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Interval Exchange Dynamics and Wandering Intervals

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Dissertação apresentada à Universidade Estadual Paulista (UNESP), Instituto de Biociências, Letras e Ciências Exatas, São José do Rio Preto, para obtenção do título de Mestre em Matemática.

Área de Concentração: Geometria e Sistemas Dinâmicos

Orientador: Prof. Dr. Ali Messaoudi

Coorientador: Prof. Dr. Pascal Hubert

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dedico.

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“Se a noite inventa a escuridão, a luz inventa o luar”
(Gilberto Gil, 1981).

RESUMO

Este trabalho tem como objetivo apresentar os conceitos e resultados necessários para a compreensão de um resultado sobre sistemas dinâmicos relacionados a intercâmbios de intervalos. Mais precisamente, se T é um intercâmbio de intervalos autossimilar com Θ_1 autovalor de Perron-Frobenius da matriz associada ao loop da indução de Rauzy sobre T e existe um autovalor real Θ_2 conjugado a Θ_1 tal que $\Theta_2 > 1$, então existe um intercâmbio de intervalos afim f, semi-conjugado a T, que possui intervalos errantes. Este resultado foi obtido por X. Bressaud, P. Hubert e A. Maass. Para construir o conhecimento necessário, são estudadas duas áreas principais: a dinâmica de intercâmbios de intervalos e a dinâmica simbólica, destacando as conexões entre essas teorias.

Palavras-chave: intercâmbios de intervalos; dinâmica simbólica; substituições; fractal de Rauzy; intervalos errantes.

ABSTRACT

This work aims to present the concepts and results necessary for understanding a result concerning dynamical systems related to interval exchange transformations. More precisely, if T is a self-similar interval exchange transformation with Θ_1 the Perron–Frobenius eigenvalue of the matrix associated with the Rauzy induction loop of T, and if there exists a real eigenvalue Θ_2 conjugate to Θ_1 such that $\Theta_2 > 1$, then there exists an affine interval exchange transformation f, semi-conjugate to T, that possesses wandering intervals. This result was obtained by X. Bressaud, P. Hubert, and A. Maass. In order to develop the necessary background, two main areas are studied: interval exchange dynamics and symbolic dynamics, emphasizing the connections between these theories.

Keywords: interval exchange transformation; symbolic dynamics; substitutions; Rauzy fractal; wandering intervals.

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Sumário

1 INTRODUCTION

The study of discrete dynamical systems is primarily based on the analysis of the iterates of a function, called dynamics, defined on a nonempty set. Because this definition is quite general, the field connects with many other areas of mathematics, enabling a wide range of diverse and interesting investigations. In this work, we focus mainly on two types of dynamical systems: Interval Exchange Dynamics and Symbolic Dynamics.

It follows from immediate observations that interval exchange transformations do not have wandering intervals. However, in the 1980s, Levitt posed the following question in [3]: do there exist uniquely ergodic affine interval exchange transformations with wandering intervals? Levitt himself found examples in which the transformations were not uniquely ergodic. In [4], Camelier and Gutierrez constructed uniquely ergodic examples with wandering intervals that were also semi-conjugate to interval exchange transformations. Their work subsequently achieved some improvements by Cobo, as reported in [5]. The article we study in detail is [2] by Bressaud, Hubert, and Maass, in which they establish conditions on an interval exchange transformation that guarantee the existence of an affine interval exchange transformation, semi-conjugate to it, with wandering intervals. The result is the following:

Theorem 1.1. ***(Bressaud-Hubert-Maass)*** *Let $T(\lambda, \pi)$ be a self-similar interval exchange transformation and R the associated matrix obtained by Rauzy induction. Let θ_1 be the Perron-Frobenius eigenvalue of R. Assume that R has an eigenvalue θ_2 such that:*

1) θ_2 is a conjugate of θ_1;

2) θ_2 is a real number;

3) $1 < \theta_2 (< \theta_1)$.

Then there exists an affine interval exchange transformation f with wandering intervals that is semi-conjugated to $T(\lambda, \pi)$.

To fully understand the result mentioned above, it is necessary to study not only the theory of interval exchange transformations but also symbolic dynamics and substitutions, since these play an important role in the proof of the result.

The so-called interval exchange transformations have attracted significant interest due to their connections with a wide variety of mathematical topics, such as geodesic flows on translation surfaces, multidimensional continued fraction algorithms, Teichmüller flow, polygonal billiards, substitutions, and Rauzy fractals. In Chapter 3, we study the basic definitions and fundamental results related to interval exchange transformations, with particular emphasis on understanding the Rauzy induction, which plays a central role in many results in this area, including the main article [2]. The main reference used is [6]

The theory of substitutive dynamical systems originated with the work of A. Hedlund and M. Morse and has since developed significantly, becoming connected to various topics across different areas of mathematics, such as ergodic theory, fractals, tiling theory, one- and multidimensional continued fractions, and, naturally, interval exchange dynamics. In Chapter 4, we introduce several concepts related to symbolic and substitutive dynamics that are essential to this line of research. A large portion of these results will be used in the proof of the main article [2]. The primary reference used in this chapter is [7].

In Chapter 5, we finally bring together the knowledge developed earlier in order to understand in detail the proof of [2], which is not trivial; however, it is based on a very interesting construction that uses several tools from symbolic dynamics.

2 PRELIMINARIES

In this chapter, we present preliminary concepts that will be extremely important throughout this work. A **discrete dynamical system** is a pair (X, f), where X is a nonempty set, and $f: X \to X$ is a map which is called the **dynamics**. We often call it just **dynamical system**. One of the main interests in the study of discrete dynamical systems is the asymptotic behavior of the orbits of points $x \in X$, that is, the behavior of the sets

$$O(x) = \{f^n(x) \mid n \in \mathbb{N}\},$$

where $f^n = f \circ \cdots \circ f$ (n times). The generality of the definition of a discrete dynamical system allows this area of study to be applied to a wide variety of objects, including those investigated in this work.

Definition 2.1. *Let (X, f) be a dynamical system. We say that (X, f) is **minimal** if the orbit of every point $x \in X$ is dense in X, that is,*

$$\overline{O(x)} = X, \quad \forall x \in X.$$

Given a dynamical system, it is possible to endow it with various types of structure that allow for very interesting studies. In what follows, we present one of the most important theorems in Ergodic Theory, known as the *Poincaré Recurrence Theorem*, which is obtained when we equip our set with the structure of a measure space.

Definition 2.2. *Let $(X, \mathcal{B}, \mu)$ be a positive measure space, where $\mathcal{B}$ is a σ-algebra and $\mu: \mathcal{B} \to [0, +\infty]$ is a measure, and let $T: X \to X$ be a measurable transformation of $(X, \mathcal{B})$. We say that*

*(i) T is a **measure-preserving transformation** of $(X, \mathcal{B}, \mu)$, or that μ is **invariant under** T, if*

$$\mu(A) = \mu\Big(T^{-1}(A)\Big), \quad \forall A \in \mathcal{B};$$

*(ii) μ is **ergodic** if*

$$\mu(A) = 0 \quad or \quad \mu(X \setminus A) = 0,$$

for every T-invariant set $A \in \mathcal{B}$, that is, A such that $T^{-1}(A) = A$;

*(iii) T is **uniquely ergodic** if there exists a unique invariant probability measure under T.*

Let $(X, \mathcal{B})$ be a measurable space and let $T: X \to X$ be a measurable transformation. Let $A \in \mathcal{B}$ and, for $y \in A$, define

$$\Gamma_y = \{n \in \mathbb{N}^* : T^n(y) \in A\}.$$

If $\Gamma_y \neq \emptyset$, we say that y is a **recurrent point of** A **under** T.

We denote by

$$\Omega_A = \{y \in A : \Gamma_y = \emptyset\}$$

the set of non-recurrent points of A under T.

Theorem 2.3. ***(Poincaré Recurrence Theorem)*** *Let $(X, \mathcal{B}, \mu)$ be a finite positive measure space, let $T\colon X \to X$ be a measurable measure-preserving transformation, and let $A \in \mathcal{B}$. Then almost every point of A is recurrent in A under T.*

3 INTERVAL EXCHANGE TRANSFORMATIONS

In this chapter, we will present the basic definitions concerning interval exchange transformations, as well as some examples for better geometric visualization. We will then define the important Rauzy induction, followed by important results that, in particular, relate the theory developed to the property of minimality, which is of great interest in the study of discrete dynamical systems. The main reference used is [6].

3.1 INTERVAL EXCHANGE TRANSFORMATIONS

Let $X \subset \mathbb{R}$ be an interval and $\mathcal{A} = \{1, 2, \ldots, r\}$, in which $r \in \mathbb{N} = \{0, 1, \ldots\},\ r \geq 2$. Consider $\{X_j, j \in \mathcal{A}\}$ to be a partition of X, where each X_j is a subinterval of the form $X_j = [a_j, b_j[$.

Definition 3.1. *An* ***interval exchange transformation*** *(IET) is a bijective map* $T : X \to X$ *that translates each subinterval* X_j*.*

Example 3.2. *An IET is often represented visually in the following ways:*

1. Graph

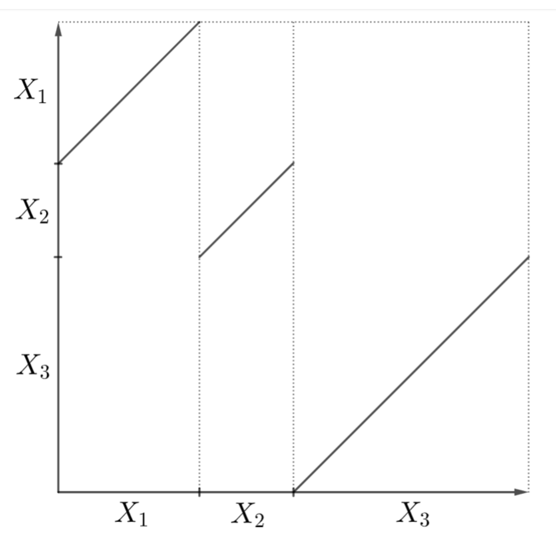

Figura 3.1: Graph of an IET.
Source: Author's own work

2. Diagram

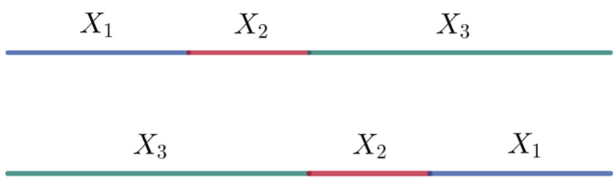

Figura 3.2: Diagram of an IET.
Source: Author's own work

Note that such function can be determined by a vector that describes the lengths of these subintervals and a combination in the order of the subintervals of the partition. Indeed, let $\Lambda_r,\ r \in \mathbb{N},\ r \geq 2$ be the positive cone in $\mathbb{R}^r$, that is, $\Lambda_r = \{\lambda = (\lambda_1, ..., \lambda_r) \in \mathbb{R}^r | \lambda_i > 0, \forall i = 1, \ldots, r\}$. Let us consider then a vector $\lambda = (\lambda_j)_{j\in\mathcal{A}} \in \Lambda_r$, where λ_j represents the length of the subinterval X_j.

Let $\pi = (\pi_0, \pi_1)$ be a pair of bijections $\pi_\epsilon : \mathcal{A} \longrightarrow \mathcal{A}$ in which π_0 describes the order of the subintervals X_j before an iterated of the map T and π_1 describes the order of the subintervals after it. Thus, we can represent π in the following way:

$$\pi = \begin{pmatrix} \alpha_1^0 & \alpha_2^0 & \ldots & \alpha_r^0 \\ \alpha_1^1 & \alpha_2^1 & \ldots & \alpha_r^1 \end{pmatrix},$$

in which $\alpha_j^\epsilon = \pi_\epsilon^{-1}(j)$ for $\epsilon \in \{0, 1\}$ and $j \in \mathcal{A}$.

Sometimes, to simplify the notation, we use $\tau \in G_r, \tau = \pi_1 \circ \pi_0^{-1}$, where G_r is the set of permutations of $\mathcal{A} = \{1, 2, \ldots, r\}$.

Given this construction, we use the pair (λ, π) to represent an interval exchange transformation T. We often denote it by $T(\lambda, \pi)$.

Example 3.3. *The following interval exchange transformation*

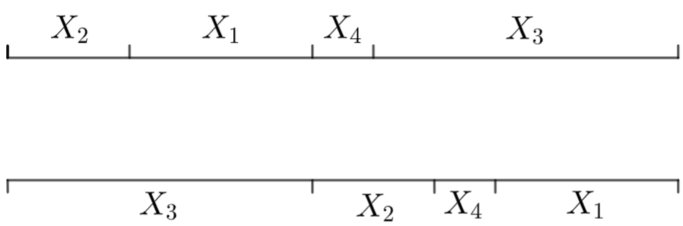

Figura 3.3: IET of four intervals.
Source: Author's own work

is given by $T(\lambda, \pi)$, in which

$$\lambda = (3, 2, 5, 1) \quad \text{and} \quad \pi = \begin{pmatrix} 2 & 1 & 4 & 3 \\ 3 & 2 & 4 & 1 \end{pmatrix}.$$

Observe that $\tau = \pi_1 \circ \pi_0^{-1} = (2, 4, 3, 1)$.

Definition 3.4. *We call the* ***monodromy invariant of the pair*** $\pi = (\pi_0, \pi_1)$ *the point*

$$p = (\pi_1 \circ \pi_0^{-1}(1), \pi_1 \circ \pi_0^{-1}(2), \ldots, \pi_1 \circ \pi_0^{-1}(r)).$$

Note that given an interval exchange transformation (λ, π) we can normalize the order choosing (λ', π') with $\pi_0' = id$.

This notation is very useful, although it hides the symmetric roles of π_0 and π_1, in addition to not being invariant to the induction algorithm that will be presented soon. Nevertheless, whenever possible, it will be used to simplify notation.

For each $\lambda \in \Lambda_r$ consider $\beta_0 = 0$, $\beta_j = \sum_{i=1}^{j} \lambda_i$, $X_j = [\beta_{j-1}, \beta_j[$, $j = 1, \ldots, r$ and $X = \bigcup_{j=1}^{r} X_j = [0, \beta_r[$.

For each $\lambda \in \Lambda_r$ and $\tau \in G_r$, consider $\lambda' = (\lambda_1', \lambda_2', \ldots, \lambda_r') \in \Lambda_r$, in which $\lambda_i' = \lambda_{\tau^{-1}(i)}$. Consequently, we define β_j' and X_j'. Consider the map $T : X \longrightarrow X$ defined by

$$T(x) = x - \beta_{j-1} + \beta'_{\tau(j)-1}, \qquad x \in X_j, \qquad j = 1, \ldots, r.$$

Thus, T permutes the subintervals X_j following the permutation τ, that is, $T(X_j) = X'_{\tau(j)}$. In other words, T is the explicit form of (λ, π).

Definition 3.5. *Let $\tau \in G_r$ be the permutation associated to a pair $\pi = (\pi_0, \pi_1)$, that is, $\tau = \pi_1 \circ \pi_0^{-1}$. We say τ (or π) is* ***reducible*** *if there exists $k \in \{1, 2, \ldots, r-1\}$ such that*

$$\tau(\{1, 2, \ldots, k\}) = \{1, 2, \ldots, k\}.$$

Otherwise, we say that τ is irreducible.

In Example 3.3, τ is irreducible. Let us see an example of a reducible permutation.

Example 3.6. *Let $\tau = (2, 1) \in G_3, \lambda \in \Lambda_3$ and suppose that*

$$\pi_0(1) = 1, \quad \pi_0(2) = 2 \quad \textit{and} \quad \pi_0(3) = 3.$$

By definition,

$$\alpha_1^0 = \pi_0^{-1}(1) = 1, \quad \alpha_2^0 = \pi_0^{-1}(2) = 2 \quad \textit{and} \quad \alpha_3^0 = \pi_0^{-1}(3) = 3;$$

and by the permutation τ, we have

$$\pi_1(1) = \pi_1\left(\pi_0^{-1}(1)\right) = \tau(1) = 2, \quad \pi_1(2) = \pi_1\left(\pi_0^{-1}(2)\right) = \tau(2) = 1$$

$$\textit{and} \quad \pi_1(3) = \pi_1\left(\pi_0^{-1}(3)\right) = \tau(3) = 3.$$

In this case, there exists $k = 2 < 3 = r$ such that

$$\tau(\{1, 2, \ldots, k\}) = \tau(\{1, 2\}) = \{\tau(1), \tau(2)\} = \{2, 1\} = \{1, 2\} = \{1, 2, \ldots, k\}.$$

Therefore, τ is reducible.

Thus, if τ is reducible, for any $\lambda \in \Lambda_r$, the subinterval

$$J = \bigcup_{\pi_0(j) \le k} X_j = \bigcup_{\pi_1(j) \le k} X_j$$

and its complementary are invariant by T. This means that T can be decomposed into two transformations. Therefore, given a pair π, we can decompose it into irreducible pairs, which means that we can restrict our studies to the set of irreducible pairs.

Obviously, an interval exchange map (λ, π) cannot be minimal if π is reducible.

3.2 TRANSLATION VECTORS

Given $\pi = (\pi_0, \pi_1)$, we define $\Omega_\pi : \mathbb{R}^{\mathcal{A}} \longrightarrow \mathbb{R}^{\mathcal{A}}$ by:

$$\Omega_\pi(\lambda) = \omega \quad \text{with} \quad \omega_\alpha = \sum_{\beta,\ \pi_1(\beta)<\pi_1(\alpha)} \lambda_\beta - \sum_{\beta,\ \pi_0(\beta)<\pi_0(\alpha)} \lambda_\beta, \quad \forall \alpha = 1, \ldots, r. \tag{3.1}$$

Then, the corresponding interval exchange transformation T is given by:

$$T(x) = x + \omega_\alpha, \text{ for } x \in X_\alpha.$$

Example 3.7. *Consider the interval exchange given by the following figure*

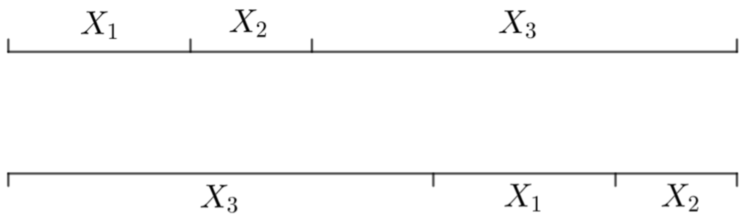

Figura 3.4: IET of three intervals.
Source: Author's own work

We have

$$\pi_0(1) = 1, \quad \pi_0(2) = 2, \quad \pi_0(3) = 3, \quad \pi_1(1) = 2, \quad \pi_1(2) = 3 \quad \textit{and} \quad \pi_1(3) = 1.$$

Hence, we have

$$\Omega_\pi(\lambda) = \omega = (\omega_1, \omega_2, \omega_3),$$

in which,

$$\omega_1 = \sum_{\pi_1(\beta)<\pi_1(1)=2} \lambda_\beta - \sum_{\pi_0(\beta)<\pi_0(1)=1} \lambda_\beta = \lambda_3,$$

$$\omega_2 = \sum_{\pi_1(\beta)<\pi_1(2)=3} \lambda_\beta - \sum_{\pi_0(\beta)<\pi_0(2)=2} \lambda_\beta = \lambda_1 + \lambda_3 - \lambda_1 = \lambda_3 \quad \textit{and}$$

$$\omega_3 = \sum_{\pi_1(\beta)<\pi_1(3)=1} \lambda_\beta - \sum_{\pi_0(\beta)<\pi_0(3)=3} \lambda_\beta = -\lambda_1 - \lambda_2.$$

We call ω **translation vector of** T. Note that the matrix $(\Omega_{\alpha,\beta})_{\alpha,\beta\in\mathcal{A}}$ of Ω_π is given by

$$\Omega_{\alpha,\beta} = \begin{cases} +1, & \text{if } \ \pi_1(\alpha) > \pi_1(\beta) \ \text{ and } \ \pi_0(\alpha) < \pi_0(\beta) \\ -1, & \text{if } \ \pi_1(\alpha) < \pi_1(\beta) \ \text{ and } \ \pi_0(\alpha) > \pi_0(\beta) \\ 0, & \text{for all other cases.} \end{cases} \tag{3.2}$$

Lemma 3.8. *The inner product $\lambda \cdot \omega = 0$.*

Proof:: Indeed,

$$\lambda \cdot \omega = \sum_{\alpha \in \mathcal{A}} \lambda_\alpha \omega_\alpha = \sum_{\alpha \in \mathcal{A}} \lambda_\alpha \left(\sum_{\pi_1(\beta) < \pi_1(\alpha)} \lambda_\beta - \sum_{\pi_0(\beta) < \pi_0(\alpha)} \lambda_\beta \right) =$$

$$= \sum_{\alpha \in \mathcal{A}} \sum_{\pi_1(\beta) < \pi_1(\alpha)} \lambda_\alpha \lambda_\beta - \sum_{\alpha \in \mathcal{A}} \sum_{\pi_0(\beta) < \pi_0(\alpha)} \lambda_\alpha \lambda_\beta. \tag{3.3}$$

Let $\epsilon \in \{0, 1\}$ and $\pi_\epsilon : \mathcal{A} \longrightarrow \mathcal{A}$ bijections. Consider $A_1^\epsilon = \{(\alpha, \beta) \in \mathcal{A} \times \mathcal{A} : \pi_\epsilon(\alpha) > \pi_\epsilon(\beta)\}$ and $A_2^\epsilon = \{(\gamma, \theta) \in \mathcal{A} \times \mathcal{A} : \pi_\epsilon(\gamma) < \pi_\epsilon(\theta)\}$.

Thus, $A_1^\epsilon \bigcap A_2^\epsilon = \emptyset$, $A_1^\epsilon \bigcup A_2^\epsilon = \mathcal{A} \times \mathcal{A} \setminus \{(\alpha, \alpha) : \alpha \in \mathcal{A}\}$ and

$$\sum_{\alpha \in \mathcal{A}} \sum_{\pi_\epsilon(\beta) < \pi_\epsilon(\alpha)} \lambda_\alpha \lambda_\beta = \sum_{(\alpha,\beta) \in A_1^\epsilon} \lambda_\alpha \lambda_\beta. \tag{3.4}$$

On the other hand,

$$\sum_{\alpha \neq \beta} \lambda_\alpha \lambda_\beta = \sum_{(\alpha,\beta) \in A_1^\epsilon \bigcup A_2^\epsilon} \lambda_\alpha \lambda_\beta = \sum_{(\alpha,\beta) \in A_1^\epsilon} \lambda_\alpha \lambda_\beta + \sum_{(\gamma,\theta) \in A_2^\epsilon} \lambda_\gamma \lambda_\theta =$$

$$= \sum_{(\alpha,\beta) \in A_1^\epsilon} \lambda_\alpha \lambda_\beta + \sum_{(\theta,\gamma) \in A_1^\epsilon} \lambda_\theta \lambda_\gamma = 2 \sum_{(\alpha,\beta) \in A_1^\epsilon} \lambda_\alpha \lambda_\beta. \tag{3.5}$$

Therefore, from (3.3), (3.4) and (3.5) it follows that

$$\lambda \cdot \omega = \frac{1}{2} \sum_{\alpha \neq \beta} \lambda_\alpha \lambda_\beta - \frac{1}{2} \sum_{\alpha \neq \beta} \lambda_\alpha \lambda_\beta = 0.$$

■

3.3 RAUZY INDUCTION

Introduced by Gérard Rauzy in 1979, Rauzy induction emerged on the basis of works from the early 1970s as a powerful renormalization procedure. This algorithm generalizes the classical continued fraction algorithm from rotations (which are exchanges of two intervals) to arbitrary IETs with more intervals. The historical significance of Rauzy's work lies not only in producing an effective analytical tool for IETs, but also in linking different areas of mathematics, including Teichmüller theory, flat surfaces, measured foliations and symbolic dynamics.

In this section, we will consider a pair (λ, π) with $\pi = (\pi_0, \pi_1)$. For each $\epsilon \in \{0, 1\}$, we denote by $\alpha(\epsilon)$ the last symbol in the vector π_ϵ, that is,

$$\alpha(\epsilon) = \pi_\epsilon^{-1}(r) = \alpha_r^\epsilon.$$

The Rauzy induction can only be defined if the intervals $X_{\alpha(0)}$ and $X_{\alpha(1)}$ have different lengths, so we will assume this condition by now. We say that

1) (λ, π) **has type** 0 if $\lambda_{\alpha(0)} > \lambda_{\alpha(1)}$;

2) (λ, π) **has type** 1 if $\lambda_{\alpha(0)} < \lambda_{\alpha(1)}$.

In this way, let Y be the subinterval of X obtained by removing the smallest of those two intervals, that is,

$$Y = \begin{cases} X \setminus T(X_{\alpha(1)}), & \text{if } (\lambda, \pi) \text{ has type } 0, \\ X \setminus X_{\alpha(0)}, & \text{if } (\lambda, \pi) \text{ has type } 1. \end{cases}$$

The Rauzy induction of T is the first return map $\widehat{R}(T)$ for the subinterval Y. We then have a new interval exchange transformation. Let us define this induction.

If (λ, π) has type 0, we define $Y_i = X_i$ for $i \neq \alpha(0)$ and $Y_{\alpha(0)} = X_{\alpha(0)} \setminus T(X_{\alpha(1)})$. Such intervals form a partition of Y and $T(Y_i) \subset Y$, for all $i \neq \alpha(1)$. In this way, $\widehat{R}(T) = T$ restricted to these $Y_i's$, with $i \neq \alpha(1)$. On the other hand, $T(Y_{\alpha(1)}) = T(X_{\alpha(1)}) \subset X_{\alpha(0)}$ $\Longrightarrow T^2(Y_{\alpha(1)}) \subset T(X_{\alpha(0)}) \subset Y$. Thus, we have $\widehat{R}(T) = T^2$ restricted to $Y_{\alpha(1)}$.

See the following figure.

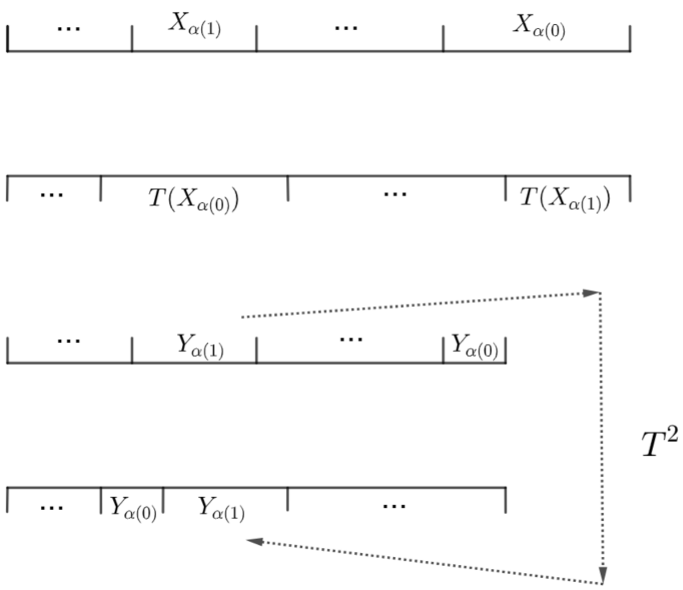

Figura 3.5: Rauzy Induction for type 0
Source: Author's own work

An example of this case is given below.

If (λ, π) has type 1, we define $Y_{\alpha(0)} = T^{-1}(X_{\alpha(0)})$, $Y_{\alpha(1)} = X_{\alpha(1)} \setminus Y_{\alpha(0)}$ and $Y_i = X_i$, for all other values of i. In this way, $T(Y_i) \subset Y$ for all $i \neq \alpha(0)$ and, therefore, $\widehat{R}(T) = T$ restricted to these $Y_i's$, with $i \neq \alpha(0)$. On the other hand, $T^2(Y_{\alpha(0)}) = T(X_{\alpha(0)}) \subset Y$.

Hence, we have $\widehat{R}(T) = T^2$ restricted to $Y_{\alpha(0)}$.

See the figure below.

An example of this case is given by:

Now, we will express the map $T \mapsto \widehat{R}(T)$ in terms of coordinates (λ, π) in the space of interval exchange transformations. This analysis will be extremely useful in later chapters.

1) If (λ, π) has type 0, then the transformation $\widehat{R}(T)$ is described by (λ', π'), such that: $\lambda' = (\lambda'_j)_{j \in \mathcal{A}}$, where

$$\lambda'_j = \lambda_j, \quad \text{for} \quad j \neq \alpha(0), \quad \text{and} \quad \lambda'_{\alpha(0)} = \lambda_{\alpha(0)} - \lambda_{\alpha(1)}. \tag{3.6}$$

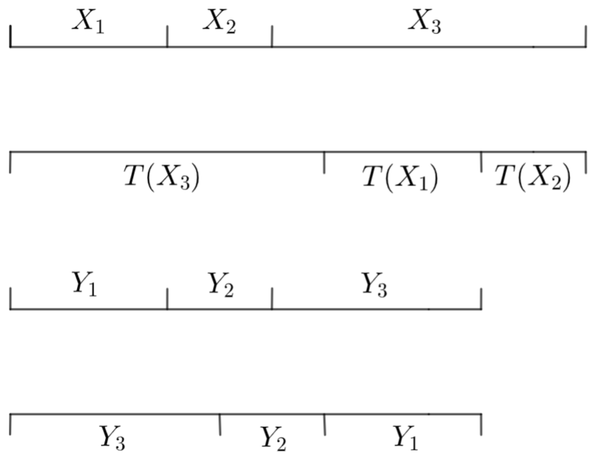

Figura 3.6: Example of Rauzy Induction for type 0
Source: Author's own work

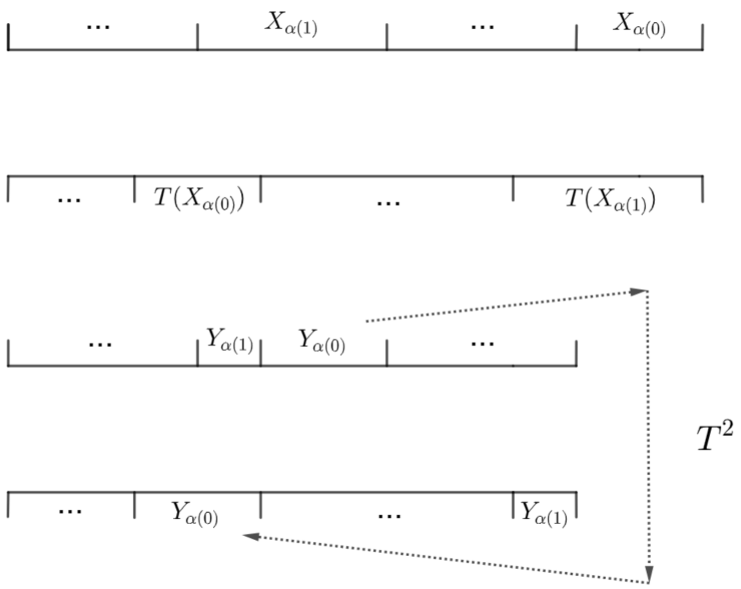

Figura 3.7: Rauzy Induction for type 1
Source: Author's own work

and

$$\pi' = \begin{pmatrix} \pi'_0 \\ \pi'_1 \end{pmatrix} = \begin{pmatrix} \alpha_1^0 & \dots & \alpha_{k-1}^0 & \alpha_k^0 & \alpha_{k+1}^0 & \dots & \dots & \alpha(0) \\ \alpha_1^1 & \dots & \alpha_{k-1}^1 & \alpha(0) & \alpha(1) & \alpha_{k+1}^1 & \dots & \alpha_{r-1}^1 \end{pmatrix},$$

that is,

$$\alpha_j^{0\prime} = \alpha_j^0 \quad \text{and} \quad \alpha_j^{1\prime} = \begin{cases} \alpha_j^1, & \text{if} \quad j \leq k; \\ \alpha(1), & \text{if} \quad j = k+1; \\ \alpha_{j-1}^1, & \text{if} \quad j > k+1. \end{cases} \tag{3.7}$$

in which $k \in \{1, \dots, r-1\}$ is defined by $\alpha_k^1 = \alpha(0)$.

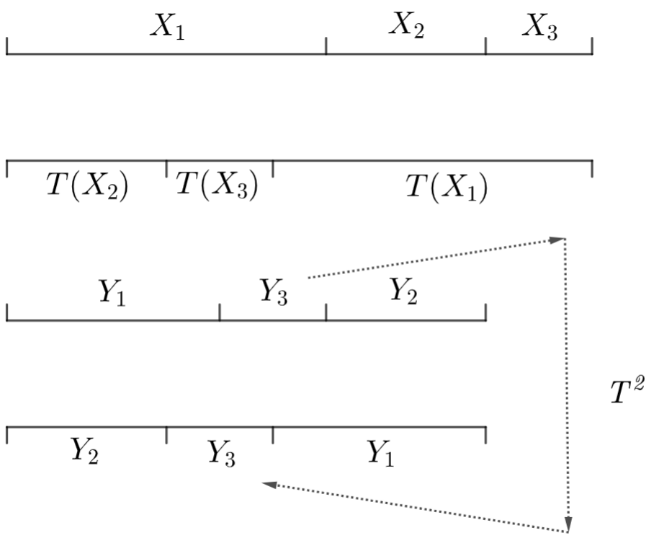

Figura 3.8: Example of Rauzy Induction for type 1
Source: Author's own work

2) Analogously, if (λ, π) has type 1, then the transformation $\widehat{R}(T)$ is described by (λ', π'), such that:
$\lambda' = (\lambda'_j)_{j\in\mathcal{A}}$, where

$$\lambda'_j = \lambda_j, \quad \text{for} \quad j \neq \alpha(1), \quad \text{and} \quad \lambda'_{\alpha(1)} = \lambda_{\alpha(1)} - \lambda_{\alpha(0).} \tag{3.8}$$

and

$$\pi' = \begin{pmatrix} \pi'_0 \\ \pi'_1 \end{pmatrix} = \begin{pmatrix} \alpha^0_1 & \dots & \alpha^0_{k-1} & \alpha(1) & \alpha(0) & \alpha^0_{k+1} & \dots & \alpha^0_{r-1} \\ \alpha^1_1 & \dots & \alpha^1_{k-1} & \alpha^1_k & \alpha^1_{k+1} & \dots & \dots & \alpha(1) \end{pmatrix},$$

that is,

$$\alpha^{0'}_j = \begin{cases} \alpha^0_j, & \text{if} \quad j \leq k; \\ \alpha(0), & \text{if} \quad j = k+1; \\ \alpha^0_{j-1}, & \text{if} \quad j > k+1; \end{cases} \quad \text{and} \quad \alpha^{1'}_j = \alpha^1_j, \tag{3.9}$$

in which $k \in \{1, \dots, r-1\}$ is defined by $\alpha^0_k = \alpha(1)$.

Example 3.9. *Let $T(\lambda, \pi)$ be an interval exchange transformation, with $\pi = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$. Depending on the type of (λ, π), that is, if it has type 0 or type 1, there are two possibilities for π':*

1) If (λ, π) has type 0, $\pi' = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$,

2) If (λ, π) has type 1, $\pi' = \begin{pmatrix} 1 & 3 & 2 \\ 3 & 2 & 1 \end{pmatrix}$.

Following this logic, we understand the concept called "Rauzy diagram".

Given pairs π and π', we say that π' is a **successor** of π if there exist λ and λ' such that $\widehat{R}(\pi, \lambda) = (\pi', \lambda')$. Of course, any pair π has exactly two successors corresponding to types 0 and 1. Similarly, each π' is the successor of exactly two pairs π. Notice that π is irreducible if and only if π' is irreducible. Thus, this relation defines a partial order in the set of irreducible pairs, which we may represent as a direct graph G. If an IET $T(\lambda,\pi)$ belongs to the graph G, we say that G is the **Rauzy diagram of** T.

The class $\mathcal{R}$ of irreducible permutations belonging to the same diagram is called a **Rauzy class**.

Example 3.10. *For three intervals, there are only three irreducible IETs - the ones presented in Example 3.9 - and all of them are connected by the graph mentioned.*

$$1 \circlearrowleft \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \underset{0}{\overset{0}{\rightleftarrows}} \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \underset{1}{\overset{1}{\rightleftarrows}} \begin{pmatrix} 1 & 3 & 2 \\ 3 & 2 & 1 \end{pmatrix} \circlearrowleft 0$$

3.4 KEANE CONDITION

Observe that the induction map $\widehat{R}(T)$ is not defined when the intervals $X_{\alpha(0)}$ and $X_{\alpha(1)}$ have the same length, that is, when $\lambda_{\alpha(0)} = \lambda_{\alpha(1)}$. In order to view $\widehat{R}$ as a dynamical system in the space of interval exchange transformations, we must therefore restrict $\widehat{R}$ to an invariant subset of (λ,π) for which the iterates $\widehat{R}^n(\lambda,\pi)$ are well defined; that is, such that $\lambda^n_{\alpha(0)} \neq \lambda^n_{\alpha(1)}$, for all $n \geq 1$. A possible choice for this subset is the set of irrational vectors, which we now define.

Definition 3.11. *We say the vector $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r) \in \Lambda_r$ is* ***irrational****, or that $\lambda_j, j \in \mathcal{A} = \{1,\dots,r\}$ are* ***rationally independent****, if $\sum_{j=1}^{r} n_j\lambda_j \neq 0$, for every nonzero integer vector $(n_j)_{j\in\mathcal{A}} \in \mathbb{Z}^r$.*

As mentioned above, this definition gives us the following result.

Proposition 3.12. *If (λ,π) is such that λ is irrational, then $\widehat{R}^n(\lambda,\pi)$ is well defined for every $n \geq 1$, that is, $\lambda^n_{\alpha(0)} \neq \lambda^n_{\alpha(1)}$, for every $n \geq 1$.*

Proof: First, let us show that if λ is irrational, then $\widehat{R}(\lambda,\pi)$ is well defined.

Suppose, by contradiction, that $\lambda_{\alpha(0)} = \lambda_{\alpha(1)}$. Then

$$1 \cdot \lambda_{\alpha(0)} + (-1) \cdot \lambda_{\alpha(1)} = 0.$$

Taking

$$n = (0, 0, \dots, \underbrace{1}_{\alpha(0)}, \dots, \underbrace{-1}_{\alpha(1)}, 0, \dots),$$

we obtain

$$\sum_{j=1}^{r} n_j\lambda_j = 0,$$

which is a contradiction.

Therefore, if λ is irrational, then $\widehat{R}(\lambda,\pi)$ is well defined.

It remains to prove that if λ is irrational, then λ' is also irrational.

Assume that (λ, π) has type 0; the case of type 1 is analogous. Let $n = (n_j)_{j\in\mathcal{A}} \in \mathbb{Z}^r$ be a nonzero vector. Then

$$\begin{aligned}\sum_{j\in\mathcal{A}} n_j\lambda'_j &= \sum_{j\neq\alpha(0)} n_j\lambda_j + n_{\alpha(0)}(\lambda_{\alpha(0)} - \lambda_{\alpha(1)}) \\ &= \sum_{j\neq\alpha(0),\alpha(1)} n_j\lambda_j + n_{\alpha(1)}\lambda_{\alpha(1)} + n_{\alpha(0)}(\lambda_{\alpha(0)} - \lambda_{\alpha(1)}) \\ &= \sum_{j\neq\alpha(0),\alpha(1)} n_j\lambda_j + n_{\alpha(0)}\lambda_{\alpha(0)} + \lambda_{\alpha(1)}\Big(n_{\alpha(1)} - n_{\alpha(0)}\Big).\end{aligned}$$

(i) If $n_{\alpha(0)} = n_{\alpha(1)} = 0$, then

$$\sum_{j\in\mathcal{A}} n_j\lambda'_j = \sum_{j\neq\alpha(0),\alpha(1)} n_j\lambda_j = \sum_{j\in\mathcal{A}} n_j\lambda_j \neq 0,$$

since $\sum_{j\in\mathcal{A}} n_j\lambda_j \neq 0$ (by the irrationality of λ) and the last two terms $n_{\alpha(0)}\lambda_{\alpha(0)}$ and $n_{\alpha(1)}\lambda_{\alpha(1)}$ vanish.

(ii) If $n_{\alpha(0)} = 0$ and $n_{\alpha(1)} \neq 0$, then

$$\sum_{j\in\mathcal{A}} n_j\lambda'_j = \sum_{j\neq\alpha(0),\alpha(1)} n_j\lambda_j + n_{\alpha(1)}\lambda_{\alpha(1)} = \sum_{j\in\mathcal{A}} n_j\lambda_j \neq 0,$$

again because $\sum_{j\in\mathcal{A}} n_j\lambda_j \neq 0$ and the term $n_{\alpha(0)}\lambda_{\alpha(0)}$ vanishes.

(iii) If $n_{\alpha(0)} \neq 0$, then the vector

$$(n_1, \ldots, n_{\alpha(0)}, \ldots, n_{\alpha(1)} - n_{\alpha(0)}, \ldots)$$

is nonzero, since $n_{\alpha(0)} \neq 0$. Therefore,

$$\sum_{j\in\mathcal{A}} n_j\lambda'_j \neq 0,$$

because λ is irrational.

Hence, if λ is irrational, then λ' is also irrational, and consequently $\widehat{R}$ is well defined for all $n \geq 1$.

■

Keane observed, however, that non-irrationality is not always an obstruction to further iterations of $\widehat{R}^n(\lambda, \pi)$.

Thus, we shall introduce a condition known as the **Keane condition** and prove that irrationality is stronger than this condition.

Let ∂X_γ denote the left endpoint of each subinterval X_γ, $\gamma \in \mathcal{A}$. Recalling that the left endpoint of X coincides with the origin, we have

$$\partial X_\gamma = \sum_{\substack{\eta\in\mathcal{A} \\ \pi_0(\eta)<\pi_0(\gamma)}} \lambda_\eta.$$

Note that if $\pi_0(\beta) = 1$, then

$$T(\partial X_\alpha) = \partial X_\beta \quad \text{for } \alpha = \pi_1^{-1}(1).$$

Definition 3.13. *We say a pair (λ, π) satisfies the **Keane condition** if the orbits of these endpoints are as disjoint as they can possibly be, that is,*

$$T^m(\partial X_\alpha) \neq \partial X_\beta, \quad \text{for all } m \geq 1 \text{ and } \alpha, \beta \in \mathcal{A}, \text{ with } \pi_0(\beta) \neq 1. \tag{3.10}$$

Proposition 3.14. *The following properties are true:*

1) *If the pair (λ, π) satisfies the Keane condition, then $\widehat{R}(\lambda, \pi) = (\lambda', \pi')$ is well defined and π is irreducible.*

2) *The Keane condition is not affected if we restrict it to the case of $\pi_1(\alpha) > 1$ instead of $\pi_0(\beta) \neq 1$.*

Proof: **(1)** Suppose that $\widehat{R}$ is not well defined. Then we would have $\lambda_{\alpha(0)} = \lambda_{\alpha(1)}$, which would imply that

$$T(\partial X_{\alpha(1)}) = \partial X_{\alpha(0)},$$

contradicting the fact that (λ, π) satisfies the Keane condition.

Now suppose that π is reducible. Then there exists $k \in \{1, 2, \ldots, r-1\}$ such that

$$\pi_1 \circ \pi_0^{-1}(\{1, 2, \ldots, k\}) = \{1, 2, \ldots, k\}.$$

Thus, there exists $\alpha \in \{k+1, k+2, \ldots, r\}$ such that

$$T(\partial X_\alpha) = \partial X_{k+1},$$

which again contradicts the fact that the pair (λ, π) satisfies the Keane condition.

(2) Indeed, assume as a hypothesis that

$$T^m(\partial X_\alpha) \neq \partial X_\beta, \quad \text{for all } m \geq 1 \text{ and } \alpha, \beta \in \mathcal{A} \text{ with } \pi_1(\alpha) > 1.$$

Arguing as in the previous item, we conclude that π is irreducible.

Let $\alpha, \beta \in \mathcal{A}$ be such that $\pi_0(\beta) \neq 1$.

(i) If $\pi_1(\alpha) > 1$, then condition (3.10) holds by hypothesis.

(ii) If $\pi_1(\alpha) = 1$, suppose by contradiction that

$$T^m(\partial X_\alpha) = \partial X_\beta \quad \text{for some } m \geq 1.$$

Then $T(\partial X_\alpha) \neq \partial X_\beta$, since $T(\partial X_\alpha) = 0$ and $\partial X_\beta > 0$. Hence $m > 1$, and therefore $m - 1 > 0$.

Now let $\gamma \in \mathcal{A}$ be such that

$$T(\partial X_\alpha) = 0 = \partial X_\gamma.$$

Then

$$T^{m-1}(\partial X_\gamma) = \partial X_\beta.$$

Moreover, we have $\pi_1(\gamma) > 1$, since $\pi_0(\gamma) = 1$ and π is irreducible.

Therefore, there exist $\gamma, \beta \in \mathcal{A}$ such that

$$T^{m-1}(\partial X_\gamma) = \partial X_\beta \quad \text{with } \pi_1(\gamma) > 1,$$

which contradicts the hypothesis.

■

Observe that property (3.10) is invariant under iterates of $\widehat{R}$, since the orbits of $\widehat{R}(T)$ are contained in the orbits of T. Therefore, the Keane condition is sufficient to guarantee the existence of all iterates

$$(\lambda^n, \pi^n) = \widehat{R}^n(\lambda, \pi), \quad n \geq 0.$$

The next result shows that, assuming the irreducibility of π, the Keane condition is indeed more general than rational independence.

Proposition 3.15. *If λ is irrational and π is irreducible, then (λ, π) satisfies the Keane condition.*

Proof: Suppose, by contradiction, that the pair (λ, π) does not satisfy the Keane condition. Then there exist $m \geq 1$ and $\alpha, \beta \in \mathcal{A}$, with $\pi_0(\beta) > 1$, such that

$$T^m(\partial X_\alpha) = \partial X_\beta.$$

Let β_j, $0 \leq j \leq m$, be such that $T^j(\partial X_\alpha) \in X_{\beta_j}$. Thus $\beta_0 = \alpha$ and $\beta_m = \beta$. Note that

$$\begin{aligned} T(\partial X_\alpha) &= \partial X_\alpha + \omega_\alpha, \\ T^2(\partial X_\alpha) = T(T(\partial X_\alpha)) &= T(\partial X_\alpha) + \omega_{\beta_1} = \partial X_\alpha + \omega_\alpha + \omega_{\beta_1}, \\ &\vdots \\ \partial X_\beta = T^m(\partial X_\alpha) &= \partial X_\alpha + \omega_\alpha + \omega_{\beta_1} + \cdots + \omega_{\beta_{m-1}}, \end{aligned}$$

where $\omega = (\omega_\gamma)_{\gamma \in \mathcal{A}}$ is the translation vector defined in (3.1). Hence,

$$\partial X_\beta - \partial X_\alpha = \sum_{j=0}^{m-1} \omega_{\beta_j}.$$

That is,

$$\begin{aligned} \sum_{\pi_0(\gamma)<\pi_0(\beta_m)} \lambda_\gamma - \sum_{\pi_0(\gamma)<\pi_0(\beta_0)} \lambda_\gamma &= \sum_{j=0}^{m-1} \left(\sum_{\pi_1(\gamma)<\pi_1(\beta_j)} \lambda_\gamma - \sum_{\pi_0(\gamma)<\pi_0(\beta_j)} \lambda_\gamma \right) \\ &= \sum_{j=0}^{m-1} \sum_{\pi_1(\gamma)<\pi_1(\beta_j)} \lambda_\gamma - \sum_{j=0}^{m-1} \sum_{\pi_0(\gamma)<\pi_0(\beta_j)} \lambda_\gamma. \end{aligned}$$

Thus, we obtain

$$\sum_{\pi_0(\gamma)<\pi_0(\beta_m)} \lambda_\gamma - \sum_{\pi_0(\gamma)<\pi_0(\beta_0)} \lambda_\gamma = \sum_{j=0}^{m-1} \sum_{\pi_1(\gamma)<\pi_1(\beta_j)} \lambda_\gamma - \sum_{j=1}^{m-1} \sum_{\pi_0(\gamma)<\pi_0(\beta_j)} \lambda_\gamma - \sum_{\pi_0(\gamma)<\pi_0(\beta_0)} \lambda_\gamma,$$

and hence

$$\sum_{j=0}^{m-1} \sum_{\pi_1(\gamma)<\pi_1(\beta_j)} \lambda_\gamma - \sum_{j=1}^{m} \sum_{\pi_0(\gamma)<\pi_0(\beta_j)} \lambda_\gamma = 0.$$

This last equality can be written as

$$\sum_{\gamma \in \mathcal{A}} n_\gamma \lambda_\gamma = 0,$$

where

$$n_\gamma = \#\{0 \leq j < m : \pi_1(\beta_j) > \pi_1(\gamma)\} - \#\{0 < j \leq m : \pi_0(\beta_j) > \pi_0(\gamma)\}.$$

However, by hypothesis, λ is irrational. Therefore, it follows that $n_\gamma = 0$ for all $\gamma \in \mathcal{A}$. Let

$$B = \max_{0<j\leq m} \{\pi_0(\beta_j)\}, \qquad C = \max_{0\leq j<m} \{\pi_1(\beta_j)\}, \qquad D = \max\{B, C\}.$$

In this way, we have

$$D \geq B \geq \pi_0(\beta_m) = \pi_0(\beta) > 1.$$

Since π is irreducible, there must exist γ such that

$$\pi_0(\gamma) < D \leq \pi_1(\gamma).$$

Now, for all $0 \leq j < m$, we have

$$\pi_1(\beta_j) \leq C \leq D \leq \pi_1(\gamma),$$

and thus

$$\{0 \leq j < m : \pi_1(\beta_j) > \pi_1(\gamma)\} = \emptyset.$$

Since $n_\gamma = 0$, we obtain

$$\{0 < j \leq m : \pi_0(\beta_j) > \pi_0(\gamma)\} = \emptyset,$$

and hence

$$\pi_0(\beta_j) \leq \pi_0(\gamma) < D \quad \text{for all } 0 < j \leq m.$$

In an analogous way, one proves that

$$\pi_1(\beta_j) < D, \quad \text{for all } 0 \leq j < m.$$

Thus, we have shown that $\pi_0(\beta_j) < D$ for all $0 < j \leq m$ and that $\pi_1(\beta_j) < D$ for all $0 \leq j < m$, which is a contradiction, since it violates the definition of D.

Therefore, (λ, π) satisfies the Keane condition.

■

The next result is of fundamental importance, as it relates the Keane condition to the strong notion of minimality in dynamical systems.

Proposition 3.16. *If (λ, π) satisfies the Keane condition, then T is minimal.*

In order to prove this proposition, we first establish the following results.

Lemma 3.17. *Given any subinterval $J = [a, b[\subset X_\alpha$, for some $\alpha \in \mathcal{A}$, there exists a finite partition $\{J_j : 1 \leq j \leq k\}$ and integers $n_1, \ldots, n_k \geq 1$ such that:*

1. *$T^i(J_j) \cap J = \emptyset$ for all $0 < i < n_j$ and $1 \leq j \leq k$;*
2. *each restriction $T^{n_j}|_{J_j}$ is a translation of J_j onto some subinterval of J;*
3. *the intervals $T^{n_j}(J_j)$, $1 \leq j \leq k$, are pairwise disjoint.*

Proof: Let A be the union of the set of endpoints $\{a, b\}$ of J with the set of endpoints of the intervals X_α. Let $B \subset J$ be the set of points $z \in J$ for which there exists some $m \geq 1$ such that $T^i(z) \notin J$ for all $0 < i < m$ and $T^m(z) \in A$.

The map $B \ni z \mapsto T^m(z) \in A$ is injective. Indeed, let $z_1, z_2 \in J$ and $m_1, m_2 \geq 1$ be such that $T^{m_1}(z_1) = T^{m_2}(z_2) \in A$. If $m_1 = m_2$, then, since T is bijective, it follows that $z_1 = z_2$. Without loss of generality, assume that $m_1 < m_2$. Then

$$T^{m_2 - m_1}(z_2) = z_1 \in J \quad \text{with} \quad 1 \leq m_2 - m_1 < m_2,$$

which contradicts the choice of m_2. Hence $m_1 = m_2$, and the map is injective.

Consequently, since A is finite, it follows that B is also finite. Therefore, consider the partition of J determined by the points of B.

By the Poincaré Recurrence Theorem (2.3), for each element $J_j = [a_j, b_j[$ of this partition, there exists $n_j > 0$ such that $T^{n_j}(J_j)$ intersects J. We take n_j to be the smallest such integer. Observe that, since $J_j \subset J \subset X_\alpha$, $T^{n_j}|_{J_j}$ is a translation.

Claim. $T^{n_j}(J_j) \subset J$.

Indeed, suppose that $T^{n_j}(J_j) \not\subset J$. Since the intervals intersect, we must have $a \in T^{n_j}(J_j)$ or $b \in T^{n_j}(J_j)$. Without loss of generality, assume that $a \in T^{n_j}(J_j)$. Then there exists $x \in J_j$, with $a_j < x < b_j$, such that

$$T^{n_j}(x) = a \in A,$$

and, by the choice of n_j, we have $T^i(x) \notin J$ for all $i < n_j$. Thus $x \in B$, which is a contradiction, since x lies in the interior of the interval $[a_j, b_j[$, which is partitioned by the points of B.

Therefore, $T^{n_j}(J_j) \subset J$.

Finally, the intervals $T^{n_j}(J_j)$, $1 \leq j \leq k$, are pairwise disjoint. Indeed, if there existed J_i and J_j such that $y \in T^{n_i}(J_i) \cap T^{n_j}(J_j)$, then there would exist $x_i \in J_i$ and $x_j \in J_j$ such that

$$T^{n_i}(x_i) = T^{n_j}(x_j) = y.$$

Now $n_i \neq n_j$, since otherwise the bijectivity of T would imply $x_i = x_j$, which is impossible because $J_i \cap J_j = \emptyset$ by construction. Without loss of generality, assume that $n_i > n_j$. Then

$$T^{n_i - n_j}(x_i) = x_j \in J,$$

which contradicts the definition of n_i. Hence, the desired result follows.

■

Observe that this result shows that the first return map of T to an interval $J \subset X_\alpha$ is again an interval exchange transformation.

Corollary 3.18. *Under the hypotheses of Lemma (3.17), the set $\hat{J}$, defined as the union of all future iterates of J, is a finite union of intervals and is also an invariant set, that is, $T(\hat{J}) = \hat{J}$.*

Proof: The first statement follows directly from the first part of Lemma (3.17):

$$\hat{J} = \bigcup_{n=0}^{\infty} T^n(J) = \bigcup_{j=1}^{k} \bigcup_{i=0}^{n_j-1} T^i(J_j).$$

From the bijectivity of T, observe that

$$\sum_{j=1}^{k} |T^{n_j}(J_j)| = \sum_{j=1}^{k} |J_j| = |J|,$$

where $|\cdot|$ denotes the length of an interval.

Moreover, using items **2.** and **3.** of the previous lemma, it follows that

$$J = \bigcup_{j=1}^{k} T^{n_j}(J_j).$$

Thus,

$$\begin{aligned} T(\hat{J}) &= \bigcup_{j=1}^{k} \bigcup_{i=1}^{n_j} T^i(J_j) \\ &= \left[\bigcup_{j=1}^{k} \bigcup_{i=1}^{n_j-1} T^i(J_j) \right] \cup \left[\bigcup_{j=1}^{k} T^{n_j}(J_j) \right] \\ &= \left[\bigcup_{j=1}^{k} \bigcup_{i=1}^{n_j-1} T^i(J_j) \right] \cup \left[\bigcup_{j=1}^{k} J_j \right] \\ &= \bigcup_{j=1}^{k} \bigcup_{i=0}^{n_j-1} T^i(J_j) \\ &= \hat{J}. \end{aligned}$$

Therefore, $\hat{J}$ is an invariant set.

■

Lemma 3.19. *If (λ, π) satisfies the Keane condition, then T has no periodic points.*

Proof: Suppose, by contradiction, that there exist $m \geq 1$ and $x \in X$ such that $T^m(x) = x$.

Define $\beta_j \in \mathcal{A}$, for $0 \leq j \leq m$, such that $T^j(x) \in X_{\beta_j}$.

Let J be the set of all points $y \in X$ such that $T^j(y) \in X_{\beta_j}$ for all $0 \leq j < m$, that is,

$$J = \{y \in X : T^j(y) \in X_{\beta_j},\ 0 \leq j < m\}.$$

Then J is an interval contained in X_{β_0}. Since $T^m(x) = x$, it follows that $T^m|_J = \mathrm{id}$. In particular,

$$T^m(\partial J) = \partial J.$$

By the construction of J, there exist $1 \leq k \leq m$ and $\beta \in \mathcal{A}$ such that

$$T^k(\partial J) = \partial X_\beta.$$

Therefore,

$$T^m(\partial X_\beta) = T^m(T^k(\partial J)) = T^k(T^m(\partial J)) = T^k(\partial J) = \partial X_\beta,$$

that is, $T^m(\partial X_\beta) = \partial X_\beta$.

Now we consider two cases:

(i) If $\pi_0(\beta) > 1$, this contradicts the Keane condition, which is absurd.

(ii) If $\pi_0(\beta) = 1$, then there exists $\alpha \in \mathcal{A}$ such that

$$T(\partial X_\alpha) = 0 = \partial X_\beta.$$

Observe that $\alpha \neq \beta$, and hence $\partial X_\alpha > 0$, since π is irreducible. Consequently,

$$T\Big(T^m(\partial X_\alpha)\Big) = T^m\Big(T(\partial X_\alpha)\Big) = T^m(\partial X_\beta) = \partial X_\beta.$$

By the injectivity of T, it follows that

$$T^m(\partial X_\alpha) = \partial X_\alpha,$$

which again contradicts the Keane condition.

Therefore, T has no periodic points.

■

Proof of Proposition (3.16)**.** Suppose that there exists $x \in X$ such that the set $\{T^n(x) : n \geq 0\}$ is not dense in X, that is,

$$\overline{\{T^n(x) : n \geq 0\}} \neq X.$$

Let $J = [a, b[\subset X_\alpha$, for some $\alpha \in \mathcal{A}$, such that

$$J \cap \{T^n(x) : n \geq 0\} = \emptyset,$$

and let $\hat{J}$ denote the union of all future iterates of J. By Corollary (3.18), $\hat{J}$ is a finite union of intervals and is invariant under T.

Claim. $\hat{J}$ cannot be of the form $[0, \hat{b}[$.

Indeed, suppose that this is the case. Let $\mathcal{B} = \{\alpha \in \mathcal{A} : X_\alpha \subset \hat{J}\}$.

(i) If $\mathcal{B} \neq \emptyset$, then

$$\pi_0(\mathcal{B}) = \{1, 2, \ldots, k\}$$

for some $k \in \mathcal{A}$ with $k < r$, since $\hat{J}$ does not intersect the orbit of x. Since $\hat{J}$ is invariant, we also have

$$\pi_1(\mathcal{B}) = \{1, 2, \ldots, k\},$$

and thus

$$\pi_0^{-1}(\{1,2,\dots,k\}) = \mathcal{B} = \pi_1^{-1}(\{1,2,\dots,k\}).$$

This contradicts the irreducibility of π, since irreducibility is a consequence of the Keane condition.

(ii) If $\mathcal{B} = \emptyset$, then $\hat{J} \subset X_\alpha$, where $\pi_0(\alpha) = 1$. By invariance, we also have $\hat{J} \subset T(X_\alpha)$, and therefore $\pi_1(\alpha) = 1$. This again contradicts the irreducibility of π.

Thus, the claim is proved.

As a consequence, and using Corollary (3.18), the set $\hat{J}$ is a finite disjoint union of subintervals of X, and there exists at least one subinterval of the form $[\hat{a}, \hat{b}[$ with $\hat{a} > 0$.

If $T^n(\hat{a}) \neq \partial X_\beta$ for all $n \geq 0$ and all $\beta \in \mathcal{A}$, then by continuity of T and invariance of $\hat{J}$, the point $T^n(\hat{a})$ must be an endpoint of one of the disjoint subintervals of $\hat{J}$ for all $n \geq 0$. Since $\hat{J}$ has finitely many connected components, it follows that T must have a periodic point, which is impossible by Lemma (3.19).

Analogously, it cannot happen that

$$T^n(\hat{a}) \neq T(\partial X_\alpha) \quad \text{for all } n \leq 0 \text{ and } \alpha \in \mathcal{A}.$$

Therefore, there exist integers $n_1 \leq 0 \leq n_2$ and $\alpha, \beta \in \mathcal{A}$ such that

$$T^{n_1}(\hat{a}) = T(\partial X_\alpha) \qquad \text{and} \qquad T^{n_2}(\hat{a}) = \partial X_\beta. \tag{3.11}$$

If $\partial X_\beta > 0$, then the Keane condition is violated, since

$$T^{n_2 - n_1 + 1}(\partial X_\alpha) = T^{n_2}(T^{1-n_1}(\partial X_\alpha)) = T^{n_2}(\hat{a}) = \partial X_\beta.$$

If $\partial X_\beta = 0$, then $n_2 > 0$, since $\hat{a} > 0$. Moreover, $\partial X_\beta = T(\partial X_\gamma)$ for some $\gamma \in \mathcal{A}$ with $\pi_1(\gamma) = 1$. In this case, equation (3.11) remains valid if we replace β by γ and n_2 by $n_2 - 1$. By irreducibility, we have $\gamma \neq \beta$, and hence $\partial X_\gamma > 0$. Once again, the Keane condition is violated, since

$$T^{n_2 - n_1}(\partial X_\alpha) = T^{n_2 - 1}(T^{1-n_1}(\partial X_\alpha)) = T^{n_2 - 1}(\hat{a}) = \partial X_\gamma, \qquad \pi_0(\gamma) \neq 1.$$

Therefore, there does not exist any $x \in X$ such that

$$\overline{\{T^n(x) : n \geq 0\}} \neq X.$$

Hence, T is minimal.

■

3.5 AFFINE INTERVAL EXCHANGE TRANSFORMATIONS

We now present a generalization of the concept of interval exchange transformation, called an *affine interval exchange transformation*. The motivation for introducing this definition arises from the main results that will be studied.

Definition 3.20. *A bijective map $T\colon [a,b) \to [a,b)$ is said to be an* ***affine interval exchange transformation*** *(AIET) if there exists a finite sequence*

$$a = y_0 < y_1 < y_2 < \cdots < y_{m-1} < y_m = b$$

such that, for every $i \in \{0,1,\ldots,m-1\}$, the restriction $T|_{[y_i,y_{i+1})}$ is continuously differentiable and its derivative is identically equal to a constant, say $\beta_i > 0$.

When T is discontinuous exactly at the points $y_1, y_2, \ldots, y_{m-1}$, we associate to T the pair of vectors

$$(y, \tilde{\gamma}) = \Big((y_1, y_2, \ldots, y_m), (\gamma_1, \gamma_2, \ldots, \gamma_m)\Big),$$

where $\gamma_i = \log \beta_i$.

Example 3.21. *As in the case of IETs, AIETs can also be represented by graphs and diagrams.*

1. Graph

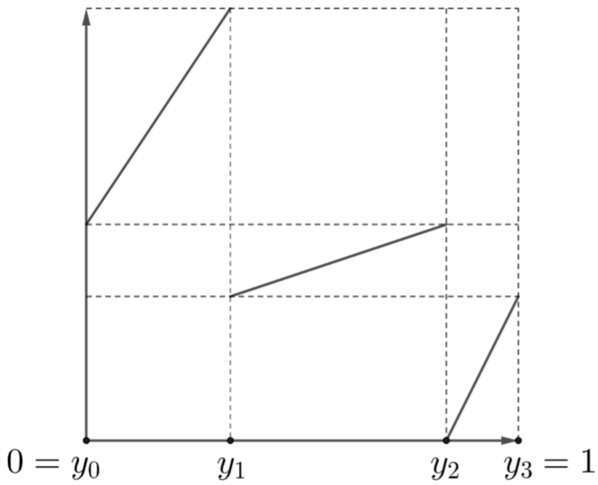

Figura 3.9: Graph of an AIET
Source: Author's own work

2. Diagram

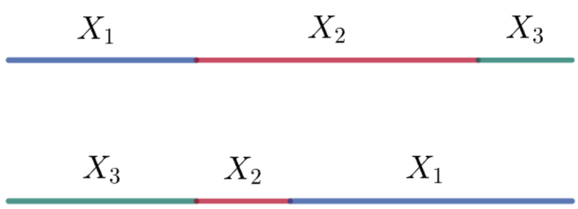

Figura 3.10: Diagram of an AIET
Source: Author's own work

Remark 3.22. *Observe that an AIET permutes the intervals and may allow changes in their lengths. An IET, on the other hand, is a particular case in which the intervals do not change their lengths. In other words, an IET E is an AIET such that $\beta_i = 1$ for all $1 \leq i \leq m$. Then, of course, $\tilde{\gamma}$ is the zero vector. In this case, only one vector is needed to describe an IET, which we have denoted by λ until now, but which we shall denote by*

$$x = (x_0, x_1, \ldots, x_m)$$

from now on.

Let N be a finite subinterval of either $\mathbb{R}$ or the circle S^1. Let $f\colon N \to N$ be a piecewise continuous map and let $J \subset N$ be an interval. We say that J is a **wandering interval** for the map f if:

(i) the intervals $J, f(J), f^2(J), \ldots$ are pairwise disjoint;

(ii) the ω-limit set of J is infinite.

It is easy to verify that an IET cannot have wandering intervals, since the lengths of the iterates of any interval always remain the same. It might therefore seem that the same would hold for AIETs. However, Levitt [L] constructed an example of a non–uniquely ergodic AIET with wandering intervals, and later Camelier and Gutierrez [4] exhibited a uniquely ergodic AIET with wandering intervals semi-conjugated to an IET. Inspired by these works, Bressaud, Hubert, and Maass [2] established conditions under which this last example occur. The last two results will be presented in this work, with the final one discussed in detail.

In order to properly understand these results, the study of symbolic dynamics and substitutions is essential. For this reason, the next chapter introduces the necessary notions.

4 SYMBOLIC DYNAMICAL SYSTEMS AND SUBSTITUTIONS

In this chapter, we introduce the basic notions of words and languages in order to construct the so-called *symbolic dynamical systems* and relate them to the concept of minimality. We then define *substitutions* and their associated matrices, which play a fundamental role not only in this theory but also in extending the study to other topics.

Finally, we present some important results, the main one being a theorem that establishes a strong connection between this theory and the concepts of *minimality* and *unique ergodicity*, both of which are central and powerful notions in ergodic theory. These concepts will be essential for proving the results developed in the subsequent chapters. The main reference for this chapter is [7].

4.1 WORDS, LANGUAGES, AND CYLINDERS

The objective of this section is to introduce several basic and important definitions in the context of symbolic dynamics.

Definition 4.1. *Let $n \in \mathbb{N}$ and let $\mathcal{A} = \{w_0, \ldots, w_n\}$ be a finite set.*

*(i) We call $\mathcal{A}$ an **alphabet**, and each w_i a **letter** of the alphabet $\mathcal{A}$;*

*(ii) A **finite word** (respectively, an **infinite word**) over $\mathcal{A}$ is a finite (respectively, infinite) sequence of letters from $\mathcal{A}$. The set of all finite words over $\mathcal{A}$ is denoted by $\mathcal{A}^*$, and the set of all infinite words by $\mathcal{A}^{\mathbb{N}}$;*

*(iii) The **length** of a word w is the number of letters in w, and it is denoted by $|w|$;*

*(iv) Given a word $w = w_0w_1w_2\cdots$ in $\mathcal{A}^{\mathbb{N}}$, we say that a finite word u is a **factor** of w if there exists $k \in \mathbb{N}$ such that*

$$u = w_k w_{k+1} \cdots w_{k+|u|-1};$$

*(v) The **language** of a word $w \in \mathcal{A}^{\mathbb{N}}$ is the set of all factors of w, denoted by $\mathcal{L}(w)$. For $n \in \mathbb{N}^*$, we denote by $\mathcal{L}_n(w)$ the set of all factors of w of length n;*

(vi) Let $a_0, a_1, \ldots, a_n \in \mathcal{A}$. We say that the set

$$\mathcal{C} = [a_0, a_1, \ldots, a_n] = \{(x_i)_{i\geq 0} \in \mathcal{A}^{\mathbb{N}} : x_i = a_i \text{ for all } 0 \leq i \leq n\}$$

*is a **cylinder** in $\mathcal{A}^{\mathbb{N}}$.*

Example 4.2. *Let* $\mathcal{A} = \{1, 2\}$*,* $u = 121 \in \mathcal{A}^*$*, and* $w = 121212\cdots \in \mathcal{A}^{\mathbb{N}}$*. Then* u *is a factor of* w*, that is,* $u \in \mathcal{L}(w)$*. Moreover,* $w \in [u]$*.*

Remark 4.3. *We identify the set of infinite words* $\mathcal{A}^{\mathbb{N}}$ *with the set of sequences whose entries belong to* $\mathcal{A}$*.*

4.2 METRIC ON $\mathcal{A}^{\mathbb{N}}$

We can define a metric on $\mathcal{A}^{\mathbb{N}}$, thus gaining the advantage of studying this space from a topological point of view, by the following.

Consider the map

$$\begin{aligned} d: \quad \mathcal{A}^{\mathbb{N}} \times \mathcal{A}^{\mathbb{N}} &\longrightarrow \mathbb{R}_+ \\ (u, v) &\longmapsto \begin{cases} e^{-\min\{\, i \in \mathbb{N} : u_i \neq v_i \,\}}, & \text{if } u \neq v, \\ 0, & \text{if } u = v. \end{cases} \end{aligned}$$

Proposition 4.4. d *is a metric.*

Proof: Let $x, y, z \in \mathcal{A}^{\mathbb{N}}$.

(i) $d(x, y) \geq 0$ and $d(x, y) = 0$ if and only if $x = y$, by the very definition of d;

(ii) $d(x, y) = e^{-\min\{\, i \in \mathbb{N} : x_i \neq y_i \,\}} = e^{-\min\{\, i \in \mathbb{N} : y_i \neq x_i \,\}} = d(y, x)$;

(iii) We now prove the triangle inequality.
Assume $x \neq y$ and consider $n_0 = \min\{i, x_i \neq y_i\}$. Then

$$x_{n_0} \neq z_{n_0} \quad \text{or} \quad y_{n_0} \neq z_{n_0}.$$

Assume without loss of generality that $x_{n_0} \neq z_{n_0}$, thus

$$d(x, y) \leq d(y, z) \Rightarrow d(x, y) \leq d(x, z) + d(z, y).$$

The other cases are analogous or follow immediately.

■

From now on, we shall consider $\mathcal{A}^{\mathbb{N}}$ as a metric space endowed with the metric d. The following results characterize this space, in particular showing that cylinders generate the topology induced by d.

Proposition 4.5. *Cylinders are both open and closed sets.*

Proof: Consider the cylinder

$$\mathcal{C} = [a_0, \ldots, a_n] = \{(x_i)_{i \in \mathbb{N}} \; ; \; x_i = a_i \text{ for all } 0 \leq i \leq n\},$$

with $a_0, \ldots, a_n \in \mathcal{A}$.

(i) $\mathcal{C}$ is open.

Indeed, let $x \in \mathcal{C}$ and set $\varepsilon = e^{-n}$. We claim that $B(x,\varepsilon) \subset \mathcal{C}$. Let $y \in B(x,\varepsilon)$. Then

$$e^{-\min\{i\in\mathbb{N}:x_i\neq y_i\}} < e^{-n}.$$

Hence,

$$\min\{i \in \mathbb{N} : x_i \neq y_i\} > n.$$

Therefore $x_i = y_i$ for all $0 \leq i \leq n$, which implies

$$y_0 = a_0,\ y_1 = a_1,\ \ldots,\ y_n = a_n,$$

and thus $y \in \mathcal{C}$. This shows that $\mathcal{C}$ is open.

(ii) $\mathcal{C}$ is closed.

Consider the complement

$$\mathcal{C}^c = \mathcal{A}^{\mathbb{N}} \setminus \mathcal{C} = \{(x_i)_{i\in\mathbb{N}} \in \mathcal{A}^{\mathbb{N}}\ ;\ x_i \neq a_i \text{ for some } i \in \{0,\ldots,n\}\}.$$

Let $x \in \mathcal{C}^c$ and define

$$k_0 = \min\{i \in \{0,\ldots,n\} : x_i \neq a_i\}.$$

Set $\varepsilon = e^{-k_0}$. We claim that $B(x,\varepsilon) \subset \mathcal{C}^c$. Indeed, if $y \in B(x,\varepsilon)$, then $d(x,y) < \varepsilon$, which implies

$$\min\{i \in \mathbb{N} : x_i \neq y_i\} > k_0.$$

Thus $x_i = y_i$ for all $0 \leq i \leq k_0$, and in particular $x_{k_0} = y_{k_0}$. Since $x_{k_0} \neq a_{k_0}$, we obtain $y_{k_0} \neq a_{k_0}$, hence $y \in \mathcal{C}^c$.

Therefore $\mathcal{C}^c$ is open, and consequently $\mathcal{C}$ is closed.

■

Proposition 4.6. *Cylinders generate the topology τ_d induced by d.*

Proof: Let G be a nonempty open set in $\mathcal{A}^{\mathbb{N}}$. We show that for every $x \in G$ there exists a cylinder $\mathcal{C} = [a_0,\ldots,a_n]$, with $n \in \mathbb{N}$, such that

$$x \in \mathcal{C} \subset G.$$

Let $x \in G$. Since G is open, there exists $r > 0$ such that $B(x,r) \subset G$. Choose $p \in \mathbb{N}$ such that $e^{-p} < r$. Then

$$B(x,e^{-p}) \subset B(x,r) \subset G.$$

Consider the cylinder $\mathcal{C} = [x_0,\ldots,x_p]$ and let $y \neq x$ be such that $y \in \mathcal{C}$. Then

$$\min\{i \in \mathbb{N} : x_i \neq y_i\} \geq p+1.$$

Hence,

$$d(x,y) = e^{-\min\{i\in\mathbb{N}:x_i\neq y_i\}} \leq e^{-(p+1)} \leq e^{-p} < r,$$

which implies $y \in B(x,r) \subset G$.

Therefore, $\mathcal{C} \subset G$ and $x \in \mathcal{C}$, proving the claim.

■

Proposition 4.7. *Every open set can be written as a countable union of cylinders.*

Proof: Let G be a nonempty open subset of $\mathcal{A}^{\mathbb{N}}$. By the previous proposition, for every $x \in G$ there exists a cylinder $\mathcal{C}_x$ such that $x \in \mathcal{C}_x \subset G$. Therefore,

$$G = \bigcup_{x \in G} \{x\} \subset \bigcup_{x \in G} \mathcal{C}_x \subset G,$$

and hence

$$G = \bigcup_{x \in G} \mathcal{C}_x.$$

On the other hand, the collection of all cylinders in $\mathcal{A}^{\mathbb{N}}$ can be written as

$$\mathcal{C} = \bigcup_{n \in \mathbb{N}} I_n, \quad \text{where} \quad I_n = \{[a_0, \ldots, a_n] \; ; \; a_i \in \mathcal{A}\}.$$

Since I_n is finite for every $n \in \mathbb{N}$, it follows that $\mathcal{C}$ is countable. Consequently, G is a countable union of cylinders.

■

Corollary 4.8. *The σ-algebra generated by the open sets (the Borel σ-algebra) coincides with the σ-algebra generated by the cylinders.*

Proposition 4.9. *The topology τ_d induced by the metric d coincides with the product topology of the discrete topologies on $\mathcal{A}$.*

Proof: Denote by τ the product topology of the discrete topologies on $\mathcal{A}$.

(i) $\tau_d \subset \tau$**.** Consider a cylinder $\mathcal{C} = [u_0, \ldots, u_n]$, which is a basic open set of τ_d. We can write

$$\mathcal{C} = \{u_0\} \times \cdots \times \{u_n\} \times \mathcal{A} \times \mathcal{A} \times \cdots .$$

Since each singleton $\{u_i\}$ is open in the discrete topology on $\mathcal{A}$, it follows that $\mathcal{C} \in \tau$.

(ii) $\tau \subset \tau_d$**.** Let $n \in \mathbb{N}$ and consider a basic open set of τ of the form

$$\mathcal{O}_n = \prod_{i \in \mathbb{N}} \mathcal{A}_i,$$

where

$$\mathcal{A}_i = \begin{cases} \mathcal{A}, & \text{if } i > n, \\ \mathcal{A}_i \subset \mathcal{A}, & \text{if } 0 \leq i \leq n. \end{cases}$$

Let $x = x_0 x_1 x_2 \cdots \in \mathcal{O}_n$. Then

$$x \in \mathcal{C} = [x_0, \ldots, x_n],$$

and clearly $\mathcal{C} \subset \mathcal{O}_n$. Since $\mathcal{C}$ is open in τ_d, it follows that $\mathcal{O}_n \in \tau_d$. Therefore, the two topologies coincide.

■

By Tychonoff's theorem, we deduce that

Proposition 4.10. $\mathcal{A}^{\mathbb{N}}$ *is a compact space.*

4.3 SHIFT MAP

We now define the map responsible for the dynamics that we will study on the space of infinite words.

Definition 4.11. *We call the* ***shift*** *(or* ***shift map****) the mapping* $S : \mathcal{A}^{\mathbb{N}} \longrightarrow \mathcal{A}^{\mathbb{N}}$ *defined by*

$$S(a_0 a_1 \cdots) = a_1 a_2 \cdots .$$

A word $w \in \mathcal{A}^{\mathbb{N}}$ is said to be **periodic under the shift** (or S**-periodic**) if there exists a positive integer k such that $S^k(w) = w$.

Let $w \in \mathcal{A}^{\mathbb{N}}$. The **orbit of** w **under** S is the set defined by

$$O(w) = \{S^n(w) : n \in \mathbb{N}\}.$$

Proposition 4.12. *The shift map* $S : \mathcal{A}^{\mathbb{N}} \longrightarrow \mathcal{A}^{\mathbb{N}}$ *is continuous.*

Proof: Let $x = x_0 x_1 \cdots$ and $y = y_0 y_1 \cdots$ be elements of $\mathcal{A}^{\mathbb{N}}$. If $x = y$, then clearly $d(x, y) = d(S(x), S(y)) = 0$.

Assume now that $x \neq y$ and that

$$d(x, y) = e^{-(n+1)}, \quad n > 1,$$

since we want x close to y. By definition of the metric, this means that

$$x_0 \cdots x_n = y_0 \cdots y_n \quad \text{and} \quad x_{n+1} \neq y_{n+1}.$$

Since

$$S(x) = x_1 x_2 \cdots \quad \text{and} \quad S(y) = y_1 y_2 \cdots ,$$

we have

$$x_1 \cdots x_n = y_1 \cdots y_n,$$

and therefore

$$d(S(x), S(y)) = e^{-n}.$$

Consequently,

$$d(S(x), S(y)) = e\, d(x, y).$$

Thus, S is continuous.

■

Remark 4.13. *S is surjective. Moreover if w is S-periodic, then the closure of the orbit of w under the shift is finite; that is, $\overline{O(w)}$ is finite.*

4.4 SYMBOLIC DYNAMICAL SYSTEMS

Let $\Omega_w = \overline{O(w)}$ denote the closure of the orbit of a word $w \in \mathcal{A}^{\mathbb{N}}$ under the shift map S. Then:

Proposition 4.14. $S(\Omega_w) \subset \Omega_w$.

Proof: Let $x \in \Omega_w$. Then there exists a sequence $(z_k)_{k\geq 0}$ of elements of $O(w)$ such that

$$x = \lim_{k\to\infty} z_k.$$

Since $z_k \in O(w)$ for each $k \geq 0$, we may write $z_k = S^{n_k}(w)$ for some $n_k \in \mathbb{N}$. Hence,

$$x = \lim_{k\to\infty} S^{n_k}(w).$$

By continuity of the shift map S, we obtain

$$S(x) = S\Big(\lim_{k\to\infty} S^{n_k}(w)\Big) = \lim_{k\to\infty} S^{n_k+1}(w).$$

Therefore, $S(x) \in \Omega_w$, which implies that $S(\Omega_w) \subset \Omega_w$.

■

The **symbolic dynamical system** generated by a word $w \in \mathcal{A}^{\mathbb{N}}$ is the pair (Ω_w, S). Since $\Omega_w = \overline{\{S^n(w)\ ;\ n \in \mathbb{N}\}}$, Ω_w is a closed subset of $\mathcal{A}^{\mathbb{N}}$. As $\mathcal{A}^{\mathbb{N}}$ is compact, it follows that Ω_w is compact.

4.4.1 Minimal Words

Before defining the important notion of minimal words, we present some results and concepts that will be of great help.

Proposition 4.15. *Let $u \in \mathcal{A}^{\mathbb{N}}$ be a word. Then*

$$\Omega_u = \{w \in \mathcal{A}^{\mathbb{N}}\ :\ \mathcal{L}(w) \subset \mathcal{L}(u)\}.$$

Proof: Let $w \in \Omega_u$. Then there exists a sequence $(n_k)_{k\geq 0}$ such that

$$w = \lim_{k\to\infty} S^{n_k}(u).$$

Hence, for every $n \in \mathbb{N}$, there exists $k_n \in \mathbb{N}$ such that

$$w_0 \cdots w_n = u_{k_n} \cdots u_{k_n+n},$$

and the sequence $(k_n)_{n\geq 0}$ can be chosen to be increasing.

Let $v = w_p \cdots w_{p+m} \in \mathcal{L}(w)$, with $p, m \in \mathbb{N}$. Taking $n > p+m$, we see that v is a factor of $w_0 \cdots w_n$. Since $w = \lim_{k\to\infty} S^{n_k}(u)$, it follows that $w_p \cdots w_{p+m}$ is a factor of u. Hence, $v \in \mathcal{L}(u)$, and therefore $\mathcal{L}(w) \subset \mathcal{L}(u)$. Thus,

$$\Omega_u \subset \{w \in \mathcal{A}^{\mathbb{N}} : \mathcal{L}(w) \subset \mathcal{L}(u)\}.$$

Conversely, let $w \in \mathcal{A}^{\mathbb{N}}$ be such that $\mathcal{L}(w) \subset \mathcal{L}(u)$. Since $w_0 \cdots w_n \in \mathcal{L}(w)$ for every $n \in \mathbb{N}$, there exists $p_n \in \mathbb{N}$ such that

$$w_0 \cdots w_n = u_{p_n} \cdots u_{p_n+n}.$$

Therefore, for each $n \in \mathbb{N}$, the word $S^{p_n}(u)$ coincides with w on $w_0 \cdots w_n$. In particular,

$$d\Big(S^{p_n}(u), w\Big) \leq e^{-(n+1)},$$

which implies that

$$w = \lim_{n\to\infty} S^{p_n}(u).$$

Since $(S^{p_n}(u))_{n\geq 0}$ is a sequence of elements of the orbit $O(u)$, we conclude that

$$w \in \overline{O(u)} = \Omega_u.$$

Thus,

$$\{w \in \mathcal{A}^{\mathbb{N}} : \mathcal{L}(w) \subset \mathcal{L}(u)\} \subset \Omega_u.$$

■

Definition 4.16. *Let $(w^{(n)})_{n\geq 0}$ be a sequence of elements of $\mathcal{A}^*$, and let $x \in \mathcal{A}$ and $w \in \mathcal{A}^{\mathbb{N}}$. We say that $w^{(n)}$* ***converges*** *to w, and we write $\lim\limits_{n\to\infty} w^{(n)} = w$, if the sequence $\left(w^{(n)}xxx\cdots\right)_{n\geq 0}$ of elements of $\mathcal{A}^{\mathbb{N}}$ converges to w with respect to the metric d.*

Example 4.17. *The sequence $w^{(n)} = u_1 \cdots u_{2n} = 1212\cdots 12$ converges to $w = 1212\cdots$.*

Remark 4.18. *We define $\mathcal{A}^{\mathbb{Z}}$ as the set of bi-infinite words over $\mathcal{A}$, that is,*

$$(x_i)_{i\in\mathbb{Z}} = \cdots x_{-2}x_{-1}.x_0x_1x_2\cdots.$$

In an analogous way, we define a metric

$$d : \mathcal{A}^{\mathbb{Z}} \times \mathcal{A}^{\mathbb{Z}} \longrightarrow \mathbb{R}_+$$

by

$$d(u,v) = \begin{cases} e^{-\min\{\,n\in\mathbb{N}: u_{[-n,n]}\neq v_{[-n,n]}\,\}}, & \text{if } u \neq v, \\ 0, & \text{if } u = v, \end{cases}$$

where $u_{[-n,n]}$ denotes the cylinder

$$[u_{-n}\cdots u_{-1}.u_0\cdots u_n].$$

Considering the cylinders

$$[a_{-n}\cdots a_{-1}.a_0\cdots a_p], \quad n, p \in \mathbb{N},$$

one can prove the same properties as in the one-sided case $\mathcal{A}^{\mathbb{N}}$. The main advantage of working in $\mathcal{A}^{\mathbb{Z}}$ is that the shift map $S\big((x_i)_{i\in\mathbb{Z}}\big) = (x_{i+1})_{i\in\mathbb{Z}}$ is a homeomorphism on $\mathcal{A}^{\mathbb{Z}}$. Moreover, in this setting the converse of Proposition 4.13 holds.

The following definition is extremely important, as it influences several key features of symbolic dynamical systems and is closely related to the notions of minimality and unique ergodicity introduced earlier.

Definition 4.19. *An infinite word u is said to be* ***uniformly recurrent*** *if every finite word occurring in u occurs infinitely many times with bounded gaps. That is, for every factor w of u, there exists $s \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, the word w appears as a factor of $u_nu_{n+1}\cdots u_{n+s-1}$.*

Lemma 4.20. *If u is uniformly recurrent and $w \in \Omega_u$, then $u \in \Omega_w$.*

Proof: Since u is uniformly recurrent, for every $p \in \mathbb{N}$ the word $u_0 \cdots u_p$ occurs infinitely many times in u with bounded gaps. Let $w \in \Omega_u$. By Proposition 4.15, we have $\mathcal{L}(w) \subset \mathcal{L}(u)$.

Hence, for every $n \in \mathbb{N}$ there exists $k_n \in \mathbb{N}$ such that

$$w_0 \cdots w_n = u_{k_n} \cdots u_{k_n+n}.$$

Taking n sufficiently large, we see that $u_0 \cdots u_p$ is a factor of

$$u_{k_n} \cdots u_{k_n+n} = w_0 \cdots w_n \in \mathcal{L}(w),$$

for every $p \in \mathbb{N}$. Therefore, $\mathcal{L}(u) \subset \mathcal{L}(w)$, which implies that $u \in \Omega_w$.

■

Proposition 4.21. *The dynamical system (Ω_u, S) is minimal if and only if the word u is uniformly recurrent.*

Proof: Assume that u is uniformly recurrent. Let us show that $\Omega_w = \Omega_u$.

By the previous lemma, $u \in \Omega_w$. Since $S(\Omega_w) \subset \Omega_w$ and $u \in \Omega_w$, it follows that $S(u) \in \Omega_w$, and hence $S^n(u) \in \Omega_w$, for all $n \in \mathbb{N}$. Therefore,

$$\Omega_u = \overline{\{S^n(u)\ ;\ n \geq 0\}} \subset \overline{\Omega_w} = \Omega_w.$$

On the other hand, since $w \in \Omega_u$, we have

$$S(w) \in S(\Omega_u) = \Omega_u,$$

and hence $O(w) \subset \Omega_u$. Taking closures, we obtain

$$\Omega_w = \overline{O(w)} \subset \overline{\Omega_u} = \Omega_u.$$

which proves the desired result.

Conversely, assume that the system (Ω_u, S) is minimal. Then, for every $w \in \Omega_u$, we have $\Omega_w = \Omega_u$. In particular, $u \in \Omega_w$, which implies

$$\mathcal{L}_n(u) \subset \mathcal{L}_n(w) \quad \text{for all } n \geq 1.$$

Claim 1. Let W be a finite factor of u. Then W appears in every $w \in \Omega_u$.
Proof of Claim 1. Since W appears in u, we may write

$$W = u_i \cdots u_{i+n-1} \in \mathcal{L}_n(u)$$

for some $i \in \mathbb{N}$. As $\mathcal{L}_n(u) \subset \mathcal{L}_n(w)$, it follows that $W \in \mathcal{L}_n(w)$, that is, W is a factor of w for every $w \in \Omega_u$. □

Claim 2.

$$\Omega_u = \bigcup_{n=0}^{\infty} S^{-n}([W]).$$

Proof of Claim 2. Since $S : \Omega_u \to \Omega_u$, we clearly have

$$\bigcup_{n=0}^{\infty} S^{-n}([W]) \subset \Omega_u.$$

Conversely, let $w \in \Omega_u$. By Claim 1, the word W appears in w, so there exist $n, p \in \mathbb{N}$ such that

$$W = w_n \cdots w_{n+p}.$$

Thus,

$$S^n(w) = w_n w_{n+1} \cdots \in [W],$$

which implies $w \in S^{-n}([W])$. Hence,

$$\Omega_u \subset \bigcup_{n=0}^{\infty} S^{-n}([W]).$$

□

Since Ω_u is compact and $\bigcup_{n=0}^{\infty} S^{-n}([W])$ is an open cover of Ω_u, there exist $n_0, \ldots, n_p$ such that

$$\Omega_u = \bigcup_{i=0}^{p} S^{-n_i}([W]).$$

Therefore, for every $k \geq 0$, we have

$$S^k(u) \in \bigcup_{i=0}^{p} S^{-n_i}([W]),$$

and hence there exists n_{i_k} such that

$$S^{n_{i_k}+k}(u) \in [W].$$

This shows that W appears infinitely many times in u. Moreover, the gaps are bounded, since

$$(n_{i_k} + k) - (n_{i_{k-1}} + k - 1) = n_{i_k} - n_{i_{k-1}} + 1 \leq \max_{i=0,\ldots,p} n_i + 1.$$

Thus, u is uniformly recurrent, that is, minimal.

■

4.5 SUBSTITUTIONS

A **substitution** is a map $\sigma : \mathcal{A} \longrightarrow \mathcal{A}^*$ that assigns to each letter $a \in \mathcal{A}$ a nonempty finite word $\sigma(a) \in \mathcal{A}^*$. A substitution σ can be extended to the set of finite (respectively, infinite) words by concatenation, that is,

$$\sigma(w_0 w_1 \cdots w_n) = \sigma(w_0)\sigma(w_1) \cdots \sigma(w_n)$$

(respectively,

$$\sigma(w_0 w_1 \cdots) = \sigma(w_0)\sigma(w_1) \cdots).$$

In this section, we consider only substitutions satisfying the property that

$$\lim_{n\to\infty} |\sigma^n(a)| = +\infty \quad \text{for every } a \in \mathcal{A},$$

where $|\sigma^n(a)|$ denotes the length of the n-th iterate of a under σ.

Examples 4.22. *Consider the alphabet* $\mathcal{A} = \{1, 2\}$*. The* ***Fibonacci substitution*** σ *is defined by*

$$\sigma(1) = 12 \quad \textit{and} \quad \sigma(2) = 1,$$

and the ***Morse substitution*** τ *is defined by*

$$\tau(1) = 12 \quad \textit{and} \quad \tau(2) = 21.$$

Let $w \in \mathcal{A}^{\mathbb{N}}$ be a word. We say that w is a **periodic point** of the substitution σ if there exists $k \in \mathbb{N}^*$ such that $\sigma^k(w) = w$, and we say that w is a **fixed point** of the substitution σ if $\sigma(w) = w$.

Example 4.23. *Let* σ *be the Fibonacci substitution, defined by*

$$\sigma(1) = 12 \quad \textit{and} \quad \sigma(2) = 1.$$

For every $n \in \mathbb{N}$*, we have*

$$\sigma^{n+1}(1) = \sigma^n(\sigma(1)) = \sigma^n(1)\,\sigma^n(2).$$

In particular, the word $\sigma^n(1)$ *is a prefix of* $\sigma^{n+1}(1)$ *for all* $n \in \mathbb{N}$*. Hence, the sequence* $(\sigma^n(1))_{n\geq 0}$ *converges to an infinite word* $w \in \mathcal{A}^{\mathbb{N}}$*.*

This limit satisfies $\sigma(w) = w$*, so* w *is a fixed point of the substitution* σ*. This point is given by* $w = 12112\cdots$*.*

Example 4.24. *Let* σ *be the substitution defined by*

$$\sigma(1) = 21 \quad \textit{and} \quad \sigma(2) = 12.$$

We have

$$u = 1221\cdots = \lim_{n\to\infty} \sigma^{2n}(1) \quad \textit{and} \quad v = 2112\cdots = \lim_{n\to\infty} \sigma^{2n}(2),$$

and both u *and* v *are periodic points of the substitution. Indeed,*

$$\sigma^2(u) = u \quad \textit{and} \quad \sigma^2(v) = v.$$

Proposition 4.25. *Every substitution* $\sigma : \mathcal{A}^{\mathbb{N}} \longrightarrow \mathcal{A}^{\mathbb{N}}$ *is a continuous map.*

Proof: Let $x = (x_i)_{i\in\mathbb{N}}$ and $y = (y_i)_{i\in\mathbb{N}}$ be elements of $\mathcal{A}^{\mathbb{N}}$, and let σ be a substitution. Suppose that $x \neq y$ and that $d(x, y) = e^{-n},\ n \in \mathbb{N}^*$. Then

$$x_0x_1\cdots x_{n-1} = y_0y_1\cdots y_{n-1}.$$

Write

$$\sigma(x) = x'_0x'_1\cdots \quad \text{and} \quad \sigma(y) = y'_0y'_1\cdots .$$

It follows that

$$\sigma(x_0)\sigma(x_1)\cdots\sigma(x_{n-1}) = \sigma(y_0)\sigma(y_1)\cdots\sigma(y_{n-1}).$$

Since $|\sigma(a)| \geq 1$ for all $a \in \mathcal{A}$, we have

$$\Big|\sigma(x_0)\cdots\sigma(x_{n-1})\Big| \geq n.$$

Therefore, the first position at which $\sigma(x)$ and $\sigma(y)$ may differ satisfies

$$\min\{\, i \in \mathbb{N} : x'_i \neq y'_i \,\} \geq n + 1.$$

Consequently,

$$d(\sigma(x), \sigma(y)) = e^{-\min\{\, i : x'_i \neq y'_i \,\}} \leq e^{-(n+1)} = e^{-1}e^{-n} = e^{-1}d(x, y).$$

Thus, σ continuous.

■

The following result characterizes periodic points.

Proposition 4.26. *Let $u \in \mathcal{A}^{\mathbb{N}}$. The word u is periodic for the substitution σ if and only if*

$$u = \lim_{n\to\infty} \sigma^{nk}(a),$$

where $a \in \mathcal{A}$, $k \in \mathbb{N}$, and $\sigma^k(a)$ begins with the letter a.

Proof: Let $u = u_0u_1\cdots$ be a periodic point of the substitution σ. Then there exists $k \in \mathbb{N}^*$ such that $\sigma^k(u) = u$. In particular, there exists a finite word $v \in \mathcal{A}^*$ such that $\sigma^k(u_0) = u_0v$.

We claim that $u = \lim_{n\to\infty} \sigma^{nk}(u_0)$.

Indeed, for every $n \in \mathbb{N}^*$ we have

$$\sigma^{nk}(u) = \sigma^{(n-1)k}(\sigma^k(u)) = \sigma^{(n-1)k}(u) = \cdots = \sigma^k(u) = u.$$

Thus,

$$u = \sigma^{nk}(u) = \sigma^{nk}(u_0)\,\sigma^{nk}(u_1)\cdots.$$

Since, by assumption,

$$\lim_{m\to\infty} |\sigma^m(a)| = \infty \quad \text{for all } a \in \mathcal{A},$$

it follows in particular that

$$\lim_{n\to\infty} |\sigma^{nk}(u_0)| = \infty.$$

Therefore, the sequence $(\sigma^{nk}(u_0))_{n\geq 1}$ converges to u, that is,

$$u = \lim_{n\to\infty} \sigma^{nk}(u_0).$$

Conversely, suppose that

$$u = \lim_{n\to\infty} \sigma^{nk}(a),$$

where $a \in \mathcal{A}$ and $\sigma^k(a)$ begins with the letter a. Then, by continuity of σ,

$$\sigma^k(u) = \sigma^k\left(\lim_{n\to\infty} \sigma^{nk}(a)\right) = \lim_{n\to\infty} \sigma^{(n+1)k}(a) = u.$$

Hence, $\sigma^k(u) = u$, and therefore u is a periodic point of σ.

■

Example 4.27. *The example 4.24 shows two points of period* 2.

Proposition 4.28. *Every substitution has at least one periodic point.*

Proof: Suppose, by contradiction, that σ has no periodic point. By proposition 4.26, this implies that for every $a \in \mathcal{A}$ and every $k \in \mathbb{N}^*$, the word $\sigma^k(a)$ does not start with a.

Fix a letter $a \in \mathcal{A}$ and an integer $k \in \mathbb{N}^*$. For each $n \in \mathbb{N}$, write

$$\sigma^{nk}(a) = x_nv_n,$$

where $x_n \in \mathcal{A}$ is the first letter of $\sigma^{nk}(a)$ and $v_n \in \mathcal{A}^*$. By assumption, we have $x_n \neq a$ for all $n \in \mathbb{N}$.

Since the alphabet $\mathcal{A}$ is finite, there exist integers $n < m$ such that $x_n = x_m$. On the other hand,

$$\sigma^{mk}(a) = \sigma^{(m-n)k}(\sigma^{nk}(a)) = \sigma^{(m-n)k}(x_n v_n).$$

Therefore, the first letter of $\sigma^{mk}(a)$ is the first letter of $\sigma^{(m-n)k}(x_n)$. Since $x_n = x_m$, this implies that $\sigma^{(m-n)k}(x_n)$ begins with x_n, which contradicts the assumption that no iterate of σ maps a letter to a word beginning with itself.

This contradiction shows that σ must admit a periodic point.

■

4.6 DYNAMICAL SYSTEMS ASSOCIATED WITH SUBSTITUTIONS

In this section, we study the dynamical system associated with a primitive substitution. In particular, we show that this system is minimal and does not depend on the choice of the periodic point.

4.6.1 Results on Primitivity

Definition 4.29. *A substitution σ defined on an alphabet $\mathcal{A}$ is said to be **primitive** if there exists an integer $k > 0$ such that, for every $a, b \in \mathcal{A}$, the letter a occurs in $\sigma^k(b)$.*

Example 4.30. *In the Fibonacci substitution, we have that* 1 *and* 2 *appear in $\sigma^2(1)$ and in $\sigma^2(2)$. Therefore, this substitution is primitive, satisfying the definition of primitivity for $k = 2$.*

Let us now consider some results involving primitivity.

Proposition 4.31. *If σ is a primitive substitution, then there exists $N \in \mathbb{N}$ such that for all $k \geq N$ and for all $a, b \in \mathcal{A}$, a appears in $\sigma^k(b)$.*

Proof: Since σ is primitive, there exists $N \in \mathbb{N}$ such that for all $a, b \in \mathcal{A}$, a appears in $\sigma^N(b)$. Let $k \geq N$; then there exists $p \geq 0$ such that $k = N+p$. Hence, $\sigma^k(b) = \sigma^N(\sigma^p(b))$.

Let $c \in \mathcal{A}$ be such that c appears in $\sigma^p(b)$. Then $\sigma^N(c)$ appears in $\sigma^N(\sigma^p(b))$. By the initial hypothesis, we have that a appears in $\sigma^N(c)$. Therefore, a appears in $\sigma^N(\sigma^p(b)) = \sigma^k(b)$.

■

Proposition 4.32. *If σ is primitive, then any of its periodic points is minimal.*

Proof: Let $u = u_0u_1u_2\cdots$ be a periodic point of σ. Then there exists $k \in \mathbb{N}^*$ such that $u = \sigma^k(u)$. Since σ is primitive, by the previous proposition there exists $N \in \mathbb{N}$ such that for all $k \geq N$ and for all $a, b \in \mathcal{A}$, a appears in $\sigma^k(b)$. Hence, for every $i \in \mathbb{N}$ we have that u_0 appears in $\sigma^{kN}(u_i)$.

We know that

$$u = \sigma^{kN}(u) = \sigma^{kN}(u_0)\sigma^{kN}(u_1)\cdots,$$

and thus u_0 appears infinitely many times in u. Let

$$M = \max\{|\sigma^{kN}(a)| : a \in \mathcal{A}\}.$$

Then u_0 appears infinitely many times with gaps bounded by $2M$.

Let V be a finite word that appears in u. Since

$$u = \lim_{n\to\infty} \sigma^{nk}(u_0),$$

there exists n sufficiently large such that V appears in $\sigma^{nk}(u_0)$. Then we have

$$u = \sigma^{(n+N)k}(u) = \sigma^{(n+N)k}(u_0)\sigma^{(n+N)k}(u_1)\cdots = \sigma^{nk}(\sigma^{Nk}(u_0))\sigma^{nk}(\sigma^{Nk}(u_1))\cdots.$$

Since u_0 appears in every $\sigma^{Nk}(u_i)$, it follows that $\sigma^{nk}(u_0)$ appears infinitely many times in u, and therefore so does V, with gaps bounded by $2M'$, where

$$M' = \max\{|\sigma^{nk}(a)| : a \in \mathcal{A}\}.$$

Therefore, u is minimal.

■

Proposition 4.33. *If σ is a primitive substitution and u and v are periodic points of σ, then $\mathcal{L}(u) = \mathcal{L}(v)$.*

Proof: Since u and v are periodic points of the substitution σ, there exist $k, k' \in \mathbb{N}^*$ such that

$$u = \sigma^k(u), \qquad v = \sigma^{k'}(v).$$

Let $m = kk'$. Then

$$\sigma^m(u) = \sigma^{kk'}(u) = \sigma^{(k'-1)k}\sigma^k(u) = \cdots = \sigma^k(u) = u.$$

In the same way, $\sigma^m(v) = v$. Let us denote $\tau = \sigma^m$; then $\tau(u) = u$ and $\tau(v) = v$. We also know that

$$u = \lim_{n\to\infty} \tau^n(a),$$

where a is a letter such that $\tau(a)$ begins with a, and

$$v = \lim_{n\to\infty} \tau^n(b),$$

where b is a letter such that $\tau(b)$ begins with b.

We show that $\mathcal{L}(u) \subset \mathcal{L}(v)$. Let $W \in \mathcal{L}(u)$. Since $u = \lim_{n\to\infty} \tau^n(a)$, there exists N sufficiently large such that W appears in

$$\tau^n(a) = \sigma^{mn}(a)$$

for all $n \geq N$. Since σ is primitive, there exists $p \in \mathbb{N}^*$ such that a appears in $\sigma^p(b)$. By Proposition 4.31, a appears in $\sigma^{pm}(b)$. Hence, $\sigma^{mn}(a)$ appears in

$$\sigma^{mn+pm}(b) = \sigma^{m(n+p)}(b).$$

Therefore, W appears in

$$\sigma^{m(n+p)}(b) = \tau^{n+p}(b),$$

that is, $W \in \mathcal{L}(v)$.

Analogously, one shows that $\mathcal{L}(v) \subset \mathcal{L}(u)$. Hence,

$$\mathcal{L}(u) = \mathcal{L}(v).$$

■

The following theorem is one of the most important results of this chapter.

Theorem 4.34. *If σ is primitive and u is a periodic point of σ, then the system (Ω_u, S) is minimal and does not depend on the periodic point u, that is, if v is also a periodic point, then $\Omega_u = \Omega_v$.*

Proof: Minimality follows from Propositions 4.21 and 4.32, and the fact that (Ω_u, S) does not depend on u follows from Propositions 4.15 and 4.33.

■

Remark 4.35. *From now on, if σ is primitive, we will denote Ω_u by Ω.*

Proposition 4.36. *If σ is primitive, then $\Omega = \{u \in \mathcal{A}^{\mathbb{N}}$, each factor of u occurs in $\sigma^n(a)$ for some $n \in \mathbb{N},\ a \in \mathcal{A}\}$.*

4.6.2 Substitution Matrix

The concepts defined in this section are of extreme importance and lie at the core of the tools that will be used in the proof of the main result of this work. By associating matrices with substitutions, we incorporate into our theory several well-known results from Linear Algebra, such as the Perron–Frobenius Theorem.

Let σ be a substitution defined on the alphabet $\mathcal{A} = \{0, 1, \ldots, d-1\}$. If B and C are two words in $\mathcal{A}^*$, we denote by $L_C(B)$ the number of occurrences of C in B. In particular, if $i \in \mathcal{A}$, $L_i(B)$ denotes the number of occurrences of i in B. We call the **σ-matrix**, and denote it by $M = M_\sigma$, the matrix whose entries are

$$\ell_{ij} = L_i(\sigma(j)), \qquad i, j \in \mathcal{A}.$$

The matrix M is an $d \times d$ positive matrix (with nonnegative entries, not all zero), whose entries are integers greater than or equal to 0.

Example 4.37. *If σ is the Fibonacci substitution, then*

$$M_\sigma = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

One of the advantages of working with substitution matrices is due to the following propositions:

Proposition 4.38. *For all $a \in \mathcal{A}$ and $n \in \mathbb{N}$,*

$$L(\sigma^n(a)) = M^n L(a),$$

where for every $B \in \mathcal{A}^*$ we define

$$L(B) = \big(L_0(B), \ldots, L_{d-1}(B)\big)^t.$$

Proof: The proof will be by induction on n.

For the case $n = 1$, we must show that $L(\sigma(a)) = M(L(a))$, where

$$M = M_\sigma = \begin{pmatrix} \ell_{00} & \cdots & \ell_{0(d-1)} \\ \vdots & \ddots & \vdots \\ \ell_{(d-1)0} & \cdots & \ell_{(d-1)(d-1)} \end{pmatrix}, \qquad \ell_{ij} = L_i(\sigma(j)), \quad 0 \le i, j \le d-1.$$

The vector $L(a)$ is the s-dimensional vector with all coordinates equal to zero, except at position $(a+1)$, where it is equal to 1. Therefore, $M(L(a))$ is the column vector

$$\left(L_{0a}, \dots, L_{(d-1)a}\right)^t,$$

which is exactly the vector $L(\sigma(a))$.

Now suppose that $L(\sigma^{n-1}(a)) = M^{n-1}(L(a))$. Then

$$L(\sigma^n(a)) = L\Big(\sigma^{n-1}(\sigma(a))\Big) = M^{n-1}\Big(L(\sigma(a))\Big) = M^{n-1}\Big(M(L(a))\Big) = M^n(L(a)).$$

■

Proposition 4.39. *σ is primitive if and only if M_σ is primitive, that is, there exists $k > 0$ such that all entries of the matrix M_σ^k are strictly positive.*

Proof: σ is primitive if and only if there exists $k > 0$ such that for all $a, b \in \mathcal{A}$, a appears in $\sigma^k(b)$, which is equivalent to saying that the s-dimensional vector

$$L(\sigma^k(a)) = \Big(L_0(\sigma^k(a)), \dots, L_{s-1}(\sigma^k(a))\Big)$$

has all coordinates positive for every $a \in \mathcal{A}$. But the s vectors $L(\sigma^k(a))$ are exactly the s columns of the matrix M_σ^k, since by the previous proposition,

$$L(\sigma^n(a)) = M^n(L(a)) \quad \text{for all } n \in \mathbb{N}.$$

Therefore, this is equivalent to saying that there exists $k > 0$ such that the matrix M_σ^k has all entries strictly positive, that is, M_σ is primitive.

■

One of the most important results applied in Ergodic Theory and Dynamical Systems is the well-known Perron–Frobenius Theorem. Proved by Oskar Perron and Ferdinand Georg Frobenius, this result also proves useful in the areas of probability, economics, and even demography. In this work, it is heavily used in the mentioned articles and, in particular, in the main article in Chapter 5.

Theorem 4.40. ***(Perron–Frobenius Theorem)*** *Let M be a $d \times d$ nonnegative and primitive matrix. Then:*

(i) There exists an eigenvalue $\theta > 0$ such that $\theta > |\lambda|$ for every eigenvalue λ of M;

(ii) The eigenvalue θ has algebraic multiplicity 1;

(iii) There exist left and right eigenvectors l and r such that

$$Mr = \theta r \quad \text{and} \quad lM = \theta l.$$

Moreover, l and r are unique up to multiplication by a constant and their coordinates are positive.

4.7 RAUZY FRACTAL

The Rauzy fractal was discovered by G. Rauzy in 1981 and has since been studied by many mathematicians across different research centers around the world. In addition to providing a geometric representation of dynamical systems, the Rauzy fractal induces a periodic tiling of the space $\mathbb{R}^{d-1}$ and has a boundary that is a fractal set. In this section, we shall define it and understand a few of its relations with substitutions, based on the work presented in [8].

4.7.1 Tribonacci Substitution

We previously defined the Fibonacci substitution

$$\tau(1) = 12 \quad \text{and} \quad \tau(2) = 1,$$

on the alphabet $\mathcal{A} = \{1, 2\}$, and we verified that its associated matrix is

$$M_\tau = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

It is possible to observe that the eigenvalues of this matrix are

$$\phi = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad -\frac{1}{\phi} = \frac{1-\sqrt{5}}{2},$$

which is one of the reasons why this substitution has this name.

G. Rauzy generalized in [9] the dynamical properties of the Fibonacci substitution to a tree-letter alphabet substitution, called Tribonacci substitution or Rauzy substitution. Let $\mathcal{A} = \{1, 2, 3\}$, we define

$$\sigma(1) = 12, \quad \sigma(2) = 13, \quad \sigma(3) = 1.$$

The associated matrix is

$$M_\sigma = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Its characteristic polynomial is $x^3 - x^2 - x - 1$, and its roots consist of a real number $\beta > 1$ and two complex conjugates α and $\overline{\alpha}$. In reference to the Fibonacci equation $x^2 - x - 1 = 0$, α is called a Tribonacci number.

In particular, the matrix M_σ admits as eigenspaces in $\mathbb{R}^3$ an expanding one-dimensional direction and a contracting plane.

4.7.2 Geometric construction of the Rauzy Fractal

Let u denote a bi-infinite word which is a periodic point of σ, for example,

$$u = \lim_{n\to\infty} \sigma^{3n}(1) \,.\, \sigma^n(1).$$

This infinite word is embedded as a broken line in $\mathbb{R}^3$ by replacing each letter in the word by the corresponding vector of the canonical basis (e_1, e_2, e_3) of $\mathbb{R}^3$.

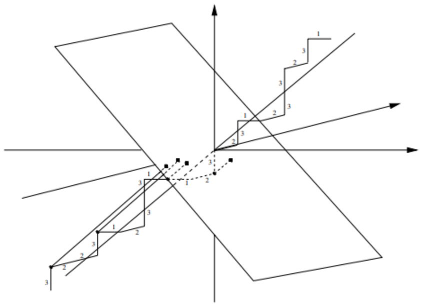

Figura 4.1: Broken line
Source: Author's own work

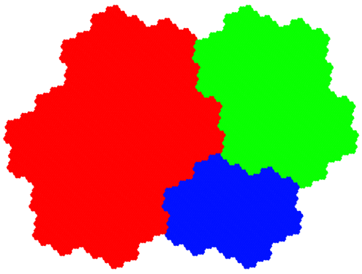

Figura 4.2: Rauzy Fractal
Source: Author's own work

An interesting property of this broken line is that it remains at a bounded distance from the expanding direction of M_σ, turning around this line. When one projects the vertices of the broken line onto the contracting plane, parallel to the expanding direction, one obtains a bounded set in a two-dimensional vector space. The closure of this set is a compact set denoted by $\mathcal{R}$ and called the **Rauzy fractal**.

To be more precise, denote by π the linear projection in $\mathbb{R}^3$, parallel to the expanding direction, on the contracting plane, identified with the complex plane $\mathbb{C}$. If $u = (u_i)_{i\in\mathbb{Z}}$ is the periodic point of the substitution, then the Rauzy Fractal is

$$\mathcal{R} = \overline{\left\{\pi\left(\sum_{i=0}^{n} e_{u_i}\right); n \in \mathbb{Z}\right\}}.$$

The figure above presents the broken line and the Rauzy Fractal, respectively.

5 PERSISTENCE OF WANDERING INTERVALS IN AFFINE INTERVAL EXCHANGE TRANSFORMATIONS

In this chapter, we will develop a thorough understanding of the article [2], which generalizes the work studied in the previous chapter.

The proof of the main theorem is given by a construction based on several results and requires careful, highly constructive work. We divide the proof into sections that facilitate and organize the understanding of each of its details.

5.1 SUBSTITUTIONS ASSOCIATED WITH RAUZY INDUCTION

In Section 3.3 we studied the mechanism of Rauzy induction defined for interval exchange transformations. It is possible to associate to each step of Rauzy induction a substitution defined on an alphabet whose cardinality equals the number of subintervals of the space.

Let $T(\lambda, \pi)$ be an IET on an interval X, and let (λ', π') be the IET obtained from $\widehat{R}(T)$ on the interval X'. The substitution obtained by coding, under T, the itinerary of the intervals of π'_1 until they return to X' is called the **substitution associated with** $\widehat{R}(T)$.

We now present an example for a type 0 IET (the same idea applies to type 1).

Example 5.1. *Consider*

$$\pi = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix},$$

with $\lambda_3 > \lambda_1$*. Applying the Rauzy induction, we obtain*

$$\pi' = \begin{pmatrix} 1' & 2' & 3' \\ 3' & 1' & 2' \end{pmatrix}.$$

Now observe that, with respect to (λ, π)*:*

(a) $1'$ *starts in* 1*, moves to* 3*, and then reaches* X'*. Hence, its itinerary is* $1 \to 3$*.*

(b) $2'$ *starts in* 2 *and goes directly to* X'*. Thus, its itinerary is simply* 2*.*

(c) $3'$ starts in 3 and also goes directly to X'. Hence, its itinerary is 3.

Therefore, we define the substitution σ associated with $\widehat{R}(T)$ by

$$\sigma(1) = 13, \qquad \sigma(2) = 2, \qquad \sigma(3) = 3.$$

Remark 5.2. *Observe that we then have the following cases:*

(a) If (λ, π) is of type 0, then

$$\sigma(a) = a \quad \text{for all } a \in \mathcal{A},\ a \neq \alpha(1), \qquad \text{and} \qquad \sigma(\alpha(1)) = \alpha(1)\alpha(0).$$

(b) If (λ, π) is of type 1, then

$$\sigma(a) = a \quad \text{for all } a \in \mathcal{A},\ a \neq \alpha(0), \qquad \text{and} \qquad \sigma(\alpha(0)) = \alpha(1)\alpha(0).$$

In the same section, we also defined the Rauzy diagrams. By following a path in the diagram, we can compose the substitutions obtained at each step, thereby obtaining a new substitution corresponding to the entire path. Let us use the example given in 3.10 to illustrate this concept.

Example 5.3. *As mentioned before, the diagram below contains all irreducible interval exchange transformations on three intervals.*

$$1 \circlearrowright \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \underset{0}{\overset{0}{\rightleftarrows}} \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \underset{1}{\overset{1}{\rightleftarrows}} \begin{pmatrix} 1 & 3 & 2 \\ 3 & 2 & 1 \end{pmatrix} \circlearrowleft 0.$$

Consider the loop 10100 that starts at the center of the graph above. By composing the substitutions along this loop, we obtain a substitution σ given by

$$\sigma(1) = 1213, \qquad \sigma(2) = 12213, \qquad \sigma(3) = 13,$$

and its associated matrix

$$M_\sigma = \begin{pmatrix} 2 & 2 & 1 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix}.$$

With these concepts understood, the next definition is of fundamental importance for the main theorem of the article.

Definition 5.4. *We say that an IET $T(\lambda, \pi)$ is **self-similar** if there exists a finite loop in the Rauzy diagram of T such that λ is an eigenvector of the Perron-Frobenius matrix associated to the corresponding substitution.*

Example 5.5. *Considering*

$$\pi = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

and the loop from Example 5.3, the vector

$$\lambda = (\sqrt{3}, 1, 1)$$

is an eigenvector of M_σ. Thus, the IET $T(\lambda, \pi)$ is self-similar.

Remark 5.6. *Observe that the definition of self-similarity is equivalent to saying that an interval exchange transformation returns to the original transformation after finitely many steps of Rauzy induction, with the same proportions among the subintervals.*

Now we have all the information necessary to understand the main result of [2].

Theorem 5.7. ***(Bressaud-Hubert-Maass)*** *Let $T(\lambda, \pi)$ be a self-similar interval exchange transformation and R the associated matrix obtained by Rauzy induction. Let θ_1 be the Perron-Frobenius eigenvalue of R. Assume that R has an eigenvalue θ_2 such that:*

1) θ_2 *is a conjugate of* θ_1*;*

2) θ_2 *is a real number;*

3) $1 < \theta_2 (< \theta_1)$.

Then there exists an affine interval exchange transformation f with wandering intervals that is semi-conjugated to $T(\lambda, \pi)$.

5.2 PRELIMINARIES

5.2.1 Words and Sequences

We will consider all the notions introduced in Chapter 4, with the addition and modification of some definitions.

Definition 5.8. *Let $\mathcal{A}$ be an alphabet.*

*(a) We denote by ε the **empty word**, that is, a word with no symbols ($|\varepsilon| = 0$).*

(b) From now on, we let $\mathcal{A}^*$ denote the set of finite words over the alphabet $\mathcal{A}$, and we define $\mathcal{A}^+ = \mathcal{A}^* \setminus \{\varepsilon\}$.

(c) We also consider finite words indexed by integers, that is,

$$w = w_{-m} \cdots w_{-1} \cdot w_0 \cdots w_\ell,$$

*where $\ell, m \in \mathbb{N}$ and the dot separates negative and non-negative coordinates. We often call such words **dotted words**.*

*(d) A point $x \in \mathcal{A}^{\mathbb{N}}$ is said to be **ultimately periodic** if it can be written as*

$$x = uvvv \cdots$$

*with $u, v \in \mathcal{A}^*$. The word v is called a **generating word** of x, its length $|v|$ is called a **period** of x, and $|u|$ is called the **preperiod**. If u is the empty word, then x is said to be **periodic**.*

(e) Given a word $x \in \mathcal{A}^+$, $\mathcal{A}^{\mathbb{N}}$, or $\mathcal{A}^{\mathbb{Z}}$, we denote by $x[i, j]$ the subword of x appearing between indices i and j. Similarly, we define $x(-\infty, i]$ and $x[i, \infty)$.

*(f) A **subshift** is any shift invariant and closed (for the product topology) subset of $\mathcal{A}^{\mathbb{Z}}$ or $\mathcal{A}^{\mathbb{N}}$. A subshift is **minimal** if all of its orbits by the shift are dense.*

In what follows, we use the shift map in several contexts, in particular, restricted to a subshift. To simplify notation, we denote it by S throughout.

5.2.2 Substitutions and Minimal Points

The definitions previously studied in connection with substitutions remain valid, with the addition of a few details.

We conventionally set $\sigma(\varepsilon) = \varepsilon$ for every substitution σ.

Since there is no standard convention for the definition of the matrix associated with a substitution, it is common to find references that adopt the transpose of the matrix previously defined. This will be the case in this chapter. Namely, given a substitution σ, we define the matrix $M_\sigma = (M_\sigma(a,b))_{a,b\in\mathcal{A}}$, where $M_\sigma(a,b)$ is the number of times the letter b appears in $\sigma(a)$.

Let $X_\sigma \subset \mathcal{A}^{\mathbb{Z}}$ be the **subshift defined by** σ. That is, a point $x \in X_\sigma$ if and only if every subword of x is a subword of $\sigma^N(a)$ for some $N \in \mathbb{N}$ and some $a \in \mathcal{A}$.

Under primitivity, one can assume without loss of generality that $M = M_\sigma > 0$.

Let θ_1 be the Perron–Frobenius eigenvalue of M. Let

$$\lambda = (\lambda(a) : a \in \mathcal{A})^{\mathsf{t}}$$

be a strictly positive right eigenvector of M associated with θ_1. We also assume the following algebraic property, which we call **(AH)**: the matrix M has an eigenvalue θ_2 that is a conjugate of θ_1. Note that this property coincides with Hypothesis (1) of the main theorem.

The following lemmas are important consequences of the algebraic property **(AH)**.

Lemma 5.9. *Let $\eta\colon \mathbb{Q}[\theta_1] \to \mathbb{Q}[\theta_2]$ be the field homomorphism that sends θ_1 to θ_2. The vector*

$$\gamma = \eta(\lambda) = (\eta(\lambda(a)) : a \in \mathcal{A})^{\mathsf{t}}$$

is an eigenvector of M associated with the eigenvalue θ_2.

Proof: The field homomorphism η naturally extends (up to normalization) to $\mathbb{Q}[\theta_1]^{|\mathcal{A}|}$. Since $\lambda \in \mathbb{Q}[\theta_1]^{|\mathcal{A}|}$ and $\eta(\theta_1) = \theta_2$, we have

$$M\lambda = \theta_1\lambda \implies \eta(M\lambda) = \eta(\theta_1\lambda) \implies M\eta(\lambda) = \eta(\theta_1)\eta(\lambda) \implies M\gamma = \theta_2\gamma.$$

Since θ_2 is an eigenvalue of M, it follows that γ is an eigenvector associated with θ_2.

■

Lemma 5.10. *Let γ be the eigenvector of M associated with θ_2 as in Lemma 5.9. Then, for any $|\mathcal{A}|$-tuple of non-negative integers $(n_a : a \in \mathcal{A})$, the equality*

$$\sum_{a\in\mathcal{A}} n_a\,\gamma(a) = 0$$

implies that $n_a = 0$ for all $a \in \mathcal{A}$.

Proof: Assume that $\sum_{a\in\mathcal{A}} n_a\,\gamma(a) = 0$.

Since $\gamma = \eta(\lambda)$, applying η^{-1} yields

$$\sum_{a\in\mathcal{A}} n_a\,\lambda(a) = 0.$$

Because the coordinates of λ are strictly positive, this implies that $n_a = 0$ for every $a \in \mathcal{A}$.

■

The following concepts are crucial in what follows. Let $\gamma = \eta(\lambda)$ be as in Lemma 5.9. Then the following definitions apply.

(1) For a word $w = w_0 \cdots w_{l-1} \in \mathcal{A}^+$, define

$$\gamma(w) = \gamma(w_0) + \cdots + \gamma(w_{l-1}).$$

(2) For $x \in X_\sigma$, define

$$\gamma_0(x) = 0, \qquad \gamma_n(x) = \sum_{i=0}^{n-1} \gamma(x_i) \quad \text{for } n > 0,$$

and

$$\gamma_n(x) = -\sum_{i=n}^{-1} \gamma(x_i) \quad \text{for } n < 0.$$

Set

$$\Gamma(x) = \{\gamma_n(x) \mid n \in \mathbb{Z}\}.$$

(3) Given a dotted word

$$w = w_{-m} \cdots w_{-1} \cdot w_0 \cdots w_{l-1},$$

define

$$\gamma_0(w) = 0, \qquad \gamma_n(w) = \sum_{i=0}^{n-1} \gamma(w_i) \quad \text{for } 0 < n \leq l,$$

and

$$\gamma_n(w) = -\sum_{i=n}^{-1} \gamma(w_i) \quad \text{for } -m \leq n < 0.$$

Set

$$\Gamma(w) = \{\gamma_n(w) \mid -m \leq n \leq l\}.$$

Definition 5.11. *The **best occurrence** of a letter $a \in \mathcal{A}$ in a (dotted) word*

$$w = w_{-m} \cdots w_{-1} \cdot w_0 \cdots w_{l-1}$$

is the symbol w_i such that $w_i = a$ and

$$\gamma_{i+1}(w) = \min\Big\{\gamma_{j+1}(w) \mid -m \leq j < l,\ w_j = a\Big\}.$$

*By Lemma 5.10, under hypothesis **(AH)**, this symbol is well defined and unique.*

Definition 5.12. *A point $x \in X_\sigma$ is said to be **minimal for** γ if*

$$\gamma_n(x) \geq 0 \quad \textit{for all } n \in \mathbb{Z}.$$

The set of points that are minimal for γ is denoted by $\mathcal{M}_\sigma(\gamma)$.

*It is important to note that if x is minimal for γ and the substitution satisfies hypothesis **(AH)**, then, by Lemma 5.10, we have*

$$\gamma_n(x) > 0 \quad \textit{for all } n \neq 0.$$

5.2.3 Interval Exchange Transformations

Let $T = T(\lambda, \pi)$ be an interval exchange transformation on $[0, 1)$, with intervals indexed by the alphabet $\mathcal{A} = \{1, \ldots, r\}$. There is a natural symbolic coding of the orbit of any point $t \in [0, 1)$ under T.

Consider the partition

$$\alpha = \{[0, a_1), \ldots, [a_{i-1}, a_i), \ldots, [a_{r-1}, 1)\},$$

which corresponds to the subintervals of T, and define the map

$$\phi(t) = (t_i)_{i \in \mathbb{Z}} \in \mathcal{A}^{\mathbb{Z}}$$

by setting $t_i = j$ if and only if $T^i(t) \in [a_{j-1}, a_j)$. The set $\phi([0, 1))$ is invariant under the shift map, but it is not necessarily closed. We therefore consider its closure $X = \overline{\phi([0, 1))}$.

This construction yields a semi-conjugacy (factor map)

$$\varphi \colon (X, S) \to ([0, 1), T).$$

If t does not belong to the orbit of the endpoints $0, a_1, \ldots, 1$, then it has a unique preimage under φ. Otherwise, it has at most two preimages, corresponding to the codings of $\left(\lim_{s \to t^-} T^i(s)\right)_{i \in \mathbb{Z}}$.

For a self-similar IET T, there is a direct relationship between the subshift X and the matrix R associated with T. Indeed, there exists a substitution $\sigma \colon \mathcal{A} \to \mathcal{A}^+$ with associated matrix $M = R^{\mathrm{t}}$ such that $X_\sigma = X$ (see [4]). If the IET T is minimal, then the subshift X_σ is also minimal.

In what follows, we use the fact that the substitution σ is primitive, which implies that X_σ is minimal. Nevertheless, no further specific properties of substitutions arising from T will be required for our purposes.

5.3 CONSTRUCTION OF MINIMAL POINTS

Let $\sigma : \mathcal{A} \to \mathcal{A}^+$ be the primitive substitution with associated matrix $M > 0$. Let $\theta_1, \theta_2, \lambda$ and γ be as in subsection 5.2.2. In addition, assume that θ_2 verifies the hypotheses of the main Theorem. By Perron-Frobenius theorem, γ has negative and positive coordinates. The main objective of the section is to give a combinatorial construction of minimal points in this case.

5.3.1 Existence of Minimal Points

The following two lemmas follow from the equality $M\gamma = \theta_2 \gamma$.

Lemma 5.13. *Let $m \in \mathbb{N}$ and $w \in \mathcal{A}^+$. Then $\gamma(\sigma^m(w)) = \theta_2^m \gamma(w)$.*

Proof: We first prove that

$$\gamma(\sigma(a)) = \theta_2 \gamma(a) \quad \text{and consequently} \quad \gamma(\sigma^m(a)) = \theta_2^m \gamma(a),$$

for every $a \in \mathcal{A}$.

Indeed, write $\sigma(a) = b_1 \cdots b_k$. Then

$$\gamma(\sigma(a)) = \gamma(b_1) + \cdots + \gamma(b_k) = n_1 \gamma(1) + \cdots + n_r \gamma(r),$$

where $(n_1, \ldots, n_r) = M^{\mathrm{t}} e_a$, since n_i counts the number of occurrences of the letter i in $\sigma(a)$ and M is the matrix associated with σ.

Therefore,

$$\begin{aligned} n_1\gamma(1) + \cdots + n_r\gamma(r) &= \langle M^{\mathrm{t}} e_a, \gamma\rangle \\ &= \langle e_a, M\gamma\rangle \\ &= \langle e_a, \theta_2\gamma\rangle \\ &= \theta_2\gamma(a). \end{aligned}$$

This proves that $\gamma(\sigma(a)) = \theta_2\gamma(a)$. Repeating the argument, by induction we obtain

$$\gamma(\sigma^m(a)) = \theta_2^m\gamma(a) \quad \text{for all } m \geq 1.$$

Now, let $w = w_0 \cdots w_l$ be a finite word. Then

$$\begin{aligned} \gamma(\sigma^m(w)) &= \gamma\Big(\sigma^m(w_0)\sigma^m(w_1)\cdots\sigma^m(w_l)\Big) \\ &= \gamma(\sigma^m(w_0)) + \cdots + \gamma(\sigma^m(w_l)) \\ &= \theta_2^m\gamma(w_0) + \cdots + \theta_2^m\gamma(w_l) \\ &= \theta_2^m\Big(\gamma(w_0) + \cdots + \gamma(w_l)\Big) \\ &= \theta_2^m\gamma(w). \end{aligned}$$

■

Lemma 5.14. *Let $w = w_0 \cdots w_{l-1} \in \mathcal{A}^+$ and let $m \in \mathbb{N}$. Write*

$$\sigma^m(w) = \sigma^m(w_0)\cdots\sigma^m(w_{l-1}).$$

Then the minimum of $\Gamma(\sigma^m(w))$ is attained at a coordinate belonging to $\sigma^m(w_i)$, where w_i is the best occurrence of its symbol in w.

This follows directly from the previous lemma.

Lemma 5.15. *Let $a \in \mathcal{A}$ such that $\gamma(a) > 0$ and $m \in \mathbb{N}$. Write $\sigma^m(a) = p_m s_m$ where the minimum of $\Gamma(\sigma^m(a))$ is attained at $\gamma_i(\sigma^m(a))$ and $i = |p_m|$. Then $\gamma(s_m) \geq \theta_2^m\gamma(a)$. In particular, $|s_m|$ grows exponentially fast with m.*

Proof: From Lemma 5.13 we have that

$$\gamma(\sigma^m(a)) = \theta_2^m\gamma(a). \tag{5.1}$$

Now note that, if we write $p_m = c_1 \cdots c_i$ and $s_m = c_{i+1} \cdots c_j$, then

$$\begin{aligned} \gamma(\sigma^m(a)) &= \gamma(p_m s_m) = \gamma(c_1 \cdots c_j) \\ &= \gamma(c_1) + \cdots + \gamma(c_i) + \gamma(c_{i+1}) + \cdots + \gamma(c_j) \\ &= \gamma(p_m) + \gamma(s_m). \end{aligned} \tag{5.2}$$

From (5.1) and (5.2), we conclude that

$$\gamma(p_m) + \gamma(s_m) = \theta_2^m\gamma(a). \tag{5.3}$$

Observe that

$$\gamma(p_m) = \gamma(c_1 \cdots c_i) = \gamma(c_1) + \cdots + \gamma(c_i) = \gamma_i(\sigma^m(a)),$$

which corresponds to the minimum of $\Gamma(\sigma^m(a))$.

However, by definition we have $\gamma_0(\sigma^m(a)) = 0$, and therefore $\gamma(p_m) \leq 0$. From this fact and (5.3), we conclude that $\gamma(s_m) \geq \theta_2^m\gamma(a)$.

■

Proposition 5.16. *The set of minimal points is nonempty, that is,* $\mathcal{M}_\sigma(\gamma) \neq \emptyset$.

Proof: **Claim 1.** There exist $b, c \in \mathcal{A}$ such that bc is a subword of a point in X_σ and $\gamma(b) < 0$ and $\gamma(c) > 0$.

Proof of Claim 1. Since γ is an eigenvector distinct from the Perron–Frobenius eigenvector, it cannot have constant sign. Hence, there exist $a^-, a^+ \in \mathcal{A}$ such that

$$\gamma(a^-) < 0 \quad \text{and} \quad \gamma(a^+) > 0.$$

Because σ is primitive, there exists $n \in \mathbb{N}^*$ such that for every $a, b \in \mathcal{A}$, the letter b appears in $\sigma^n(a)$. In particular, there exist $n_0, n_1 \in \mathbb{N}^*$ and $a \in \mathcal{A}$ such that a^- appears in $\sigma^{n_0}(a)$ and a^+ appears in $\sigma^{n_1}(a)$. Therefore, both a^- and a^+ occur in words of X_σ.

Since (X_σ, S) is minimal, it is topologically transitive. Hence, there exists $n \in \mathbb{N}$ such that

$$S^n([a^-]) \cap [a^+] \neq \emptyset.$$

Equivalently, there exists a finite word w such that $a^- w a^+$ occurs in some word of X_σ.

Consequently, since $\gamma(a^-) < 0$ and $\gamma(a^+) > 0$, there exist letters b and c such that bc is a subword of $a^- w a^+$, with

$$\gamma(b) < 0 \quad \text{and} \quad \gamma(c) > 0.$$

This concludes the proof of Claim 1.

Fix $n \geq 0$ and define the dotted word $u_n = \sigma^n(b).\sigma^n(c)$. Let N_n be the index at which $\Gamma(u_n)$ attains its minimum, where $N_n \in \{-|\sigma^n(b)|, \ldots, -1, 0, 1, \ldots, |\sigma^n(c)|\}$.

Define the dotted word

$$v_n = u_n[-|\sigma^n(b)|,\, N_n - 1]\,.\,u_n[N_n,\, |\sigma^n(c)| - 1] = v_n^-.v_n^+.$$

Claim 2. The set $\Gamma(v_n)$ attains its minimum at coordinate zero, and this minimum is equal to zero.

Proof of Claim 2. First observe that $N_n \neq -|\sigma^n(b)|$ and $N_n \neq |\sigma^n(c)|$. Indeed,

$$\gamma_{-|\sigma^n(b)|}(u_n) = -\gamma(\sigma^n(b)) = -\theta_2^n \gamma(b) > 0,$$

and

$$\gamma_{|\sigma^n(c)|}(u_n) = \gamma(\sigma^n(c)) = \theta_2^n \gamma(c) > 0.$$

Since $\gamma_0(u_n) = 0$, the minimum of $\Gamma(u_n)$ cannot be attained at the endpoints.

We distinguish two cases.

(i) $N_n < 0$.

By definition of N_n, $-\gamma(u_{n_{-1}}) - \cdots - \gamma(u_{n_{N_n}}) < 0$.

On the other hand, $-\gamma(u_{n_{-1}}) - \cdots - \gamma(u_{n_{-|\sigma^n(b)|}}) = \gamma_{-|\sigma^n(b)|}(v_n) > 0$.

Therefore,

$$-\gamma(u_{n_{N_n - 1}}) - \cdots - \gamma(u_{n_{-|\sigma^n(b)|}}) > 0.$$

Moreover, for every $j < N_n$, $-\gamma(u_{n_{-1}}) - \cdots - \gamma(u_{n_j}) > 0$, otherwise the minimum of $\Gamma(u_n)$ would be attained at j. This implies that $\gamma_i(v_n) \geq 0$ for all $i \leq 0$.

Now, for $N_n < j < 0$, we have

$$\gamma_j(u_n) = -\gamma(u_{n_{-1}}) - \cdots - \gamma(u_{n_j}) > -\gamma(u_{n_{-1}}) - \cdots - \gamma(u_{n_{N_n}}),$$

which implies

$$-\gamma(u_{n_{j-1}}) - \cdots - \gamma(u_{n_{N_n}}) < 0,$$

and hence

$$\gamma(u_{n_{N_n}}) + \gamma(u_{n_{N_n+1}}) + \cdots + \gamma(u_{n_{j-1}}) > 0.$$

For $0 < i < -N_n$, taking $j = N_n + i$, we obtain

$$\gamma_i(v_n) = \gamma(u_{n_{N_n}}) + \cdots + \gamma(u_{n_{j-1}}) > 0.$$

For $i = -N_n$, since $\gamma_{N_n}(u_n) < 0$, we also have $\gamma_i(v_n) > 0$.

Finally, if $i > -N_n$ and $\gamma_i(v_n) \leq 0$, then, since $\gamma_{-N_n}(v_n) > 0$ and $\gamma_{|\sigma^n(c)|}(u_n) > 0$, it would follow that $\gamma_{i+N_n}(u_n) < \gamma_{N_n}(u_n)$, contradicting the minimality of N_n. Thus $\gamma_i(v_n) > 0$ for all $i > -N_n$.

(ii) $N_n > 0$.

This case is completely analogous to the previous one and can be proved by symmetric arguments.

Therefore, $\gamma_i(v_n) \geq 0$ for all $-|v_n^-| \leq i \leq |v_n^+|$, and the minimum of $\Gamma(v_n)$ is attained at coordinate zero, with value zero. This concludes the proof of Claim 2.

Claim 3. There exists a subsequence $(n_i)_{i\in\mathbb{N}}$ such that

$$\lim_{i\to\infty} |v_{n_i}^-| = \lim_{i\to\infty} |v_{n_i}^+| = \infty.$$

Proof of Claim 3. Suppose that for infinitely many indices n_i, the set $\Gamma(u_{n_i})$ attains its minimum at a positive index. In this case, this minimum is also the minimum of $\Gamma(\sigma^{n_i}(c))$. Therefore, Lemma 5.15 applies and implies that $|v_{n_i}^+|$ grows as $n_i \to \infty$.

Notice that Lemma 5.15 can also be applied to $\sigma^{n_i}(b)$ by interchanging the roles of p_{n_i} and s_{n_i}. In other words, the growing part is now the beginning of the word, which implies that $|v_{n_i}^-|$ also grows.

If $\Gamma(u_n)$ attains its minimum at a positive index only for finitely many values of n, then it must attain its minimum at negative indices for infinitely many n. Taking such a subsequence (n_i) and applying the same argument as above, we conclude that both $|v_{n_i}^-|$ and $|v_{n_i}^+|$ diverge to infinity.

This concludes the proof of Claim 3.

Considering this subsequence (n_i), we obtain a sequence of points $(x_{n_i}) \subset X_\sigma$ such that each x_{n_i} contains the dotted word $v_{n_i}^-.v_{n_i}^+$, where both $|v_{n_i}^-|$ and $|v_{n_i}^+|$ diverge as $n_i \to \infty$.

Since X_σ is compact, there exists a further subsequence (n_{i_k}) such that $x_{n_{i_k}} \to x \in X_\sigma$. By construction, we have $\gamma_n(x) \geq 0$ for every $n \in \mathbb{Z}$. Therefore, x is minimal for γ.

■

5.3.2 The Best Strategy Algorithm

In what follows we develop a procedure to construct minimal points for γ that will become useful later.

The Basic Procedure

The following procedure will allow us to construct what we call the prefix-suffix decomposition of a minimal point x for γ, that is, a sequence

$$(p_i, c_i, s_i)_{i\in\mathbb{N}} \in (\mathcal{A}^* \times \mathcal{A} \times \mathcal{A}^*)^{\mathbb{N}}$$

such that, for each $i \in \mathbb{N}$, $\sigma(c_{i+1}) = p_i c_i s_i$ and

$$\cdots \sigma^3(p_3)\sigma^2(p_2)\sigma(p_1)p_0 \cdot c_0 s_0 \sigma(s_1)\sigma^2(s_2)\sigma^3(s_3)\cdots$$

is the central part of x, where the dot separates negative and non-negative coordinates.

Step 0: For each $a \in \mathcal{A}$ write

$$\sigma(a) = p_0^{a,0} c_0^{a,0} s_0^{a,0}$$

where $\Gamma(\sigma(a))$ attains its minimum at $\gamma_{|p_0^{a,0}|}(\sigma(a))$.

Step 1: Let $a \in \mathcal{A}$. By Lemma 5.14, the minimum of $\Gamma(\sigma^2(a))$ comes from $\sigma(b)$ for some $b \in \mathcal{A}$ in its best occurrence in $\sigma(a)$. Write

$$\sigma(a) = p_1^{a,1} c_1^{a,1} s_1^{a,1}$$

where $c_1^{a,1} = b$ is the best occurrence of b in $\sigma(a)$. Put

$$w_1(a) = \sigma(p_1^{a,1}) p_0^{b,0} . c_0^{b,0} s_0^{b,0} \sigma(s_1^{a,1}),$$

where the dot separates negative and non-negative coordinates. Let

$$p_0^{a,1} = p_0^{a,0}, \quad c_0^{a,1} = c_0^{b,0} \quad \text{and} \quad s_0^{a,1} = s_0^{b,0}.$$

The sequence $(p_i^{a,1}, c_i^{a,1}, s_i^{a,1})_{i=0}^1$ is called **the best strategy for symbol a at step 1**. By construction $\Gamma(w_1(a)) \subset \mathbb{R}^+$ and the minimum is equal to zero at coordinate zero.

Step n+1: Assume that in the previous step we have constructed for each symbol $a \in \mathcal{A}$ the best strategy $(p_i^{a,n}, c_i^{a,n}, s_i^{a,n})_{i=0}^n$. This sequence verifies the following results.

(i) For $0 \le i \le n$, $\sigma(c_{i+1}^{a,n}) = p_i^{a,n} c_i^{a,n} s_i^{a,n}$ (here $c_{n+1}^{a,n} = a$). Moreover, each $c_i^{a,n}$ is the best occurrence of this symbol in $\sigma(c_{i+1}^{a,n})$.

(ii) We have $\Gamma(w_n(a)) \subset \mathbb{R}^+$ and its minimum is zero at zero coordinate, where

$$w_n(a) = \sigma^n(p_n^{a,n}) \cdots \sigma(p_1^{a,n}) p_0^{a,n} . c_0^{a,n} s_0^{a,n} \sigma(s_1^{a,n}) \cdots \sigma^n(s_n^{a,n}).$$

Observe that as a non-dotted word $w_n(a)$ is equal to $\sigma^{n+1}(a)$.

Now we proceed as in step 1. Consider $a \in \mathcal{A}$. By Lemma 5.14, the minimum of $\Gamma(\sigma^{n+2}(a))$ comes from $\sigma^{n+1}(b)$ for some $b \in \mathcal{A}$ in its best occurrence in $\sigma(a)$. Write

$$\sigma(a) = p_{n+1}^{a,n+1} c_{n+1}^{a,n+1} s_{n+1}^{a,n+1},$$

where $c_{n+1}^{a,n+1} = b$ is the best occurrence of b in $\sigma(a)$.

The finite sequence $(p_i^{a,n+1}, c_i^{a,n+1}, s_i^{a,n+1})_{i=0}^{n+1}$, where

$$(p_i^{a,n+1}, c_i^{a,n+1}, s_i^{a,n+1}) = (p_i^{b,n}, c_i^{b,n}, s_i^{b,n}),$$

for $0 \le i \le n$ is a **best strategy for a at step n+1** and verifies conditions (i) and (ii) by construction.

The following scheme helps us to understand steps 0 and 1, and consequently the following ones.

$$a \to \sigma(a) = \underbrace{p_0^{a,0}}_{\min\Gamma(\sigma(a))} c_0^{a,0} s_0^{a,0} = p_1^{a,1} c_1^{a,1} s_1^{a,1}$$

$c_1^{a,1} = b$, such that $\min\Gamma(\sigma^2(a))$ comes from $\sigma(b)$, b in its best occurrence in $\sigma(a)$.

$$\Rightarrow \sigma^2(a) = \sigma(p_1^{a,1}) p_0^{b,0} c_0^{b,0} s_0^{b,0} \sigma(s_1^{a,1});$$

$$w_1(a) = \sigma(p_1^{a,1}) p_0^{b,0} . c_0^{b,0} s_0^{b,0} \sigma(s_1^{a,1})$$

$$\Rightarrow p_0^{a,1} := p_0^{b,0}, \ c_0^{a,1} := c_0^{b,0}, \ s_0^{a,1} := s_0^{b,0}$$

$$\Rightarrow (p_i^{a,1}, c_i^{a,1}, s_i^{a,1})_{i=0}^{1}.$$

Finitely many Minimal Points for γ

For each $a \in \mathcal{A}$ and $n \in \mathbb{N}$ consider the cylinder set $C^{a,n} = [w_n(a)]$, where $w_n(a)$ is the dotted word defined in the previous subsection. It is clear from the basic procedure that for any $a \in \mathcal{A}$ and $n \in \mathbb{N}$ there exists a unique $b \in \mathcal{A}$ such that $C^{a,n+1} \subset C^{b,n}$ since $w_n(b)$ is inside $w_{n+1}(a)$. Thus, by compactness, there exist at most $|\mathcal{A}|$ infinite decreasing sequences of the form $(C^{a_n,n})_{n\in\mathbb{N}}$. Let $C_1, \ldots, C_l,\ l \leq |\mathcal{A}|$ be the collection of intersections of such sequences. Note that such sets are finite.

Given $x \in X_\sigma$ a minimal point for γ with prefix-suffix decomposition $(p_i, c_i, s_i)_{i\in\mathbb{N}}$ and $n \in \mathbb{N}$, there is $a_n \in \mathcal{A}$ such that $(p_i, c_i, s_i) = (p_i^{a_n,n}, c_i^{a_n,n}, s_i^{a_n,n})$ for $0 \leq i \leq n$. Therefore, $x \in C_j = \bigcap_{n\in\mathbb{N}} C^{a_n,n}$ for some $1 \leq j \leq l$.s

The following proposition is immediate.

Proposition 5.17. *There are finitely many minimal points for γ.*

We will see later that minimal points for γ have ultimately periodic prefix-suffix decomposition. This fact yields an alternative proof of the previous proposition.

5.3.3 Series Associated with a Minimal Point

Define

$$\overline{\mathcal{S}} = \{(p_i, c_i, s_i)_{i\in\mathbb{N}} |\ \forall i > 0,\ \sigma(c_i) = p_{i-1}c_{i-1}s_{i-1}\}$$

and

$$\underline{\mathcal{S}} = \{(p_i, c_i, s_i)_{i\in\mathbb{N}} |\ \forall i > 0,\ \sigma(c_i) = p_{i+1}c_{i+1}s_{i+1}\}.$$

Observe that finite sequences taken from sequences in $\overline{\mathcal{S}}$ and $\underline{\mathcal{S}}$ coincide once reversed.

Let $a \in \mathcal{A}$ and $n \geq 1$. Then $\sigma^n(a)$ can be decomposed as

$$\sigma^n(a) = \sigma^{n-1}(p_1) \cdots \sigma(p_{n-1}) p_n c_n s_n \sigma(s_{n-1}) \cdots \sigma^{n-1}(s_1)$$

where for all $1 \leq i \leq n$, $\sigma(c_{i-1}) = p_i c_i s_i$ (we have considered $c_0 = a$). This decomposition is not unique. To a and the finite sequence $(p_i, c_i, s_i)_{i=1}^n$ one associates the finite sum:

$$v(a; (p_i, c_i, s_i)_{i=1}^n) = \sum_{i=1}^{n} \theta_2^{-1} \gamma(p_i).$$

Clearly, given $\boldsymbol{x} = (p_i^{\boldsymbol{x}}, c_i^{\boldsymbol{x}}, s_i^{\boldsymbol{x}})_{i\in\mathbb{N}} \in \underline{\mathcal{S}}$ with $c_0^{\boldsymbol{x}} = a$, the series

$$v(a; \boldsymbol{x}) = \lim_{n\to\infty} v(a; (p_i^{\boldsymbol{x}}, c_i^{\boldsymbol{x}}, s_i^{\boldsymbol{x}})_{i=1}^n) = \sum_{i\geq 1} \theta_2^{-i}\gamma(p_i^{\boldsymbol{x}})$$

exists, since $\theta_2 > 1$.

Let $v(a) = \min\{v(a; \boldsymbol{x}) \mid \boldsymbol{x} \in \underline{\mathcal{S}}$ with $c_0^{\boldsymbol{x}} = a\}$. A sequence $\boldsymbol{x} \in \underline{\mathcal{S}}$ with $c_0^{\boldsymbol{x}} = a$ such that $v(a; \boldsymbol{x}) = v(a)$ is said to be **minimal for** a.

The best strategy for symbol a at step $n \geq 1$ given by the algorithm produces a finite sequence $(p_i^{a,n}, c_i^{a,n}, s_i^{a,n})_{i=0}^n$. Set

$$v_n(a) = \sum_{i=0}^{n} \theta_2^{-n+i-1}\gamma(p_{n-i+1}^{a,n}).$$

It follows that $v_n(a) = v(a; (p_{n-i+1}^{a,n}, c_{n-i+1}^{a,n}, s_{n-i+1}^{a,n})_{i=0}^n$.

Lemma 5.18. *For every $a \in \mathcal{A}$ and $n \geq 1$, we have that $v_n(a)$ is minimal among the $v(a; (p_i, c_i, s_i)_{i=1}^{n+1})$ and $v(a) = \lim_{n\to\infty} v_n(a)$.*

Proof: The first fact is analogous to saying that $(p_i^{a,n}, c_i^{a,n}, s_i^{a,n})_{i=0}^n$ is the best strategy. Moreover, $|v_n(a) - v(a)| \leq K\theta_2^{-n}$ for some constant $K > 0$. This implies the desired result.

■

Lemma 5.19. *Let $a \in \mathcal{A}$ and $l \in \mathbb{N}$. Assume that there is a finite sequence $(\overline{p}_j, \overline{c}_j, \overline{s}_j)_{j=1}^l$ such that for infinitely many $n \in \mathbb{N}$, $(p_{n-j+1}^{a,n}, c_{n-j+1}^{a,n}, s_{n-j+1}^{a,n})_{j=1}^l = (\overline{p}_j, \overline{c}_j, \overline{s}_j)_{j=1}^l$ (when $l = 0$ this hypothesis is void). Then, there exists $\boldsymbol{y} = (p_i^{\boldsymbol{y}}, c_i^{\boldsymbol{y}}, s_i^{\boldsymbol{y}})_{i\in\mathbb{N}} \in \underline{\mathcal{S}}$ such that $(\boldsymbol{y}_j)_{j=1}^l = (\overline{p}_j, \overline{c}_j, \overline{s}_j)_{j=1}^l$, $c_0^{\boldsymbol{y}} = a$ and $v(a) = v(c_0^{\boldsymbol{y}}; \boldsymbol{y})$, that is, $\boldsymbol{y}$ is minimal for a.*

Proof: For any $n \in \mathbb{N}$ where the property of the lemma holds consider the sequence

$$\boldsymbol{y}^{(n)} = \boldsymbol{y}_0^{(n)} \cdots \boldsymbol{y}_{n+1}^{(n)} = (p, a, s)(p_n^{a,n}, c_n^{a,n}, s_n^{a,n}) \cdots (p_0^{a,n}, c_0^{a,n}, s_0^{a,n}),$$

where $\sigma(b) = pas$ for some $b \in \mathcal{A}$.

Let $\boldsymbol{y} = (p_i^{\boldsymbol{y}}, c_i^{\boldsymbol{y}}, s_i^{\boldsymbol{y}})_{i\in\mathbb{N}}$ be the limit of a subsequence $(\boldsymbol{y}^{(n_i)})_{i\in\mathbb{N}}$. It follows by construction that $(\boldsymbol{y}_j)_{j=1}^l = (\overline{p}_j, \overline{c}_j, \overline{s}_j)_{j=1}^l$, $c_0^{\boldsymbol{y}} = a$ and $\sigma(c_i^{\boldsymbol{y}}) = p_{i+1}^{\boldsymbol{y}} c_{i+1}^{\boldsymbol{y}} s_{i+1}^{\boldsymbol{y}}$ for any $i \geq 0$. Also, $c_{i+1}^{\boldsymbol{y}}$ is the best occurence of this symbol in $\sigma(c_i^{\boldsymbol{y}})$.

Let $i, L \in \mathbb{N}$ such that $(p_j^{\boldsymbol{y}}, c_j^{\boldsymbol{y}}, s_j^{\boldsymbol{y}}) = (p_{n_i-j+1}, c_{n_i-j+1}, s_{n_i-j+1})$ for $1 \leq j \leq L$. Since $\gamma(p_k^{a,n})$ and $\gamma(p_k^{\boldsymbol{y}})$ are bounded independently of n, k and $\boldsymbol{y}$, there is $C > 0$ such that

$$\left| v(a) - \sum_{k\geq 1} \theta_2^{-k}\gamma(p_k^{\boldsymbol{y}}) \right| \leq |v(a) - v_{n_i}(a)| + C\theta_2^{-L}.$$

Let $\epsilon > 0$. By Lemma 5.18, considering i and L large enough onde deduces that

$$\left| v(a) - \sum_{k\geq 1} \theta_2^{-k}\gamma(p_k^{\boldsymbol{y}}) \right| \leq |v(a) - v_{n_i}(a)| + C\theta_2^{-L} \leq \epsilon.$$

Since ϵ is arbitrary we conclude that $v(c_0^{\boldsymbol{y}}) = v(a) = \sum_{k\geq 1} \theta_2^{-k}\gamma(p_k^{\boldsymbol{y}})$.

■

We say that a point $\boldsymbol{y} = (p_i^{\boldsymbol{y}}, c_i^{\boldsymbol{y}}, s_i^{\boldsymbol{y}})_{i\in\mathbb{N}} \in \underline{\mathcal{S}}$ verifies the **continuation property** if $v(c_i^{\boldsymbol{y}}) = v(c_i^{\boldsymbol{y}}; S^i(\boldsymbol{y}))$ for all $i \geq 0$, where S is the shift map. It is clear that $S^i(\boldsymbol{y})$ also has the continuation property, for any $i \in \mathbb{N}$. In fact, as proven in the next lemma, if $\boldsymbol{y}$ is minimal for $c_0^{\boldsymbol{y}}$, then it satisfies the continuation property.

Lemma 5.20. *If $\boldsymbol{y} = (p_i^{\boldsymbol{y}}, c_i^{\boldsymbol{y}}, s_i^{\boldsymbol{y}})_{i\in\mathbb{N}} \in \underline{\mathcal{S}}$ is minimal for $c_0^{\boldsymbol{y}}$ (that is, $v(c_0^{\boldsymbol{y}}) = v(c_0^{\boldsymbol{y}}; \boldsymbol{y})$), then $\boldsymbol{y}$ verifies the continuation property.*

Proof: Let $a = c_0^{\boldsymbol{y}}$, $b = c_1^{\boldsymbol{y}}$, and let $\boldsymbol{z} = (p_i^{\boldsymbol{z}}, c_i^{\boldsymbol{z}}, s_i^{\boldsymbol{z}})_{i\in\mathbb{N}} \in \underline{\mathcal{S}}$ be such that $c_0^{\boldsymbol{z}} = b$ and $v(b; \boldsymbol{z}) = v(b)$, as given by Lemma 5.19 (taking $l = 0$). Then the sequence $\boldsymbol{w} = \boldsymbol{y}_0\boldsymbol{y}_1 S(\boldsymbol{z})$ belongs to $\underline{\mathcal{S}}$ and satisfies

$$v(a; \boldsymbol{w}) = \theta_2^{-1}\gamma(p_1^{\boldsymbol{y}}) + \theta_2^{-1}v(b).$$

Thus, if $v(b; S(\boldsymbol{y})) > v(b)$, it follows from

$$v(a) = v(a; \boldsymbol{y}) = \theta_2^{-1}\gamma(p_1^{\boldsymbol{y}}) + \theta_2^{-1}v(b; S(\boldsymbol{y}))$$

that $v(a; \boldsymbol{w}) < v(a)$, which is a contradiction.

■

In particular, this lemma proves that sequences $\boldsymbol{y}$ constructed in Lemma 5.19 verify the continuation property.

5.3.4 Minimal Points for γ have Ultimately Periodic Prefix-Suffix Decomposition

In this section, we prove that any minimal point for $\gamma, x \in X_\sigma$ has ultimately periodic prefix-suffix decomposition. That is, if $\overline{x} = (p_i, c_i, s_i)_{i\in\mathbb{N}}$ is the prefix-suffix decomposition of x, then $S^{p+q}(\overline{x}) = S^q(\overline{x})$ for some $p > 0, q \geq 0$. If $q = 0$ we say that x is a periodic minimal point for γ.

In order to prove this, we need the following lemmas.

Lemma 5.21. *For every $a \in \mathcal{A}$ there exists an ultimately periodic point $\boldsymbol{x}(a) = (p_i^{\boldsymbol{x}(a)}, c_i^{\boldsymbol{x}(a)}, s_i^{\boldsymbol{x}(a)})_{i\in\mathbb{N}} \in \underline{\mathcal{S}}$ with $c_0^{\boldsymbol{x}(a)} = a$ and $v(a; \boldsymbol{x}(a)) = v(a)$ (so, $\boldsymbol{x}(a)$ has the continuation property).*

Proof: Let $a \in \mathcal{A}$ and $\boldsymbol{y} = (p_i^{\boldsymbol{y}}, c_i^{\boldsymbol{y}}, s_i^{\boldsymbol{y}})_{i\in\mathbb{N}} \in \underline{\mathcal{S}}$ with $c_0^{\boldsymbol{y}} = a$ and $v(a; \boldsymbol{y}) = v(a)$ given by Lemma 5.19 (considering $l = 0$). We are going to construct another point with ultimately periodic decomposition.

Let $0 < q < p$ be such that $\boldsymbol{y}_q = \boldsymbol{y}_p$ and $c_{q-1}^{\boldsymbol{y}} = c_{p-1}^{\boldsymbol{y}} = b$. The ultimately periodic sequence $\boldsymbol{x}(a) = \boldsymbol{y}_0 \cdots \boldsymbol{y}_{q-1}\boldsymbol{y}_q \cdots \boldsymbol{y}_{p-1} \cdots \in \underline{\mathcal{S}}$ since $\sigma(c_{p-1}^{\boldsymbol{y}}) = p_q^{\boldsymbol{y}} c_q^{\boldsymbol{y}} s_q^{\boldsymbol{y}}$ by hypothesis. We are going to prove that $v(a; \boldsymbol{x}(a)) = v(a)$.

■

Observe that, by Lemmas 5.19 and 5.20,

$$v(b) = \sum_{i\geq q} \theta_2^{-(i-q+1)}\gamma(p_i^{\boldsymbol{y}}) \quad \text{and} \quad v(b) = \sum_{i\geq p} \theta_2^{-(i-p+1)}\gamma(p_i^{\boldsymbol{y}}).$$

Thus,

$$v(b)=\sum_{i=q}^{p-1}\theta_2^{-(i-q+1)}\gamma(p_i^{\boldsymbol{y}})+\sum_{i\geq p}\theta_2^{-(i-q+1)}\gamma(p_i^{\boldsymbol{y}})=\sum_{i=q}^{p-1}\theta_2^{-(i-q+1)}\gamma(p_i^{\boldsymbol{y}})+\theta_2^{-(p-q)}v(b).$$

If we denote $B=\sum_{i=q}^{p-1}\theta_2^{-(i-q+1)}\gamma(p_i^{\boldsymbol{y}})$, then $v(b)=B\sum_{i\geq 0}\theta_2^{-(-p-q)i}$. Consequently,

$$v(a)=\sum_{i=1}^{q-1}\theta_2^{-i}\gamma(p_i^{\boldsymbol{y}})+\theta_2^{-(q-1)}B\sum_{i\geq 0}\theta_2^{-(-p-q)i}.$$

On the other hand, a direct computation yields the following

$$v(a;\boldsymbol{x}(a))=\sum_{i=1}^{q-1}\theta_2^{-i}\gamma(p_i^{\boldsymbol{y}})+\theta_2^{-(q-1)}\left(\sum_{i\geq 0}\theta_2^{-(p-q)i}B\right),$$

which implies $v(a;\boldsymbol{x}(a))=v(a)$.

To each ultimately periodic sequence $\boldsymbol{x}(a)$ constructed in the previous lemma we can associate a point x in the symbolic space X_σ with periodic prefix-suffix decomposition with generating word

$$(p_0,c_0,s_0),\ldots,(p_{p-q},c_{p-q},s_{p-q})=(p_{p-1}^{\boldsymbol{x}(a)},c_{p-1}^{\boldsymbol{x}(a)},s_{p-1}^{\boldsymbol{x}(a)}),\ldots,(p_q^{\boldsymbol{x}(a)},c_q^{\boldsymbol{x}(a)},s_q^{\boldsymbol{x}(a)}).$$

Even if, by construction, this point is associated with the minimal value $v(b)$, there is no reason for it to be a minimal point for γ.

Without loss of generality, we perform the following simplification. By iterating σ sufficiently many times, we may assume that every ultimately periodic sequence constructed in Lemma 5.21 has period 1 and preperiod 1. That is, for each letter $a\in\mathcal{A}$, we have $c_0^{\boldsymbol{x}(a)}=a$ and $\boldsymbol{x}_i(a)=(p^{(a)},\hat{a},s^{(a)})$ for all $i\geq 1$. A letter $a\in\mathcal{A}$ is said to be **periodic** if $\hat{a}=a$, and we denote by $\hat{\mathcal{A}}$ the subset of periodic letters. Since the construction in Lemma 5.21 implies that $v(c_i^{\boldsymbol{x}(a)})=v(c_i^{\boldsymbol{x}(a)};S^i(\boldsymbol{x}(a)))$ for $0\leq i\leq p-1$, under this simplification we obtain $v(\hat{a})=v(\hat{a};S(\boldsymbol{x}(a)))$.

Lemma 5.22. *Let $\boldsymbol{y}\in\underline{S}$ be a sequence that verifies the continuation property. Then, for any $i\geq 1$ the point $\boldsymbol{y}^{(i)}=\boldsymbol{y}_0\cdots\boldsymbol{y}_iS(\boldsymbol{x}(c_i^{\boldsymbol{y}}))$ also has the continuation property.*

Proof: Let $i\geq 1$ and $1\leq j\leq i$. From the continuation property we deduce that

$$v(c_j^{\boldsymbol{y}})=\sum_{k=1}^{i-j}\theta_2^{-k}\gamma(p_{k+j}^{\boldsymbol{y}})+\theta_2^{-(i-j)}v(c_i^{\boldsymbol{y}}).$$

However,

$$v(c_i^{\boldsymbol{y}})=v(c_i^{\boldsymbol{y}};\boldsymbol{x}(c_i^{\boldsymbol{y}})\quad\text{and}\quad v(\hat{c}_i^{\boldsymbol{y}})=v(\hat{c}_i^{\boldsymbol{y}};S(\boldsymbol{x}(c_i^{\boldsymbol{y}}))),$$

then $\boldsymbol{y}^{(i)}=\boldsymbol{y}_0\cdots\boldsymbol{y}_{i-1}S(\boldsymbol{x}(c_i^{\boldsymbol{y}}))$ also has the continuation property.

■

Lemma 5.23. *Let $\boldsymbol{x},\boldsymbol{y}\in\underline{S}$ such that $(\boldsymbol{x}_i)_{i\geq l+1}=(\boldsymbol{y}_i)_{i\geq l+1}$ and $c_0^{\boldsymbol{x}}=c_0^{\boldsymbol{y}}=a$. If $v(a;\boldsymbol{x})=v(a;\boldsymbol{y})$, then $(\boldsymbol{x}_i)_{i\geq 1}=(\boldsymbol{y}_i)_{i\geq 1}$.*

Proof: Let $\boldsymbol{x} = (p_i^{\boldsymbol{x}}, c_i^{\boldsymbol{x}}, s_i^{\boldsymbol{x}})_{i\in\mathbb{N}}$ and $\boldsymbol{y} = (p_i^{\boldsymbol{y}}, c_i^{\boldsymbol{y}}, s_i^{\boldsymbol{y}})_{i\in\mathbb{N}}$. From the hypothesis, we deduce that

$$\sum_{i=1}^{l} \theta_2^{-i}\gamma(p_i^{\boldsymbol{x}}) = \sum_{i=1}^{l} \theta_2^{-i}\gamma(p_i^{\boldsymbol{y}})$$

and consequently,

$$\gamma(\sigma^{l-1}(p_1^{\boldsymbol{x}})\cdots p_l^{\boldsymbol{x}}) = \gamma(\sigma^{l-1}(p_1^{\boldsymbol{y}})\cdots p_l^{\boldsymbol{y}}).$$

However, words $\sigma^{l-1}(p_1^{\boldsymbol{x}})\cdots p_l^{\boldsymbol{x}}$ and $\sigma^{l-1}(p_1^{\boldsymbol{y}})\cdots p_l^{\boldsymbol{y}}$ are both prefixes of $\sigma^l(a)$. Then, without loss of generality we can suppose that $\sigma^{l-1}(p_1^{\boldsymbol{x}})\cdots p_l^{\boldsymbol{x}}$ is a prefix of $\sigma^{l-1}(p_1^{\boldsymbol{y}})\cdots p_l^{\boldsymbol{y}}$. This implies that

$$\gamma(\sigma^{l-1}(p_1^{\boldsymbol{y}})\cdots p_l^{\boldsymbol{y}}) - \gamma(\sigma^{l-1}(p_1^{\boldsymbol{x}})\cdots p_l^{\boldsymbol{x}}) = \sum_{a\in\mathcal{A}} n_a\gamma(a) = 0$$

with $n_a \geq 0$ for any $a \in \mathcal{A}$. Therefore, by the algebraic condition (Lemma 5.10),

$$\gamma(\sigma^{l-1}(p_1^{\boldsymbol{y}})\cdots p_l^{\boldsymbol{y}}) = \gamma(\sigma^{l-1}(p_1^{\boldsymbol{x}})\cdots p_l^{\boldsymbol{x}}).$$

This implies that $(p_i^{\boldsymbol{x}}, c_i^{\boldsymbol{x}}, s_i^{\boldsymbol{x}}) = (p_i^{\boldsymbol{y}}, c_i^{\boldsymbol{y}}, s_i^{\boldsymbol{y}})$, for $1 \leq i \leq l$.

■

We now have the necessary results to prove the following theorem.

Theorem 5.24. *The prefix-suffix decomposition of any minimal point for γ is ultimately periodic.*

Proof: Let $x \in X_\sigma$ be a minimal point with prefix-suffix decomposition $(p_i, c_i, s_i)_{i\in\mathbb{N}}$. There exists a finite sequence $(\overline{p}_j, \overline{c}_j, \overline{s}_j)_{j=0}^l$ such that $(\overline{p}_0, \overline{c}_0, \overline{s}_0) = (\overline{p}_l, \overline{c}_l, \overline{s}_l)$ and for infinitely many $i \in \mathbb{N}$, $(p_{i-j}, c_{i-j}, s_{i-j})_{j=0}^l = (\overline{p}_j, \overline{c}_j, \overline{s}_j)_{j=0}^l$.

Let $a = \overline{c}_0 = \overline{c}_l$. By Lemma 5.19, there is a point $\boldsymbol{y} \in \underline{\mathcal{S}}$ verifying the continuation property such that $(\boldsymbol{y}_j)_{j=0}^l = (\overline{p}_j, \overline{c}_j, \overline{s}_j)_{j=0}^l$, $v(a;\boldsymbol{y}) = v(a)$ and $v(a; S^l(\boldsymbol{y})) = v(a)$. Since $v(a) = v(a;\boldsymbol{x}(a))$, by Lemma 5.22, the sequence $\boldsymbol{z} = \boldsymbol{y}_0\cdots\boldsymbol{y}_l S(\boldsymbol{x}(a))$ has the continuation property, and $v(a) = v(a;\boldsymbol{z})$ holds. Therefore, by Lemma 5.23, we conclude that $(\boldsymbol{x}(a)_i)_{i\geq 1} = (\boldsymbol{z}_i)_{i\geq 1}$.

We have proved that $a \in \hat{A}$, that is, $a = \hat{a}$, and that the word $(p^{(a)}, a, s^{(a)})(p^{(a)}, a, s^{(a)})$ appears infinitely many times in the prefix-suffix decomposition of x. Now we shall prove, by contradiction, that $(p_i, c_i, s_i)_{i\in\mathbb{N}}$ is ultimately periodic with generating word $(p^{(a)}, a, s^{(a)})$.

Assume that this result does not hold. Then there is $b \neq a$ in $\mathcal{A}$ such that

$$(p_i, c_i, s_i)(p_{i-1}, c_{i-1}, s_{i-1})(p_{i-2}, c_{i-2}, s_{i-2}) = (p^{(a)}, a, s^{(a)})(p^{(a)}, a, s^{(a)})(p, b, s)$$

for infinitely many $i \in \mathbb{N}$.

By Lemma 5.19, there is a point $\boldsymbol{w} \in \underline{\mathcal{S}}$ verifying the continuation property and such that

$$\boldsymbol{w}_0\boldsymbol{w}_1\boldsymbol{w}_2 = (p^{(a)}, a, s^{(a)})(p^{(a)}, a, s^{(a)})(p, b, s).$$

Since $v(b) = v(b; S^2(\boldsymbol{w})$ and $v(b) = v(b;\boldsymbol{x}(b))$, by Lemma 5.22, the points $\boldsymbol{u} = \boldsymbol{w}_0\boldsymbol{w}_1\boldsymbol{w}_2 S(\boldsymbol{x}(b))$ and $\boldsymbol{v} = \boldsymbol{x}(a)\boldsymbol{x}(b)$ have the continuation property. Since $\boldsymbol{u}$ and $\boldsymbol{v}$ are ultimately equal, then, by Lemma 5.23, we conclude that $a = b$, which is a contradiction. Therefore, the theorem is proved.

■

5.3.5 Convergence of Series Associated with Minimal Points for γ

In order to prove the main results, we need the following general property of prefix-suffix decomposition.

Lemma 5.25. *Let $x \in X_\sigma$ be a point with prefix-suffix decomposition $(p_i, c_i, s_i)_{i\in\mathbb{N}}$ and consider $n \geq 0$ an integer. Let $N \geq 1$ be the smallest integer such that $x_1 \cdots x_{n+1}$ is a prefix of $s_0\sigma(s_1)\cdots\sigma^{N-1}(s_{N-1})$. Let $(p_i^{(n)}, c_i^{(n)}, s_i^{(n)})_{i\in\mathbb{N}}$ be the prefix-suffix decomposition of $S^{n+1}(x)$. Then, $(p_i^{(n)}, c_i^{(n)}, s_i^{(n)})_{i\geq N} = (p_i, c_i, s_i)_{i\geq N}$ and*

$$\sigma^{N-1}(p_{N-1})\cdots p_0c_0x_1\cdots x_n = \sigma^{N-1}(p_{N-1}^{(n)})\cdots p_0^{(n)}.$$

Proof: By hypothesis, $\sigma(c_N) = p_{N-1}c_{N-1}s_{N-1} = p_{N-1}^{(n)}c_{N-1}^{(n)}s_{N-1}^{(n)}$ where $p_{N-1}c_{N-1}$ is a prefix of $p_{N-1}^{(n)}$. This implies that

$$(p_i^{(n)}, c_i^{(n)}, s_i^{(n)})_{i\geq N} = (p_i, c_i, s_i)_{i\geq N}.$$

Also, from the same equality we deduce that

$$\sigma^{N-1}(p_{N-1})\cdots p_0c_0s_0\cdots\sigma^{N-1}(s_{N-1}) = \sigma^{N-1}(p_{N-1}^{(n)})\cdots p_0^{(n)}c_0^{(n)}s_0^{(n)}\cdots\sigma^{N-1}(s_{N-1}^{(n)}).$$

However, by construction, x_{n+1} corresponds to $c_0^{(n)}$ and thus $x_1\cdots x_n$ is a suffix of $\sigma^{N-1}(p_{N-1}^{(n)})\cdots p_0^{(n)}$. Therefore, we conclude that

$$\sigma^{N-1}(p_{N-1})\cdots p_0c_0x_1\cdots x_n = \sigma^{N-1}(p_{n-1}^{(n)})\cdots p_0^{(n)}$$

as desired.

■

Lemma 5.26. *Let $\boldsymbol{y} \in \underline{\mathcal{S}}$ such that $c_0^{\boldsymbol{y}} = a \in \hat{A}$ and $v(a;\boldsymbol{y}) = v(a)$. Then $\boldsymbol{y}_1 = (p^{(a)}, a, s^{(a)})$.*

Proof: First we prove the following.
Claim: $v(c_1^{\boldsymbol{y}}) = v(c_1^{\boldsymbol{y}}; S(\boldsymbol{y}))$.
Indeed, let $\boldsymbol{z} = \boldsymbol{y}_0\boldsymbol{y}_1S(\boldsymbol{x}(c_1^{\boldsymbol{y}})) \in \underline{\mathcal{S}}$. If the assertion does not hold, then

$$v(a) = \theta_2^{-1}(\gamma(p_1^{\boldsymbol{y}}) + v(c_1^{\boldsymbol{y}}; S(\boldsymbol{y}))) > \theta_2^{-1}(\gamma(p_1^{\boldsymbol{y}}) + v(c_1^{\boldsymbol{y}})) = v(a;\boldsymbol{z}) \geq v(a)$$

which is a contradiction. Thus, $v(c_1^{\boldsymbol{y}}) = v(c_1^{\boldsymbol{y}}; S(\boldsymbol{y}))$ and furthermore $v(a) = v(a;\boldsymbol{z})$.
Then, the point $\boldsymbol{w} = (p^{(a)}, a, s^{(a)})(p^{(a)}, a, s^{(a)})S(\boldsymbol{z})$ verifies $v(a) = v(a;\boldsymbol{w})$. However, $\boldsymbol{w}$ and $\boldsymbol{x}(a)$ are ultimately equal, then by Lemma 5.23, $\boldsymbol{y}_1 = (p^{(a)}, a, s^{(a)})$.

■

Lemma 5.27. *Let $x \in X_\sigma$ be a minimal point for γ. Then,*

$$\liminf_{n\to\infty}\frac{\gamma(x_0\cdots x_n)}{n^{\log(\theta_2)/\log(\theta_1)}} > 0 \quad \textit{and} \quad \liminf_{n\to\infty}\frac{-\gamma(x_{-n}\cdots x_{-1})}{n^{\log(\theta_2)/\log(\theta_1)}} > 0.$$

Proof: We will only prove the firs inequality, the other one can be shown analogously. Assume, by contradiction, that the result does not hold. Then, for a subsequence $(n_i)_{i\in\mathbb{N}}$,

$$\lim_{i\to\infty} \frac{\gamma(x_0\cdots x_{n_i})}{n_i^{\log(\theta_2)/\log(\theta_1)}} = 0.$$

Let $(p_i, c_i, s_i)_{i\in\mathbb{N}}$ be the prefix-suffix decomposition of x and let $a \in \hat{A}$ such that $(p^{(a)}, a, s^{(a)})$ is its generating word (recall that it is ultimately periodic).

(1) First we assume that $s^{(a)}$ is not the empty word. For i large enough, let $N_i \geq 1$ be the minimal integer such that $x_1\cdots x_{n_i+1}$ is the prefix of $s_0\cdots\sigma^{N_i-1}(s^{(a)})$ (recall that $s_n = s^{(a)}$ for n large enough).

Consider the prefix-suffix decomposition $(p_j^{(n_i)}, c_j^{(n_i)}, s_j^{(n_i)})_{j\in\mathbb{N}}$ of $S^{n_i+1}(x)$. By Lemma 5.25,

$$\sigma^{N_i-1}(p_{N_i-1}^{(n_i)})\cdots\sigma(p_1^{(n_i)})p_0^{(n_i)} = \sigma^{N_i-1}(p_{N_i-1})\cdots\sigma(p_1)p_0x_0\cdots x_{n_i}.$$

Then,

$$\sum_{j=N_i-1}^{0} \theta_2^j\gamma(p_j^{(n_i)}) = \sum_{j=N_i-1}^{0} \theta_2^j\gamma(p_j) + \gamma(x_0\cdots x_{n_i}).$$

Dividing by $\theta_2^{N_i}$ we obtain

$$\sum_{j=1}^{N_i} \theta_2^{-j}\gamma(p_{N_i-j}^{(n_i)}) = \sum_{j=1}^{N_i} \theta_2^{-j}\gamma(p_{N_i-j}) + \theta_2^{-N_i}\gamma(x_0\cdots x_{n_i}). \tag{5.4}$$

Observe that n_i behaves like $\theta_1^{N_i}$, so $\lim\limits_{i\to\infty}\theta_2^{-N_i}\gamma(x_0\cdots x_{n_i}) = 0$. Also, since $(p_i, c_i, s_i)_{i\in\mathbb{N}}$ is ultimately periodic with generating word $(p^{(a)}, a, s^{(a)})$, then

$$\lim_{i\to\infty}\sum_{j=1}^{N_i} \theta_2^{-j}\gamma(p_{N_i-j}) = \sum_{j\geq 1}\theta_2^{-j}\gamma(p^{(a)}) = v(a;\boldsymbol{x}(a)) = v(a).$$

Thus, taking the limit when $i\to\infty$ in 5.4, we obtain

$$\lim_{i\to\infty}\sum_{j=1}^{N_i} \theta_2^{-j}\gamma(p_{N_i-j}^{(n_i)}) = v(a). \tag{5.5}$$

Now, for i large enough, consider the finite sequences:

$$\boldsymbol{y}^{(n_i)} = (p^{(a)}, a, s^{(a)})(p_{N_i-1}^{(n_i)}, c_{N_i-1}^{(n_i)}, s_{N_i-1}^{(n_i)})\cdots(p_0^{(n_i)}, c_0^{(n_i)}, s_0^{(n_i)}).$$

By construction, $p^{(a)}a$ is a stric prefix of $p_{N_i-1}^{(n_i)}c_{N_i-1}^{(n_i)}$ unless $n_i+1=0$ which is not the case. As in the proof of Lemma 5.19 bu taking a subsequence one constructs from the finite sequence $\boldsymbol{y}^{(n_i)}$ a limit point $\boldsymbol{y} = (p_i^{\boldsymbol{y}}, c_i^{\boldsymbol{y}}, s_i^{\boldsymbol{y}})_{i\in\mathbb{N}} \in \underline{\mathcal{S}}$.

Let $L\geq 1$. Then, for i large enough,

$$\begin{aligned}
v(a;\boldsymbol{y}) &= \sum_{j\geq 1}\theta_2^{-j}\gamma(p_j^{\boldsymbol{y}}) = \sum_{j=1}^{L}\theta_2^{-j}\gamma(p_j^{\boldsymbol{y}}) + \sum_{j>L}\theta_2^{-j}\gamma(p_j^{\boldsymbol{y}}) \\
&= \sum_{j=1}^{L}\theta_2^{-j}\gamma(p_{N_i-j}^{(n_i)}) + \sum_{j>L}\theta_2^{-j}\gamma(p_j^{\boldsymbol{y}}) \\
&= \sum_{j=1}^{N_i}\theta_2^{-j}\gamma(p_{N_i-j}^{(n_i)}) - \sum_{j=L+1}^{N_i}\theta_2^{-j}\gamma(p_{N_i-j}^{(n_i)}) + \sum_{j>L}\theta_2^{-j}\gamma(p_j^{\boldsymbol{y}}).
\end{aligned}$$

Let $\epsilon > 0$ and consider L such that $\sum_{j>L} \theta_2^{-j} \leq \epsilon/C$ where $C = \max\{|\gamma(p)| : p$ is a prefix of $\sigma(b)$ for some $b \in \mathcal{A}\}$. Now, for i big enough, we obtain that

$$|v(a;\boldsymbol{y}) - v(a)| \leq \left|\sum_{j=1}^{N_i} \theta_2^{-j}\gamma(p_{N_i-j}^{(n_i)}) - v(a)\right| + 2\epsilon.$$

Finally, from 5.5 we conclude that

$$|v(a;\boldsymbol{y}) - v(a)| \leq 2\epsilon$$

and, thus. $v(a;\boldsymbol{y}) = v(a)$. This implies that $(p^{(a)}, a, s^{(a)}) = (p_{N_i-1}^{(n_i)}, c_{N_i-1}^{(n_i)}, s_{N_i-1}^{(n_i)})$ for some large i, which is a contradiction.

(2) Now suppose that $s^{(a)}$ is the empty word. Then, considering a power of σ if necessary we can assume that

$$(x_n)_{n\geq N} = \lim_{m\to\infty} \sigma^m(b)$$

for some $N \in \mathbb{N}$ and $b \in \mathcal{A}$. If we write $\sigma(b) = bs$ we obtain

$$x_N x_{N+1} \cdots = bs\sigma(s)\sigma^2(s)\cdots.$$

Let $\boldsymbol{x} = (\varepsilon, b, s)_{i\in\mathbb{N}} \in \underline{\mathcal{S}}$. We have that $v(b;\boldsymbol{x}) = 0$.

Claim: $v(b) = 0$.

Suppose, by contradiction, that this is not true. Then $v(b) < 0$ and for $k \in \mathbb{N}$ large enough we have

$$\sum_{i=1}^{k} \theta_2^{k-i}\gamma(p_i^{\boldsymbol{x}(b)}) \leq K\theta_2^k$$

with $K < 0$. That is, γ applied to the prefix

$$\sigma^k(p_1^{\boldsymbol{x}(b)})\sigma^{k-1}(p_2^{\boldsymbol{x}(b)})\cdots p_k^{\boldsymbol{x}(b)}$$

of $\sigma^{k+1}(b)$ can be negative as we want if k increases. This implies that $\gamma_n(x) < 0$ for some $n \in \mathbb{N}$, which is impossible since x is a minimal point for γ, by hypothesis. Then $v(b) = 0$ as we claimed.

On the other hand, $v(b) = \sum_{j\geq 1} \theta_2^{-j}\gamma(p^{(b)})$ which implies $\gamma(p^{(b)}) = 0$. However, $\gamma(p^{(b)}) = \sum_{c\in\mathcal{A}} n_c\gamma(c)$, where n_c is the number of times the symbol c appears in $p^{(b)}$. Then, by Lemma 5.10,

$$p^{(b)} = \varepsilon \quad \text{and} \quad \boldsymbol{x}(b) = \boldsymbol{x}.$$

Finally, we shall prove that $\gamma(x_N \cdots x_{N+i}) > 0$ for all $i \geq 1$. Suppose that $\gamma(x_N \cdots x_{N+i}) < 0$ for some $i \geq 1$. Let $l \geq 1$ such that $x_N \cdots x_{N+i} = b\sigma(s)\cdots\sigma^{l-1}(s)s^-$ with s^- a prefix of $\sigma^l(s)$. By Lemma 5.13,

$$\gamma(\sigma(b)) = \theta_2\gamma(b) = \gamma(b) + \gamma(s)$$
$$\Rightarrow \gamma(s) = (\theta_2 - 1)\gamma(b).$$

We deduce that

$$\gamma(x_n \cdots x_{N+i}) = \gamma(b) + \sum_{j=1}^{l-1} \theta_2^j\gamma(s) + \gamma(s^-) = \theta_2^l\gamma(b) + \gamma(s^-).$$

Let $k \geq 0$ be an integer and write $\sigma^l(s) = s^- s^+$. Then,

$$\sigma^{l+k}(s) = \sigma^k(s^- s^+) = \sigma^k(s^-)\sigma^k(s^+)$$

and $b\sigma(s)\cdots\sigma^{l+k-1}(s)\sigma^k(s^-)$ is a prefix of $b\sigma(s)\cdots\sigma^{l+k}(s)$. We have

$$\begin{aligned}\gamma(b\sigma(s)\cdots\sigma^{l+k-1}(s)\sigma^k(s^-)) &= \gamma(b) + \sum_{j=1}^{l+k-1} \theta_2^j \gamma(s) + \theta_2^k \gamma(s^-) \\ &= \theta_2^k(\theta_2^l \gamma(b) + \gamma(s^-)).\end{aligned}$$

However $\theta_2^k(\theta_2^l\gamma(b)+\gamma(s^-))$ can be as negative as we want, so for some $M \in \mathbb{N}$, $\gamma(x_0\cdots x_M) < 0$, which contradicts the fact that x is a minimal point for γ. Therefore, $\gamma(x_N\cdots x_{N+i}) > 0$ for all $i \geq 1$.

To conclude, we use the proof of part **(1)** with b instead of a to deduce that

$$\liminf_{n\to\infty} \frac{\gamma(x_N\cdots x_{N+n})}{n^{\log(\theta_2)/\log(\theta_1)}} > 0$$

and then

$$\liminf_{n\to\infty} \frac{\gamma(x_0\cdots x_n)}{n^{\log(\theta_2)/\log(\theta_1)}} = \liminf_{n\to\infty} \frac{\gamma(x_0\cdots x_{N-1}) + \gamma(x_N\cdots x_{N+n})}{n^{\log(\theta_2)/\log(\theta_1)}} > 0.$$

■

The following proposition is plain.

Proposition 5.28. *Let $x \in X_\sigma$ be a minimal point for γ. Then,*

$$\sum_{n\geq 1} \exp(-\gamma(x_0\cdots x_{n-1})) < \infty \quad \textit{and} \quad \sum_{n\geq 1} \exp(\gamma(x_{-n}\cdots x_{-1})) < \infty.$$

5.4 PROOF OF THE MAIN THEOREM

We conclude this chapter with the proof of the main theorem. The arguments used follow the strategy developed in the works of [4] and [10].

Let $T(\lambda, \pi)$ be a self-similar IET and R its associated matrix. Assume R verifies the hypotheses of Theorem 5.7. To simplify notations, we shall use $T = T(\lambda, \pi)$ throughout.

Let X_σ be the substitutive system associated to T and let $M = R^t$ be the associated matrix. Consider a minimal poinit $x \in X_\sigma$. By Proposition 5.28,

$$K = \sum_{n\geq 1} \exp(\gamma(x_{-n}\cdots x_{-1})) + 1 + \sum_{n\geq 1} \exp(-\gamma(x_0\cdots x_{n-1})) < \infty.$$

Let $t = \varphi(x)$. That is, x is the coding of t or x is the coding of $(\lim_{s\to t^-} T^i(s))_{i\in\mathbb{Z}}$ in the case t is in the orbit of one of the a_i. To simplify notation we assume that the first case holds, the other is analogous.

Define the probability measure μ_t on $[0, 1)$ by

$$\mu_t = \frac{1}{K}\left(\sum_{n\geq 1} \exp(\gamma(x_{-n}\cdots x_{-1}))\delta_{T^{-n}(t)} + \delta_t + \sum_{n\geq 1} \exp(-\gamma(x_0\cdots x_{n-1}))\delta_{T^n(t)}\right).$$

Lemma 5.29. *For everey Borel set $I \subset [0,1)$*

$$\mu_t(T(I)) = \sum_{i=1}^{r} e^{-\gamma_i}\mu_t(I \cap [a_{i-1}, a_i)).$$

Proof: It is enough to consider $I = [a_{i-1}, a_i)$ for $i \in \mathcal{A}$. Thus, we have

$$\begin{aligned}
\mu_t(T(I)) &= \frac{1}{K}\left(\sum_{n\geq 1}\exp(\gamma(x_{-n}\cdots x_{-1}))\delta_{T^{-n}(t)} + \delta_t + \sum_{n\geq 1}\exp(-\gamma(x_0\cdots x_{n-1}))\delta_{T^n(t)}\right)(T(I))\\
&= \frac{1}{K}\left(\sum_{n\geq 1}\exp(\gamma(x_{-n}\cdots x_{-1}))\delta_{T^{-n-1}(t)} + \delta_{T^{-1}(t)} + \sum_{n\geq 1}\exp(-\gamma(x_0\cdots x_{n-1}))\delta_{T^{n-1}(t)}\right)(I)\\
&= \frac{1}{K}\left(\sum_{n\geq 1}\exp(-\gamma(x_{-n}))\exp(\gamma(x_{-n}\cdots x_{-1}))\delta_{T^{-n}(t)} + e^{-\gamma(x_0)}\delta_t\right.\\
&\left.+\sum_{n\geq 1}\exp(-\gamma(x_n))\exp(-\gamma(x_0\cdots x_{n-1}))\delta_{T^n(t)}\right)(I)\\
&= e^{-\gamma_i}\mu_t(I).
\end{aligned}$$

The last equality uses the fact that $T^n(t) \in I$ if and only if $\gamma(x_n) = \gamma_i$.

∎

Define $g : [0,1) \to [0,1)$ by $g(s) = \mu_t([0,s])$. This function is non-decreasing, right continuous and has left limits. Let $i \in \mathcal{A}$, denote $a_i' = T(a_i)$ and define $b_i = \lim_{a\to a_i^-} g(a)$ and $b_i' = \lim_{a'\to(a_i')^-} g(a')$. Then at interval $[b_{i-1}, b_i)$ define linearly the AIET f with image $[b_{i-1}', b_i')$. The slope vector of f is $w = (e^{-\gamma_1}, \ldots, e^{-\gamma_r})$. Indeed,

$$\frac{b_i' - b_{i-1}'}{b_i - b_{i-1}} = \frac{\mu_t([a_{i-1}', a_i'))}{\mu_t([a_{i-1}, a_i))} = e^{-\gamma_i},$$

where the last equality follows from Lemma 5.29.

Let $h : [0,1) \to [0,1)$ be the map defined by:

$$\begin{cases} h(v) = u, & \text{if} \quad g(u) = v;\\ h(v) = u, & \text{if} \quad \lim_{w\to u^-} g(w) \leq v \leq g(u). \end{cases}$$

Clearly h is surjective, continuous and non-decreasing. Since μ_t has atoms, then h is not injective.

The following lemma alows us to conlude Theorem 5.7

Lemma 5.30. *The map h defines a semi-conjugacy between the AIET f and T. Moreover, f has wandering intervals.*

Proof: The semi-conjugacy follows from construction. The interval

$$I = \left(\lim_{s\to t^-} g(s), g(t)\right]$$

is a wandering interval for f.

∎

6 RELATED RESULTS

In this chapter, we study some results and concepts related to the theory developed so far, which may be useful and of interest to the reader.

In the first section, we examine the results of [4], previously mentioned as motivation for the work carried out in the preceding chapter. In the second section, we discuss the Rauzy–Veech induction, and in the final section, we present the result of [11].

6.1 CAMELIER-GUTIERREZ

In this section, we study the main results of [4] and the ideas behind its construction. It should be noted that this article served as the basis for the main article [2], so understanding the former is a nice way to follow the reasoning developed in the latter.

The idea, as mentioned previously, is to prove the existence of a uniquely ergodic affine interval exchange transformation with wandering intervals that is semi-conjugate to an interval exchange transformation. The results of this article will be presented briefly, solely for the purpose of providing context for the type of construction that will be studied in detail in the next chapter.

6.1.1 Existence of the Affine Interval Exchange Transformation

Before stating the first results, let us introduce some important preliminary concepts.

Let $I = [a, b) \subset \mathbb{R}$ be a finite left-closed, right-open interval, and let $T\colon I \to I$ be an interval exchange transformation (IET) such that $(a = x_0, \ldots, x_m = b)$ are the endpoints of the subintervals in the associated partition. For notational convenience, we write $T = (x_0, \ldots, x_m)$.

(i) Let $I_1 = [c, d)$ be a proper subinterval of I. We say that the IET T is **renormalizable** (on I_1) if there exists an orientation-preserving affine map $L\colon \mathbb{R} \to \mathbb{R}$ such that $L(I_1) = I$ and

$$L \circ T_1 \circ L^{-1} = T,$$

where $T_1\colon I_1 \to I_1$ denotes the IET induced by T on I_1.

(ii) Let $\mathcal{M}$ be an $m \times m$ nonnegative matrix whose entries are given by

$$\mathcal{M}_{ji} = \#\Big\{\, 0 \le k \le N_i \;:\; T^k\Big([z_{i-1}, z_i)\Big) \subset [x_{j-1}, x_j) \,\Big\},$$

where N_i is the smallest nonnegative integer such that, for some $z \in [z_{i-1}, z_i)$ (and hence for all $z \in [z_{i-1}, z_i)$), one has

$$E^{N_i+1}(z) \in I_1.$$

We shall refer to $\mathcal{M}$ as the **matrix associated** with the pair (T, I_1).

Let $T\colon I \to I$ be a renormalizable IET on $I_1 \subset I$, and let Λ_m denote the cone of positive vectors in $\mathbb{R}^m$. It can be shown that the matrix $\mathcal{M}$ associated with (T, I_1) is aperiodic. Then, by the Perron–Frobenius theorem, $\mathcal{M}$ admits a unique probability right eigenvector $\alpha \in \Lambda_m$. Moreover, the corresponding eigenvalue μ is simple, real, and greater than 1, and all other eigenvalues of $\mathcal{M}$ have absolute value less than μ.

Veech and Masur proved that, under these conditions, T is uniquely ergodic. In addition, one can conclude that

$$\alpha = (x_1 - x_0, x_2 - x_1, \dots, x_m - x_{m-1}).$$

As previously seen, we say that an AIET $f\colon [0,1) \to [0,1)$ is **semi-conjugate** (respectively, **conjugate**) to an IET T if there exists a non-decreasing surjective (respectively, bijective) continuous map $h\colon [0,1) \to [0,1)$ such that

$$T \circ h = h \circ f.$$

The first result is the following.

Theorem 6.1. ***(Camelier-Gutierrez)*** *Let $T : [0,1) \to [0,1)$ be a renormalizable IET with exactly $m-1$ discontinuities and associated $m \times m$ matrix $\mathcal{M}$. Let $\mathcal{M}^*$ be the adjoint operator of $\mathcal{M}$. Let $\mathcal{G}^s$ be the $\mathcal{M}^*$-invariant stable subspace of $\mathbb{R}^m$ and let $\mathcal{G}^*$ be the $(m-1)$-dimensional $\mathcal{M}^*$-invariant subspace of $\mathbb{R}^m$ such that the spectral radius of $\mathcal{M}^*|_{\mathcal{G}^*}$ is less than the spectral radius of $\mathcal{M}^*$.*

a) Let $\tilde{\gamma} = (\gamma_1, \gamma_2, \dots, \gamma_m) \in \mathbb{R}^m$. There exists an AIET $f : [0,1) \to [0,1)$ semi-conjugate to T and with associated pair of vectors $(y, \tilde{\gamma})$ if and only if $\tilde{\gamma} \in \mathcal{G}^*$.

b) Let $\tilde{\gamma} = (\gamma_1, \gamma_2, \dots, \gamma_m) \in \mathcal{G}^s$. Let $f : [0,1) \to [0,1)$ be an AIET semi-conjugate to E and with associated pair of vectors $(y, \tilde{\gamma})$. Then f is conjugate to T.

The proof of 6.1 (a) is based on the following lemmas:

Lemma 6.2. *Let f be a uniquely ergodic AIET such that all of its orbits are infinite sets. Let μ be the f-invariant measure, and let $(y, \tilde{\gamma})$ be the pair of vectors associated with f, where $y = (y_0, y_1, \dots, y_m)$ and $\tilde{\gamma} = (\gamma_1, \gamma_2, \dots, \gamma_m)$. Then the vector $(\mu_1, \mu_2, \dots, \mu_m)$ is orthogonal to $\tilde{\gamma}$, where*

$$\mu_i = \mu([y_{i-1}, y_i]).$$

Here, $\mathbb{R}^m$ is endowed with the usual inner product.

For the next lemma, let μ be the spectral radius of $\mathcal{M}$. Denote by $\mathcal{F}$ (respectively, $\mathcal{F}^*$) the Perron–Frobenius eigenspace associated with the eigenvalue μ of $\mathcal{M}$ (respectively, of $\mathcal{M}^*$). Let

$$\mathbb{R}^m = \mathcal{F} \oplus \mathcal{G} \quad \text{and} \quad \mathbb{R}^m = \mathcal{F}^* \oplus \mathcal{G}^*$$

be the Jordan direct sum decompositions invariant under $\mathcal{M}$ and $\mathcal{M}^*$, respectively. In this way, the spectral radius ρ of both $\mathcal{M}|_{\mathcal{G}}$ and $\mathcal{M}^*|_{\mathcal{G}^*}$ is the same and satisfies

$$0 < \rho < \mu.$$

Lemma 6.3. *Under the above conditions, the orthogonal complement $\mathcal{F}^\perp$ of $\mathcal{F}$ with respect to the usual inner product $\langle\, , \,\rangle$ on $\mathbb{R}^m$ is $\mathcal{G}^*$.*

Lemma 6.4. *Let $\varphi = \{1, 2, \ldots, m\}$ and let $\gamma\colon \varphi \to \mathbb{R}$ be a map such that*

$$\tilde{\gamma} = (\gamma(1), \gamma(2), \ldots, \gamma(m)) \in \mathcal{G}^*.$$

Then there exists an AIET $f\colon [0,1) \to [0,1)$ that is semi-conjugate to T and whose associated pair of vectors is $(y, \tilde{\gamma})$.

The idea behind the proof of this lemma (and, consequently, of the first part of the theorem) is to construct the required AIET starting from a probability measure ν. This construction is carried out as follows:

Idea of the proof. It is possible to show that, under the above conditions, if $u\colon [0,1) \to \varphi$ is defined by

$$u(x) = i \iff x \in [x_{i-1}, x_i),$$

then there exists $x \in [0,1)$ such that

$$\limsup_k \left(\sum_{j=0}^{k} \gamma\Big(u(T^j(x))\Big) \right) \geq 0 \geq \liminf_k \left(\sum_{j=0}^{k} \gamma\Big(u(T^j(x))\Big) \right),$$

where $(x_1, x_2, \ldots, x_m)$ is the vector associated with T.

Now, let us denote $\vartheta_0 = \delta_x$ and, for all integers $k \geq 1$,

$$\vartheta_k = \exp\Big(\gamma(u(x)) + \gamma(u(T(x))) + \cdots + \gamma(u(T^{k-1}(x)))\Big)\, \delta_{T^k(x)},$$

where δ_x is the Dirac probability measure concentrated at x. Let

$$\nu_n = c_n(\vartheta_0 + \vartheta_1 + \cdots + \vartheta_{a_n+1}),$$

where c_n is the positive constant that makes ν_n a probability measure. It is not difficult to prove that there exists a probability measure ν which is a weak limit of the sequence $\{\nu_n\}$.

We now define the map $\phi\colon [0,1) \to [0,1)$ by

$$\phi(x) = \nu([0,x)), \qquad \phi(0) = \nu(\{0\}).$$

It is clear that ϕ is non-decreasing, and hence it has at most countably many discontinuities and is right-continuous. Therefore, the set

$$G = [0,1) \setminus \phi([0,1))$$

is the union of at most countably many intervals of the form $[a,b)$, which we call *gaps*.

Define the map $h\colon [0,1) \to [0,1)$ as follows:

(i) $h = \phi^{-1}$ on $[0,1) \setminus G$;

(ii) $h(x) = \phi^{-1}(b)$ for $a \leq x < b$, for each gap $[a,b) \subset G$.

Since ϕ^{-1} is also right-continuous and non-decreasing, it follows that h is continuous, non-decreasing, and satisfies $h([0,1)) = [0,1)$.

We now define the affine map $f\colon [0,1) \to [0,1)$. First, define the vector $(y_0, y_1, \dots, y_m)$ by

$$\begin{cases} y_0 = 0, \\ y_i = \phi(x_i), \quad i = 1, 2, \dots, m-1, \\ y_m = 1. \end{cases}$$

On each subinterval $[y_{i-1}, y_i)$ we define f as follows:

(i) On $[y_{i-1}, y_i) \setminus G$, the map h is one-to-one, so we define

$$f(x) = h^{-1} \circ T \circ h(x).$$

(ii) On G, if J is a gap, then

$$J' = h^{-1}(T(h(J)))$$

is also a gap and satisfies

$$|J'| = \exp(\gamma(i))\, |J|.$$

In this case, define $f|_J\colon J \to J'$ to be linear and orientation preserving.

The map $f\colon [0,1) \to [0,1)$ defined above is an AIET semi-conjugate to T.

■

The next theorem concerns the existence of wandering intervals. Its proof, although studied, will be omitted, as it is highly technical.

To state the result, we restrict ourselves to a particular IET. Let

$$\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in \Lambda_4$$

be the probability Perron–Frobenius right eigenvector of the matrix

$$M = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 0 & 0 & 2 & 1 \\ 1 & 1 & 2 & 2 \end{pmatrix},$$

that is, each $\alpha_i > 0$ and

$$|\alpha| = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 1.$$

We consider Rauzy's IET $R\colon [0,1) \to [0,1)$, which is determined by the following conditions. The map R has associated vector

$$x = (x_0, x_1, x_2, x_3, x_4),$$

where

$$x_0 = 0, \quad x_1 = \alpha_1, \quad x_2 = \alpha_1 + \alpha_2, \quad x_3 = \alpha_1 + \alpha_2 + \alpha_3, \quad x_4 = 1,$$

and satisfies

$$R(x_0) > R(x_2) > R(x_3) > R(x_1) = 0.$$

The map R is renormalizable on the interval $[0, \alpha_1)$, and A is precisely the matrix associated with the induced transformation $(R, [0, \alpha_1))$. It is numerically observed that

$\alpha_1 < \alpha_4$, and hence R is also induced on the interval $R([0, \alpha_1))$, with A again being the matrix associated with $(R, R([0, \alpha_1)))$.

The characteristic polynomial of A is the reciprocal polynomial

$$x^4 - 7x^3 + 13x^2 - 7x + 1,$$

which has four real roots

$$\rho^{-1},\ \lambda^{-1},\ \lambda,\ \rho$$

satisfying

$$0 < \rho^{-1} < \lambda^{-1} < 1 < \lambda < \rho.$$

Let A^* denote the adjoint operator of A, and let G^* (respectively, G^s) be the A^*-invariant vector subspace of dimension 3 (respectively, 2) generated by the eigenvectors corresponding to the eigenvalues $\rho^{-1}, \lambda^{-1}, \lambda$ (respectively, ρ^{-1}, λ^{-1}).

Theorem 6.5. ***(Camelier-Gutierrez)*** *One of the connected components, say C of $G^* \setminus G^s$ is such that, for all $\tilde{\gamma} = (\gamma_1, \gamma_2, \gamma_3, \gamma_4) \in C$, there exists a uniquely ergodic AIET $f : [0, 1) \to [0, 1)$ with associated pair of vectors $(y, \hat{\gamma})$ of the form $y = (y_0, y_1, y_2, y_3, y_4)$ and $\hat{\gamma} = (\delta, \gamma_1, \gamma_2, \gamma_3, \gamma_4)$ satisfying the following:*

a) $[y_0, y_1) = [0, y_1)$ is a wandering interval of f and, for all $n \in \mathbb{Z}$, $f^n([0, y_1))$ is an interval;

b) If $\theta = \bigcup_{n\in\mathbb{Z}} f^n([0, y_1))$, there exists a continuous bijective map

$$h : [0, 1) \setminus \theta \to [0, 1)$$

such that, for all $y \in [0, 1) \setminus \theta$, $R \circ h(y) = h \circ f(y)$.

6.2 RAUZY-VEECH INDUCTION

There exists a generalization of the Rauzy induction known as the Rauzy–Veech induction. This extension establishes a deep connection between interval exchange transformations and translation surfaces, the latter serving as a bridge that allows one to transfer results and techniques from one theory to the other.

Moreover, unlike the classical Rauzy induction, for which a given permutation admits two distinct preimages, the Rauzy–Veech induction defines a bijection on the space of irreducible IETs with permutations belonging to some Rauzy class.

In this section, we outline the main ideas underlying this theory in order to present a result from [11] that is closely related to the works studied in the previous chapters. The main reference used is [12].

6.2.1 Suspension Data over an Interval Exchange Transformation

We now describe the construction of a *suspension* over an interval exchange transformation T, that is, a flat surface for which T arises as the first return map of the vertical flow to a suitably chosen segment.

Let $T(\lambda, \pi)$ be an interval exchange transformation. A **suspension data** for T is a collection of vectors $(v_\alpha)_{\alpha\in\mathcal{A}}$ such that:

(i) $\mathrm{Re}(v_\alpha) = \lambda_\alpha, \ \forall \alpha \in \mathcal{A}$;

(ii) $\mathrm{Im}\left(\sum_{\pi_0(a)\le k} v_\alpha\right) > 0, \ \forall 1 \le k \le d-1$;

(iii) $\mathrm{Im}\left(\sum_{\pi_1(a)\le k} v_\alpha\right) < 0, \ \forall 1 \le k \le d-1$.

Given a suspension datum v, consider the broken lines L_ε, $\varepsilon \in \{0,1\}$, in $\mathbb{C} \simeq \mathbb{R}^2$, obtained by concatenating the vectors $v_{\pi_\varepsilon^{-1}(j)}$ in this order for $j = 1, \dots, d$, starting at the origin.

If the broken lines L_0 and L_1 intersect only at their common endpoints, then one can construct a translation surface S as follows: for each $\alpha \in \mathcal{A}$, identify the side corresponding to v_α in L_0 with the corresponding side in L_1 by translation. Let $X \subset S$ be the horizontal segment defined by

$$X = \left(0, \sum_{\alpha\in\mathcal{A}} \lambda_\alpha\right) \times \{0\}.$$

Then the associated interval exchange transformation T coincides with the first return map of the vertical flow on S to the transversal segment X.

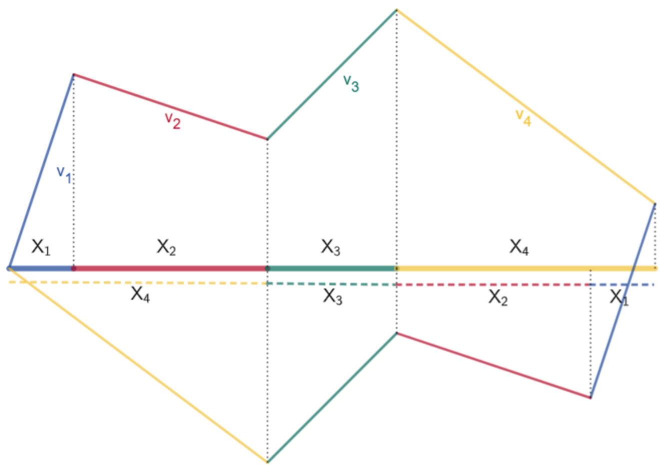

Figura 6.1: IET and suspension data
Source: Author's own work

Remark 6.6. *A necessary condition for an interval exchange transformation T (with permutation π) to admit suspension data is that the permutation π be irreducible.*

Using the Euler characteristic, it is possible to associate a surface $\mathcal{S}$ with the given interval exchange transformation.

Example 6.7. *In the figure above, the associated surface $\mathcal{S}$ is a bi-torus. Indeed, we have*

$$\chi(\mathcal{S}) = V - E + F = 1 - 4 + 1 = -2,$$

and therefore

$$2 - 2g = -2,$$

which implies that $g = 2$.

6.2.2 Rauzy-Veech Induction on Suspensions

Let $T(\lambda, \pi)$ be an interval exchange transformation and let v be suspension data over T. We define $\widehat{R}(v, \pi) = (v', \pi')$ as follows.

First, we set

$$(\mathrm{Re}(v'), \pi') = \widehat{R}(\mathrm{Re}(v), \pi),$$

where the right-hand side denotes the classical Rauzy induction.

Let $\varepsilon \in \{0, 1\}$ be such that $X_{\pi_\varepsilon^{-1}(d)}$ is the winner for $T(\mathrm{Re}(v), \pi)$. Then we define

$$\begin{cases} v'_{\pi_\varepsilon^{-1}(d)} = v_{\pi_\varepsilon^{-1}(d)} - v_{\pi_{1-\varepsilon}^{-1}(d)}, \\ v'_\alpha = v_\alpha, \quad \text{for all } \alpha \neq \pi_\varepsilon^{-1}(d). \end{cases}$$

The following figures illustrate how the induction works for the previous example.

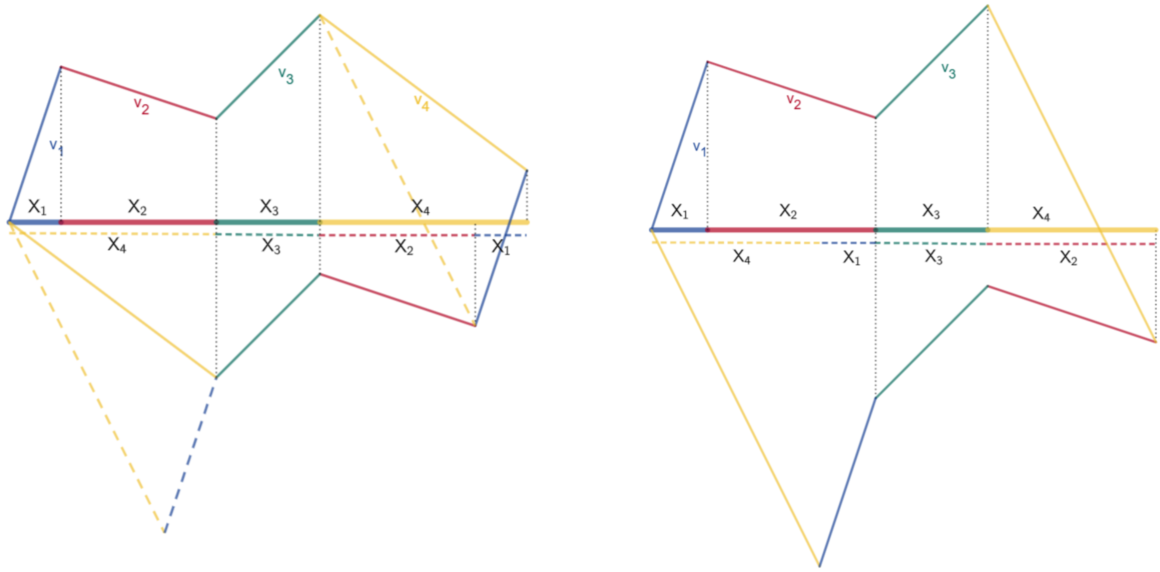

Figura 6.2: Rauzy-Veech Induction
Source: Author's own work

Remark 6.8. *Observe that the last point lies below the interval because the permutation arises from a Rauzy induction of type 0. If it were above the interval, it would correspond to an induction of type 1. This distinction explains the bijectivity of the Rauzy-Veech induction on Rauzy classes.*

6.3 MARMI-MOUSSA-YOCCOZ

With the material developed so far in this chapter, we are now in a position to understand the result established in [MMY].

Theorem 6.9. ***(Marmi-Moussa-Yoccoz)*** *For almost every interval exchange transformation T_0 whose associated surface has genus $g \geq 2$, there exists an affine interval exchange transformation T that is semi-conjugate to T_0 and possesses a wandering interval.*

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