International Journal of Quality & Reliability Management

Use of copula functions for the reliability of series systems

Jorge Alberto Achcar, Fernando Antonio Moala,

RELIABILITY PAPER

Use of copula functions for the reliability of series systems

Jorge Alberto Achcar
Department of Social Medicine, Universidade de São Paulo, Ribeirão Preto, Brazil, and
Fernando Antonio Moala
Department of Statistics, Universidade Estadual Paulista, UNESP, Presidente Prudente, Brazil

Received 3 October 2013
Revised 4 February 2015
Accepted 5 February 2015

Abstract

Purpose – The purpose of this paper is to provide a new method to estimate the reliability of series system by using copula functions. This problem is of great interest in industrial and engineering applications.

Design/methodology/approach – The authors introduce copula functions and consider a Bayesian analysis for the proposed models with application to the simulated data.

Findings – The use of copula functions for modeling the bivariate distribution could be a good alternative to estimate the reliability of a two components series system. From the results of this study, the authors observe that they get accurate Bayesian inferences for the reliability function considering large samples sizes. The Bayesian parametric models proposed also allow the assessment of system reliability for multicomponent systems simultaneously.

Originality/value – Usually, the studies of systems reliability engineering assume independence among the component lifetimes. In the approach the authors consider a dependence structure. Using standard classical inference methods based on asymptotical normality of the maximum likelihood estimators for the parameters the authors could have great computational difficulties and possibly, not accurate inference results, which there is not found in the approach.

Keywords Bayesian approach, Reliability, Copula, Farlie-Gumbel-Morgenstern, Gumbel, Series system

Paper type Research paper

1. Introduction

Usually, the studies of systems reliability engineering assume independence among the component lifetimes. In practice, however, the components of a system are usually dependent. Indeed, two components assembled in a system structure may share the same load, may be subject to the same set of stresses or similar environment and hence leading to similar performances. This way, the correlated lifetimes can modify the reliability of the systems. This problem is of great interest in industrial and engineering applications.

Different bivariate lifetime models could be assumed for correlated lifetimes (see Kotz *et al.*, 2000). The bivariate exponential has been the most popular choice for estimation of the reliability systems (see Freund, 1961; Marshall and Olkin, 1967; Block and Basu, 1974; Franco and Vivo, 2002).

An another popular lifetime distribution is given by the Weibull family (Weibull, 1951; Murthy *et al.*, 2004), since this model presents a great flexibility of fzit: constant, increasing or decreasing hazard function $h(t)=f(t)/R(t)$ where $f(t)$ is the probability density function for lifetime T and $R(t)$ is the reliability function, that is, $R(t)=P[T>t]$ for fixed t.

International Journal of Quality & Reliability Management
Vol. 32 No. 6, 2015
pp. 617-634

0265-671X
DOI 10.1108/IJQRM-10-2013-0161

The Weibull distribtion has been used extensively by reliability engineers (see, e.g. Cran, 1988; Teimouri and Gupta, 2013) and mainly with censored lifetime as right censored data (Lawless, 1982; Sirvanci and Yang, 1984; Wang and Keats, 1995; Kundu and Raqab, 2012), randomly censored data (Danish and Aslam, 2014), interval-censored data (Pradhan and Kundu, 2014), progressively type-II censored samples (Ng, 2005), and others.

In this paper the bivariate Weibull distribution is derived from two different copula functions, the Gumbel (1960) copula and the Farlie-Gumbel-Morgenstern copula, denoted by FGM (see Morgenstern, 1956; Gumbel, 1960; Farley, 1960), in order to model the dependent lifetimes of two-components series system.

Copula function is an alternative and flexible way to construct different multivariate lifetime distributions with pre-specified marginal distributions (see, e.g. Nelsen, 2006; Trivedi and Zimmer, 2005a, b).

The main goal of this paper is to describe two approaches for the construction of the bivariate Weibull distribution where the marginal distributions are univariate Weibull distributions. The parameters of the copulas and reliability function of the system are estimated by using the Bayesian methods to get the posterior summaries of interest. In this way, we use Markov Chain Monte Carlo (MCMC) methods as the popular Gibbs sampling algorithm (see, e.g. Gelfand and Smith, 1990; Casella and George, 1992) and the Metropolis-Hastings algorithm (see, e.g. Chib and Grenberg, 1995).

We also study how the dependency influences the performance of the system. We extend some comparison results obtained in the case of independent components to the case of two dependent components for each proposed copula function.

The paper is organized as follows: Section 2 presents a review of series system; in Section 3, we introduce copula functions and provides the bivariate Weibull distributions; in Section 4, we consider a Bayesian analysis for the proposed models; in Section 5, we introduce simulated examples; finally in Section 6, we present some conclusions.

2. A series system with two components

Let us assume a series system with two components, where the system works if and only if both components work (see Figure 1).

The reliability of a series system with two components in a fixed time t is given by:

$$R(t) = P[\min(t_1, t_2) > t] \tag{1}$$

where T_1 and T_2 are random variables denoting the lifetimes of components 1 and 2, respectively.

From (1), we observe that $R(t) = P[T_1 > t,\ T_2 > \ t]$; under the usual independence assumption, the reliability of the system is given by:

$$R(t) = P[T_1 > t]\, P[T_2 > t] = R_1(t)\, R_2(t) \tag{2}$$

where $R_j(t) = P[T_j > t]$ is the reliability for the component j, $j = 1, 2$.

In practical reliability engineering studies, usually we have correlated lifetimes T_1 and T_2; in this case we could have different reliability as assuming independence between the lifetimes (see, e.g. Aggarwal, 1993; Smith, 1972; Nelson, 1982).

Figure 1.
Series system

In this way, the reliability for the series system at a fixed time t is given by:

$$R(t) = P[T_1 > t, T_2 > t] = S_{1,2}(t_1, t_2) \tag{3}$$

where $S_{12}(t_1, t_2)$ denotes the joint survival function for the lifetimes T_1 and T_2 and definite as:

$$S_I(t_1, t_2) = P(T_1 > t_1, T_2 > t_2) = 1 - F_1(t_1) - F_2(t_2) + F(t_1, t_2) \tag{4}$$

where $F_j(t) = P[T_j \leqslant t]$ is the distribution function for the component j, $j = 1, 2$ and $F(t_1, t_2)$ is the joint distribution function.

3. Bivariate Weibull based on copula functions

In this section, we introduce two new formulations for the bivariate Weibull distribution based on copula functions.

Copula functions have been studied by many authors (Nelsen, 2006, is a classical book in this topic). Copula functions can be used to link marginal distributions with a joint distribution. For specified univariate marginal distribution functions $F_1(t_1), \ldots, F_m(t_m)$, the function $C[F_1(t_1), \ldots, F_m(t_m)]$ which is defined using a copula function C, results in a multivariate distribution specified as $F_1(t_1), \ldots, F_m(t_m)$. It is important to point out that any multivariate distribution function F can be written in the form of a copula function (Sklar, 1959), that is, if $F(t_1, \ldots, t_m)$ is a joint multivariate distribution function with univariate marginal distribution functions $F_1(t_1), \ldots, F_m(t_m)$, thus exists a copula function $C[u_1, \ldots, u_m]$ such that:

$$F(t_1, \ldots, t_m) = C[F_1(t_1), \ldots, F_m(t_m)] \tag{5}$$

For the special case of bivariate distributions, we have $m = 2$.

Observe that the probability transformations $U_1 = F_1(t_1)$ and $U_2 = F_2(t_2)$ where U_1 and U_2 have uniform (0, 1) distributions, are dependent if T_1 and T_2 are dependent (T_1 and T_2 independent implies that U_1 and U_2 are independent).

Specifying, dependence between T_1 and T_2 is the same as specifying dependence between U_1 and U_2, thus the problem reduces to specifying a bivariate distribution between two uniform variables, that is, a copula. Our first model considered for the study of dependence structure for two components is based in the FGM copula.

The FGM copula is defined by:

$$C_I(u_1, u_2) = u_1 u_2 [1 + \theta(1 - u_1)(1 - u_2)] \tag{6}$$

where $u_1 = F_1(t_1)$ and $u_2 = F_2(t_2)$ and $-1 < \theta < 1$. Observe that θ measures the dependence between two marginals, that is, if $\theta = 0$, we have independent random variables. This copula is appropriated to model weak dependence structures.

The association parameter θ can take different values depending on the copula, whereas measures of association, such as Pearson's correlation coefficient are usually bounded. The parameter θ is related to the well-known association coefficients Kendall's τ and Spearman's ρ, by the equations:

$$\tau = 4 \iint C(u_1, u_2)\, dC(u_1, u_2) - 1 = 4[\theta/18 + 1/4] - 1 = 2\theta/9 \tag{7}$$

and:

$$\rho = 12\iint C(u_1,u_2)\,dC(u_1,u_2) - 3 = 12\left[\theta/36 + 1/4\right] - 3 = \theta/3$$

From (6), the cumulative joint distribution function for the random variables T_1 and T_2 is given by:

$$F(t_1,t_2) = C_{\mathrm{I}}(F_1(t_1),F_2(t_2)) = F_1(t_1)F_2(t_2)[1+\theta(1-F_1(t_1))(1-F_2(t_2))] \quad (8)$$

that is:

$$F_{\mathrm{I}}(t_1,t_2) = F_1(t_1)F_2(t_2)[1+\theta R_1(t_1)R_2(t_2)] \quad (9)$$

The joint bivariate survival function for the lifetimes T_1 and T_2 obtained from (4) and (9) is given by:

$$S_I(t_1,t_2) = R_1(t_1)R_2(t_2)[1+\theta F_1(t_1)F_2(t_2)] \quad (10)$$

Let us assume the marginal Weibull distributions given by:

$$F_1(t_1) = 1-\exp\left[-\left(t_1/\lambda_1\right)^{\alpha_1} \text{ and } F_2(t_2) = 1-\exp\left[-\left(\left(t_2/\lambda_2\right)^{\alpha_2}\right)\right] \quad (11)$$

and the reliability functions at fixed times t_1 and t_2 are given by:

$$R_1(t_1) = \exp\left[-\left(t_1/\lambda_1\right)^{\alpha_1}\right] \text{ and } R_2(t_2) = \exp\left[-\left(t_2/\lambda_2\right)^{\alpha_2}\right] \quad (12)$$

where λ_1, λ_2, α_1 and α_2 are all positives.

Considering the bivariate Weibull distribution derived from the FGM copula (model I), the joint distribution function for T_1 and T_2 at fixed times t_1 and t_2 (see (8)) is given by:

$$F_{\mathrm{I}}(t_1,\ t_2) = \left\{1-\exp\left[-\left(t_1/\lambda_1\right)^{\alpha_1}\right]\right\}\left\{1-\exp\left[\left(t_2/\lambda_2\right)^{\alpha_2}\right]\right\}$$
$$\left\{1+\theta\exp\left[-\left(t_1/\lambda_1\right)^{\alpha_1}-\left(t_2/\lambda_2\right)^{\alpha_2}\right]\right\} \quad (13)$$

and the joint survival function for T_1 and T_2 at fixed times t_1 and t_2 (see (10)) is given by:

$$S_{\mathrm{I}}(t_1,t_2) = \exp\left[-\left(t_1/\lambda_1\right)^{\alpha_1}-\left(t_2/\lambda_2\right)^{\alpha_2}\right]\left\{1+\theta\left(1-\exp\left[-\left(t_1/\lambda_1\right)^{\alpha_1}\right]\right)\left(1-\exp\left[-\left(t_2/\lambda_2\right)^{\alpha_2}\right]\right)\right\} \quad (14)$$

Assuming a two component series system (see (3)), the reliability of the system at a fixed time t is given by:

$$R(t) = S_{\mathrm{I}}(t,t) \quad (15)$$

that is:

$$R(t) = \exp\left[-\left(t/\lambda_1\right)^{\alpha_1}-\left(t/\lambda_2\right)^{\alpha_2}\right)\right]\left\{1+\theta\left(1-\exp\left[-\left(t/\lambda_1\right)^{\alpha_1}\right]\right)\left(1-\exp\left[-\left(t/\lambda_2\right)^{\alpha_2}\right]\right)\right\} \quad (16)$$

Let us denote this model derived from FGM copula, as "copula-I model" or "model I."

A second copula function considered for the two-component series system, which is denoted as "copula-II model" or "model II," is the Gumbel copula function (1960), defined as:

$$C_{II}(u_1, u_2) = u_1 + u_2 - 1 + (1-u_1)(1-u_2)\exp[-\Phi\ln(1-u_1)\ln(1-u_2)] \quad (17)$$

where $0 < \Phi < 1$.

In this model, the joint cumulative distribution function for the random variables T_1 and T_2 is given by:

$$F_{II}(t_1, t_2) = F_1(t) + F_2(t) - 1 + (1-F_1(t))(1-F_2(t))\exp[-\Phi\ln(1-F_1(t))\ln(1-F_2(t))] \quad (18)$$

and corresponding survival function given by:

$$S_{II}(t_1, t_2) = R_1(t_1)R_2(t_2)\exp[-\Phi\ln(R_1(t_1))\ln(R_2(t_2))] \quad (19)$$

As pointed out by Gumbel (1960), for this copula model, when $\Phi = 1$, the Pearson correlation linear coefficient takes the value −0.40365. When the two variables are independent, Φ takes the 0 value.

Considering the bivariate Weibull distribution derived from the Gumbel copula function (model II), the joint distribution function for T_1 and T_2 at fixed times t_1 and t_2 (see (18)) is given by:

$$F_{II}(t_1, t_2) = 1 - \exp\left\{-\left(t_1/\lambda_1\right)^{\alpha 1}\right\} - \exp\left\{-\left(t_2/\lambda_2\right)^{\alpha_2}\right\}$$
$$+ \exp\left\{-\left(t_1/\lambda_1\right)^{\alpha_1} - \left(t_2/\lambda_2\right)^{\alpha_2} - \Phi\left(\mathrm{t}_1/\lambda_1\right)^{\alpha_1}\left(\mathrm{t}_2/\lambda_2\right)^{\alpha_2}\right\} \quad (20)$$

and the joint survival function for T_1 and T_2 at fixed times t_1 and t_2 (see (19)) is given by:

$$S_{II}(t_1, t_2) = \exp\left\{-\left(t_1/\lambda_1\right)^{\alpha_1} - \left(t_2/\lambda_2\right)^{\alpha_2} - \Phi\left(t_1/\lambda_1\right)^{\alpha_1}\left(t_2/\lambda_2\right)^{\alpha_2}\right\} \quad (21)$$

Assuming a two-component series system (see (3)), the reliability of the system at a fixed time t is given from (21), by $R(t) = S_{II}(t, t)$, that is:

$$R(t) = \exp\left\{-\left(t/\lambda_1\right)^{\alpha_1} - \left(t/\lambda_2\right)^{\alpha_2} - \Phi\left(t/\lambda_1\right)^{\alpha_1}\left(t/\lambda_2\right)^{\alpha_2}\right\} \quad (22)$$

It is important to point out that many other copula functions are introduced in the literature which are appropriate to model different degrees of dependence (see, e.g. Amblard and Girard, 2002, 2009; Chan and Li, 2008; Czado, 2010; Frahm *et al.*, 2003; Genest, 1987; Genest and MacKay, 1986a, b; Genest and Rivest, 1993; Joe, 1993, 1997; Kurowicka and Joe, 2011; McNeil and Nešlehová, 2009; Nelsen, 1986; Schweizer, 1991; Shih and Louis, 1995; Song, 2000; Zheng and Klein, 1995).

4. A Bayesian analysis assuming Weibull distributions for the component lifetimes

Let us assume a random sample of size n of a two-component series system with lifetimes T_1 and T_2 related to the two components. We only observe the minimum lifetime beween T_1 and T_2, that is, $T = \min(T_1, T_2)$. Thus, we have a data set denoted by (t_i, δ_i), where $\delta_i = 1$ if $t_{1i} < t_{2i}$ or $\delta_i = 0$ if $t_{1i} \geqslant t_{2i}$, $i = 1, \ldots, n$. Observe that, if $\delta_i = 1$, the contribution for the likelihood function is given by $P[T_{1i} = t_i, T_{2i} > t_i]$, if $\delta_i = 0$, the contribution for the likelihood function is given by $P[T_{1i} > t_i, T_{2i} = t_i]$.

The likelihood function (see, e.g. Lawless, 1982) is given by:

$$L = \prod_{i=1}^{n} \left(-\frac{\partial S(t_{1i}, t_{2i})}{\partial t_{1i}} \right)^{\delta_i} \left(-\frac{\partial S(t_{1i}, t_{2i})}{\partial t_{2i}} \right)^{1-\delta_i} \tag{23}$$

where $S(t_{1i}, t_{2i})$ is the joint survival function for the lifetimes T_{1i} and T_{2i}.

In order to determine the likelihood for the parameters of the bivariate Weibull distribution derived from the FGM copula function (model I), we need the first derivatives of $S_{\mathrm{I}}(t_1, t_2)$, given in (14), with respect to t_1 and t_2 which are given, respectively, by:

$$-\partial S_{\mathrm{I}}(t_1, t_2)/\partial t_1 = [\alpha_1 (t_1)^{\alpha_1 - 1}]/(\lambda_1)^{\alpha_1} \exp\left[-(t_1/\lambda_1)^{\alpha_1} - (t_2/\lambda_2)^{\alpha_2}\right]$$
$$\left\{1 + \theta - 2\theta \exp\left[-(t_1/\lambda_1)^{\alpha_1}\right] - \theta \exp\left[-(t_2/\lambda_2)^{\alpha_2}\right] + 2\theta \exp\left[-(t_1/\lambda_1)^{\alpha_1} - (t_2/\lambda_2)^{\alpha_2}\right]\right\} \tag{24}$$

and:

$$\begin{aligned} -\partial S_{\mathrm{I}}(t_{1,2})/\partial t_2 = {} & [\alpha_2 (t_2)^{\alpha_2 - 1}]/(\lambda_2)^{\alpha_2} \exp\left[-(t_1/\lambda_1)^{\alpha_1} - (t_2/\lambda_2)^{\alpha_2}\right] \\ & \times \left\{1 + \theta - \theta \exp\left[-(t_1/\lambda_1)^{\alpha_1}\right] - 2\theta \exp\left[-(t_2/\lambda_2)^{\alpha_2}\right] \right. \\ & \left. + 2\theta \exp\left[-(t_1/\lambda_1)^{\alpha_1} - (t_2/\lambda_2)^{\alpha_2}\right]\right\} \end{aligned}$$

and replacing (24) in (23) we have the likelihood function L_{I}.

Now, considering the bivariate Weibull distribution derived from the Gumbel copula function (model II), the likelihood function L_{II} is obtained with the first derivatives of $S_{\mathrm{II}}(t_1, t_2)$ given in (21) with respect to t_1 and t_2, that is:

$$\begin{aligned} -\partial S_{\mathrm{II}}(t_1, t_2)/\partial t1 = {} & [\alpha_1 (t_1)^{\alpha_1 - 1}]/(\lambda_1)^{\alpha_1} \left\{1 + \Phi (t_2/\lambda_2)^{\alpha_2}\right\} \\ & \exp\left\{-(t_1/\lambda_1)^{\alpha_1} - (t_2/\lambda_2)^{\alpha_2} - \Phi (t_1/\lambda_1)^{\alpha_1} (t_2/\lambda_2)^{\alpha_2}\right\} \end{aligned} \tag{25}$$

and:

$$\begin{aligned} -\partial S_{\mathrm{II}}(t_1, t_2)/\partial t_2 = {} & [\alpha_2 (t_2)^{\alpha_2 - 1}]/(\lambda_2)^{\alpha_2} \left\{1 + \Phi (t_1/\lambda_1)^{\alpha_1}\right\} \\ & \exp\left\{-(t_1/\lambda_1)^{\alpha_1} - (t_2/\lambda_2)^{\alpha_2} - \Phi (t_1/\lambda_1)^{\alpha_1} (t_2/\lambda_2)^{\alpha_2}\right\} \end{aligned}$$

From (23) and (25) we have the likelihood function for the model II.

For a Bayesian analysis of "model I" and "model II," we assume uniform $U(a, b)$ prior distributions for the parameters λ_1, λ_2, α_1, α_2, θ and Φ, with known hyperparameters a and b. We further assume prior independence among the parameters. Other prior distributions also could be considered, as γ distributions. The joint posterior distribution for the parameters of the model is obtained using the Bayes formula.

Due to the complex analytical form of the obtained joint posterior distributions from where we cannot analytically obtain the marginal posterior distributions, we use MCMC techniques as the popular Gibbs sampling algorithm or the Metropolis-Hastings algorithm to get samples of the joint posterior distribution of interest.

Specifically, we run an algorithm to simulate a long chain of draws from the joint posterior distribution through the conditional posterior distributions $\Pi(\theta_j/\theta_{(j)}, \mathbf{t_1}, \mathbf{t_2})$ where $\theta_{(j)} = (\theta_1, \ldots, \theta_{j-1}, \theta_{j+1}, \ldots, \theta_k)$, $\mathbf{t_1} = (t_{11}, \ldots, t_{1n})$ and, $\mathbf{t_2} = (t_{21}, \ldots, t_{2n})$, $j = 1, \ldots, k$, and we get Monte Carlo Bayesian point estimates or credible intervals of interest based on the posterior summaries calculated from the simulated Gibbs samples.

5. A numerical illustration

We show in this section how one can apply the previous obtained theoretical results.

Figure 2 displays the survival functions given by (14) and (21), regarding FGM and Gumbel copulas, respectively. It is assumed the parameter values $\lambda_1 = 1$, $\lambda_2 = 1.5$, $\alpha_1 = 2$, $\alpha_2 = 3$, $\theta = 0.8$ and $\Phi = 0.8$ for the both copulas. Through the contour plots we can verify the existence of dependence between the random variables T_1 and T_2.

We simulate a connected two components series system. It is assumed in this simulation that n systems were put on the life test and the lifetime of each component that may cause the system failure was observed. The data set was generated, respectively, from the models I and II with parameters $\lambda_1 = 3.5$, $\lambda_2 = 3.0$, $\alpha_1 = 1.5$, $\alpha_2 = 2.5$, $\theta = 0.8$ and $\Phi = 0.8$.

For all considered sample sizes, we assume $\Gamma(0.1, 0.1)$ prior distributions for λ_j and α_j, $j = 1, 2$ where $\Gamma(a, b)$ denotes a γ distribution with mean equals to a/b and variance

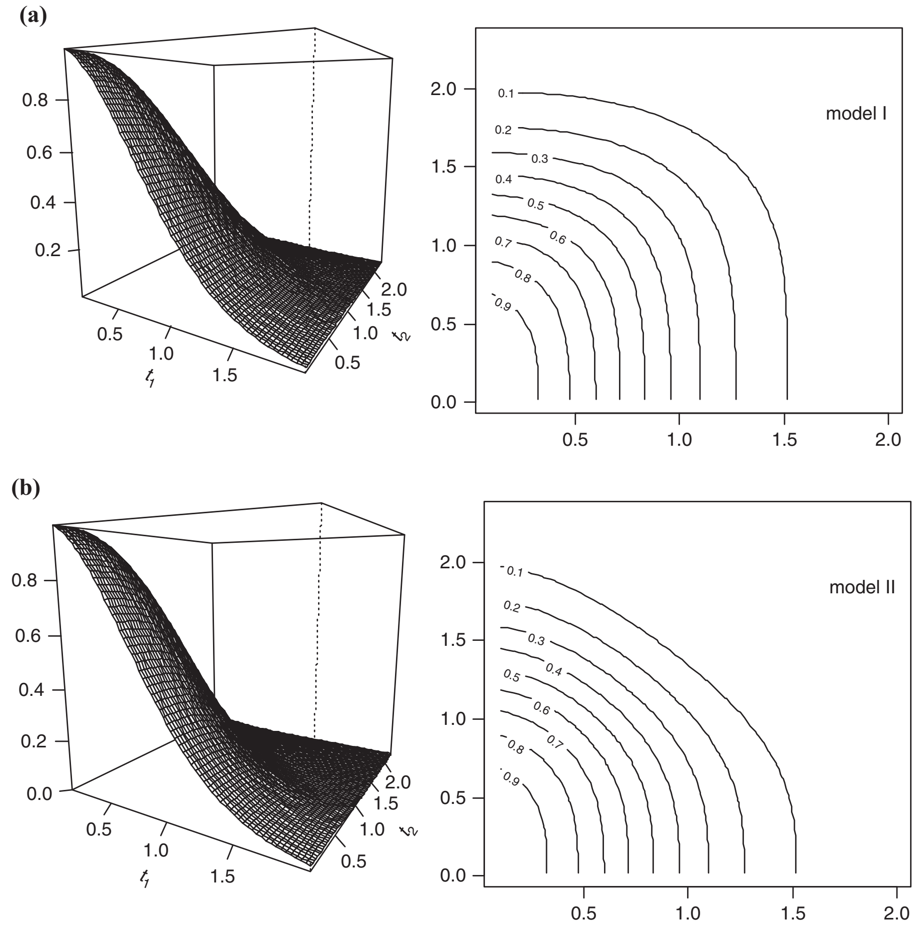

Figure 2. Contour plots of bivariate survival function for the models I and II with parameters $\theta = 0.8$ and $\Phi = 0.8$

equals to a/b^2, and $\theta \sim U(-1, 1)$, where $U(a, b)$ denotes an uniform distribution on the interval (a, b). We also assume prior independence among the parameters. We use MCMC algorithms to simulate a sample of values for the parameters from the joint posterior distribution for λ_j, α_j and θ. In our application, the chain was run for 25,000 iterations with a burn-in period of size 5,000. Convergence of the MCMC algorithm was monitored using standard existing methods as time series for the simulated samples.

Figure 3 illustrates the performance of the marginal posterior densities for each parameter λ_1, λ_2, α_1, α_2 and θ when the sample size increases ($n = 10$, 50 and 100) considering the model I.

In the plots of Figure 3 we observe that as the sample size increases, we have more accurate Bayesian inferences (see also Table I). In the situation of small sample sizes we could get very noninformative inferences. Note that the plot of the posterior distribution for the parameter θ shows a considerable skewness.

In Table I, we have the posterior means, the posterior standard deviations and 95 percent credible intervals for the parameters of "model I."

From the results of Table I, we observe that we get accurate Bayesian inferences for the parameters considering large samples sizes. Based on this simulation data, the reliability function for the system given by (10) can be also estimated. For the specified mission time $t = 1.5$ the true reliability value is 0.6530. The Bayes estimators of $R(1.5)$ are computed and the obtained results are presented in Table II.

The plot of the posterior distribution for R(1.5) is shown in Figure 4.

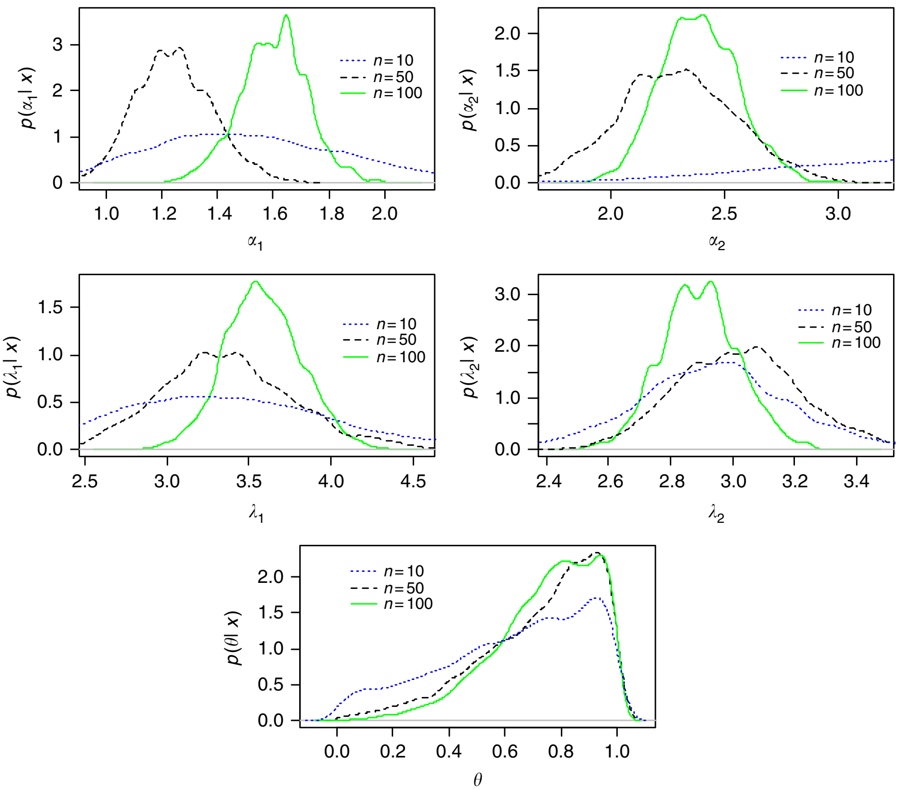

Figure 3.
Marginal posterior densities for each parameter of the model I for $n = 10$, 50 and 100

Sample size	Parameter	Mean	SD	95% credible interval
$n=10$	α_1	1.5860	0.4614	(0.9525, 2.6304)
	α_2	2.6305	0.7411	(1.6282, 4.4678)
	λ_1	3.5492	0.7544	(2.1834, 5.0785)
	λ_2	3.0190	0.3985	(2.2281, 3.7656)
	θ	0.5607	0.0717	(0.4135, 0.6811)
$n=50$	α_1	1.5264	0.1652	(1.2293, 1.9121)
	α_2	2.534	0.2849	(2.0617, 3.0998)
	λ_1	3.5252	0.3422	(2.8733, 4.1871)
	λ_2	3.0125	0.1728	(2.6856, 3.3528)
	θ	0.6298	0.1254	(0.3213, 0.8230)
$n=100$	α_1	1.5054	0.1214	(1.2980, 1.7758)
	α_2	2.5062	0.1966	(2.1374, 2.9038)
	λ_1	3.4748	0.2459	(2.9880, 3.9686)
	λ_2	2.9926	0.1268	(2.7425, 3.2431)
	θ	0.6728	0.1400	(0.3374, 0.8620)

Table I.
Posterior summaries (model I)

	Mean	SD	Interval
$n=10$	0.6039	0.1068	(0.4031, 0.8108)
$n=50$	0.6445	0.0501	(0.5447, 0.7464)
$n=100$	0.6420	0.0383	(0.5670, 0.7157)

Table II.
Estimation of reliability function for $t=1.5$ with model I

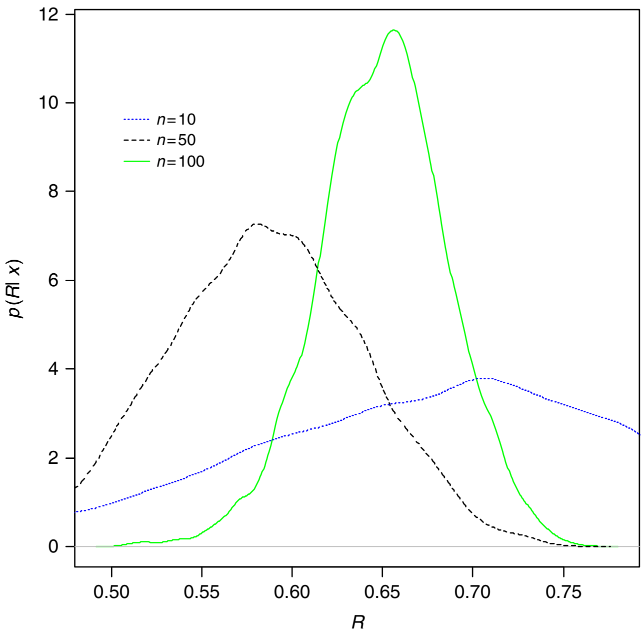

Figure 4.
Plot of the posterior reliability function R for $t=1.5$

From Figure 4, we observe that theposterior distribution for R is very noninformative for small values of n as $n=10$. The plots of mean and pointwise 95 percent credible Bayesian interval for the reliability function are shown in Figure 5. The credible intervals illustrate the degree of uncertainty surrounding the reliability function $R(t)$.

The 95 percent credibility Bayesian intervals show that there is rather little remaining uncertainty concerning the reliability function for large n.

As a second analysis, let us assume "model II" defined by the joint distribution function (12). For a Bayesian analysis of "model II," let us assume the same priors for the parameters λ_1, λ_2, α_1, α_2 considered for "model I" and an uniform $U(0, 1)$ prior for the dependence parameter Φ. Following the same simulation procedure considered for "model I," we have in Table III, the posterior summaries of interest. Convergence of the MCMC algorithm was also monitored using time series plots of the simulated samples.

Figure 6 illustrates the performance of the marginal posterior densities for each parameter λ_1, λ_2, α_1, α_2 and Φ with $n=10$, 50 and 100. We also observe a significant skewness for the posterior distribution of the parameter Φ.

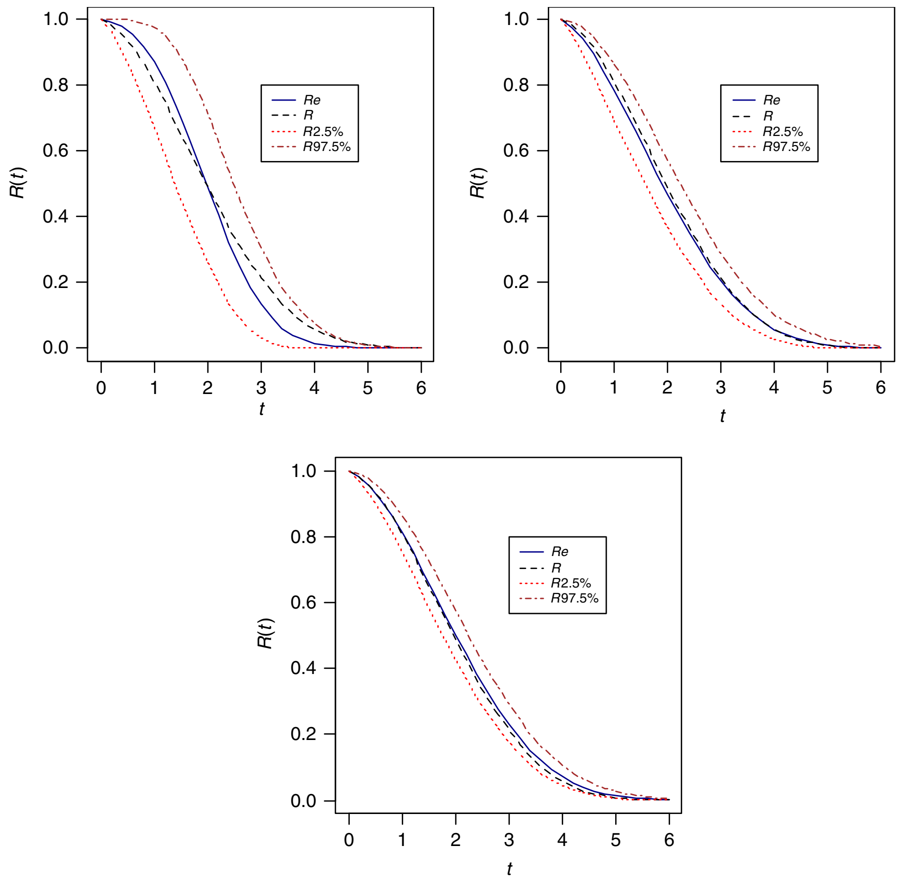

Figure 5.
The mean and 95 percent credible intervals for the reliability function considering model I for $n=10$, 50 and 100

Sample size	Parameter	Mean	SD	95% credible interval
$n=10$	α_1	0.6111	0.4661	(0.9478, 2.7528)
	α_2	2.6852	0.7686	(1.5762, 4.5887)
	λ_1	3.6025	0.7361	(2.3004, 5.1265)
	λ_2	2.9858	0.3744	(2.2320, 3.7383)
	Φ	0.5801	0.1061	(0.3369, 0.7331)
$n=50$	α_1	1.5474	0.1689	(1.2454, 1.9200)
	α_2	2.5585	0.2698	(2.0934, 3.1762)
	λ_1	3.4982	0.3275	(2.8917, 4.1149)
	λ_2	2.9957	0.1764	(2.6528, 3.3526)
	Φ	0.6921	0.1210	(0.3917, 0.8710)
$n=100$	α_1	1.5238	0.1156	(1.3268, 1.7635)
	α_2	2.5316	0.1778	(2.1987, 2.9034)
	λ_1	3.5079	0.2430	(3.0508, 4.0216)
	λ_2	2.9887	0.1280	(2.7261, 3.2354)
	Φ	0.7358	0.1162	(0.4592, 0.8966)

Table III. Posterior summaries (model II)

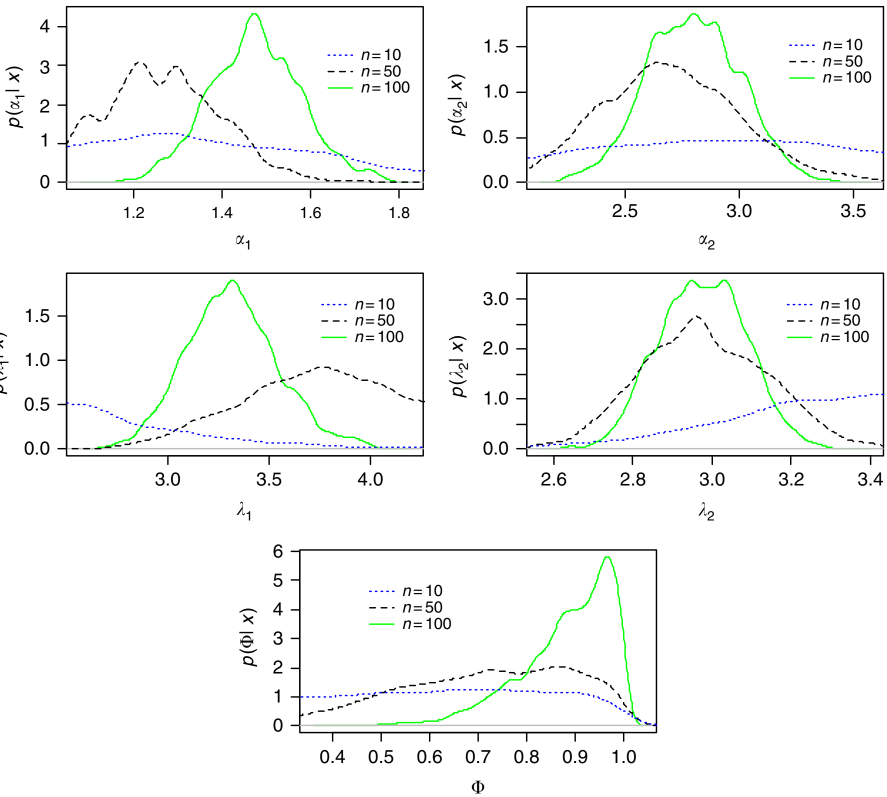

Figure 6. Marginal posterior densities for each parameter of the model II for $n=10$, 50 and 100

Table III presents the posterior summaries of interest obtained from the simulation study.

From the results of Table III considering "model II," we observe similar results for some parameters as obtained considering "model I" (see Table II), except for the dependence parameter Φ, where the 95 percent credible interval is more accurate, especially for $n = 100$ when compared to the 95 percent credible interval for θ assuming FGM copula. It is interesting to point out that FGM copula is only appropriated for weak dependence. In our simulation with simulated data, we have a strong dependence, that is, "model II" is more appropriated to analyze the data set.

In Table IV we have the posterior summaries for $R(1.5)$ assuming the model II for different values of n. Given the fixed values of the parameters for the simulation study we have the true value of $R(1.5)$ given by 0.6083. We observe accurate Bayesian inferences for the reliability function $R(1.5)$, especially for large sample sizes ($n = 50$ and 100).

The posterior means of the reliability function of the system are plotted in Figure 7. We observe that we get accurate Bayesian inferences for the reliabilityfunction for the model II, especially considering large samples sizes.

Table IV.
Estimation of the reliability $R(t_0)$ for $t_0 = 1.5$ (model II)

	Mean	SD	Interval
$n = 10$	0.5739	0.1028	(0.3759, 0.7722)
$n = 50$	0.6083	0.0501	(0.5070, 0.7043)
$n = 100$	0.6085	0.0338	(0.5408, 0.6711)

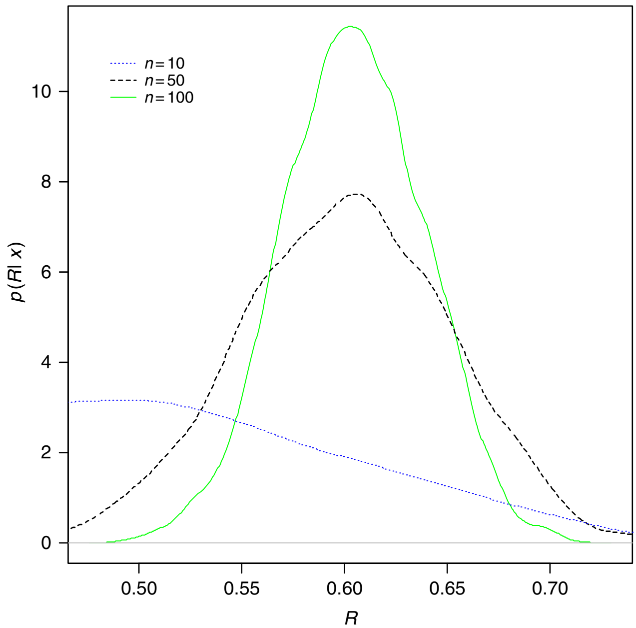

Figure 7.
Plot of the posterior reliability function $R(t_0)$ for $t_0 = 1.5$

In evaluating the reliability of a system, it is often assumed that the components are independent. However, this assumption is not often valid in practice. Then it is important to check the impact of dependence in the reliability of a system. Tables V-VIII show the estimators and credibility Bayesian intervals for the system reliability at time $t = 1.5$, under the assumption of dependence and independence of the components in sample sizes

Sample size	Parameter	Mean	SD	95% credible interval
$n = 10$	α_1	1.6664	0.5454	(0.9301, 3.1558)
	α_2	2.6755	0.8425	(1.5572, 4.6744)
	λ_1	3.5795	0.7613	(2.3233, 5.2678)
	λ_2	3.0004	0.3795	(2.2555, 3.6967)
	θ	–	–	–
$n = 50$	α_1	1.5283	0.1779	(1.2142, 1.9544)
	α_2	2.5261	0.2992	(2.0303, 3.2313)
	λ_1	3.5165	0.3347	(2.8549, 4.1600)
	λ_2	3.0075	0.1745	(2.6669, 3.3525)
	θ	–	–	–
$n = 100$	α_1	1.5075	0.1203	(1.2732, 1.7580)
	α_2	2.5117	0.2058	(2.1518, 2.9803)
	λ_1	3.5065	0.2319	(3.0779, 3.9687)
	λ_2	2.9973	0.1306	(2.7534, 3.2392)
	θ	–	–	–

Table V. Posterior summaries (model I) with independent data

	Dependence		Independence	
	Mean (SD)	Interval	Mean (SD)	Interval
$n = 10$	0.6039 (0.1068)	(0.4031, 0.8108)	0.5977 (0.1182)	(0.3770, 0.8416)
$n = 50$	0.6445 (0.0501)	(0.5447, 0.7464)	0.6267 (0.0557)	(0.5209, 0.7314)
$n = 100$	0.6420 (0.0383)	(0.5670, 0.7157)	0.6279 (0.0392)	(0.5462, 0.7018)

Table VI. Estimation of reliability function (model I) for $t = 1.5$ with independent data

Sample size	Parameter	Mean	SD	95% credible interval
$n = 10$	α_1	1.5766	0.4352	(0.9477, 2.6496)
	α_2	2.6495	0.7334	(1.6139, 4.4778)
	λ_1	3.6157	0.7969	(2.2609, 5.4654)
	λ_2	3.0057	0.4037	(2.1958, 3.7439)
	Φ	–	–	–
$n = 50$	α_1	1.5284	0.1728	(1.2266, 1.9229)
	α_2	2.5333	0.3262	(2.0429, 3.2019)
	λ_1	3.5262	0.3211	(2.8987, 4.1201)
	λ_2	2.9898	0.1770	(2.6502, 3.3275)
	Φ	–	–	–
$n = 100$	α_1	1.5051	0.1197	(1.3057, 1.7549)
	α_2	2.5095	0.2206	(2.1530, 2.9435)
	λ_1	3.5048	0.2581	(2.9839, 4.0037)
	λ_2	2.9961	0.1236	(2.7569, 3.2240)
	ϕ	–	–	–

Table VII. Posterior summaries (model II) with independent data

$n = 10$, 50 and 100. The estimations indicate a large gain with both dependence models mainly with model II. To get more insight in the dependence structure given by models I and II, Figures 8 and 9 display the contour plots of the densities obtained by FGM and Gumbel copulas and assuming independence between the variables (Figure 10).

6. Conclusions

The use of copula functions for modeling the bivariate distribution could be a good alternative to estimate the reliability of a two components series system. Bayesian

Table VIII.
Estimation of reliability function (model II) for $t = 1.5$ with independent data

	Dependence		Independence	
	Mean (SD)	Interval	Mean (SD)	Interval
$n = 10$	0.5739 (0.1028)	(0.3759, 0.7722)	0.5869 (0.0935)	(0.3956, 0.7686)
$n = 50$	0.6083 (0.0501)	(0.5070, 0.7043)	0.6253 (0.0465)	(0.5387, 0.7109)
$n = 100$	0.6085 (0.0338)	(0.5408, 0.6711)	0.6263 (0.0327)	(0.5672, 0.6881)

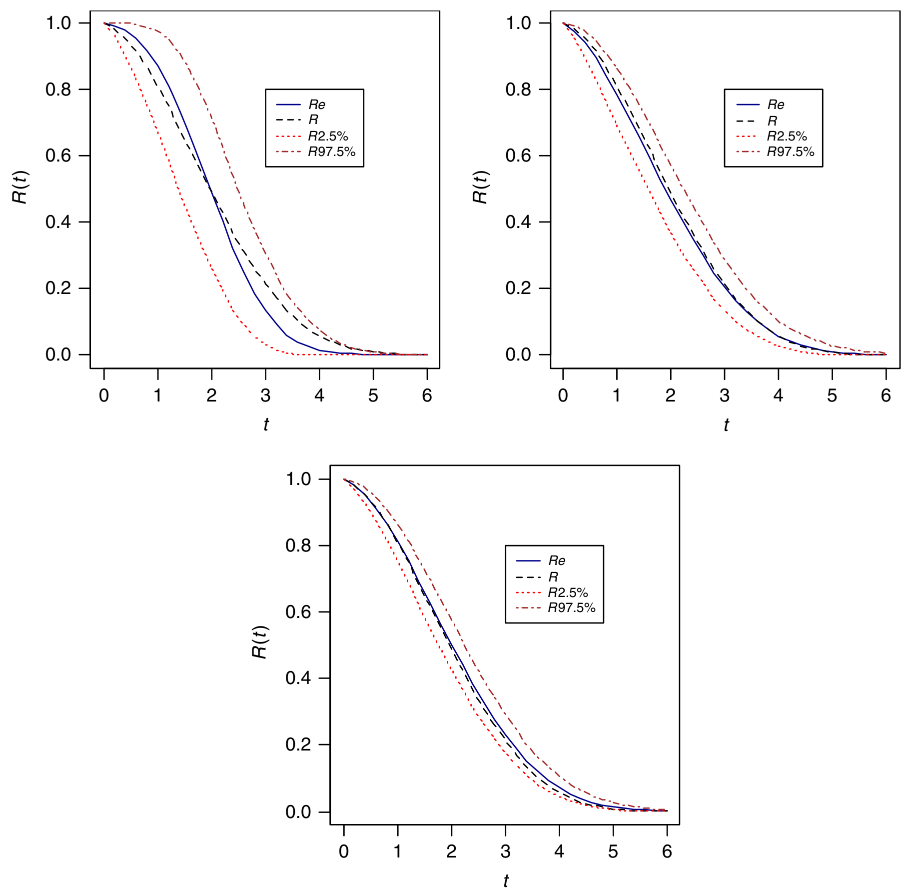

Figure 8.
The mean and 95 percent credible intervals for the reliability function for $n = 10$, 50 and 100

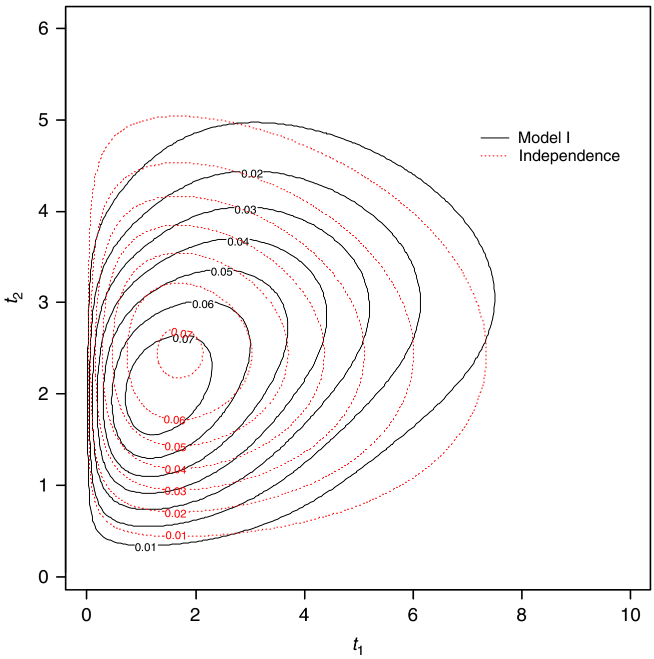

Figure 9.
Plot of the joint Weibull density assuming a FGM copula function and independence

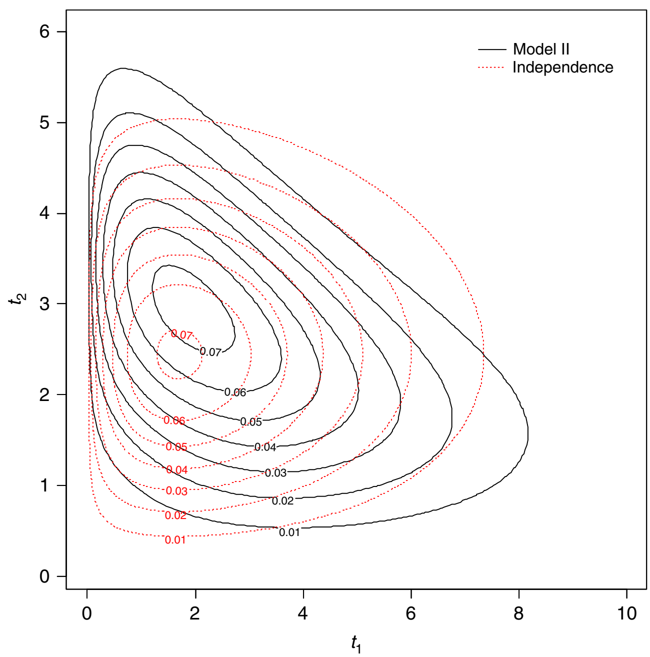

Figure 10.
Plot of the joint Weibull density assuming a Gumbel copula function and independence

inference for this class of models using standard existing MCMC methods is not difficult to apply providing accurate inference results.

Using standard classical inference methods based on asymptotical normality of the maximum likelihood estimators for the parameters λ_1, λ_2, α_1, α_2, θ and Φ we could have great computational difficulties and possibly, not accurate inference results. From the results of this study, we observe that we get accurate Bayesian inferences for the reliability function considering large samples sizes.

The Bayesian parametric models proposed also allow the assessment of system reliability for multicomponent systems simultaneously. For a next study, we are also interested to estimate the reliability of the series system in the presence of censored observations and covariates.

It is important to point out that the choice of appropriate copula function is crucial to analyze the reliability of the systems. Many other copula functions are introduced in literature. A carefully preliminary analysis of the lifetimes in each application or even physical interpretations could be used in the choice of the copula.

The results of this paper could be generalized for any lifetime distribution as, for example, γ, log-normal, log-logistic or generalized γ distributions (see, e.g. Lawless, 1982).

We also could get a generalization of our approach, assuming complex systems with many two dependent component subsystems. These subsystems could be further assumed as independents and also could be of different components with different lifetime probability distributions. These results could be of great practical interest to have better reliability inferences.

References

Aggarwal, K. (1993), *Reliability Engineering*, Kluwer Academic Publishers, Boston, MA.

Amblard, C. and Girard, S. (2002), "Symmetry and dependence properties within a semiparametric family of bivariate copulas", *Journal of Nonparametric Statistics*, Vol. 14 No. 6, pp. 715-727.

Amblard, C. and Girard, S. (2009), "A new extension of bivariate FGM copulas", *Metrika*, Vol. 70 No. 1, pp. 1-17.

Block, H.W. and Basu, A.P. (1974), "A continuous bivariate exponential extension", *Journal of the American Statistical Association*, Vol. 69 No. 348, pp. 1031-1037.

Casella, G. and George, E.I. (1992), "Explaining the gibbs sampler", *American Statistician*, Vol. 46 No. 3, pp. 167-174.

Chan, Y. and Li, H. (2008), "Tail dependence for multivariate t-copulas and its monotonicity", *Insurance: Mathematics and Economics*, Vol. 42 No. 1, pp. 763-770.

Chib, S. and Grenberg, E. (1995), "Understanding the Metropolis-Hastings algorithm", *American Satistician*, Vol. 49 No. 4, pp. 327-335.

Cran, G.W. (1988), "Moment estimators for the 3-parameter Weibull distribution", *IEEE Transactions on Reliability*, Vol. 37 No. 4, pp. 360-363.

Czado, C. (2010), "Pair-copula constructions of multivariate copulas", in Jaworski, P., Durante, F., Hardle, W. and Rychlik, T. (Eds), *Copula Theory and Its Applications*, Springer-Verlag, Berlin, pp. 93-109.

Danish, M.Y. and Aslam, M. (2014), "Bayesian inference for the randomly censored Weibull distribution", *Journal of Statistical Computation and Simulation*, Vol. 84 No. 1, pp. 215-230.

Farley, J.C. (1960), "The performance of some correlation coefficients for a general bivariate distribution", *Biometrika*, Vol. 47, pp. 307-323.

Frahm, G., Junker, M. and Szimayer, A. (2003), "Elliptical copulas: applicability and limitations", *Statistics & Probability Letters*, Vol. 63 No. 3, pp. 275-286.

Franco, M. and Vivo, J.M. (2002), "Reliability properties of series and parallel systems from bivariate exponential models", *Commun. Statist. Theor. Meth*, Vol. 31 No. 12, pp. 2349-2360.

Freund, J.E. (1961), "A bivariate extension of the exponential distribution", *Journal of the American Association Statistical*, Vol. 56 No. 296, pp. 971-977.

Gelfand, A.E. and Smith, A.F.M. (1990), "Sampling based approaches to calculating marginal densities", *Journal of the American Statistical Association*, Vol. 85 No. 410, pp. 398-409.

Genest, C. (1987), "Frank's family of bivariate distributions", *Biometrika*, Vol. 74 No. 3, pp. 549-555.

Genest, C. and MacKay, R.J. (1986a), "Copules archimediennes et familles de lois bidimensionnelles dont les marges sont donnees", *The Canadian Journal of Statistics*, Vol. 14 No. 2, pp. 145-159.

Genest, C. and MacKay, R.J. (1986b), "The joy of copulas: bivariate distributions with uniform marginals", *The American Statistician*, Vol. 40 No. 4, pp. 280-283.

Genest, C. and Rivest, L. (1993), "Statistical inference procedures for bivariate Archimedean copulas", *Journal of the American Statistical Association*, Vol. 88 No. 423, pp. 1034-1043.

Gumbel, E.J. (1960), "Bivariate exponential distributions", *Journal of the American Statistical Association*, Vol. 55 No. 292, pp. 698-707.

Joe, H. (1993), "Parametric families of multivariate distributions with given margins", *Journal of Multivariate Analysis*, Vol. 46 No. 2, pp. 262-282.

Joe, H. (1997), *Multivariate Models and Dependence Concepts*, Chapman and Hall, London.

Kotz, S., Balakrishnan, N. and Johnson, N.L. (2000), *Continuous Multivariate Distributions, Volume 1: Models and Applications*, 2nd ed., John Wiley and Sons, New York, NY.

Kundu, D. and Raqab, M.Z. (2012), "Bayesian inference and prediction for a type-II censored Weibull distribution", *Journal of Statistical Planning & Inference*, Vol. 142 No. 1, pp. 41-47.

Kurowicka, D. and Joe, H. (2011), *Dependence Modeling: Vine Copula Handbook*, World Scientific Publishing Co., Singapore.

Lawless, J.F. (1982), *Statistical Models and Methods for Lifetime Data, Wiley Series in Probability and Mathematical Statistics*, John Wiley & Sons Inc., New York, NY.

McNeil, A.J. and Nešlehová, J. (2009), "Multivariate Archimedean copulas, d-monotone functions and l_1-norm symmetric distributions", *The Annals of Statistics*, Vol. 37 No. 5B, pp. 3059-3309.

Marshall, A.W. and Olkin, I. (1967), "A multivariate exponential distribution", *Journal of American Statistical Association*, Vol. 62 No. 317, pp. 30-41.

Morgenstern, D. (1956), "Einfache beispiele zweidimensionaler verteilungen", *Mitteilurgsb. Math. Statistcs*, Vol. 8, pp. 234-235 (in German).

Murthy, D.N.P., Xie, M. and Jiang, R. (2004), *Weibull Models*, John Wiley, New York, NY.

Nelsen, R.B. (1986), "Properties of a one-parameter family of bivariate distributions with specified marginals", *Communications in Statistics-Theory and Methods*, Vol. 15 No. 11, pp. 3277-3285.

Nelsen, R.B. (2006), *An Introduction to Copulas*, Springer Verlag, New York, NY.

Nelson, W. (1982), *Applied Life Data Analysis*, John Wiley & Sons, New York, NY.

Ng, H.K.T. (2005), "Parameter estimation for a modified weibull distribution for progressively type-II censored samples", *IEEE Transactions on Reliability*, Vol. 54 No. 3, p. 374.

Pradhan, B. and Kundu, D. (2014), "'Thirty years of copulas'", in Dall'Aglio, G., Kotz, S. and Salinetti, G. (Eds), *Advances in Probability Distributions with Given Marginals*, Kluwer Academic Publishers, Dordrecht, pp. 13-50.

Schweizer, B. (1991), "Thirty years of copulas. In advances in probability distributions with given marginals: beyond the copulas", in Dall'Aglio, G., Kotz, S. and Salinetti, G. (Eds), Kluwer Academic Publishers.

Shih, J.H. and Louis, T.A. (1995), "Inferences on the association parameter in copula models for bivariate survival data", *Biometrics*, Vol. 51 No. 4, pp. 1384-1399.

Sirvanci, M. and Yang, G. (1984), "Estimation of the weibull parameters under type I censoring", *Journal of the Americam Statistical Association*, Vol. 79 No. 385, pp. 183-187.

Sklar, M. (1959), "Functions de repartition a n-dimensions et leurs marges", *Publ. Inst. Statist. Univ. Paris*, Vol. 8, pp. 229-231 (in French).

Smith, D.J. (1972), *Reliability Engineering*, Pitman, London.

Song, P.X. (2000), "Multivariate dispersion models generated from gaussian copula", *Scandinavian Journal of Statistics*, Vol. 27 No. 2, pp. 305-320.

Teimouri, M. and Gupta, A.K. (2013), "On the three-parameter Weibull distribution shape parameter estimation", *Journal of Data Science*, Vol. 11 No. 3, pp. 403-414.

Trivedi, P.K. and Zimmer, D.M. (2005a), *Copula Modelling*, New Publishers, New York, NY.

Trivedi, P.K. and Zimmer, D.M. (2005b), "Copula modeling: an introduction to practitioners", *Foundations and Trends in Econometrics*, Vol. 1 No. 1, pp. 1-111.

Wang, F.K. and Keats, J.B. (1995), "Improved percentile estimation for the 2-parameter weibull distribution", *Microelectronics and Reliability*, Vol. 35 No. 5, pp. 883-892.

Weibull, W. (1951), "A statistical distribution function of wide applicability", *J Appl Mech*, Vol. 18 No. 3, pp. 293-297.

Zheng, M. and Klein, J.P. (1995), "Estimates of marginal survival for dependent competing risks based on an assumed copula", *Biometrika*, Vol. 82 No. 1, pp. 127-138.

Corresponding author
Dr Fernando Antonio Moala can be contacted at: femoala@fct.unesp.br

This article has been cited by:

1. OliveiraRicardo Puziol, Ricardo Puziol Oliveira, AchcarJorge Alberto, Jorge Alberto Achcar. 2019. Use of Basu-Dhar bivariate geometric distribution in the analysis of the reliability of component series systems. *International Journal of Quality & Reliability Management* **36**:4, 569-586. [Abstract] [Full Text] [PDF]
2. Hemantha S. B. Herath. 2019. Postauditing and Cost Estimation Applications: An Illustration of MCMC Simulation for Bayesian Regression Analysis. *The Engineering Economist* **64**:1, 40-67. [Crossref]
3. TemrazNeama, Neama Temraz. 2018. Improvement of the reliability of a dependent system under copula. *International Journal of Quality & Reliability Management* **35**:10, 2414-2435. [Abstract] [Full Text] [PDF]