop level the thermal contributions to the fermion self-energy spectral function and on the complex poles. We found that although temperature changes the spectral properties of the propagators it does not eliminate the poles but close to the chiral restoration temperature there are significant qualitative changes in those quantities. We conclude with considerations about the implications of our results for the interpretation of the appearance of complex-mass poles in QCD propagators as signaling quark-gluon confinement.

Keywords: Quantum Chromodynamics; Chiral Symmetry; Spontaneous Symmetry Breaking; Effective Theories; Linear Sigma Model.

Areas of Knowledge: Particle Physics; Nuclear Physics; Quantum Field Theory.

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Chapter 1

Introduction

Quantum Chromodynamics (QCD) is the fundamental theory describing the strong interactions among quarks and gluons, the elementary constituents of hadrons such as protons and neutrons. The theory is governed by the non-Abelian gauge group $SU(3)$ and describes the dynamics of the strong force, which is responsible for binding quarks together to form protons, neutrons, and other hadrons. Unlike the electromagnetic force, which diminishes with increasing distance, the strong force exhibits a unique property known as confinement. Confinement refers to the phenomenon where quarks and gluons are perpetually bound within hadrons, preventing their isolation as free particles [1]. This aspect of QCD is not only central to our understanding of nuclear physics but also remains one of the most challenging aspects to study both theoretically and experimentally [1, 2, 3, 4].

The mathematical aspects of confinement are intricately linked to the unresolved question of proving the existence of a mass gap in Yang-Mills theory, a problem that stands at the frontier of both physics and mathematics. Specifically, it requires showing that in a non-Abelian gauge theory, such as $SU(3)$ Yang-Mills theory, there exists a finite energy gap between the vacuum state and the first excited state, which is a necessary condition for confinement. [5] The mass gap in Yang-Mills theory refers to the existence of a positive lower bound on the energy of any possible excitation above the vacuum state. In simpler terms, it implies that the lightest possible particle in the theory (in this case, a hypothetical glueball) would have a non-zero mass, creating a gap between the energy of the vacuum and the energy of this particle. This is a crucial aspect of confinement because if quarks or gluons could exist as free particles, there would be no such gap—massless or nearly massless particles would dominate, as is the case in QED with photons. The mass gap is essential for ensuring that quarks and gluons remain bound within hadrons. Despite the compelling physical evidence and numerical simulations supporting the existence of this gap, a rigorous mathematical proof within the framework of Yang-Mills theory remains one of the most significant open problems in theoretical physics [5, 6].

Another manifestation of confinement in QCD is the formation of confining flux tubes. When quarks are pulled apart, the strong force between them does not diminish as it does with other forces like electromagnetism. Instead, the field lines of the strong force, carried by gluons, become concentrated into a narrow tube or string between the quarks. This tube is known as a confining flux tube, and its tension is responsible for the increasing potential energy between quarks as they are separated [7]. If quarks are separated beyond a certain point, the energy in the flux tube becomes sufficient to create a new quark-antiquark pair, effectively preventing the isolation of individual quarks. This phenomenon is beautifully captured by lattice QCD simulations, where the flux tubes are directly observed, and the linear rise of potential with distance is clearly demonstrated [8, 9]. The study of these flux tubes is not only essential for understanding the mechanism of confinement but also provides insights into the behavior of QCD at different temperatures. As the temperature increases, the energy in the system can reach a point where these flux tubes cease to exist, leading to the deconfinement of quarks and gluons.

Exploring QCD under extreme conditions, such as at finite temperature, reveals many intriguing phenomena. At sufficiently high temperatures, quarks and gluons become deconfined, forming a quark-gluon plasma (QGP)—a state of matter that existed shortly after the Big Bang. A number of works have investigated this transition and its implications for understanding the early universe and the nature of phase transitions in the strong interaction regime. We refer the reader to the recent review [10] which provides a comprehensive overview of the current theoretical and experimental approaches to studying these effects.

Experimental efforts, particularly in heavy-ion collisions at facilities like the Large Hadron Collider (LHC) and the Relativistic Heavy Ion Collider (RHIC), have been instrumental in probing these extreme conditions. These experiments aim to recreate the environment of the early universe by colliding nuclei at near-light speeds, generating temperatures high enough to potentially produce the quark-gluon plasma (QGP). Observations from these collisions have provided valuable information on the properties of QGP, such as its near-perfect fluidity. [11, 12]. Furthermore, lattice QCD simulations and other theoretical approaches complement these experimental studies, offering predictions and a deeper understanding of the behavior of QCD under such extreme conditions and the nature of the transition between the confined and deconfined phases. Together, these experimental and theoretical investigations continue to refine our understanding

of the strong force and the early moments of our universe.

Understanding the transition from confinement to deconfinement, and the behavior of QCD at finite temperatures is therefore a key focus in high-energy physics, providing deep insights into the fundamental processes and the conditions that shaped the formation of matter as we know it today. A contemporary challenge for theorists, in addition to providing guidance to the experimental efforts, is to build on results from the first-principles lattice QCD numerical simulations to develop effective theories and models to understand how QCD works. The present dissertation builds on one piece of knowledge brought from lattice QCD on quark-gluon correlation functions, namely that quark and gluon propagators exhibit a violation of positivity of the associated spectral functions [13, 14, 15]. Such positivity violation has long been associated with the confinement of quarks and gluons [16, 17, 18]. Since the spectral function is related to the probability of measuring a determined state, the violation of positivity implies the existence of states of negative norm which lie outside the Hilbert space of the theory and are thus not associated with observable particles.

This dissertation is concerned with one aspect of the connection between positivity violation of spectral functions and confinement, namely if such violation is also manifest in effective models of QCD which lie outside of the confining regime. This was first shown, for example, in meson nuclear models [19] and pursued later on in [20, 21, 22]. In the work [22], in particular, the linear-sigma model (LSM), an effective field theory was investigated for describing pion nuclear processes (pN) and the spectral functions have been shown to exhibit complex-conjugated mass poles and violation of the positivity of spectral densities. The negativity that arises in this case is usually related to a pair of complex poles found in the propagator, which invalidate the Källén–Lehmann spectral representation, since such poles correspond to eigenstates of the system that have complex energy and probability, and are thus unphysical. Our main objective in this dissertation is to analyze whether such effects are maintained in the model at finite temperature and whether this carries some implications to the interpretation on the quark-gluon confinement phenomenon in QCD. Alternatively, it is equally important to consider whether the observed complex structure could represent an artifact of the approximation scheme employed. To address these issues, our strategy involves computing the one-loop contributions to the fermion propagator in the LSM fermion propagator. This approach, while not entirely novel, has a solid foundation, having been applied in similar contexts, such as the Walecka model [23]

and the quark-meson-coupling model [24]. These studies demonstrated the utility of employing nonperturbative effective masses within exchange self-energies, a technique often utilized in frameworks like optimized perturbation theory or the δ -expansion method [25]. However, none of these previous applications addressed the issue of complex mass poles at finite temperature, a question that remains mostly unexplored.

Our original contribution lies precisely in this area, filling a gap in the existing literature. By focusing on the fermion self-energy, both in vacuum and under finite-temperature conditions, we aim to elucidate the significance of these complex poles and their broader implications for the underlying physics of QCD. Our calculations are performed within the finite temperature field theory formalism in real time [26, 27, 28], which is the formalism required to calculate dynamical processes and transport properties of strongly interacting matter such as shear and bulk viscosities and thermal and electric conductivities [29, 30, 31]. Unlike the imaginary-time formalism, which is formulated in Euclidean space, the real-time formalism remains in Minkowski space, allowing direct access to time-dependent observables and real-time dynamics.

We carry out our study by inspecting the poles in the fermion propagator of the LSM at finite temperature. We employ two versions of the LSM, one in which the fermions are nucleons and the other in which they are quarks. By studying these two versions, we explore the model across different scales, from low to high temperatures. This will allow us to assess how temperature influences the existence and nature of the complex-mass poles and draw conclusions about, first, potential flaws in the interpretations about confinement in QCD and, second, on using the LSM to compute finite-temperature observables.

We structure the the dissertation as follows. After briefly reviewing the LSM and its origins in Chapter 2, we introduce the spectral representation of propagators in Chapter 3 and prove the important property that the physical poles of the propagator must lie in the real axis. In Chapter 4, we briefly review the real-time formalism of thermal field theory which we will be using to perform most of the calculations. In Chapter 5, we compute the one-loop corrections to nucleon and propagators first in vacuum and then at finite temperature, and present numerical results. Our conclusions and perspectives are presented in Chapter 6.

Chapter 2

The Linear Sigma Model

2.1 Historical Aspects

As mentioned in the introduction, the linear sigma model (LSM) was originally conceived to describe pion-nucleon processes (pN). The model was firstly developed by M. Gell-Mann and M. Levy [32] in the context of beta decay processes, as an extension to a previous model developed by Schwinger [33]. The Lagrangian density of the model is as follows:

$$\begin{aligned}\mathcal{L} = & \bar{\psi} [i\gamma^\mu \partial_\mu + g\bar{\psi}(\sigma + i\tau \cdot \pi \gamma_5)] \psi + \frac{1}{2}(\partial_\mu \sigma)^2 + \frac{1}{2}(\partial_\mu \pi)^2 \\ & + \frac{\mu^2}{2}(\sigma^2 + \pi^2) - \frac{\lambda}{4!}(\sigma^2 + \pi^2)^2\end{aligned}\tag{2.1}$$

The scalar meson field σ and a pseudoscalar meson field π couple to fermion field ψ in a chirally invariant way. Gell-Mann and Levy introduced the model to describe the dynamics of beta decay, in which case the fermion field represents the nucleon. The phenomenon of spontaneous symmetry breaking inherent in the model explains the emergence of masses for these particles and successfully reproduce a number of experimental observations [34, 35].

Major developments were made during the 60s related to the LSM and its extensions, such as the discovery of the soft-pion theorems by Adler, Weisenberger and Weinberg [36, 37, 38] which allowed for easier calculation of amplitudes in decay processes. It was also during this time that the renormalizability of the model was investigated in detail [39]. After the advent of QCD, it was realized that the linear-sigma model is an effective model useful to describe the dynamics of the strong dynamics at scales above the confining scale. Indeed, effective chiral models of QCD are widely used to study the QCD low-energy phenomenology. Since the literature is too extensive to cite here, we simply quote some very recent work related to Brazilian institutions, namely [40, 41, 42, 43, 43].

2.2 Spontaneous Chiral Symmetry Breaking

In particle physics, spontaneous chiral symmetry breaking plays an essential role in explaining the generation of mass for particles such as nucleons. The LSM is an effective model of QCD that elegantly explains this symmetry breaking mechanism. In its simplest form, the model consists of a scalar and a pseudoscalar meson fields, and a fermion field. When chiral symmetry is spontaneously broken, the model predicts the emergence of massive particles from originally massless ones, offering an explanation for the origins of mass in the strong interaction sector.

Let us analyze in detail how this can be achieved. The Lagrangian of Eq. (2.1) is invariant under the following $SU(2)$ symmetry transformation:

$$\begin{aligned}\sigma &\rightarrow \sigma' = \sigma \\ \pi &\rightarrow \pi' = \pi + \alpha \times \pi \\ \psi &\rightarrow \psi' = \psi - i\alpha \cdot \frac{\tau}{2}\psi\end{aligned}\tag{2.2}$$

where $\alpha \ll 1$. This allows us to define the following conserved current:

$$J_\mu^a = \bar{\psi}\gamma_\mu \frac{\tau^a}{2}\psi + \epsilon^{abc}\pi^b\partial_\mu\pi^c\tag{2.3}$$

The corresponding conserved charge given in terms of the J_0 component by:

$$Q^a = \int J_0^a(x)d^3x\tag{2.4}$$

The Lagrangian is also invariant under the axial $SU(2)$ transformations:

$$\begin{aligned}\sigma &\rightarrow \sigma' = \sigma + \beta \cdot \pi \\ \pi &\rightarrow \pi' = \pi - \beta\sigma \\ \psi &\rightarrow \psi' = \psi + i\beta \cdot \frac{\tau}{2}\gamma_5\psi\end{aligned}\tag{2.5}$$

with the conserved current this time given by:

$$A_\mu^a = \bar{\psi}\gamma_\mu\gamma_5\frac{\tau^a}{2}\psi + (\partial_\mu\sigma)\pi^a - (\partial_\mu\pi^a)\sigma\tag{2.6}$$

and the charge:

$$Q_5^a = \int A_0^a(x) d^3x \quad (2.7)$$

This allows us to define the $SU(2)_L \times SU(2)_R$ algebra [44]:

$$\begin{aligned} [Q^a, Q^b] &= i\epsilon^{abc} Q^c \\ [Q^a, Q_5^b] &= i\epsilon^{abc} Q_5^c \\ [Q_5^a, Q_5^b] &= i\epsilon^{abc} Q^c \end{aligned} \quad (2.8)$$

When the σ field acquires a quantum vacuum expectation value, one says that this chiral symmetry is spontaneously broken. The π field has zero vacuum expectation value due to its pseudoscalar (and also isovector) nature. One can understand this in classical physics. Consider the classical potential term in the Lagrangian:

$$V(\sigma^2 + \pi^2) = -\frac{\mu^2}{2}(\sigma^2 + \pi^2) + \frac{\lambda}{4!}(\sigma^2 + \pi^2)^2 \quad (2.9)$$

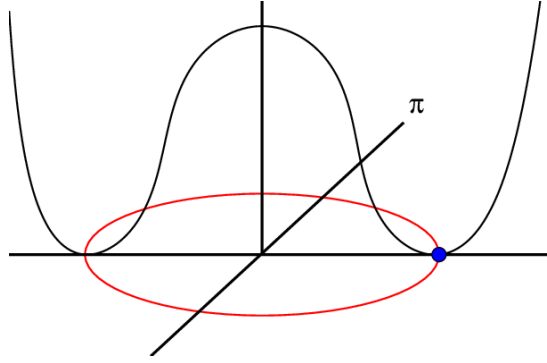


Figure 2.1: Double-well potential of Eq. (2.9)

This potential is minimized for:

$$\sigma^2 + \pi^2 = v^2 \quad \text{where} \quad v = 3\sqrt{\frac{\mu^2}{\lambda}} \quad (2.10)$$

That is, the minimum-energy state (the “vacuum” state) is given by the σ and π fields satisfying (2.10). In a quantum language we refer to those fields as vacuum expectation values, $\langle 0|\pi|0\rangle$ and $\langle 0|\sigma|0\rangle$. Because in the quantum theory $\langle 0|\pi|0\rangle = 0$, the natural choice is $\langle 0|\sigma|0\rangle = v$. To explore fluctuations around the new vacuum v , we define the shifted field:

$$\sigma' = \sigma + v \quad (2.11)$$

This has the effect of giving the fermion in the model an emergent mass $m_\psi = g\nu$, namely:

$$g\bar{\psi}(\sigma + i\boldsymbol{\tau} \cdot \boldsymbol{\pi}\gamma_5)\psi \rightarrow g\nu\bar{\psi}\psi + \bar{\psi}(\sigma' + i\boldsymbol{\tau} \cdot \boldsymbol{\pi}\gamma_5)\psi \quad (2.12)$$

It can be checked explicitly that this term breaks the $SU(2)_L \times SU(2)_R$ chiral symmetry. The pion masses under the scheme on the other hand are massless; they are Goldstone bosons, predicted by Goldstone's theorem [45]. In the case of chiral symmetry breaking in QCD, if up and down quarks were exactly massless, the chiral $SU(2)_L \times SU(2)_R$ symmetry would be exact and the pions would emerge as true Goldstone bosons with zero mass. In the real world, however, the u and d quarks are not exactly massless; they have small but finite masses. This makes the chiral symmetry only an approximate symmetry of the QCD Lagrangian. When the symmetry is explicitly broken, the resulting bosons acquire a small mass, making them pseudo-Goldstone bosons rather than exact Goldstone bosons. In QCD, pions are such pseudo-Goldstone bosons. Their mass is not zero, but is significantly lighter than that of the other hadrons such as protons and neutrons, which is a hallmark of their pseudo-Goldstone nature. The smallness of their mass compared to other QCD scale phenomena is a consequence of the approximate chiral symmetry breaking.

To describe more faithfully QCD, one needs that the $SU(2) \times SU(2)$ symmetry of the LSM Lagrangian is explicitly broken from the beginning:

$$\begin{aligned} \mathcal{L} = & \bar{\psi} [i\gamma^\mu \partial_\mu + g(i\boldsymbol{\tau} \cdot \boldsymbol{\pi}\gamma_5 + \sigma)] \psi + \frac{1}{2}(\partial_\mu \sigma)^2 + \frac{1}{2}(\partial_\mu \boldsymbol{\pi})^2 \\ & + \frac{\mu^2}{2}(\sigma^2 + \boldsymbol{\pi}^2) - \frac{\lambda}{4!}(\sigma^2 + \boldsymbol{\pi}^2)^2 + c\sigma \end{aligned} \quad (2.13)$$

in which the $c\sigma$ term breaks the symmetry. The potential is now shifted and has a new minimum determined by the condition:

$$\nu \left(\mu^2 + \frac{\lambda}{6} \nu^2 \right) - c = 0 \quad (2.14)$$

For small c , ν can be approximated as:

$$\nu \approx \nu_0 + \frac{c}{\mu^2 + \frac{\lambda}{2}\nu_0^2} \quad (2.15)$$

where ν_0 is the vacuum expectation value in the absence of explicit symmetry breaking (i.e., $c = 0$). Upon performing the σ field shift as in Eq. (2.11) to explore

fluctuations around the minimum of the potential, this time we obtain, up to additive constants and dropping the prime in σ' :

$$\begin{aligned} \mathcal{L} = & \bar{\psi}[i\not{\partial} - m_\psi - g(\sigma + i\gamma_5 \boldsymbol{\tau} \cdot \boldsymbol{\pi})]\psi + \frac{1}{2}[(\partial_\mu \sigma)^2 + (\partial_\mu \boldsymbol{\pi})^2] - \frac{1}{2}m_\pi^2 \boldsymbol{\pi}^2 \\ & - \frac{1}{2}m_\sigma^2 \sigma^2 - \frac{\lambda}{4!}(\sigma^2 + \boldsymbol{\pi}^2)^2 - 4\nu \frac{\lambda}{4!} \sigma(\sigma^2 + \boldsymbol{\pi}^2) + \sigma[\nu(\mu^2 + \frac{1}{6}\lambda\nu^2) - c] \end{aligned} \quad (2.16)$$

where, as previously, $m_\psi = g\nu$, and the meson masses are now given by:

$$m_\pi^2 = \frac{c}{\nu} \quad (2.17)$$

$$m_\sigma^2 = \mu^2 + \frac{1}{2}\lambda\nu^2 \quad (2.18)$$

The new vacuum expectation value ν is related to the pion decay constant f_π . This relationship arises because ν characterizes the spontaneous breaking of chiral symmetry in the linear sigma model, which gives rise to the pion as a pseudo-Goldstone boson. In the model, f_π is directly associated with the amplitude for the pion to couple to the axial current and this amplitude is determined by ν [44]. In essence, f_π reflects the strength of the symmetry-breaking effect, which is governed by ν . Given that ν is determined by the parameters of the potential in the model, f_π is also influenced by these parameters. Consequently, the physical observable f_π , which can be experimentally measured, provides insights into the underlying symmetry-breaking mechanism and the values of the parameters in the theory.

2.3 Quantum Chromodynamics

As mentioned earlier the LSM is an effective model of QCD. We provide in this section a brief review of the theory and elaborate on the intricacies that lead us to consider effective models when dealing with its dynamics. The origins of QCD can be traced back to the quark model of Gell-Mann and Ne'eman [46][47] which was introduced in the 60s to account for the large number of particles that were being discovered at the time. They proposed an SU(3) invariant theory in which the particles would be made up of more fundamental constituents called quarks

which were placed in the fundamental representation of the group:

$$u = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad d = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad s = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad (2.19)$$

The quarks then distinct themselves into three flavors: up, down and strange. The generators of the SU(3) group consist of a set of 8 matrices called Gell-Mann matrices:

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (2.20)$$

$$\lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad (2.21)$$

$$\lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad (2.22)$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \quad (2.23)$$

From these generators, one defines the quantum numbers isospin I_3 and hypercharge Y as the eigenvalues of the operators:

$$\hat{I}_3 = \frac{1}{2}\lambda_3, \quad \hat{Y} = \frac{2}{\sqrt{3}}\lambda_8 \quad (2.24)$$

Particles in this scheme are characterized by their hypercharge and isospin. Quarks placed in the fundamental representation have the following quantum numbers displayed in the weight diagram below:

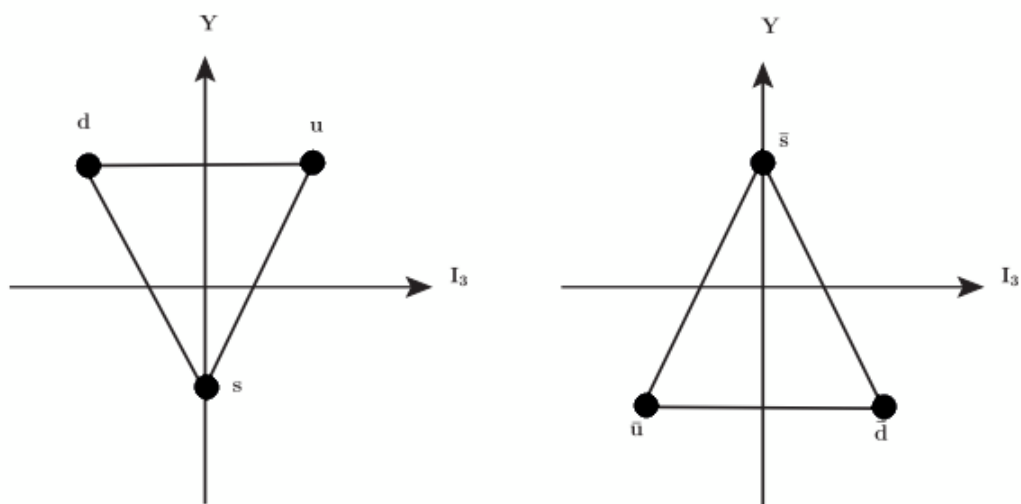


Figure 2.2: Quarks and their respective quantum numbers.

The direct product between quark representations allows us to identify two distinct kinds of particles:

$$\text{Baryons: } \mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} = \mathbf{1} \oplus \mathbf{8} \oplus \mathbf{8} \oplus \mathbf{10} \quad (2.25)$$

$$\text{Mesons: } \mathbf{3} \otimes \bar{\mathbf{3}} = \mathbf{1} \oplus \mathbf{8} \quad (2.26)$$

States composed of three quarks form the two baryon octets and the baryon decuplet and states composed of one quark and one antiquark form the meson octet:

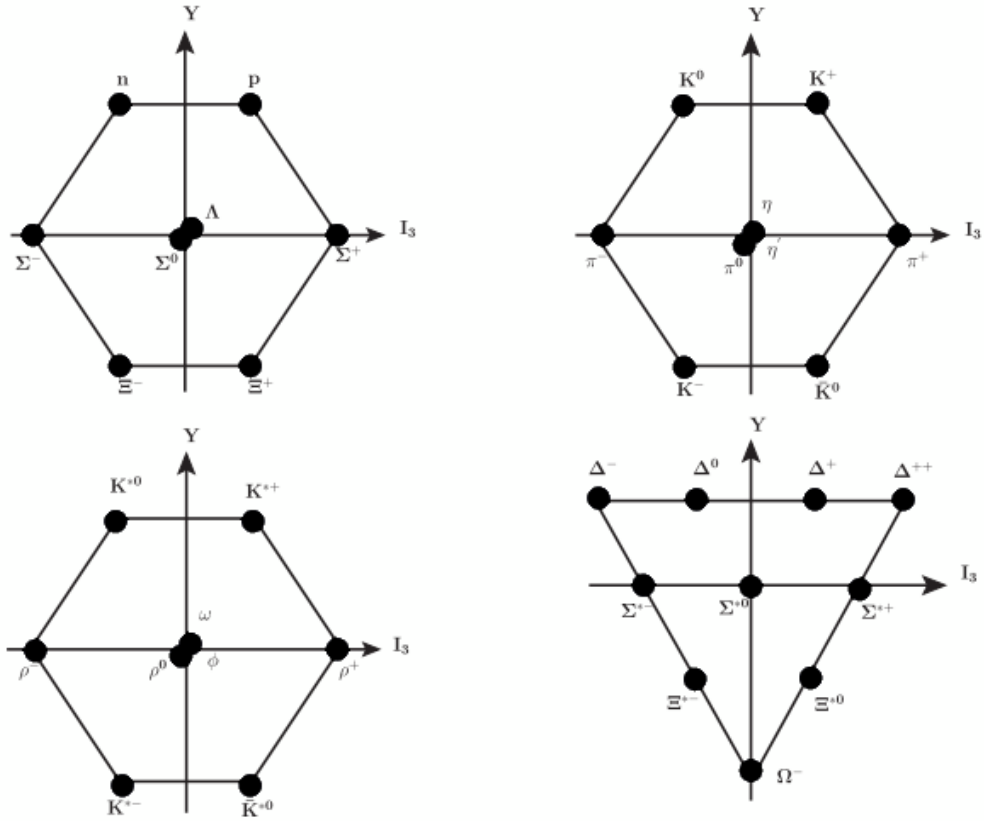


Figure 2.3: Hadrons and their respective quantum numbers.

The confirmation of Gell-Mann's model came with the discovery of the omega minus particle, detected at Brookhaven National Laboratory in 1964 [48]. The subsequent discovery of additional particles and the necessity to satisfy Pauli's exclusion principle led to some new developments. Already in 1964 O. W. Greenberg [49] had noted that if the quarks are fermions, states composed of three quarks would seemingly violate Pauli's exclusion principle which states that no two identical fermions can occupy the same quantum state simultaneously. To circumvent this he proposed that quarks possess an additional quantum number called "color," in this framework, each quark flavor (up, down, strange, etc.) comes in three different "colors": red, green, and blue. It was later realized that the SU(3) of flavor was not the key underlying symmetry of the strong force, but rather an SU(3) of color. The SU(3) color invariant Lagrangian is expressed as:

$$\mathcal{L} = -\frac{1}{4}G_{\mu\nu}G^{\mu\nu} + \bar{\psi}(i\gamma^\mu D_\mu - m)\psi, \quad (2.27)$$

where D_μ is the SU(3) gauge covariant derivative, which incorporates the interac-

tion between quarks and gluons:

$$D_\mu = \partial_\mu + igA_\mu \quad (2.28)$$

with $A_\mu = \lambda^a A_\mu^a$ representing the gluon fields and λ^a being the Gell-Mann matrices corresponding to the SU(3) generators and the term $G_{\mu\nu}$ denotes the field strength tensor. It is important to note that the λ^a matrices here are not the same as the matrices discussed previously. The former are related to the generators of the SU(3) group in the context of the Eightfold Way, which describes the symmetry of hadrons in terms of flavor. The latter λ^a matrices correspond to the generators of the SU(3) group in the context of color charge in Quantum Chromodynamics (QCD). The explicit form of the field-strength tensor is given by:

$$G_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu] \quad (2.29)$$

The fields ψ in the Lagrangian denote the quark fields, which belong to the fundamental representation of the SU(3) color group. These quarks are SU(2) isospin doublets and SU(3) color triplets. Here, we specifically mention only two flavors, the up (u) and down (d) quarks, because they form the basis of the SU(2) isospin symmetry, which is relevant in contexts such as the study of low-energy hadron physics. In low-energy regimes, the decoupling theorem implies that the heavy quarks (such as strange, charm, bottom, and top) do not directly influence the dynamics and can be effectively decoupled from the theory. [50]. The quark doublet can therefore be written as:

$$\psi = \begin{pmatrix} u^c \\ d^c \end{pmatrix}, \quad \text{where: } u^c = \begin{pmatrix} u^r \\ u^b \\ u^g \end{pmatrix} \quad (2.30)$$

The superscript c indicates that these fields are color triplets, consisting of the three color charges (red r , blue b , and green g) represented by u^r , u^b , and u^g . For massless quarks, the QCD Lagrangian exhibits an exact chiral symmetry, which can be decomposed into two parts: $SU(2)_L \times SU(2)_R$. This symmetry implies that left-handed and right-handed quarks do not mix and can be transformed independently. Quarks, however, possess a finite mass, which explicitly breaks chiral symmetry. Despite this, an approximate chiral symmetry is still preserved, as the masses of the up and down quarks are relatively small compared to the

QCD scale. Experimentally, the up quark mass is around 2.2 MeV, and the down quark mass is approximately 4.7 MeV. This approximate symmetry plays a crucial role in understanding the low-energy dynamics of hadrons, as it allows for the classification of hadrons into multiplets and the prediction of their interactions.

In the vacuum, this chiral symmetry is spontaneously broken due to the formation of a quark condensate, $\langle \bar{\psi}\psi \rangle$, which serves as an order parameter [44]. The spontaneous breaking of chiral symmetry leads to the appearance of massless Goldstone bosons, which correspond to the pions in the real world. These pions acquire a small mass due to the explicit breaking of chiral symmetry by the quark masses. This aspect is significant because it explains why hadrons, such as pions, are much lighter than the scale of QCD and why they play a dominant role in low-energy hadron physics.

Another key property of QCD worth mentioning is that the coupling between quarks and gluons becomes small at high energies, a phenomenon called asymptotic freedom which was discovered by David Gross and Frank Wilczek [51] and independently by David Politzer [52] in the early 70s. At low energies or long distances, the interaction between quarks becomes strong, leading to confinement and the formation of bound states like protons, neutrons, and other hadrons. It is in this low-energy regime that chiral symmetry breaking occurs, however as the coupling strength increases at low energies the theory becomes strongly coupled, making perturbative methods unsuitable. We thus seek an effective theory whose symmetries mimic the ones from QCD to describe the dynamics at those scales.

One such effective theory is the Linear Sigma Model (LSM). The LSM is particularly valuable because it mirrors the symmetries of QCD, especially chiral symmetry and its spontaneous breaking. In QCD, spontaneous chiral symmetry breaking is responsible for generating the masses of the light pseudoscalar mesons (pions) that act as Goldstone bosons. The LSM captures this aspect by introducing a scalar field (the sigma field) and pseudoscalar fields (pions) that behave analogously to the scalar and pseudoscalar mesons in QCD. The connection between the LSM and QCD's spontaneous chiral symmetry breaking lies in the role of these fields. In QCD, the quark condensate $\langle \bar{\psi}\psi \rangle$ acts as the order parameter for chiral symmetry breaking, which spontaneously breaks the $SU(2)_L \times SU(2)_R$ symmetry down to $SU(2)_V$, giving rise to the pions as massless Goldstone bosons. The LSM models this behavior by incorporating a potential that leads to spontaneous symmetry breaking, where the sigma field acquires a vacuum expectation value, and the pions emerge as Goldstone bosons in the process [44].

However, while the LSM captures the essence of chiral symmetry breaking in QCD, it does not account for confinement, the phenomenon that binds quarks together into hadrons. Confinement is a non-perturbative effect in QCD that arises from the behavior of the strong force at long distances, something that the LSM, being an effective theory at lower energies, does not directly address. Instead, the LSM focuses on modeling the mass spectrum and interactions of mesons resulting from chiral symmetry breaking, providing a useful framework for understanding hadronic physics without delving into the complexities of confinement.

In summary, the LSM effectively captures the spontaneous breaking of chiral symmetry in QCD by modeling the generation of meson masses and the appearance of Goldstone bosons. However, it does not address the confinement aspect of QCD, which requires a different approach to fully understand the binding of quarks into hadrons. This distinction is important for the discussion on the relation between confinement and complex poles that will be explored in the upcoming chapters.

Chapter 3

Spectral Representation

In this chapter, we briefly review the spectral representation of propagators. More specifically, we consider the Källén-Lehmann (KL) representation. We first outline the derivation of the KL representation for the case of real scalar fields. In subsection 3.2 we outline the derivation of the corresponding representation for spin-1/2 fermions field. A complete and detailed derivation can be found in quantum field theory textbooks, e.g. [53, 54, 55].

3.1 Spectral Representation of Real Scalar Fields

The propagator for a free scalar field is defined by the following time-ordered product:

$$iG(x-y) = \langle 0|T\{\phi(x)\phi(y)\}|0\rangle = \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\epsilon} e^{-ik \cdot (x-y)} \quad (3.1)$$

where m stands for the mass of the particle and k is its momentum. In momentum space this expression can be written as:

$$iG(k) = \frac{i}{k^2 - m^2 + i\epsilon} \quad (3.2)$$

One can construct a Källén-Lehmann spectral representation for the full propagator by starting from the matrix element $\langle 0|\phi(x)\phi(y)|0\rangle$. By inserting a complete set of positive energy eigenstates between the two field operators [55]:

$$\langle 0|\phi(x)\phi(y)|0\rangle = \sum_n \int dk \langle 0|\phi(x)|k,n\rangle \langle k,n|\phi(y)|0\rangle \quad (3.3)$$

and then using translation invariance, we can write:

$$\langle k,n|\phi(x)|0\rangle = e^{-ikx} \langle k,n|\phi(0)|0\rangle \quad (3.4)$$

we obtain that (3.3) becomes:

$$\langle 0|\phi(x)\phi(y)|0\rangle = \sum_n \int dk e^{-ik(x-y)} |\langle k, n|\phi(0)|0\rangle|^2 \quad (3.5)$$

We are free to introduce a δ -function in this expression such that:

$$\int \frac{d^4 p}{(2\pi)^4} e^{-ip \cdot (x-y)} \sum_n \int dk (2\pi)^4 \delta^4(p-k) |\langle k, n|\phi(0)|0\rangle|^2 \quad (3.6)$$

Defining:

$$\sum_n \int dk (2\pi)^4 \delta^4(p-k) |\langle k, n|\phi(0)|0\rangle|^2 = 2\pi \theta(p^0) \rho(p^2) \quad (3.7)$$

we reach:

$$\langle 0|\phi(x)\phi(y)|0\rangle = \int \frac{d^4 p}{(2\pi)^3} e^{-ip \cdot (x-y)} \theta(p^0) \rho(p^2) \quad (3.8)$$

Given this first step, one then can show that the time-ordered propagator can written as [53, 54, 55]:

$$\begin{aligned} \langle 0|T\{\phi(x)\phi(y)\}|0\rangle &= \langle 0|\phi(x)\phi(y)|0\rangle \theta(x^0 - y^0) + \langle 0|\phi(x)\phi(y)|0\rangle \theta(y^0 - x^0) \\ &= \int \frac{d^4 p}{(2\pi)^4} e^{ip \cdot (x-y)} iG(p^2) \end{aligned} \quad (3.9)$$

where

$$G(p^2) = \int_0^\infty dm^2 \frac{\rho(m^2)}{p^2 - m^2 + i\epsilon} \quad (3.10)$$

The quantity ρ is called the spectral density. It is a Lorentz-invariant quantity and must be positive definite given the modulus contained in its defining expression in (3.7).

We note that the spectral function $\rho(p^2)$ is given by the imaginary part of the propagator, namely:

$$\rho(p^2) = \frac{1}{2\pi i} [G(p) - G^*(p)] \quad (3.11)$$

where $G(p)^*$ denotes the complex conjugation. The proof is straightforward:

$$\begin{aligned}
 G(p) - G^*(p) &= \int dm^2 \rho(m^2) \left[\frac{1}{p^2 - m^2 + i\epsilon} - \frac{1}{p^2 - m^2 - i\epsilon} \right] \\
 &= \int dm^2 \rho(m^2) \frac{p^2 - m^2 - i\epsilon - p^2 + m^2 - i\epsilon}{(p^2 - m^2)^2 + \epsilon^2} \\
 &= \int dm^2 \rho(m^2) \frac{-2i\epsilon}{(p^2 - m^2)^2 + \epsilon^2}
 \end{aligned} \tag{3.12}$$

by using the delta function representation:

$$\lim_{\epsilon \rightarrow 0} \frac{\epsilon}{x^2 + \epsilon^2} = \pi \delta(x) \tag{3.13}$$

we obtain:

$$\frac{1}{2\pi i} [G(p) - G^*(p)] = \int dm^2 \rho(m^2) \delta(p^2 - m^2) = \rho(p^2) \tag{3.14}$$

Therefore the spectral function can be computed by evaluating:

$$\rho(p) = -\frac{1}{\pi} \text{Im } G(p) \tag{3.15}$$

3.2 Spectral Representation of Fermions

The propagator for a free fermion is given by:

$$iS(x - y) = \langle 0 | T \{ \psi(x) \bar{\psi}(y) \} | 0 \rangle = \int \frac{d^4 p}{(2\pi)^4} e^{-ip \cdot (x-y)} \frac{i(\not{p} + m)}{p^2 - m^2 + i\epsilon} \tag{3.16}$$

which in momentum space equals:

$$iS(p) = \frac{i(\not{p} + m)}{p^2 - m^2 + i\epsilon} = \frac{i}{\not{p} - m + i\epsilon} \tag{3.17}$$

The spectral representation for it can be constructed by once again introducing a complete set of positive momentum eigenstates into the expression:

$$\langle 0 | \psi(x) \bar{\psi}(y) | 0 \rangle = \sum_n \int dk \langle 0 | \psi(x) | k, n \rangle \langle k, n | \bar{\psi}(y) | 0 \rangle \tag{3.18}$$

We may define the spectral density this time as:

$$\sum_n \int dk (2\pi)^4 \delta^4(p - k) |\langle 0 | \psi(x) | k, n \rangle \langle k, n | \bar{\psi}(y) | 0 \rangle| = 2\pi \theta(p^0) \rho(p^2) \quad (3.19)$$

We now note that the spectral density can be expanded in the basis below:

$$\rho(p^2) = \rho_1(p^2) \not{p} + \rho_2(p^2) + \rho_3(p^2) \not{p} \gamma^5 + \rho_4(p^2) \gamma^5 \quad (3.20)$$

However, requiring invariance under parity transformations leads to:

$$\rho_3(p^2) = \rho_4(p^2) = 0 \quad (3.21)$$

So that the spectral representation may be written as:

$$S(p) = \int_0^\infty dm^2 \frac{\not{p} \rho_1(m^2) + \rho_2(m^2)}{p^2 - m^2 + i\epsilon} \quad (3.22)$$

The two spectral functions can be further reduced to a single one, via the transformation:

$$\rho_1(m^2) = \frac{\rho(m) + \rho(-m)}{2m}, \quad \rho_2(m^2) = \frac{\rho(m) - \rho(-m)}{2} \quad (3.23)$$

and by extending the domain of integration, we get:

$$S(p) = \int_{-\infty}^\infty \rho(m) \frac{\not{p} + m}{p^2 - m^2 + i\epsilon} \quad (3.24)$$

3.3 The Effect of Interactions

3.3.1 Dressed Scalar Propagator

Although our interest in the present dissertation is in the fermion propagator, for completeness let us first recall how interactions modify the free-boson propagators. When interactions are taken into account, the propagator is modified due to the effect of quantum fluctuations. The dressed propagator includes contributions from virtual particle-antiparticle pairs, which effectively dress the bare particle propagator. For ϕ^4 theory for example this can be represented diagrammatically in the following way [53]:

$$-i\Sigma(p^2) = \text{[diagram: tadpole]} + \text{[diagram: tadpole with loop]} + \text{[diagram: bubble]} + \dots = \text{[diagram: circle labeled 1PI]}$$

where we denoted by $-i\Sigma(p^2)$ all the possible one-particle-irreducible (1PI) insertions into the scalar propagator.

$$\text{[diagram: shaded circle]} = \text{[diagram: empty circle]} + \text{[diagram: circle labeled 1PI]} + \text{[diagram: two circles labeled 1PI]} + \dots$$

Therefore, the dressed propagator can be written as a geometric series:

$$\begin{aligned} iG(p) &= \frac{i}{p^2 - m^2} + \frac{i}{p^2 - m^2} (-i\Sigma) \frac{i}{p^2 - m^2} + \dots \\ &= \frac{i}{p^2 - m^2 - \Sigma(p^2)} \end{aligned} \quad (3.25)$$

The spectral density will now be given by:

$$\begin{aligned} \rho(p^2) &= -\frac{1}{\pi} \frac{\text{Im } \Sigma(p^2)}{(p^2 - m^2 - \text{Re } \Sigma(p^2))^2 - \text{Im } \Sigma(p^2)^2} \\ &= -\frac{1}{\pi} \frac{\text{Im } \Sigma(p^2)}{(p^2 - m_{\text{eff}}^2)^2 - \text{Im } \Sigma(p^2)^2} \end{aligned} \quad (3.26)$$

where m_{eff}^2 is called the effective mass.

In general, one has to address the issue of renormalization because quantum fluctuations introduce ultraviolet divergences in $\Sigma(p^2)$. Since our interest is not in the boson propagators, we do not discuss the issue here. We will come to renormalization in the next section, when we consider the fermion propagators. We conclude this subsection observing that by calculating the contribution of the self-energy to the propagator one can deduce the modified spectral functions. Specifically, our interest will be on the presence of complex-mass poles in the propagator. It has been shown that when a pole of the propagator in euclidean space is complex, it implies an oscillatory behavior in the Schwinger function due to the exponential terms $e^{ip\tau}$ associated with propagation [18]. This can lead to negative contributions to the propagator, especially off the real axis in the complex p^2 plane. Consequently, $\rho(p^2)$ might attain negative values for some range of p^2 , violating the positivity condition. This is particularly problematic because it suggests that some states contribute negatively to the probability densities, which

is not physically meaningful. In this dissertation we will not consider further dressed boson propagators.

3.3.2 Dressed Fermion Propagator

In this section we follow very closely the presentation in [56]. For the case of the fermion propagator, the self-energy corrections modify the propagator in the following way:

$$S(p) = \frac{1}{\not{p} - m - \Sigma(p) + i\epsilon} = \frac{1}{A(p^2)\not{p} - B(p^2) + i\epsilon} \quad (3.27)$$

where $A(p^2)$ and $B(p^2)$ are the dressing functions that account for the corrections from the interactions to the propagator. Letting $\psi_\Lambda(x)$ and m_Λ denote the bare field operator and bare mass of the fermion, respectively, where Λ represents a regularization cutoff. The relationship between the bare quantities and the renormalized field ψ and renormalized mass m is given by the renormalization factors as follows:

$$\psi_\Lambda(x) = \sqrt{Z_\psi} \psi(x) \quad m_\Lambda = Z_m m \quad (3.28)$$

The dressed renormalized propagator can then be related to the bare one, via the definition:

$$iS(x-y) = \langle \Omega | T[\psi(x)\bar{\psi}(y)] | \Omega \rangle = Z_\psi^{-1} iS_\Lambda(x-y) \quad (3.29)$$

This allows us to write the bare propagator as:

$$\begin{aligned} S_\Lambda(p) &= \frac{1}{A_\Lambda(p^2)\not{p} - B_\Lambda(p^2) + i\epsilon} = \frac{1}{A_\Lambda(p^2)} \frac{1}{\not{p} - M_\Lambda(p^2) + i\epsilon} \\ &= Z_\psi S(p) = Z_\psi \frac{1}{A(p^2)\not{p} - B(p^2) + i\epsilon} \\ &= \frac{Z_\psi}{A(p^2)} \frac{1}{\not{p} - M(p^2) + i\epsilon} \end{aligned} \quad (3.30)$$

where the dressing functions $A(p^2)$ and $B(p^2)$ of the renormalized propagator, $S(p)$, are related to the corresponding unrenormalized ones by $A(p^2) = Z_\psi A_\Lambda(p^2)$ and $B(p^2) = Z_\psi B_\Lambda(p^2)$, which imply $M(p^2) = M_\Lambda(p^2)$. The noninteracting

propagator is given by:

$$S_{\Lambda}^{(0)}(p) = \frac{1}{\not{p} - m_{\Lambda} + i\epsilon} = \frac{\not{p} + m_{\Lambda}}{p^2 - m_{\Lambda}^2 + i\epsilon} \quad (3.31)$$

The inverse of $S_{\Lambda}(p)$ is conveniently written as

$$S_{\Lambda}^{-1}(p) = [S_{\Lambda}^{(0)}(p)]^{-1} - \Sigma_{\Lambda}(p) \quad (3.32)$$

where $\Sigma_{\Lambda}(p) = (1 - A_{\Lambda}(p)) \not{p} + (m_{\Lambda} + B_{\Lambda}(p))$ is the self-energy. The bare propagator of Eq. (3.31) has a spectral representation like the one defined in Eq. (3.24) of the previous section. In order to evaluate the spectral function, it is useful to introduce the following projection operators:

$$P_{\pm}(p) = \frac{1}{2} \left(1 \pm \frac{\not{p}}{w(p)} \right) \quad \text{where} \quad w(p) \equiv \begin{cases} \sqrt{p^2} = \sqrt{(p^0)^2 - \vec{p}^2} & p^2 > 0 \\ i\sqrt{-p^2} = i\sqrt{\vec{p}^2 - (p^0)^2} & p^2 < 0 \end{cases} \quad (3.33)$$

As they allow us to project out the Dirac structure of the propagator:

$$S_{\Lambda}(p) = P_{+}(p) \tilde{S}_{\Lambda}(w(p) + i\epsilon) + P_{-}(p) \tilde{S}_{\Lambda}(-w(p) - i\epsilon) \quad (3.34)$$

where $\tilde{S}_{\Lambda}(z)$ is the scalar function

$$\tilde{S}_{\Lambda}(z) = \int_{-\infty}^{+\infty} d\kappa \frac{\rho_{\Lambda}(\kappa)}{z - \kappa} \quad (3.35)$$

with $z = \pm(w(p) + i\epsilon)$. The corresponding spectral representation for renormalized propagator, written in terms of a renormalized spectral function $\rho(\kappa)$, is given by

$$\tilde{S}(z) = \int_{-\infty}^{+\infty} d\kappa \frac{\rho(\kappa)}{z - \kappa} \quad (3.36)$$

where $\rho_{\Lambda}(\kappa)$ and $\rho(\kappa)$ are related by the field renormalization constant Z_{ψ} as:

$$\rho_{\Lambda}(\kappa) = Z_{\psi} \rho(\kappa) \quad (3.37)$$

Since $\rho_{\Lambda}(\kappa)$ is a probability density, it must satisfy the following property:

$$\int_{-\infty}^{+\infty} d\kappa \rho_{\Lambda}(\kappa) = 1 \quad (3.38)$$

which allows us to relate the renormalization constant Z_ψ to the renormalized spectral function $\rho(\kappa)$:

$$Z_\psi^{-1} = \int_{-\infty}^{+\infty} d\kappa \rho(\kappa) \quad (3.39)$$

This expression also implies that $\tilde{S}_\Lambda(z)$ has no zeros or poles off the real axis for positive $\rho_\Lambda(\kappa)$. This can be shown as follows. Letting $z = x + iy$, with x and y real, we get:

$$\tilde{S}_\Lambda(z) = \tilde{S}_\Lambda(x + iy) = \int_{-\infty}^{+\infty} d\kappa \frac{\rho_\Lambda(\kappa)}{x + iy - \kappa} = (x - \kappa - iy) \int_{-\infty}^{+\infty} d\kappa \frac{\rho_\Lambda(\kappa)}{(x - \kappa)^2 + y^2} \quad (3.40)$$

The imaginary part of $\tilde{S}_\Lambda(z)$ will then be given by

$$\text{Im } \tilde{S}_\Lambda(z) = -y \int_{-\infty}^{+\infty} d\kappa \frac{\rho_\Lambda(\kappa)}{(x - \kappa)^2 + y^2} \quad (3.41)$$

This expression is zero only if $y = 0$ when $\rho_\Lambda(\kappa) > 0$; therefore, any zero in $\tilde{S}_\Lambda(z)$ must be located on the real axis. When $\rho_\Lambda(\kappa)$ is not positive, singularities can appear off the real axis. The absence of zeros off the real axis in $\tilde{S}_\Lambda(z)$ indicates that the inverse of the propagator does not have poles off the real axis. This property allows us to write a spectral representation for the inverse of $\tilde{S}_\Lambda(z)$:

$$\begin{aligned} \tilde{S}_\Lambda^{-1}(z) &= z - m_\Lambda - \tilde{\Sigma}(z) \\ &= z - m_\Lambda - \int_{-\infty}^{+\infty} d\kappa \frac{\sigma_\Lambda(\kappa)}{z - \kappa} \end{aligned} \quad (3.42)$$

To write this, we used the fact that $S_\Lambda^{-1}(p)$ has the same Dirac structure that $S_\Lambda(p)$ and so we can make use of the projection operators $P_\pm(p)$:

$$S_\Lambda^{-1}(p^2) = P_+(p) \tilde{S}^{-1}(w(p) + i\epsilon) + P_-(p) \tilde{S}^{-1}(-w(p) - i\epsilon) \quad (3.43)$$

It is possible to prove that $\tilde{S}_\Lambda^{-1}(z)$ also does not have zeros off the real axis if $\sigma_\Lambda(\kappa) > 0$; in this case, $\tilde{S}_\Lambda(z)$ has no poles off the real axis. The renormalized inverse is

$$\tilde{S}^{-1}(z) = Z_\psi \tilde{S}_\Lambda^{-1}(z) = Z_\psi(z - Z_m m) - \int_{-\infty}^{+\infty} d\kappa \frac{\sigma(\kappa)}{z - \kappa} \quad (3.44)$$

where the renormalized spectral function $\sigma(\kappa)$ is given by

$$\sigma_\Lambda(\kappa) = Z_\psi^{-1} \sigma(\kappa) \quad (3.45)$$

From (3.44) one has that

$$\sigma(\kappa) = \frac{1}{2\pi i} \left[\tilde{S}^{-1}(\kappa + i\epsilon) - \tilde{S}^{-1}(\kappa - i\epsilon) \right] = -\frac{1}{\pi} \text{Im } \tilde{\Sigma}(\kappa) \quad (3.46)$$

One can obtain a relationship between the $\sigma(\kappa)$ and $\rho(\kappa)$ spectral functions using the identity

$$\tilde{S}^{-1}(\kappa + i\epsilon) - \tilde{S}^{-1}(\kappa - i\epsilon) = \tilde{S}^{-1}(\kappa + i\epsilon) \tilde{S}^{-1}(\kappa - i\epsilon) \left[\tilde{S}(\kappa - i\epsilon) - \tilde{S}(\kappa + i\epsilon) \right] \quad (3.47)$$

and taking into account (3.46) one obtains

$$\sigma(\kappa) = |\tilde{S}^{-1}(\kappa + i\epsilon)|^2 \rho(\kappa) \quad (3.48)$$

The inverse relationship is obtained as follows: since $[\tilde{S}^{-1}(z)]^{-1} = \tilde{S}(z)$, one can write using (3.46)

$$\rho(\kappa) = \frac{i}{2\pi} \left[\tilde{S}^{-1}(\kappa + i\epsilon) \right]^{-1} - \left[\tilde{S}^{-1}(\kappa - i\epsilon) \right]^{-1} = R(M_p) \delta(\kappa - M_p) + \bar{\rho}(\kappa) \quad (3.49)$$

where

$$\bar{\rho}(\kappa) = |\tilde{S}^{-1}(\kappa + i\epsilon)|^{-2} \sigma(\kappa) \quad (3.50)$$

In (3.49), M_p is a mass pole and $R(M_p)$ the corresponding residue. This pole is identified as a zero of $\tilde{S}^{-1}(z)$. If there are multiple poles, one must sum over all of them and if there are no poles, then $R(M_p) = 0$.

Before concluding, let us discuss renormalization further. In this dissertation, we will consider an on-shell renormalization scheme in that we fix the renormalization constants Z_ψ and Z_m by imposing that the propagator has a pole at some timelike momentum $p^2 = \mu^2 \equiv M_p^2 > 0$ with unit residue. In this renormalization scheme, the renormalized inverse propagator, $\tilde{S}^{-1}(z, M_p) = \tilde{S}^{-1}(z, \mu = M_p)$ is given by [57, 56]

$$\tilde{S}^{-1}(z, M_p) = (z - M_p) \left[1 - (z - M_p) \int_{-\infty}^{+\infty} d\kappa \frac{\sigma(\kappa, M_p)}{(z - \kappa)(\kappa - M_p)^2} \right] \quad (3.51)$$

We do not need the explicit expressions for Z_ψ and Z_m because we are not interested in this dissertation on the renormalization group flows of the parameters of the model; the explicit expressions are given in [56].

To conclude this chapter, we note that the question of whether the propagator has complex poles can then be easily answered by searching the zeros of the function

$$F(z) = 1 - (z - M_p) \int_{-\infty}^{+\infty} d\kappa \frac{\sigma(\kappa, M_p)}{(z - \kappa)(\kappa - M_p)^2} \quad (3.52)$$

for complex values of z . We then need to compute the spectral function $\sigma(\kappa, M_p)$ in the model. Building upon this discussion of mass poles and residues, in the following chapter we will delve into the finite temperature field formalism. This method will allow us to evaluate self-energy corrections influenced by finite temperatures, relevant for understanding changes in the pole structure under thermal conditions.

Chapter 4

Finite Temperature Field Theory

4.1 Finite Temperature Formalism

There are mainly three formalisms for describing quantum field theory at finite temperature: the imaginary time formalism (ITF)[58], the real time formalism (RTF)[59], and thermofield dynamics[60]. In this work we shall be adopting the real-time formalism. Let us begin by briefly reviewing the main aspects of the theory.

Consider a quantum mechanical system in contact with a heat reservoir, which may or may not be in thermal equilibrium. The density matrix that describes such a system is given by:

$$\rho = \sum_n p_n |\psi_n(t)\rangle \langle \psi_n(t)| \quad (4.1)$$

where p_n denotes the associated probabilities of finding the system in a determined state $|\psi_n(t)\rangle$, and which must satisfy:

$$\sum p_n = 1 \quad (4.2)$$

The expectation value of any observable related to the system may be calculated through:

$$\langle A \rangle(t) = \text{Tr} \rho(t) A = \sum_n p_n \langle \psi_n(t) | A | \psi_n(t) \rangle \quad (4.3)$$

Consider now the time evolution of the density matrix which is governed by the Liouville equation ($\hbar = 1$):

$$i \frac{\partial \rho(t)}{\partial t} = [H, \rho(t)] \quad (4.4)$$

In the most general case in which the Hamiltonian is not time-independent, the solution of Eq. (4.4) may be written as:

$$\rho(t) = U(t, 0) \rho(0) U^\dagger(t, 0) \quad (4.5)$$

where $U(t, t')$ is the time evolution operator, defined as:

$$U(t, t') = T \left(e^{-i \int_{t'}^t dt'' H(t'')} \right) \quad (4.6)$$

We now wish to describe a situation for which the system is prepared at equilibrium and evolve through a time-dependent Hamiltonian. This can be accomplished by choosing:

$$H(t) = \begin{cases} H_i & \text{for } \text{Re } t \leq 0 \\ \mathcal{H}(t) & \text{for } \text{Re } t \geq 0 \end{cases} \quad (4.7)$$

The density matrix at an initial time for this case can be written as:

$$\rho(0) = \frac{e^{-\beta H_i}}{\text{Tr } e^{-\beta H_i}} = \frac{U(T - i\beta, T)}{\text{Tr } U(T - i\beta, T)} \quad (4.8)$$

Substituting this into the expression for the expectation value of an observer at Eq. (4.3), we reach:

$$\langle A \rangle(t) = \text{Tr } \rho(t) A = \frac{\text{Tr } U(t, 0) U(T - i\beta, T) U(0, t) A}{\text{Tr } U(T - i\beta, T)} \quad (4.9)$$

And using the property that $U(t_1, t_2) U(t_2, t_1) = 1$, we can write:

$$\langle A \rangle(t) = \frac{\text{Tr } U(T - i\beta, T) U(0, t) A U(t, 0) U(0, T) U(T, 0)}{\text{Tr } U(T - i\beta, T)} \quad (4.10)$$

Next, applying the cyclic property and using the fact that for $T < 0$, $U(T - i\beta, T)$ and $U(T, 0)$ commute we get:

$$\langle A \rangle(t) = \frac{\text{Tr } U(T - i\beta, T) U(T, 0) U(0, t) A U(t, 0) U(0, T)}{\text{Tr } U(T - i\beta, T)} \quad (4.11)$$

Lastly, we can introduce the term $U(T, T')$ into the expression, where T' denotes a large positive time:

$$\begin{aligned} \langle A \rangle(t) &= \frac{\text{Tr } U(T - i\beta, T) U(T, T) U(T', T) U(T, t) A U(t, T)}{\text{Tr } U(T - i\beta, T)} \\ &= \frac{\text{Tr } U(T - i\beta, T) U(T, T') U(T', t) A U(t, T)}{\text{Tr } U(T - i\beta, T) U(T, T') U(T', T)} \end{aligned} \quad (4.12)$$

We can interpret this expression as the system evolving from a negative time T

to a time t where the operator A was inserted and then, evolving from this time t to a positive time T' , after which it evolves back to T and along an imaginary direction. The contour in the complex plane which describes this situation is displayed below.

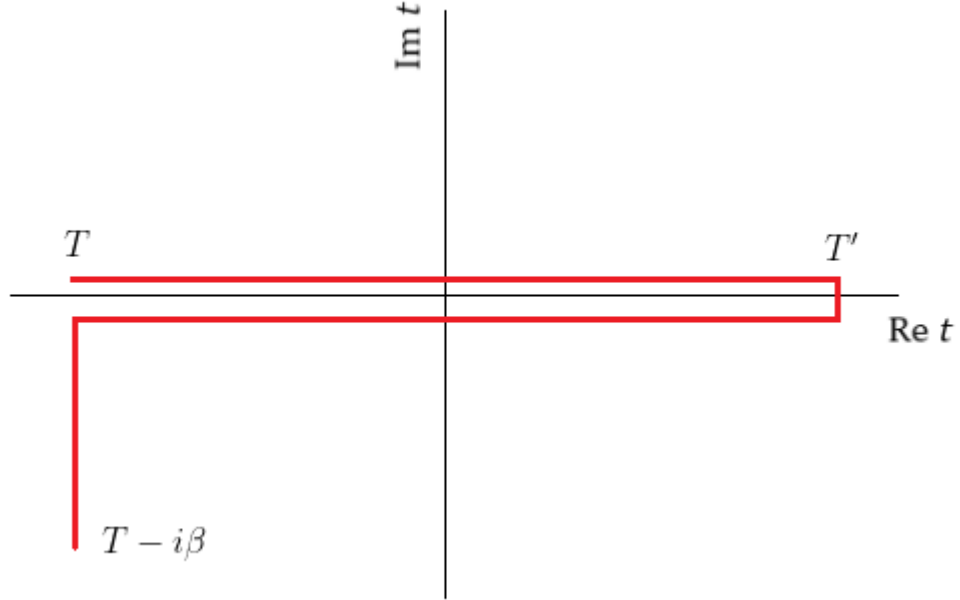


Figure 4.1: Contour representation of the time evolution in the complex plane.

There is some degree of arbitrariness in the choice of this contour, as one is free to select a different contour that satisfies the required conditions [28, 27]. Specifically, the path should originate at an initial time T , terminate at a final time $T - i\beta$ and must traverse the real axis. These conditions, however, still allow for a wide range of possible paths. One such possibility is the path displayed below:

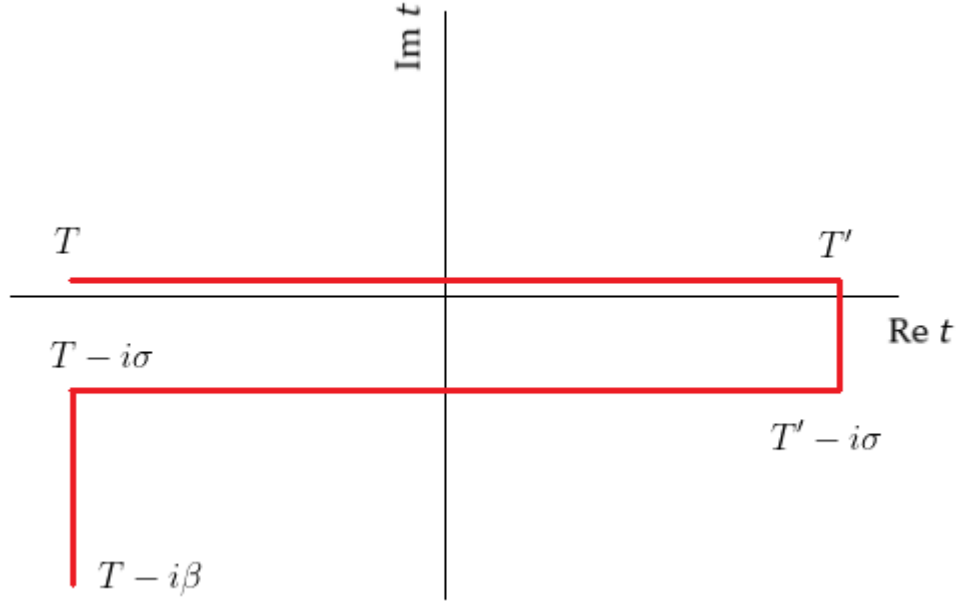


Figure 4.2: Generalization of the contour from Fig (3.1) with $\sigma = \beta/2$. Propagators defined on this contour coincide with those derived in the thermofield dynamics formalism [27].

The contour starts at T and goes along the real axis to T' , we call this segment C_1 . From T' , the contour continues along the imaginary direction in time to $T' - i\frac{\beta}{2}$ (segment C_3). From $T' - i\frac{\beta}{2}$, it goes parallel to the real axis to $T - i\frac{\beta}{2}$ (segment C_2), and finally parallel to the imaginary axis in time to $T - i\beta$ (segment C_4). We shall be adopting this contour convention in our calculations in the upcoming sections. The motivation for using such a different contour is that there is a well-documented literature [27, 26, 61] on this approach that allows us make consistency checks of our results. It is also worth mentioning that the propagators formulated on the contour of Fig. 4.2 coincide with the one derived using the thermofield dynamics formalism, which is a Hilbert-space based approach [27].

4.2 Scalar Fields at Finite Temperature

In this section we show how the formalism just presented is used for the case of quantum field theory. Consider the Hamiltonian density for a scalar field theory

given by:

$$\mathcal{H}[\phi, \pi] = \frac{1}{2}\pi^2 + \frac{1}{2}\phi(-\nabla^2 + m^2)\phi + V[\phi] \quad (4.13)$$

where ϕ denotes a bosonic field and π is the canonical conjugate momentum such that the following commutation relation is valid:

$$[\phi(x), \pi(x')] = i\delta(x - x') \quad (4.14)$$

In the Heisenberg representation the fields evolve with time according to:

$$\phi(x, t') = e^{iH(t'-t)}\phi(x, t)e^{-iH(t'-t)} \quad (4.15)$$

$$\pi(x, t') = e^{iH(t'-t)}\pi(x, t)e^{-iH(t'-t)} \quad (4.16)$$

and we may define the eigenstates of the operators as:

$$\phi(x, t)|\phi(x), t\rangle = \phi(x)|\phi(x), t\rangle \quad (4.17)$$

$$\pi(x, t)|\pi(x), t\rangle = \pi(x)|\pi(x), t\rangle \quad (4.18)$$

such that the eigenvalues $\phi(x)$ and $\pi(x)$ in the Schrödinger representation are independent of time, given that the time evolution of the eigenstates is of the form:

$$|\phi(x), t'\rangle = e^{iH(t'-t)}|\phi(x), t\rangle \quad (4.19)$$

$$|\pi(x), t'\rangle = e^{iH(t'-t)}|\pi(x), t\rangle \quad (4.20)$$

We're interested in evaluating the thermal n -point functions of the theory defined as the expectation value of the time-ordered operators along a contour C in the complex plane:

$$iG_c(x_1, \dots, x_n) = \frac{1}{\text{Tr}e^{-\beta H}} \text{Tr}e^{-\beta H} T[\phi(x_1) \dots \phi(x_n)] \quad (4.21)$$

This definition is equivalent to the following one with an external source J :

$$iG_c(x_1, \dots, x_n) = \frac{1}{Z[0]} \frac{\delta}{\delta J(x_n)} \dots \frac{\delta}{\delta J(x_1)} Z[J] \Big|_{J=0} \quad (4.22)$$

where the generating functional $Z[J]$ is given by:

$$Z[J] = \text{Tr} \left\{ e^{-\beta H} T_c \left[e^{i \int_c d\tau \int d^3x J(x, \tau) \phi(x, \tau)} \right] \right\} \quad (4.23)$$

and T_c denotes time ordering along a generalization of the contour C in the complex plane described in the previous section. Time-ordering along this contour is defined according to:

$$T_c[\phi(x)\phi(x')] = \theta_c(\tau - \tau')\phi(x)\phi(x') + \theta_c(\tau' - \tau)\phi(x')\phi(x) \quad (4.24)$$

$$\partial_\tau T_c[\phi(x)\phi(x')] = \delta_c(\tau - \tau')[\phi(x), \phi(x')] + T_c[\partial_\tau \phi(x)\phi(x')] \quad (4.25)$$

where we introduced the following functions which can be defined in terms of the contour parametrization $\tau = z(\nu)$ with ν real and monotonically increasing:

$$\theta_c(t - t') = \theta(\nu - \nu') \quad (4.26)$$

$$\delta_c(t - t') = \left(\frac{\partial z}{\partial \nu} \right)^{-1} \delta(\nu - \nu') \quad (4.27)$$

in such a way that the ordering along the path will correspond to the ordering in ν and the following properties hold true:

$$\int_c d\tau' (\tau - \tau') f(\tau) = f(\tau') \quad (4.28)$$

$$\frac{\delta J(x)}{\delta J(x')} = \delta_c(\tau - \tau') \delta(x - x') \quad (4.29)$$

Let us now evaluate the generating functional of Eq. (3.23). Inserting a complete set of eigenstates:

$$\int D\phi |\phi, t\rangle \langle \phi, t| = 1 \quad (4.30)$$

at $t = 0$ and evaluating the trace, we reach:

$$Z[J] = \int D\phi \langle \phi, 0 | e^{-\beta H} T_c [e^{i \int_c J \phi}] | \phi, 0 \rangle \quad (4.31)$$

which we may write as:

$$Z[J] = \int \mathcal{D}\phi \langle \phi, 0 | e^{-iHT} e^{iHT} e^{-\beta H} T_c \left[e^{i \int J\phi} \right] | \phi, 0 \rangle \quad (4.32)$$

$$= \int \mathcal{D}\phi \langle \phi, 0 | e^{iHT} e^{-\beta H} T_c \left[e^{i \int J\phi} \right] e^{-iHT} | \phi, 0 \rangle \quad (4.33)$$

$$= \int \mathcal{D}\phi \langle \phi, -T - i\beta | T_c \left[e^{i \int J\phi} \right] | \phi, -T \rangle \quad (4.34)$$

Inserting once again complete sets of eigenstates along the ends of C_1 , C_2 and C_3

$$\begin{aligned} Z[J] &= \int \mathcal{D}\phi''' \mathcal{D}\phi'' \mathcal{D}\phi' \mathcal{D}\phi \langle \phi''', -T - i\beta | T_c \left[e^{i \int_{C_4} J\phi} \right] | \phi'', -T \rangle \\ &\quad \times \langle \phi'', -T - \frac{i\beta}{2} | T_c \left[e^{i \int_{C_3} J\phi} \right] | \phi'', -T - \frac{i\beta}{2} \rangle \\ &\quad \times \langle \phi'', -T - \frac{i\beta}{2} | T_c \left[e^{i \int_{C_2} J\phi} \right] | \phi', +T \rangle \\ &\quad \times \langle \phi', +T | T_c \left[e^{i \int_{C_1} J\phi} \right] | \phi, -T \rangle \end{aligned} \quad (4.35)$$

This generating functional can be shown to equal:

$$Z[J] = \int_{\tau \in C} \frac{D\phi(\tau) D\pi(\tau)}{2\pi} e^{i \int_C (\pi\dot{\phi} - H + J\phi) d\tau} \quad (4.36)$$

and by substituting the Hamiltonian we can write:

$$\begin{aligned} Z[J] &= \int_{\tau \in C} \frac{D\phi(\tau) D\pi(\tau)}{2\pi} \exp \left\{ i \left[\int_C \left(-\frac{1}{2} (-2\pi\dot{\phi} + \pi^2 + \dot{\phi}^2 - \dot{\pi}^2) \right) \right. \right. \\ &\quad \left. \left. + \left(-\frac{1}{2} \phi(-\nabla^2 + m^2)\phi - V[\phi] \right) + J\phi \right] \right\} \\ &= \int_{\tau \in C} \frac{D\phi(\tau) D\pi(\tau)}{2\pi} \exp \left\{ i \left[\int_C \left(-\frac{1}{2} (\pi - \dot{\phi})^2 \right) \right] \right\} \\ &\quad \times \exp \left\{ i \left[\int_C \left(\frac{\dot{\pi}^2}{2} - \frac{1}{2} \phi(-\nabla^2 + m^2)\phi - V[\phi] \right) + J\phi \right] \right\} \end{aligned} \quad (4.37)$$

such that:

$$\begin{aligned} Z[J] &= \int_{\tau \in C} \frac{D\phi(\tau) D\pi(\tau)}{2\pi} \exp \left\{ i \left[\int_C \left(-\frac{1}{2} (\pi - \dot{\phi})^2 \right) \right] \right\} \\ &\quad \times \exp \left(i \left[\int_C \left(-\frac{\phi \partial_0^2 \phi}{2} - \frac{1}{2} \phi(-\nabla^2 + m^2)\phi - V[\phi] \right) + J\phi \right] \right) \end{aligned} \quad (4.38)$$

and manipulating terms further we reach:

$$\begin{aligned}
Z[J] &= \int_{\tau \in \mathcal{C}} \frac{D\phi(\tau)D\pi(\tau)}{2\pi} \exp \left\{ i \left[\int_{\mathcal{C}} \left(-\frac{1}{2} (\pi - \dot{\phi})^2 \right) \right] \right\} \\
&\quad \times \exp \left(i \left[\int_{\mathcal{C}} \left(-\frac{\phi \partial_0^2 \phi}{2} + \frac{1}{2} \phi (-\nabla^2) \phi - \frac{1}{2} m^2 \phi^2 - V[\phi] + J\phi \right) \right] \right) \\
&= \int_{\tau \in \mathcal{C}} \frac{D\phi(\tau)D\pi(\tau)}{2\pi} \exp \left\{ i \left[\int_{\mathcal{C}} \left(-\frac{1}{2} (\pi - \dot{\phi})^2 \right) \right] \right\} \\
&\quad \times \exp \left(i \left[\int_{\mathcal{C}} \left(-\frac{1}{2} \phi (\ddot{\nabla}^2 - \nabla^2 - m^2) \phi - V[\phi] + J\phi \right) \right] \right) \\
&= \int_{\tau \in \mathcal{C}} \frac{D\phi(\tau)D\pi(\tau)}{2\pi} \exp \left\{ i \left[\int_{\mathcal{C}} \left(-\frac{1}{2} (\pi - \dot{\phi})^2 \right) \right] \right\} \\
&\quad \times \exp \left(i \left[\int_{\mathcal{C}} \left(\frac{1}{2} \phi (-\square - m^2) \phi - V[\phi] + J\phi \right) \right] \right) \tag{4.39}
\end{aligned}$$

the Lagrangian density is given by:

$$\mathcal{L}(\phi) = \frac{1}{2} \phi (-\square - m^2) \phi - V[\phi] \tag{4.40}$$

which allow us to write:

$$Z[J] = \int D\phi \exp \left[i \int_{\mathcal{C}} \mathcal{L}(\phi) + J\phi \right] \tag{4.41}$$

It is clear then that using property in Eq. (4.28) we can write the generating functional as:

$$Z[J] = \exp \left\{ -i \int_{\mathcal{C}} V \left[\frac{1}{i} \frac{\delta}{\delta J} \right] \right\} \exp \left\{ -\frac{1}{2} \int_{\mathcal{C}} J(x) G_c(x-y) J(y) \right\} \tag{4.42}$$

Performing the shift:

$$\phi'(x) = \phi(x) - i \int_{\mathcal{C}} G_c(x-y) J(y) \tag{4.43}$$

where $G_c(x-y)$ is defined as the Green function along the contour $\mathcal{C}$:

$$(-\square_c - m^2) i G_c(x-y) = i \delta_c(x-y) \tag{4.44}$$

We may substitute the shifted field in our original expression:

$$Z[J] = \exp \left[-i \int_c V \left[\frac{1}{i} \frac{\delta}{\delta J} \right] d^4x \right] \int D\phi \exp \left[\frac{i}{2} \int_c d^4x \phi(x) (-\square_c - m^2) \phi(x) \right] \quad (4.45)$$

Reaching the following:

$$\begin{aligned} Z[J] = & \exp \left[-i \int_c V \left[\frac{1}{i} \frac{\delta}{\delta J} \right] d^4x \right] \int \mathcal{D}\phi \times \exp \left[\frac{i}{2} \int d^4x \phi(x) (-\square - m^2) \phi(x) \right] \\ & \times \exp \left[-\frac{1}{2} \int d^4x d^4y \phi(x) (-\square - m^2) G_c(x-y) J(y) \right] \\ & \times \exp \left[-\frac{1}{2} \int d^4x d^4y G_c(x-y) J(y) (-\square - m^2) \phi(x) \right] \\ & \times \exp \left[-\frac{i}{2} \int d^4x d^4z \int d^4y G_c(x-y) J(y) (-\square - m^2) G_c(x-z) J(z) \right] \\ & \times \exp \left[\frac{i}{2} \int d^4x J(x) \phi(x) - \int d^4x d^4y J(x) G_c(x-y) J(y) \right] \end{aligned} \quad (4.46)$$

and after integrating by parts we reach:

$$\begin{aligned} Z[J] = & \exp \left[-i \int_c V \left[\frac{1}{i} \frac{\delta}{\delta J} \right] d^4x \right] \int \mathcal{D}\phi \\ & \times \exp \left[-\frac{1}{2} \int_c d^4y \int_c d^4x J(x) (-\square - m^2) J(y) \right] \end{aligned} \quad (4.47)$$

Consider now the following ansatz for G_c :

$$G_c(x-y) = G_c^>(x-y) \theta_c(\tau_x - \tau_y) + G_c^<(x-y) \theta_c(\tau_y - \tau_x) \quad (4.48)$$

the solution of Eq. (4.44) is found to be [27]:

$$\begin{aligned} G_c(\tau - \tau', \omega_k) = & \frac{1}{2\omega_k} \frac{1}{1 - e^{-\beta\omega_k}} \left\{ \left[e^{-i\omega_k(\tau-\tau')} + e^{-\beta\omega_k + i\omega_k(\tau-\tau')} \right] \theta(\tau - \tau') \right. \\ & \left. + \left[e^{i\omega_k(\tau-\tau')} + e^{-\beta\omega_k - i\omega_k(\tau-\tau')} \right] \theta(\tau' - \tau) \right\} \end{aligned} \quad (4.49)$$

where $\omega_k = \sqrt{\vec{k}^2 + m^2}$. When we consider this propagator evaluated along the contour of Fig. (3.2), we note that in the limit of $T \rightarrow -\infty$, the contribution from curves C_3 and C_4 that connect a real time to a imaginary time and vice-versa, will be of the form:

$$\lim_{T \rightarrow -\infty} (e^{-i\omega(t-T+is)} + e^{\beta\omega + i\omega(t-T+is)}) \rightarrow 0 \quad (4.50)$$

and the relevant contour curves are reduced to C_1 and C_2 . We can write therefore:

$$\int_{C_+C_-} J(x)G_c(x-y)J(y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dt dt' \int dx dy J_a(t,x)G_{a,b}(t-t',x-y)J_b(t',y) \quad (4.51)$$

where $a, b = 1, 2$:

$$J_1 = J(x), \quad J_2 = J\left(t - \frac{i\beta}{2}, x\right) \quad (4.52)$$

To find the thermal propagators we must take functional derivatives with respect to J_1 and J_2 . This allows us to define four non-trivial propagators depending on where x_0 and y_0 lie along the contour:

$$iG_c(x-y) = \begin{cases} iG_{11}(x-y) = iG_c(x-y) & x_0 \in C_1, y_0 \in C_1 \\ iG_{12}(x-y) = iG_c^<(x_0 - y_0 + \frac{i\beta}{2}, x-y) & x_0 \in C_1, y_0 \in C_2 \\ iG_{21}(x-y) = iG_c^>(x_0 - y_0 - \frac{i\beta}{2}, x-y) & x_0 \in C_2, y_0 \in C_1 \\ iG_{22}(x-y) = iG_c(y-x) & x_0 \in C_2, y_0 \in C_2 \end{cases} \quad (4.53)$$

which in momentum space can be written as:

$$iG_c(p) = \begin{cases} iG_{11} = \frac{i}{p^2 - m^2 + i\epsilon} + 2\pi n_B(|p_0|)\delta(p^2 - m^2) \\ iG_{12} = 2\pi\delta(p^2 - m^2)e^{\beta|p_0|/2}n_B(|p_0|) \\ iG_{21} = 2\pi\delta(p^2 - m^2)e^{\beta|p_0|/2}n_B(|p_0|) \\ iG_{22} = -\frac{i}{p^2 - m^2 + i\epsilon} + 2\pi n_B(|p_0|)\delta(p^2 - m^2) \end{cases} \quad (4.54)$$

It is often convenient to simplify this representation through a matrix notation of the form:

$$iG_c = \begin{pmatrix} iG_{11} & iG_{12} \\ iG_{21} & iG_{22} \end{pmatrix} \quad (4.55)$$

In this notation, one doubles the degrees of freedom associated to the field, with one field defined above the real axis and another field defined below it:

$$\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \quad (4.56)$$

4.3 Fermion Fields at Finite Temperature

The procedure for obtaining the propagator for fermions is analogous to the one derived before for scalar fields, however now the thermal distribution function is given by:

$$n_F(\omega_k) = \frac{1}{e^{\beta\omega_k} + 1} \quad (4.57)$$

and the momentum-space propagators defined along the contour are given by [26, 27]:

$$iS_c(p) = \begin{cases} iS_{11} = (\not{p} + m) \left(\frac{i}{p^2 - m^2 + i\epsilon} - 2\pi n_F(|p_0|) \delta(p^2 - m^2) \right) \\ iS_{12} = -2\pi(\not{p} + m) \varepsilon(p_0) e^{\beta|p_0|/2} n_F(|p_0|) \delta(p^2 - m^2) \\ iS_{21} = 2\pi(\not{p} + m) \varepsilon(p_0) e^{\beta|p_0|/2} n_F(|p_0|) \delta(p^2 - m^2) \\ iS_{22} = (\not{p} + m) \left(-\frac{i}{p^2 - m^2 + i\epsilon} - 2\pi n_F(|p_0|) \delta(p^2 - m^2) \right) \end{cases} \quad (4.58)$$

4.4 Self-Energy Corrections at Finite Temperature

The doubling of degrees of freedom has another consequence which is worth mentioning. Pinch singularities that might arise in the perturbative calculation of Green's functions or self-energies at finite temperature when products of propagators with poles on the same side of the complex energy plane appear in the integrands get cancelled.

These singularities occur because, in the naive perturbative expansion, terms that look like $\frac{1}{p^2 - m^2 + i\epsilon} \frac{1}{p^2 - m^2 - i\epsilon}$ can appear, where p is the four-momentum and m is the mass of the particle. When integrated over all energies, such terms can lead to divergences, called pinch singularities, since the $i\epsilon$ prescription does not

adequately regularize the integral. The matrix structure of Eq. (4.55) prevents this however. Noting that the matrix can be put into diagonal form as follows:

$$iG_0(p) = U(\beta, p) \begin{pmatrix} \frac{i}{p^2 - m^2 + i\epsilon} & 0 \\ 0 & \frac{-i}{p^2 - m^2 - i\epsilon} \end{pmatrix} U(\beta, p) \quad (4.59)$$

where $U(\beta, p)$ is given by

$$U(\beta, p) = \begin{pmatrix} \cosh \theta_p & \sinh \theta_p \\ \sinh \theta_p & \cosh \theta_p \end{pmatrix} \quad (4.60)$$

and we have used the following redefinitions:

$$\cosh(\theta_p) = \frac{1}{\sqrt{1 - e^{-\beta|p_0|}}} \quad \text{and} \quad \sinh(\theta_p) = \frac{e^{-\beta|p_0|/2}}{\sqrt{1 - e^{-\beta|p_0|}}} \quad (4.61)$$

We note that even after the propagator acquires self-energy corrections:

$$iG^{ab}(p) = iG_0^{ab}(p) + iG_0^{ac}(-i\Sigma^{cd}(p))iG_0^{db}(p) + \dots \quad (4.62)$$

the expression remains normalized due to:

$$-i\Sigma^{ab}(p) = U^{-1}(\beta, p) \begin{pmatrix} -i\Sigma(p) & 0 \\ 0 & i\Sigma(p)^* \end{pmatrix} U^{-1}(\beta, p) \quad (4.63)$$

Therefore terms like $G_0(p)G_0(p)$ never appear in the expansion. From the matrix equation at (4.63) we may derive the following relations for the real and imaginary parts of the self-energy [26]:

$$\text{Re } \Sigma = \text{Re } \Sigma_{11} \quad (4.64)$$

$$\begin{aligned} \text{Im}\Sigma &= \text{sech}(2\theta_p)\text{Im}\Sigma_{11} \\ &= \epsilon(p_0) \tanh\left(\frac{1}{2}\beta p_0\right)\text{Im}\Sigma_{11} \end{aligned} \quad (4.65)$$

For fermions, a similar expression is valid. In matrix form, the fermion propagator can be put into diagonal form [26]:

$$iS(p) = V(\beta, p) \begin{pmatrix} \frac{i}{p - m + i\epsilon} & 0 \\ 0 & \frac{-i}{p - m - i\epsilon} \end{pmatrix} V(\beta, p) \quad (4.66)$$

where $V(\beta, p)$ is given by:

$$V(\beta, p) = \begin{pmatrix} \cos \phi_p & -\epsilon(p_0) \sin \phi_p \\ \epsilon(p_0) \sin \phi_p & \cos \phi_p \end{pmatrix} \quad (4.67)$$

and:

$$\cos(\phi_p) = \frac{1}{\sqrt{1 + e^{-\beta|p_0|}}} \quad \text{and} \quad \sin(\phi_p) = \frac{e^{-\beta|p_0|/2}}{\sqrt{1 + e^{-\beta|p_0|}}} \quad (4.68)$$

When interactions are added the following relations are valid:

$$\text{Re } \Sigma = \text{Re } \Sigma_{11} \quad (4.69)$$

$$\begin{aligned} \text{Im } \Sigma &= \sec(2\phi_p) \text{Im } \Sigma_{11} \\ &= \epsilon(p_0) \coth\left(\frac{1}{2}\beta p_0\right) \text{Im } \Sigma_{11} \end{aligned} \quad (4.70)$$

The derivations are shown in detail in Appendix C.

Chapter 5

Temperature effects on unphysical fermion mass-poles in the LSM

This chapter contains the central part of this dissertation: we study the effects of temperature in the complex-mass poles that appear in the propagator of the fermion propagator in the linear sigma model (LSM). Our interest is in the unphysical poles, i.e. poles of the propagator with negative or complex residues.

The Lagrangian density of the model was discussed in Chapter 2. When the fermions of the model are nucleons, the LSM model realizes spontaneous breaking of chiral symmetry (SSB χ) in the confined phase of QCD and when the fermions are quarks, the model realizes SSB χ in the QCD deconfined phase. In this dissertation, we consider both cases. Recall that SSB χ generates the mass of the fermion; at the classical level, the generated mass is given by $m_f = g\nu$, where ν is the value of the σ field minimum of the classical potential. When considering quantum and thermal fluctuations, the appropriate quantity to minimize is the effective potential or free energy. This question of how quantum and thermal effects affect the fermion mass has been addressed along the last decades in terms of a mean-field approximation (MFA), in that the σ and π fields remain constants even when including quantum and thermal fluctuations as constants. The MFA, a.k.a. the Hartree approximation, is very efficient in providing qualitative insight on many nonperturbative effects driving SSB and phase transitions in the LSM, and has become a standard in the field.

Minimization of the LSM free energy leads to a cubic nonlinear equation for the σ field. In the LSM with nucleons, for sufficiently low temperatures away from the QCD phase transition, there is a simpler and insightful way to implement the MFA, which allows us to make contact with the well-known MFA to the Walecka model [23]. Specifically, one takes $\pi = 0$ and writes the σ field as previously $\sigma = \sigma_c + \sigma'$, where $\sigma_c = \nu$ and σ' are also constant, but carry quantum and thermal fluctuations. Therefore, the fermion mass is given by the sum of the classical mass

$m_f = g\nu$ and a constant self-energy correction, Σ_H , such that $m_f + \Sigma_H \equiv m_f^*$; the index H stands for Hartree. In the language of Feynman diagrams, a constant self-energy at the one-loop level is a tadpole. Note that there is no tadpole from the coupling of the pion to the fermion, which is consistent with taking $\pi = 0$ in the MFA. The tadpole can be computed either in the imaginary time formalism or the real time formalism, the result for m_f^* is the same in both formalisms [62, 61], given by:

$$m_f^* = m_f - 2\gamma g \int \frac{d^3k}{(2\pi)^3} \frac{m_f^*}{\omega_f(k)} \left[1 + 2n_f^*(k) \right] \quad (5.1)$$

where m_f is the classical mass and $\omega_f(k) = (k^2 + m_f^{*2})^{1/2}$ and $n_f^*(k)$ is the Fermi-Dirac distribution

$$n_f^*(k) = \frac{1}{e^{\beta\omega_f^*(k)} + 1} \quad \text{with} \quad \beta = 1/T \quad (5.2)$$

and the factor γ in (5.1) is an isospin degeneracy factor; only neutrons or protons $\gamma = 1$, neutrons and protons $\gamma = 2$. The factor 2 multiplying the Fermi-Dirac distribution in (5.1) is due to the fact that the Fermi-Dirac distributions of the fermions and antifermions are the same, as we are considering zero chemical potential. Note that to lighten the notation, we have written $n_f^*(\omega_f^*(k))$ as $n_f^*(k)$.

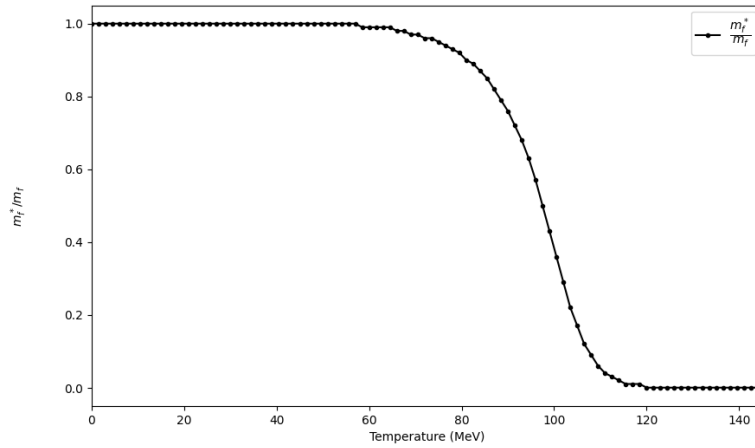


Figure 5.1: Temperature dependence of the nucleon effective mass.

The term involving $n_f^*(k)$ in (5.1) is finite, but the other, proportional to 1, is divergent but the divergence is eliminated by renormalization (it is absorbed by m_f). The result for m_f^*/m_f for the case of nucleons is shown in Fig. 5.1. We used the experimental value [63] of the coupling, namely $g^2/4\pi = 14.6$, and the vacuum nucleon mass $m_f = 939$ MeV, and for the vacuum meson masses $m_\pi = 138$ MeV

and $m_\sigma = 550$ MeV, and $\gamma = 1$. The figure reveals that when the temperature increases, m_f^* decreases, which is a sign of chiral restoration. This temperature dependence will affect the spectral function of the nucleon self-energy propagator and, as we will show shortly, affects the complex poles. We note that we extended the temperature in the above graph far beyond the validity of the model. That is, we recall that the LSM formulated in terms of nucleons is not an adequate model for the QCD chiral transition because, for temperatures $T \geq 100$ MeV, the relevant fermionic degrees of freedom are not nucleons but rather quarks.

As noted above, the appropriate MFA for the LSM with quarks is implemented by minimizing the free energy. The minimization leads to a cubic nonlinear equation for the constant field σ , which contains besides the classical contribution (2.14), thermal and quantum tadpole contribution, namely [64, 40]:

$$\sigma \left(\mu^2 + \frac{\lambda}{6} \sigma^2 \right) - c - 2\gamma g^2 \sigma \int \frac{d^3k}{(2\pi)^3} \frac{1}{\omega_f(k)} [1 + 2n_f(k)] = 0 \quad (5.3)$$

where $\gamma = 3 \times 2 = 6$, degeneracy factor due color and u and d quarks. Solving this equation for σ leads to the effective fermion mass $m_f^* = g\sigma$.

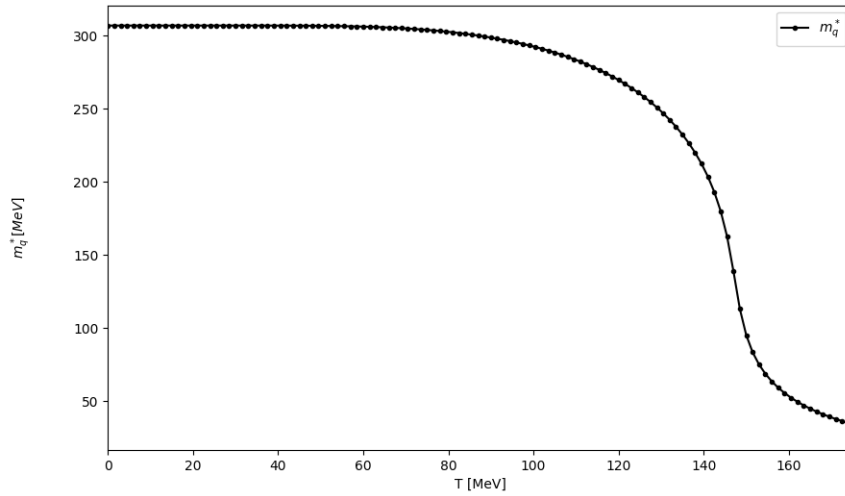


Figure 5.2: Temperature dependence of the quark effective mass.

We analyze the solutions of this equation for the LSMq parameters corresponding to a crossover transition [64]: $g = 3.3$ and $\lambda = 60$ (note that in [64] the coupling λ enters as $\lambda/4$ while in (2.16) it enters as $\lambda/4!$), and $c = m_\pi^2 f_\pi^2$, with $m_\pi = 138$ MeV and $f_\pi = 93$ MeV. These parameter values lead to a σ mass $m_\sigma = 604$ MeV and a quark mass $m_f = m_q = 306.9$ MeV. The temperature depen-

dence of the quark effective mass is shown in Fig. 5.2, in which we restored the physical dimension of m_q^* so that one can see that the crossover transition happens at $T_{\text{crov}} \simeq 150$ MeV and at this temperature $(m_q^*)_{\text{crov}} \simeq 95$ MeV.

Note that (5.1) and (5.3) are nonperturbative results. They are nonperturbative because m_f^* (or σ) are not given as a power series expansion in the coupling g , they are nonlinear equations in that the right side of (5.1) depends non-linearly on m_f^* . The tadpole is not the only contribution to the self-energy of the fermion; there are also exchange diagrams, a.k.a. Fock terms; see Fig. 5.1, both σ and π couplings to the fermion contribute to the exchange terms. But if one keeps only the tadpole, then all of the thermodynamics of the model are computed with a fermion propagator with the structure of a free propagator, with m_f replaced by m_f^* , where m_f^* , we recall, is the nonperturbative solution of (5.1). For example, the real-time fermion propagators are those given in (4.58) with m_f replaced by m_f^* everywhere, including in the Fermi-Dirac distributions.

Therefore, if only the tadpole is kept, there are no unphysical fermion complex mass poles; there is only one pole, the one at m_f^* . The natural question is then whether such unphysical complex poles appear in the exchange contributions, since they are present at zero temperature, as we show in the next section. Our strategy to answer this question is to compute in one loop the exchange contribution to the fermion propagator in the LSM using MFA propagators. This approach is not new; it has been used before with the Walecka model [23] in [61] and in the quark-meson-coupling model [24] in [65]. This kind of approach, using nonperturbative effective masses in exchange self-energies, is also common in the context of optimized perturbation theory or δ expansion method; for extensive lists of references, see [25, 66, 67]. In none of these applications however, was the question of complex mass poles addressed. Our original contribution covers precisely this question. In the next section, we consider the fermion self-energy in vacuum, and in the following section we consider temperature effects.

5.1 Self-Energy in the Vacuum

Given the interaction term in the LSM Lagrangian density in (2.16), the expressions for the exchange contributions, illustrated by the graphs shown in Fig. 5.1,

are given by (we use m_f to denote the mass of either the nucleon or the quark):

$$\Sigma_\pi(p) = 3ig^2 \int \frac{d^4k}{(2\pi)^4} \frac{(k - m_f)}{k^2 - m_f^2 + i\epsilon} \frac{1}{(k - p)^2 - m_\pi^2 + i\epsilon} \quad (5.4)$$

$$\Sigma_\sigma(p) = ig^2 \int \frac{d^4k}{(2\pi)^4} \frac{(k + m_f)}{k^2 - m_f^2} \frac{1}{(k - p)^2 - m_\sigma^2 + i\epsilon} \quad (5.5)$$



Figure 5.3: Exchange contributions to the fermion self-energy.

Our aim is to obtain an expression for the renormalized spectral function $\sigma(\kappa)$ that appears in (3.51). Once we have this spectral function, we can search for complex poles by solving equation $F(z) = 0$, with $F(z)$ given in (3.52). To compute $\sigma(\kappa)$, we will compute (5.4) and (5.5) for $p = 0$. Since here we are considering the vacuum case, we have Lorentz invariance and the full p dependence of the self-energies can be easily restored at the end of the calculations. When considering finite temperatures, one usually works in the system's rest frame, and as such the Lorentz covariance is not explicit; our results will have validity for $p = 0$ only. Of course, one could use an explicitly covariant description taking into account the Lorentz transformation properties of the temperature [68]. We do not pursue such an approach here because it would deviate us considerably from our main purposes.

We explicitly compute the spectral function for $\Sigma_\pi(p)$, from which one can immediately obtain the corresponding expression of the spectral function for $\Sigma_\sigma(p)$. This is so because the only differences between (5.4) and (5.5) are the factor of 3 coming from the isospin and the relative sign between p and m_f and, of course, replacing m_π by m_σ . We obtain the spectral function by computing the imaginary part of the self-energy.

Using the projection operators $P_\pm(p)$, one can write for the unrenormalized $\tilde{\Sigma}_{\pi\Lambda}(z)$ for $z = p_0 + i\epsilon$ as (to lighten the notation, we omit the subindex Λ that

indicates a regularized but unrenormalized quantity):

$$\tilde{\Sigma}_\pi(p_0 + i\epsilon) = 3 \frac{g^2}{(4\pi)^2} \int_0^\infty dk k^2 K_\pi(p_0, k) \quad (5.6)$$

where the kernel $K_\pi(p_0, k)$ contains the terms coming from the product of the propagators, written in a convenient form to perform the integral over k_0 :

$$\begin{aligned} K_\pi(p_0, k) &= \frac{2i}{\pi} \int_{-\infty}^\infty dk_0 \frac{k_0 - m_f}{[k_0 - (\omega_f(k) - i\epsilon)][k_0 + (\omega_f(k) - i\epsilon)]} \\ &\times \frac{1}{[(k_0 - p_0) - (\omega_\pi(k) - i\epsilon)][(k_0 - p_0) + (\omega_\pi(k) - i\epsilon)]} \end{aligned} \quad (5.7)$$

with $\omega_f(k) = (k^2 + m_f^2)^{1/2}$ and $\omega_\pi(k) = (k^2 + m_\pi^2)^{1/2}$. Then, the unrenormalized spectral function is given by

$$\begin{aligned} \sigma_\pi(p_0) &= -\frac{1}{\pi} \text{Im} \tilde{\Sigma}(p_0) = \frac{1}{2i} [\tilde{\Sigma}_\pi(p_0 + i\epsilon) - \tilde{\Sigma}_\pi^*(p_0 + i\epsilon)] \\ &= 3 \frac{g^2}{(4\pi)^2} \int_0^\infty dk k^2 \frac{1}{2i} [K_\pi(p_0, k) - K_\pi^*(p_0, k)] \end{aligned} \quad (5.8)$$

We now integrate over k_0 using the residue theorem to obtain $K_\pi(p_0, k)$ to obtain

$$\begin{aligned} K_\pi(p_0, k) - K_\pi^*(p_0, k) &= \frac{4\pi i}{\omega_f(k)\omega_\pi(k)} \varepsilon(p_0) \left\{ [m_f - \omega_f(k)] \delta(p_0 - \omega_f(k) - \omega_\pi(k)) \right. \\ &\times [m_f + \omega_f(k)] \delta(p_0 + \omega_f(k) + \omega_\pi(k)) \left. \right\} \end{aligned} \quad (5.9)$$

where $\varepsilon(p_0) = \theta(p_0) - \theta(-p_0) = \text{sign}(p_0)$. The delta functions here allow us to perform the integrals over k analytically in (5.8). To do so, we use the expression

$$\delta(f(x)) = \sum_i \frac{1}{|f'(x_i)|} \delta(x - x_i) \quad (5.10)$$

where the x_i are the roots of $f(x) = 0$ and $f'(x_i)$ is the derivative of $f(x)$ w.r.t. x evaluated at the root x_i . For $p_0 > 0$ and $p_0 < 0$, there are two roots, one giving $k > 0$ and the other $k < 0$. Since the integral is over positive k , the relevant root is:

$$k = \frac{1}{2|p_0|} \left[p_0^4 - 2p_0^2(m_f^2 + m_\pi^2) + (m_f^2 - m_\pi^2)^2 \right]^{1/2} \quad (5.11)$$

Replacing k in $\omega_f(k)$ and $\omega_\pi(k)$, the final result for the unrenormalized spectral

function can be put in the form:

$$\begin{aligned} \sigma_\pi(p_0) = & 3 \frac{g^2}{(4\pi)^2} \frac{1}{2|p_0|^3} \left[p_0^4 - 2p_0^2(m_f^2 + m_\pi^2) + (m_f^2 - m_\pi^2)^2 \right]^{1/2} \\ & \times \left[(p_0 - m_f)^2 - m_\pi^2 \right] \theta \left(p_0^2 - (m_f + m_\pi)^2 \right) \end{aligned} \quad (5.12)$$

This expression is valid for $p_0 > 0$ and $p^0 < 0$. The presence of the theta function reflects the possibility that the state running in the loop comes on-shell, a consequence of the unitarity of the theory. At finite temperature, as we will see, there is another branch cut due to the presence of a hot medium which allows the fermion under consideration scatter with the fermions and the bosons in the medium. That branch is called Landau cut. Note that since this one-loop result is finite, the unrenormalized and renormalized spectral functions are the same. Therefore, it can be plugged into the expression for the renormalized expression for the inverse of the propagator in (3.51).

The spectral function for $\Sigma_\sigma(p)$ is obtained from (5.12) by removing the factor 3 coming from isospin and reversing the relative sign between p_0 and m_f and replacing m_π by m_σ :

$$\begin{aligned} \sigma_\sigma(p_0) = & \frac{g^2}{(4\pi)^2} \frac{1}{2|p_0|^3} \left[p_0^4 - 2p_0^2(m_f^2 + m_\sigma^2) + (m_f^2 - m_\sigma^2)^2 \right]^{1/2} \\ & \times \left[(p_0 + m_f)^2 - m_\sigma^2 \right] \theta \left(p_0^2 - (m_f + m_\sigma)^2 \right) \end{aligned} \quad (5.13)$$

We note that the above computation could be considerably simplified by using the Cutkosky cutting rules [69], which are based on the unitarity of the S-matrix. Chapter 5 of the book [27] presents the derivation of these rules in a very detailed and didactic way, in vacuum and at finite temperature. We do not reproduce that derivation here; it would deviate us from our main objectives in this dissertation. We simply give the rule for the one-loop graphs, which are the only ones that we need. In vacuum, they state that to obtain the imaginary part of the graph:

1. Take a the Feynman diagram for the self-energy Σ and cut it in a way that can put the propagators in the loop on-shell;
2. Replaced each propagator term $1/(p^2 - m^2 + i\epsilon)$ entering each vertex of the graph by $2\pi \theta(p^0) \delta(p^2 - m^2)$ and that leaving the vertex by $2\pi \theta(-p^0) \delta(p^2 - m^2)$;
3. The imaginary part of the Feynman graph is then the sum over all the cuts:

$$2 \operatorname{Im} \Sigma = \sum_{\text{cuts}} \Sigma.$$

One can check very easily that one recovers the result above obtained by an explicit calculation. We will use such rules to compute the imaginary part of the self-energy at finite temperature in the following section.

In the Appendix B, we detail the evaluation of the diagrams in Fig. 5.1 using dimensional regularization. There, we also discuss two different renormalization schemes. That appendix serves as motivation for future work regarding a potential renormalization scheme dependence on the appearance of ghost poles. In the next sections, we shall focus on the finite-temperature corrections to the self-energy.

5.2 Self-Energy at Finite Temperature

We need to compute the imaginary part of the self-energy. A complication arises at finite temperatures, in that the zero-temperature Cutkosky cutting rules need modifications [26, 27]. However, at one one-loop, one can use the traditional cutting rules and simply compute the self-energy following the same steps as in the vacuum, using, of course, the finite-temperature propagators instead of the vacuum propagators. This means that there is an extra imaginary contribution coming from the thermal part, which is purely imaginary. As we have seen, knowing the imaginary part of $\Sigma_{11}(p)$, one can obtain all other components.

At the level of one-loop order, Σ_{11} for the contributions from π and σ are given by expressions with the same structure as those in vacuum, (5.4) and 5.5, but now including the extra finite-temperature terms of the fermion and boson propagators. Using the Cutkosky rules for the one-loop self-energy and taking $\mathbf{p} = 0$, one obtains for the pion loop contribution:

$$\begin{aligned} \operatorname{Im} \Sigma_{11\pi}(p_0) = & -\frac{1}{2} \times 3 g^2 (2\pi)^2 \int \frac{d^4 k}{(2\pi)^4} \left(\gamma_0 k_0 + m_f^* \right) \left\{ \left[\theta(k_0) - n_f^*(k) \right] \right. \\ & \times \left[\theta(-k_0 + p_0) + n_\pi(k - p) \right] + \left[\theta(-k_0) - n_f^*(k) \right] \left[\theta(k_0 - p_0) + n_\pi(k - p) \right] \Big\} \\ & \times \delta(k^2 - m_f^{*2}) \delta((k - p)^2 - m_\pi^2) \end{aligned} \quad (5.14)$$

where $n_f^*(k)$ indicates the Fermi-Dirac distribution computed with the energy $\omega_f^*(k) = (k^2 + m_f^{*2})^{1/2}$. Again as in the vacuum case, since we are taking $\mathbf{p} = 0$, there is no relative angle between the $\mathbf{p}$ and $\mathbf{k}$. One can easily identify the Cutkosky rules at work in the above expression: one of the terms comes from the fermion propagator leaving and the π propagator entering one of the vertices and the other

term represents the reverse situation, fermion entering and π leaving the other vertex. Next, using the projection operators $P_{\pm}$ to remove the Dirac γ_0 matrix and integrating over k_0 to eliminate one of the delta functions, one can manipulate the thermal distributions in a way to extract an overall $\tanh(p_0)$ factor and write $\text{Im } \tilde{\Sigma}_{11\pi}(p_0)$ in terms of the diagonal self-energy $\tilde{\Sigma}_{\pi}(p_0)$ as:

$$\text{Im } \tilde{\Sigma}_{11\pi}(p_0) = \varepsilon(p_0) \tanh(\beta p_0) \text{Im } \tilde{\Sigma}_{\pi}(p_0) \quad (5.15)$$

where

$$\begin{aligned} \text{Im } \tilde{\Sigma}_{\pi}(p_0) = & -3 \frac{g^2}{(4\pi)^2} \times (2\pi) \int \frac{d^3k}{(2\pi)^3} \frac{1}{\omega_f^*(k) \omega_{\pi}(k)} \left\{ \left[m_f^* - \omega_f^*(k) \right] \right. \\ & \times \left[\left(1 - n_f^*(k) + n_{\pi}(k) \right) \delta(p_0 - \omega_f^*(k) - \omega_{\pi}(k)) \right. \\ & + \left. \left(n_f^*(k) + n_{\pi}(k) \right) \delta(p_0 - \omega_f^*(k) + \omega_{\pi}(k)) \right] + \left[m_f^* + \omega_f^*(k) \right] \\ & \times \left[\left(1 + n_f^*(k) - n_{\pi}(k) \right) \delta(p_0 + \omega_f^*(k) + \omega_{\pi}(k)) \right. \\ & + \left. \left. \left(n_f^*(k) + n_{\pi}(k) \right) \delta(p_0 + \omega_f^*(k) - \omega_{\pi}(k)) \right] \right\} \end{aligned} \quad (5.16)$$

Again, the delta functions allow us to perform the integral over k analytically and the result can be put in an essentially the same structure as the vacuum result. Specifically, the spectral density $\sigma_{11\pi}(p_0) = -\text{Im } \tilde{\Sigma}_{11\pi}(p_0)$ is given by

$$\begin{aligned} \sigma_{11\pi}(p_0) = & 3 \frac{g^2}{(4\pi)^2} \tanh(\beta|p_0|) \frac{1}{2|p_0|^3} \left[p_0^4 - 2p_0^2(m_f^{*2} + m_{\pi}^2) + (m_f^{*2} - m_{\pi}^2)^2 \right]^{1/2} \\ & \times \left[(p_0 - m_f^*)^2 - m_{\pi}^2 \right] \left\{ \left[1 - n_f^*(k) + n_{\pi}^*(k) \right] \theta \left(p_0^2 - (m_f^* + m_{\pi})^2 \right) \right. \\ & + \left. \left[n_f^*(k) + n_{\pi}^*(k) \right] \theta \left(p_0^2 - (m_f^* - m_{\pi})^2 \right) \right\} \end{aligned} \quad (5.17)$$

in which we used $\varepsilon(p_0) \tanh(\beta p_0) = \tanh(\beta|p_0|)$, and k is given by the same expression as in (5.11) with m_f replaced by m_f^* ; the reason for $*$ on the pion Bose-Einstein distribution is because of the m_f^* dependence of k .

In (5.17) one observes the presence of two types of Heaviside theta functions multiplying Fermi-Dirac and Bose-Einstein distributions: $\theta(p_0^2 - (m_f^* + m_{\pi})^2)$, which was present in the vacuum case, and a new one, $\theta(p_0^2 - (m_f^* - m_{\pi})^2)$. The latter, which comes from the Dirac deltas $\delta(p_0 \pm \omega_f(k) \mp \omega_{\pi}(k))$, leads to what is known as the Landau cut. Physically, it represents the possibility that the fermion can scatter with the fermions and bosons of the medium created by the finite-

temperature heat bath. The other cut, which starts at $|p_0| \geq m_f + m_\pi$, is called the unitarity cut.

As in the vacuum case, the contribution of the σ loop to the self-energy is obtained from the π loop above by changing the relative sign between p_0 and m_f^* in the first term of the second line in (5.17) and, of course, replacing the π mass by the σ mass.

5.3 Numerical Results

In this section, we present numerical results for the LSM with nucleons and LSM with quarks. We recall the values used to obtain the results. The pion mass is $m_\pi = 138$ MeV for both versions of the LSM. For the nucleon LSM, the coupling is $g = 14.6$, the vacuum nucleon mass is $m_f = m_N = 939$ MeV and the σ mass is fixed to $m_\sigma = 550$ MeV. For LSMq, the coupling is $g_c = 3.3$, which leads to a crossover transition, and the vacuum masses are $m_f = m_q = 306.9$ MeV and $m_\sigma = 604$ MeV. We start by presenting vacuum results in the next subsection and finite temperature results in the subsequent one.

5.3.1 Vacuum results

We start by showing in Table 5.1 the results for poles in vacuum for the LSM with nucleons for $m_\sigma = 550$ MeV and $m_\sigma = 600$ MeV. We searched for the zeros of $F(z)$ in (3.52) using a Newton-Raphson algorithm.

Table 5.1: Pole positions and residues for LSM with nucleons in the vacuum.

Meson loop	Pole position (units of m_f)	Pole residue (dimensionless)
$m_\sigma = 550$ MeV		
π	$0.732 \pm i 1.252$	$-0.750 \pm i 0.321$
σ	$0.068 \pm i 0.000$	$-1.763 \pm i 0.000$
$\pi + \sigma$	$0.216 \pm i 0.000$	$-5.761 \pm i 0.000$
$m_\sigma = 600$ MeV		
π	$0.732 \pm i 1.252$	$-0.750 \pm i 0.321$
σ	$-0.133 \pm i 0.000$	$-1.893 \pm i 0.000$
$\pi + \sigma$	$0.130 \pm i 0.303$	$-0.637 \pm i 1.801$

As one sees from the table, these poles have complex-valued or negative residues, which makes it clear their unphysical nature. When changing the σ mass,

the situation does not change much qualitatively, although quantitatively there are some changes. We recall that changing the σ mass implies changes in the effective nucleon mass and the σ propagator.

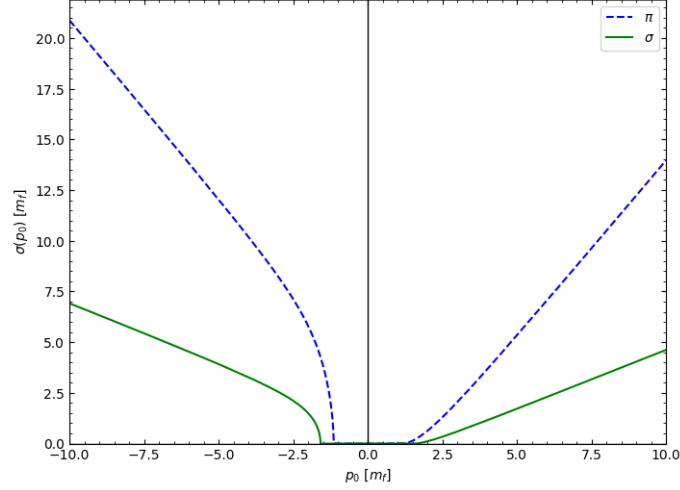


Figure 5.4: Comparison of the nucleon spectral functions from the π and σ loops for $m_\sigma = 550$ MeV.

To gain a qualitative understanding of these results, we have plotted in Fig. 5.4 the individual vacuum nucleon spectral functions due to the π loop $\sigma_\pi(p_0)$ and the σ loop $\sigma_\sigma(p_0)$. One sees that σ_π is much larger than σ_σ , a feature that seems to imply that a higher value of z in the denominator of the integral in $F(z)$ is necessary to make $F(z) = 0$.

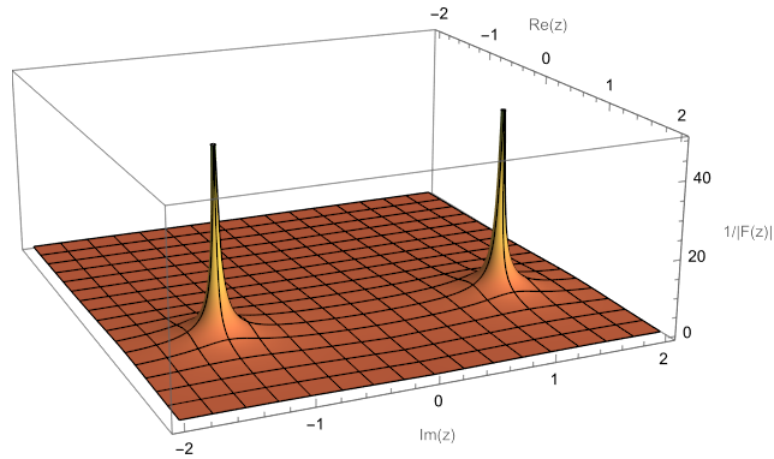


Figure 5.5: Complex poles in the nucleon propagator.

To have a pictorial view of the location of the poles in the complex p_0 -plane,

we have plotted in Fig. 5.5 the function $1/|F(z)|$. In the following subsection, we plot them together with the poles at finite temperature.

Next, we present the results for LSMq. Table 5.2 shows the complex-mass poles. The table reveals results that are very different from those for the LSM with nucleons. We observe that both the real and imaginary parts of the pole due to the π loop are considerably larger than the corresponding ones for the nucleon. The residues are complex, indicating their unphysical nature. Another difference from the nucleon case is the absence of complex poles due to the σ loop. When both π and σ loops are considered, the poles are present, with a relatively small value for the real part but still with a very large value for the imaginary part; recall that the results are given in units of the vacuum quark mass.

Table 5.2: Pole positions and residues for LSM with quarks in the vacuum.

Meson loop	Pole position (units of m_f)	Pole residue (dimensionless)
π	$4.518 \pm i 259.515$	$-4.424 \pm i 0.058$
σ	No complex pole	--
$\pi + \sigma$	$0.559 \pm i 90.706$	$-3.609 \pm i 0.017$

Figure 5.6 displays the spectral functions. It should be noted that their values are considerably lower than the spectral functions of the nucleon.

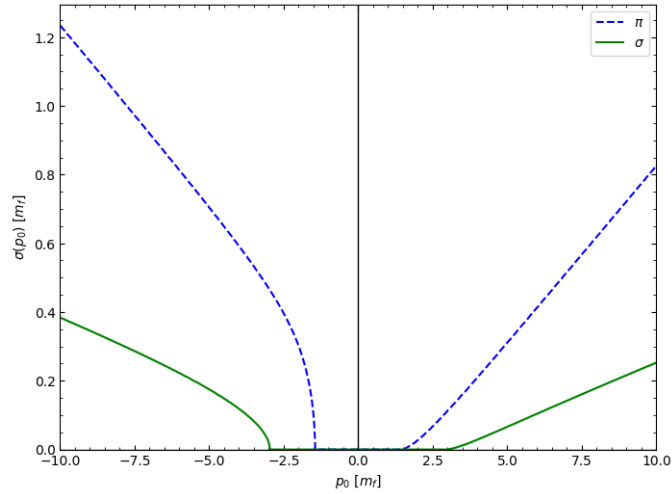


Figure 5.6: Comparison of the quark spectral functions from the π and σ loops.

In the next section we analyze the impact of thermal corrections to the pole structure for different values of temperature.

5.3.2 Thermal Contributions to the Pole Structure

We now turn our attention to the situation at finite temperature. The dependence of these poles on temperature can reveal important insights about the particle medium's response to thermal fluctuations. Temperature affects enter through the Fermi-Dirac and Bose-Einstein distributions and also in the modification of the masses. The latter modify the location of the unitarity and Landau cuts.

Figure 5.7 gives a pictorial view of the movement of the poles in the complex poles due to temperature, for a temperature $T = 145.5$ MeV. We used this temperature value so that the movement of poles is visible on the scale of the figure, although this temperature value is beyond the applicability of the model.

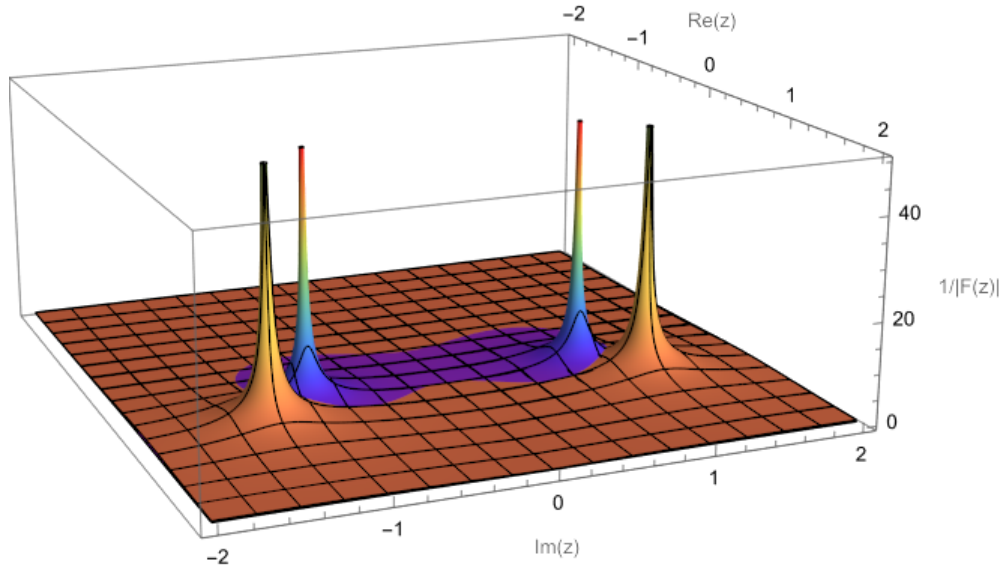


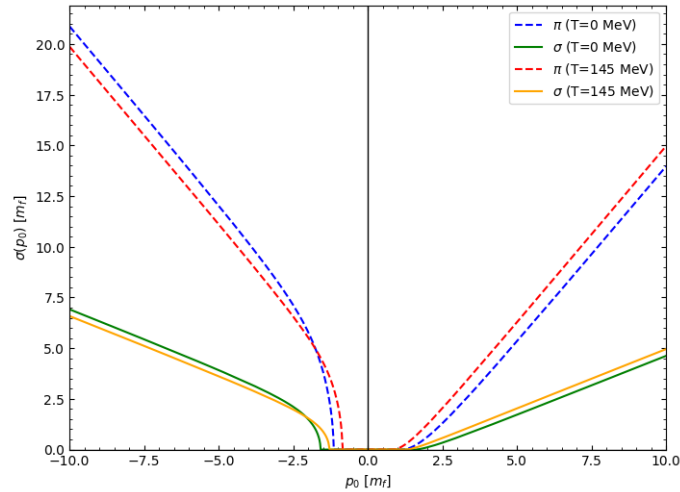
Figure 5.7: Comparison between the poles of the nucleon propagator in the vacuum and at $T = 145$ MeV.

Table 5.3 presents the values of the complex masses and the corresponding residues for selected temperatures. One sees that the same features seen for the vacuum case are maintained in that the σ loop gives smaller values for the complex masses than the π loop. In addition, the effect of the temperature on the masses and residues is not large but clearly visible.

These features can be well understood by examining the spectral functions shown in Fig. 5.8: the effects of the temperature on the π and σ loops are small. Another conclusion is that the contributions from the Landau cut are very small, not being visible on the scale of that figure.

Table 5.3: Masses and residues at selected temperatures for the LSM with nucleons.

Meson loop	Temperature (MeV)	Pole position	Pole residue
$m_\sigma = 550 \text{ MeV}$			
π	50	$0.733 \pm i 1.251$	$-0.751 \pm i 0.321$
	75	$0.735 \pm i 1.247$	$-0.752 \pm i 0.320$
	100	$0.736 \pm i 1.236$	$-0.756 \pm i 0.320$
σ	50	$0.068 \pm i 0.000$	$-1.763 \pm i 0.000$
	75	$0.068 \pm i 0.000$	$-1.764 \pm i 0.000$
	100	$0.066 \pm i 0.000$	$-1.778 \pm i 0.000$
$\pi + \sigma$	50	$0.215 \pm i 0.000$	$-5.745 \pm i 0.000$
	75	$0.217 \pm i 0.000$	$-7.668 \pm i 0.000$
	100	$0.226 \pm i 0.001$	$-5.288 \pm i 2.007$
$m_\sigma = 600 \text{ MeV}$			
π	50	$0.733 \pm i 1.251$	$-0.751 \pm i 0.321$
	75	$0.735 \pm i 1.247$	$-0.752 \pm i 0.320$
	100	$0.737 \pm i 1.237$	$-0.756 \pm i 0.320$
σ	50	$-0.133 \pm i 0.000$	$-1.893 \pm i 0.000$
	75	$-0.133 \pm i 0.000$	$-1.893 \pm i 0.000$
	100	$-0.137 \pm i 0.000$	$-1.903 \pm i 0.000$
$\pi + \sigma$	50	$0.131 \pm i 0.304$	$-0.637 \pm i 1.799$
	75	$0.135 \pm i 0.304$	$-0.643 \pm i 1.795$
	100	$0.145 \pm i 0.305$	$-0.670 \pm i 1.783$

Figure 5.8: Impact of the temperature on the nucleon spectral function from the π and σ loops.

Next, we move on to the LSMq results. Table 5.4 shows the values of the complex masses and residues. For temperatures well below the crossover temperature T_{crov} , the temperature effects are significant but not very large. Just below T_{crov} , there is no complex pole when one considers both the π and σ loops. However, looking closely at our numerical data file, we observed that for temperatures close and above T_{crov} there is no obvious trend in the behavior of the masses, in that for some temperatures there are complex mass poles and for others there are none. We recall that the quark mass becomes very small for $T > T_{\text{crov}}$ and, as such, numerical instabilities might occur with our root-finding algorithm, which is based on the Newton-Raphson method. It is likely that with a more refined algorithm we can achieve a better understanding of the temperature impact on the existence of complex mass poles.

Table 5.4: Pole positions and residues for the LSMq.

Meson loop	Temperature (MeV)	Pole position	Pole residue
π	75	$4.492 \pm i 253.787$	$-4.437 \pm i 0.058$
	100	$4.341 \pm i 242.177$	$-4.453 \pm i 0.059$
	145	$2.431 \pm i 174.383$	$-4.429 \pm i 0.046$
σ	75	No complex pole	--
	75	"	--
	145	"	--
$\pi + \sigma$	75	$0.573 \pm i 89.498$	$-3.609 \pm i 0.017$
	100	$0.575 \pm i 187.270$	$-3.499 \pm i 0.007$
	145	No complex pole	--

To conclude, we show in Fig. 5.9 the impact of the temperature on the quark spectral function due to the π and σ loops. Again, the Landau cut is not visible on the scale of the figure. In addition, nothing particularly different happens for the temperature close to T_{crov} to suggest the absence of complex mass poles we found for that temperature. This might suggest that some interesting dynamics is happening around the crossover temperature.

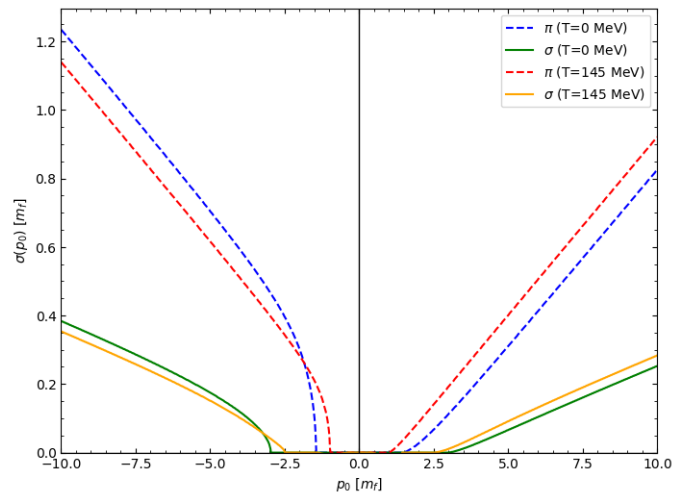


Figure 5.9: Impact of the temperature on the quark spectral function due to π and σ loops.

Chapter 6

Conclusions

We have investigated the effects of temperature on the pole structure of fermions within the linear sigma model, employing the real-time formalism and one-loop propagators. The main motivation for this work comes from the varied nature of pole interpretations. Although complex poles in QCD propagators are often associated with quark-gluon confinement in some theoretical contexts [18, 70, 71, 72], their appearance in models that do not relate to confinement phenomena [19, 57, 73, 21] raises questions about this interpretation. In these latter instances, the complex pole structure might be attributed as being an artifact of the approximation methods employed, rather than an indication of physical properties. Another interpretation [74] is that theories not asymptotically free, such as the LSM, are not nonperturbatively renormalizable, in the sense that quantum fluctuations at strong coupling destabilize the theory.

Given this understanding, our analysis emphasizes the need to discern the origins and implications of complex poles. In effective theories that are not asymptotically free, our results are significant because unphysical spectral properties ruin thermodynamic quantities, such as the equation of state and transport coefficients [29, 30]. By quantifying the influence of thermal corrections on the pole structure, we ensure a deeper and more accurate interpretation of the fermion dynamics within the linear sigma model at finite temperatures. Our findings reveal that the complex structure of the fermion propagator is maintained at finite temperature, reinforcing the need for further studies to settle the question whether the appearance of the complex poles is an artifact of the one-loop mean field approximation or is intrinsic to the model because it is not asymptotically free. Although the complex poles aren't present when one employs solely the one-loop Hartree approximation, they nevertheless become apparent when including one-loop exchange contributions to the fermion self-energy. There is no a priori reason to neglect exchange contributions and keep only the Hartree contribution since both are one-loop contributions. However, in some cases, such as in the large- N limit, where N refers to the number of colors, the Hartree contribution scales as

N , while the exchange contribution scales as $1/N$. This disparity in scaling could allow one to neglect the exchange diagrams in this limit, simplifying the analysis.

The appearance of complex poles become less problematic when considering the quark-level linear sigma model, as they appear at scales much higher than the validity of the model. This suggests that they might be neglected. Nevertheless, the presence of complex poles remains problematic and could potentially influence observable quantities evaluated at finite temperature, as there still remains the difficulty of how to control the ultraviolet of such nonasymptotically free theories.

More work in this investigation is needed to provide more numerical results to accurately characterize the pole structures. To address the limitations of the one-loop approximation, future research should incorporate higher-loop orders and explore self-consistent calculations using the Dyson-Schwinger equations [20]. Furthermore, from a phenomenological point of view, the inclusion of the vector meson ω in future studies involving nucleons could provide valuable information on the dynamics within the linear sigma model under thermal conditions, such as the potential violation of positivity in the spectral function. Such studies would help in refining our understanding of non-perturbative effects and provide a more robust framework for analyzing nucleon-meson dynamics at finite temperatures.

Appendix A

Metric conventions and Dirac matrices

A.1 Metric conventions

We adopt the metric convention $(+ - - -)$ for Minkowski spacetime which in matrix form is given by:

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad (\text{A.1})$$

This leads to the inner product between two vectors A and B in spacetime geometry being given by:

$$A \cdot B = g_{\mu\nu} A^\mu B^\nu = A^0 B^0 - A^1 B^1 - A^2 B^2 - A^3 B^3 \quad (\text{A.2})$$

Here, $g_{\mu\nu}$ is the Minkowski metric tensor with the signature previously defined and A^μ and B^ν are the components of the vectors A and B in spacetime. The indices μ and ν run over the spacetime coordinates (0 for the time component and 1,2,3 for the spatial components).

A.2 Dirac Matrices

In the context of relativistic quantum mechanics and field theory, an important role is played by the gamma matrices γ^μ , which are used in the formulation of the Dirac equation. These matrices are crucial for describing the behavior of spin- $\frac{1}{2}$ particles, like electrons, in a relativistic framework. Throughout the dissertation we make use of a number of properties of Dirac matrices under the metric convention defined in the previous section. Recalling that the gamma matrices satisfy the

anticommutation relation:

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} I_4 \quad (\text{A.3})$$

where $g^{\mu\nu}$ is the Minkowski metric tensor. Adopting the Dirac representation:

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \quad (\text{A.4})$$

where in this representation γ^0 corresponds to the time component, while γ^i (with $i = 1, 2, 3$), we shall be proving a number of useful identities in the following sections.

A.2.1 $\gamma^\mu \gamma_\mu = I_4$

To prove that $\gamma^\mu \gamma_\mu = I_4$, where I_4 is the 4x4 identity matrix, we start by writting:

$$\gamma^\mu \gamma_\mu = \gamma^\mu g_{\mu\nu} \gamma^\nu = g_{\mu\nu} \gamma^\mu \gamma^\nu \quad (\text{A.5})$$

And using that g is symmetric, we reach:

$$\gamma^\mu \gamma_\mu = \frac{1}{2} (g_{\mu\nu} + g_{\nu\mu}) \gamma^\mu \gamma^\nu \quad (\text{A.6})$$

$$= \frac{1}{2} (g_{\mu\nu} \gamma^\mu \gamma^\nu + g_{\nu\mu} \gamma^\mu \gamma^\nu) \quad (\text{A.7})$$

Finally, relabelling indices allows us to reach:

$$\gamma^\mu \gamma_\mu = \frac{1}{2} (g_{\mu\nu} \gamma^\mu \gamma^\nu + g_{\mu\nu} \gamma^\nu \gamma^\mu) \quad (\text{A.8})$$

$$= \frac{1}{2} g_{\mu\nu} \{\gamma^\mu, \gamma^\nu\} \quad (\text{A.9})$$

$$= \frac{1}{2} g_{\mu\nu} (2g^{\mu\nu} I_4) = g_{\mu\nu} g^{\mu\nu} I_4 = 4I_4. \quad (\text{A.10})$$

A.2.2 $\gamma^\mu \gamma^\nu \gamma_\mu = -2\gamma^\nu$

The proof is straightfoward:

$$\gamma^\mu \gamma^\nu \gamma_\mu = \gamma^\mu (2g_\mu^\nu I_4 - \gamma_\mu \gamma^\nu) \quad (\text{A.11})$$

$$= 2\gamma^\mu g_\mu^\nu - \gamma^\mu \gamma_\mu \gamma^\nu \quad (\text{A.12})$$

$$= 2\gamma^\nu - 4\gamma^\nu = -2\gamma^\nu \quad (\text{A.13})$$

A.2.3 $\text{Tr}(\gamma^\mu) = 0$

From the definition of the gamma matrices,

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu} I \quad (\text{A.14})$$

we get:

$$\gamma^\mu \gamma^\mu = g^{\mu\mu} I \quad (\text{A.15})$$

or equivalently,

$$\frac{\gamma^\mu \gamma^\mu}{g^{\mu\mu}} = I \quad (\text{A.16})$$

where $g^{\mu\mu}$ is a number, and $\gamma^\mu \gamma^\mu$ is a matrix. This allows us to write:

$$\text{tr}(\gamma^\nu) = \frac{1}{g^{\mu\mu}} \text{tr}(\gamma^\nu \gamma^\mu \gamma^\mu) \quad (\text{A.17})$$

$$= -\frac{1}{g^{\mu\mu}} \text{tr}(\gamma^\mu \gamma^\nu \gamma^\mu) \quad (\text{A.18})$$

Finally, using the cyclic property:

$$= -\frac{1}{g^{\mu\mu}} \text{tr}(\gamma^\nu \gamma^\mu \gamma^\mu) \quad (\text{A.19})$$

$$= -\text{tr}(\gamma^\nu) \quad (\text{A.20})$$

Proving that $\text{tr}(\gamma^\nu) = 0$.

A.2.4 $\text{Tr}(\gamma^5) = 0$

The proof is similar to the previous one, using the anticommutation relation and the cyclic property we can show that:

$$\text{tr}(\gamma^5) = \text{tr}(\gamma^0 \gamma^0 \gamma^5) \quad (\text{A.21})$$

$$= -\text{tr}(\gamma^0 \gamma^5 \gamma^0) \quad (\text{A.22})$$

$$= -\text{tr}(\gamma^0 \gamma^0 \gamma^5) \quad (\text{A.23})$$

$$= -\text{tr}(\gamma^5) \quad (\text{A.24})$$

A.2.5 $\gamma^5 \not{p} \gamma^5 = -\not{p}$

We start by writing $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$:

$$\gamma^5 \not{p} \gamma^5 = \gamma^5 (\gamma^\mu p_\mu) \gamma^5 \quad (\text{A.25})$$

$$= (i\gamma^0\gamma^1\gamma^2\gamma^3) (\gamma^\mu p_\mu) (i\gamma^0\gamma^1\gamma^2\gamma^3) \quad (\text{A.26})$$

$$= -\gamma^0\gamma^1\gamma^2\gamma^3 \gamma^\mu p_\mu \gamma^0\gamma^1\gamma^2\gamma^3 \quad (\text{A.27})$$

We then move p_μ to the right and use the anticommutation relation between gamma-matrices:

$$\gamma^5 \not{p} \gamma^5 = -\gamma^0\gamma^1\gamma^2\gamma^3 \gamma^\mu \gamma^0\gamma^1\gamma^2\gamma^3 p_\mu \quad (\text{A.28})$$

$$= -\gamma^\mu p_\mu \quad (\text{A.29})$$

$$= -\not{p} \quad (\text{A.30})$$

Appendix B

Self-energy computation using dimensional regularization

For completeness, in this appendix we detail step by step the calculation of the self-energy of the fermion in the vacuum using dimensional regularization. We evaluate the contributions of the sigma only. We also show how one can impose different renormalization conditions, but we do not search for ghost poles using these expressions. It would be interesting to see whether there is renormalization scheme dependence in the appearance of ghost poles. We do not expect to find differences with our vacuum results when using the on-shell renormalization scheme, as the final result should be expressible in terms of the spectral function of the self-energy, which is a finite quantity. The situation might be different when dealing with the $\overline{MS}$ scheme. We leave these questions as a motivation for a future work.

B.1 Loop Integrals for Fermions

Consider the bosonic correction to the fermion self-energy. Starting with the sigma boson contribution:

$$-i\Sigma_\sigma(p) = (-ig)^2 \int \frac{d^4k}{(2\pi)^4} \frac{i(\not{k} + m_f)}{k^2 - m_f^2 + i\epsilon} \frac{i}{(k-p)^2 - m_\sigma^2 + i\epsilon} \quad (\text{B.1})$$

We now turn our attention to the denominator. Using the Feynman parametrization trick:

$$\frac{1}{AB} = \int_0^1 dx \frac{1}{[xA + (1-x)B]^2} \quad (\text{B.2})$$

and performing the shift $k \rightarrow l$ with $l = k + xp$ we reach:

$$-i\Sigma(p) = g^2 \int \frac{d^4l}{(2\pi)^4} \int_0^1 dx \frac{x\not{p} + m_f}{(l^2 - \Delta)^2} \quad (\text{B.3})$$

where $\Delta = -x(1-x)p^2 + (1-x)m_f + xm_\sigma^2$ and we neglected the terms that are linear in k . Next, we perform a Wick rotation on the integral by letting $l_0 = il_E$ which allows us to write:

$$-i\Sigma(p) = ig^2 \int \frac{d^4 l_E}{(2\pi)^4} \int_0^1 dx \frac{x\not{p} + m_f}{(l_E^2 + \Delta)^2} \quad (\text{B.4})$$

The next step is performing dimensional regularization. In order to keep the dimension of the Lagrangian consistent in d -dimensional spacetime, we must rescale the coupling by $\mu^{\frac{4-d}{2}}$:

$$-i\Sigma(p) = i\mu^{4-d} g^2 \int \frac{d^4 l_E}{(2\pi)^4} \int_0^1 dx \frac{x\not{p} + m_f}{(l_E^2 + \Delta)^2} \quad (\text{B.5})$$

Next, we shall prove that:

$$\int \frac{d^d l_E}{(2\pi)^d} \frac{1}{(l_E^2 + \Delta)^2} = \frac{1}{(4\pi)^{d/2}} \frac{\Gamma(2 - \frac{d}{2})}{\Gamma(2)} \left(\frac{1}{\Delta}\right)^{2 - \frac{d}{2}} \quad (\text{B.6})$$

Let us begin by breaking the integral into:

$$\int \frac{d^d l_E}{(2\pi)^d} \frac{1}{(l_E^2 + \Delta)^2} = \int \frac{d\Omega_d}{(2\pi)^d} \int_0^\infty dl_E \frac{l_E^{d-1}}{(l_E^2 + \Delta)^2} \quad (\text{B.7})$$

The first factor can be evaluated straight-forward by noting that:

$$\begin{aligned} (\pi)^d &= \left(\int dx e^{-x^2} \right)^d = \int d^d x \exp\left(-\sum_{i=1}^d x_i^2\right) = \int d\Omega_d \int_0^\infty dx x^{d-1} e^{-x^2} \\ &= \int d\Omega_d \frac{1}{2} \int_0^\infty dx^2 (x^2)^{\frac{d}{2}-1} e^{-x^2} = \int d\Omega_d \frac{1}{2} \Gamma(d/2) \end{aligned}$$

which leads to:

$$\int d\Omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)} \quad (\text{B.8})$$

The remaining factor is:

$$\int_0^\infty dl \frac{l^{d-1}}{(l^2 + \Delta)^2} = \frac{1}{2} \int_0^\infty dl^2 \frac{(l^2)^{\frac{d}{2}-1}}{(l^2 + \Delta)^2} \quad (\text{B.9})$$

And by introducing $x = \Delta/(l^2 + \Delta)$ we reach:

$$\int_0^\infty dl \frac{l^{d-1}}{(l^2 + \Delta)^2} = \frac{1}{2} \left(\frac{1}{\Delta} \right)^{2-\frac{d}{2}} \int_0^1 dx x^{1-\frac{d}{2}} (1-x)^{\frac{d}{2}-1} \quad (\text{B.10})$$

Using the definition of the beta function:

$$B(\alpha, \beta) = \int_0^1 dx x^{\alpha-1} (1-x)^{\beta-1} = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \quad (\text{B.11})$$

We can evaluate the integral, obtaining:

$$\int \frac{d^d l_E}{(2\pi)^d} \frac{1}{(l_E^2 + \Delta)^2} = \frac{1}{(4\pi)^{d/2}} \frac{\Gamma(2-\frac{d}{2})}{\Gamma(2)} \left(\frac{1}{\Delta} \right)^{2-\frac{d}{2}} \quad (\text{B.12})$$

We now use the following approximation

$$\Gamma\left(2 - \frac{d}{2}\right) = \Gamma(\epsilon/2) = \frac{2}{\epsilon} - \gamma + O(\epsilon) \quad (\text{B.13})$$

where γ is the Euler-Mascheroni constant. Therefore, the integral for $d = 4$ becomes:

$$\mu^{4-d} \int \frac{d^4 l_E}{(2\pi)^4} \frac{1}{(l_E^2 + \Delta)^2} = \frac{1}{(4\pi)^2} \left(\frac{2}{\epsilon} - \log(\Delta) + \log \mu^2 - \gamma + \log(4\pi) + O(\epsilon) \right) \quad (\text{B.14})$$

It can be proved from this property that:

$$\int \frac{d^d l_E}{(2\pi)^d} \frac{l_E^2}{(l_E^2 + \Delta)^n} = \frac{1}{(4\pi)^{d/2}} \frac{d}{2} \frac{\Gamma(n - \frac{d}{2} - 1)}{\Gamma(n)} \left(\frac{1}{\Delta} \right)^{n-\frac{d}{2}-1} \quad (\text{B.15})$$

And using the recurrence relation:

$$\Gamma\left(1 - \frac{d}{2}\right) = \frac{\Gamma(2 - \frac{d}{2})}{1 - \frac{d}{2}} \quad (\text{B.16})$$

Eq. (B.5) becomes:

$$\Sigma(p) = -\frac{g^2}{16\pi^2} \int_0^1 dx (x\not{p} + m_f) \left[\frac{2}{\epsilon} - \log \Delta + \log \mu^2 - \gamma + \log 4\pi + O(\epsilon) \right] \quad (\text{B.17})$$

In order to reach a finite result we must renormalize the fields and the mass in the lagrangian. Letting ψ_Λ and m_Λ denote the bare field and mass respectively we can write:

$$\psi_\Lambda(x) = \sqrt{Z_\psi} \psi(x) \quad \bar{\psi}_\Lambda(x) = \sqrt{Z_\psi} \bar{\psi}(x) \quad m_\Lambda = Z_m m \quad (\text{B.18})$$

and letting:

$$Z_\psi = \delta_\psi + 1 \quad Z_m = 1 + \delta_m \quad (\text{B.19})$$

The renormalized fermion propagator $\langle T\psi(x)\bar{\psi}(y) \rangle$ in momentum space will then be given by:

$$\begin{aligned} iS(p) &= \frac{1}{Z_\psi} iS_\Lambda(p) \\ &= \left(\frac{1}{1 + \delta_\psi} \right) \left(\frac{i}{\not{p} - m - \delta_m m} \right) + \text{loops} \\ &= \frac{i}{\not{p} - m} + \frac{i}{\not{p} - m} [i(\delta_\psi \not{p} - (\delta_\psi + \delta_m)m)] \frac{i}{\not{p} - m} + \text{loops} \end{aligned} \quad (\text{B.20})$$

Adding the 1-loop contribution from Eq. (B.17), we may write:

$$iS(p) = \frac{i}{\not{p} - m} + \frac{i}{\not{p} - m} [i(\delta_\psi \not{p} - (\delta_\psi + \delta_m)m - \Sigma(\not{p}))] \frac{i}{\not{p} - m} + \text{loops} \quad (\text{B.21})$$

Such that:

$$iS(p) = \frac{i}{\not{p} - m - \Sigma_R(\not{p})} \quad (\text{B.22})$$

where $\Sigma_R(\not{p}) = \Sigma(\not{p}) - \delta_\psi \not{p} + (\delta_m + \delta_\psi)m$. This expression can then be written as:

$$iS(p) = \frac{i}{A(p^2)\not{p} - B(p^2) + i\epsilon} \quad (\text{B.23})$$

where $A(p^2)$ and $B(p^2)$ are Lorentz scalar functions. The pole mass of the renormalized propagator will depend on the subtraction scheme used. In the following sections we present a number of different schemes and determine the functions $A(p^2)$ and $B(p^2)$ for each.

B.2 Zero-momentum Subtraction Scheme

Renormalizing the theory at $p = 0$ gives us the following condition:

$$S^{-1}(0) = -m \Rightarrow \Sigma_R(0) = 0 \quad (\text{B.24})$$

and having residue i implies that:

$$i = \lim_{p \rightarrow m} (p - m) \frac{i}{p - m - \Sigma_R(p)} = \lim_{p \rightarrow m} \frac{i}{1 - \frac{d}{dp} \Sigma_R(p)} \quad (\text{B.25})$$

Therefore:

$$\left. \frac{d}{dp} \Sigma_R(p) \right|_{p=0} = 0 \quad (\text{B.26})$$

Solving Eqs. (B.24) and (B.26) allows us to determine the functions $A(p^2)$ and $B(p^2)$ to be:

$$A(p^2) = 1 - \frac{g^2}{(4\pi)^2} \int_0^1 dx \, x \ln \left(\frac{\Delta}{\Delta_0} \right) \quad (\text{B.27})$$

$$B(p^2) = m \left[1 + \frac{g^2}{(4\pi)^2} \int_0^1 dx \, \ln \left(\frac{\Delta}{\Delta_0} \right) \right] \quad (\text{B.28})$$

where $\Delta_0 = (1 - x)m + xm_\sigma^2$.

B.3 On-shell Scheme

In the on-shell scheme we renormalize the theory at $p = m$ such that:

$$S^{-1}(m) = 0 \Rightarrow \Sigma_R(m) = 0 \quad (\text{B.29})$$

the residue remains at i such that:

$$\left. \frac{d}{dp} \Sigma_R(p) \right|_{p=m} = 0 \quad (\text{B.30})$$

Solving the equations above allows us to determine the functions $A(p^2)$ and $B(p^2)$ for this case to be:

$$A(p^2) = 1 - \frac{g^2}{(4\pi)^2} \int_0^1 dx \, x \ln \left(\frac{\Delta}{\Delta_m} \right) + \frac{g^2}{(4\pi)^2} \int_0^1 dx \, (xm + m) \left(\frac{\Delta'_m}{\Delta_m} \right) \quad (\text{B.31})$$

$$B(p^2) = m \left[1 + \frac{g^2}{(4\pi)^2} \int_0^1 dx \, \ln \left(\frac{\Delta}{\Delta_M} \right) + \frac{g^2}{(4\pi)^2} \int_0^1 dx \, (xm + m) \left(\frac{\Delta'_M}{\Delta_M} \right) \right] \quad (\text{B.32})$$

where $\Delta_m = -x(1 - x)m^2 + (1 - x)m + xm_\sigma^2$ and $\Delta'_m = -2mx(1 - x)$.

B.4 $\overline{MS}$ Scheme

Let us redefine μ as:

$$\mu = \tilde{\mu} \left(\frac{e^\gamma}{4\pi} \right)^{1/2}$$

this has the effect of cancelling the last terms in Eq. (B.17) such that:

$$\Sigma(\not{p}) = -\frac{g^2}{16\pi^2} \int_0^1 dx (x\not{p} + m_f) \left[\frac{2}{\epsilon} - \log \Delta + \log \tilde{\mu}^2 + O(\epsilon) \right] \quad (\text{B.33})$$

And the functions $A(p^2)$ and $B(p^2)$ will be given by:

$$A(p^2) = 1 - \frac{g^2}{(4\pi)^2} \int_0^1 dx x \ln \left(\frac{\Delta}{\tilde{\mu}^2} \right) \quad (\text{B.34})$$

$$B(p^2) = m \left[1 + \frac{g^2}{(4\pi)^2} \int_0^1 dx \ln \left(\frac{\Delta}{\tilde{\mu}^2} \right) \right] \quad (\text{B.35})$$

Appendix C

Self-Energy Relations at Finite Temperature

C.1 Self-Energy Relations for Bosons

The unitary matrix at Eq. (4.59) is given by:

$$U(\beta, p) = \begin{pmatrix} \cosh \theta_p & \sinh \theta_p \\ \sinh \theta_p & \cosh \theta_p \end{pmatrix} \quad (\text{C.1})$$

where:

$$\cosh \theta_p = \frac{1}{\sqrt{1 - e^{-\beta|p_0|}}} \quad \text{and} \quad \sinh \theta_p = \frac{e^{-\beta|p_0|/2}}{\sqrt{1 - e^{-\beta|p_0|}}} \quad (\text{C.2})$$

the inverse matrix is given by:

$$U^{-1}(\beta, p) = \begin{pmatrix} \cosh \theta_p & -\sinh \theta_p \\ -\sinh \theta_p & \cosh \theta_p \end{pmatrix} \quad (\text{C.3})$$

Performing the matrix product on the left from Eq. (4.59) gives us:

$$\begin{pmatrix} \cosh \theta_p & -\sinh \theta_p \\ -\sinh \theta_p & \cosh \theta_p \end{pmatrix} \begin{pmatrix} -i\Sigma(p) & 0 \\ 0 & i\Sigma^*(p) \end{pmatrix} = \begin{pmatrix} -i\Sigma(p) \cosh \theta_p & -i\Sigma^*(p) \sinh \theta_p \\ i\Sigma(p) \sinh \theta_p & i\Sigma^*(p) \cosh \theta_p \end{pmatrix} \quad (\text{C.4})$$

Multiplying the result by $U^{-1}(\theta_p)$:

$$\begin{aligned} & \begin{pmatrix} -i\Sigma(p) \cosh \theta_p & -i\Sigma^*(p) \sinh \theta_p \\ i\Sigma(p) \sinh \theta_p & i\Sigma^*(p) \cosh \theta_p \end{pmatrix} \begin{pmatrix} \cosh \theta_p & -\sinh \theta_p \\ -\sinh \theta_p & \cosh \theta_p \end{pmatrix} \\ &= \begin{pmatrix} -i\Sigma(p) \cosh^2 \theta_p + i\Sigma^*(p) \sinh^2 \theta_p & (i\Sigma(p) - i\Sigma^*(p)) \cosh \theta_p \sinh \theta_p \\ (i\Sigma(p) - i\Sigma^*(p)) \cosh \theta_p \sinh \theta_p & -i\Sigma(p) \sinh^2 \theta_p + i\Sigma^*(p) \cosh^2 \theta_p \end{pmatrix} \end{aligned}$$

We then concentrate on the first entry

$$-i\Sigma^{11} = -i\Sigma(p) \cosh^2 \theta_p + i\Sigma^*(p) \sinh^2 \theta_p \quad (\text{C.5})$$

Separating it into real and imaginary parts:

$$-i\Sigma^{11} = -i(i\text{Im}\Sigma + \text{Re}\Sigma) \cosh^2 \theta_p + i(-i\text{Im}\Sigma + \text{Re}\Sigma) \sinh^2 \theta_p \quad (\text{C.6})$$

Using $\cosh^2 \theta_p - \sinh^2 \theta_p = 1$, it's immediate to see that:

$$\text{Re } \Sigma^{11} = \text{Re } \Sigma \quad (\text{C.7})$$

and for the imaginary part, using that $\cosh^2 \theta_p + \sinh^2 \theta_p = \cosh 2\theta_p$:

$$\text{Im } \Sigma^{11} = \text{Im } \Sigma \cosh 2\theta_p \quad (\text{C.8})$$

therefore:

$$\text{Im } \Sigma = \text{sech} 2\theta_p \text{Im } \Sigma^{11} \quad (\text{C.9})$$

We can simplify the expression further by noting that:

$$\cosh 2\theta_p = 2 \cosh^2 \theta_p - 1 \quad (\text{C.10})$$

therefore:

$$\cosh 2\theta_p = 2 \left(\frac{1}{1 - e^{\beta|p_0|}} \right) - 1 = \frac{2 - 1 + e^{\beta|p_0|}}{1 - e^{\beta|p_0|}} = \frac{1 + e^{\beta|p_0|}}{1 - e^{\beta|p_0|}} \quad (\text{C.11})$$

and since:

$$\tanh \left(\frac{1}{2} \beta p_0 \right) = \frac{e^{\frac{1}{2} \beta p_0} - e^{-\frac{1}{2} \beta p_0}}{e^{\frac{1}{2} \beta p_0} + e^{-\frac{1}{2} \beta p_0}} = \frac{1 - e^{\beta p_0}}{1 + e^{\beta p_0}} \quad (\text{C.12})$$

We may write:

$$\text{Im } \Sigma = \epsilon(p_0) \tanh \left(\frac{1}{2} \beta p_0 \right) \text{Im } \Sigma^{11} \quad (\text{C.13})$$

where we introduced the signal function $\epsilon(p_0)$ to account for the module of p_0 at Eq. (C.11). We may also derive the additional relations for the off-diagonal components and for Σ^{22} :

$$\Sigma^{12} = \Sigma^{21} = -i \tanh 2\theta_p \text{Im } \Sigma^{11} \quad (\text{C.14})$$

$$\Sigma^{22} = -\Sigma^{11*} \quad (\text{C.15})$$

C.2 Self-Energy Relations for Fermions

The unitary matrix at Eq. (4.66) is given by:

$$V(\beta, p) = \begin{pmatrix} \cos \phi_p & -\epsilon(p_0) \sin \phi_p \\ \epsilon(p_0) \sin \phi_p & \cos \phi_p \end{pmatrix} \quad (\text{C.16})$$

where:

$$\cos \phi_p = \frac{1}{\sqrt{1 + e^{-\beta|p_0|}}} \quad \text{and} \quad \sin \phi_p = \frac{e^{-\beta|p_0|/2}}{\sqrt{1 + e^{-\beta|p_0|}}} \quad (\text{C.17})$$

the inverse matrix is given by:

$$V^{-1}(\beta, p) = \begin{pmatrix} \cos \phi_p & \epsilon(p_0) \sin \phi_p \\ -\epsilon(p_0) \sin \phi_p & \cos \phi_p \end{pmatrix} \quad (\text{C.18})$$

Performing the matrix product on the left from Eq. (4.66) gives us:

$$\begin{aligned} & \begin{pmatrix} \cos \phi_p & \epsilon(p_0) \sin \phi_p \\ -\epsilon(p_0) \sin \phi_p & \cos \phi_p \end{pmatrix} \begin{pmatrix} -i\Sigma(p) & 0 \\ 0 & i\Sigma^*(p) \end{pmatrix} \\ &= \begin{pmatrix} -i\Sigma(p) \cos \phi_p & \epsilon(p_0) i\Sigma^*(p) \sin \phi_p \\ -\epsilon(p_0) i\Sigma(p) \sin \phi_p & i\Sigma^*(p) \cos \phi_p \end{pmatrix} \end{aligned} \quad (\text{C.19})$$

Multiplying the result by $V^{-1}(\phi_p)$:

$$\begin{aligned} & \begin{pmatrix} -i\Sigma(p) \cos \phi_p & \epsilon(p_0) i\Sigma^*(p) \sin \phi_p \\ -\epsilon(p_0) i\Sigma(p) \sin \phi_p & i\Sigma^*(p) \cos \phi_p \end{pmatrix} \begin{pmatrix} \cos \phi_p & \epsilon(p_0) \sin \phi_p \\ -\epsilon(p_0) \sin \phi_p & \cos \phi_p \end{pmatrix} \\ &= \begin{pmatrix} -i\Sigma(p) \cos^2 \phi_p - i\Sigma^*(p) \sin^2 \phi_p & (-i\Sigma(p) + i\Sigma^*(p)) \cos \phi_p \sin \phi_p \\ (-i\Sigma(p) + i\Sigma^*(p)) \cos \phi_p \sin \phi_p & -i\Sigma(p) \sin^2 \phi_p + i\Sigma^*(p) \cos^2 \phi_p \end{pmatrix} \end{aligned}$$

We then concentrate on the first entry

$$-i\Sigma^{11} = -i\Sigma(p) \cos^2 \phi_p - i\Sigma^*(p) \sin^2 \phi_p \quad (\text{C.20})$$

Separating it into real and imaginary parts:

$$-i\Sigma^{11} = -i(i\text{Im}\Sigma + \text{Re}\Sigma) \cosh^2 \phi_p - i(-i\text{Im}\Sigma + \text{Re}\Sigma) \sin^2 \phi_p \quad (\text{C.21})$$

Using $\cos^2 \phi_p + \sin^2 \phi_p = 1$, it's immediate to see that:

$$\text{Re } \Sigma^{11} = \text{Re } \Sigma \quad (\text{C.22})$$

and for the imaginary part, using that $\cos^2 \phi_p - \sin^2 \phi_p = \cos 2\phi_p$:

$$\text{Im } \Sigma^{11} = \text{Im } \Sigma \cos 2\phi_p \quad (\text{C.23})$$

therefore:

$$\text{Im } \Sigma = \sec 2\phi_p \text{Im } \Sigma^{11} \quad (\text{C.24})$$

We can simplify the expression further by noting that:

$$\cos 2\phi_p = 2 \cos^2 \phi_p - 1 \quad (\text{C.25})$$

therefore:

$$\cos 2\phi_p = 2 \left(\frac{1}{1 + e^{\beta|p_0|}} \right) - 1 = \frac{2 - 1 - e^{\beta|p_0|}}{1 + e^{\beta|p_0|}} = \frac{1 - e^{\beta|p_0|}}{1 + e^{\beta|p_0|}} \quad (\text{C.26})$$

and since:

$$\coth \left(\frac{1}{2} \beta p_0 \right) = \frac{e^{\frac{1}{2} \beta p_0} + e^{-\frac{1}{2} \beta p_0}}{e^{\frac{1}{2} \beta p_0} - e^{-\frac{1}{2} \beta p_0}} = \frac{1 + e^{\beta p_0}}{1 - e^{\beta p_0}} \quad (\text{C.27})$$

We may write:

$$\text{Im } \Sigma = \epsilon(p_0) \coth \left(\frac{1}{2} \beta p_0 \right) \text{Im } \Sigma^{11} \quad (\text{C.28})$$

where we introduced the signal function $\epsilon(p_0)$ to account for the module of p_0 at Eq. (C.26). We may also derive the additional relations for the off-diagonal components and for Σ^{22} :

$$\Sigma^{12} = \Sigma^{21} = -i \tan 2\phi_p \text{Im } \Sigma^{11} \quad (\text{C.29})$$

$$\Sigma^{22} = -\Sigma^{11*} \quad (\text{C.30})$$

C.3 Alternative derivation of the propagator at finite temperature.

Let us now evaluate the propagator for a scalar particle at finite temperature:

$$i\Delta_F(x - y) = \langle \phi(x)\phi(y) \rangle_\beta \quad (\text{C.31})$$

In order to evaluate the propagator we must compute the expectation value given by:

$$i\Delta_F(x - y) = \frac{1}{Z(\beta)} \text{Tr} \left[e^{-\beta H} T \{ \phi(x)\phi(y) \} \right] \quad (\text{C.32})$$

where T denotes the time ordering along the path described in the previous section and $Z(\beta) = \text{Tr} e^{-\beta H}$. We then make use of the field expansion:

$$\phi(x) = \int \frac{d^3k}{(2\pi)^{3/2}} \frac{1}{(2\omega_k)^{1/2}} [a(k)e^{-ik \cdot x} + a^\dagger(k)e^{ik \cdot x}]|_{k_0=\omega_k} \quad (\text{C.33})$$

plugging it back into Eq. (C.31), obtaining:

$$\begin{aligned} i\Delta_F(x - y) &= \frac{1}{Z} \int \frac{d^3k}{(2\pi)^{3/2}} \frac{d^3k'}{(2\pi)^{3/2}} \frac{1}{(2\omega_k)^{1/2}} \frac{1}{(2\omega_{k'})^{1/2}} \\ &\times \sum_n e^{-\beta E_n} \langle n | [a(k)e^{-ik \cdot x} + a^\dagger(k)e^{ik \cdot x}] [a(k')e^{-ik' \cdot y} + a^\dagger(k')e^{ik' \cdot y}] | n \rangle \end{aligned} \quad (\text{C.34})$$

where we introduced the energy eigenstates given by:

$$|n\rangle = |n_1(k_1), n_2(k_2), \dots\rangle = \prod_i \frac{[a^\dagger(k_i)]^{n_i(k_i)}}{\sqrt{n_i(k_i)!}} |0\rangle \quad (\text{C.35})$$

we note that Eq. (C.33) contains terms involving the following:

$$a(k)|n\rangle = \sqrt{n_i(k)} |n_1(k_1), n_2(k_2), \dots, n_i(k_i) - 1, \dots\rangle \quad (\text{C.36})$$

$$a^\dagger(k)|n\rangle = \sqrt{n_i(k_i) + 1} |n_1(k_1), n_2(k_2), \dots, n_i(k_i) + 1, \dots\rangle \quad (\text{C.37})$$

this leads to:

$$\begin{aligned}
 i\Delta_F(x-y) &= \frac{1}{Z} \int \frac{d^3k}{(2\pi)^{3/2}} \frac{d^3k'}{(2\pi)^{3/2}} \frac{1}{(2\omega_k)^{1/2}} \frac{1}{(2\omega_{k'})^{1/2}} \\
 &\quad \times \sum_n e^{-\beta E_n} \{ [n(k) + 1] \delta^3(k - k') e^{-ik \cdot x + ik' \cdot y} + n(k) \delta^3(k - k') e^{ik \cdot x - ik' \cdot y} \} \\
 &= \frac{1}{Z} \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k} \sum_n e^{-\beta E_n} \{ [n(k) + 1] e^{-ik \cdot (x-y)} \} |_{k_0=\omega_k} \quad (C.38)
 \end{aligned}$$

and using the property:

$$\frac{1}{Z} \sum_n n(k) e^{-\beta E_n} = \frac{1}{e^{\beta \omega_k} - 1} = n_B(\omega_k) \quad (C.39)$$

we reach:

$$i\Delta_F(x-y) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k} \{ [1 + n_B(\omega_k)] e^{-ik(x-y)} + n_B(\omega_k) e^{ik(x-y)} \} |_{k_0=\omega_k} \quad (C.40)$$

As it turns out this expression is equivalent to:

$$i\Delta(x-y) = \int \frac{d^4k}{(2\pi)^4} \left[\frac{i}{k^2 - m^2 + i\epsilon} + 2\pi n_B(|k_0|) \delta(k^2 - m^2) \right] \quad (C.41)$$

The proof is straightforward, integrating the first term over k_0 gives the $T = 0$ part and integrating the second term gives:

$$\begin{aligned}
 &\int \frac{d^4k}{(2\pi)^4} 2\pi n_B(|k_0|) \delta(k^2 - m^2) e^{-ik(x-y)} \\
 &= \int \frac{d^3k}{(2\pi)^3} dk_0 n_B(|k_0|) \frac{1}{2\omega_k} [\delta(k_0 - \omega_k) + \delta(k_0 + \omega_k)] e^{-ik_0(x_0 - y_0)} e^{ik(x-y)} \\
 &= \int \frac{d^3k}{(2\pi)^3} \frac{n_B(\omega_k)}{2\omega_k} [e^{-i\omega_k(x_0 - y_0)} + e^{i\omega_k(x_0 - y_0)}] e^{ik(x-y)} \\
 &= \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k} [n_B(\omega_k)] e^{-ik(x-y)} + n_B(\omega_k) e^{ik(x-y)} |_{k_0=\omega_k} \quad (C.42)
 \end{aligned}$$

which proves the equivalence of Eqs. (C.40) and (C.41).

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