New tool to detect inhomogeneous chiral-symmetry breaking

Theo F. Motta,[1,2,3] Julian Bernhardt,[1] Michael Buballa,[2,4] and Christian S. Fischer[1,5]

[1]*Institut für Theoretische Physik, Justus-Liebig-Universität Gießen, 35392 Gießen, Germany*
[2]*Technische Universität Darmstadt, Fachbereich Physik, Institut für Kernphysik, Theoriezentrum, Schlossgartenstraße 2D-64289 Darmstadt, Germany*
[3]*Instituto de Física Teórica, Universidade Estadual Paulista, 01140-070 São Paulo, SP, Brazil*
[4]*Helmholtz Forschungsakademie Hessen für FAIR (HFHF), GSI Helmholtzzentrum für Schwerionenforschung, Campus Darmstadt, 64289 Darmstadt, Germany*
[5]*Helmholtz Forschungsakademie Hessen für FAIR (HFHF), GSI Helmholtzzentrum für Schwerionenforschung, Campus Gießen, 35392 Gießen, Germany*

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In this paper, we discuss a novel method to search for inhomogeneous chiral symmetry breaking in theories with fermions. The prime application we have in mind is QCD, but the method is also applicable for other theories, including solid state applications. It is based on an extension of the chiral susceptibility to inhomogeneous phases, and it works as a stability analysis. Our method allows us to determine when homogeneous solutions have the tendency to create spatially modulated condensates. As proof of principle, we apply this technique to a rainbow-ladder QCD model and find that its phase diagram contains an inhomogeneous region.

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I. INTRODUCTION

Chiral-symmetry breaking (χSB) is one of the most consequential phenomena associated with strong interactions. With massless quarks, the QCD Lagrangian is manifestly chirally symmetric; however the ground state breaks this symmetry. For two massless quark flavours, at high temperatures the system undergoes a second-order phase transition to the symmetric phase, which becomes a crossover at finite quark masses [1,2]. What happens to chiral symmetry at large chemical potentials, however, is less clear. One possibility is that the system undergoes a transition to a phase where chiral symmetry is broken *inhomogeneously*, that is, the order parameter is a periodic function of space. These phases have been seen in models of strong interactions [3–6]. If real, these phases might occur in a region of the phase diagram that is connected to the physics of neutron stars and neutron-star-mergers as well as with low-energy heavy-ion collisions. In fact, it has already been argued [7–10] that moat regimes (a broader but correlated phenomenon to inhomogeneous phases, see also [11,12]) would leave experimental signatures in scattering experiments.

From a theoretical perspective, it is highly nontrivial to obtain solid information on the existence of inhomogeneous phases at large chemical potential. Lattice calculations of QCD are notoriously difficult if not prohibited in this region because of the sign problem [13]. Therefore, some attention has been paid to simpler (and partly lower-dimensional) models, such as Gross-Neveu or Nambu-Jona-Lasinio (NJL) type models [14–28], which are accessible to both, continuum and lattice methods.

In general, two strategies have been employed so far. Using a specific *Ansatz* for the shape of the inhomogeneous phase, one can calculate the free energy in this configuration and compare with the homogeneous one. One of the most commonly used *Ansätze* is the so-called chiral density wave (CDW) [4].[1] Here, the condensates oscillate with spatial variable $\vec{x}$,

$$\begin{aligned}\langle\bar{\psi}(x)\psi(x)\rangle &\propto \cos(2\vec{q}\cdot\vec{x})\\ \langle\bar{\psi}(x)i\gamma_5\tau_3\psi(x)\rangle &\propto \sin(2\vec{q}\cdot\vec{x}),\end{aligned} \tag{1}$$

where τ_3 is the third Pauli matrix in flavour space and $2\vec{q}$ is the wave vector of the modulation. Alternatively, one can perform what is called a stability analysis which probes the stability of the homogeneous phase with respect to very

[1]The CDW is closely related to the (quarkyonic) chiral spiral in $1+1$ dimensions [29,30]. Other commonly used *Ansatz* functions are the real-kink-crystal [3,20], the twisted kink crystal [31,32], and similar hybrid condensates [33].

small inhomogeneous perturbations. This technique has the benefit that it is agnostic with respect to the shape of the inhomogeneous phase. On the other hand, since it is based on small perturbations, it could miss instabilities with respect to large fluctuations. Nevertheless, these two techniques typically agree on second-order phase boundaries.

Although much has been learned from models, in the end one needs to study the problem within QCD. Both, the *Ansatz* and the stability analysis techniques face severe additional difficulties in QCD compared to their application in simple models. Due to the nonlocality of the self-energy the expectation values of quark bilinears are functions of two momentum variables in an inhomogeneous phase. To our knowledge, Ref. [34] is the only attempt so far to deal with these difficulties in a functional approach to QCD via Dyson-Schwinger equations (DSEs). Using an *Ansatz* for the quark propagator which yields condensates like Eq. (1), they show that spatially inhomogeneous solutions to their set of (truncated) DSEs appear at high chemical potentials.

The stability analysis framework has only recently been generalized from model applications to QCD [35,36]. Although agnostic in general, the QCD formulation of the framework relied on the specific shape of the perturbations (what is called the “test-function”). Furthermore, so far, it has been possible to be applied to a very limited region of the phase diagram, namely the left spinodal line. The details are explained in Refs. [35,36], here we just note that the shape of the test-functions has constraints, which one can exactly enforce only in this region. Thus, although this version of the stability analysis is very promising and has led to first evidence of a inhomogeneous phase in a simple truncation of QCD [36], it suffers somewhat from these limitations.

In this paper, we overcome these shortcomings by introducing an alternative and novel technique to perform a stability analysis. The technique does not depend on any test-functions, and it is not restricted to any region of the phase diagram. It is based on a well-known method to identify the second-order homogeneous phase boundaries, namely calculating the chiral susceptibility (see, e.g., Refs. [37,38]). However, by performing a space-dependent chiral rotation of the quark fields, we are able to compute an *inhomogeneous chiral susceptibility* and determine when the symmetric solution is unstable with respect to inhomogeneous chiral symmetry breaking.

II. INHOMOGENEOUS CHIRAL SUSCEPTIBILITY IN NJL

Before we generalize it to QCD, let us illustrate the idea in the NJL model, defined by the Lagrangian,[2]

$$\mathcal{L} = \bar{\psi}(\not{\partial} + m)\psi + G((\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma_5\tau_a\psi)^2). \qquad (2)$$

We will work in the chiral limit of zero bare mass m. In mean field, the inverse dressed fermion propagator reads

$$S^{-1} = i\not{p} + M, \qquad (3)$$

where M is the dynamically generated mass, which depends on the chiral condensate and is an order parameter for χSB. One way to resolve the phase diagram of the theory is to calculate the derivative of this order parameter with respect to the bare quark mass. For the symmetric phase ($M \to 0$) and N_c colors, this is simply

$$\chi_{\mathrm{NJL}} = \frac{dM}{dm} = \frac{1}{1 - 16N_cG\sum\!\!\!\!\!\!\int_k 1/k^2}, \qquad (4)$$

where we denote the combination of the Matsubara sum of the frequencies ω_k and 3-momentum $\vec{k}$ integration as $\sum\!\!\!\!\!\!\int_k \coloneqq T\sum_{\omega_k}\int d^3k/(2\pi)^3$. The susceptibility is negative where the symmetric phase is unstable with respect to small homogeneous breakings of chiral symmetry, and it is positive otherwise. In a second-order chiral transition, χ_{NJL} changes its sign and diverges on the phase boundary. Around a first-order transition, however, there is a region where both the restored and the broken solution are (meta-) stable. This is known as the spinodal region, and its edges are known as the spinodal lines. In Fig. 1, we show how the

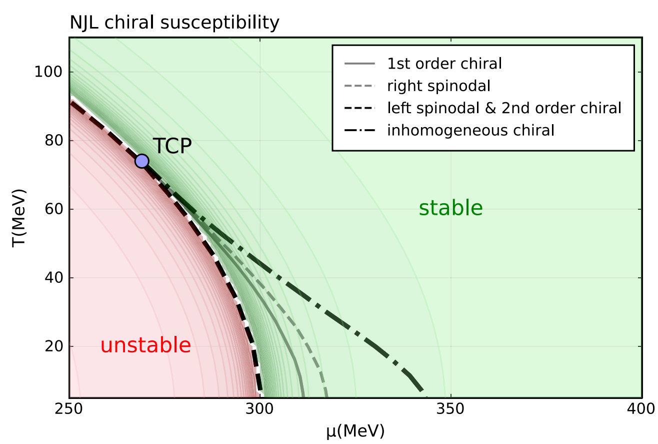


FIG. 1. NJL example of the homogeneous chiral susceptibility χ_{NJL}, representing a probe for the chiral phase transition. We show the phase diagram in the setup of Ref. [3]; see text for details. Green shows positive values of χ_{NJL}, calculated with Eq. (4). Values in red show negativity of χ_{NJL}. The dashed black line represents the second-order transition line at small chemical potential and the left spinodal of the first-order transition region for $\mu > \mu_{\mathrm{TCP}}$. The light gray lines show the first-order phase transition (solid) and the right spinodal (dashed). The dashed-dotted black line shows the second-order transition to the inhomogeneous phase found in Ref. [3]. The spinodals converge on the tri-critical point (TCP).

[2]We will use the same Euclidian-space conventions as in Ref. [35] and as for the numerical values for the couplings, cutoff, and regularization scheme, we will take the setup of Ref. [3].

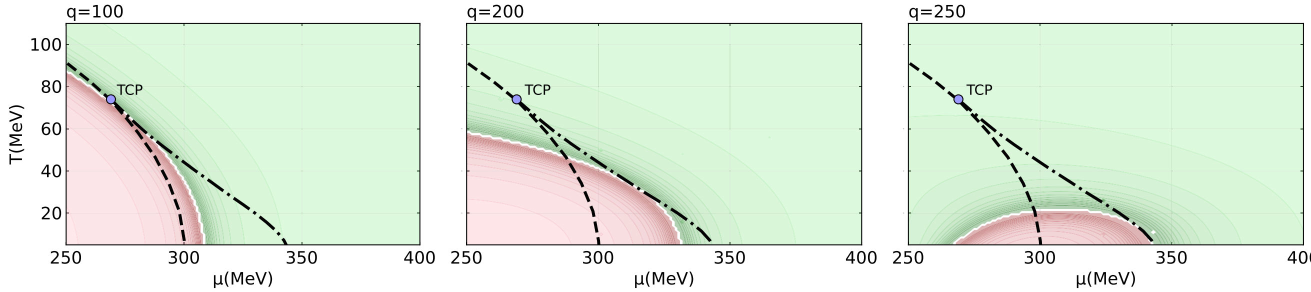


FIG. 2. Inhomogeneous chiral susceptibility $\chi_{\mathrm{NJL}}(q)$ for different values of q. In every diagram, the red region indicates instability of the symmetric phase with respect to the momentum scale q indicated above the figures. Below the critical point, the boundary of the union of all regions where $\chi_{\mathrm{NJL}}(q)$ is negative for some value of q reproduces exactly the right boundary of the region where in Ref. [3] an inhomogeneous phase is found. The physically relevant result here is that we find negative χ for finite q where for $q=0$ it is positive.

sign of χ_{NJL} neatly resolves the second-order transition and left-spinodal lines in NJL, i.e., the regions where the restored solution changes from unstable to (meta-)stable.

Take now the following chiral-rotated quark fields [39]:

$$\xi = e^{i\gamma_5\tau_3 q\cdot x}\psi. \tag{5}$$

The NJL-Lagrangian, Eq. (2) with $m=0$, then transforms to

$$\mathcal{L} = \bar{\xi}(\partial\!\!\!/ + i\gamma_5\tau_3 q\!\!\!/)\xi + G((\bar{\xi}\xi)^2 + (\bar{\xi}i\gamma_5\tau_a\xi)^2). \tag{6}$$

For nonzero q, assuming a homogeneous condensate $\langle\bar{\xi}\xi\rangle = \mathrm{const}$ and $\langle\bar{\xi}\gamma_5\tau_3\xi\rangle = 0$, it is easy to show that one obtains a modulated $\langle\bar{\psi}(x)\psi(x)\rangle$ just like Eq. (1),

$$\begin{aligned}\langle\bar{\psi}(x)\psi(x)\rangle &= \cos(2q\cdot x)\langle\bar{\xi}\xi\rangle,\\ \langle\bar{\psi}(x)i\gamma_5\tau_3\psi(x)\rangle &= \sin(2q\cdot x)\langle\bar{\xi}\xi\rangle.\end{aligned} \tag{7}$$

Under this transformation, the inverse fermion propagator becomes

$$S^{-1} = ip\!\!\!/ + M + i\gamma_5\tau_3 q\!\!\!/. \tag{8}$$

It can still be inverted and an inhomogeneous chiral susceptibility in the symmetric phase can be calculated[3] according to

$$\chi_{\mathrm{NJL}}(q) = \frac{dM}{dm} = \frac{1}{1 - 16N_c G\sum_k \frac{k^2-q^2}{(k^2+q^2)^2 - 4(k\cdot q)^2}}. \tag{9}$$

This expression is general, however, in practice we take the wave vector q_ν taken to be fixed in the z direction, i.e., $q_\nu = (0,0,q,0)$. In Fig. 2 we show how this resolves exactly the phase boundary of the inhomogeneous phase as well: By varying q we manage to find where the symmetric solution is unstable with respect to inhomogeneous χSB, in regions where it is *not* unstable with respect to homogeneous χSB. The union of the red regions in Fig. 2, that is, regions of negative $\chi_{\mathrm{NJL}}(q)$ for some value of q, reproduces exactly the unstable region in Ref. [3]. Note that the denominator of Eq. (9) is exactly the stability condition found in standard stability analysis, so this correspondence is exact.

Having verified that our new method works and reproduces known results within the NJL-model, let us now proceed extending this idea to QCD.

III. QCD

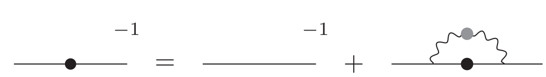
In QCD the setup is completely analogous. Under the chiral rotation, Eq. (5), the QCD Lagrangian,

$$\mathcal{L}_{\mathrm{QCD}} = \bar{\psi}(D\!\!\!\!/)\psi + \frac{1}{4}F^a_{\mu\nu}F^a_{\mu\nu}, \tag{10}$$

with covariant derivative D_μ and gluon field-strength tensor $F^a_{\mu\nu}$, transforms to

$$\mathcal{L}_{\mathrm{QCD}} = \bar{\xi}(D\!\!\!\!/ + i\gamma_5\tau_3 q\!\!\!/)\xi + \frac{1}{4}F^a_{\mu\nu}F^a_{\mu\nu}. \tag{11}$$

As in NJL, the procedure is to first find the chiral symmetric solution for the quark propagator of the nonrotated system, described by Eq. (10). To this end we will use the DSE framework. Different approaches to obtain fully dressed n-point functions, as, e.g., the functional renormalization group are equally suitable. In this paper, in order to illustrate our new method we employ a simple rainbow-ladder truncation of the DSE for the quark propagator, given by



(12)

[3]For this we temporarily reintroduce a small bare quark mass m to the gap equation *after* the chiral rotation and send it to zero after taking the derivative.

Here, the bare line denotes the (inverse) bare quark propagator, whereas the line with the black dot represents the (inverse) dressed quark propagator including quantum fluctuations. It has the general structure,

$$S^{-1}(p) = i\vec{p}\!\!\!/A(p) + i\gamma_4\tilde{\omega}_p C(p) + B(p), \tag{13}$$

with two chiral-symmetry preserving terms parametrized by dressing functions A and C (where $\tilde{\omega}_p = \omega_p + i\mu$) and a symmetry-breaking dressing function B which vanishes in the symmetric phase. The self-interaction diagram contains two bare quark-gluon vertices. The dressing of one of these vertices has already been combined with the dressing of the gluon propagator (wiggly line) to an effective running coupling. For this coupling we will use the Qin-Chang model [40] in the same setup as detailed in [36]. This type of model has been applied to aspects of the QCD phase diagram before; see, e.g., the review Ref. [41].

We now induce the inhomogeneous modulation by the chiral rotation. In an *Ansatz*-based approach, one would have to go through the complicated and extremely costly exercise of solving the resulting set of DSEs self-consistently to get the stable CDW solution. On the contrary, it is much simpler and numerically cheaper to determine the inhomogeneous chiral susceptibility $\chi(\omega_p, \vec{p}|q)$. Proceeding analogously to the NJL case yields the following expression (in the symmetric phase):

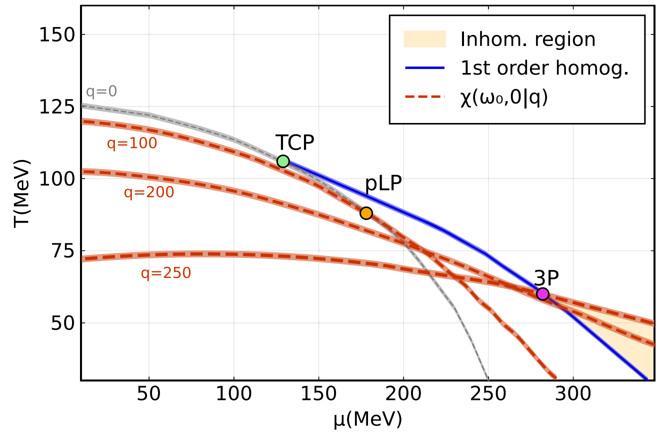


FIG. 3. Equivalent of Fig. 2 for our rainbow-ladder QCD model. Rather than showing a full heatmap of χ, we show the lines in which χ diverges (dashed lines). The first-order chiral homogeneous phase boundary is shown by the solid line. The gray dashed line is the homogeneous chiral solution stability boundary, i.e., the combination of the second-order transition line with the left spinodal, which, as in Fig. 1, agrees with the $q = 0$ result of Eq. (14). The highest point where the instabilities cross the first-order transition is our triple-point (3P) and the light shaded region shows where inhomogeneous phases appear. The proto-Lifschitz Point (pLP) agrees perfectly with Ref. [36] All curves are shown with a width representing a systematic error (see Ref. [35]).

$$\chi(\omega_p, \vec{p}|q) = \frac{\partial B}{\partial m} = 1 + \frac{4}{3}Z_2^2 \sum\!\!\!\!\!\!\!\int_k \frac{\chi(\omega_k, \vec{k}|q)(A(\omega_k, \vec{k})^2\vec{k}^2 + C(\omega_k, \vec{k})^2\tilde{\omega}_k^2 - q^2)}{(A(\omega_k, \vec{k})^2\vec{k}^2 + C(\omega_k, \vec{k})^2\tilde{\omega}_k^2 + q^2)^2 - 4(A(\omega_k, \vec{k})\vec{k}\cdot\vec{q})^2} \times \frac{4\pi(\alpha_L((k-p)^2) + 2\alpha_T((k-p)^2))}{(k-p)^2}, \tag{14}$$

where $\alpha_{T,L}$ are the transverse and longitudinal components of the effective running coupling of the Qin-Chang model (see Ref. [36] for details). Again, a nontrivial *modulated* chiral condensate is indicated if χ goes negative at finite q. Comparing Eq. (14) with the corresponding one for the NJL-model, Eq. (9), we find the slight complication that the susceptibility now depends on the quark momentum $p_\nu = (\omega_p, \vec{p})$ and appears on both sides such that the equation has to be solved self-consistently. Note, however, that the quark dressing functions $A(\omega, \vec{k})$ and $C(\omega, \vec{k})$ on the right-hand side do not change under the chiral rotation in the symmetric phase and are therefore the same as those calculated without the extra $\gamma_5\tau_a q\!\!\!/$ term present in the theory. This is a subtle and important point. In this analysis, we do not need to solve the DSE with the extra term present. Instead, we study whether the addition of $\gamma_5\tau_a q\!\!\!/$ induces the formation of a Dirac scalar term to the quark propagator. If so, then the susceptibility diverges at the second-order phase boundary and an inhomogeneous phase will be formed.

In Fig. 3 we show the lines where the susceptibility diverges for different values of q. We confirm our finding from Ref. [36] that near $T = 85$ MeV there is a "proto-Lifschitz point" (pLP), i.e., a point where the stability boundary of the symmetric phase wiith respect to inhomogeneous fluctuations meets the left spinodal of the homogeneous first-order transition. Below this point there is a region where the symmetric phase is unstable against developing inhomogeneous structures. However, this instability is without consequences as long as it occurs to the left of the first-order homogeneous phase boundary (blue solid line), since in this region the symmetric phase is also disfavored against the chiral broken phase. In Ref. [36] the technical complexity of the applied test-function method prevented us from moving away very far from the left spinodal, and therefore we could only speculate about the existence of an inhomogeneous phase at higher chemical potential. Our new method, on the other hand, can be applied at arbitrarily high chemical potential. Varying the value of q we find a triple-point (3P) at $T \simeq 60$ MeV where the instability line of the symmetric phase crosses the homogeneous first-order phase boundary. In particular we establish that, while for temperatures above 60 MeV, the instability region does not extend beyond the homogeneous first-order transition, at lower temperatures it does. Hence, in this regime the symmetric phase is unstable against inhomogeneous fluctuations in a region where it is favoured over the homogeneous chiral broken solution. This is an unambiguous proof that, in this truncation, a crystalline phase would surface.

Although this is very exciting, the main point of this study is to demonstrate that we now have a methodology where this analysis can be performed. Our ultimate goal is to prioritize more realistic truncations as a direction for future work, where our methodology can provide more rigorous insights.

IV. CONCLUSIONS

In this work, we proposed a new method to analyse the stability of a chirally symmetric phase against the formation of an inhomogeneous condensate via chiral symmetry breaking. The method is based on an extension of the concept of chiral susceptibility toward probing the reaction of the theory to the presence of spatially modulated quark fields. As a proof of principle, we showed that our method exactly reproduces previous NJL-model results [3] on the presence of inhomogeneous phases, we confirmed previous results for the presence of a proto-Lifschitz Point in a rainbow-ladder model of QCD [36], and we now also find a true instability of the symmetric phase in a regime where it is favored over the homogeneous broken phase.

However, the main point of this paper is not this specific result but rather that the proposed method is indeed suited to perform such analyses. In fact, with this methodology we were able to obtain substantially more conclusive results than with the test-function based stability analysis applied in Ref. [36]. The latter requires numerically searching for a saddle point in the function's parameter space, which can be very expensive. Moreover, choosing the parameter space too small bears the risk of wrong results. For that reason this analysis is so far only feasible close to the left spinodal line [36] where the sufficiency of the parameter space can be ensured by analyzing the homogeneous transition. Our new method requires no external parameters, and the *only* real numerically costly operation is that of solving the homogeneous DSE.

The only restriction of the method is that, as it is based on a chiral rotation, it primarily signals instabilities toward CDW-like condensates; see Eq. (7). However, as known from model studies, even though the CDW is typically not the most favored inhomogeneous phase, in the chiral limit its second-order phase boundary to the symmetric phase coincides with those for other modulations. In this context one should keep in mind that a stability analysis in general does not say much about the fully self-consistent solution of the system. Here it just shows that there is the tendency to break chiral symmetry once we introduce a modulation, in this instance a CDW-like condensation. The actual shape of the modulation would eventually to be determined by a self-consistent solution of the DSEs for all dressing functions of the quark. This is a tremendous effort, but one that should be undertaken once our method does show instabilities in more advanced truncations to QCD than used in this work.

Some more care is required to generalise the method to be applicable away from the chiral limit. Then the chiral rotation (5) is not a symmetry transformation of the QCD Lagrangian but generates nonstandard mass terms that need to be dealt with. This is work in progress.

On the other hand, the new method can easily be generalized to any theory, including simple models, but also effective theories in solid state physics, to study the inhomogeneous breaking of any exact symmetry of that theory. For instance, in the QCD context it could also be used to study inhomogeneous color superconductivity [42]. Also, it is equally applicable to any truncation of QCD, with no computational costs in addition to solving the homogeneous DSE. Since it is not necessary to know the full momentum dependence of χ, we expect this should also be calculable with the functional renormalization group.

Some studies suggest that additional quantum fluctuations could destabilise inhomogeneous condensates [30,43–45]. In that case, we expect the homogeneous solution to the DSE to capture the energetically preferred, stable configuration. Other studies indicate that the boundary between inhomogeneous and restored phases may be first order [46], limiting our ability to precisely determine the transition line. Nevertheless, our approach should still identify the corresponding spinodal.

Finally, it is important to realise that such instabilities (or even their associated phenomenon, the moat regimes) are highly relevant as they might leave signatures in heavy-ion collisions [7]. Even if these phases turn out not to be exactly stable in full QCD, these signatures might still be present due to long-lived meta-stable realizations of these phases as the system moves towards equilibrium [47].

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