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Cohomological Field Theory in the BV Formalism

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Resumo

O formalismo de Batalin e Vilkovisky (BV) é um dos principais ingredientes de muitas das abordagens que temos para formulações matematicamente precisas de teoria quântica de campos (QFT) perturbativa. Vamos expor este formalismo através de uma reinterpretação cohomologica da equação de Schwinger-Dyson para uma teoria sem simetria de calibre. Isto nos levará a um ponto de vista alternativo das integrais de caminho. Seguindo em analogia com teorias de calibre, vamos codificar de maneira cohomológica as equações de movimento e a estrutura de calibre de teorias clássicas de campos. Os complexos de cocadeias resultantes vão ter como diferenciais os campos vetoriais Hamiltonianos de uma extensão BV da ação. Fixar o calibre nesta nos levará aos análogos quânticos dos complexos clássicos. Os elementos algébricos relevantes vão ser discutidos no contexto de "toy theories" de dimensão finita. Com estas vamos poder proceder de maneira análoga ao formalismo BV de uma partícula relativística, teoria de Yang-Mills, teoria de Chern-Simons, e teoria BF.

Palavras Chaves: Formalismo BV; Simetrias de calibre; Espaços simpléticos ímpares; Cohomologia.

Áreas do conhecimento: Física; Física-Matemática; Teorias de Calibre.

Abstract

The Batalin-Vilkovisky (BV) formalism is a major ingredient in several approaches towards mathematically sound formulations of perturbative quantum field theory (QFT). We will develop this framework by interpreting the Schwinger-Dyson equation for a theory without gauge symmetries in a cohomological fashion. This will lead us to an alternative point of view towards path integration. Mirroring this construction for gauge theories we will be able to encode the equations of motion and the gauge structure of classical field theories cohomologically. The differentials of the resulting cochain complexes will be Hamiltonian vector fields for an extended BV action. We will describe a method to gauge fix this action and obtain a quantum theory. The lifting of gauge invariance to the quantum theory will lead us to the quantum analogues of the classical cochain complexes we constructed. The relevant algebraic elements of this formalism will be developed in the simple case of finite dimensional QFT analogues. These will allow us to proceed by analogy to the BV treatment of the relativistic particle, Yang-Mills theory, Chern-Simons theory, and BF theory.

Keywords: BV Formalism; Gauge symmetries; Odd symplectic spaces; Cohomology.

Areas of knowledge: Physics; Mathematical Physics; Gauge Theories.

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Chapter 1

Introduction

The standard model of particle physics (SM) has provided some of the most accurate predictions to date and some claim it is the most well-tested theory of physics (see for example 11.1 of [1], which makes the claim for quantum electrodynamics, a theory included in the SM). It is a particular instance of what has come to be known as quantum field theory (QFT). Applications of QFT can be found in other fields, such as condensed matter physics and cosmology [2, 3]. This makes QFT one of the pinnacles of modern physics. Much like other theories, practitioners develop familiarity with the theory by examining different examples which have varying degrees of similarity among themselves. However, unlike theories such as quantum mechanics (QM), there is no universally accepted definition of what QFT actually is. A manifestation of this is that in QM we are able to describe sets of axioms [4, 5, 6] which accommodate several known examples and allow us to treat them within the same framework. These axioms are not only of structural interest, for they have allowed us to make general statements about the quantum behaviour of the universe. A simple example of this is Heisenberg's Uncertainty Principle, which relates the noncommutative structure of quantum observables with the incompatibility of their measurements (see Theorem 12.4 of [4]). Moreover, a happy by-product of this exercise has been the development of areas within mathematics. In our running example, functional analysis benefited greatly from its applications to QM. It is thus of fundamental importance to develop rigorous mathematical foundations for QFT that are flexible enough to accommodate the well known applications we currently have, but stringent enough to allow us to probe the theory further.

There have been several attempts in this direction. Let us remark some of them and mention an incomplete account of the important consequences that have followed from each. The Wightman axioms [7] point out general properties that a QFT should have. These accommodate several free field theories while still allowing the study of general properties of QFTs, such as the CPT theorem, the connection between spin and statistics, and the reconstruction of full QFTs from the knowledge of vacuum expectation values, to which a large amount of the applications in particle physics reduces to. The Osterwalder-Schrader axioms [8] yield a Euclidean counterpart to these, allowing the possibility of studying Wick rotations and the relationship between QFTs in Minkowski and Euclidean signatures. Finally, let us also remark the Haag-Kastler axioms [9, 10], which provide a generalization based on nets of algebras of observables now known as algebraic quantum field theory (AQFT). This formalism provides a suitable framework for the study of observables in the presence of gauge symmetries [11, 12], superselection sectors and particle statistics [13, 14], and the relationship between thermal states and time [15]. However, there is currently no proof that some of the important interacting field theories, like the SM, can be described through any of these frameworks.

Although no universally accepted mathematically rigorous definition of QFT is currently available, the perturbative aspects of the theory are much better developed. Alternative approaches to axiomatic QFT have focused on devising rigorous foundations for these. Such attempts have proven more successful at accommodating the existing examples. Among these we can mention the framework of factorization algebras, developed by Costello and Gwilliam [16, 17, 18], and the framework of perturbative algebraic quantum field theory (pAQFT), developed by Brunetti, Dütsch, Fredenhagen and Rejzner [19, 20, 21, 22, 23]. The former combines the assignment of factorization algebras to local regions in spacetime with the renormalization theory developed by Costello [16]. The latter extends the nets of observables considered in AQFT and treats them using Epstein-Glaser renormalization [24]. These approaches share several common ingredients [25]. Among them is the treatment of gauge symmetries through the Batalin-Vilkovisky (BV) formalism [26, 27, 28, 29].

The BV formalism is one of the most general methods we have (if not the most, as claimed in 9.3 of Chapter 1 of [16]) to study quantum theories with gauge symmetries. In what follows we will attempt to give an account of this formalism as a tool for the treatment of such theories in a unified fashion. Roughly speaking, the formalism tackles two problems: the lack of rigorous foundations for path integrals and the evaluation of said integrals on theories with gauge redundancies.

At the classical level, it reduces to a Lagrangian reformulation of the BRST procedure. In it, the usual phase spaces are replaced by the spaces of field configurations, with constraint surfaces of the former being instantaneous snapshots at some spatial hypersurface of the on-shell configurations in the latter. Since no such surface is required in the Lagrangian picture, the BV formalism provides an approach to quantization that deals better with covariance. The space of field configurations is extended to include ghosts and their corresponding antifields, which gives it the structure of a shifted Poisson manifold. As a consequence, the action will induce a shifted Hamiltonian vector field on the extended manifold of field configurations, which plays the distinguished role of encoding the equations of motion, symmetry structures, gauge redundancies, and their representations on the theory. This is done by requiring that the action satisfies the classical master equation. Solving this equation will lead to a modified action from which the gauge-fixed actions obtained by more heuristic methods can be recovered.

At the quantum level, the formalism does an order $\hbar$ perturbation on this Hamiltonian vector field. Requiring that the classical gauge symmetry lifts to the quantum theory, i.e. that the path integral measure be gauge invariant, leads to the quantum master equation. Finding solutions of this will in general require quantum corrections to the solution of the classical master equation. Obstructions to finding solutions then correspond to gauge anomalies in the theory. Moreover, naive interpretations of the quantum master equation will in general lead to ill-defined quantities. Instead, making sense of the quantum master equation will require a renormalization procedure. Much of the work in the frameworks of factorization algebras and pAQFT has been focused on the proper mathematical formulation of the coupling between the BV formalism and renormalization.

Both at the classical and quantum levels, the corresponding master equations are equivalent to the requirement that the Hamiltonian vector field induced by the action, or its quantum perturbation, are differentials in the cohomological sense. Consequently, the BV formalism naturally brings the techniques of cohomological algebra into field theory. In particular, the resulting cohomology groups will encode the structure of the classical and quantum gauge invariant observables, as well as the information needed to study perturbative expansions. The computational power of homological algebra has played a major role in new developments in field theory. An example of this is a proof of the perturbative renormalizability of Yang-Mills (YM) theory on flat four-manifolds with coefficients in semi-simple Lie algebras (see [30] or Theorem 1.0.1 of [16]).

In the BV formalism we assign two kind of graduations to our fields. One of them determines whether the field will behave as a commuting or anticommuting variable. The other determines the physical origin of the field and will play an important role in extracting cohomological information from our theories. The associated mathematical structures that describe these type of objects are those of graded algebra. We will start our discussion in Chapter 2 by giving a thorough review of this language. In here most of the structures that will appear in the BV formalism are presented and the essential theorems regarding the mathematical interpretation of the classical and quantum master equations are proven.

In Chapter 3, we begin by introducing the point of view of the BV formalism towards path integrals. This will be done by trying to find a method to compute finite dimensional gaussian integrals without actually needing to integrate. We will stick to the finite dimensional case throughout this chapter. Not all of the important physical and mathematical aspects of the BV formalism can be described in this case. However, we can justify our approach in several ways. On the one hand, there are several aspects that such models still exhibit and are most easily understood without the difficult subtleties that accompany more realistic field theories. On the other, the formal "generalized Einstein summation convention" of DeWitt [31] makes several of the formulas that will follow identical to the ones presented in usual expositions of the formalism in the physics literature (cf. [32, 33]). Finally, our considerations will be enough to proceed by analogy with several examples of proper field theories. Our approach will be based on a cohomological interpretation of the Schwinger-Dyson equation [see 32, (15.25)]. The classical limit of this construction will yield a cohomology theory which encodes both the equations of motion of the theory and its infinitesimal symmetry structure. This is known as the Koszul complex. Mirroring this we will try to build a cohomological theory of the Lie algebras of our gauge symmetries. This will lead us to the Chevalley-Eilenberg complex. The classical theory will be complete once we unify this two cohomology theories and construct the BV action. Finally, we discuss a procedure for gauge fixing such an action through a gauge fixing fermion. Independence from the gauge fixing fermion of the path integral measure will lead us to the quantum BV formalism.

Finally, we will discuss the applications of this formalism in Chapter 4. After a quick finite dimensional example, we will discuss the extension of the formalism to infinite dimensional theories. We will do so in a rigorous fashion for the scalar field. Afterwards, we will do a brief heuristic discussion of this extension for more general field theories. We will conclude by exhibiting examples of the BV formalism as an alternative of the BRST formalism for diffeomorphism invariant theories and theories of connections on principal bundles. For the first, we will discuss the theory of extended quantum objects. We will particularly focus on the relativistic particle to exemplify the BV formalism. For the second we will study Yang-Mills theory and Chern-Simons theory. We will also discuss abelian BF theory to see how the BV formalism deals with reducible gauge symmetries.

Before we start, let me comment on some important topics that will not be discussed in this work. First of all, the cohomological interpretation of path integrals in the BV formalism will only be discussed for theories without gauge symmetries. Its purpose will be to make natural the construction of the Koszul complex from a physics viewpoint. In particular, once we have constructed the classical BV complex, we will introduce its quantum counterpart and its physical significance using heuristically the usual point of view towards the path integral. Secondly, although motivated by the use of the BV formalism in factorization algebras and pAQFT, a discussion of these is outside the scope of this thesis. Finally, we will not attempt to find solutions of the quantum master equation or study the problem of renormalization in this context.

Chapter 2

Graded Linear Algebra

2.1 Graded Vector Spaces

The formalism we will explore is based on an extension of linear algebra based on a grading. After giving a general definition we will follow with some foresight to the reader on its importance.

Definition 2.1.1 ([Special case of 34, Definition 3])**.** Let G be an Abelian group. A (G-)graduation on a vector space V is a collection of subspaces $\{ V^i \mid i \in G \}$ such that $V = \bigoplus_{i \in G} V^i$. A vector space equipped with such a graduation is said to be a (G-)graded vector space. The elements in V^i are called homogeneous of degree i. This yields a map $p_V : \bigcup_{i \in G} V^i \setminus \{0\} \to G$, defined by $v \in V^{p_V(v)}$ for all homogeneous v. A basis α of V consisting solely of homogeneous elements is said to be adapted. If there is a subset $S \subseteq G$ such that $V^i = \{0\}$ for all $i \in G \setminus S$, we say that V is concentrated in S.

Remark 2.1.1. To avoid cluttering our notation, we will use $p \equiv p_V$ whenever the graded vector space in question is clear.

Example 2.1.1. ${}^{\mathbb{Z}}\!/\!{}_{2\mathbb{Z}}$-graded vector spaces are called super vector spaces. We will prefer the snotation $|\cdot| \equiv p$ for these spaces and we will use the term parity instead of degree. We will also denote the graduation using lower indices $\{V_0, V_1\}$. The elements of V_0 are said to even bosonic while the elements of V_1 are said to be fermionic. Having this structure will be useful when we construct algebras from V. We will use the bosonic elements as commuting and the fermionic elements as anticommuting.

Remark 2.1.2. We should note that during this work our use of bosonic and fermionic is not tied in principle to any notion of spin. These two are usually related by the spin-statistics theorem [see 35, Theorem 4-10]. However, we will encounter in our procedure auxiliary fields that do not comply with this theorem.

Example 2.1.2. $\mathbb{Z}$-graded vector spaces are at the core of the theory of cohomology. We will see this in Definition 2.1.3. In these we usually use the notation $\deg \equiv p$. We will use them to distinguish fields with different physical meanings. The elements in degree 0 will correspond to the fields we start with. In terms of these we will have our initial description of the physics of the system. Associated to each of these we will build new fields, called antifields, which will sit in degree 1. This will be of importance in order to give us a Poisson structure to work with. These antifields will play a role with respect to the fields similar to the one played by the momenta with respect to the positions in classical mechanics. Their grading will be chosen so that the cohomology will capture certain physically important data.

In order to couple symmetries to our theory, we will interpret the parameters of such symmetries as new fields, called ghosts. These will sit in degree -1, guaranteeing that they fit along with the other infinitesimal transformations of the theory (see (3.82)). In order to recover a Poisson structure, we will introduce antifields for these too, which will in turn sit in degree 2. Redundancies in these symmetries, their redundancies, and so on, will populate the degrees less than -1, while their antifields will sit in degrees bigger than 2.

Example 2.1.3. The reader may by now suspect we will need $\mathbb{Z}/_{2\mathbb{Z}} \times \mathbb{Z}$-graduations. This is indeed the case. However, since $\mathbb{Z}$ and $\mathbb{Z}/_{2\mathbb{Z}}$ graduations are interesting on their own, we will simply develop the theory for general Abelian groups, which adds little difficulty. Having however both graduations is in fact crucial since they will have to interact together in our theory. For example, in order to obtain a nice interpretation to our cohomologies, we will assign to the antifields a different parity than their corresponding fields. On the other hand, bosonic symmetries of some theory (see Definition 2.1.2 for the degree/parity of a transformation) will be described by fermionic ghosts.

$\mathbb{Z}/_{2\mathbb{Z}} \times \mathbb{Z}$-graded vector spaces will be called bigraded vector spaces. The degree map has two projections. We will use the notation $|\cdot|$ for the $\mathbb{Z}/_{2\mathbb{Z}}$ projection and deg for the $\mathbb{Z}$ one. These are then defined by $p(v) = (|v|, \deg v)$ for all homogeneous $v \in V$.

Every super vector space can be understood as a bigraded vector space by setting $\deg = 0$. From now on, we will always do this implicitly whenever no other bigraduation is given. Similarly, every $\mathbb{Z}$-graded vector space can be understood as a bigraded vector space by setting $|\cdot| = 0$. However, for the signs below to coincide with those usually used in the derived geometry literature, it is better to set $|\cdot| \equiv \deg \mod 2$ in these type of vector spaces.

Let us go back to consider a general Abelian group G. We will always use additive notation for these.

Example 2.1.4. Any vector space V can be regarded as a G-graded vector space by setting $V^0 = V$ and $V^i = \{0\}$ for all $i \in G \setminus \{0\}$. We will do this whenever a vector space is not already equipped with a grading.

Whenever we introduce new mathematical objects we would like to identify the structure preserving maps.

Definition 2.1.2. A linear map $T : V \to W$ between two G-graded vector spaces is said to be homogeneous of degree $p(T) = k \in G$ if $TV^i \subseteq W^{i+k}$. The set of such maps is denoted by $\mathrm{Hom}^k(V, W)$. If the degree of a homogeneous map is omitted, we imply it has degree 0.

Theorem 2.1.1. Given two G-graded vector spaces V and W,

$$\mathrm{Hom}(V, W) = \bigoplus_{k\in G} \mathrm{Hom}^k(V, W) \tag{2.1}$$

makes $\mathrm{Hom}(V, W)$ a graded vector space.

Proof. Let $v \in V^i$. The grading of W guarantees that for every $j \in G$ there exist unique $w^j \in W^j$ such that $\left\{ w^j \,\middle|\, j \in G \quad \text{and} \quad w^j \neq 0 \right\}$ is finite and $Tv = \sum_{j\in G} w^j$. For every $k \in G$ let $T^k v := w^{i+k}$. We then have $\sum_{k\in G} T^k v = \sum_{k\in G} w^{i+k} = \sum_{j\in G} w^j = Tv$. Repeating this for every $v \in V$ we obtain a degree k linear $T^k : V \to W$ for every $k \in \mathbb{Z}$ such that $T = \sum_{k\in G} T^k$. This shows that the space of linear maps is indeed the sum of the grading we are trying to impose. This sum is clearly direct since it is impossible for a non-zero homogeneous linear map to have two degrees. $\square$

Corollary 2.1.2. Let V be a G-graded vector space. Then $(V^*)^i \cong (V^{-i})^*$ canonically.

Proof. First, recall that $V^* := \mathrm{Hom}(V, \mathbb{F})$, with $\mathbb{F}$ the field where V is defined. Since $\mathbb{F}$ is a G-graded vector space concentrated at degree 0, the previous theorem establishes the grading on V^*. Now, every $f \in (V^{-i})^*$ can be canonically extended to V by setting it equal to 0 at all other degrees, i.e. declaring that $f(v) = 0$ if $p(v) \neq -i$. The resulting map is in $(V^*)^i$ since $f(V^{-i}) \subseteq \mathbb{F} = \mathbb{F}^0 = \mathbb{F}^{-i+i}$, while $f(V^j) = \{0\} = \mathbb{F}^{j+i}$ for all $j \in G \setminus \{i\}$.

On the other hand, every map $f \in (V^*)^i$ satisfies the above inclusions. It can thus be restricted to a map in $(V^{-i})^*$. □

Remark 2.1.3. Let V, W, Z be graded vector spaces and $T \in \mathrm{Hom}(V, W)^i$ while $S \in \mathrm{Hom}(W, Z)^j$. Then it is clear that $S \circ T \in \mathrm{Hom}(V, Z)^{i+j}$.

At the core of cohomology theory is the notion of a cochain complex. In order to continue our discussion in general terms we will assume from now on that our graded vector spaces are equipped with a homomorphism $\deg : G \to \mathbb{Z}$. Precomposing this with the degree map, we obtain a map $\deg : \bigcup_{i\in G} V^i \setminus \{0\} \to \mathbb{Z}$, which we will denote by the same symbol. We will call this the cohomological degree. This of course induces a $\mathbb{Z}$-graduation on V. The resulting $\mathbb{Z}$-graded vector space will be denoted $V_{\deg}$ and we have

$$V^i_{\deg} = \operatorname{span} \deg^{-1}(i). \tag{2.2}$$

In agreement with Example 2.1.3, in the case of $G = {\mathbb{Z}}/{2\mathbb{Z}} \times \mathbb{Z}$ we will always assume this deg map is the projection of the degree map onto the second component.

Definition 2.1.3. Let V be a graded vector space. A differential is a degree k map $\mathrm{d} : V \to V$ such that $\deg k = 1$ annd $\mathrm{d}^2 = 0$. The pair (V, d) is called a cochain complex. We will use diagrams of the form

$$\cdots V^{i-1}_{\deg} \xrightarrow{\mathrm{d}^{i-1}} V^i_{\deg} \xrightarrow{\mathrm{d}^i} V^{i+1}_{\deg} \xrightarrow{\mathrm{d}^{i+1}} \cdots, \tag{2.3}$$

to describe them. Elements in $\operatorname{im} \mathrm{d}$ are called exact, while elements in $\ker \mathrm{d}$ are called closed. Note that $B^k := \operatorname{im} \mathrm{d} \cap V^k_{\deg} \subseteq \ker \mathrm{d} \cap V^k_{\deg} =: Z^k$ for all $k \in \mathbb{Z}$. We thus define the cohomology $H^k(V, \mathrm{d}) := {Z^k}/{B^k}$ in degree k.

We are also interested in creating new such objects out of previous ones. We have already seen how to do this by taking spaces of linear maps. We will now consider direct sums, tensor products, and shifts.

Theorem 2.1.3. Given two graded vector spaces V and W, the following gives the direct sum the structure of a graded vector space

$$(V \oplus W)^i = V^i \oplus W^i \tag{2.4}$$

Proof. This is clear from the associativity and commutativity of the direct sum. □

Theorem 2.1.4. Given two graded vector spaces V and W, the following decomposition defines a grading of their tensor product

$$(V \otimes W)^i := \bigoplus_{j \in G} (V^j \otimes W^{i-j}). \tag{2.5}$$

Proof. This is clear from the distributivity of the tensor product over the direct sum [see 36, Corollary 2.2 of Chapter XVI]. □

The tensor product will be our most useful tool to produce algebras out of graded vector spaces. In order to do this we need to be able to multiply several elements together. This is achieved through the isomorphism

$$\begin{aligned} (U \otimes V) \otimes W &\to U \otimes (V \otimes W) \\ (u \otimes v) \otimes w &\mapsto u \otimes (v \otimes w). \end{aligned} \tag{2.6}$$

Moreover, we want our algebras to reflect the fermionic and bosonic nature of their elements. For this we will need to modify the usual commutativity isomorphism to a *graded commutativity* isomorphism. In order to do this, we need a super vector space structure. Following the same procedure we used to introduce cochain complexes, will from now on assume our graded vector spaces are equipped with a homomorphism $|\cdot| : G \to {}^{\mathbb{Z}}\!/_{2\mathbb{Z}}$. Precomposing this with the degree map, we obtain a map $|\cdot| : \bigcup_{i \in G} V^i \to {}^{\mathbb{Z}}\!/_{2\mathbb{Z}}$, which we will denote by the same symbol. We will call this the parity. Then the graded commutativity isomorphism takes the form

$$\begin{aligned} V \otimes W &\to W \otimes V \\ v \otimes w &\mapsto (-1)^{|w||v|} w \otimes v \end{aligned} \tag{2.7}$$

Thus far we have given the category of super vector spaces with even linear maps the structure of a symmetric monoidal category [see 37, Chapter XI]. Although we will refrain from the use of this language, this is the mathematical structure behind the intricate sign manipulations that will follow.

As a simple example of how this signs appear, consider the identification of $V^{**} \cong V$. In order for us to do this, we have to stablish the action of V on V^* via a pairing

$$V \otimes V^* \to \mathbb{F}. \tag{2.8}$$

However, we already have an action of V^* on V, given by the pairing

$$\begin{aligned} V^* \otimes V &\to \mathbb{F} \\ f \otimes v &\mapsto f(v). \end{aligned} \tag{2.9}$$

Thus, we obtain the map we are seeking by application of (2.7)

$$V \otimes V^* \to V^* \otimes V \to \mathbb{F}, \tag{2.10}$$

under which

$$v \otimes f \mapsto (-1)^{|f||v|} f \otimes v \mapsto (-1)^{|f||v|} f(v) =: v(f). \tag{2.11}$$

A similar example is in the identification of $V^* \otimes W^*$ with $(V \otimes W)^*$. In this case, the pairing is obtained by application of assocativity and graded commutativity

$$(V^* \otimes W^*) \otimes (V \otimes W) \to (V^* \otimes V) \otimes (W^* \otimes W) \to \mathbb{F}, \tag{2.12}$$

under which

$$\begin{aligned} (f \otimes g) \otimes (v \otimes w) &\mapsto (-1)^{|g||v|} (f \otimes v) \otimes (g \otimes w) \\ &\mapsto (-1)^{|g||v|} f(v) g(w) =: (f \otimes g)(v \otimes w). \end{aligned} \tag{2.13}$$

As a last example for this remark, note the fact that $\mathrm{Hom}(V, W)$ is isomorphic to both $W \otimes V^*$ and $V^* \otimes W$ whenever the dimensions of V and W are finite. We will make the identification $\mathrm{Hom}(V, W) \cong W \otimes V^*$, so that our linear maps will *act from the left*. Indeed, if $(v_1, \ldots, v_n)$ is a basis of V and $(v^1, \ldots, v^n)$ is its dual, this identification takes the form

$$f(v) = v^i(v) f(v_i) =: (f(v_i) \otimes v^i)(v). \tag{2.14}$$

On the other hand, it introduces a sign when consider the dual of a linear map. Indeed,

$$\mathrm{Hom}(V, W) \to W \otimes V^* \to V^* \otimes W \to V^* \otimes W^{**} \to \mathrm{Hom}(W^*, V^*) \tag{2.15}$$

defines the dual $f^* \in \mathrm{Hom}(W^*, V^*)$ of $f \in \mathrm{Hom}(V, W)$. In the second and third arrow of this chain we have to be careful to introduce the correct signs. In particular, we obtain from the first two arrows

$$f \mapsto f(v_i) \otimes v^i \mapsto (-1)^{|v_i|(|f|+|v_i|)} v^i \otimes f(v_i). \tag{2.16}$$

In here we used the fact that $|v^i| = |v_i|$, which is clear form Corollary 2.1.2 and the fact that $|\cdot| : G \to \mathbb{Z}/2\mathbb{Z}$ is a homomorphism. Through the third and fourth arrows, this defines a map f^* whose action on a $g \in W^*$ is

$$\begin{aligned} f^*(g) &= (-1)^{|v_i|(|f|+|v_i|)} f(v_i)(g) v^i = (-1)^{(|v_i|+|g|)(|f|+|v_i|)} g(f(v_i)) v^i \\ &= (-1)^{|f||g|} g(f(v_i)) v^i = (-1)^{|f||g|} g \circ f. \end{aligned} \tag{2.17}$$

In the last manipuation of the signs we used that $g(f(v_i)) \neq 0$ implies $|g| + |f| + |v_i| = 0$.

Remark 2.1.4. As consistency of notation suggests, in the $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}$ case we will choose the homomorphism $G \to \mathbb{Z}/2\mathbb{Z}$ so that $|\cdot|$ coincides with the projection of p onto the $\mathbb{Z}/2\mathbb{Z}$ factor.

Definition 2.1.4. Let V be a graded vector space and $k \in G$. We define the shifted graded vector space $V[k]$ as the vector space V with the grading

$$V[k]^i := V^{i+k}, \tag{2.18}$$

for all $i \in G$.

Remark 2.1.5. Given a graded vector space V and $k \in \mathbb{Z}$, we have

$$V^{p_V(v)} \ni v \in V[k]^{p_{V[k]}(v)} = V^{p_{V[k]}(v)+k}, \tag{2.19}$$

that is, $p_{V[k]} = p_V - k$. In super vector spaces this is trivial since addition and subtraction coincide. In this case the only possible shift we have is $\Pi V \equiv V[1]$. However, the sign in this formula is an important (although arbitrary) choice we have made for $\mathbb{Z}$-graded vector spaces.

Example 2.1.5. Let V be a vector space with no grading. Then $V[-k]$ is the graded vector space which has V in degree k and $\{0\}$ in all other degrees.

Let V be a graded vector space and $k \in G$. In the following several signs that will appear stem from noticing that $V[k]$ is isomorphic to both $\mathbb{F}[k] \otimes V$ and $V \otimes \mathbb{F}[k]$. However, the isomorphism between the latter is fixed by the isomorphism (2.7). We will adopt by convention to make the identification $V[k] \cong \mathbb{F}[k] \otimes V$. We can keep track of the signs coming from this choice by the use of a symbol s^k which generates $\mathbb{F}[k]$ and thus has $p(s^k) = -k$. This symbol is fermionic or bosonic depending on the parity $|k|$. It implements the identification by denoting the counterpart in $V[k]$ of $v \in V$ by $s^k \otimes v \in \mathbb{F}[k] \otimes V$. The fact that $V[k][l] = V[k+l]$ can be obtained by identifying $s^k \otimes s^l \cong s^{k+l}$.

As a first consequence of this choice, note that the identification $V[k]^* \cong V^*[-k]$ is obtained through the pairing

$$\begin{aligned} V^*[k] \otimes V[-k] &\to \mathbb{F}[k] \otimes V^* \otimes \mathbb{F}[-k] \otimes V \to \mathbb{F}[k] \otimes \mathbb{F}[-k] \otimes V^* \otimes V \\ &\to V^* \otimes V \to \mathbb{F}, \end{aligned} \tag{2.20}$$

which yields

$$\begin{aligned} f \otimes v &\to s^k \otimes f \otimes s^{-k} \otimes v \to (-1)^{|k||f|_{V^*}} s^k \otimes s^{-k} \otimes f \otimes v \\ &\to (-1)^{|k||f|_{V^*}} f \otimes v \to (-1)^{|k||f|_{V^*}} f(v). \end{aligned} \tag{2.21}$$

In other words, we have the identification

$$\begin{aligned} V^*[k] &\to V[k]^* \\ f &\mapsto (-1)^{|k||f|} f. \end{aligned} \tag{2.22}$$

As a second consequence, note the isomorphism of finite dimensional vector spaces

$$\begin{aligned} \mathrm{Hom}(V[k], W[l]) &\to W[l] \otimes (V[k])^* \to W[l] \otimes V^*[-k] \\ &\to \mathbb{F}[l] \otimes W \otimes \mathbb{F}[-k] \otimes V^* \to \mathbb{F}[l-k] \otimes W \otimes V^* \\ &\to \mathrm{Hom}(V, W)[l-k]. \end{aligned} \tag{2.23}$$

Using a basis $(v_1, \ldots, v_n)$ for V and taking $(v^1, \ldots, v^n)$ to be its dual, we have

$$\begin{aligned} f &\mapsto f(v_i) \otimes v^i \mapsto (-1)^{|k||v_i|_V} f(v_i) \otimes v^i \\ &\mapsto (-1)^{|k||v_i|_V} s^l \otimes f(v_i) \otimes s^{-k} \otimes v^i \\ &\mapsto (-1)^{\cancel{|k||v_i|_V} + |k|(|f|_{\mathrm{Hom}(V,W)} + \cancel{|v_i|_V})} s^{l-k} \otimes f(v_i) \otimes v^i \\ &\mapsto (-1)^{|k||f|_{\mathrm{Hom}(V,W)}} f. \end{aligned} \tag{2.24}$$

Remark 2.1.6. For bigraded vector spaces V a shift by $k = (k_1, k_2) \in \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}$

will be denoted by $V[k] \equiv V[(k_1, k_2)] =: V[k_1, k_2]$. Most of the shifts that will appear in the BV formalism will in fact satisfy $k_1 \equiv k_2 \mod 2$. Therefore, we will from now on use the notation $(n) = (n, n)$ for all $n \in \mathbb{Z}$. Similarly, we will use $V[n] := V[n, n]$.

2.2 Graded Symplectic Vector Spaces

We will begin by introducing the symplectic structure from which we will develop the BV formalism.

Definition 2.2.1 ([slightly generalized version of 38, Definition 2.4.1]). A symplectic structure of degree $k \in G$ on a graded vector space V over a field $\mathbb{F}$ is a bilinear map $\omega : V \times V \to \mathbb{F}$ such that:

1.
$$\omega(V^i, V^j) \subseteq \mathbb{F}^{i+j+k} := \begin{cases} \mathbb{F}, & i+j+k=0 \\ \{0\}, & i+j+k \neq 0, \end{cases} \tag{2.25}$$

2. for all homogeneous $v, u \in V$ we have
$$\omega(v, u) = -(-1)^{|v||u|}\omega(u, v), \tag{2.26}$$
and

3. for all homogeneous $v \in V \setminus \{0\}$ the map $\omega(v, \cdot) : V^{-p(v)-k} \to \mathbb{F}$ in $(V^*)^{p(v)+k}$ is not zero.

Remark 2.2.1 ([extension of 38, Remark 2.4.2]). Note that in Definition 2.2.1 we do not require that the degree k map

$$\begin{aligned} \cdot^\flat : V &\to V^* \\ v &\mapsto v^\flat := \omega(v, \cdot), \end{aligned} \tag{2.27}$$

is invertible. In the finite dimensional case, the fact that the map is not null already assures that it is an isomorphism via the dimension theorem [see 39, Theorem 2.3]. In particular, we get the constraint

$$\dim V^i = \dim(V^*[k])^i = \dim(V^*)^{i+k} = \dim(V^{-i-k})^* = \dim V^{-i-k}, \tag{2.28}$$

and the degree $-k$ isomorphism

$$\cdot^{\sharp} := (\cdot^{\flat})^{-1} : V^* \to V. \tag{2.29}$$

Remark 2.2.2. By the definition of a tensor product, establishing a bilinear map $\omega : V \times V \to \mathbb{F}$ is equivalent to establishing a linear map usually denoted by the same symbol $\omega : V \otimes V \to \mathbb{F}$. With the grading we have given to tensor products, condition 1. corresponds to this linear map being of degree k.

Example 2.2.1. Let $\mathcal{V}$ be a bigraded vector space. Then $\mathcal{E} = \mathcal{V} \oplus \mathcal{V}^*[-1]$ has a degree (-1) symplectic form

$$\sigma(\Phi + \Phi^*, \Psi + \Psi^*) := \Psi^*(\Phi) - (-1)^{|\Phi^*|_{\mathcal{V}^*[-1]}|\Psi|_{\mathcal{V}}}\Phi^*(\Psi) \tag{2.30}$$

for all homogeneous $\Phi, \Psi \in \mathcal{V}$ and $\Phi^*, \Psi^* \in \mathcal{V}^*$. To study its degree, let us further assume that the entries are homogeneous

$$\begin{aligned} p_{\mathcal{V}}(\Phi) &= p_{\mathcal{V}^*[-1]}(\Phi^*) = k, \\ p_{\mathcal{V}}(\Psi) &= p_{\mathcal{V}^*[-1]}(\Psi^*) = l. \end{aligned} \tag{2.31}$$

notice that if the symplectic form doesn't vanish we must have

$$p_{\mathcal{V}^*[-1]}(\Psi^*) + (-1) = p_{\mathcal{V}^*}(\Psi^*) = -p_{\mathcal{V}}(\Phi), \tag{2.32}$$

or

$$p_{\mathcal{V}^*[-1]}(\Phi^*) + (-1) = p_{\mathcal{V}^*}(\Phi^*) = -p_{\mathcal{V}}(\Psi). \tag{2.33}$$

Of course both equations are equivalent due to the restrictions in the degrees above. This means that the symplectic form is non-trivial only for elements satisfying $k + l + (-1) = 0$. We thus conclude that $p(\sigma) = (-1)$. On the other hand, the sign between both terms is chosen so that the correct skew symmetry is attained.

Let $(\Phi_1^*, \ldots, \Phi_n^*)$ be an adapted basis for $\mathcal{V}$ and $(\Phi^1, \ldots, \Phi^n)$ be its dual. The notation in here is chosen since the first basis yields linear coordinates on $\mathcal{V}^*$ while the second yields linear coordinates on V. Then

$$\begin{aligned} \sigma(\Phi_A^*, \Phi^B) = \delta^B{}_A &= -(-1)^{|\Phi_A^*|_{\mathcal{V}}|\Phi^B|_{\mathcal{V}^*[-1]}}\sigma(\Phi^B, \Phi_A) \\ &= -(-1)^{|\Phi_A^*|_{\mathcal{V}}(|\Phi^B|_{\mathcal{V}^*}+1)}\sigma(\Phi^B, \Phi_A^*) = -\sigma(\Phi^B, \Phi_A^*), \end{aligned} \tag{2.34}$$

while $\sigma(\Phi_A^*, \Phi_B^*) = \sigma(\Phi^A, \Phi^B) = 0$. To express the musical isomorphism induced by this symplectic structure, let $(\Phi^1, \ldots, \Phi^n, \Phi^*_1, \ldots, \Phi^*_n)$ be the basis of $\mathcal{E}^*$ dual to $(\Phi^*_1, \ldots, \Phi^*_n, \Phi^1, \ldots, \Phi^n)$. Then, this symplectic form induces the isomorphism of degree (-1).

$$\begin{aligned} \flat : \mathcal{E} &\to \mathcal{E}^* \\ \Phi^*_A &\mapsto \Phi^*_A \\ \Phi^A &\mapsto -\Phi^A \end{aligned} \tag{2.35}$$

$\mathcal{E}$ is called the shifted cotangent bundle of $\mathcal{V}$. Let us anticipate that $\mathcal{V}$ will be the structure containing the fields we will start with and the ghosts in various degrees. On the other hand $\mathcal{V}^*[-1]$ will contain all of the antifields.

2.3 Graded Algebras

Definition 2.3.1. A graded algebra $\mathcal{A}$ is an algebra with a collection of subspaces $\{ \mathcal{A}^i \mid i \in G \}$ that makes it a graded vector space and that satisfies $\mathcal{A}^i\mathcal{A}^j \subseteq \mathcal{A}^{i+j}$.

A trivial example of these algebras is the ring of polynomial functions on a vector space.

Example 2.3.1. Let V be a vector space. Then $S^\bullet V := \oplus_{k\in\mathbb{N}} S^n V$ is a commutative bigraded algebra with the symmetric tensor product and the grading obtained by declaring it concentrated in $\{0\} \times \mathbb{Z}$ and setting

$$(S^\bullet V)^{(0,i)} = \begin{cases} S^i V & i \geq 0 \\ \{0\} & i < 0. \end{cases} \tag{2.36}$$

Once a basis $(v_1, \ldots, v_n)$ has been chosen for V, one can identify this graded algebra with the algebra

$$S^\bullet V \cong \mathbb{F}[v_1, \ldots, v_n] \big/ \langle v_i v_j - v_j v_i \rangle \tag{2.37}$$

of polynomials with coefficients in $\mathbb{F}$, the field over which V is defined, on the symbols $v_1, \ldots, v_n$ which satisfy $v_i v_j = v_j v_i$ for all $i, j \in \{1, \ldots, n\}$. The ring of polynomial function on V is defined to be $S^\bullet(V^*)$. Given elements $f_1, \ldots, f_k \in V^*$, the element $f_1 \cdots f_k \in S^\bullet(V^*)$ is usually interpreted as a map

$$\begin{aligned} F : V &\to \mathbb{F} \\ v &\mapsto f_1(v) \cdots f_n(v). \end{aligned} \tag{2.38}$$

This is however not adequate for an extension to anticommuting objects. A more convenient interpretation is obtained as

$$\begin{aligned} G : S^k V &\to \mathbb{F} \\ v_1 \cdots v_k &\mapsto \frac{1}{k!} \sum_{\sigma \in S_n} f_1(v_{\sigma(1)}) \cdots f_k(v_{\sigma(k)}). \end{aligned} \tag{2.39}$$

Both actions carry the same information. Indeed, $F(v) = G(v \cdots v)$. On the other hand, $G(v_1 \cdots v_k) = \frac{1}{k!} D_{v_1} \cdots D_{v_k} F$. We can proof this by induction on n. Taking $k = 1$, we have that $F = f_1$ is linear and thus $D_{v_1} F = f_1(v_1) = G(v_1)$. Now, assume that we have proven it for $k-1$. Then

$$\begin{aligned} D_{v_1} \cdots D_{v_k} F &= D_1 \cdots D_{v_{k-1}} \sum_{i=1}^{k} f_1 \cdots f_{i-1} f_i(v_k) f_{i+1} \cdots f_k \\ &= \sum_{i=1}^{k} \sum_{\sigma \in S_{k-1}} f_1(v_{\sigma(1)}) \cdots f_{i-1}(v_{\sigma(i-1)}) f_i(v_k) f_{i+1}(v_{\sigma(i)}) \cdots f_n(v_{\sigma(k-1)}). \end{aligned} \tag{2.40}$$

Now, for every $i \in \{1, \ldots, k\}$ and $\sigma \in S_{k-1}$ we can define $\mu \in \{1, \ldots, k\}$ by

$$\mu(j) := \begin{cases} \sigma(j) & j \in \{1, \ldots, i-1\} \\ k & j = i \\ \sigma(j-1) & j \in \{i+1, \ldots, k\}, \end{cases} \tag{2.41}$$

which gives us a bijection $\{1, \ldots, k\} \times S_{k-1} \to S_k$. We thus conclude

$$D_{v_1} \cdots D_{v_n} F = \sum_{\mu \in S_n} f_1(v_{\mu(1)}) \cdots f_n(v_{\mu(k)}) = k! G(v_1 \cdots v_n). \tag{2.42}$$

Another standard example of such algebras comes from the exterior algebra. This example shows how the grading helps us keep track of which elements commute and which anticommute.

Example 2.3.2. Let V be a vector space. Then $\bigwedge^\bullet V := \bigoplus_{k \in \mathbb{N}} \bigwedge^k V$ is a commutative graded algebra with the wedge product and the graduation obtained by

$$\left(\bigwedge\nolimits^\bullet V\right)^{(i,j)} = \begin{cases} \bigwedge^j V, & j \geq 0 \quad \text{and} \quad i \equiv j \mod 2 \\ \{0\} & i < 0 \quad \text{or} \quad i \not\equiv j \mod 2. \end{cases} \tag{2.43}$$

Once a basis $(v_1, \ldots, v_n)$ has been chosen for V, one can identify this graded algebra with the algebra

$$\bigwedge^{\bullet} V \cong \mathbb{F}[v_1, \ldots, v_n] \big/ \langle v_i v_j + v_j v_i \rangle \tag{2.44}$$

of polynomials with coefficients in $\mathbb{F}$, the field over which V is defined, on the symbols $v_1, \ldots, v_n$ which satisfy $v_i v_j = -v_j v_i$ for all $i, j \in \{1, \ldots, n\}$. In particular, $\bigwedge^{\bullet} V^*$ mimics the ring of polynomial functions over a vector space V whose symbols anticommute. Heuristically speaking, in the commuting case this ring of polynomial functions usually only approximates the algebra of smooth functions on the vector space. However, in the anticommuting case functions always have to be of this form. This is because their Taylor series expansion in any of the variables terminates at order 1

$$f(v) = a + bv, \tag{2.45}$$

since $v^2 = 0$.

Example 2.3.3. The previous examples generalize to graded vector spaces V. We define their symmetric algebra to be the commutative graded algebra

$$S^{\bullet} V := V^{\otimes \bullet} \Big/ \left\langle \left\{ v \otimes w - (-1)^{|v||w|} w \otimes v \;\middle|\; v, w \in \bigcup_{i \in G} V^i \right\} \right\rangle. \tag{2.46}$$

We will systematically avoid any tensor product signs in such a space from now on. Given homogeneous $v_1, \ldots, v_n \in V$, we declare $p(v_1 \cdots v_n) = \sum_{i=1}^{n} p(v_i)$. This gives $S^{\bullet} V$ its grading. Moreover, we define the subspaces

$$S^k V = \operatorname{span} \{ v_1 \cdots v_k \mid v_1, \ldots, v_k \in V \text{ are homogeneous} \}. \tag{2.47}$$

Given the intuition gained from the previous example, we define the ring of polynomial functions on V to be $\mathcal{O}(V) := S^{\bullet}(V^*)$. We will use this algebra extensively as an approximation to the algebra of smooth functions on V. It is when speaking of this space that algebraic considerations will be most transparent.

Similarly, we can define the antisymmetric algebra of graded vector spaces

$$\bigwedge^{\bullet} V := V^{\otimes \bullet} \Big/ \left\langle \left\{ v \otimes w + (-1)^{|v||w|} w \otimes v \;\middle|\; v, w \in \bigcup_{i \in \mathbb{Z}} V^i \right\} \right\rangle. \tag{2.48}$$

Remark 2.3.1 ([see 40, Section 2.1]). Let V be a graded vector space with an adapted basis $(v_1, \ldots, v_n)$. As is standard in discussions of the BV formalism [see 23, for a counterexample], if we use the generic symbol $\phi \in V$ for an element of V, then we denote the elements of the basis dual to $(v_1, \ldots, v_n)$ by $(\phi^1, \ldots, \phi^n)$. One should be careful with this convention, specially in field theory. In this case the discrete indices transform to continuous indices, so that the coordinates on V will be written as $\phi(x)$. It is then common to omit the x symbol from this coordinates to simplify formulas. This however leads us to using the same symbol ϕ for generic elements of V and the coordinates on V. The interpretation should be clear from context.

Remark 2.3.2. Note that although $S^\bullet V = \bigoplus_{k\in\mathbb{N}} S^k V$, this is not the grading we have given to this space above.

Remark 2.3.3. Using the antisymmetric algebra we can recast the definition of a graded symplectic form in a simpler fashion. Namely, it is a linear map $\omega : V \wedge V \to V$ of degree k such that for all $v \in V \setminus \{0\}$ the map $\omega(v, \cdot) \neq 0$.

Remark 2.3.4. Let V be a bigraded vector space. Notice that using the graded commutativity (2.7), we have the isomorphisms

$$\begin{aligned}(V[1])^{\otimes n} &\to \underbrace{\mathbb{F}[1] \otimes V \otimes \cdots \otimes \mathbb{F}[1] \otimes V}_{n \text{ times}} \\ &\to \cdots \to \underbrace{\mathbb{F}[1] \otimes V \otimes \cdots \otimes \mathbb{F}[1] \otimes V}_{n-2 \text{ times}} \otimes \mathbb{F}[2,2] \otimes V \otimes V \\ &\to \mathbb{F}[n] \otimes V^{\otimes n} \to V^{\otimes n}[n],\end{aligned} \tag{2.49}$$

under which

$$\begin{aligned}&v_1 \otimes \cdots \otimes v_n \to s^{(1)} \otimes v_1 \otimes \cdots \otimes s^{(1)} \otimes v_n \mapsto \\ &(-1)^{|v_{n-1}|} s^{(1)} \otimes v_1 \otimes \cdots \otimes s^{(1)} \otimes v_{n-2} \otimes s^{(2)} \otimes v_{n-1} \otimes v_n \mapsto \\ &(-1)^{|v_{n-1}|} s^{(1)} \otimes v_1 \otimes \cdots \otimes s^{(1)} \otimes v_{n-3} \otimes s^{(3)} \otimes v_{n-2} \otimes v_{n-1} \otimes v_n \\ &\mapsto \cdots \to (-1)^{|v_{n-1}|+|v_{n-3}|+\cdots} s^{(n)} \otimes v_1 \otimes \cdots \otimes v_n \\ &\mapsto (-1)^{\sum_{k=1}^{n}(n-k)|v_k|} v_1 \otimes \cdots \otimes v_n.\end{aligned} \tag{2.50}$$

This is known as the décalage (shift in French) isomorphism [see 41, Section 1]. Notice that symmetrization using the degrees in $V[1]$ corresponds to antisymmetrization using the degree in V. This descends into an isomorphism

$$S^n(V[1]) \to \left(\bigwedge\nolimits^n V\right)[n]. \tag{2.51}$$

Making use of this identification we can obtain a degree 0 isomorphism

$$\mathrm{dec} : \mathrm{Hom}\left(\bigwedge^n V, W\right) \to \mathrm{Hom}(S^n(V[1]), W[1])[n-1]. \tag{2.52}$$

Indeed, we already found a map (2.23) which assigns to $f \in \mathrm{Hom}\left(\bigwedge^n V, W\right)$ the function $(-1)^{n|f|}f \in \mathrm{Hom}\left((\bigwedge^n V)[n], W[1]\right)[n-1]$. This can then be precomposed with the décalage isomorphism to obtain a map in $\mathrm{Hom}(S^n(V[1]), W[1])[n-1]$ given by

$$\mathrm{dec}(f)(v_1 \cdots v_n) = (-1)^{n|f|+\sum_{k=1}^{n}(n-k)|v_k|} f(v_1 \cdots v_n). \tag{2.53}$$

This map will be instrumental in order to understand gauge symmetries withing the BV formalism. For example, in the case of a bosonic gauge symmetry described by some Lie algebra, this map allows us to understand the Lie bracket at the level of the fermionic ghosts that enter the theory.

Much like on functions, we can pullback the regular functions on a graded vector space via homogeneous linear transformations.

Definition 2.3.2. Let V and W be graded vector spaces and $\varphi : V \to W$ be a homogeneous linear transformation. Then, let $f_1 \cdots f_n \in W^*$ be homogeneous. We define the pullback of $f_1 \cdots f_n \in \mathcal{O}(V)$ via φ to be $(f_1 \circ \varphi) \cdots (f_n \circ \varphi) \in \mathcal{O}(V)$. This extends to a homogeneous map of graded algebras $\varphi^* : \mathcal{O}(W) \to \mathcal{O}(V)$

Definition 2.3.3. A left derivation X of degree $k \in G$ of a graded algebra $\mathcal{A}$ is a homogeneous linear map $X : \mathcal{A} \to \mathcal{A}$ of degree k such that for all homogeneous $a, b \in \mathcal{A}$

$$X(ab) = (Xa)b + (-1)^{|k||a|}a(Xb) \tag{2.54}$$

A right derivation satisfies

$$X(ab) = a(Xb) + (-1)^{|k||b|}(Xa)b \tag{2.55}$$

instead. The graded vector space of left derivations is denoted by $\mathrm{Der}\,\mathcal{A}$.

Theorem 2.3.1. Given a left derivation X of a graded algebra $\mathcal{A}$, we define $X_r : \mathcal{A} \to \mathcal{A}$ by

$$X_r a = (-1)^{|X|(|a|+1)} X a \tag{2.56}$$

for all $a \in \mathcal{A}$. Then this is a right derivation of degree $|X|$.

Proof. Indeed, for all $a, b \in \mathcal{A}$ we have

$$\begin{aligned} X_r(ab) &= (-1)^{|X|(|a|+|b|+1)}((Xa)b + (-1)^{|X||a|}a(Xb)) \\ &= (-1)^{|X||b|}(X_r a)b + aX_r b. \end{aligned} \tag{2.57}$$

□

Remark 2.3.5. Heuristically speaking, we want to have

$$\begin{aligned} \frac{\partial}{\partial\theta}(\psi\theta) &= (-1)^{|\psi||\theta|}\psi\frac{\partial\theta}{\partial\theta} = (-1)^{|\psi||\theta|}\psi = (-1)^{(|\psi\theta|+|\theta|)|\theta|}\psi \\ &= (-1)^{(|\psi\theta|+1)|\theta|}\psi, \\ \frac{\partial_r}{\partial\theta}(\psi\theta) &= \psi\frac{\partial\theta}{\partial\theta} = \psi. \end{aligned} \tag{2.58}$$

In the last step of the first requirement we used the fact that for $x \in \mathbb{Z}_2$ we have $x^2 = x$. This heuristics is the reason behind our definition of X_r.

2.4 Graded Lie Algebras

Another example of graded algebraic structures we will encounter is that of a graded Lie algebra.

Definition 2.4.1 ([42, see, Definition 3.1])**.** A graded Lie bracket of degree $k \in G$ on a graded vector space $\mathfrak{g}$ is a bilinear map $[\,\cdot\,,\,\cdot\,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ such that

1. $[\mathfrak{g}^i, \mathfrak{g}^j] \subseteq \mathfrak{g}^{i+j+k}$,
2. $[X, Y] = -(-1)^{(|X|+|k|)(|Y|+|k|)}[Y, X]$ for all homogeneous $X, Y \in \mathfrak{g}$, and
3. for all homogeneous $X, Y, Z \in \mathfrak{g}$ we have the graded Jacobi identity

$$\begin{aligned} 0 = &(-1)^{(|X|+|k|)(|Z|+|k|)}[X, [Y, Z]] \\ &+ (-1)^{(|Y|+|k|)(|X|+|k|)}[Y, [Z, X]] \\ &+ (-1)^{(|Z|+|k|)(|Y|+|k|)}[Z, [X, Y]]. \end{aligned} \tag{2.59}$$

A graded vector space equipped with a graded Lie bracket of degree k is called a graded Lie algebra of degree k. If no degree is mentioned, we will assume that graded Lie algebras are of degree 0.

Remark 2.4.1. Super Lie algebras corresponding to the choice of $G = {}^{\mathbb{Z}}\!/_{2\mathbb{Z}}$ are extremely important to supersymmetric theories. Using these we can not only describe bosonic symmetries which preserve the parity of a field, but also fermionic symmetries which mix bosonic and fermionic components. On the other hand, $\mathbb{Z}$-graded Lie algebras allow us to characterize symmetries that may be trivial in a way we will make precise in Section 3.3. Indeed, an element with a certain degree i will characterize the trivial symmetries corresponding to the degree $i+1$. In order to do this however we need a map between these two degrees. This leads us immediately to the notion of a differential graded Lie algebra.

Definition 2.4.2 ([see 43, Definition 3.1])**.** A dg (differential graded) Lie algebra $\mathfrak{g}$ is a graded Lie algebra with a differential $\mathrm{d} : \mathfrak{g} \to \mathfrak{g}$ of degree $k \in G$ which makes $\mathfrak{g}$ a cochain complex, and

$$\mathrm{d}[X, Y] = [\mathrm{d}X, Y] + (-1)^{|X|}[X, \mathrm{d}Y], \tag{2.60}$$

for all $X, Y \in \mathfrak{g}$.

Remark 2.4.2. Much like in Remark 2.3.3, we can define a graded Lie bracket of degree k more simply as a degree 0 map $[\,\cdot\,,\,\cdot\,] : \mathfrak{g}[-k] \wedge \mathfrak{g}[-k] \to \mathfrak{g}[-k]$ satisfying the graded Jacobi identity

$$\begin{aligned} 0 = {} & (-1)^{|X|_{\mathfrak{g}[-k]}|Z|_{\mathfrak{g}[-k]}}[X, [Y, Z]] + (-1)^{|Y|_{\mathfrak{g}[-k]}|X|_{\mathfrak{g}[-k]}}[Y, [Z, X]] \\ & + (-1)^{|Z|_{\mathfrak{g}[-k]}|Y|_{\mathfrak{g}[-k]}}[Z, [X, Y]]. \end{aligned} \tag{2.61}$$

Remark 2.4.3. Notice that a graded Lie algebra of degree 0 is almost a graded algebra by viewing the Lie bracket as a product. Property 1. ensures that the degrees work as expected in a graded algebra. However, Lie algebras are distinguished by the algebras we are studying in that associativity fails. Indeed, this failure is precisely measured through the graded Jacobi identity. To see this, let us put it in the form

$$\begin{aligned} [X, [Y, Z]] &= -(-1)^{|Y||X|+|X||Z|}[Y, [Z, X]] - (-1)^{|Z||Y|+|X||Z|}[Z, [X, Y]] \\ &= (-1)^{|Y||X|}[Y, [X, Z]] + [[X, Y], Z]. \end{aligned} \tag{2.62}$$

In this form we also see that the Jacobi identity guarantees that $[X, \,\cdot\,]$ is a derivation of degree $|X|$ of this "algebra".

All graded Lie algebras can be obtained through graded Lie algebras of degree 0 via the following theorem.

Theorem 2.4.1. Let L be a graded Lie algebra of degree k. Then $L[s]$ is a graded Lie algebra of degree $k+s$.

Proof. This is a simple consequence of Remark 2.4.2. □

Graded algebras induce two important examples of a graded Lie algebras.

Theorem 2.4.2 ([see 42, Proposition 4.1])**.** Let $\mathcal{A}$ be a graded algebra, and define its commutator

$$\begin{aligned} [\,\cdot\,,\,\cdot\,] : \mathcal{A} \times \mathcal{A} &\to \mathcal{A} \\ (a,b) &\mapsto ab - (-1)^{|a||b|} ba. \end{aligned} \tag{2.63}$$

Then $[\,\cdot\,,\,\cdot\,]$ is a graded Lie bracket on $\mathcal{A}$.

Theorem 2.4.3. Let $\mathcal{A}$ be a graded algebra and define

$$\begin{aligned} [\,\cdot\,,\,\cdot\,] : \operatorname{Der}\mathcal{A} \times \operatorname{Der}\mathcal{A} &\to \operatorname{Der}\mathcal{A} \\ (X,Y) &\mapsto XY - (-1)^{|X||Y|} YX. \end{aligned} \tag{2.64}$$

Then $[\,\cdot\,,\,\cdot\,]$ is a graded Lie bracket on $\operatorname{Der}\mathcal{A}$.

Proof. From Remark 2.1.3 it is clear that the degrees work. We just need to show that the Leibniz rule remains valid. Consider homogeneous $a, b \in \mathcal{A}$ and $X, Y \in \operatorname{Der}\mathcal{A}$. We then have

$$\begin{aligned} (XY)(ab) =& X((Ya)b + (-1)^{|Y||a|} a(Yb)) \\ =& (X(Ya))b + (-1)^{|X|(|a|+|Y|)} (Ya)(Xb) \\ &+ (-1)^{|Y||a|} (Xa)(Yb) \\ &+ (-1)^{(|Y|+|X|)|a|} a(X(Yb)). \end{aligned} \tag{2.65}$$

To compute $[X,Y](ab)$ we need to mirror the above computation exchanging the roles of X and Y, multiplying by $(-1)^{|X||Y|}$ and subtracting the result. The mirror of

$$(X(Ya))b + (-1)^{(|X|+|Y|)|a|} a(X(Yb)), \tag{2.66}$$

yields the Leibniz rule

$$([X,Y]a)b + (-1)^{(|X|+|Y|)|a|} a([X,Y]b), \tag{2.67}$$

during the computation of $[X,Y](ab)$. We thus only need to show that the terms coming from $(-1)^{|X|(|a|+|Y|)}(Ya)(Xb) + (-1)^{|Y||a|}(Xa)(Yb)$ vanish.

The mirror of $(-1)^{|X|(|a|+|Y|)}(Ya)(Xb)$ is of the form

$$(-1)^{|Y|(|a|+|X|)+|X||Y|}(Xa)(Yb) = (-1)^{|Y||a|}(Xa)(Yb). \tag{2.68}$$

This will cancel the term $(-1)^{|Y||a|}(Xa)(Yb)$. Conversely, the mirror of this term is $(-1)^{|X||a|+|X||Y|}(Ya)(Xb)$, which cancels $(-1)^{|X|(|a|+|Y|)}(Ya)(Xb)$.

□

2.5 Graded Vector Fields

Following the trend in noncommutative geometry, we proceed to define vector fields on graded vector spaces by extending their algebraic description on usual manifolds [see 44, Problem 19.12].

Definition 2.5.1 ([see 45, Definition 4.7.2])**.** Let V be a graded vector space. Graded derivations of $\mathcal{O}(V)$ are called graded vector fields on V.

Theorem 2.5.1. Let V be a graded vector space and $X : V^* \to \mathcal{O}(V)$ a homogeneous map of degree $k \in G$. Then there is a unique extension of X into a left vector field of degree k on V. Similarly, there is a unique extension into a right vector field X_r.

Proof. We will proof this only for left derivations since the proof for right derivations is identical. The extension is easily constructed by induction on m in the decomposition $\mathcal{O}(V) = S^\bullet V^* = \bigoplus_{m\in\mathbb{N}} S^m V^*$. Namely, it acts trivially on $S^0V^* = \mathbb{F}$ and we have already defined it on $S^1V^* = V^*$. Assuming that it has been defined for $k = m-1$, recall that S^mV^* is spanned by elements of the form fF, with $f \in V^*$ and $F \in S^{m-1}v^*$. We then define

$$X(fF) := (Xf)F + (-1)^{|k||f|}fX(F). \tag{2.69}$$

The fact that the resulting linear map has degree k is immediate. We can now show the graded Leibniz rule also via induction. Indeed, if it has been shown for all pairs $(F,G) \in S^mV^* \times S^lV^*$ with $m + l < N$, then we have for $f \in V^*$

$$\begin{aligned} X(fFG) &= (Xf)FG + (-1)^{|X||f|}fX(FG) \\ &= (Xf)FG + (-1)^{|X||f|}fX(F)G + (-1)^{|X|(|f|+|F|)}fFXG \\ &= X(fF)G + (-1)^{|X|||fF|}fFXG. \end{aligned} \tag{2.70}$$

□

Example 2.5.1. Let $(x^1, \ldots, x^n)$ be an adapted basis of V^* with V a bigraded vector space over a field $\mathbb{F}$. Consider $\partial/\partial x^i : V^* \to \mathbb{F}$ defined by

$$\frac{\partial x^i}{\partial x^j} = \delta^i{}_j. \tag{2.71}$$

In other words, let $(\partial/\partial x^1, \ldots \partial/\partial x^n)$ be the dual basis. Notice that $\deg \partial/\partial x^i = -\deg x^i$ while $|\partial/\partial x^i| = |x^i|$. We will always extend this to a left vector field on V. The extension onto a right vector field will be denoted

$$\frac{\partial_r}{\partial x^i} := \left(\frac{\partial}{\partial x^i}\right)_r \tag{2.72}$$

In fact, the most generic vector fields can be built from this example.

Theorem 2.5.2. Let X be a graded vector field on a graded vector space V with a graded basis $(x^1, \ldots, x^n)$ of its dual. Then

$$X = X(x^i)\frac{\partial}{\partial x^i}. \tag{2.73}$$

Proof. In light of Theorem 2.5.1 this is clear from the trivial computation

$$X(x^i)\frac{\partial x^j}{\partial x^i} = X(x^j). \tag{2.74}$$

□

Remark 2.5.1. Given Theorem 2.4.3, we see that the graded vector fields on a graded manifold form a graded Lie algebra.

Theorem 2.5.3. Let V be a graded vector space and $(x^1, \ldots, x^n)$ be a basis of its dual. We then have $[\partial/\partial x^i, \partial/\partial x^j] = 0$.

Proof. Due to Theorem 2.5.1, we only need to prove this on V^*. We then have

$$\left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right] x^k = \frac{\partial}{\partial x^i}\delta^j{}_k - (-1)^{|x|^i|x|^j}\frac{\partial}{\partial x^j}\delta^i{}_k = 0. \tag{2.75}$$

□

Theorem 2.5.4. Let V be a graded vector space with $(x^1, \ldots, x^n)$ an adapted basis of its dual. Then a vector field $F := F^i \, \partial / \partial x^i$ induces the right vector field

$$F_r G = (-1)^{F^i(|x^i|+1)} \frac{\partial G}{\partial x^i} F^i, \tag{2.76}$$

for all $G \in \mathcal{O}(V)$

Proof.

$$\left(F^i \frac{\partial}{\partial x^i} \right)_r G = (-1)^{|F|(|G|+1)} F^i \frac{\partial G}{\partial x^i} \tag{2.77}$$

Reordering to put the left hand side in the form $\partial_r G / \partial x^i \, F^i$ yields an overall sign with exponent

$$|F|(|G|+1) + |F^i|(|G| + |x^i|) + |x^i|(|G|+1). \tag{2.78}$$

Using the fact that in $\mathbb{Z}/2\mathbb{Z}$ we have $x^2 = x$, we have $|x^i|(|G|+1) = |x^i|(|G| + |x^i|)$. We can thus rewrite this exponent in the form

$$|F|(|G|+1) + (|F^i| + |x^i|)(|G| + |x^i|). \tag{2.79}$$

Noticing that $|F^i| + |x^i| = |F|$, the exponent reduces to

$$|F|(\cancel{|G|} + 1 + \cancel{|G|} + |x^i|). \tag{2.80}$$

□

Definition 2.5.2 ([see 46, Definition 3.3])**.** A cohomological vector field is a graded vector field of degree (1) which commutes with itself.

Remark 2.5.2 ([see 46, Remark 3.4]). Since for a cohomological vector field δ we have $0 = [\delta, \delta] = \delta\delta - (-1)^{1\times 1}\delta\delta = 2\delta^2$, we conclude that δ is a differential for the graded algebra $\mathcal{O}(V)$. Notice that the choice of degree (1) is fundamental for this result. Indeed, $|(1)| = 1$ guarantees that a cohomological vector field squares to 0, while $\deg(1) = 1$ guarantees that it raises the cohomological degree by 1.

Definition 2.5.3 ([see 46, Definition 3.6])**.** A graded vector space equipped with a cohomological vector field is called a differential graded space or dg space for short.

Remark 2.5.3. A dg space (V, δ) induces a cochain complex (see Definition 2.1.3)

$$\cdots \mathcal{O}(V)^{i-1}_{\deg} \xrightarrow{\delta^{i-1}} \mathcal{O}(V)^{i}_{\deg} \xrightarrow{\delta^{i}} \mathcal{O}(V)^{i+1}_{\deg} \xrightarrow{\delta^{i+1}} \cdots . \tag{2.81}$$

In other words, the space $\mathcal{O}(V)$ equipped with δ and the graduation given by deg instead of p, is a cochain complex.

2.6 Graded Poisson Vector Spaces

Definition 2.6.1 ([see 47, Section 2.3])**.** A graded Poisson bracket $\{\,\cdot\,,\,\cdot\,\}$ of degree $k \in G$ on a graded algebra $\mathcal{A}$ is a graded Lie bracket of degree k such that for all homogeneous $a \in \mathcal{A}$ the induced map $\{a, \cdot\,\} : \mathcal{A} \to \mathcal{A}$ is a derivation of degree $p(a) + k$. A graded algebra equipped with a Poisson bracket of degree k is called a graded Poisson algebra of degree k.

The sign conventions between the Lie brackets and derivations are compatible in the Poisson bracket.

Theorem 2.6.1. Let $\mathcal{A}$ be a graded Poisson algebra of degree k. Then for all homogeneous $a \in \mathcal{A}$ the map $\{\,\cdot\,, a\} : \mathcal{A} \to \mathcal{A}$ is a right derivation of degree $p(a) + k$.

Proof. This is clear from the fact that for all $b \in \mathcal{A}$

$$\begin{aligned} \{b, a\} &= -(-1)^{(|a|+|k|)(|b|+|k|)}\{a, b\} \\ &= -(-1)^{(|a|+|k|)(|b|+1)+(|a|+|k|)(|k|-1)}\{a, b\} \\ &= -(-1)^{(|a|+k)(|k|-1)}\{a, \cdot\,\}_r b, \end{aligned} \tag{2.82}$$

i.e. $\{\,\cdot\,, a\} = -(-1)^{(|a|+|k|)(|k|-1)}\{a, \cdot\,\}_r$. ☐

Example 2.6.1. Let V be a graded symplectic vector space of degree k. We then have the musical isomorphism of degree $-k$, $\sharp : V^* \to V$ of (2.29). We define the bilinear

$$\begin{aligned} &\{\,\cdot\,, \cdot\,\} : V^* \times V^* \to \mathbb{F} \\ &(f, g) \mapsto \omega(f^\sharp, g^\sharp) \in \mathbb{F}^{\cancelto{|f|-\cancel{k}+|g|-k+\cancel{k}}{|f|^\sharp+|g|^\sharp+k}} \subseteq \mathcal{O}(V)^{|f|+|g|-k}. \end{aligned} \tag{2.83}$$

We can extend this to a Poisson bracket of degree $-k$ via the Leibniz rule. Namely, for all $f \in V^*$ we have the degree $p(f) - k$ linear map $\{f, \cdot\,\} : V^* \to \mathbb{F} \subseteq \mathcal{O}(V)$.

Via Theorem 2.5.1, this extends to a derivation $\{f, \cdot\} : \mathcal{O}(V) \to \mathcal{O}(V)$. Using the symmetry of the Poisson bracket we want to obtain, this defines a degree $p(F) - k$ linear map

$$\begin{aligned} \{F, \cdot\} : V^* &\to \mathcal{O}(V) \\ f &\mapsto \{F, f\} := -(-1)^{(|F|-k)(|f|-k)}\{f, F\}. \end{aligned} \tag{2.84}$$

for every $F \in \mathcal{O}(V)$. Via Theorem 2.5.1, this extends to a vector field $\{F, \cdot\} : \mathcal{O}(V) \to \mathcal{O}(V)$. This defines the Poisson bracket $\{\cdot, \cdot\}$ of $\mathcal{O}(V)$. Notice that it satisfies the correct skew symmetry since for all $f, g \in V^*$

$$\begin{aligned} \{f, g\} &= \omega(f^\sharp, g^\sharp) = -(-1)^{|f^\sharp|_V |g^\sharp|_V}\omega(g^\sharp, f^\sharp) \\ &= -(-1)^{(|f|_{V^*}-k)(|g^*|_V-k)}\{g, f\} \end{aligned} \tag{2.85}$$

Definition 2.6.2. A graded Poisson vector space of degree $k \in G$ is a graded vector space V equipped with a graded Poisson bracket $\{\cdot, \cdot\}$ of degree k on $\mathcal{O}(V)$.

Definition 2.6.3. Let V be a graded Poisson vector space of degree $k \in G$ and $f \in \mathcal{O}(V)$. The graded vector field $X_f := \{f, \cdot\}$ of degree $p(f) + k$ is called the Hamiltonian vector field of f.

Theorem 2.6.2 (Classical Master Equation)**.** Let $\mathcal{A}$ be a bigraded Poisson algebra of degree (1). The Hamiltonian vector field to a smooth "function" $S \in \mathcal{A}^0$ is cohomological if and only if the master equation $\{S, S\} = 0$ is satisfied.

Proof. By the graded Jacobi identity

$$\begin{aligned} 0 &= (-1)^{|f|+1}\{S, \{S, f\}\} + (-1)^{|f|+1}\{f, \{S, S\}\} - \{S, \{f, S\}\} \\ &= 2(-1)^{|f|+1}\{S, \{S, f\}\} + (-1)^{|f|+1}\{f, \{S, S\}\}, \end{aligned} \tag{2.86}$$

i.e. $\{S, \cdot\}^2 = -\frac{1}{2}\{\cdot, \{S, S\}\}$. Thus, $\delta^2 = 0$ if and only if $\{S, S\} = 0$. □

Example 2.6.2. Continuing with Example 2.6.1, let us now consider the degree (-1) graded symplectic vector space $\mathcal{E}$ of Example 2.2.1. Given the musical isomorphism, we characterize the graded Poisson bracket on $\mathcal{E}$ by

$$\{\Phi^A, \Phi^*_B\} = -\sigma(\Phi^A, \Phi^*_B) = \delta^A{}_B, \tag{2.87}$$

with all other combinations of the basis elements equal to 0. Thus,

$$\{\Phi^A, \cdot\} = \frac{\partial}{\partial \Phi^*_A}, \quad \{\Phi^*_A, \cdot\} = -\frac{\partial}{\partial \Phi^A}. \tag{2.88}$$

We then conclude that for $F \in \mathcal{O}(\mathcal{E})$ we have

$$\begin{aligned}\{F, \Phi^A\} &= -(-1)^{(|F|+1)(|\Phi^A|+1)} \frac{\partial F}{\partial \Phi^*_A} = -\frac{\partial_r F}{\partial \Phi^*_A} \\ \{F, \Phi^*_A\} &= (-1)^{(|F|+1)(|\Phi^*_A|+1)} \frac{\partial F}{\partial \Phi^A} = \frac{\partial_r F}{\partial \Phi^A},\end{aligned} \tag{2.89}$$

so that

$$\{F, \cdot\} = \frac{\partial_r F}{\partial \Phi^A} \frac{\partial}{\partial \Phi^*_A} - \frac{\partial_r F}{\partial \Phi_A} \frac{\partial}{\partial \Phi^A}. \tag{2.90}$$

In other words, if $F, G \in \mathcal{O}(\mathcal{E})$

$$\{F, G\} = \frac{\partial_r F}{\partial \Phi^A} \frac{\partial G}{\partial \Phi^*_A} - \frac{\partial_r F}{\partial \Phi^*_A} \frac{\partial G}{\partial \Phi^A}. \tag{2.91}$$

In particular, for $S \in \mathcal{O}(\mathcal{E})^0$ we have

$$\begin{aligned}\frac{\partial_r S}{\partial \Phi^*_A} \frac{\partial S}{\partial \Phi^A} &= (-1)^{|\Phi^*_A|+|\Phi^A|} \frac{\partial S}{\partial \Phi^*_A} \frac{\partial_r S}{\partial \Phi^A} \\ &= (-1)^{|\Phi^*_A|+|\Phi^A|+|\Phi^A||\Phi^*_A|} \frac{\partial_r S}{\partial \Phi^A} \frac{\partial S}{\partial \Phi^*_A} \\ &= (-1)^{-1+|\Phi^A|(|\Phi^A|-1)} \frac{\partial_r S}{\partial \Phi^A} \frac{\partial S}{\partial \Phi^*_A} = -\frac{\partial_r S}{\partial \Phi^A} \frac{\partial S}{\partial \Phi^*_A},\end{aligned} \tag{2.92}$$

so that its master equation is

$$0 = \frac{\partial_r S}{\partial \Phi^A} \frac{\partial S}{\partial \Phi^*_A}. \tag{2.93}$$

2.7 BV Algebras

Definition 2.7.1 ([see 48, p. 2.2.2])**.** A BV algebra is a graded algebra $\mathcal{A}$ equipped with a degree (1) map $\Delta : \mathcal{A} \to \mathcal{A}$, the BV Laplacian, which satisfies $\Delta^2 = 0$ and is second order, i.e. for all $F, G, H \in \mathcal{A}$ we have

$$\begin{aligned}\Delta(FGH) = \Delta(FG)H &+ (-1)^{|F|} F\Delta(GH) + (-1)^{(|F|+1)|G|} G\Delta(FH) \\ &- \Delta FGH - (-1)^{|F|} F\Delta GH - (-1)^{|F|+|G|} FG\Delta H.\end{aligned} \tag{2.94}$$

Every BV algebra has a graded Poisson bracket of degree 1.

Theorem 2.7.1. Let $\mathcal{A}$ be a BV algebra. Define $\{\,\cdot\,,\,\cdot\,\} : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ by

$$\{F, G\} = (-1)^{|F|}\Delta(FG) - (-1)^{|F|}\Delta FG - F\Delta G \tag{2.95}$$

for all $F, G \in \mathcal{A}$. Then for all $F, G \in \mathcal{A}$ we have

$$\Delta\{F, G\} = \{\Delta F, G\} + (-1)^{|F|+1}\{F, \Delta G\}. \tag{2.96}$$

Proof. For all $F, G \in \mathcal{A}$ we can express the BV Laplacian of FG in terms of the new bracket

$$\Delta(FG) = \Delta FG + (-1)^{|F|}F\Delta G + (-1)^{|F|}\{F, G\}. \tag{2.97}$$

Applying this expansion twice we then obtain

$$\begin{aligned} 0 &= \Delta^2(FG) \\ &= \cancelto{0}{(\Delta^2 F)}G + \cancel{(-1)^{|F|+1}\Delta F\Delta G} + (-1)^{|F|+1}\{\Delta F, G\} + \cancel{(-1)^{|F|}\Delta F\Delta G} \\ &\quad + F\cancelto{0}{(\Delta^2 G)} + \{F, \Delta G\} + (-1)^{|F|}\Delta\{F, G\}. \end{aligned} \tag{2.98}$$

$\square$

Theorem 2.7.2. Let $\mathcal{A}$ be a BV algebra. Then $\{\,\cdot\,,\,\cdot\,\}$ is a graded Poisson bracket of degree (1) on $\mathcal{A}$.

Proof. The bilinearity is clear from the linearity of Δ. The degree is clear from the degree of Δ. The second order guarantees that for every $F \in \mathcal{A}$ the map $\{F,\,\cdot\,\}$ is a derivation. Indeed, applying the definition of the Poisson bracket we have

$$\{F, GH\} = (-1)^{|F|}\Delta(FGH) - (-1)^{|F|}\Delta FGH - F\Delta(GH). \tag{2.99}$$

The second order condition on the first term then yields

$$\begin{aligned} \{F, GH\} = (-1)^{|F|}\Delta(FG)H &+ \cancel{F\Delta(GH)} + (-1)^{(|F|+1)|G|+|F|}G\Delta(FH) \\ &- (-1)^{|F|}\Delta FGH - F\Delta GH - (-1)^{|G|}FG\Delta H \\ &- (-1)^{|F|}\Delta FGH - \cancel{F\Delta(GH)}. \end{aligned} \tag{2.100}$$

We notice that we have two terms $(-1)^{|F|}\Delta FGH$. Let us rewrite one of them as $(-1)^{|F|+|G|(|F|+1)}G\Delta FH$. The remaining terms proportional to H on the right yield precisely

$$\Big((-1)^{|F|}\Delta(FG)-(-1)^{|F|}\Delta FG-F\Delta G\Big)H=\{F,G\}H. \tag{2.101}$$

If we rewrite $(-1)^{|G|}FG\Delta H$ as $(-1)^{|G|+|F||G|}GF\Delta H$, the other terms will be proportional to G on the left and equal to

$$\begin{aligned}(-1)^{(|F|+1)|G|}G\Big((-1)^{|F|}\Delta(FH)-(-1)^{|F|}\Delta FH-F\Delta H\Big)&\\ =(-1)^{(|F|+1)|G|}G\{F,H\}.&\end{aligned} \tag{2.102}$$

We conclude that

$$\{F,GH\}=\{F,G\}H+(-1)^{(|F|+1)|G|}G\{F,H\}. \tag{2.103}$$

The graded antisymmetry can be directly computed

$$\begin{aligned}\{F,G\}&=(-1)^{|F|}\Delta(FG)-(-1)^{|F|}\Delta FG-F\Delta G\\ &=(-1)^{|F|+|F||G|}\Delta(GF)-(-1)^{|F|+(|F|+1)|G|}G\Delta F\\ &\quad-(-1)^{(|G|+1)|F|}\Delta GF\\ &=-(-1)^{(|F|+1)(|G|+1)+|G|}\Delta(GF)+(-1)^{(|F|+1)(|G|+1)}G\Delta F\\ &\quad+(-1)^{(|G|+1)(|F|+1)+|G|}\Delta GF\\ &=-(-1)^{(|F|+1)(|G|+1)}\{G,F\}.\end{aligned} \tag{2.104}$$

Finally, the Jacobi identity is a consequence of applying the BV Laplacian to the Leibniz rule (2.96). Indeed, we can use the definition of the bracket on the second term of

$$\Delta(\{F,GH\})=\{\Delta F,GH\}+(-1)^{|F|+1}\{F,\Delta(GH)\}, \tag{2.105}$$

to obtain

$$\begin{aligned}\Delta(\{F,GH\})=\{\Delta F,GH\}&+(-1)^{|F|+1}\{F,(\Delta G)H\}\\ &+(-1)^{|F|+1+|G|}\{F,G(\Delta H)\}+(-1)^{|F|+1+|G|}\{F,\{G,H\}\}\end{aligned} \tag{2.106}$$

We can now use (2.103) to expand the first, second and third brackets. In particular, in the expansion of the first and second bracket we will recover

$$\{\Delta F,G\}H+(-1)^{|F|+1}\{F,\Delta G\}H=\Delta\{F,G\}H. \tag{2.107}$$

Adding in the term $(-1)^{|F|+|G|+1}\{F,G\}\Delta H$ coming from the expansion of the third term leads us to

$$\Delta(\{F,G\}H) - (-1)^{|F|+|G|+1}\{\{F,G\},H\}. \tag{2.108}$$

Similarly, using the terms left from the expansion of the first and third bracket we will also recover

$$\begin{aligned}(-1)^{|F||G|}G\{\Delta F,H\} + (-1)^{|F|+\cancel{|G|}+1+(|F|+\cancel{1})|G|}G\{F,\Delta H\} \\ = (-1)^{|F||G|}G\Delta\{F,H\}.\end{aligned} \tag{2.109}$$

Then, adding the term $(-1)^{|F|+1+(|F|+1)(|G|+1)}\Delta G\{F,H\}$ left from the expansion of the second bracket we have

$$(-1)^{|F||G|+|G|}\Delta(G\{F,H\}) - (-1)^{|F||G|}\{G,\{F,H\}. \tag{2.110}$$

Addition of (2.108) and (2.110) leads us to

$$\Delta\{F,GH\} - (-1)^{|F|+|G|+1}\{\{F,G\},H\} - (-1)^{|F||G|}\{G,\{F,H\}. \tag{2.111}$$

Putting everything together we have

$$\begin{aligned}\Delta\{F,GH\} = \Delta(\{F,GH\}) - (-1)^{|F|+|G|+1}\{\{F,G\},H\} \\ - (-1)^{|F||G|}\{G,\{F,H\}\} + (-1)^{|F|+|G|+1}\{F,\{G,H\}\}.\end{aligned} \tag{2.112}$$

From this we conclude that

$$\{F,\{G,H\}\} = \{\{F,G\},H\} + (-1)^{(|F|+1)(|G|+1)}\{G,\{F,H\}\}, \tag{2.113}$$

which is another way to express the graded Jacobi identity (2.59) for $|k| = 1$ (see (3.2) of [42]). □

From now on we will always assume that every BV algebra is equipped with this bracket, called the BV bracket. Even though the BV Laplacian is not a derivative for the ordinary product, it does satisfy a graded Leibniz rule for the BV bracket (2.96). Therefore, we will be able to use it to deform cohomological Hamiltonian vector fields.

Theorem 2.7.3. Let $\mathcal{A}$ be a BV algebra and $S \in \mathcal{A}^0$. Then $Q = \{S,\cdot\} - \hbar\Delta$ is a differential if and only if the quantum master equation

$$\frac{1}{2}\{S,S\} + \hbar\Delta S = 0 \tag{2.114}$$

is satisfied.

Proof. As we already computed in the proof of Theorem 2.6.2

$$\{S,\cdot\}^2 = -\frac{1}{2}\{\cdot,\{S,S\}\}. \tag{2.115}$$

Therefore

$$\begin{aligned} Q^2 &= -\frac{1}{2}\{\cdot,\{S,S\}\} - \hbar\{S,\Delta\cdot\} - \hbar\Delta\{S,\cdot\} + \hbar^2\cancelto{0}{\Delta^2} \\ &= -\frac{1}{2}\{\cdot,\{S,S\}\} - \cancel{\hbar\{S,\Delta\cdot\}} - \hbar\{\Delta S,\cdot\} + \cancel{\hbar\{S,\Delta\cdot\}}. \end{aligned} \tag{2.116}$$

Finally, note that $\deg\{S,S\} = 1$ and for all $f \in \mathcal{A}$ we have

$$\{f,\{S,S\}\} = -(-1)^{(1+1)(|f|+1)}\{\{S,S\},f\}. \tag{2.117}$$

We conclude that $Q^2 = \{\frac{1}{2}\{S,S\} - \hbar\Delta S,\cdot\}$. ☐

Definition 2.7.2 ([see 38, Construction 2.4.9])**.** Let V be a bigraded Poisson vector space of degree (1). We will define the BV Laplacian inductively, as the linear map $\Delta_{\mathrm{BV}} : \mathcal{O}(V) \to \mathcal{O}(V)$ satisfying the initial data

1. $\Delta_{\mathrm{BV}} f = \Delta_{\mathrm{BV}} 1 = 0$,
2. $\Delta_{\mathrm{BV}}(fg) = (-1)^{|f|}\{f,g\}$,

for all $f,g \in V^*$, and the identity

$$\Delta_{\mathrm{BV}}(FG) = \Delta_{\mathrm{BV}}FG + (-1)^{|F|}F\Delta_{\mathrm{BV}}G + (-1)^{|F|}\{F,G\}, \tag{2.118}$$

for all $F,G \in \mathcal{O}(V)$.

Theorem 2.7.4. Let V be a Poisson vector space of degree (1). Then Δ_{BV} makes $\mathcal{O}(V)$ a BV algebra.

Proof. Trivially, Δ_{BV} is of degree (1) on $S^0V^* = \mathbb{F}$ and $S^1V^* = V^*$. It inherits its degree from the Poisson bracket in S^2V^*. Assuming that it acts with degree (1) on

$S^k V^*$, consider $f \in V^*$ and $F \in S^k V^*$. Then,

$$\Delta_{\mathrm{BV}}(fF) = \cancelto{0}{(\Delta_{\mathrm{BV}} f) F} + (-1)^{|f|} f \Delta_{\mathrm{BV}} F + (-1)^{|f|}\{f, F\}. \tag{2.119}$$

We conclude that it acts with degree (1) on $S^{k+1}V^*$ by noticing that due to the degree of the Poisson bracket

$$p(\{f, F\}) = p(f) + p(F) + (1) = p(fF) + (1), \tag{2.120}$$

and due to our induction hypothesis

$$p(f\Delta_{\mathrm{BV}}F) = p(f) + p(\Delta_{\mathrm{BV}}F) = p(f) + p(F) + (1) = p(fF) + (1). \tag{2.121}$$

We thus establish that $p(\Delta_{\mathrm{BV}}) = 1$.

Next, we need to show that $\Delta_{\mathrm{BV}}^2 = 0$. We will do this using an induction like the one above. It is trivial to see that $\Delta_{\mathrm{BV}}^2 F = 0$ for all $F \in S^k V^*$ with $k \in \{1, 2, 3\}$. Now, assume that $\Delta_{\mathrm{BV}}^2 F = 0$ for every $F \in S^k(V)$. Take $f \in V^*$. We then have

$$\begin{aligned} \Delta_{\mathrm{BV}}^2(fF) &= (-1)^{|f|}\Delta_{\mathrm{BV}}(f\Delta_{\mathrm{BV}}F + \{f, F\}) \\ &= \cancelto{0}{f\Delta_{\mathrm{BV}}^2 F} + \{f, \Delta_{\mathrm{BV}}F\} + (-1)^{|f|}\Delta_{\mathrm{BV}}\{f, F\}. \end{aligned} \tag{2.122}$$

We thus see that $\Delta_{\mathrm{BV}}^2(fF) = 0$ is equivalent to

$$\Delta_{\mathrm{BV}}\{f, F\} = (-1)^{|f|+1}\{f, \Delta_{\mathrm{BV}}F\}. \tag{2.123}$$

This should not come as a suprise, given the proof of Theorem 2.7.1. To proof the latter, we note that F will be a linear combination of elements of the form gG, with $g \in V^*$ and $G \in S^{k-1}V^*$. Due to linearity, we will proof (2.123) for $F = gG$.

The left hand side is of the form

$$\begin{aligned} \Delta_{\mathrm{BV}}\{f, gG\} &= \Delta_{\mathrm{BV}}(\{f, g\}G + (-1)^{(|f|+1)|g|}g\{f, G\}) \\ &= (-1)^{|f|+|g|+1}\{f, g\}\Delta_{\mathrm{BV}}G + (-1)^{|f|+|g|+1}\{\{f, g\}, G\} \\ &\quad + (-1)^{(|f|+\cancel{1})|g|+\cancel{|g|}}g\Delta_{\mathrm{BV}}\{f, G\} \\ &\quad + (-1)^{(|f|+\cancel{1})|g|+\cancel{|g|}}\{g, \{f, G\}\}. \end{aligned} \tag{2.124}$$

On the other hand, the right hand side is of the form

$$\begin{aligned}(-1)^{|f|+1}\{f, \Delta_{\mathrm{BV}}(gG)\} = (-1)^{|f|+|g|+1}&\{f, g\Delta_{\mathrm{BV}}G\} \\ &+ (-1)^{|f|+|g|+1}\{f, \{g, G\}\}.\end{aligned} \tag{2.125}$$

We thus get 6 terms in total, with half depending on Δ_{BV}. Let us first deal with the independent ones.

Let us group the terms independent of Δ_{BV} on one side. The term proportional to $\{\{f, g\}, G\}$ can be rewritten in terms of $\{G, \{f, g\}\}$. The resulting sign has exponent

$$\begin{aligned}&|f| + |g| + \cancel{1} + (|G|+1)(|f|+|g|) + \cancel{1} \\ &\qquad = \cancel{|f| + |g|} + |G||f| + \cancel{|f|} + |G||g| + \cancel{|g|}.\end{aligned} \tag{2.126}$$

Multiplying this term by a sign with exponent $|f||G| + |g| + |G| + 1$, yields an exponent

$$|G||g| + |g| + |G| + 1 = (|G|+1)(|g|+1). \tag{2.127}$$

Similarly, the term propotional to $\{g, \{f, G\}\}$ can be rewritten in terms of $\{g, \{G, f\}\}$. This yields an overall sign with exponent $|f||g| + (|G|+1)(|f|+1) + 1$. Multiplying this term with the sign we used before then yields a total sign with exponent

$$\begin{aligned}|f||g| + (|G|+1)(|f|+1) &+ \cancel{1} + |f||G| + |g| + |G| + \cancel{1} \\ &= |f||g| + |f| + |g| + 1 = (|g|+1)(|f|+1).\end{aligned} \tag{2.128}$$

Finally, the term proportional to $\{f, \{g, G\}\}$ has to be moved to the other side of the equation. Multiplying it by the sign we have used so far yields the sign with exponent

$$|f| + \cancel{|g|} + |f||G| + \cancel{|g|} + |G| + 1 = (|f|+1)(|G|+1). \tag{2.129}$$

The grouping of these three terms together then yields

$$\begin{aligned}&(-1)^{(|G|+1)(|g|+1)}\{G, \{f, g\}\} + (-1)^{(|g|+1)(|f|+1)}\{g, \{G, f\}\} \\ &\qquad + (-1)^{(|f|+1)(|G|+1)}\{f, \{g, G\}\},\end{aligned} \tag{2.130}$$

which vanishes due to the Jacobi identity.

We are thus left with three Δ_{BV}-dependent terms. Multiplying by $|f| + |g| + 1$

and noting that $|f|+|g|+1+|f||g| = (|f|+1)(|g|+1)$, we see that what remains to be shown is

$$\{f,g\}\Delta_{\mathrm{BV}}G + (-1)^{(|f|+1)(|g|+1)} g\Delta_{\mathrm{BV}}\{f,G\} = \{f, g\Delta_{\mathrm{BV}}G\}. \tag{2.131}$$

Using the Leibniz rule, the right hand side can be expanded to

$$\{f, g\Delta_{\mathrm{BV}}G\} = \{f,g\}\Delta_{\mathrm{BV}}G + (-1)^{(|f|+1)|g|} g\{f, \Delta_{\mathrm{BV}}G\}. \tag{2.132}$$

We are thus left with showing that

$$(-1)^{|f|+1} g\Delta_{\mathrm{BV}}\{f,G\} = g\{f, \Delta_{\mathrm{BV}}G\}. \tag{2.133}$$

Using computation (2.122) with $F \mapsto G$, we conclude this is equivalent to $g\Delta_{\mathrm{BV}}^2(fG) = 0$. This is true due to our induction hypothesis since $fG \in S^kV$.

Finally, we will show that it is a second order differential operator. This is an immediate consequence of the graded Leibniz rule of the graded Poisson bracket. Indeed, one can solve (2.118) to express the Poisson bracket in terms of Δ_{BV}. Namely, for all $F, G \in \mathcal{O}(V)$ we have that

$$\{F,G\} = (-1)^{|F|}\Delta_{\mathrm{BV}}(FG) - (-1)^{|F|}\Delta_{\mathrm{BV}}FG - F\Delta_{\mathrm{BV}}G \tag{2.134}$$

Then, by comparing the outcomes of

$$\{F,GH\} = (-1)^{|F|}\Delta_{\mathrm{BV}}(FGH) - (-1)^{|F|}\Delta_{\mathrm{BV}}FGH - F\Delta_{\mathrm{BV}}(GH), \tag{2.135}$$

and the Leibniz rule

$$\begin{aligned} \{F,GH\} &= \{F,G\}H + (-1)^{(|F|+1)|G|}G\{F,H\} \\ &= (-1)^{|F|}\Delta_{\mathrm{BV}}(FG)H - (-1)^{|F|}\Delta_{\mathrm{BV}}FGH - F\Delta_{\mathrm{BV}}GH \\ &\quad + (-1)^{|F|+(|F|+1)|G|}G\Delta_{\mathrm{BV}}(FH) - (-1)^{|F|+(|F|+1)|G|}G\Delta_{\mathrm{BV}}FH \\ &\quad - (-1)^{(|F|+1)|G|}GF\Delta_{\mathrm{BV}}H, \end{aligned} \tag{2.136}$$

we conclude that Δ_{BV} is second order. □

Example 2.7.1. Continuing with Example 2.6.2, the Laplacian obtained in the construction above can be written in coordinates as

$$\Delta_{\mathrm{BV}} = (-1)^{|\Phi^A|}\frac{\partial^2}{\partial\Phi^A\partial\Phi^*_A}. \tag{2.137}$$

Indeed, we have

$$\begin{aligned}\Delta_{\mathrm{BV}}(\Phi^B\Phi^*_C) &= (-1)^{|\Phi^A|}\frac{\partial^2}{\partial\Phi^A\partial\Phi^*_A}\left(\Phi^B\Phi^*_C\right)\\ &= (-1)^{|\Phi^A|+(|\Phi^A|+1)|\Phi^B|}\delta^B{}_A\delta^A{}_C = (-1)^{|\Phi^B|}\delta^B{}_C,\end{aligned} \tag{2.138}$$

which coincides with $(-1)^{|\Phi^B|}\{\Phi^B,\Phi^*_C\}$. Moreover, we can expand

$$\begin{aligned}\Delta_{\mathrm{BV}}(FG) &= (-1)^{|\Phi^A|}\frac{\partial}{\partial\Phi^A}\left(\frac{\partial F}{\partial\Phi^*_A}G + (-1)^{|\Phi^*_A||F|}F\frac{\partial G}{\partial\Phi^*_A}\right)\\ &= (-1)^{|\Phi^A|}\frac{\partial^2 F}{\partial\Phi^A\partial\Phi^*_A}G + (-1)^{|\Phi^A|+|\Phi^A|(|F|+|\Phi^*_A|)}\frac{\partial F}{\partial\Phi^*_A}\frac{\partial G}{\partial\Phi^A}\\ &+ (-1)^{|\Phi^*_A||F|+|\Phi^A|}\frac{\partial F}{\partial\Phi^A}\frac{\partial G}{\partial\Phi^*_A}\\ &+ (-1)^{|\Phi^*_A||F|+|\Phi^A||F|+|\Phi^A|}F\frac{\partial^2 G}{\partial\Phi^A\partial\Phi^*_A}\end{aligned} \tag{2.139}$$

We immediately recognize the actions of Δ_{BV} in the first and last terms. With respect to the last term, we can also use $|\Phi^A| + |\Phi^*_A| = 1$ to simplify the exponent in the sign. As for the second and third term we can modify the derivatives acting on F to right derivatives in the hope of recovering the BV bracket. Doing so yields

$$\begin{aligned}\Delta_{\mathrm{BV}}(FG) = (\Delta_{\mathrm{BV}}F)G &+ (-1)^{|F|}F(\Delta_{\mathrm{BV}}G)\\ &+ (-1)^{|\Phi^A|+|\Phi^A|(|F|+|\Phi^*_A|)+|\Phi^*_A|(|F|+1)}\frac{\partial_r F}{\partial\Phi^*_A}\frac{\partial G}{\partial\Phi^A}\\ &+ (-1)^{|\Phi^*_A||F|+|\Phi^A|+|\Phi^A|(|F|+1)}\frac{\partial_r F}{\partial\Phi^A}\frac{\partial G}{\partial\Phi^*_A}\end{aligned} \tag{2.140}$$

Studying the exponent of the signs in the last two factors we see that the $|F|$ dependent term will be $|F|(|\Phi^A| + |\Phi^*_A|) = |F|$. In the second term the remaining exponent is

$$|\Phi^A| + |\Phi^A||\Phi^*_A| + |\Phi^*_A| = (|\Phi^A|+1)(|\Phi^*_A|+1) + 1 = 1, \tag{2.141}$$

while the exponent of the remaining exponent of the last term is $|\Phi^A| + |\Phi^A| = 0$.

We thus recover the BV bracket and we have

$$\Delta_{\mathrm{BV}}(FG) = (\Delta_{\mathrm{BV}}F)G + (-1)^{|F|}F(\Delta_{\mathrm{BV}}G) + (-1)^{|F|}\{F, G\}. \tag{2.142}$$

Given our realization that any degree (1) vector space is a BV algebra, we may now wonder whether any degree (1) Poisson algebra leads to a BV algebra. Answers to this question and the uniqueness of the associated BV Laplacian can be found in [49].

Chapter 3

BV Formalism in Finite Dimensions

3.1 Gaussian Integration

In the path integral formulation of quantum field theory, expectation values are of the form

$$\langle F \rangle = \frac{1}{Z} \int \mathcal{D}\phi \, e^{-\frac{1}{\hbar}S} F, \tag{3.1}$$

with the normalization factor

$$Z = \int \mathcal{D}\phi \, e^{-\frac{1}{\hbar}S}. \tag{3.2}$$

This integral only has heuristic meaning as we shall now explain. Every field configuration ϕ has a probabilistic weight $\frac{1}{Z}e^{-\frac{1}{\hbar}S}$, where S is the Euclidean action functional. Thus, if F is an observable, i.e. a suitable (in a sense that will be discussed latter) function of the fields, its expectation value is given by the sum of its value at each field configuration weighed by the probability of such a configuration. Of course, the space of all field configuration is continuous and the sum needs to be interpreted as an integral over this space against some measure $\mathcal{D}\phi$. This is where the pitfall of the path integral formulation lies in. Such a measure in general does not exist.

Before seeing how the BV formalism addresses this issue, let us reflect on why path integrals may seem reasonable at a first glance. For this purpose, we recall the following theorem:

Theorem 3.1.1 (Riesz's Representation Theorem)**.** Let X be a locally compact Hausdorff space. If I is a positive linear functional on the compactly supported continuous functions $C_c(X)$ on X. Then, there exists a measure space structure (X, Σ, μ) such that

$$I(f) = \int \mathrm{d}\mu f \tag{3.3}$$

for all $f \in C_c(X)$.

Proof. [see 50, Theorem 12.36] □

Thus, if $\langle \cdot \rangle$ is to be a positive linear functional, it seems that the above theorem guarantees the existence of a measure that will aid us in its calculation. However, the devil lies in the details. A topological vector space, as we expect our space of field configurations to be, is locally compact if and only if it is finite dimensional [see 51, Theorems 1.21 and 1.22]. This is not the case unless our spacetime has a finite number of points. Thus, Theorem 3.1.1 does not guarantee the existence of such a measure in field theory.

Thus, we are now faced against the problem of finding another way to calculate such a functional. Our approach is based on the fact that linear functionals are determined, up to a constant, by their kernel [see 52, Proposition 1.1.1]. In the case where the integral does make sense, the divergence gives us a method of calculating elements of such a kernel through Stokes' theorem [see 53, equation $(*)$]. Thus, such a divergence allows us to study these functionals. The treatment below is based on the work in [54].

Let us illustrate this idea by studying the case where Theorem 3.1.1 applies. We will follow [17, 54]. Namely, consider a scalar field theory on a finite spacetime $M = \{ p_1, \dots, p_n \}$. The field configurations are given by real functions on M. This space is clearly identifiable with a real vector space V of dimension[1] n. Since this space is finite dimensional, we can associate to our expectation value a probability measure μ. Let $(\phi^1, \dots, \phi^n)$ be a basis for the dual of V for the rest of this chapter. Let us further assume μ is absolutely continuous with respect to $\mathrm{d}\lambda := \mathcal{D}\phi \equiv \mathrm{d}^n\phi := \mathrm{d}\phi^1 \wedge \cdots \wedge \mathrm{d}\phi^n$ and that the Radon-Nykodim derivative $\mathrm{d}\mu/\mathrm{d}\lambda$ is always positive . Then, taking our action to be

$$S := -\hbar \log\left(\mathcal{N}\frac{\mathrm{d}\mu}{\mathrm{d}\lambda}\right), \tag{3.4}$$

for some $\mathcal{N} \in (0, \infty)$, we obtain[2]

$$\langle F \rangle = \frac{1}{Z}\int \mathcal{D}\phi\, e^{-\frac{1}{\hbar}S} F, \qquad Z := \int \mathcal{D}\phi\, e^{-\frac{1}{\hbar}S}. \tag{3.5}$$

[1] Of course, such a structure can arise in several different ways. For example, n-fields living in a spacetime consisting of only one point would be described by the same type of vector space.

[2] For the reader that has not yet been introduced to measure theory, this can be taken as the starting point of our discussion. The material above served the purpose of illustrating how in this case Theorem 3.1.1 yields to a generic integral of the form (3.1). All of the required material is however found in [50].

The constant $\mathcal{N}$ is added because we want to identify S with the action of our system, which usually doesn't lead to automatically normalized measures like μ.

As usual in field theory, let us now specialize to the case of

$$S = Q + b, \tag{3.6}$$

for some positive-definite quadratic form Q and some polynomial b of order higher[3] than 3. We will also assume that S is increasing at infinity and we will denote by A the positive-definite symmetric matrix for which $Q = \frac{1}{2}\phi^i A_{ij}\phi^j$. We want to define a divergence div_μ such that

$$e^{-\frac{1}{\hbar}S}\operatorname{div}_\mu F = \operatorname{div}\left(e^{-\frac{1}{\hbar}S}F\right) \tag{3.7}$$

for suitable vector fields F on V. In particular, if $F = F^i\,\partial/\partial\phi^i$ is a vector field with polynomial components, the rapid decrease of the exponential guarantees

$$\langle \operatorname{div}_\mu F\rangle = \frac{1}{Z}\int \mathcal{D}\phi\, e^{-\frac{1}{\hbar}S}\operatorname{div}_\mu F = \frac{1}{Z}\int \mathcal{D}\phi\, \operatorname{div}\left(e^{-\frac{1}{\hbar}S}F\right) = 0. \tag{3.8}$$

Equation (3.7) is immediately solved for these vector fields

$$\operatorname{div}_\mu F = e^{\frac{1}{\hbar}S}\operatorname{div}\left(e^{-\frac{1}{\hbar}S}F\right) = -\frac{1}{\hbar}\frac{\partial S}{\partial\phi^i}F^i + \frac{\partial F^i}{\partial\phi^i}. \tag{3.9}$$

Thus, if we let $\operatorname{Vect}(V)$ be the set of all polynomial vector fields, this defines a map

$$\operatorname{div}_\mu : \operatorname{Vect}(V) \to \mathcal{O}(V). \tag{3.10}$$

Applying (3.8), we conclude that

$$\left\langle \frac{\partial S}{\partial\phi^i}F^i\right\rangle = \hbar\left\langle\frac{\partial F^i}{\partial\phi^i}\right\rangle. \tag{3.11}$$

This is known as the Schwinger-Dyson equation ([see 32, (15.25)] or, for its appearance in pAQFT, [see 22, Remark 7.7]).

As an example of the use of this formula, let us take $b = 0$, which corresponds to a free theory. Now consider a set of indices $i_1, \dots, i_n, j \in \{1, \dots, n\}$. We can

[3] At this level considering polynomials of order one is futile since we can always complete the square.

compute

$$-\hbar \operatorname{div}_\mu \left(\phi^{i_1} \cdots \phi^{i_n} \frac{\partial}{\partial \phi^j} \right) = \phi^i A_{ij} \phi^{i_1} \cdots \phi^{i_n} \\ - \hbar \sum_{r=1}^{n} \phi^{i_1} \cdots \widehat{\phi^{i_r}} \cdots \phi^{i_n} \delta^{i_r}{}_j. \tag{3.12}$$

Thus, if we let A^{ij} be the inverse matrix to A_{ij}, i.e. $A^{ij}A_{jk} = \delta^i{}_k$, we conclude

$$\phi^i \phi^{i_1} \cdots \phi^{i_n} = -\hbar \operatorname{div}_\mu \left(\phi^{i_1} \cdots \phi^{i_n} A^{ij} \frac{\partial}{\partial \phi^j} \right) \\ + \hbar \sum_{r=1}^{n} A^{i i_r} \phi^{i_1} \cdots \widehat{\phi^{i_r}} \cdots \phi^{i_n}. \tag{3.13}$$

By taking the expectation value of both sides and doing a simple relabeling, we go back to a special case of the Schwinger-Dyson equation

$$\left\langle \phi^{i_1} \cdots \phi^{i_n} \right\rangle = \hbar \sum_{r=2}^{n} A^{i_1 i_r} \left\langle \phi^{i_2} \cdots \widehat{\phi^{i_r}} \cdots \phi^{i_n} \right\rangle. \tag{3.14}$$

This recurrence relation instructs us to pair the element ϕ^{i_1} which any element ϕ^{i_r} in the list $(\phi^{i_2}, \ldots, \phi^{i_n})$, remove these two and replace them by $A^{i_1 i_r}$, and take the expectation value of the rest. By the symmetry of our μ, it is clear that if n is odd, the expectation value is null. On the other hand, if n is even, given that $\langle 1 \rangle = 1$ we can continue our recursion relation to

$$\left\langle \phi^{i_1} \cdots \phi^{i_n} \right\rangle = \hbar^{n/2} \sum_{P \in \operatorname{Pair}(n)} \prod_{\{r,s\} \in P} A^{i_r i_s}, \tag{3.15}$$

where the set of pairings $\operatorname{Pair}(n)$ is defined to be the set of partitions of $\{1, \ldots, n\}$ into sets of 2 elements. This is of course just Wick's theorem! We have found a way of computing expectation values without actually needing to integrate. Although there is a measure μ in this case, knowledge of div_μ was the only thing we needed to proof this theorem.

3.2 Homological Picture of Integration: Antifields

Let us now build a homological picture of this construction. To do this note that we can understand the map $\langle \cdot \rangle : \mathcal{O}(V) \to \mathbb{R}$ through the induced diagram

$$\begin{tikzcd} \mathcal{O}(V) \arrow[r, two heads] \arrow[rr, bend right, "\langle \cdot \rangle"'] & \mathcal{O}(V)/\ker\langle \cdot \rangle \arrow[r, "{[1]\mapsto 1}"] & \mathbb{R}, \end{tikzcd} \tag{3.16}$$

In here the first arrow is the canonical projection into the quotient, while the second is the isomorphism induced by $\langle \cdot \rangle$ using the first isomorphism theorem [see 52, Proposition 1.1.1]. The latter is fixed by the normalization $\langle 1 \rangle = 1$. Thus, we see that the computation of $\langle \cdot \rangle$ can be achieved by computing $\mathcal{O}(V)/\ker\langle \cdot \rangle$. In the previous section we attempted to do this using the operator div_μ, for which $\mathrm{im}\,\mathrm{div}_\mu \subseteq \ker\langle \cdot \rangle$. The computation of the quotient $\mathcal{O}(V)/\mathrm{im}\,\mathrm{div}_\mu$ then corresponded to the Schwinger-Dyson equation (3.11). Comparing with the Definition 2.1.3, we see this would correspond to the cohomology in the cohomological degree of $\mathcal{O}(V)$ of a cochain complex with

$$\cdots \mathrm{Vect}(V) \to \mathcal{O}(V) \to 0 \cdots . \tag{3.17}$$

To obtain such a complex, note that we can identify

$$\mathrm{Vect}(V) \cong \mathcal{O}(V) \otimes V \tag{3.18}$$

by identifying vectors with the directional derivatives along them. Thus, if $F \in \mathcal{O}(V)$ and $\phi \in V$, we identify the vector field $F\partial_\phi$ with $F \otimes \phi$. Moreover $\mathcal{O}(V)$ and $\mathrm{Vect}(V) \cong \mathcal{O}(V) \otimes V$ are the smooth functions in cohomological degree 0 and -1 of the degree (-1) symplectic vector space $\mathcal{E} = V \oplus V^*[-1]$. In light of Example 2.7.1, we have that $\mathcal{O}(\mathcal{E})$ is a BV algebra and in particular has a degree (1) Poisson structure. In this identification we can use coordinates $\phi_i^* := \partial/\partial\phi^i$ for $V^*[-1]$ such that our vector fields on V can be written as $\phi_i^* F^i$. We will keep this convention for the rest of the chapter. These are called antifields to the fields ϕ^i. Under these identifications, our divergence (3.9) is a degree (1) map

$$-\hbar\,\mathrm{div}_\mu = \frac{\partial S}{\partial \phi^i}\frac{\partial}{\partial \phi_i^*} - \hbar\frac{\partial^2}{\partial\phi^i\partial\phi_i^*} = \{S, \cdot\} - \hbar\Delta_{\mathrm{BV}} : \mathcal{O}(\mathcal{E})^{-1} \to \mathcal{O}(E)^0 \tag{3.19}$$

Via this formula, it extends to a degree (1) algebra homomorphism Q on $\mathcal{O}(\mathcal{E})$. The corresponding cochain complex is then of the form

$$\cdots \mathcal{O}(V) \otimes V \wedge V \xrightarrow{Q} \mathcal{O}(V) \otimes V \xrightarrow{Q} \mathcal{O}(V) \xrightarrow{Q} 0 \cdots, \tag{3.20}$$

which we will call the quantum Koszul complex. In fact, Q can be decomposed into the Hamiltonian graded vector field $\delta := \partial S/\partial\phi^i \; \partial/\partial\phi_i^* = \{S, \cdot\}$ and the BV Laplacian $\Delta_{\mathrm{BV}} = \partial^2/\partial\phi^i\partial\phi_i^*$. Via Theorem 2.6.2, δ is cohomological. In this case it is clear that the classical master equation is satisfied since S only depends on the ϕ coordinates. By this same fact, the quantum master equation is also satisfied. We conclude by Theorem 2.7.3 that Q is a differential.

In the free case we showed in (3.13) that

$$H^0(\mathcal{O}(\mathcal{E}), Q) = {}^{\ker Q^0}\!\big/_{\mathrm{im}\, Q^{-1}} = {}^{\mathcal{O}(V)}\!\big/_{\mathrm{im}\, Q^{-1}} \cong \mathbb{R}. \tag{3.21}$$

Then we conclude our expectation value map is precisely the projection

$$\langle \cdot \rangle : \mathcal{O}(\mathcal{E}) \to H^0(\mathcal{O}(\mathcal{E}), Q), \tag{3.22}$$

once we fix $\langle 1 \rangle = 1$.

Let us remark that in our finite dimensional case, the cochain complex $\mathcal{O}(\mathcal{E})$ is nothing but a disguise for the de Rham complex. Namely, the path integral measure yields a map

$$\mathcal{O}(V) \otimes \overbrace{V \wedge \cdots \wedge V}^{m \quad \text{times}} \to \Omega^{N-m}(V), \tag{3.23}$$

given by contracting a given multivector field with the top form $\mathcal{D}\phi\, e^{-\frac{1}{\hbar}S}$. For example, under this map the vector field $\phi_i^* F^i$ is mapped to

$$\sum_{i=1}^{N} (-1)^{i-1} e^{-\frac{1}{\hbar}S} F^i \,\mathrm{d}\phi^1 \wedge \cdots \wedge \widehat{\mathrm{d}\phi^i} \wedge \cdots \wedge \mathrm{d}\phi^N . \tag{3.24}$$

Applying $-\hbar\mathrm{d}$ to this form we obtain

$$\sum_{i=1}^{N} (-1)^{i-1} e^{-\frac{1}{\hbar}S} \left(\frac{\partial S}{\partial \phi^j} F^i - \hbar \frac{\partial F^i}{\partial \phi^j} \right) \mathrm{d}\phi^j \wedge \mathrm{d}\phi^1 \wedge \cdots \wedge \widehat{\mathrm{d}\phi^i} \wedge \cdots \wedge \mathrm{d}\phi^N , \tag{3.25}$$

which is simply $\mathcal{D}\phi\, e^{-\frac{1}{\hbar}S} Q(\phi_i^* F^i)$. Since both Q and $-\hbar \mathrm{d}$ are extended to $\mathcal{O}(E)$ and $\Omega(V)$ using the Leibniz rule, we see that we have a commutative diagram

$$\begin{tikzcd} \cdots & \mathcal{O}(V)\otimes V\wedge V \arrow[r, "Q"] \arrow[d] & \mathcal{O}(V)\otimes V \arrow[r, "Q"] \arrow[d] & \mathcal{O}(V) \arrow[d] \\ & \Omega^{N-2}(V) \arrow[r, "-\hbar \mathrm{d}"] & \Omega^{N-1}(V) \arrow[r, "-\hbar \mathrm{d}"] & \Omega^{N}(V) \end{tikzcd} \tag{3.26}$$

However, in infinite dimensions we do not have a notion of top form. This is the reason why this change of perspective plays an important role in QFT.

At this point, we note that the quantum Koszul complex makes sense even in the case where V is a bigraded vector space concentrated in $\mathbb{Z}/2\mathbb{Z} \times \{0\}$ (i.e. a super vector space in cohomological degree 0) as long as we introduce the proper sign conventions to the BV bracket (2.91)

$$\{S, \cdot\} = \frac{\partial_r S}{\partial \phi^i} \frac{\partial}{\partial \phi_i^*}, \tag{3.27}$$

and Laplacian (2.137)

$$\Delta_{\mathrm{BV}} = (-1)^{|\phi^i|} \frac{\partial^2}{\partial \phi^i \partial \phi_i^*}. \tag{3.28}$$

As a consequence, the vector fields $\phi_i^* F^i$ act naturally on $G \in \mathcal{O}(V)$ from the right via

$$\{G, \phi_i^* F^i\} = \frac{\partial_r G}{\partial \phi^i} F^i. \tag{3.29}$$

We will also define $\epsilon^i := |\phi^i|$ and $\epsilon_i^* := |\phi_i^*| = \epsilon^i + 1$. We will extend to this case from now on to include fermionic theories into our discussion.

3.3 Koszul Complex: Going On-Shell

Let us first begin with a theorem from differential geometry which will proof useful for us.

Theorem 3.3.1 ([see 32, Theorem 1.1])**.** Let N be a regular zero set of a smooth map $x : M \to \mathbb{R}^n$ on a manifold M of dimension $n+m$. Then every $F \in C^\infty(M)$ vanishing on N is of the form $F = x^i G_i$ for some $G_1, \ldots, G_n \in C^\infty(M)$.

Proof. First, note that by the Regular Level Set Theorem [see 44, Theorem 9.9], N is a regular submanifold. Moreover, every $p \in N$ is covered by a chart $(U, \phi =$

$(x, y))$ adapted to N [see 44, Lemma 9.10]. Consider the coordinate representation $F_\phi := F \circ \phi^{-1}$ of F. Then for all $a \in x(U) \subseteq \mathbb{R}^n$ and $b \in y(U) \subseteq \mathbb{R}^m$ we have

$$\begin{aligned} F_\phi(a,b) &= \int_0^1 \mathrm{d}t \, \frac{\mathrm{d}}{\mathrm{d}t} F_\phi(ta,b) = \int_0^1 \mathrm{d}t \, \frac{\partial F}{\partial x^i}\Big(\phi^{-1}(ta,b)\Big) a^i \\ &= (x^i \circ \phi^{-1})(a,b) \int_0^1 \mathrm{d}t \, \frac{\partial F}{\partial x^i}\Big(\phi^{-1}(ta,b)\Big). \end{aligned} \tag{3.30}$$

Thus, if one defines

$$G_i(p) := \int_0^1 \mathrm{d}t \, \frac{\partial F}{\partial x^i}\Big(\phi^{-1}(tx(p), y(p))\Big), \tag{3.31}$$

We conclude that $F = x^i G_i$ on U.

On the other hand, if $p \in M$ is not in N, there must be a component x^i which does not vanish at this point. In particular, there must a neighborhood of p where the component x^i does not vanish (take for example $(x^i)^{-1}(\mathbb{R} \setminus \{0\})$). We thus have that $F = x^i G$, where $G = F/x^i$, on this neighborhood. We have thus shown that every $p \in M$ has a neighborhood $U \subseteq M$ where $F = x^i G_i^{(U)}$ for some $G_1, \dots, G_n \in C^\infty(U)$.

Recall that every manifold is paracompact [see 55, Theorem 4.77]. We thus have a locally finite open cover $\{ U_\alpha \mid \alpha \in I \}$ of M for which in each $\alpha \in I$ there are $G_1^\alpha, \dots, G_n^\alpha \in C^\infty(U_\alpha)$ such that $F = x^i G_i^\alpha$ on U_α. Moreover, we have a partition of unity $\{ \rho_\alpha \mid \alpha \in I \}$ subbordinate to this cover [see 44, Theorem 13.7]. Using this we have that for each $\alpha \in I$ the support of $\rho_\alpha G^\alpha$ is in U_α, and thus the function can be extended to all of M [see 44, Proposition 13.2]. We can then define the global functions $G_i := \sum_{\alpha \in A} \rho_\alpha G_i^\alpha \in C^\infty(M)$. Finally, we have

$$x^i G_i = \sum_{\alpha \in A} x^i G_i^\alpha \rho_\alpha = \sum_{\alpha \in A} F \rho_\alpha = F. \tag{3.32}$$

□

Now, consider the limit $\hbar \to 0$ of our previous discussion. In this limit our differential Q reduces to the cohomological Hamiltonian vector field $\delta := \{S, \cdot\} = \partial_r S / \partial \phi^i \, \partial / \partial \phi_i^*$. Let us study the cohomology in degree 0. First of all, note that every element $\mathcal{O}(\mathcal{E})^0_{\mathrm{deg}} = \mathcal{O}(V)$ is automatically closed since it is independent of the antifields. On the other hand, for a generic vector field $\phi_i^* F^i \in \mathcal{O}(V) \otimes V$, the element $\delta(\phi_i^* F^i) = \partial_r S / \partial \phi^i \, F^i$ corresponds simply to the infinitesimal change due to this vector field on the action function from the right. In particular, by definition of the equations of motion[4] $\partial S/\partial\phi^i = 0$, the image of δ in degree 0 consists of the functions on $\mathcal{O}(V)$ which vanish on-shell[5]. Moreover, by Theorem 3.3.1, under certain regularity assumptions, we can assume that all functions which vanish on-shell are of the form $\partial_r S/\partial\phi^i\, F^i$. We thus conclude that $H^0(\mathcal{O}(\mathcal{E}),\delta)$ corresponds to identifying the functions of our fields in V if they coincide on-shell. As a consequence, we will call these the on-shell functions. This construction is known as the Koszul complex of the ideal generated by the equations of motion $\partial S/\partial\phi^i$ in $\mathcal{O}(V)$.

Let us now turn to the study of $H^{-1}(\mathcal{O}(\mathcal{E}),\delta)$. As we already noticed, the action of δ on the vector fields $\mathcal{O}(\mathcal{E})^{-1}_{\text{deg}} = \mathcal{O}(V)\otimes V$ is simply to let the vector field act on the action functional S. Therefore, the closed elements Z^{-1}, which is the set of vector fields $\phi_i^* F^i$ for which $\partial_r S/\partial\phi^i\, F^i = 0$, are precisely symmetries of the action. On the other hand, a generic bivector field $\frac{1}{2}\phi_i^*\phi_j^* F^{ij}$, where we can choose $F^{ij} = (-1)^{\epsilon_i^*\epsilon_j^*} F^{ji}$ without loss of generality, is sent to the symmetry $\delta(\frac{1}{2}\phi_i^*\phi_j^* F^{ij}) = \partial_r S/\partial\phi^i\, \phi_j^* F^{ij} = (-1)^{\epsilon_j^*\epsilon^i}\phi_j^*\, \partial_r S/\partial\phi^i\, F^{ij}$. The fact that this is indeed a symmetry is a consequence of $\delta^2 = 0$, i.e. the classical master equation. We can however explicitly check this by noting that

$$\delta^2\left(\frac{1}{2}\phi_i^*\phi_j^* F^{ij}\right) = \delta\left(\frac{\partial_r S}{\partial\phi^i}\phi_j^* F^{ij}\right) = (-1)^{\epsilon_j^*\epsilon^i}\frac{\partial_r S}{\partial\phi^j}\frac{\partial_r S}{\partial\phi^i}F^{ij}. \tag{3.33}$$

Then the coefficient $\mu^{ij} := (-1)^{\epsilon_j^*\epsilon^i} F^{ij}$ is graded antisymmetric

$$\begin{aligned}\mu^{ij} &= (-1)^{\epsilon_j^*\epsilon^i} F^{ij} = (-1)^{\epsilon^j\epsilon^i+\epsilon^i} F^{ij} = (-1)^{\epsilon^j\epsilon^i+\epsilon^i+\epsilon_j^*\epsilon_i^*} F^{ji} \\ &= (-1)^{\epsilon^j\epsilon^i+\cancelto{1}{(\epsilon^i+\epsilon_i^*)}+\epsilon^j\epsilon_i^*} F^{ji} = -(-1)^{\epsilon^j\epsilon^i}\mu^{ji},\end{aligned} \tag{3.34}$$

while the term $\partial_r S/\partial\phi^j\ \partial_r S/\partial\phi^i$ is graded symmetric. Therefore, the contraction vanishes. These symmetries are trivial because they vanish on-shell and act trivially on the solutions of the equations of motion. The degree -1 cohomology $H^{-1}(\mathcal{O}(\mathcal{E}),\delta)$ then yields all of the non-trivial symmetries.

Let us now remark on the graded Lie algebra structure of this complex. By the Definition 2.6.1, the BV bracket gives $\mathcal{O}(\mathcal{E})$ the structure of a graded Lie algebra of degree (1). In particular, the set of vector fields $\mathcal{O}(\mathcal{E})^{-1}_{\text{deg}}$ is a super Lie algebra. The BV bracket structure is compatible with the usual structure of super Lie algebra on

[4]Remember, the equations of motion are obtained by demanding that the action be stationary under an arbitrary local variation (vector field).

[5]We will reserve the use of "on-shell" for the solutions of the equations of motion

vector fields $\mathcal{O}(V) \otimes V$ obtained by the commutator. Indeed, consider $F = \phi_i^* F^i$ and $G = \phi_j^* F^j$

$$\{\phi_i^* F^i, \phi_j^* G^j\} = \phi_i^* \frac{\partial_r F^i}{\partial \phi_j} G^j - (-1)^{\epsilon_i^* |F^i| + \epsilon^i \epsilon_j^*} F^i \phi_j^* \frac{\partial G^j}{\partial \phi^i}. \tag{3.35}$$

If we attempt to put the last term into the form $\phi_j^* \, \partial_r G^j / \partial \phi^i \, F^i$, the overall exponent in the sign becomes (recall the definition in Theorem 2.3.1)

$$\epsilon_i^* |F^i| + \epsilon^i \epsilon_j^* + |F^i|(\epsilon_j^* + |G^j| + \epsilon^i) + \epsilon^i(|G^j| + 1). \tag{3.36}$$

If we now assume that our vector fields are homogeneous, so that $|F| = |F^i| + \epsilon^i$ and $|G| = |G^i| + \epsilon^i$, we have

$$\begin{aligned} \epsilon_i^*(|F| + \epsilon^i) + \cancel{\epsilon^i \epsilon_j^*} + (|F| + \epsilon^i)(|G| + \epsilon^i + 1) + \epsilon^i(|G| + \cancel{\epsilon^j + 1}) = \\ \epsilon_i^*(\cancel{|F|} + \cancel{\epsilon^i}) + |F|(|G| + \cancel{\epsilon^i + 1}) + \epsilon^i(\cancel{|G|} + \cancel{\epsilon^i + 1}) + \cancel{\epsilon^i |G|} = |F||G|. \end{aligned} \tag{3.37}$$

We thus conclude that

$$\{\phi_i^* F^i, \phi_j^* G^j\} = \phi_i^* \frac{\partial_r F^i}{\partial \phi_j} G^j - (-1)^{|F||G|} \phi_j^* \frac{\partial_r G^j}{\partial \phi^i} F^i = \{\,\cdot\,, [G, F]\}, \tag{3.38}$$

where $[G, F]$ is the commutator of the right vector fields $G = \{\,\cdot\,, \phi_i^* G^i\}$ and $F = \{\,\cdot\,, \phi_i^* F^i\}$. On top of this, $\mathcal{O}(\mathcal{E})[1]$ is in fact a dg Lie algebra with differential $\{S, \cdot\}$.

3.4 Lie Algebras: Ghosts

In physics, Lie groups usually represent symmetries of physical systems. These are usually referred to as continuous symmetries. The set of infinitesimal deviations from the identity of such groups has the structure of a Lie algebra. In the case that fermionic symmetries are present, one in fact requires a super Lie algebra. One can reconstruct some of the Lie group elements by a process of infinite iteration of these infinitesimal deviations. This process is known as exponentiation. Depending on the group, this process yields more or less information. However, it is always true that this process reconstructs a neighborhood of the identity.

To continue our excursion into the BV formalism, we will recast the definition of a super Lie algebra in the language of graded manifolds. In particular, we will see that the super Lie brackets on a super vector space $\mathfrak{g}$ are in one to one correspondance with the cohomological vector fields on $\mathfrak{g}[1]$. This section and the next are extensions of the material found in [56].

For the rest of this chapter, let us fix $\mathfrak{g}$ to be a super Lie algebra. As we saw, the Lie algebra of vector fields in the Koszul complex was in cohomological degree -1. Consequently, we would like to reinterpret the Lie bracket on $\mathfrak{g}$ as a structure in $\mathfrak{g}[1]$ by making use of (2.53). This yields the degree (1) map

$$\operatorname{dec}([\cdot,\cdot]) : S^2(g[1]) \to \mathfrak{g}[1] \tag{3.39}$$

We can now consider the dual of this map using (2.17)

$$\delta_{[\cdot,\cdot]} := \operatorname{dec}([\cdot,\cdot])^* : \mathfrak{g}^*[-1] \to S^2(g[1])^*. \tag{3.40}$$

Replacing the codomain by $S^2(\mathfrak{g}[1]^*) \subseteq \mathcal{O}(\mathfrak{g}[1])$ using (2.13) and applying Theorem 2.5.1, this extends to a degree (1) vector field on $\mathfrak{g}[1]$.

To be explicit, let us calculate this map on a basis $\left(X_1, \dots, X_{m:=\dim \mathfrak{g}}\right)$ of $\mathfrak{g}$. We will also denote the dual basis by $\left(c^1, \dots, c^m\right)$ and define $\epsilon^a = |c^a|_{\mathfrak{g}[1]^*} = |X_a|_{\mathfrak{g}[1]} = |X_a|_{\mathfrak{g}} + 1$. We will refer to the elements $c^a \in \mathcal{O}(\mathfrak{g}[1])$ as ghosts. Let $f^a{}_{bc}$ be the associated structure constants, i.e.

$$f^a{}_{bc} = c^a([X_b, X_c]). \tag{3.41}$$

We will keep these conventions throughout the rest of this chapter. Given Definition 2.4.1, the structure constants are graded antisymmetric in the bottom indices

$$\begin{aligned} f^a{}_{bc} &= c^a([X_b, X_c]) = -(-1)^{(\epsilon^b+1)(\epsilon^c+1)} c^a([X_c, X_b]) \\ &= -(-1)^{(\epsilon^b+1)(\epsilon^c+1)} f^a{}_{cb} = -(-1)^{(\epsilon^b+1)(\epsilon^c+1)} f^a{}_{cb} \end{aligned} \tag{3.42}$$

and satisfy the Jacobi identity

$$\begin{aligned} 0 &= c^a((-1)^{(\epsilon^b+1)(\epsilon^d+1)}[X_b,[X_c,X_d]] + (-1)^{(\epsilon^d+1)(\epsilon^c+1)}[X_d,[X_b,X_c]] \\ &\quad + (-1)^{(\epsilon^c+1)(\epsilon^b+1)}[X_c,[X_d,X_b]]) \\ &= (-1)^{(\epsilon^b+1)(\epsilon^d+1)} f^a{}_{be} f^e{}_{cd} + (-1)^{(\epsilon^d+1)(\epsilon^e+1)} f^a{}_{de} f^e{}_{bc} \\ &\quad + (-1)^{(\epsilon^c+1)(\epsilon^b+1)} f^a{}_{ce} f^e{}_{db}. \end{aligned} \tag{3.43}$$

According to (2.53), the resulting shifted Lie bracket is then

$$\operatorname{dec}([\cdot,\cdot])(X_b X_c) = (-1)^{\epsilon^a+1}[X_b, X_c] = (-1)^{\epsilon^b+1} f^a{}_{bc} X_a. \tag{3.44}$$

Following (2.17), its dual is then

$$(\operatorname{dec}([\cdot,\cdot])^* c^a)(X_b X_c) = (-1)^{\epsilon^a+\epsilon^b+1} f^a{}_{bc} = f^a{}_{bc}(-1)^{\epsilon^c}. \tag{3.45}$$

In here we used the fact that $[\cdot,\cdot]$ has degree (0) and thus $f^a{}_{bc} = 0$ unless

$$0 = \epsilon^a + 1 + \epsilon^b + 1 + \epsilon^c + 1 = \epsilon^a + \epsilon^b + \epsilon^c + 1. \tag{3.46}$$

One can express $\operatorname{dec}([\cdot,\cdot])^* c^a \in S^2(\mathfrak{g}[1])^*$ as an element of $S^2(\mathfrak{g}[1]^*)$ following (2.13). Let us start by computing

$$\begin{aligned}\frac{1}{2} f^a{}_{de}(-1)^{\epsilon^e} c^e c^d (X_b X_c)& \\ &= \frac{1}{2} f^a{}_{de}(-1)^{\epsilon^e}\left((-1)^{\epsilon^b\epsilon^d}\delta^e{}_b\delta^d{}_c + (-1)^{\cancel{\epsilon^e\epsilon^d}+\cancel{\epsilon^e\epsilon^b}}\delta^d{}_b\delta^e{}_c\right). \end{aligned}\tag{3.47}$$

The cancellation of the signs on the second term of the right hand side is due to the constraints imposed by the Kronecker deltas. Now, using the graded antisymmetry of the structure constants, we can see that the coefficient $f^a{}_{de}(-1)^{\epsilon^e}$ is in fact graded symmetric. This should be the case to ensure that the term on the left hand side makes sense in $\mathcal{O}(\mathfrak{g}[1])$. This is easily seen by multiplying (3.42) by $(-1)^{\epsilon^c}$ and noticing that the overall sign on the right hand side will end up having exponent

$$(\epsilon^b+1)(\epsilon^c+1) + 1 + \epsilon^c = \epsilon^b\epsilon^c + \epsilon^b. \tag{3.48}$$

As a result, both of the terms in the right hand side of (3.47) yield the same outcome. In particular, focusing on the form obtained from the second term, we conclude that

$$\frac{1}{2} f^a{}_{de}(-1)^{\epsilon^e} c^e c^d (X_b X_c) = f^a{}_{cb}(-1)^{\epsilon^c} = \operatorname{dec}([\cdot,\cdot])^* c^a)(X_b X_c). \tag{3.49}$$

The resulting vector field is

$$\delta_{[\cdot,\cdot]} = \frac{1}{2} f^a{}_{bc}(-1)^{\epsilon^c} c^c c^b \frac{\partial}{\partial c^a}. \tag{3.50}$$

It turns out that $\delta_{\mathfrak{g}}$ extends to a graded Hamiltonian vector field. To obtain a

Poisson bracket, we first extend to the space $\mathcal{E} = \mathfrak{g}[1] \oplus \mathfrak{g}^*[-2]$. Much like in section 3.2, this is a symplectic vector space of degree (-1). In particular, we have the coordinates $(c^1, \ldots, c^m, c_1^*, \cdots, c_m^*)$ on $\mathcal{E}$ dual to the basis $(X_1, \ldots, X_m, c^1, \ldots, c^m)$, in which we can define

$$S_{[\cdot,\cdot]} = \frac{1}{2} c_a^* f^a{}_{bc} (-1)^{\epsilon^b+1} c^c c^b. \tag{3.51}$$

The new elements $c_a^* \in \mathcal{O}(\mathcal{E})$ will be referred to as the antifields of our ghosts c^a. Let us also set $\epsilon_a^* := |c_a^*|_{\mathfrak{g}^*[-2]^*} = \epsilon^a + 1$. This function allows us to us to define the graded Hamiltonian vector field $\delta_\mathfrak{g} = \{S_\mathfrak{g}, \cdot\}$ on $\mathcal{E}$. Notice that the extended $\delta_\mathfrak{g}$ agrees with the previous one on $\mathcal{O}(\mathfrak{g}[1])$, which can be considered to be a graded subalgebra of $\mathcal{O}(\mathcal{E})$ via pullback with the projection $\mathfrak{g}[1] \oplus \mathfrak{g}^*[-2] \to \mathfrak{g}[1]$. Indeed, we have

$$\{S_{[\cdot,\cdot]}, c^a\} = -\frac{1}{2} f^a{}_{bc} (-1)^{\epsilon^b+1+\epsilon_a^*(\epsilon^b+\epsilon^c)} c^c c^b. \tag{3.52}$$

Using the conditions (3.46) we can reduce the exponent of the overall sign to

$$\epsilon^b + 1 + \epsilon_a^*(\epsilon^a + 1) + 1 = \epsilon^b + (\epsilon^a + 1)^2 = \epsilon^b + \epsilon^a + 1 = \epsilon^c, \tag{3.53}$$

which yields

$$\{S_{[\cdot,\cdot]}, c^a\} = \frac{1}{2} f^a{}_{bc} (-1)^{\epsilon^c} c^c c^b = \delta_{[\cdot,\cdot]} c^a. \tag{3.54}$$

This new graded vector field is cohomological due to Theorem 2.6.2 since $S_{[\cdot,\cdot]}$ satisfies the classical master equation. The latter is easily computed in the form (2.93). To begin with, note that the coefficient $f^a{}_{bc}(-1)^{\epsilon^b+1} = f^a{}_{bc}(-1)^{\epsilon^c+\epsilon^a}$ inherits the graded symmetry of $f^a{}_{bc}(-1)^{\epsilon^c}$. Consequently, the two terms in the derivative

$$\frac{\partial_r S_{[\cdot,\cdot]}}{\partial c^d} = \frac{1}{2} c_a^* (f^a{}_{dc} (-1)^{\epsilon^d+1} c^c + f^a{}_{bd} (-1)^{\epsilon^b+1+\epsilon^d\epsilon^b} c^b), \tag{3.55}$$

coincide. This yields

$$\frac{\partial_r S_{[\cdot,\cdot]}}{\partial c^d} = c_a^* f^a{}_{dc} (-1)^{\epsilon^d+1} c^c, \tag{3.56}$$

and

$$\begin{aligned} \{S_{[\cdot,\cdot]}, S_{[\cdot,\cdot]}\} &= c_a^* f^a{}_{bc} (-1)^{\epsilon^b+1} c^c f^b{}_{de} (-1)^{\epsilon^d+1} c^e c^d \\ &= c_a^* (-1)^{\epsilon^a+\epsilon^c+\epsilon^d+1} f^a{}_{bc} f^b{}_{de} c^c c^e c^d \end{aligned} \tag{3.57}$$

In here we have used (3.46) to make the overall sign independent of the index b. To take the graded symmetrization in (c, e, d) we need to sum over the permutations of these indices. In fact, since the structure constants $f^b{}_{de}(-1)^{\epsilon^d+1}$ are already graded symmetric in its two lower indices, we only need to sum over cyclic permutations. We conclude that the classical master equation is satisfied if and only if

$$(-1)^{\epsilon^a+\epsilon^c+\epsilon^d+1} f^a{}_{bc} f^b{}_{de} + (-1)^{\epsilon^d(\epsilon^c+\epsilon^e)+\epsilon^a+\epsilon^d+\epsilon^e+1} f^a{}_{bd} f^b{}_{ec} \\ + (-1)^{\epsilon^c(\epsilon^e+\epsilon^d)+\epsilon^a+\epsilon^e+\epsilon^c+1} f^a{}_{be} f^b{}_{cd} \tag{3.58}$$

vanishes. Multiplying the whole expression by $(-1)^{\epsilon^a+\epsilon^d\epsilon^c}$, we obtain the graded Jacobi identity (3.42).

The resulting cochain complex $(\mathcal{O}(\mathfrak{g}[1]), \delta_{[\cdot,\cdot]})$ is called the Chevalley-Eilenberg complex for cohomology of $\mathfrak{g}$. It is useful to also remark that in the case where all symmetries are bosonic the expression for the action simplifies to

$$S_{[\cdot,\cdot]} = -\frac{1}{2} c^*_a f^a{}_{bc} c^b c^c. \tag{3.59}$$

3.5 DG Lie algebras: Ghosts for ghosts

Recall that the Koszul complex discussed in Section 3.3 described the structure of the Lie algebra of all infinitesimal symmetries as its closed elements in cohomological degree -1. Correspondingly, in the previous section we explored Lie algebras that correspond to infinitesimal gauge symmetries. In preparation to see how these fit into the Koszul complex, we studied their Chevalley-Eilenberg complex, which shifted them to be concentrated in cohomological degree -1 and encoded their structure through an action. However, in Section 3.3 we also found that the Koszul complex forms a shifted degree dg Lie algebra. The cohomological structure in this case encoded the possibility of having trivial symmetries. Now, our gauge theories may in general also contain symmetries that act trivially in this sense. Therefore, we have to extend our description of gauge structures to allow for dg Lie algebras.

Let $\mathfrak{g}$ be a dg Lie algebra concentrated in $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}_{\leq 0}$ with Lie bracket $[\cdot,\cdot]$ and a degree (1) differential d. We can encode the Lie bracket using the same action (3.51). To encode d into an action, we can follow the same procedure: use (2.53) to obtain a degree (1) map

$$\mathrm{dec}(\mathrm{d}) : \mathfrak{g}[1] \to \mathfrak{g}[1], \tag{3.60}$$

use (2.17) to construct its dual,

$$\delta_{\mathrm{d}} := \operatorname{dec}(\mathrm{d})^* : (\mathfrak{g}[1])^* \to (\mathfrak{g}[1])^* \subseteq \mathcal{O}(\mathfrak{g}[1]), \tag{3.61}$$

and, finally, apply Theorem 2.5.1 to obtain a degree (1) vector field on $\mathfrak{g}[1]$.

Let us compute this using the same coordinates as above. However, now we have the possibility $\deg c^a \neq 1$. The coordinates with $\deg c^a = i$ are said to be the ghosts for the ghosts of cohomological degree $i+1$. Let us define the matrix representation $D^a{}_b$ by $\mathrm{d}X_b = D^a{}_b X_a$. Since d has degree (1), we know that $D^a{}_b$ vanishes unless

$$0 = \epsilon^a + 1 + \epsilon^b + 1 + 1 = \epsilon^a + \epsilon^b + 1. \tag{3.62}$$

Application of (2.53) yields [see 41, Section 2]

$$\operatorname{dec}(\mathrm{d})(X_a) = -\,\mathrm{d}X_a = -D^b{}_a X_b. \tag{3.63}$$

Its dual is then given according to (2.17)

$$\begin{aligned}(\operatorname{dec}(\mathrm{d})^* c^a)(X_b) &= (-1)^{\epsilon^a} c^a(\operatorname{dec}(\mathrm{d})(X_b)) = (-1)^{\epsilon^a+1} D^a{}_b \\ &= (-1)^{\epsilon^a+1} D^a{}_b = (-1)^{\epsilon^a+1} D^a{}_c c^c(X_b).\end{aligned} \tag{3.64}$$

Finally, Theorem 2.5.1 yields the degree (1) vector field

$$\delta_{\mathrm{d}} := (-1)^{\epsilon^a+1} D^a{}_b c^b \frac{\partial}{\partial c^a}. \tag{3.65}$$

We can extend this to a cohomological vector field on $\mathcal{E} := \mathfrak{g}[1] \oplus \mathfrak{g}^*[-2]$ by considering the action

$$S_{\mathrm{d}} := -c^*_a D^a{}_b c^b. \tag{3.66}$$

Indeed, with this action we have

$$\{S_{\mathrm{d}}, c^a\} = (-1)^{\epsilon^b \epsilon^*_a} D^a{}_b c^b. \tag{3.67}$$

Using (3.62), we can set $\epsilon^b = \epsilon^a + 1 = \epsilon^*_a$, so that $\epsilon^b \epsilon^*_a = (\epsilon^a + 1)^2 = \epsilon^a + 1$. This shows that $\{S_{\mathrm{d}}, c^a\} = \delta_{\mathrm{d}} c^a$. Moreover, this vector field is cohomological since

$$\{S_{\mathrm{d}}, S_{\mathrm{d}}\} = 2c^*_a D^a{}_b D^b{}_c c^c = 0, \tag{3.68}$$

given that $\mathrm{d}^2 = 0$.

Finally, let us combine the actions obtained so far into

$$S_{\mathfrak{g}} := S_{\mathrm{d}} + S_{[\cdot,\cdot]}. \tag{3.69}$$

To see that this action satisfies the classical master equation we only need to show that $\{S_{\mathrm{d}}, S_{[\cdot,\cdot]}\} = 0$. The left hand side can be expanded to

$$-\, c_a^* D^a{}_b f^b{}_{cd} (-1)^{\epsilon^c+1} c^d c^c + 2(-1)^{\epsilon^b \epsilon_a^* + \epsilon^a \epsilon_c^*} D^a{}_b c^b c_c^* f^c{}_{da} (-1)^{\epsilon^d+1} c^d. \tag{3.70}$$

We have used the graded symmetry of $f^c{}_{de}(-1)^{\epsilon^d+1}$ to see that the two terms obtained in $\partial S_{[\cdot,\cdot]} \big/ \partial c^a$ are in fact equal. On the second term we can use (3.62) to set $\epsilon_a^* = \epsilon^a + 1 = \epsilon^b$, so that $\epsilon^b \epsilon_a^* = (\epsilon^b)^2 = \epsilon^b$. Relabelling and differentiating away the antifield of the ghost we obtain that the part of the classical master equation at hand is

$$D^c{}_a f^a{}_{db} (-1)^{\epsilon^d+1} c^b c^d = 2(-1)^{\epsilon^b + \epsilon^a \epsilon_c^* + \epsilon^b \epsilon_c^*} D^a{}_b f^c{}_{da} (-1)^{\epsilon^d+1} c^b c^d \tag{3.71}$$

The overall sign of the right hand side ends up having an exponent

$$\epsilon^b + \cancelto{1}{(\epsilon^a + \epsilon^b)}(\epsilon^c + \cancel{1}) + \epsilon^d + \cancel{1} = \epsilon^a + \epsilon^c + \epsilon^d + 1 = 0, \tag{3.72}$$

using (3.46) and (3.62). Therefore, differentiating away the last two ghosts we obtain

$$D^c{}_a f^a{}_{db} (-1)^{\epsilon^d+1} = D^a{}_b f^c{}_{da} + (-1)^{\epsilon^b \epsilon^d} D^a{}_d f^c{}_{ba}. \tag{3.73}$$

Multiplying this equation by the sign $(-1)^{\epsilon^d+1}$ and switching the order of (a, b) in the second term of the right hand side, we obtain a sign for this term with exponent

$$\epsilon^b \epsilon^d + \epsilon^d + \cancel{1} + (\epsilon^a + 1)(\epsilon^b + 1) + \cancel{1} = 0, \tag{3.74}$$

given that we can set $\epsilon^a + 1 = \epsilon^d$ in view of (3.62). We conclude that the vanishing of this term is equivalent to

$$D^c{}_a f^a{}_{db} = (-1)^{\epsilon^d+1} D^a{}_b f^c{}_{da} + D^a{}_d f^c{}_{ab}. \tag{3.75}$$

This is precisely the condition (2.60) for d to act as a derivation for the Lie bracket.

We have managed to describe the structure of the dg Lie algebra $\mathfrak{g}$ completely in terms of the action $S_{\mathfrak{g}}$. The resulting cochain complex $(\mathcal{O}(\mathfrak{g}[1]), \delta_{\mathfrak{g}} = \{S_{\mathfrak{g}}, \cdot\})$ is the Chevalley-Eilenberg complex for the cohomology of $\mathfrak{g}$.

Let us now take a small detour to present an application of the theory we have developed so far. First, we managed to find a one to one correspondance between super Lie brackets on a super vector space $\mathfrak{g}$ and actions $S_\mathfrak{g}$ on $\mathfrak{g}[1] \oplus \mathfrak{g}^*[-2]$ which satisfy the classical master equation and are linear in the ghosts and their antifields. Afterwards, we managed to find a one to one correspondance between dg Lie algebra structures on a bigraded vector space $\mathfrak{g}$ and actions $S_\mathfrak{g}$ on $\mathfrak{g}[1] \oplus \mathfrak{g}^*[-2]$ which satisfy the classical master equation and are at most quadratic in the ghosts and linear in the antifields. In here, the linear part in the ghosts corresponded to the differential, whose domain is $\mathfrak{g}$, while the bilinear part corresponded to the Lie bracket, whose domain is $\bigwedge^2 \mathfrak{g}$. A natural next step is to consider general (degree 0) actions $S_\mathfrak{g}$ on $\mathfrak{g}[1] \oplus \mathfrak{g}^*[-2]$ which satisfy the master equation and are linear in the antifields of the ghosts

$$S_\mathfrak{g} = \sum_{k=1}^{\infty} c_a^* F^a{}_{b_1 \cdots b_k} c^{b_1} \cdots c^{b_k} \tag{3.76}$$

Focusing on a term with k ghosts, we can invert the machinery we have used thus far to recover a map $\bigwedge^k \mathfrak{g} \to \mathfrak{g}$. This results in a k-bracket on $\mathfrak{g}$. Along with the relations induced on these k-brackets by the classical master equation, the resulting structure is known as an L_∞-algebra [57].

Physically, these structures have been found to encode the gauge structure of several field theories [58]. In particular, it has played an important role in the formulation of string field theory [59, 60, 61]. Mathematically, the theory has its foundations in the work of [62, 63], and is related to deformation theory [64]. Moreover, recent work shows that field theories generically give rise to these algebras and are classified by them [40]. We thus see that the BV formalism is a natural framework for the study of these subjects.

3.6 Classical BV Formalism: Gauge Invariance and Observables

Let G be a Lie group and M be a smooth manifold. Assume we have an action in form of a smooth map

$$\begin{aligned} \triangleright : G \times M &\to M \\ (g, p) &\mapsto g \triangleright p. \end{aligned} \tag{3.77}$$

This induces a map

$$\begin{aligned} \cdot \triangleright p : G &\to M \\ g &\mapsto g \triangleright p \end{aligned} \tag{3.78}$$

at every point $p \in M$. The action of the infinitesmial elements of this group at this point assigns to every element of the Lie algebra a tangent vector in a linear fashion

$$(\cdot \triangleright p)_{\star,e} : \mathfrak{g} \to T_pM. \tag{3.79}$$

This in turn maps every element of the Lie algebra into a vector field on M

$$\begin{aligned} R : \mathfrak{g} &\to \mathfrak{X}(M) \\ X &\mapsto R(X) \end{aligned} \tag{3.80}$$

defined as $R(X)_p := (\cdot \triangleright p)_{\star,e}X$. This assignment turns out to be a Lie algebra antihomomorphism [see 65, Theorem 20.18].

In the context of gauge theories, the group G that we want to consider is that of gauge transformations. It then acts on $M = V$ our space of field configurations. Infinitesimally, this action is then described by a map (3.80) into the vector fields on V. As we saw in Section 3.3, these vector fields are contained in the Koszul complex. In fact, we argued that this complex describes all of the symmetry structure of our theory. Having the structure of a dg Lie algebra, we saw that in general the gauge transformations of our theory should be described in these terms as well. In this case, the representation of $\mathfrak{g}$ in our theory is given by a map $R : \mathfrak{g} \to \mathcal{O}(V \oplus V^*[-1])[-1]$ which preserves the dg Lie algebra structures

$$\begin{aligned} \{R(X_b), R(X_c)\} &= R([X_b, X_c]), \\ \{S, R(X_a)\} &= R(\mathrm{d}X_a). \end{aligned} \tag{3.81}$$

Some remarks are in order. On one hand, note that the first equation of (3.81) and (3.38) indicate precisely that the action of $\mathfrak{g}$ is an antihomorphism with respect to the commutator of vector fields whenever $\mathfrak{g}$ is a the Lie algebra of some Lie group and every element of V is bosonic. On the other hand, the second equation of (3.81) indicates that the following diagram commutes

$$
\begin{tikzcd}
\cdots & \mathcal{O}(V)\otimes(V\wedge V) \arrow[r, "\{S,\cdot\}"] & \mathcal{O}(V)\otimes V \arrow[r, "\{S,\cdot\}"] & \mathcal{O}(V) \\
& \mathfrak{g}^{-1} \arrow[u, "R"] \arrow[r, "\mathrm{d}"] & \mathfrak{g}^{0} \arrow[u, "R"] \arrow[r, "\mathrm{d}"] & 0 \arrow[u, "R"]
\end{tikzcd}
\tag{3.82}
$$

In particular, the commutativity of the rightmost square yields

$$0 = \{S, R(X_a)\} = \frac{\partial_r S}{\partial \phi^i} R(X_a)^i, \tag{3.83}$$

that is, $\mathfrak{g}^0$ is represented through symmetries of the action. The commutativity of the next square amounts to the gauge symmetries $R(\mathrm{d}X_a)$ coming from $X_a \in \mathfrak{g}^{-1}$ being, in fact, trivial

$$\{S, R(X_a)\} = \frac{\partial_r S}{\partial \phi^i} \phi^*_j R(X_a)^{ij}. \tag{3.84}$$

In order to encode the representation conditions (3.81) in the formalism we have developed we will introduce the action

$$S_R := R(X_a)c^a, \tag{3.85}$$

on a space of fields which includes all of the fields we have encountered previously. This is $\mathcal{E} := \mathcal{V} \oplus \mathcal{V}[-1]$, where $\mathcal{V} := \mathfrak{g}[1] \oplus V$. The total BV action obtained is then of the form

$$S_{\mathrm{BV}} = S + S_{\mathfrak{g}} + S_R. \tag{3.86}$$

The classical master equation is then of the form

$$
\begin{aligned}
0 = \{S_{\mathrm{BV}}, S_{\mathrm{BV}}\} &= \{S, S\} + \{S_{\mathfrak{g}}, S_{\mathfrak{g}}\} + 2\{S, S_{\mathfrak{g}}\} \\
&\quad + \{S_R, S_R\} + 2\{S, S_R\} + 2\{S_R, S_{\mathfrak{g}}\}.
\end{aligned}
\tag{3.87}
$$

The terms on the rightmost side of the top row all vanish. The first one does because S is independent of antifields, the second one because $\mathfrak{g}$ is a dg Lie algebra, and the third because S is independent of ghosts and their antifields. We then only have to explore the terms in the bottom row. The first one of this is

$$
\begin{aligned}
\{S_R, S_R\} &= \{R(X_b)c^b, R(X_c)c^c\} \\
&= (-1)^{(|R(X_c)|+1)\epsilon^b}\{R(X_b), R(X_c)\}c^b c^c.
\end{aligned}
\tag{3.88}
$$

In here we used the derivation properties of the BV bracket. The condition $|R(X_a)| = \epsilon^a$ required for S to be bosonic then yields

$$\{S_R, S_R\} = (-1)^{(\epsilon^c+1)\epsilon^b}\{R(X_b), R(X_c)\}c^b c^c \tag{3.89}$$

Similarly, the second term is of the form

$$\{S, S_R\} = \{S, R(X_a)c^a\} = \{S, R(X_a)\}c^a. \tag{3.90}$$

The third term can be further decomposed into two parts. One of them is

$$\{S_R, S_{\mathrm{d}}\} = -\{R(X_a)c^a, c_b^* D^b{}_c c^c\} = -R(X_a)D^a{}_b c^b = -R(\mathrm{d}X_a)c^a. \tag{3.91}$$

The final one is

$$\begin{aligned}\{S_R, S_{[\cdot,\cdot]}\} &= \frac{1}{2}\{R(X_a)c^a, c_b^* f^b{}_{cd}(-1)^{\epsilon^c+1}c^d c^c\} \\ &= \frac{1}{2}R(X_a)f^a{}_{bc}(-1)^{\epsilon^b+1}c^c c^b = \frac{1}{2}R([X_b, X_c])(-1)^{\epsilon^b+1}c^c c^b \\ &= \frac{1}{2}R([X_b, X_c])(-1)^{\epsilon^b+1+\epsilon^b\epsilon^c}c^b c^c\end{aligned} \tag{3.92}$$

Comparison with $\{S_R, S_R\}$ will be easier if we reverse the order of the factors $c^c c^b$. This yields an overall sign with exponent

$$\epsilon^b + 1 + \epsilon^b\epsilon^c = \epsilon^b(\epsilon^c + 1) + 1, \tag{3.93}$$

that is,

$$\{S_R, S_{[\cdot,\cdot]}\} = -\frac{1}{2}R([X_b, X_c])(-1)^{\epsilon^b(\epsilon^c+1)}c^b c^c. \tag{3.94}$$

The classical master equation can now be studied by looking at the vanishing of the terms with an equal number of ghost fields. The terms linear in the ghost fields are

$$0 = \{S, S_R\} + \{S_R, S_{\mathrm{d}}\} = (\{S, R(X_a)\} - R(\mathrm{d}X_a))c^a, \tag{3.95}$$

which corresponds precisely to the first condition of (3.81) or, equivalently, the commutation of (3.82). Similarly, the terms quadratic in the ghosts are

$$\begin{aligned}0 &= \{S_R, S_R\} + 2\{S_R, S_{[\cdot,\cdot]}\} \\ &= (-1)^{(\epsilon^c+1)\epsilon^b}(\{R(X_b), R(X_c)\} - R([X_b, X_c]))c^b c^c,\end{aligned} \tag{3.96}$$

which corresponds precisely to the second equation of (3.81). In this regard it is important to note that both terms inside the parenthesis are of a form T_{bc} satisfying

$$T_{bc} = -(-1)^{(\epsilon^b+1)(\epsilon^c+1)} T_{cb}, \tag{3.97}$$

which leads to a graded symmetric $(-1)^{(\epsilon^c+1)\epsilon^b} T_{bc}$. Indeed, interchanging the lower indices would yield a sign with exponent

$$\begin{aligned}(\epsilon^b+1)(\epsilon^c+1) + (\epsilon^c+1)\epsilon^b + 1 &= \epsilon^b\epsilon^c + \cancel{\epsilon^b} + \epsilon^c + (\epsilon^c + \cancel{1})\epsilon^b \\ &= \epsilon^b\epsilon^c + \epsilon^c(\epsilon^b+1).\end{aligned} \tag{3.98}$$

It is for this reason that differentiating away the ghosts does not alter the formula inside the parenthesis.

Let us do a remark, restricting ourselves to the case where $\mathfrak{g}$ is an ordinary Lie algebra and V is a vector space for simplicity. We will encounter examples where the Lie group has a right action on our space of fields. In this the map $\mathfrak{g} \to \mathfrak{X}(M)$ ends up being a homomorphism rather than an antihomomorphism. The treatment however is entirely analogue. The only difference in the result is that instead a relative minus sign has to be inserted between S_R and $S_\mathfrak{g}$.

In this way we have managed to encode the structure of the equations of motion, symmetries, gauge symmetries and their representation into the solution of the classical master equation S_{BV} and its corresponding cohomological vector field $\delta := \{S_{\mathrm{BV}}, \cdot\}$. The resulting cochain complex $(\mathcal{O}(\mathcal{E}), \delta)$ is known as the classical BV complex. Via the projection $\mathcal{E} \to \mathcal{V}$, this induces a cohomological vector field on $\mathcal{V}$. The resulting cochain complex is called the Chevalley-Eilenberg complex for the representation R. Let us study the significance of $H^0(\mathcal{O}(\mathcal{V}), \delta)$. This complex is of the form

$$0 \xrightarrow{\delta^{-1}} \mathcal{O}(V) \xrightarrow{\delta^0} \mathcal{O}(V) \otimes (\mathfrak{g}^0)^*, \tag{3.99}$$

near the 0th cohomological degree. The zeroth degree cohomology of this complex is then

$$\begin{aligned} H^0(\mathcal{O}(\mathcal{V}), \delta) &:= {}^{\ker \delta^0}\!\big/_{\operatorname{im} \delta^{-1}} \cong \ker \delta^0 \\ &= \left\{ F \in \mathcal{O}(V) \;\middle|\; 0 = \delta F = \{S_R, F\} = \frac{\partial_r F}{\partial \phi^i} R(X_a)^i c^a \right\}, \end{aligned} \tag{3.100}$$

i.e. the gauge invariant functions on V.

Let us finish this section mentioning two possible extensions. On the one hand, one could consider vector fields where at each field configuration we act with a different gauge transformation. This leads us to the notions of action groupoids and Lie algebroids. These are important in theories where the gauge transformations depend on the field configurations. On the other hand, we could demand that (3.82) only holds at a cohomological level, given that it is in fact the cohomology of the Koszul complex what contains the symmetry information of the theory. This leads us to the notion of an L_∞ map. For a discussion of these extensions in a mathematical setting, see [66].

3.7 Gauge Fixing Fermion

At this point we could circle back to our starting point in Section 3.2 and attempt to explore the cohomology of the quantum complex obtained by equipping $\mathcal{O}(\mathcal{E})$ with the analogue of (3.19) for the BV action

$$Q = \{S_{\text{BV}}, \cdot\} - \hbar\Delta_{\text{BV}} \tag{3.101}$$

This however requires techniques in homological algebra that are beyond the scope of this essay. We will instead show the importance of the developments done so far by systematically obtaining some well known results in the applications studied in Chapter 4. To do this we will construct a method to obtain a gauge-fixed action from S_{BV} which is suitable for path integral formulations of QFT. This will lead us to the quantum master equation, yielding the corrections needed to ensure that Q is a differential.

Our first step along this path is to note that the original path integral (3.1) coincides with the path integral against the measure given by S_{BV} over the Lagrangian subspace defined by the vanishing of the antifields $\Phi^* = 0$. In the presence of gauge symmetries we are unable to assign expectation values with this measure since we do not have propagators. However, now that we have the whole field space $\mathcal{E}$, we can choose several Lagrangian submanifolds $\mathcal{L} \subseteq \mathcal{E}$ on which to perform our path integrals. These may be better behaved. Moreover, if the obtained integral is independent of deformations of the Lagrangian submanifold that we choose to replace the original one with, we can interpret the resulting expectation values as those of the original problem. A pictorial representation of this idea is

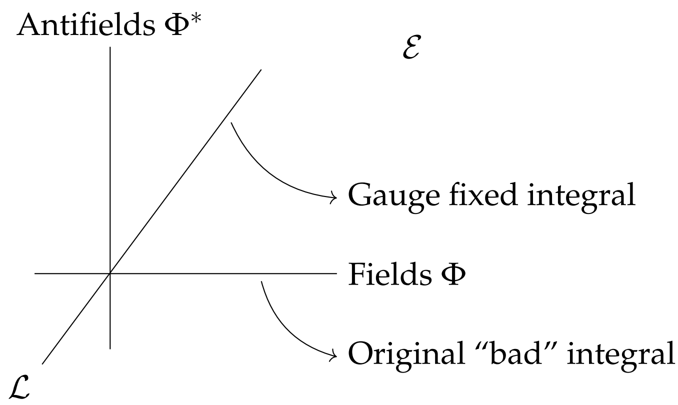

Figure 3.1: A pictorial depiction of $\mathcal{E}$ showing two Lagrangian subspaces, the $\Phi^* = 0$ subspace where the original "bad" integral was taken and a new submanifold $\mathcal{L}$ where the integral can be better behaved.

found in Figure 3.1.

Perturbations of the original Lagrangian submanifold $\Phi^* = 0$ can be obtained by considering a gauge fixing fermion $\Psi \in \mathcal{O}(\mathcal{V})$ and taking

$$\Phi^*_A = \frac{\partial \Psi}{\partial \Phi^A}, \tag{3.102}$$

as the equations defining the perturbation. Of course, comparing the degrees of both sides shows that $\deg \Psi = -1$. This is problematic since $\mathcal{V}$ is concentrated in non-positive degrees, which means that its coordinates have all non-negative degrees. This can be remedied by introducing new fields into our theory in the form of trivial pairs. These will be fields of the form $\bar{c}$ and b, along with their corresponding antifields $\bar{c}^*$ and b^*, which enter the action in the term $b\bar{c}^*$. The fields $\bar{c}$ and b will be called antighosts and Lagrange multipliers, respectively. We can then choose to place $\bar{c}$ in degree -1, so that we can use it to construct Ψ. Declaring b in degree 0 then ensures that the action remains in degree 0. Under the BV differential these new fields transform like

$$\{S_{\mathrm{BV}}, \bar{c}\} = -b, \qquad \{S_{\mathrm{BV}}, b\} = 0, \tag{3.103}$$
$$\{S_{\mathrm{BV}}, b^*\} = \bar{c}^*, \qquad \{S_{\mathrm{BV}}, \bar{c}^*\} = 0. \tag{3.104}$$

We thus see that in each of the equations above the closed elements that we obtain are exact. Therefore, they are trivialized in cohomology. We conclude that the addition of these new terms does not modify the BV complex in cohomology (see exercise 11.9 and sections 11.3.1 and 17.3.5 of [32]).

3.8 Quantum BV Formalism

Let us now discuss how we may use a gauge-fixed action in the computation of path integrals. We will follow the discussion found in [33]. In particular assume we have some functional $\mathcal{I}(\Phi, \Phi^*)$ for which we want to compute the integral

$$\int \mathcal{D}\Phi \, \mathcal{I}\left(\Phi, \frac{\partial \Psi}{\partial \Phi}\right). \tag{3.105}$$

Under a variation $\Psi \to \Psi + \delta\Psi$, this integral varies by

$$\int \mathcal{D}\Phi \, \frac{\partial \delta\Psi}{\partial \Phi^A} \frac{\partial \mathcal{I}}{\partial \Phi^*_A}\left(\Phi, \frac{\partial \Psi}{\partial \Phi}\right) = -\int \mathcal{D}\Phi \, \delta\Psi \Delta_{\mathrm{BV}} \mathcal{I}\left(\Phi, \frac{\partial \Psi}{\partial \Phi}\right), \tag{3.106}$$

up to boundary terms. We thus conclude that the integral is invariant under local variations of the gauge fixing fermion if $\Delta_{\mathrm{BV}}\mathcal{I} = 0$. The quantum master equation of Theorem 2.7.3 appears now in this context by demanding the invariance of the path integral measure $\mathcal{I} = e^{-\frac{1}{\hbar}S_{\mathrm{BV}}}$. Indeed,

$$\begin{aligned}
\Delta_{\mathrm{BV}}\left(e^{-\frac{1}{\hbar}S_{\mathrm{BV}}}\right) &= (-1)^{\epsilon^A} \frac{\partial}{\partial \Phi^A}\left(-\frac{1}{\hbar} e^{-\frac{1}{\hbar}S_{\mathrm{BV}}} \frac{\partial S_{\mathrm{BV}}}{\partial \Phi^*_A}\right) \\
&= (-1)^{\epsilon^A} e^{-\frac{1}{\hbar}S_{\mathrm{BV}}}\left(-\frac{1}{\hbar}\frac{\partial^2 S_{\mathrm{BV}}}{\partial \Phi^A \partial \Phi^*_A} + \frac{1}{\hbar^2}\frac{\partial S_{\mathrm{BV}}}{\partial \Phi^A}\frac{\partial S_{\mathrm{BV}}}{\partial \Phi^*_A}\right) \\
&= \frac{1}{\hbar^2} e^{-\frac{1}{\hbar}S_{\mathrm{BV}}}\left(-\hbar \Delta_{\mathrm{BV}} S_{\mathrm{BV}} + \frac{\partial_r S_{\mathrm{BV}}}{\partial \Phi^A}\frac{\partial S_{\mathrm{BV}}}{\partial \Phi^*_A}\right) \\
&= \frac{1}{\hbar^2} e^{-\frac{1}{\hbar}S_{\mathrm{BV}}}\left(-\hbar \Delta_{\mathrm{BV}} S_{\mathrm{BV}} + \frac{1}{2}\{S_{\mathrm{BV}}, S_{\mathrm{BV}}\}\right).
\end{aligned} \tag{3.107}$$

Recalling Theorem 2.7.3, we see that the existence of a quantum BV complex $(\mathcal{O}(\mathcal{E}), \{S_{\mathrm{BV}}, \cdot\} - \hbar\Delta_{\mathrm{BV}})$ is equivalent to the gauge invariance of the path integral measure. In general, the BV action we have obtained will not satisfy this equation. We can however use it as the inital action $S^{(0)}$ in a perturbative expansion

$$S_{\mathrm{BV}} = S^{(0)} + \hbar S^{(1)} + \hbar^2 S^{(2)} + \cdots. \tag{3.108}$$

We will analyze this more thorougly in the next section.

Similarly, for a function F of the fields to have a well defined expectation value, it must satisfy $\Delta_{\mathrm{BV}}(e^{-\frac{1}{\hbar}W}F) = 0$. This is equivalent to being closed in the quantum

BV complex

$$\begin{aligned}\Delta_{\text{BV}}\left(e^{-\frac{1}{\hbar}S}F\right) &= e^{-\frac{1}{\hbar}S}\Delta_{\text{BV}}F + \{e^{-\frac{1}{\hbar}}, F\} = e^{-\frac{1}{\hbar}S}\left(\Delta_{\text{BV}}F - \frac{1}{\hbar}\{S, F\}\right) \\ &= -\hbar e^{-\frac{1}{\hbar}}QF.\end{aligned} \tag{3.109}$$

A general function does not have to satisfy this. Again, we can attempt to build a solution perturbatively

$$F + \hbar F^{(1)} + \hbar^2 F^{(2)} + \cdots . \tag{3.110}$$

Constructing such observables is part of the quantization procedure. Moreover, the expectation value of an exact function in our BV complex is

$$\int \mathcal{D}\Phi \; (e^{-\frac{1}{\hbar}W}QF)\Big|_{\Phi^*=\frac{\partial\Psi}{\partial\Phi}} = -\frac{1}{\hbar}\int \mathcal{D}\Phi\, \left(\Delta_{\text{BV}}e^{-\frac{1}{\hbar}W}F\right)\Big|_{\Phi^*=\frac{\partial\Psi}{\partial\Phi}} = 0, \tag{3.111}$$

since it is a total derivative. To see this, set $\mathcal{I} = e^{-\frac{1}{\hbar}W}F$ and note that

$$\frac{\partial}{\partial\Phi^A}\left(\frac{\partial\mathcal{I}}{\partial\Phi^*_A}\circ\left(\Phi, \frac{\partial\Psi}{\partial\Phi}\right)\right) = \frac{\partial^2\mathcal{I}}{\partial\Phi^A\partial\Phi^*_A}\left(\Phi, \frac{\partial\Psi}{\partial\Phi}\right) + \frac{\partial^2\Psi}{\partial\Phi^B\partial\Phi^A}\frac{\partial^2\mathcal{I}}{\partial\Phi^*_B\partial\Phi^*_A}. \tag{3.112}$$

The first terms corresponds to $\Delta_{\text{BV}}\mathcal{I}$ while the second vanishes, which can be seen by the exchange $A \leftrightarrow B$. Therefore, the expectation value descends to a function in the cohomology of the quantum BV complex. We conclude that the observables of the quantum theory will be in the cohomology of the quantum BV complex.

Finally, let us remark that, unlike our classical BV differential, the quantum one is not a vector field. As a consequence, the quantum cohomology does not form an algebra. Theorem 2.7.2 shows that the Poisson bracket measures the failure of Δ_{BV} to be a derivation. Since such a bracket vanishes for functionals supported in disjoint regions, there is no issue with multiplying such functionals. This can be viewed as a starting point for the discussion of factorization algebras (see Lemma 3.0.1 of [17]). Multiplying functionals with overlapping supports requires a renormalization procedure.

3.9 Deformations and Anomalies

Let us study in more detail the solution of the quantum master equation using the $\hbar$ expansion (3.108). We will follow [see 32, Section 18.2.1]. The quantum master equation takes the form

$$\begin{aligned}0 &= \sum_{i,j=0}^{\infty} \hbar^{i+j}\frac{1}{2}\{S^{(i)}, S^{(j)}\} - \sum_{k=0}^{\infty} \hbar^{k+1}\Delta_{\mathrm{BV}} S^{(k)} \\ &= \frac{1}{2}\{S^{(0)}, S^{(0)}\} + \sum_{k=1}^{\infty} \hbar^k \left(\frac{1}{2}\sum_{n=0}^{k}\{S^{(n)}, S^{(k-n)}\} - \Delta_{\mathrm{BV}} S^{(k-1)} \right).\end{aligned} \tag{3.113}$$

Therefore, the order $1 = \hbar^0$ term of the quantum master equation coincides with the classical master equation $\{S_0, S_0\}$ for which we have already found solutions. For the order $\hbar^k$ term, with $k \in \mathbb{Z}^+$, the dependence on $S^{(k)}$ is contained in the terms with $n = 0$ and $n = k$ in the sum. Given that the BV bracket is symmetric on bosonic functions, both of these coincide. We conclude that the quantum master equation at order $\hbar^k$ is of the form

$$\{S^{(0)}, S^{(k)}\} = \Delta_{\mathrm{BV}} S^{(k-1)} - \frac{1}{2}\sum_{n=1}^{k-1}\{S^{(n)}, S^{(k-n)}\}. \tag{3.114}$$

This equation expresses the order $\hbar^k$ correction to the action, in terms of the action at lower orders. However, it only determines its image under the classical BV differential $\delta := \{S^{(0)}, \cdot\}$. Consequently, the quantum master equation only dictates the BV action S_{BV} up to a δ-closed term at each order in $\hbar$. In particular, $H^0(\mathcal{O}(\mathcal{E}), \delta)$ yields the set of possible deformations of the action.

On the other hand, (3.114) implies that the right hand side is δ-exact. Therefore, at the very least it must be δ-closed. Fortunately, this is always the case.

Theorem 3.9.1 ([see 32, Exercise 18.6])**.** Assume that the quantum master equation (3.114) has been solved up to order $k-1$, for some $k \in \mathbb{Z}^+$. Then

$$\Delta_{\mathrm{BV}} S^{(k-1)} - \frac{1}{2}\sum_{n=1}^{k-1}\{S^{(n)}, S^{(k-n)}\} \tag{3.115}$$

is δ-closed.

Proof. We need to show that

$$0 = \{S^{(0)}, \Delta_{\mathrm{BV}} S^{(k-1)}\} - \frac{1}{2}\sum_{n=1}^{k-1}\{S^{(0)}, \{S^{(n)}, S^{(k-n)}\}\}. \tag{3.116}$$

To begin with, note that the Jacobi identity

$$0 = \{S^{(0)}, \{S^{(n)}, S^{(k-n)}\}\} + \{S^{(k-n)}, \{S^{(0)}, S^{(n)}\}\} \\ + \{S^{(n)}, \{S^{(k-n)}, S^{(0)}\}\}, \tag{3.117}$$

allows us to express the summand in terms of quantities fixed by the quantum master equation at orders less than $\hbar^k$. Therefore, the last term of (3.116) can be rewritten as

$$\frac{1}{2}\sum_{n=1}^{k-1}\Big(\{S^{(k-n)}, \{S^{(0)}, S^{(n)}\}\} + \{S^{(n)}, \{S^{(k-n)}, S^{(0)}\}\}\Big). \tag{3.118}$$

In fact, since for every $n \in \{1, \dots, k-1\}$ we have $n' = k - n \in \{1, \dots, k-1\}$, both terms have the same sum. Remembering that the BV bracket is symmetric on bosonic elements, we can rewrite this term as

$$\sum_{n=1}^{k-1}\{S^{(n)}, \{S^{(0)}, S^{(k-n)}\}\}. \tag{3.119}$$

At this stage we can use the quantum master equation at order $\hbar^{k-n}$ to rewrite this as

$$\sum_{n=1}^{k-1}\{S^{(n)}, \Delta_{\mathrm{BV}} S^{(k-1-n)}\} - \frac{1}{2}\sum_{n=1}^{k-1}\sum_{m=1}^{k-1-n}\{S^{(n)}, \{S^{(m)}, S^{(k-n-m)}\}\}. \tag{3.120}$$

The term in the double sum vanishes due to the Jacobi identity. Indeed, for fixed $n \in \{1, \dots, k-1\}$ and $m \in \{1, \dots, k-1-n\}$, let $n' := k - n - m$ and $m' := n$. Then we have

$$1 = k - (n + k - 1 - n) \leq n' = k - (n + m) \leq k - (1 + 1) = k - 2 \leq k - 1, \tag{3.121}$$

$$1 \leq n = m' \leq n + m - 1 = k - 1 - (k - n - m) = k - 1 - n', \tag{3.122}$$

and

$$k - n' - m' = k - (k - n - m) - n = m. \tag{3.123}$$

Therefore, if the term $\{S^{(n)}, \{S^{(m)}, S^{(k-n-m)}\}\}$ appears in the sum, so will the term $\{S^{(k-n-m)}, \{S^{(n)}, S^{(m)}\}\}$. By the same token, the term $\{S^{(m)}, \{S^{(k-n-m)}, S^{(n)}\}\}$ will also appear, and these three terms vanish.

Taking the results above, the theorem reduces to showing that

$$0 = \{S^{(0)}, \Delta_{\mathrm{BV}} S^{(k-1)}\} + \sum_{n=1}^{k-1} \{S^{(n)}, \Delta_{\mathrm{BV}} S^{(k-1-n)}\}. \tag{3.124}$$

If we take the first term of the right hand side and the $n = k-1$ term of the sum, we obtain

$$\{S^0, \Delta_{\mathrm{BV}} S^{k-1}\} + \{S^{(k-1)}, \Delta_{\mathrm{BV}} S^{(0)}\} = -\Delta_{\mathrm{BV}}\{S^{(0)}, S^{(k-1)}\}. \tag{3.125}$$

In here we used the graded antisymmetry of the BV bracket and (2.96). We can use the same to properties to study the remaining terms in the sum. To do this, note the change of variable of summation $n' = k-1-n$ has the same range as n

$$1 = k-1-(k-2) \leq n' = k-1-n \leq k-1-1 = k-2. \tag{3.126}$$

We thus see that

$$\sum_{n=1}^{k-2} \{S^{(n)}, \Delta_{\mathrm{BV}} S^{(k-1-n)}\} = \sum_{n'=1}^{k-2} \{S^{(k-1-n')}, \Delta_{\mathrm{BV}} S^{(n')}\}. \tag{3.127}$$

Therefore, the sum above can be written as

$$\begin{aligned} \frac{1}{2}\sum_{n=1}^{k-2} &\left(\{S^{(n)}, \Delta_{\mathrm{BV}} S^{(k-1-n)}\} + \{S^{(k-1-n)}, \Delta_{\mathrm{BV}} S^{(n)}\}\right) \\ &= -\Delta_{\mathrm{BV}}\left(\frac{1}{2}\sum_{n=1}^{k-2}\{S^{(n)}, S^{(k-1-n)}\}\right). \end{aligned} \tag{3.128}$$

The theorem then reduces to proving

$$0 = -\Delta_{\mathrm{BV}}\left(\{S^{(0)}, S^{(k-1)}\} + \frac{1}{2}\sum_{n=1}^{k-2}\{S^{(n)}, S^{(k-1-n)}\}\right). \tag{3.129}$$

The quantum master equation at order $\hbar^{k-1}$ guarantees that the term in parenthesis is equal to $\Delta_{\mathrm{BV}} S^{(k-2)}$. The result follows from $\Delta_{\mathrm{BV}}^2 = 0$. □

As a consequence of this result, we can consider the cohomology class of the right hand side of the order $\hbar^k$ quantum master equation (3.114). Since Δ_{BV} and $\{\cdot, \cdot\}$ have cohomological degree 1, this will be an element of $H^1(\mathcal{O}(\mathcal{E}), \delta)$. The existence of a solution to this equation is then equivalent to the statement that this class is $0 \in H^1(\mathcal{O}(\mathcal{E}), \delta)$. Otherwise, we conclude that there is no solution to the quantum master equation and one cannot define a gauge-invariant path integral measure. We conclude that the obstructions to find solutions to the quantum master equation are in $H^1(\mathcal{O}(\mathcal{E}), \delta)$. These are known as gauge anomalies.

Chapter 4

Applications of the BV Formalism

4.1 The Hat Potential

Consider the toy model of a complex field theory in a spacetime consisting of a point $M = \{p\}$. A configuration of the field is given by $\phi \in \mathbb{C} =: V$. Consider the action $S(\phi) = |\phi|^4 - |\phi|^2$, which is shown in 4.1. Choosing coordinates ϕ^1 and ϕ^2 on V so that $\phi^1(x+iy) = x$ and $\phi^2(x+iy) = y$, our action takes the form

$$S = ((\phi^1)^2 + (\phi^2)^2)^2 - ((\phi^1)^2 + (\phi^2)^2). \tag{4.1}$$

This action is invariant under the $U(1)$ action on V. Now, assume that this symmetry is, in fact, a gauge symmetry and the physical field is $|\phi| \in [0, \infty)$. In particular, our observables take the form of functions $f \in \mathcal{O}(V)$ which are $U(1)$ invariant. For such a function there always is a function $\tilde{f}$ on $[0, \infty)$ such that $f(\phi) = \tilde{f}(|\phi|)$. Its expectation value is then

$$\int \mathrm{d}^2\phi\, e^{-S(\phi)} f(\phi) = \int_0^{2\pi} \mathrm{d}\theta \int_0^\infty \mathrm{d}r\, r e^{-r^4+r^2} \tilde{f}(r) = \pi \int \mathrm{d}r\, r e^{-r^4+r^2} \tilde{f}(r). \tag{4.2}$$

The angular integral corresponds to the degeneracy of $U(1)$. Given any physical field configuration ϕ we have a circle of radius $|\phi|$ of configurations in V which are physically indistinguishable. In the infinite dimensional case, its value will in general be infinite. The second integral is the one corresponding to the actual expectation value we are interested in calculating. This is the integral we hope to recover using the BV formalism.

Remark 4.1.1. Gauge invariance plays an important role. Indeed, if we had started with the action $\tilde{S}$ directly on $[0, \infty)$, our path integrals would have been of the form

$$\int_0^\infty \mathrm{d}r\, e^{-r^4+r^2} \tilde{f}(r). \tag{4.3}$$

In particular, the correct Jacobian factor would have been lost. To consider the

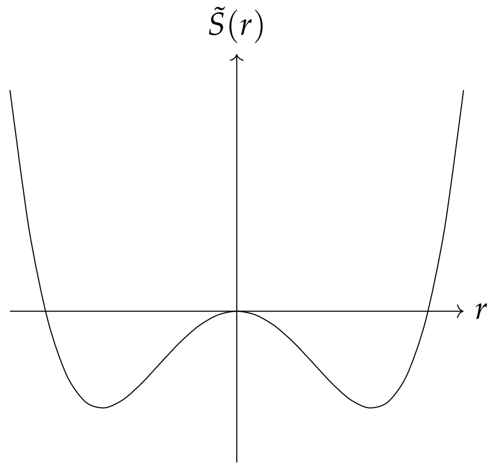

Figure 4.1: Mexican hat action

theory in terms of the physical fields from the get go requires the more complicated action $\tilde{S}(r) = r^4 - r^2 - \log r$.

To apply the BV formalism, we note that the induced action of the Lie algebra $\mathfrak{g} = \mathbb{R}$ is

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\begin{pmatrix}\cos(\theta) & -\sin(\theta)\\ \sin(\theta) & \cos(\theta)\end{pmatrix}\begin{pmatrix}\phi^1\\ \phi^2\end{pmatrix}\Bigg|_{\theta=0} = -\phi_1^*\phi^2 + \phi_2^*\phi^1. \tag{4.4}$$

and the Lie algebra is commutative. Therefore, the action is given by (3.85)

$$S_R = (\phi_2^*\phi^1 - \phi_1^*\phi^2)c. \tag{4.5}$$

Now, instead of considering the integral over the Lagrangian submanifold $\mathfrak{g}[1] \oplus V$, where the total action takes the form $S_{\mathrm{BV}}|_{\mathfrak{g}[1]\oplus V} = S$, one can obtain the same result in a more direct fashion by considering the integral over the Lagrangian submanifold $\mathcal{L}$ given by $\phi_1^* = c^* = \phi^2 = 0$. In these $S_{\mathrm{BV}}|_{\mathcal{L}} = (\phi^1)^4 - (\phi^1)^2 + \phi_2^*\phi^1 c$, and the expectation value of an observable is

$$\begin{aligned}
&\int \mathrm{d}\phi^1\,\mathrm{d}c\,\mathrm{d}\phi_2^*\, e^{-(\phi^1)^4+(\phi^1)^2-\phi_2^*\phi^1 c} f\\
&= \int \mathrm{d}\phi^1\,\mathrm{d}c\,\mathrm{d}\phi_2^*\, e^{-(\phi^1)^4+(\phi^1)^2}(1-\phi_2^*\phi^1 c) f\\
&= \int \mathrm{d}\phi^1\,\phi^1 e^{-(\phi^1)^4+(\phi^1)^2} f,
\end{aligned} \tag{4.6}$$

which recovers the integral we obtained before.

4.2 Free Real Scalar Field

Proceeding by analogy with the previous example, let us now proceed to consider the free real scalar field. We will follow [17, Section 2 of Chapter 2]. Let M be a Riemannian manifold with metric tensor g. The space of field configurations of the real scalar field is the vector space $C^\infty(M)$. Let us however organize this in a local fashion. For every open $U \subseteq M$, the space of field configurations on U is given by $C^\infty(U)$. The action for the free scalar field is the map

$$\begin{aligned} S : C_c^\infty(M) &\to \mathbb{R} \\ \phi &\mapsto -\frac{1}{2}\int \mathrm{d}\,\mathrm{vol}_g\, \phi(\Delta_g - m^2)\phi, \end{aligned} \tag{4.7}$$

with $\mathrm{d}\,\mathrm{vol}_g$ the volume form on M and Δ_g the associated Laplace-Beltrami operator. The chosen Lagrangian only differs from the usual one (after Wick rotation) by a total derivative, which does not contribute to the action for compactly supported functions. Indeed, on a chart (U, x) we have

$$-\frac{1}{2}\phi(\Delta_g - m^2)\phi = \frac{1}{2}(\partial\phi)^2 + \frac{1}{2}m^2\phi^2 - \nabla^\mu(\phi\nabla_\mu\phi). \tag{4.8}$$

However, this form is more directly comparable to the action in Section 3.1

$$\begin{aligned} i \in \{1, \dots, N\} &\mapsto x \in M, \\ \phi \in V &\mapsto \phi \in C^\infty(M), \\ \phi^i &\mapsto \phi(x), \\ \phi^i A_{ij}\phi^j &\mapsto -\int \mathrm{d}\,\mathrm{vol}_g\, \phi(\Delta_g - m^2)\phi. \end{aligned} \tag{4.9}$$

The positive definiteness of A corresponds now to the negative spectrum of Δ_g.

We wish to use the techniques developed in Sections 3.1 and 3.2 to make sense of

$$\int \mathcal{D}\phi e^{-\frac{1}{\hbar}S} F \tag{4.10}$$

However, unlike our previous case, we do not actually have a measure to define our divergence operator. Thus, we will instead propose a divergence operator that reproduces (3.9) under the dictionary (4.9). Let us begin by describing the codomain of this map, i.e. the ring of polynomial functions on our configuration space. Much like in the finite dimensional case, we will take the symmetric algebra of an appropriate dual as our first attempt of such a ring. When equipped with its usual topology, the continuous dual of $C^\infty(U)$ is given by the compactly supported distributions on U [see 67, Theorem 2.3.1]. Let us take a particularly well behaved subspace of this, namely $C_c^\infty(U)$. Each $g \in C_c^\infty(U)$ acts on $\phi \in C^\infty(U)$ via

$$g(\phi) := \int_U \mathrm{d}\,\mathrm{vol}_g\, g\phi. \tag{4.11}$$

We thus define $\tilde{\mathcal{O}}(C^\infty(U)) := S^\bullet(C_c^\infty(U))$. Omitting the tensor product symbol again, for every $g_1, \ldots, g_n \in C_c^\infty(U)$, we have that $g_1 \cdots g_n$ acts on $\phi \in C^\infty(U)$ via

$$\begin{aligned} g_1 \cdots g_n(\phi) :=& g_1(\phi) \cdots g_n(\phi) \\ =& \int_{U^n} \mathrm{d}\,\mathrm{vol}_g^n\,(x_1, \ldots, x_n) g(x_1) \cdots g(x_n) \phi(x_1) \cdots \phi(x_n). \end{aligned} \tag{4.12}$$

In here $\mathrm{d}\,\mathrm{vol}_g^n$ is the measure induced by the product metric on M^n. In light of equation (3.18), we define the polynomial vector fields as $\widetilde{\mathrm{Vect}}(C^\infty(U)) := \tilde{\mathcal{O}}(C^\infty(U)) \otimes C^\infty(U)$. In fact, let us instead consider

$$\widetilde{\mathrm{Vect}}_c(C^\infty(U)) := \tilde{\mathcal{O}}(C^\infty(U)) \otimes C_c^\infty(U). \tag{4.13}$$

This corresponds to the fact that we want to consider variations of compact support which do not provide additional boundary terms.

Every $g \in C_c^\infty(U)$ acts on a linear fashion on $C^\infty(U)$. It is thus simple to calculate its directional derivative with respect to $\phi \in C^\infty(U)$

$$\frac{\delta g}{\delta \phi}(\psi) = \lim_{t \to 0} \frac{g(\psi + t\phi) - g(\psi)}{t} = g(\phi) =: \frac{\delta g}{\delta \phi}. \tag{4.14}$$

We can then extend this definition to all of $\tilde{\mathcal{O}}(C^\infty(U))$ via the Leibniz rule. For every $g^1, \ldots, g^n, \phi \in C_c^\infty(U)$, we denote the element of $\widetilde{\mathrm{Vect}}_c(C^\infty(U))$ constructed as the tensor product of $g^1 \cdots g^n \in \tilde{F}(C^\infty(U))$ and ϕ by $g^1 \cdots g^n \frac{\delta}{\delta\phi}$. This element acts on a polynomial $F \in \tilde{O}(C^\infty(U))$ via

$$F \mapsto g^1 \cdots g^n \frac{\delta F}{\delta \phi} \in \tilde{O}(C^\infty(U)). \tag{4.15}$$

Thus, mimicking (3.12), we define the divergence

$$\begin{aligned} -\hbar\,\mathrm{div}_S : \widetilde{\mathrm{Vect}}_c(C^\infty(U)) &\to \tilde{\mathcal{O}}(C^\infty(U)) \\ g^1\cdots g^n \frac{\delta}{\delta\phi} &\mapsto g^1\cdots g^n(\Delta_g - m^2)\phi \\ &\qquad - \hbar\sum_{r=1}^n g^1\cdots\widehat{g_r}\cdots g^n \int_U \mathrm{d}\,\mathrm{vol}_g\, g^r\varphi \end{aligned} \tag{4.16}$$

Even though $-\hbar\,\mathrm{div}_S$ makes sense on $\widetilde{\mathrm{Vect}}(C^\infty(U))$, the choice of $\widetilde{\mathrm{Vect}}_c(C^\infty(U))$ is essential, for otherwise the codomain would not be $\tilde{\mathcal{O}}(C^\infty(U))$.

Thus far the domains and codomains considered are purely algebraic. We will need to consider their completions in suitable topologies. Equation (4.12) suggests a simple extension of the notion of polynomial. Indeed, every $F \in C_c^\infty(U^n)$ induces a map

$$\begin{aligned} F : C^\infty(U) &\to C^\infty(U) \\ \varphi &\mapsto F(\varphi) := \int_{U^n} \mathrm{d}\,\mathrm{vol}_g^n\,(x_1,\ldots x_n)F(x_1,\ldots,x_n)\varphi(x_1)\cdots\varphi(x_n). \end{aligned} \tag{4.17}$$

This action is invariant under the action of the permutation group S_n of n letters on $C_c^\infty(U^n)$ given by

$$\begin{aligned} S_n \times C_c^\infty(U^n) &\to C_c^\infty(U^n) \\ (\sigma, F) &\mapsto \left\{ \begin{array}{c} \sigma \triangleright F : U^n \to \mathbb{R} \\ (x_1,\ldots,x_n) \mapsto F(x_{\sigma(1)},\ldots,x_{\sigma(n)}) \end{array} \right\} \end{aligned} \tag{4.18}$$

Thus, the map is completely determined by the image of F in the space of coinvariants ${C_c^\infty(U^n)}/{S_n} =: C_c^\infty(U^n)_{S_n}$. We thus redefine the space of polynomials on $C^\infty(U)$ as

$$\mathcal{O}(C^\infty(U)) := \bigoplus_{n=0}^{\infty} C_c^\infty(U^n)_{S_n}. \tag{4.19}$$

We have that $S^\bullet(C_c^\infty(U))$ is dense in the invariant functions $C_c^\infty(U^n)^{S_n} \cong C_c^\infty(U^n)_{S_n}$. Thus, we conclude that $\tilde{\mathcal{O}}(C^\infty(U))$ lies densely in $\mathcal{O}(C^\infty(U))$. With this redefinition we are tempted to take the set polynomial vector fields as $C_c^\infty(U^n)_{S_n} \otimes C_c^\infty(U)$. We take however the extension

$$\mathrm{Vect}_c(C^\infty(U)) = C_c^\infty(U^n \times U)_{S_n}, \tag{4.20}$$

with the action on the first n entries

$$\begin{aligned} S_n \times C_c^\infty(U^n \times U) &\to C_c^\infty(U^n \times U) \\ (\sigma, X) &\mapsto \left\{ \begin{aligned} \sigma \triangleright X : U^n \times U &\to \mathbb{R} \\ (x_1, \ldots, x_n, y) &\mapsto X(x_{\sigma(1)}, \ldots, x_{\sigma(n)}, y). \end{aligned} \right\} \end{aligned} \tag{4.21}$$

Now, in the case that M is compact, we can *define* the expectation value map without the need of a path integral as the projection

$$\langle \cdot \rangle : \mathcal{O}(C^\infty(M)) \to \mathcal{O}(C^\infty(M)) \big/ \mathrm{im}(\hbar\, \mathrm{div}_S). \tag{4.22}$$

In particular, in light of equation (4.16), we get that for all $g^1, \ldots, g^n, \phi \in C_c^\infty(M)$

$$\left\langle g^1 \cdots g^n (\Delta_g - m^2)\phi \right\rangle = \hbar \sum_{r=1}^n \int \mathrm{d}\,\mathrm{vol}_g\, g^r \phi \left\langle g^1 \cdots \widehat{g^r} \cdots g^n \right\rangle \tag{4.23}$$

Taking $g^{n+1} \in \mathcal{O}(C^\infty(M))$, G to be the Greens function of $\Delta_g - m^2$ (which is unique since M is compact), and

$$\phi(x) = \int \mathrm{d}\,\mathrm{vol}_g\,(y) G(x-y) g^{n+1}(y), \tag{4.24}$$

we obtain $(\Delta_g - m^2)\phi = g^{n+1}$ and

$$\left\langle g^1 \cdots g^{n+1} \right\rangle = \hbar \sum_{r=1}^n \int_{U^2} \mathrm{d}\,\mathrm{vol}_g^2\,(x,y) g^r(x) G(x-y) g^{n+1}(y) \left\langle g^1 \cdots \widehat{g^r} \cdots g^n \right\rangle. \tag{4.25}$$

Once again, this leads to Wick's theorem.

4.3 Extension to Infinite Dimensions

Inspired by the rigorous treatment above, let us now give a heuristic treatment to see how the formalism developed in Chapter 3 can be extended to more general field theories.

Let us fix a manifold M. As we mentioned at the beginning of Chapter 3, the field theories we will discuss have spaces of field configurations given by the smooth sections $V = \Gamma(E)$ of a vector bundle $\pi : E \to M$. Our first task will be to find appropriate linear coordinates on V. The main idea comes from the insight of accompanying discrete indices with continuous ones $\phi^i \mapsto \phi^i(x)$. This is the first element of DeWitt's notation. In order to make sense out of this, let us consider a set of local sections $e_1, \ldots, e_{\operatorname{rank} E} \in \Gamma(E|_U)$ on an open $U \subseteq M$ such that at every point $x \in U$ the set $\{e_1(x), \ldots, e_N(x)\}$ is a basis for the fiber $\pi^{-1}(\{x\}) =: E_x$. This yields an evaluation map for every $x \in U$

$$\begin{aligned} \phi^i(x) : \Gamma(E) &\to \mathbb{R} \\ \varphi &\mapsto \phi^i(x)(\varphi) := \varphi^i(x), \end{aligned} \tag{4.26}$$

where $\varphi(x) = \varphi^i(x)e_i(x)$. The first step of our heuristics will be to replace ϕ^i by the set of evaluations $\{ \phi^i(x) \mid i \in \{1, \ldots, N\}, p \in U \}$. We can bundle these maps together in several ways

$$\phi(x) : \Gamma(E) \to E_x, \qquad \phi^i : \Gamma(E) \to C^\infty(U), \qquad \phi : \Gamma(E) \to \Gamma(E|_U). \tag{4.27}$$

The introduction of these may initially seem superfluous. However, they serve an important conceptual role in transferring our intuition gained in finite dimensions to the infinite dimensional case.

Let us comment on the allowed functions on V. The polynomials in $\mathcal{O}(V)$ should be spanned in some way by monomials on the evaluations. However, in field theory we are concerned with observables that can be built locally. This amounts to considering functions of the form

$$\begin{aligned} F : \Gamma(E) &\to \mathbb{R} \\ \varphi &\mapsto \int \prod_{j=1}^{k} \mathrm{d}^D x_j f_{i_1 \cdots i_k}(x_1, \ldots, x_k) \varphi^{i_1}(x_1) \cdots \varphi^{i_k}(x_k). \end{aligned} \tag{4.28}$$

Following [23], we will use the notation

$$\int \prod_{j=1}^{k} \mathrm{d}^D x_j f_{i_1 \cdots i_k}(x_1, \ldots, x_k) \phi^{i_1}(x_1) \cdots \phi^{i_k}(x_k) := F \tag{4.29}$$

for these. In order for the integration to make sense the allowed f's should be (compactly supported) densities. Their index structure also points at them taking values naturally in the dual bundle E^*. Moreover, in order to incorporate the examples below we will need to allow for them to be of a distributional nature on M^k. The space V^* is obtained by taking $k = 1$ in the above considerations. In particular, we can identify each $\phi^i(x)$ as an element of V^* via

$$\phi^i(x) = \int \mathrm{d}^D y\, \delta(x-y)\delta^i_j \phi^j(y). \tag{4.30}$$

Note that on these spaces the functional derivatives $\delta/\delta\phi^i(x)$ can be easily defined by extending $\delta\phi^j(y)/\delta\phi^i(x) = \delta^j_i \delta(y-x)$ through the Leibniz rule.

Let us discuss a quick example to show how these conventions are applied. Consider the case where M has a metric g and $E = M \times \mathbb{R}$. Then, $\Gamma(E) = C^\infty(M)$, which is the situation describing the scalar field. The class of functions we are considering allows for the construction of the action. For example,

$$\int \mathrm{d}^D x\, \mathrm{d}^D y\, \mathrm{d}^D z\, g^{\mu\nu} \frac{\partial \delta(x-y)}{\partial x^\mu} \frac{\partial \delta(x-z)}{\partial x^\nu} \sqrt{\det g_{..}(x)}\phi(y)\phi(z) \tag{4.31}$$

maps

$$\varphi \mapsto \int \mathrm{d}^D x\, \sqrt{\det g_{..}(x)} \partial^\mu \varphi(x) \partial_\mu \varphi(x). \tag{4.32}$$

We will use the notation

$$\int \mathrm{d}^D x\, \sqrt{\det g}\, \partial^\mu \phi \partial_\mu \phi \tag{4.33}$$

for such a function. Note that we will suppress the variable of differentiation as well as the arguments of our functions whenever no confusion may arise. We also sometimes use the same symbols for points in M and the coordinate functions on a chart.

In order to treat the space of antifield configurations on the same footing as the space of fields, we need to choose an appropriate subspace of smooth sections on V^*. This is obtained by restricting the allowed f's to be smooth. The resulting space can be identified as $V^! := \Gamma(E^* \otimes \mathrm{Dens}(M))$, where $\mathrm{Dens}(M)$ is the density bundle on M. Its action on V is given by the pairing

$$\begin{aligned} \langle\, \cdot\, , \cdot\, \rangle : \Gamma(E^* \otimes \mathrm{Dens}(M)) \times \Gamma(E) &\to \mathbb{R} \\ (\varphi^* \otimes \rho, \varphi) &\mapsto \int \mathrm{d}^D x\, \rho(x) \varphi^*(x)(\varphi(x)). \end{aligned} \tag{4.34}$$

Being in itself a space of sections on a vector bundle, one can equip this with coordinates following the procedure used for V. This yields the antifields

$$\begin{aligned} \phi^*_i(x) : \Gamma(E^* \otimes \mathrm{Dens}(M)) &\to \mathbb{R} \\ \varphi^* &\mapsto \varphi^*_i(x). \end{aligned} \tag{4.35}$$

These are dual to $\phi^i(x)$, as can be seen from the following heuristics. The decomposition (4.30) shows that the corresponding element "in" $V^!$ is $\delta(x - \cdot)e^i(x)$, with $e^1, \ldots, e^{\operatorname{rank} E} \in \Gamma(E^*)$ defined by $e^i(x)(e_j(x)) = \delta^i_j$. Therefore, $\phi^*_j(y)(\phi^i(x)) = \delta(x-y)\delta^i_j$. In particular, we see that we can make the identification

$$\frac{\delta}{\delta\phi^i(x)} \equiv \phi^*_i(x). \tag{4.36}$$

We will restrict to examples of gauge algebras where $\mathfrak{g}$ is the section space over a vector bundle. Following the procedure used for V one obtains the ghosts $\{\, c^a(x) \mid a \in \{1, \ldots, r\}, x \in M \,\}$, with r the rank of the relevant vector bundle. One can construct the space of antighost configurations $\mathfrak{g}^!$ mirroring the construction of $V^!$. This yields the antighosts $\{\, c^*_a \mid a \in \{1, \ldots, r\}, x \in M \,\}$. Combining these we obtain the fields corresponding to the BV extension

$$\mathcal{E} = \mathfrak{g}[1] \oplus V \oplus V^![-1] \oplus \mathfrak{g}^![-2]. \tag{4.37}$$

This is in itself the space of sections over a vector bundle. Functions on this extension are spanned by the monomials of the type (4.29), allowing for the appearence of ghosts $c^a(x)$ and fields $\phi^i(x)$, as well as their corresponding antifields $\phi^*_i(x)$ and $c^*_a(x)$. These will be taken to commute or anticommute in the same way that their counterparts in Chapter 3 did. For example, the vector fields $\mathcal{O}(V) \otimes V$ will be spanned by elements of the form

$$\int \mathrm{d}^D y \prod_{j=1}^{k} \mathrm{d}^D x_j \, \phi^*_i(y) f^i_{j_1 \cdots j_k}(y, x_1, \ldots, x_k) \phi^{j_1}(x_1) \cdots \phi^{j_k}(x_k). \tag{4.38}$$

The reader may find this expression more familiar by using the identification (4.36). Comparing this to the vector fields $\phi^*_i F^i$ found in Chapter 3, one sees the second element of DeWitt's notation: contraction of two discrete indices has to be supplemented by an integration over the corresponding continuous label.

4.4 Diffeomorphism Symmetries

The development of general relativity showed that gravity is an expression of the geometry of spacetime. Thus, in gravitational theories we strive to find actions that depend only on such geometry. This can be interpreted in two ways. On the more classic perspective, recall that coordinates on a manifold are an artifact for computations. By themselves they fail to reflect the geometry of the manifold. Thus, our actions have to be reparametrization invariant. On a more geometric view, which we will prefer for this thesis, we want to describe our geometries through the topology, differentiable structure, and tensor fields we equip our manifold with. The first two are by definition preserved under diffeomorphisms. The third, which will be part of the field content of our gravitational theories, can either be transported through the pullback and pushforward induced by such diffeomorphisms. This defines an action of our diffeomorphisms on the space of tensor fields. From now on, whenever we have a dynamical field in a theory of gravity whose geometric nature has been specified as one of these tensor fields, we will assume that the group of diffeomorphisms acts in this way.

Let us now try to be more precise about this action. Given a manifold M and a diffeomorphism $f : M \to N$, a function $F \in C^\infty(M)$ is most naturally pulled back via the inverse isomorphism $f^{-1} : N \to M$. Indeed $(f^{-1})^*F = F \circ f^{-1} \in C^\infty(N)$. On the other hand, vector fields $X \in \mathfrak{X}(M)$ are most naturally pushed forward. Indeed, we define the vector field $f_*X \in \mathfrak{X}(N)$ by $(f_*X)F = X(F \circ f^{-1})$ on every $F \in C^\infty(N)$. Finally, 1-forms α, like functions, are most naturally pulled back. Namely, $(f^{-1})^*\alpha \in \Omega^1(N)$ is defined by $((f^{-1})^*\alpha)Y = \alpha((f^{-1})_\star Y)$ for all $Y \in \mathfrak{X}(N)$. Given that every tensor field can be decomposed into a tensor product of vector fields and 1-forms, this defines the left action of the diffeomorphism group on our space of tensor fields. To have a unified notation in this space we will define the pushforward $f_* := (f^{-1})^*$ on our functions and 1-forms. We remark that we could have obtained a right action by replacing f by f^{-1} in all of our formulas above. This is sometimes done to avoid a minus signs that will appear at the infinitesimal level. We will however prefer to stick to the left action defined above.

The machinery we have developed so far is useful for the treatment of infinitesimal symmetries. We then have to understand the Lie algebra of the diffeomorphism group $\mathrm{Diff}(M)$. This will turn out to be, as a vector space, the vector fields $\mathfrak{X}(M)$. Rather than developing the theory of infinite dimensional Lie groups in order to obtain this statement, we will give a plausibility argument for it. Assume we have a "smooth" (whatever that means) one-parameter group of diffeomorphisms $\gamma : \mathbb{R} \to \mathrm{Diff}(M)$. At every $p \in M$ this induces a curve $\mu : t \mapsto \gamma(t)(p)$ on M with a tangent vector $X_p = \frac{\mathrm{d}}{\mathrm{d}t}\gamma(t)(p)\Big|_{t=0} \in T_pM$. Of course, in here we are assuming that the definition of "smooth" is such that μ is smooth. This describes the infinitesimal displacement of the point p under our curve γ. Then, all infinitesimal displacements are described by the vector field $\dot{\gamma}(0) \equiv \frac{\mathrm{d}}{\mathrm{d}t}\gamma(t)\Big|_{t=0} := X \in \mathfrak{X}(M)$.

Now, to find the action on the space of tensor fields of this Lie algebra we will use the discussion at the beginning of Section 3.6. In this case, our group $\mathrm{Diff}(M)$ acts on the manifold $V = \mathcal{T}^{\cdot}_{\cdot} M = \Gamma(T^{\cdot}_{\cdot} M)$ of tensor fields on M. Indeed, given such a tensor field F we have

$$\begin{aligned} \mathrm{Diff}(M) &\to V \\ f &\mapsto f_* F. \end{aligned} \tag{4.39}$$

In order to find the action of the Lie algebra we need to compute the pushforward of the above map at the identity. This is of course done by considering a vector field $X \in \mathrm{Lie}(\mathrm{Diff}(M))$ and the corresponding one-parameter group of diffeomorphisms γ as above. This curve on $\mathrm{Diff}(M)$ gets mapped into the curve $t \mapsto \gamma(t)_\star F$ on $\mathcal{T}^{\cdot}_{\cdot} M$. Following the philosophy above, we can map this into a curve on the finite dimensional vector space $T^{\cdot}_{\cdot} M$ by choosing a point $p \in M$. The resulting curve is $t \mapsto (\gamma(t)_* F)_p$. Assuming that γ "smooth" is enough to guarantee the smoothness of this curve, we can compute its derivative. Of course, this is nothing but the Lie derivative $-\mathcal{L}_X F$ [this is a simple extension of 44, Definition 20.3]. Let us recall that that for $F \in C^\infty(M)$ and $Y \in \mathfrak{X}(M)$ we have [see 44, Theorem 20.4 and Proposition 20.6]

$$\mathcal{L}_X f = Xf, \qquad\qquad \mathcal{L}_X Y = [X, Y]. \tag{4.40}$$

Meanwhile, for covariant vector fields $\sigma \in \mathcal{T}^0_n M$ we have the Leibniz rule [see 44, Theorem 20.10]

$$X(\sigma(Y_1, \dots, Y_n)) = (\mathcal{L}_X \sigma)(Y_1, \dots, Y_n) + \sum_{r=1}^{n} \sigma(Y_1, \dots, [X, Y_r], \dots, Y_n), \tag{4.41}$$

for all $Y_1, \dots, Y_n \in \mathfrak{X}(M)$. For the examples that we will discuss, it is convenient to compute this in a chart (U, x). Using $[X, \partial/\partial x^\mu] = -\,\partial X^\nu/\partial x^\mu\; \partial/\partial x^\nu$ and applying (4.41) to the basis vectors $\partial/\partial x^\mu$, we obtain

$$X^\nu \frac{\partial \sigma_{\mu_1 \cdots \mu_n}}{\partial x^\nu} = (\mathcal{L}_X \sigma)_{\mu_1 \cdots \mu_n} - \sum_{r=1}^{n} \sigma_{\mu_1 \cdots \mu_{r-1} \alpha \mu_{r+1} \cdots \mu_n} \frac{\partial X^\alpha}{\partial x^{\mu_r}}. \tag{4.42}$$

So far although we have claimed that $\mathfrak{X}(M)$ is the Lie algebra of $\mathrm{Diff}(M)$ because it is it tangent space at the identity, we have still to describe its Lie algebra structure. Given our description of the Lie algebra (as opposed to finding the left invariant vector fields on Diff(M)), we will do this by computing the derivative of the adjoint representation. Every $f \in \mathrm{Diff}(M)$ should induce a diffeomorphism

$$\begin{aligned} \mathrm{Diff}(M) &\to \mathrm{Diff}(M) \\ h &\mapsto f \circ h \circ f^{-1} \end{aligned} \tag{4.43}$$

The pushforward induced by this map is then called the adjoint representation of f

$$\mathrm{Ad}_f : \mathrm{Lie}(\mathrm{Diff}(M)) \to \mathrm{Lie}(\mathrm{Diff}(M)) \tag{4.44}$$

To compute this on a Lie algebra element $X \in \mathrm{Lie}(\mathrm{Diff}(M))$, let us consider the associated one-parameter group γ such that $X = \dot{\gamma}(0)$. Under the diffeomorphism induced by f, this one-parameter group gets mapper to $t \mapsto f \circ \gamma(t) \circ f^{-1}$. Ad_f is then defined to be the tangent vector to this curve at 0. Following our ongoing method, we can induce a curve on M by choosing a point $p \in M$, namely $t \mapsto f(\gamma(t)(f^{-1}(p)))$. Then tangent vector at 0 of this curve is given by

$$\frac{\mathrm{d}}{\mathrm{d}t} f(\gamma(t)(f^{-1}(p)))\bigg| = f_{*,f^{-1}(p)} X_{f^{-1}(p)} = (f_* X)_p \tag{4.45}$$

We thus declare $\mathrm{Ad}_f\, X = f_* X$. The pushforward induced by

$$\mathrm{Ad} : \mathrm{Diff}(M) \to \mathrm{Gl}(\mathrm{Diff(M)}) \tag{4.46}$$

then yields

$$\mathrm{ad} : \mathrm{Lie}(\mathrm{Diff}(M)) \to \mathrm{End}(\mathrm{Lie}(\mathrm{Diff}(M))). \tag{4.47}$$

It is obtained by

$$\mathrm{ad}_X = \frac{\mathrm{d}}{\mathrm{d}t} \mathrm{Ad}_{\gamma(t)}\bigg|_{t=0}. \tag{4.48}$$

We attempt to make sense of this derivative by evaluating it at a vector field $Y \in \mathrm{Lie}(\mathrm{Diff}(M))$

$$\mathrm{ad}_X\, Y = \frac{\mathrm{d}}{\mathrm{d}t} \mathrm{Ad}_{\gamma(t)}\, Y\bigg|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t} \gamma(t)_* Y\bigg|_{t=0} = -\mathcal{L}_X Y = -[X, Y]. \tag{4.49}$$

Given [see 68, Proposition 5.4.3], we see that the Lie bracket on Lie$(\mathrm{Diff}(M))$ that makes it the Lie algebra of Diff(M) is minus the Lie bracket of vector fields.

Rigorous foundations for the statements developed so far can be found in [69].

Let us equip with coordinates $c = c^\mu \frac{\partial}{\partial x^\mu}$ the Lie algebra Lie(Diff(M)). Then, from the bracket on $\mathfrak{X}(M)$

$$[X,Y] = X^\mu \frac{\partial Y^\nu}{\partial x^\mu}\frac{\partial}{\partial x^\nu} - Y^\mu \frac{\partial X^\nu}{\partial x^\mu}\frac{\partial}{\partial x^\nu}, \tag{4.50}$$

we see that the Lie bracket structure can be written as

$$[c,c] = -2c^\mu \frac{\partial c^\nu}{\partial x^\mu}\frac{\partial}{\partial x^\nu}. \tag{4.51}$$

Given that $\mathrm{Lie}(\mathrm{Diff}(M)) = \Gamma(TM)$, we can introduce the antifield of c as coordinatizing $\Gamma(T^*M \otimes \mathrm{Dens}(M))$ so that the action associated to the Lie algebra structure will be of the form

$$-\frac{1}{2}\langle c^*, [c,c]\rangle = \int \mathrm{d}^D x\, c^*_\nu c^\mu \frac{\partial c^\nu}{\partial x^\mu} \tag{4.52}$$

4.5 Equations of Motion for a Relativistic Extended Object

In preparation for our discussion of the relativistic particle, let us begin by studying its classical equations of motion. Motion of the particle traces out 1 dimensional trajectories on spacetime . As it turns out, the classical treatment through the Nambu-Goto action for the particle is essentially the same as the one for objects which trace trajectories of arbitrary dimension. A more general discussion also gives a clearer exposition of the geometry of the problem.

Consider a spacetime consisting of a smooth D-manifold M with a metric g of signature $(-1,1,\ldots,1)$. Recall that a vector $X \in T_pM$ is said to be time-like if $g(X,X) < 0$, space-like if $g(X,X) > 0$, and light-like if $g(X,X) = 0$. Vectors that are either light-like or time-like are called causal. The trajectory of our object is determined by a submanifold $\mathcal{T} \subseteq M$ of dimension d. This is called the worldvolume. We will restrict to worldvolumes whose tangent space at each point admit an orthonormal basis with a time-like vector. If we denote the identity embedding by $\gamma : \mathcal{T} \hookrightarrow M$, then this condition is equivalent to having an induced metric $\gamma^* g$ with signature $(-1,1,\ldots,1)$. One can therefore use this metric to measure the volume of the trajectory in spacetime. This yields the Nambu-Goto action

$$S(\mathcal{T}) := -T \int_{\mathcal{T}} \mathrm{d}\,\mathrm{vol}_{\gamma^* g}\,. \tag{4.53}$$

In here T is a parameter describing the "tension in our extended object". More precisely, in the case $d = 1$ it will describe the mass of the particle, while in the case $d = 2$ it will describe the tension of a string. This can be seen by computing the non-relativistic limit of this action [see 70, Exercise 1.1].

In its present form, this action is not suitable for the tools we have developed thus far. This is because the space of worldvolumes we are considering is far from having the structure of a theory of fields. We will solve this in two steps. The first one is to restrict to worldvolumes with a fixed topology and smooth structure. This amounts to fixing a worldvolume $\mathcal{T}$ of dimension d. This worldvolume is now an abstract object which should be thought of independently of M. Then the different worldvolumes that are embedded in M are obtained by considering different embeddings $\gamma : \mathcal{T} \to M$. The Nambu-Goto action for each of the embeddings $S(\gamma)$ is then defined by the same formula as (4.53). In this procedure something extremely subtle has happened. We have introduced a gauge symmetry. Indeed, if $f \in \mathrm{Diff}(\mathcal{T})$, then $f \triangleright \gamma := \gamma \circ f^{-1}$ yields the same embedded worldvolume in spacetime. Since only the resulting embedded worldvolume is physical, these parametrizations should be identified.

The second step is to consider the action only on a chart of $\mathcal{T}$. As far as the equations of motion are concerned, this is justified since an extremal of the action must be an extremal in every patch. Consider then charts (U, λ) and (V, y) covering a point in the worldvolume $p \in \mathcal{T}$ and the corresponding point in spacetime $\gamma(p) \in M$ respectively. This introduces the coordinate representation $\gamma_{(y)} := y \circ \gamma$

$$\begin{tikzcd} U \subseteq \mathcal{T} \arrow[r, "\gamma"] \arrow[d, "\lambda"] \arrow[dr, "\gamma_{(y)}"] & V \subseteq M \arrow[d, "y"] \\ x(U) \subseteq \mathbb{R}^2 & y(V) \subseteq \mathbb{R}^D. \end{tikzcd} \tag{4.54}$$

With this coordinate representation we can compute the action corresponding to to the patch U of the world sheet. Indeed, if $g = (g_{\mu\nu} \circ y)\, \mathrm{d}y^\mu \otimes \mathrm{d}y^\nu$, we have the induced metric

$$\begin{aligned} h := \gamma^* g &= (g_{\mu\nu} \circ y \circ \gamma)\, \mathrm{d}(y^\mu \circ \gamma) \otimes \mathrm{d}(y^\nu \circ \gamma) \\ &= (g_{\mu\nu} \circ \gamma_{(y)}) \frac{\partial \gamma_{(y)}}{\partial \lambda^a} \frac{\partial \gamma_{(y)}}{\partial \lambda^b}\, \mathrm{d}\lambda^a \otimes \mathrm{d}\lambda^b \end{aligned} \tag{4.55}$$

This allows us to define the Nambu-Goto action in terms of this coordinate representation

$$S(\gamma_{(y)}) := \int \mathrm{d}^d\lambda \, \sqrt{-h}, \tag{4.56}$$

with $\sqrt{-h}$ being short hand notation for $\sqrt{-\det h_{\cdot\cdot}}$.

Note that this action is now defined on a subset of the vector space $C^\infty(U, \mathbb{R}^D)$. Thus, we can think of it as the action of D scalar fields, where our worldvolume acts as the spacetime. The action is however at this stage not defined for all such fields because we need to ensure that our parametrizations lead to an embedding with a Lorentzian metric. However, the space of such parametrizations is open. The fields in this space will be called X. Note that this action is indeed invariant under orientation preserving diffeomorphisms of the worldvolume. This is simply the statement that $\sqrt{-h}\,\mathrm{d}^d\lambda \in \Omega^d(U)$ and that the integration of d-forms is invariant under changes of coordinates.

The equations of motion are

$$0 = \frac{\partial S}{\partial X^\mu(\lambda)} = -\int \mathrm{d}^d\sigma \, \frac{1}{2\sqrt{-h(\sigma)}} \frac{\partial \det h_{\cdot\cdot}(\sigma)}{\partial X^\mu(\lambda)}. \tag{4.57}$$

Via Jacobi's formula we can compute

$$\frac{\partial \det(h_{\cdot\cdot}(\sigma))}{\partial X^\mu(\lambda)} = \det(h_{\cdot\cdot}(\sigma)) h^{ab}(\sigma) \frac{\partial h_{ab}(\sigma)}{\partial X^\mu(\lambda)}, \tag{4.58}$$

so that the equations of motion become

$$0 = \frac{1}{2} \int \mathrm{d}^d\sigma \, \sqrt{-h(\sigma)} h^{ab}(\sigma) \frac{\partial h_{ab}(\sigma)}{\partial X^\mu(\lambda)} \tag{4.59}$$

Moreover,

$$\begin{aligned} \frac{\partial h_{ab}(\sigma)}{\partial X^\mu(\lambda)} &= \partial_\mu g_{\alpha\beta}(X(\sigma)) \delta(\sigma - \lambda) \partial_a X^\alpha(\sigma) \partial_b X^\beta(\sigma) \\ &\quad + 2 g_{\mu\beta}(X(\sigma)) \partial_a \delta(\sigma - \lambda) \partial_b X^\beta(\sigma). \end{aligned} \tag{4.60}$$

We then conclude that the equations of motion are

$$
\begin{aligned}
0 &= \frac{1}{2}\int \mathrm{d}^d\sigma \sqrt{-h(\sigma)}h^{ab}(\sigma)\partial_\mu g_{\alpha\beta}(X(\sigma))\delta(\sigma-\lambda)\partial_a X^\alpha(\sigma)\partial_b X^\beta(\sigma) \\
&\quad + \int \mathrm{d}^d\sigma \sqrt{-h(\sigma)}h^{ab}(\sigma)g_{\mu\beta}(X(\sigma))\partial_a\delta(\sigma-\lambda)\partial_b X^\beta(\sigma) \\
&= \frac{1}{2}\sqrt{-h(\lambda)}h^{ab}(\lambda)\partial_\mu g_{\alpha\beta}(X(\lambda))\partial_a X^\alpha(\lambda)\partial_b X^\beta(\lambda) \\
&\quad - \partial_a(\sqrt{-h(\lambda)}h^{ab}(\lambda)g_{\mu\beta}(X(\lambda))\partial_b X^\beta(\lambda))
\end{aligned}
\tag{4.61}
$$

The geometric meaning of these equations of motion is clearer by distributing the last derivative over the metric on spacetime

$$
\begin{aligned}
0 &= \partial_a(\sqrt{-h}h^{ab}\partial_b X^\beta)g_{\mu\beta}(X) + \sqrt{-h}h^{ab}\partial_b X^\beta\partial_\alpha g_{\mu\beta}(X)\partial_a X^\alpha \\
&\quad - \frac{1}{2}\sqrt{-h}h^{ab}\partial_\mu g_{\alpha\beta}(X)\partial_a X^\alpha\partial_b X^\beta \\
&= \partial_a(\sqrt{-h}h^{ab}\partial_b X^\beta)g_{\mu\beta}(X) \\
&\quad + \sqrt{-h}h^{ab}\partial_a X^\alpha\partial_b X^\beta\left(\partial_\alpha g_{\mu\beta}(X) - \frac{1}{2}\partial_\mu g_{\alpha\beta}(X)\right)
\end{aligned}
\tag{4.62}
$$

For the second term we notice that

$$
\begin{aligned}
&\partial_a X^\alpha\partial_b X^\beta(\lambda)\left(\partial_\alpha g_{\mu\beta}(X) - \frac{1}{2}\partial_\mu g_{\alpha\beta}(X)\right) \\
&= \frac{1}{2}\partial_a X^\alpha\partial_b X^\beta(\partial_\alpha g_{\mu\beta}(X) + \partial_\beta g_{\mu\alpha}(X) - \partial_\mu g_{\alpha\beta}(X)) \\
&= g_{\mu\nu}(X)\Gamma_\alpha{}^\nu{}_\beta(X)\partial_a X^\alpha\partial_b X^\beta = g_{\mu\nu}(X)\Gamma_a{}^\nu{}_\beta\partial_b X^\beta,
\end{aligned}
\tag{4.63}
$$

where $\Gamma_\alpha{}^\nu{}_\beta$ are the Christoffel symbols for the metric on spacetime and

$$
\Gamma_a{}^\nu{}_\beta := \partial_a X^\alpha\Gamma_\alpha{}^\nu{}_\beta(X),
\tag{4.64}
$$

are the connection coefficients of the pullback by X. For the first term, we can use the familiar expression for the covariant derivative of a vector field [see 71, equation (4.7.7)] and the covariant conservation of the metric to see that

$$
\begin{aligned}
&\partial_a(\sqrt{-h}h^{ab}\partial_b X^\beta)g_{\mu\beta}(X) \\
&= \sqrt{-h}h^{ab}\nabla_a^{(X)}(\partial_b X^\beta)g_{\mu\beta}(X),
\end{aligned}
\tag{4.65}
$$

where now $\nabla^{(X)}$ is the covariant derivative with respect to the metric h induced on the worldvolume. Placing these results together, we obtain the equations of motion

$$0 = h^{ab}\left(\nabla_a^{(X)}(\partial_b X^\nu) + \Gamma_a{}^\nu{}_\beta \partial_b X^\beta\right), \tag{4.66}$$

where we have cancelled the square root and the metric on spacetimes since they are both invertible. This can all be written in a final compact form by introducing the connection $D^{(X)}$ on $T\mathcal{T} \oplus x^*Ty(V)$ on the worldvolume. We then obtain the equations of motion [72]

$$D^{(X)a}(\partial_a X^\mu) = 0. \tag{4.67}$$

4.6 Polyakov Action

While the Nambu-Goto action was helpful to develop the classical theory of extended objects, the presence of the square root makes it difficult to understand the corresponding quantum theory. In particular, it is not clear how to do perturbation theory since there is no discernible quadratic part that describes the corresponding free theory. As it turns out, reinterpreting the Nambu-Goto action as a theory of gravity on the worldvolume has the by-product of solving this issue. The resulting action is known as the Polyakov action. This section is a review of the material found in [see 73, Section 2].

In order to interpret our theory as a theory of gravity on the world volume, we need to introduce an additional field. This will be a metric γ on $\mathcal{T}$ which describes the gravitational field on the worldvolume. Thus, we extend our space of fields to $V = C^\infty(U, \mathbb{R}^D) \oplus \Gamma(S^2(T^*U))$. We then obtain a theory equivalent to (4.53) by rewritting the action in terms of this intrinsic metric and adding a Lagrange multiplier in $\Gamma(S^2(TU) \otimes \mathrm{Dens}(M))$ which demands that this coincides with the induced metric

$$S(X, \gamma, \Lambda) = -T \int \mathrm{d}^d\lambda \,(\sqrt{-\gamma} + \Lambda^{ab}(h_{ab} - \gamma_{ab})) \tag{4.68}$$

Eliminating Λ by imposing its equations of motion shows that this action is equivalent to (4.53). On the other hand, let us consider the equations of motion for γ. These are

$$\begin{aligned} \frac{\partial S}{\partial \gamma_{ab}(\lambda)} &\propto \int \mathrm{d}^d\sigma \left(\frac{1}{2}\sqrt{-\gamma(\sigma)}\gamma^{ab}(\sigma) - \Lambda^{ab}\right)\delta(\sigma - \lambda) \\ &= \frac{1}{2}\sqrt{-\gamma(\lambda)}\gamma^{ab}(\lambda) - \Lambda^{ab}(\lambda). \end{aligned} \tag{4.69}$$

Plugging this back into the action we obtain the Polyakov action

$$\begin{aligned} S(X,\gamma) &= -T\int \mathrm{d}^d\lambda \left(\sqrt{-\gamma} + \frac{1}{2}\sqrt{-\gamma}\gamma^{ab}(h_{ab}-\gamma_{ab})\right) \\ &= -\frac{T}{2}\int \mathrm{d}^d\lambda\, \sqrt{-\gamma}\Big(\gamma^{ab}h_{ab} - d + 2\Big), \end{aligned} \tag{4.70}$$

where we have used $\gamma^{ab}\gamma_{ab} = \delta^a{}_a = d$. Notice that this action is now a polynomial on X. The first term is common to worldsheets of all dimensions. On the other hand, the second term is a dimension dependent cosmological constant in our theory [see 74, Problem 4 in Section 2.4]. This constant vanishes precisely in the case of a string $d = 2$.

At this point there is no guarantee that this action is equivalent to the Nambu-Goto action since we have forgotten the equation of motion of Λ. Fortunately, it is still equivalent. Indeed, the equations of motion $\partial S/\partial\gamma^{ab}(\lambda)$ for γ are proportional to

$$\int \mathrm{d}^d\sigma\, \sqrt{-\gamma(\sigma)}\left(-\frac{1}{2}\gamma_{ab}(\sigma)(\gamma^{cd}(\sigma)h_{cd}(\sigma) - d + 2) + h_{ab}(\sigma)\right)\delta(\sigma-\lambda). \tag{4.71}$$

We thus obtain

$$h_{ab} = \frac{1}{2}\gamma_{ab}(\gamma^{cd}h_{cd} - d + 2). \tag{4.72}$$

Multiplying this equation through by γ^{ab} we obtain

$$\gamma^{ab}h_{ab} = \frac{1}{2}d(\gamma^{de}h_{de} - d + 2), \tag{4.73}$$

i.e.

$$\gamma^{ab}h_{ab}(2-d) = (2-d)d. \tag{4.74}$$

For $d \neq 2$ this sets $\gamma^{ab}h_{ab} = d$. Plugging back into (4.72) we obtain back $\gamma_{ab} = h_{ab}$ which was the equation of motion for Λ. This shows the equivalence with the Nambu-Goto action. If $d = 2$ we will not be able to recover this equation. However, in that case, taking the determinant of (4.72) yields

$$\det h_{\cdot\cdot} = \left(\frac{1}{2}(\gamma^{cd}h_{cd})\right)^2 \det\gamma_{\cdot\cdot}. \tag{4.75}$$

Therefore, dividing through by $\sqrt{-h}$ in (4.72) yields

$$\frac{h_{ab}}{\sqrt{-h}} = \frac{\gamma_{ab}}{\sqrt{-\gamma}}, \tag{4.76}$$

which allows us to conclude that $\gamma_{ab} = \Omega^2 h_{ab}$ for some $\Omega \in C^\infty(U)$, which is weaker than the equations of motion for Λ. Still, plugging this into the Polyakov action we are able to recover the Nambu-Goto action

$$-\frac{T}{2}\int \mathrm{d}^d\lambda\, \sqrt{\cancel{\Omega^4}(-h)}\cancel{\Omega^{-2}}\cancelto{2}{h^{ab}h_{ab}} = -T\int \mathrm{d}^d\lambda\, \sqrt{-h}. \tag{4.77}$$

The reason for us not being able to retrieve the equation $\gamma_{ab} = h_{ab}$ from the Polyakov action in $d = 2$ is because in this case the action has an extra symmetry other than the diffeomorphism symmetry we have already mentioned. Namely, this action is invariant under rescalings of the worldsheet metric $\gamma_{ab} \to \Omega^2\gamma_{ab}$ since $\sqrt{\Omega^4(-\gamma)}\Omega^{-2}\gamma^{ab} = \sqrt{-\gamma}\gamma^{ab}$. This is an additional symmetry that needs to be gauged by in the BV formalism.

4.7 Relativistic Particle

Let us now study the case $d = 1$. Point particles trace these sort of trajectories in spacetime. We also remark that in this case it is customary to multiply the action by an overall factor of $-m$, with m the mass of the particle. In this way one ensures that the action has dimension of angular momentum. Let us begin by understanding their equations of motion (4.67). In order to do this, let us see what our objects look like in this case

$$\begin{aligned}
\partial_a X^\mu &= \dot{X}^\mu, \\
\Gamma_{a\ \beta}^{\ \nu} &= \dot{X}^\alpha \Gamma_{\alpha\ \beta}^{\ \nu}(X) \\
h_{ab} &= g_{\mu\nu}(X)\dot{X}^\mu\dot{X}^\nu =: h, \\
\Gamma_{a\ c}^{\ b} &= \frac{1}{2}h^{bd}(\partial_a h_{db} + \partial_b h_{da} - \partial_d h_{ab}) = \frac{1}{2}h^{-1}\dot{h} = \frac{\mathrm{d}}{\mathrm{d}\lambda}\log\left(\sqrt{-h}\right).
\end{aligned} \tag{4.78}$$

In here we have used (4.64). We thus obtain the equations of motion

$$0 = \ddot{X}^\mu + \dot{X}^\alpha \Gamma_{\alpha\ \beta}^{\ \mu}(X)\dot{X}^\beta - \dot{X}^\mu \frac{\mathrm{d}}{\mathrm{d}\lambda}\log\left(\sqrt{-g_{\mu\nu}(X)\dot{X}^\mu\dot{X}^\nu}\right). \tag{4.79}$$

The first two terms are simply the autoparallel equation. The third is a term which ensures reparametrization invariance. In particular, if we parametrize our curves in such a way that their tangent vectors have a constant length, we obtain back the autoparallel equation.

In order to proceed with the BV formalism we will now go to the $d = 1$ Polyakov action

$$S(X,\gamma) = -\frac{m}{2}\int \mathrm{d}\lambda\,\sqrt{-\gamma}(\gamma^{-1}h+1). \tag{4.80}$$

The more usual form of this action is obtained by replacing the field γ by a positive einbein $e\,\mathrm{d}\lambda \in \Omega^1(U)$, such that $\gamma = -m^2e^2$. We then have

$$S(X,e) = -\frac{m}{2}\int \mathrm{d}\lambda\, me\left(-\frac{1}{m^2e^2}h+1\right) = \frac{1}{2}\int \mathrm{d}\lambda\,\left(\frac{1}{e}h - em^2\right). \tag{4.81}$$

Notice that, unlike the Nambu-Goto action, this action makes sense even for massless particles.

Our theory, being described in terms of tensor fields on the worldline, has diffeormorphism invariance. The associated Lie algebra is $\mathfrak{g} = \mathrm{Lie}(\mathrm{Diff}(U))$. Let us use coordinates $c\,\mathrm{d}/\mathrm{d}\lambda$ on this space. The action on our space of fields is

$$\begin{aligned} \mathcal{L}_{X\frac{\mathrm{d}}{\mathrm{d}\lambda}} f &= Xf', \\ \mathcal{L}_{X\frac{\mathrm{d}}{\mathrm{d}\lambda}}(\alpha\,\mathrm{d}\lambda) &= (X\alpha' + \alpha X')\,\mathrm{d}\lambda = (X\alpha)'\,\mathrm{d}\lambda\,, \end{aligned} \tag{4.82}$$

for $X\,\mathrm{d}/\mathrm{d}\lambda \in \mathfrak{g}$, $f \in C^\infty(U)$ and $\alpha\,\mathrm{d}\lambda \in \Omega^1(U)$. Now, notice that $C^\infty(U,\mathbb{R}^d)$ is simply the space of sections of the trivial bundle $U\times\mathbb{R}^d$. The corresponding space of antifields is then the space of sections of the tensor product $(U\times(\mathbb{R}^d)^*)\otimes \mathrm{Dens}(U)$. In one dimension we can however identify T^*U and $\mathrm{Dens}(U)$ since every 1-manifold is orientable. Therefore the space of antifields are the $(\mathbb{R}^d)^*$-valued 1-forms $\Omega^1(U)\otimes(\mathbb{R}^d)^*$. Let us denote the corresponding antifields as $x^*_\mu\,\mathrm{d}\lambda$. Similarly, the space of antifields to $\Omega^1(U)$ is $C^\infty(U)$, whose fields we will denote by e^*. The BV action then takes the form

$$S = \int \mathrm{d}\lambda\,\left(\frac{1}{2}\left(\frac{1}{e}g_{\mu\nu}(X)\dot{X}^\mu\dot{X}^\nu - em^2\right) - X^*_\mu\dot{X}^\mu c - e^*(ec)' + c^*cc'\right) \tag{4.83}$$

In order to gauge fix this action, let us include the trivial pair of a Faddeev-Popov antighost $\bar{c}$ and a Lagrange multiplier b coordinatizing the new spaces $C^\infty(U)[-1]$ and $C^\infty(U)$ respectively. We also include the respective antifields $\bar{c}^*$

and b^* on $\Omega^1(U)$ and $\Omega^1(U)[-1]$. The non minimal action is then

$$S = \int \mathrm{d}\lambda \left(\frac{1}{2}\left(\frac{1}{e} g_{\mu\nu}(X)\dot{X}^\mu \dot{X}^\nu - em^2\right) \right. \\ \left. - X^*_\mu \dot{X}^\mu c - e^*(ec)' + c^* c\dot{c} + b\bar{c}^* \right) \tag{4.84}$$

We note at this point that the BRST symmetries of our fields are

$$QX^\mu = -\dot{X}^\mu c, \qquad Qe = -(ec)', \tag{4.85}$$

$$Qc = -c\dot{c}, \qquad Q\bar{c} = -b, \qquad Qb = 0. \tag{4.86}$$

We can choose the gauge fixing fermion

$$\Psi = \int \mathrm{d}\lambda\, \bar{c}(e-1). \tag{4.87}$$

This fixes our antifields to

$$\bar{c}^* = e - 1, \quad e^* = \bar{c}, \tag{4.88}$$

with the rest vanishing. The resulting action is

$$S_\Psi = \int \mathrm{d}\lambda \left(\frac{1}{2}\left(\frac{1}{e} g_{\mu\nu}(X)\dot{X}^\mu \dot{X}^\nu - em^2\right) - \bar{c}(ec)' + b(e-1) \right). \tag{4.89}$$

The equations of motion of b then set $e = 1$, while the equations of motion of e are

$$0 = \frac{1}{2}\left(-\frac{1}{e^2} g_{\mu\nu}(X)\dot{X}^\mu \dot{X}^\nu - m^2\right) + \dot{\bar{c}}c + b. \tag{4.90}$$

We conclude that these fix

$$b = \frac{1}{2}\left(g_{\mu\nu}(X)\dot{X}^\mu \dot{X}^\nu + m^2\right) - \dot{\bar{c}}c, \tag{4.91}$$

and leave us with the action

$$S = \int \mathrm{d}\lambda \left(\frac{1}{2}\left(g_{\mu\nu}(X)\dot{X}^\mu \dot{X}^\nu - m^2\right) - \bar{c}\dot{c} \right), \tag{4.92}$$

as well as the BRST transformations

$$
\begin{aligned}
QX^{\mu} &= -\dot{X}^{\mu} c, \quad Qc = -c\dot{c}, \\
Q\bar{c} &= -\frac{1}{2}\Big(g_{\mu\nu}(X)\dot{X}^{\mu}\dot{X}^{\nu} + m^2\Big) + \dot{\bar{c}}c.
\end{aligned}
\tag{4.93}
$$

The conjugate momenta are

$$
p_{\mu} := \frac{\partial S}{\partial \dot{X}^{\mu}} = g_{\mu\nu}(X)\dot{X}^{\nu}, \quad \pi := \frac{\partial S}{\partial \dot{c}} = \bar{c}, \quad \bar{\pi} \; := \frac{\partial S}{\partial \dot{\bar{c}}} = 0. \tag{4.94}
$$

We thus have the primary constraints $\chi_1 := \bar{\pi}$ and $\chi_2 := \pi - \bar{c}$. These are second order since the canonical bracket between them is

$$
\{\bar{\pi}, \pi - \bar{c}\} = -1. \tag{4.95}
$$

In other words, we have the matrix

$$
C_{\alpha\beta} := \{\chi_{\alpha}, \chi_{\beta}\} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}_{\alpha\beta}, \tag{4.96}
$$

which is its own inverse. We then have the Dirac brackets $\{F, G\}^* := \{F, G\} - \{F, \chi_{\alpha}\}C^{\alpha\beta}\{\chi_{\beta}, G\}$, which in this particular case yield

$$
\{X^{\mu}, p_{\nu}\}^* = \delta^{\mu}{}_{\nu}, \qquad \{c, \bar{c}\}^* = \{c, \chi_2\}\{\chi_1, \bar{c}\} = 1. \tag{4.97}
$$

Proceeding as usual, the Hamiltonian of our system is given by

$$
\begin{aligned}
H &= \dot{X}^{\mu} p_{\mu} + \cancel{\dot{c}\dot{\bar{c}}} - \left(\frac{1}{2}\Big(\dot{X}^{\mu} p_{\mu} - m^2\Big) - \cancel{\dot{\bar{c}}\dot{c}}\right) = \frac{1}{2}(\dot{X}^{\mu} p_{\mu} + m^2) \\
&= \frac{1}{2}(g^{\mu\nu}(X) p_{\mu} p_{\nu} + m^2).
\end{aligned}
\tag{4.98}
$$

On the other hand, the BRST symmetries generate via Noether's theorem a conserved charge. To derive this, note that we have the variations

$$
\begin{aligned}
Q\dot{X}^{\mu} &= -\int \mathrm{d}\sigma\, \dot{\delta}(\lambda - \sigma)\dot{X}^{\mu}(\sigma)c(\sigma) = -\frac{\mathrm{d}}{\mathrm{d}\lambda}(\dot{X}^{\mu} c), \\
Qg_{\mu\nu}(X) &= -\partial_{\alpha} g_{\mu\nu}(X)\dot{X}^{\alpha} c, \\
Q\dot{c} &= -\frac{\mathrm{d}}{\mathrm{d}\lambda}(c\dot{c}) = -c\ddot{c},
\end{aligned}
\tag{4.99}
$$

so that the variation of the action is by a total derivative

$$\begin{aligned}
QS &= \int \mathrm{d}\lambda \left(-\frac{1}{2}\partial_\alpha g_{\mu\nu}(X)\dot{X}^\alpha \dot{X}^\mu \dot{X}^\nu c - g_{\mu\nu}(X)\frac{\mathrm{d}}{\mathrm{d}\lambda}(\dot{X}^\mu c)\dot{X}^\nu \right. \\
&\quad \left. + \frac{1}{2}\Big(g_{\mu\nu}(X)\dot{X}^\mu \dot{X}^\nu + m^2\Big)\dot{c} - \dot{\bar{c}}c\dot{c} - \bar{c}c\ddot{c} \right) \\
&= \int \mathrm{d}\lambda \left(-\frac{1}{2}\partial_\alpha g_{\mu\nu}(X)\dot{X}^\alpha \dot{X}^\mu \dot{X}^\nu c - g_{\mu\nu}(X)\ddot{X}^\mu \dot{X}^\nu c \right. \\
&\quad \left. - \frac{1}{2}\Big(g_{\mu\nu}(X)\dot{X}^\mu \dot{X}^\nu - m^2\Big)\dot{c} - \frac{\mathrm{d}}{\mathrm{d}\lambda}(\bar{c}c\dot{c}) \right) \\
&= \int \mathrm{d}\lambda \left(-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\lambda}(g_{\mu\nu}(X)\dot{X}^\mu \dot{X}^\nu)c \right. \\
&\quad \left. - \frac{1}{2}\Big(g_{\mu\nu}(X)\dot{X}^\mu \dot{X}^\nu - m^2\Big)\dot{c} - \frac{\mathrm{d}}{\mathrm{d}\lambda}(\bar{c}c\dot{c}) \right) \\
&= \int \mathrm{d}\lambda \frac{\mathrm{d}}{\mathrm{d}\lambda}\left(-\frac{1}{2}(g_{\mu\nu}(X)\dot{X}^\mu \dot{X}^\nu - m^2)c - \bar{c}c\dot{c} \right)
\end{aligned} \tag{4.100}$$

Therefore, the conserved charge is

$$\begin{aligned}
Q &= -g_{\mu\nu}(X)\dot{X}^\nu \dot{X}^\mu c - \cancel{\bar{c}c\dot{c}} + \frac{1}{2}(g_{\mu\nu}(X)\dot{X}^\mu \dot{X}^\nu - m^2)c + \cancel{\bar{c}c\dot{c}} \\
&= -\frac{1}{2}(g_{\mu\nu}(X)\dot{X}^\mu \dot{X}^\nu + m^2)c = -Hc
\end{aligned} \tag{4.101}$$

We can proceed now to do the usual canonical quantization of this system. In this we have the canonical commutation relations induced by the Dirac brackets (4.97)

$$[X^\mu(\lambda), p_\nu(\lambda)] = i\hbar\delta^\mu{}_\nu, \quad [c(\lambda), \bar{c}(\lambda)]_+ = 1, \tag{4.102}$$

with all other commutators vanishing. Since the operators X^μ and p_μ commute with $c(\lambda)$ and $\bar{c}(\lambda)$, we can construct an irreducible representation of these commutation relations by taking the tensor product of irreducible representations of each of them. For the (exponentiated version of the) first, we have that every irreducible representation is isomorphic to $L^2(\mathbb{R}^4)$ [see 4, Theorem 14.8]. For all $\psi \in L^2(\mathbb{R}^4)$ the operators act by $(X^\mu(\lambda)\psi)(x) = x^\mu\psi(y)$ and $(p_\mu\psi)(x) = -i\hbar\partial_\mu\psi(x)$. For the second, note that $c(\lambda)^2 = 0$ and therefore there is no representation of dimension 1. However, we do have a representation on a two dimensional complex inner product space F with an orthonormal basis $\{|\uparrow\rangle, |\downarrow\rangle\}$, where

$$c(\lambda)|\downarrow\rangle = 0, \qquad \bar{c}(\lambda)|\downarrow\rangle = |\uparrow\rangle, \qquad c(\lambda)|\uparrow\rangle = |\downarrow\rangle, \qquad \bar{c}(\lambda)|\downarrow\rangle = 0. \tag{4.103}$$

Then, the Hilbert space of our quantum theory is $\mathcal{H} = L^2(\mathbb{R}^4) \otimes F$.

Not all of the states in that Hilbert space are physical. These in fact belong to the cohomology of the BRST operator Q, which satisfies $Q^2 = H(\lambda)^2(c(\lambda))^2 = 0$. Now, the closed states are precisely those $|\psi\rangle \otimes |s\rangle \in \mathcal{H}$ satisfying $H|\psi\rangle = 0$ or $s = \downarrow$. The first condition is the relativistic energy dispersion relation. In our chosen representation this is simply the Klein-Gordon equation. On the other hand,

$$Q(|\psi\rangle \otimes |\uparrow\rangle) = H|\psi\rangle \otimes |\downarrow\rangle . \tag{4.104}$$

Thus, the exact states are those $|\psi\rangle \otimes |\downarrow\rangle$ for which there is a $|\phi\rangle \in L^2(\mathbb{R}^4)$ such that $|\psi\rangle = H|\phi\rangle$. This condition is better understood in terms of the improper basis $|q\rangle \in L^2(\mathbb{R}^4)$ such that $p^\mu |q\rangle = q^\mu |q\rangle$. Then $H|q\rangle = \frac{1}{2}(q^2 + m^2)|q\rangle$. We thus see that every $|q\rangle$ satisfies the condition unless $H|q\rangle = 0$. We conclude that the exact states are those for which $|\psi\rangle \otimes |\downarrow\rangle$ for which $H|\psi\rangle \neq 0$. We conclude that the physical states can be taken to be the states $|\psi\rangle \otimes |s\rangle$ with $|\psi\rangle$ satisfying the Klein-Gordon equation.

4.8 Connection Symmetries

Let us consider a G-principal bundle $\pi : P \to M$ for some structure group G. If we picture the fibres of such a bundle as vertical, then at every $p \in P$ we have a natural definition of a vertical subspace of T_pP. This is simply the space of vectors tangent to the fibre

$$V_pP := \ker \pi_{*,p}. \tag{4.105}$$

This space is isomorphic as a vector space to the Lie algebra [see 75, Proposition 27.18]

$$\begin{aligned} \mathfrak{g} &\to V_pP \\ A &\mapsto X_p^A := \left.\frac{\mathrm{d}}{\mathrm{d}t} p \triangleleft e^{tA}\right|_{t=0} . \end{aligned} \tag{4.106}$$

However, a notion of a horizontal subspace is not readily available. Making a choice for such subspace is important in order to compare different fibres. These choices are known as connections. Yang-Mills theory is an example of a field theory which assigns dynamics to these.

These theories are of extreme importance to high-energy particle physics. There, matter is usually described by fields on M which form a multiplet. This means that they transform among themselves under such a structure group. The standard model introduces interactions between these fields by asking that such transformations be local. This means that at every point of spacetime we may decide to transform our multiplet with a different element of G. However, in order to be able to write down kinetic terms for our matter fields so they can propagate we need to be able to take spatial derivatives of them. A connection is what allows us to compare such matter fields at different points even though they may transform differently in each. In these way connections become the carriers of the interactions in the standard model. In this case, their dynamics are precisely given by Yang-Mills theory.

A useful way of describing connections is through a connection one-form. These are Lie algebra valued smooth one-forms

$$\omega_p : T_pP \to \mathfrak{g} \in \mathrm{Hom}(T_pP, \mathfrak{g}) = \mathrm{Hom}(T_pP, \mathbb{R}) \otimes \mathfrak{g} = T_p^*P \otimes \mathfrak{g} \tag{4.107}$$

satisfying

1. $\omega_p(X_p^A) = A$
2. $((\triangleleft g)^*\omega_p)(X_p) = \mathrm{Ad}_{g^{-1}}\,\omega_p(X_p)$.

In here $\triangleleft$ is the right action of G on P, X_p^A is the tangent vector of the curve $t \mapsto p \triangleleft e^{tA}$, and Ad is the adjoint representation of G on $\mathfrak{g}$. The horizontal subspace at each point is then given by $H_pP := \ker \omega_p$. The smoothness of ω guarantees that these horizontal subspaces vary smoothly. Moreover, since $X_p^A \in V_pP$ is vertical, we see that condition 1. guarantees that $T_pP = H_pP \oplus V_pP$ [simple consequence of 39, Theorem 2.3]. Finally, note that we can use the group action to push-forward horizontal subspaces along a given fibre. Condition 2. ensures that our choice of connection is compatible with this $(\triangleleft g)_{*,p}H_pP = H_{p\triangleleft g}P$.

Although this picture is geometrically satisfactory, in physics one rarely has information about the bundle P. Fortunately, the connection can also be described through a "Yang-Mills field". This will be a $\mathfrak{g}$-valued connection on spacetime M rather than on P. However, we will in general only be able to do this locally. We will thus have several Yang-Mills fields defined on different neighborhoods of M. However, these can be used to recuperate the full structure of P.

The way we construct the Yang-Mills fields is by choosing a local section $\sigma : U \to P$. The Yang-Mills field is then defined as $A := \sigma^*\omega \in \Omega^1(U) \otimes \mathfrak{g}$. We are then faced with understanding the relation between these fields on overlapping section. Assume we have two sections $\sigma_1 : U_1 \to P$ and $\sigma_2 : U_2 \to P$, each defining a Yang-Mills field respectively $A^{(1)}$ and $A^{(2)}$. To understand their relationship on the overlap $U_1 \cap U_2$ we need two ingredients. For the first, remember that on a Lie group the left multiplication ℓ_g by a given group element $g \in G$ is a diffeomorphism. It then induces an isomorphism $(\ell_g)_{*,e} : \mathfrak{g} \to T_g G$. The Maurer-Cartan form Ξ is a Lie algebra valued one-form on G such that $\Xi_g := ((\ell_g)_{*,e})^{-1} = (\ell_{g^{-1}})_{*,g} : T_g G \to \mathfrak{g}$. For the second, we have a gauge function, which is a G-valued function on spacetime $\Omega : U_1 \cap U_2 \to G$ given by $\sigma_2(p) = \sigma_1(p) \triangleleft \Omega(p)$ for all $p \in U_1 \cap U_2$. Using these two ingredients, we have

$$A^{(2)}_p = \mathrm{Ad}_{\Omega(p)^{-1}} A^{(1)}_p + (\Omega^* \Xi)_p \tag{4.108}$$

Following the general philosophy of building Quantum Field Theories from simple local building blocks, let us now assume that our Yang-Mills fields are globally defined $A \in \Omega^1(M) \otimes \mathfrak{g}$. Given such an A, the group $C^\infty(M, G)$ of smooth functions $M \to G$ acts on the space of these fields

$$\begin{aligned} A \triangleleft \cdot \, : C^\infty(M, G) &\to \Omega^1(M) \otimes \mathfrak{g} \\ \Omega &\mapsto \mathrm{Ad}_{\Omega^{-1}} A + \Omega^* \Xi, \end{aligned} \tag{4.109}$$

where the latter is defined pointwise. Notice that $C^\infty(M, G)$ is a group with pointwise product. Let us compute the pullback of the Maurer-Cartan form. For every $p \in M$ and $Y = \dot{\mu}(0) \in T_p M$, for some curve $\mu : \mathbb{R} \to M$ such that $\mu(0) = p$, we have

$$\begin{aligned} (\Omega^* \Xi)_p Y &= \Xi_{\Omega(p)} \Omega_{*,p} Y_p = \Xi_{\Omega(p)} \left. \frac{\mathrm{d}}{\mathrm{d}t} \Omega(\mu(t)) \right|_{t=0} \\ &= (\ell_{\Omega(p)^{-1}})_{*,\Omega(p)} \left. \frac{\mathrm{d}}{\mathrm{d}t} \Omega(\mu(t)) \right|_{t=0} = \left. \frac{\mathrm{d}}{\mathrm{d}t} \Omega(p)^{-1} \Omega(\mu(t)) \right|_{t=0} . \end{aligned} \tag{4.110}$$

In particular, for a matrix Lie group we have

$$(\Omega^* \Xi)_p Y = \Omega(p)^{-1} \frac{\partial \Omega}{\partial x^\mu} Y^\mu, \tag{4.111}$$

or in other words

$$(\Omega^* \Xi) = \Omega^{-1} \, \mathrm{d}\Omega \, , \tag{4.112}$$

where the product is taken pointwise.

Let us now turn into the study of the Lie algebra of $C^\infty(M, G)$ and the corresponding action. Consider a one-parameter subgroup $\gamma : \mathbb{R} \to C^\infty(M, G)$ of

$C^\infty(M,G)$. Given $p \in M$ we can induce a one-parameter subgroup $t \mapsto \gamma(t)(p)$ of G. The tangent vector to the latter at 0 is a Lie algebra element $X(p) \in \mathfrak{g}$. We will therefore assign the Lie algebra valued smooth maps $C^\infty(M,\mathfrak{g}) = C^\infty(M) \otimes \mathfrak{g}$ as the Lie algebra of $C^\infty(M,G)$. Similarly, the Lie algebra structure on $C^\infty(M) \otimes \mathfrak{g}$ is obtained from the Lie bracket of $\mathfrak{g}$ pointwise. To see this one simply notices that the derivative of the adjoint map Ad can be made sense of pointwise.

Now, consider a curve γ as above. Via our right action, this maps into the curve

$$t \mapsto \mathrm{Ad}_{\gamma(t)^{-1}} A + \gamma(t)^* \Xi \tag{4.113}$$

on $\Omega^1(M) \otimes \mathfrak{g}$. Choosing a point $p \in M$, this induces the curve

$$t \mapsto \mathrm{Ad}_{\gamma(-t)(p)} A_p + (\gamma(t)^* \Xi)_p, \tag{4.114}$$

now defined on $T^*M \otimes \mathfrak{g}$, the latter being a finite dimensional manifold. Assuming that the "smoothness" of γ is enough to guarantee the smoothness of this curve, we can now take its derivative at 0. This is of course, given by

$$\mathrm{ad}_{-X(p)} A_p + \frac{\mathrm{d}}{\mathrm{d}t}(\gamma(t)^* \Xi)_p \bigg|_{t=0}. \tag{4.115}$$

The second term hand be evaluated at a $Y \in T_pM$ as above

$$\frac{\partial^2}{\partial t \partial s} \gamma(-t)(p)\gamma(t)(\mu(s)) \bigg|_{s=t=0} = \frac{\partial^2}{\partial t \partial s} \gamma(t)(\mu(s)) \bigg|_{t=s=0}. \tag{4.116}$$

In the manipulation above we applied the Leibniz rule for curves in a Lie group [see 76, Exercise 8.30.], to the product of the constant curve $s \mapsto \gamma(-t)(p)$ and the curve $s \mapsto \gamma(t)(\mu(s))$. Switching the order of differentiation we then see that the above is equal to

$$\frac{\mathrm{d}}{\mathrm{d}s} X(\mu(s)) \bigg|_{s=0} = X_{*,p} Y. \tag{4.117}$$

In this situation it is more common to use a differential notation for the push-forward $(\mathrm{d}X)_p := X_{*,p}$. We conclude that the induced action of the Lie algebra is

$$\begin{aligned} C^\infty(M) \otimes \mathfrak{g} &\to \Omega^1(M) \otimes \mathfrak{g} \\ X &\mapsto [A, X] + \mathrm{d}X =: D_A X. \end{aligned} \tag{4.118}$$

In the examples we will consider, $\mathfrak{g}$ is equipped with an invariant symmetric non-degenerate bilinear form $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathfrak{g}} : \mathfrak{g}\times\mathfrak{g}\to\mathbb{F}$. The invariant qualifier amounts to

$$\langle [X,Y],Z\rangle_{\mathfrak{g}} + \langle X,[Y,Z]\rangle_{\mathfrak{g}} = 0. \tag{4.119}$$

An example commonly found in applications is the Killing form [see 77, Section VIII.2]

$$\langle X,Y\rangle_{\mathfrak{g}} = -\operatorname{tr}_{\mathfrak{g}}([X,[Y,\,\cdot\,]]), \tag{4.120}$$

when $\mathfrak{g}$ is semisimple. Such a pairing induces a bilinear form on $\Omega(M)\otimes\mathfrak{g}$

$$\begin{aligned}\langle\,\cdot\,,\,\cdot\,\rangle : (\Omega(M)\otimes\mathfrak{g})\times(\Omega(M)\otimes\mathfrak{g}) &\to \mathbb{F}\\ (\alpha\otimes X,\beta\otimes Y) &\mapsto \int_M (\alpha\otimes X,\beta\otimes Y),\end{aligned} \tag{4.121}$$

where

$$\begin{aligned}(\,\cdot\,,\,\cdot\,) : (\Omega(M)\otimes\mathfrak{g})\times(\Omega(M)\otimes\mathfrak{g}) &\to \Omega(M)\\ (\alpha\otimes X,\beta\otimes Y) &\mapsto \alpha\wedge\beta\langle X,Y\rangle_{\mathfrak{g}},\end{aligned} \tag{4.122}$$

In here we set the integral of a form outside of $\Omega^{D:=\dim M}(M)$ equal to 0. This pairing allows us to identify $(\Omega^k(M)\otimes\mathfrak{g})^*$ with $\Omega^{D-k}(M)\otimes\mathfrak{g}$. In the following we will set fields c, A, A^*, and c^* as coordinates for the spaces $C^\infty(M)\otimes\mathfrak{g} = \Omega^0(M)\otimes\mathfrak{g}$, $\Omega^1(M)\otimes\mathfrak{g}$, $\Omega^{D-1}(M)\otimes\mathfrak{g}$, and $\Omega^D(M)\otimes\mathfrak{g}$, respectively. These are of course the ghosts, fields, antifields, and antighosts, in that order.

Given a basis T_a for $\mathfrak{g}$, it is customary in physics to use $K_{ab} := 2\langle T_a,T_b\rangle$ to raise and lower Lie algebra indices. Then, it must be stressed that the under the identification $(\Omega^k(M)\otimes\mathfrak{g})^*\cong\Omega^{D-k}(M)\otimes\mathfrak{g}$ we choose our coordinates so that

$$\begin{aligned}\langle B^*,B\rangle &= \frac{1}{k!(D-k!)}\int \mathrm{d}^D x\,\epsilon^{\mu_1\cdots\mu_D} {B^*_{\mu_1\cdots\mu_{D-k}}}^a {B_{\mu_{D-k}\cdots\mu_D}}^b [T_a,T_b]\\ &= \frac{1}{(k!)^2}\int \mathrm{d}^D x\, {B^{*\mu_1\cdots\mu_k}}_a {B_{\mu_1\cdots\mu_k}}^a.\end{aligned} \tag{4.123}$$

Therefore, we take

$${B^{*\mu_1\cdots\mu_k}}_a = \frac{k!}{2(D-k)!}\epsilon^{\nu_1\cdots\nu_{D-k}\mu_1\cdots\mu_k} {B^*_{\nu_1\cdots\nu_{D-k}}}^b K_{ab}. \tag{4.124}$$

In particular

$$\begin{aligned}&\frac{1}{2}\epsilon^{\nu_1\cdots\nu_{D-k}\mu_1\cdots\mu_k}\{{B_{\nu_1\cdots\nu_{D-k}}}^b(x),{B_{\alpha_1\cdots\alpha_k}}^c(y)\}K_{ab}\\ &= {\delta^c}_a{\delta^{\mu_1}}_{\alpha_1}\cdots{\delta^{\mu_k}}_{\alpha_k}\delta(x-y).\end{aligned} \tag{4.125}$$

The action of the Lie algebra $\Omega^0(M) \otimes \mathfrak{g}$ on $\Omega^1(M) \otimes \mathfrak{g}$ induces the representation term in the action S_ξ

$$\begin{aligned}\langle A^*, D_A c\rangle &= \int \mathrm{d}^D x\, \epsilon^{\mu_1\cdots\mu_D} \Big(A^*{}_{\mu_1\cdots\mu_{D-1}}{}^a A_{\mu_D}{}^b c^c \langle T_a, [T_b, T_c]\rangle_{\mathfrak{g}} \\ &\qquad\qquad + A^*{}_{\mu_1\cdots\mu_{D-1}}{}^a \partial_{\mu_D} c^b \langle T_a, T_b\rangle_{\mathfrak{g}}\Big) \\ &= \frac{1}{2}\int \mathrm{d}^D x\, \epsilon^{\mu_1\cdots\mu_D} A^*{}_{\mu_1\cdots\mu_{D-1}}{}^a \left(A_{\mu_D}{}^b c^c f_{abc} + \partial_{\mu_D} c_a \right).\end{aligned} \tag{4.126}$$

On the other hand, the Lie algebra term is of the form

$$\frac{1}{2}\langle c^*, [c,c]\rangle = \frac{1}{4}\int \mathrm{d}^D x\, \epsilon^{\mu_1\cdots\mu_D} c^*{}_{\mu_1\cdots\mu_D}{}^a c^b c^c f_{abc}. \tag{4.127}$$

4.9 Yang-Mills Theory

Yang-Mills Theory is an example of a theory for the connection which is based around its curvature $F := \mathrm{d}A + \frac{1}{2}[A \wedge A] \in \Omega^2(M) \otimes \mathfrak{g}$. In here we have introduced

$$\begin{aligned}[\,\cdot\, \wedge\, \cdot\,] : (\Omega(M) \otimes \mathfrak{g}) \otimes (\Omega(M) \otimes \mathfrak{g}) &\to \Omega(M) \otimes \mathfrak{g} \\ (\alpha \otimes T_1, \beta \otimes T_2) &\mapsto (\alpha \wedge \beta) \otimes [T_1, T_2].\end{aligned} \tag{4.128}$$

Remark 4.9.1. We can make contact with some of the literature ([see 78, just before Lemma 10.2]), by defining $\alpha \wedge \alpha := \frac{1}{2}[\alpha \wedge \alpha]$ if $\alpha \in \Omega^p(M) \otimes \mathfrak{g}$ with $p \in 2\mathbb{N} + 1$. This stems from a natural extension of the wedge product to matrix-valued forms. Indeed, assume that we are in a matrix Lie algebra. Then, with such an extension one obtains the following computation

$$(\alpha^a \otimes T_a) \wedge (\alpha^b \otimes T_b) := \alpha^a \wedge \alpha^b T_a T_b. \tag{4.129}$$

Since in the degrees we chose the wedge product is antisymmetric, this yields

$$(\alpha^a \otimes T_a) \wedge (\alpha^b \otimes T_b) = \frac{1}{2}\alpha^a \wedge \alpha^b (T_a T_b - T_b T_a) = \frac{1}{2}\alpha^a \wedge \alpha^b [T_a, T_b]. \tag{4.130}$$

In particular, if $p = 1$ we have

$$(\alpha \wedge \alpha)(X, Y) = \frac{1}{2}(\alpha^a(X)\alpha^b(Y) - \alpha^a(Y)\alpha^b(X))[T_a, T_b] = [\alpha(X), \alpha(Y)], \tag{4.131}$$

for all $X, Y \in \mathfrak{X}(M)$. With this notation $F = \mathrm{d}A + A \wedge A$

In coordinates we have we have

$$F = \frac{\partial A_\nu{}^a}{\partial x^\mu} \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \otimes T_a + \frac{1}{2} A_\mu{}^b A_\nu{}^c (\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu) \otimes f^a{}_{bc} T_a. \tag{4.132}$$

Therefore $F = \frac{1}{2} F_{\mu\nu}{}^a (\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu) \otimes T_a$, with

$$F_{\mu\nu}{}^a = \frac{\partial A_\nu{}^a}{\partial x^\mu} - \frac{\partial A_\mu{}^a}{\partial x^\nu} + f^a{}_{bc} A_\mu{}^b A_\nu{}^c. \tag{4.133}$$

To study the behaviour under gauge transformations of the curvature we can compute

$$\begin{aligned} \left\{ S_R, \frac{\partial A_\nu{}^a}{\partial x^\mu} \right\} &= -\frac{\partial}{\partial x^\mu} \left(A_\nu{}^b c^c f^a{}_{bc} + \frac{\partial c^a}{\partial x^\nu} \right) \\ &= f^a{}_{bc} \frac{\partial A_\nu{}^b}{\partial x^\mu} c^c + f^a{}_{bc} A_\nu{}^b \frac{\partial c^c}{\partial x^\mu} + \frac{\partial^2 c^a}{\partial x^\mu \partial x^\nu}, \end{aligned} \tag{4.134}$$

and

$$\begin{aligned} \left\{ S_R, f^a{}_{bc} A_\mu{}^b A_\nu{}^c \right\} &= -f^a{}_{bc} \left(\left(A_\mu{}^d c^e f^b{}_{de} - \frac{\partial c^b}{\partial x^\mu} \right) A_\nu{}^c \right. \\ &\qquad \left. + A_\mu{}^b \left(A_\nu{}^d c^e f^c{}_{de} + \frac{\partial c^c}{\partial x^\nu} \right) \right) \\ &= -(f^a{}_{cb} f^b{}_{ed} + f^a{}_{db} f^b{}_{ce}) A_\mu{}^d A_\nu{}^c c^e \\ &\quad + f^a{}_{bc} \left(\frac{\partial c^b}{\partial x^\mu} A_\nu{}^c - \frac{\partial c^c}{\partial x^\nu} A_\mu{}^b \right) \\ &= -f^a{}_{be} f^b{}_{dc} A_\mu{}^d A_\nu{}^c c^e \\ &\quad + f^a{}_{bc} \left(\frac{\partial c^b}{\partial x^\mu} A_\nu{}^c - \frac{\partial c^c}{\partial x^\nu} A_\mu{}^b \right). \end{aligned} \tag{4.135}$$

Antisymmetrizing (4.134) we see that the last term vanishes while the second cancels with the second term of (4.135). We thus conclude that

$$\begin{aligned} \{S_R, F_{\mu\nu}{}^a\} &= -f^a{}_{bc} \frac{\partial A_\nu{}^b}{\partial x^\mu} c^c + f^a{}_{bc} \frac{\partial A_\mu{}^b}{\partial x^\nu} c^c - f^a{}_{be} f^b{}_{dc} A_\mu{}^d A_\nu{}^c c^e \\ &= -f^a{}_{bc} F_{\mu\nu}{}^b c^c. \end{aligned} \tag{4.136}$$

In other words, $\{S_R, F\} = -[F \wedge c]$.

This allows us to consider the action

$$S = \frac{1}{2} \langle F, \star F \rangle. \tag{4.137}$$

In here $\star$ is the hodge operator associated to g [see 68, Section 4.5]. This action is invariant due to the transformation properties of F and the fact that we chose an invariant pairing $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathfrak{g}}$. Let us put this action in a more familiar form. In light of [see 68, Remark 4.5.4], we have

$$S = \frac{(-1)^s}{4}\int \mathrm{d}^n x\,\sqrt{|\det(g_{..})|}F_{\mu\nu}{}^a F^{\mu\nu}{}_a. \tag{4.138}$$

In here s is the number of minus signs in the signature of the metric. For euclidean theories this is 0 while for Minkowski theories it is 1.

Remark 4.9.2. In physics, it is common to consider the Lie algebra as the vector space generated by $\tilde{T}_a := iT_a$. The reason for this is that in quantum mechanics symmetries one-parameter subgroups of a symmetry group are of the form $U(t) = e^{-itX}$ with X self-adjoint. Therefore, the physical interpretation of X is clearer than that of $-iX$, which is the actual Lie algebra element. The Lie bracket is extended to these elements by requiring that it is complex bilinear. The relationship between the structure constants an this algebra is then $[\tilde{T}_b, \tilde{T}_c] = if^a{}_{bc}T_a$.

We conclude that the BV action for this theory is

$$S = \frac{1}{2}\langle F, \star F\rangle + \langle A^*, D_A c\rangle + \frac{1}{2}\langle c^*, [c,c]\rangle. \tag{4.139}$$

As an initial test for the usefulness of this action, let us gauge fix it into the form familiar from the Faddeev-Popov method. This is easier to do in the point of view where our space of antighosts is $\Gamma(\mathrm{Dens}(M)) \otimes \mathfrak{g}^*$ and our space of antifields is $\Gamma(\bigwedge^k TM \otimes \mathrm{Dens}(M)) \otimes \mathfrak{g}^*$ We can then add the Faddeev-Popov antighosts $\bar{c}$ and the Lagrange multiplier b coordinatizing respectively new copies of $\mathrm{Dens}(M) \otimes \mathfrak{g}^*[-1]$ and $\mathrm{Dens}(M) \otimes \mathfrak{g}^*$. We also have to include the corresponding fields $\bar{c}^*$ and b^* coordinatizing $\Omega^0(M) \otimes \mathfrak{g}$ and $\Omega^0(M) \otimes \mathfrak{g}[-1]$ so that we still have a Poisson structure. We then add to the action the term

$$\langle b, \bar{c}^*\rangle. \tag{4.140}$$

Consider the gauge-fixing fermion

$$\Psi = \langle \bar{c}, \nabla^\mu A_\mu - \epsilon\rangle, \tag{4.141}$$

for some $\epsilon \in \Omega^0(M) \otimes \mathfrak{g}$. This yields

$$\begin{aligned} \bar{c}^* &= \nabla^\mu A_\mu - \epsilon, \\ b^* &= 0 \\ A^{*\mu}{}_a(x) &= \int \mathrm{d}^D y \, \nabla_y^\mu \delta(y-x) = -\partial^\mu \bar{c}_a(x), \\ c^* &= 0. \end{aligned} \tag{4.142}$$

The gauge-fixed action is then

$$S = \frac{1}{2}\langle F, \star F\rangle + \langle (d\bar{c})^\#, D_A c\rangle + \langle b, \nabla^\mu A_\mu - \epsilon\rangle. \tag{4.143}$$

The delta function obtained by integrating b_a can be recasted as a new term in the action through a Gaussian averaging procedure. We can obtain this alternatively by choosing a different gauge-fermion

$$\Psi = \langle \bar{c}, \nabla^\mu A_\mu - \xi b^{\sharp_\mathfrak{g}}\rangle, \tag{4.144}$$

where

$$\begin{aligned} \cdot^{\sharp_\mathfrak{g}} : \mathrm{Dens}(M) \otimes \mathfrak{g}^* &\to \Omega^0(M) \otimes \mathfrak{g} \\ b = b_a T^a &\mapsto \frac{1}{2\sqrt{|\det g_{..}|}} b^b T_b \end{aligned} \tag{4.145}$$

is the map induced by $\langle \cdot, \cdot \rangle_\mathfrak{g}$ and the hodge dual $\star$. The Lagrangian subspace this yields is defined by the same equations as the previous one except for b^* and

$$\bar{c}^* = \nabla^\mu A_\mu - \xi b^{\sharp_\mathfrak{g}}. \tag{4.146}$$

The resulting action is then

$$S = \frac{1}{2}\langle F, \star F\rangle + \langle (d\bar{c})^\#, [A, c] + \mathrm{d}c\rangle + \langle b, \nabla^\mu A_\mu - \xi b^{\sharp_\mathfrak{g}}\rangle. \tag{4.147}$$

The corresponding equations of motion for b are then

$$b^a = \frac{1}{\xi}\sqrt{|\det g_{..}|}\nabla^\mu A_\mu{}^a. \tag{4.148}$$

Integrating out b^a in the action the leaves

$$S = \frac{1}{2}\langle F, \star F\rangle + \langle (d\bar{c})^\#, [A, c] + \mathrm{d}c\rangle + \frac{1}{2\xi}\int \mathrm{d}^D x \sqrt{|\det g_{..}|}\nabla^\mu A_\mu{}^a \nabla^\nu A_{\nu a}. \tag{4.149}$$

This action can then be taken as the starting point for quantizing Yang-Mills theories.

4.10 Chern-Simons Theory

Chern-Simons theory is another theory for the connection on a principal bundle. Unlike Yang-Mills theory, it is defined specifically on a three dimensional manifold and it doesn't require a metric structure. It is of importance in several fields of mathematics and physics. Let us mention some of them. In knot theory it was found that the Jones polynomial, which is a link invariant, could be computed through the expectation values of observables known as Wilson lines of a Chern-Simons theory [79]. In fact, this technique can be used to compute other link invariants in other 3-manifolds. On the quantum gravity side, the action of this theory coincides with the action of $2+1$ gravity under a correct identification of fields [80]. Finally, in the theory of condensed matter certain aspect of the fractional quantum Hall effect have been successfully described through Chern-Simons theory [81, 82].

In order, to introduce the theory, let us remark that in $D = 4$ there is yet another way to write the Yang-Mills action. This can be done with $F_+ = \frac{1}{2}(1+\star)F \in \Omega^2(M) \otimes \mathfrak{g}$. Notice that [see 68, Proposition 4.5.3]

$$\star F_+ = \frac{1}{2}(\star + \star\star)F = \frac{1}{2}(\star + (-1)^s)F. \tag{4.150}$$

Therefore, in Euclidean theories F_+ is self dual. Let us restric to this case . We could have then considered the action

$$\langle F_+, F_+ \rangle = \frac{1}{4}\langle F, F\rangle + \frac{1}{4}\langle F, \star F\rangle + \frac{1}{4}\langle F, \star F\rangle + \frac{1}{4}\langle \star F, \star F\rangle. \tag{4.151}$$

Notice that the symmetry of $\wedge$ and $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ makes our pairing symmetric when acting on 2-forms. Then, using [see 68, Exercise 4.5.3]

$$\langle F_+, F_+ \rangle = \frac{1}{2}\langle F, F\rangle + S_{\text{YM}}. \tag{4.152}$$

We will show that $\frac{1}{2}\langle F, F\rangle$ is the integral of a total derivative. This will then allow us to conclude that $\langle F_+, F_+ \rangle$ yields a classically equivalent theory to S_{YM}.

We begin by expanding this term

$$\frac{1}{2}\langle F, F\rangle = \frac{1}{2}\bigg(\langle \mathrm{d}A\,, \mathrm{d}A\rangle + \langle \mathrm{d}A\,, [A\wedge A]\rangle + \frac{1}{4}\langle [A\wedge A], [A\wedge A]\rangle\bigg). \tag{4.153}$$

In here we used the symmetry of $\langle\,\cdot\,,\,\cdot\,,\rangle$ when acting on 2-forms. The last term in this expansion vanishes as a consequence of the symmetries of $\wedge$ and the invariance of $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathfrak{g}}$. To see this, note that the Jacobi identity yields an identity on this invariant pairing

$$\begin{aligned}\langle [X,Y],[Z,W]\rangle_{\mathfrak{g}} &= -\langle Y,[X,[Z,W]]\rangle_g \\ &= \langle Y,[W,[X,Z]]\rangle_g + \langle Y,[Z,[W,X]]\rangle_g \\ &= -\langle [W,Y],[X,Z]\rangle_g - \langle [Z,Y],[W,X]\rangle_g.\end{aligned} \tag{4.154}$$

This can be rewritten into the form

$$\langle [X,Y],[Z,W]\rangle_{\mathfrak{g}} + \langle [X,Z],[W,Y]\rangle_g + \langle [X,W],[Y,Z]\rangle_{\mathfrak{g}} = 0. \tag{4.155}$$

However

$$A^a\wedge A^b\wedge A^c\wedge A^d = A^a\wedge A^d\wedge A^b\wedge A^c = A^a\wedge A^c\wedge A^d\wedge A^b, \tag{4.156}$$

so that

$$\begin{aligned}0 = {}& A^a\wedge A^b\wedge A^c\wedge A^d \\ & \big(\langle [T_a,T_b],[T_c,T_d]\rangle_{\mathfrak{g}} + \langle [T_a,T_c],[T_d,T_b]\rangle_g + \langle [T_a,T_d],[T_b,T_c]\rangle_{\mathfrak{g}}\big) \\ ={}& 3A^a\wedge A^b\wedge A^c\wedge A^d\langle [T_a,T_b],[T_c,T_d]\rangle_{\mathfrak{g}} = 3\langle [A\wedge A],[A\wedge A]\rangle.\end{aligned} \tag{4.157}$$

We conclude that

$$\frac{1}{2}\langle F, F\rangle = \frac{1}{2}(\langle \mathrm{d}A\,, \mathrm{d}A + [A\wedge A]\rangle). \tag{4.158}$$

To show that the corresponding Lagrangian

$$\mathcal{L} = \frac{1}{2}((\mathrm{d}A\,, \mathrm{d}A + [A\wedge A])) \in \Omega^4(M) \tag{4.159}$$

is a total derivative, we will explicitly exhibit a 3-form as a witness. We propose

$$\mathcal{L}_{\mathrm{CS}} := \frac{1}{2}\bigg(A, \mathrm{d}A + \frac{1}{3}[A\wedge A]\bigg). \tag{4.160}$$

We have that $\mathrm{d}\mathcal{L}_{\mathrm{CS}}$ is given by

$$\begin{aligned}&\frac{1}{2}\Big((\mathrm{d}A\,,\mathrm{d}A)+\frac{1}{3}(\mathrm{d}A\,,[A\wedge A])-\frac{1}{3}(A,[\mathrm{d}A\wedge A])+\frac{1}{3}(A,[A\wedge \mathrm{d}A])\Big)\\&=\frac{1}{2}\Big((\mathrm{d}A\,,\mathrm{d}A)+\frac{1}{3}(\mathrm{d}A\,,[A\wedge A])+\frac{2}{3}(A,[A\wedge \mathrm{d}A])\Big).\end{aligned}\tag{4.161}$$

In here we have used the fact that the de Rham operator satisfies the same Leibniz rule over $\wedge$ when applied over $(\,\cdot\,,\,\cdot\,)$ or $[\,\cdot\,\wedge\,\cdot\,]$. We have also used the fact that $[\,\cdot\,\wedge\,\cdot\,]$ is antisymmetric when applied to a 1-form and a 2-form. Finally, we note that

$$\begin{aligned}(A,[A\wedge \mathrm{d}A])&=A^a\wedge A^b\wedge \mathrm{d}A^c\,\langle T_a,[T_b,T_c]\rangle_{\mathfrak{g}}\\&=-A^a\wedge A^b\wedge \mathrm{d}A^c\,\langle [T_b,T_a],T_c\rangle_{\mathfrak{g}}\\&=\mathrm{d}A^c\wedge A^a\wedge A^b\langle T_c,[T_a,T_b]\rangle_{\mathfrak{g}}=(\mathrm{d}A\,,[A\wedge A]).\end{aligned}\tag{4.162}$$

We thus confirm that

$$\frac{1}{2}\langle F_+,F_+\rangle=\int_M \mathrm{d}\mathcal{L}_{\mathrm{CS}}\,.\tag{4.163}$$

In particular, if M has a boundary, the action describes a theory on the three dimensional boundary

$$\frac{1}{2}\langle F_+,F_+\rangle=\int_{\partial M} \mathcal{L}_{\mathrm{CS}}.\tag{4.164}$$

Given a manifold M with $\dim M=3$, Chern-Simons theory is a theory whose field content is the space of connection 1-forms $A\in\Omega^1(M)\otimes\mathfrak{g}$ with the action

$$S_{\mathrm{CS}}=\frac{1}{2}\langle A,\mathrm{d}A\rangle+\frac{1}{6}\langle A,[A\wedge A]\rangle\tag{4.165}$$

We stress that this theory does not require a metric on M. We will also consider the gauge group described on Section 4.8. The full BV space of fields can then be considered to be

$$\begin{aligned}\mathcal{E}=(\Omega^0(M)\otimes\mathfrak{g}[1])\oplus(\Omega^1(M)\otimes\mathfrak{g})&\\ \oplus(\Omega^3(M)\otimes\mathfrak{g}[-1])\oplus(\Omega^2(M)\otimes\mathfrak{g}[-2]).&\end{aligned}\tag{4.166}$$

The corresponding BV action is then

$$S_{\mathrm{BV}}=\frac{1}{2}\langle A,\mathrm{d}A\rangle+\frac{1}{6}\langle A,[A\wedge A]\rangle+\langle A^*,D_Ac\rangle+\frac{1}{2}\langle c^*,[c,c]\rangle.\tag{4.167}$$

4.11 BF Theory

Let us now study an example which shows how dg Lie algebras may be required to encode the gauge structure of a theory. For this, let us set $D = 4$. Moreover, let us take the abelian case $\mathfrak{g} = \mathbb{R}$. Our theory is then obtained by a modification of the theory with action $\langle F, F\rangle$, from which we obtained Chern-Simons theory. Namely, let us now consider a field B on $\Omega^2(M)$. BF theory [83] is obtained by the action

$$S_{\mathrm{BF}} = \frac{1}{2}\langle F, B\rangle. \tag{4.168}$$

The space of field configurations of this theory is then $V = \Omega^1(M) \oplus \Omega^2(M)$.

We can preserve the gauge invariance of the theory above by declaring B to transform trivially under the gauge transformations above. Of course, so does F since we have restricted to the Abelian case. Moreover, this theory has an additional symmetry. Indeed, if $A + B \in \Omega^1(M) \oplus \Omega^2(M)$, consider

$$\begin{aligned} \Omega^1(M) &\to \Omega^1(M) \oplus \Omega^2(M) \\ \beta &\mapsto 0 + \mathrm{d}\beta\,. \end{aligned} \tag{4.169}$$

To see this is indeed a symmetry note that $(F, \mathrm{d}\beta) = (\mathrm{d}A\,, \mathrm{d}\beta) = -(A, \mathrm{d}^2\beta) = 0$ up to boundary terms.

Notice that the equations of motion of this theory are $\mathrm{d}A = 0$ and $\mathrm{d}B = 0$. Therefore, classically the fields are closed. We can thus declare this to be a theory for the cohomology in degree 1 and 2 of M. Therefore, the YM type symmetry (4.118) of A is still interpreted as a gauge symmetry. By the same token, the symmetry (4.169) is a gauge symmetry as well. The latter however has a degeneracy. Namely, if β is exact, the corresponding transformation leaves B invariant. The corresponding symmetry is then trivial. We will add this to the structure of our gauge transformations by considering the dg Lie algebra $\mathfrak{g}$ with

$$\mathfrak{g}^0 = \Omega^0(M)_A \oplus \Omega^1(M)_B, \tag{4.170}$$

corresponding to the gauge symmetries obtained through the identifications in the de Rham cohomology, and

$$\mathfrak{g}^{-1} = \Omega^0(M)_B, \tag{4.171}$$

corresponding to the degeneracy. The differential is then given by the de Rham differential

$$\begin{aligned} \mathfrak{g}^{-1} &\to \mathfrak{g}^0 = \Omega^0(M)_A \oplus \Omega^1(M)_B \\ \alpha &\mapsto 0 + \mathrm{d}\alpha\,. \end{aligned} \tag{4.172}$$

Since we are in the abelian case, we will take the Lie bracket to be trivial. The representation of $\mathfrak{g}$ on V is given by $R(\alpha) = 0$, $R(\beta) = \mathrm{d}\beta$ and $R(c) = \mathrm{d}c$, for all $\alpha \in \Omega^0(M)_B$, $\beta \in \Omega^1(M)_B$ and $c \in \Omega^0(M)_A$.

Let us assign coordinates in the spaces above with the same names. Let us also take coordinates α^* and β^* on $\Omega^4(M)_B$ and $\Omega^3(M)_B$, respectively. These will be the antifields of α and β. Finally, let us also consider an antifield B^* of $\Omega^2(M)$ for B. Then the BV action takes the form

$$S_{\mathrm{BV}} = \langle F, B\rangle + \langle A^*, \mathrm{d}c\rangle + \langle B^*, \mathrm{d}\beta\rangle + \langle \beta^*, \mathrm{d}\alpha\rangle. \tag{4.173}$$

Theories requiring bigger dg Lie algebras can be immediately obtained by considering the case $D > 4$.

Chapter 5

Conclusions

We have rephrased the problem of computing expectation values through path integrals to that of computing the image of an appropriate divergence operator. Not only does this new point of view provide useful insight for the rigorous formulation of the path integral methods of perturbative QFT, but it also brought cohomological structures into the problem. In particular, its classical limit provided the Koszul complex as a shifted cotangent bundle with a cohomological vector field. This complex allowed for the computation of the on-shell functions of our fields, giving a counterpart to the Euler-Lagrange equations in a cohomological language. Moreover, this Koszul complex also contained the structural information of the symmetries of our theories.

Analyzing this structure we were able to introduce dg Lie algebras as a model for possible gauge symmetries. This provided a generalization of Lie algebras that can be studied with the BV formalism. Aiming to describe this structure through a cohomological vector field we found a further generalization using L_∞-algebras and the Chevalley-Eilenberg complex. Coupling these two complexes through a representation, we found the BV extension of our action. The classical master equation, which guaranteed that the resulting Hamiltonian vector field was cohomological, encoded both the gauge structure and its representation. Moreover, the BV complex obtained gave us a method to compute the true classical observables of our theories.

Going back to the quantum theory, we presented a method to gauge fix the BV action and incorporate it into a path integral. Invariance of the resulting measure from the gauge-fixing procedure immediately led us to the quantum master equation. This ensured that the BV divergence produced a quantum BV complex. We argued that the cohomology of this complex contains the quantum observables. Moreover, we found this to be a starting point for the discussion of renormalization and gauge anomalies in QFT.

To study applications of the formalism developed, we gave a heuristic method to extend our findings to full field theories. With this we were able to probe the gauge structure of diffeomorphism invariant theories. As an example, we studied the case of a relativistic particle. Gauge-fixing our BV action we were able to recover an action that can be used for the quantization of this system. Furthermore, we studied theories for connections on principal bundles. The gauge symmetries obtained by describing these connections as fields on spacetime were treated with the BV formalism. We employed this to construct the BV extension of both YM and CS theories. The gauge fixing of the YM case immediately led us to the action obtained via the Faddeev-Popov procedure. Finally, we showcased how the theory can be implemented in cases with reducible gauge strutures by studying BF theory.

The path we have taken has led us to the development of important tools for computations in QFT. It has also brought forward important physical insight regarding structural properties of gauge symmetries and their connection to QFT. Moreover, this has been achieved by following the guiding principle of describing our mathematical structure through a cohomological picture. We hope then that the reader has found in this account of the BV formalism an example of the importance of the interaction between physics and mathematics.

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