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# Backlund transformation for integrable hierarchies: example - mKdV hierarchy

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**Abstract.** In this note we present explicitly the construction of the mKdV hierarchy and show that it decomposes into positive and negative graded sub-hierarchies. We extend the construction of the Backlund transformation for the sinh-Gordon model to all other positive and negative odd graded equations of motion generated by the same affine algebraic structure. Some simple examples of solutions are explicitly verified to satisfy, in a universal manner, the Backlund transformations for the first few odd (positive and negative) sub-hierarchies.

## 1. Introduction

A subclass of non linear integrable models underlined by an affine  $\hat{sl}(2)$  Lie algebra is well known to be connected to the mKdV equation. These are in fact higher flows obtained from the zero curvature representation [1] and by general algebraic arguments these flows are restricted to be related to certain positive odd grade generators. Another subclass of nonlinear integrable models containing for instance, the sinh-Gordon model may be formulated, also within the zero curvature representation, but now associated to negative grade generators. The relation between the mKdV and sine-Gordon models was already observed some time ago [2], [3] in terms of conservation laws and by more algebraic arguments it was generalized for the AKNS hierarchy in [4] and to other integrable hierarchies associated to mixed gradations allowing internal degrees of freedom in [5], [6] and [7]. The general construction and classification of the hierarchy maybe understood in terms of a decomposition of the affine Lie algebra into graded subspaces by a judicious choice of a grading operator. The hierarchy is further specified by choosing a constant grade one operator. These algebraic ingredients define a series of non-linear equations of motion, each corresponding to a different time evolution and hence to a hamiltonian structure (see for instance [8] and references therein).

Backlund transformation has recently been employed to extend the set of integrable models to incorporate defects. Such defects preserve the integrability when they are described by Backlund transformations connecting two distinct solutions of the same equation of motion at its location. This was firstly observed in [9] for the sine-Gordon and extended to affine Toda field theories with defects in [10] and to other non-relativistic models [11]. In this last reference the same space component of the Backlund transformation of the sine-Gordon models is employed to describe integrable defects within the mKdV model. It thus seem natural the extend the same space component of the Backlund transformation to other members of the integrable hierarchy.



In this note we start by reviewing, in section 2, the algebraic construction for general integrable hierarchies. We discuss explicitly the mKdV Hierarchy constructed out of the affine  $\hat{sl}(2)$  Kac-Moody algebra and principal gradation. We show how the *positive and negative odd* sub-hierarchies naturally arise from the zero curvature representation. An important point to notice is that, while the time component for the construction of each model varies with the gradation, the form of the space component remains the same for all models within the hierarchy. Moreover, we discuss, in section 3, the construction of Backlund transformation in terms of gauge transformation that preserves the form of the space Lax operator. By general arguments, the space Backlund transformation is understood to be universal within the hierarchy. This fact can be seen in ref. [12] where the space Backlund transformation for the sinh-Gordon and mKdV were derived by canonical transformations and shown to agree. Also, by the same procedure the space Backlund transformation for the KdV and the Sawada-Kotera equations (belonging to the same hierarchy) obtained in [12] and [13] respectively agree. Its time components can be extended to the whole hierarchy by using the appropriated equations of motion. As an explicit example, we derive the space Backlund transformation for the sinh-Gordon model and construct its time components for higher positive and negative grade time evolutions.

This, in fact, may be observed by the explicit space-time dependence of the soliton solutions. We explicitly display the 1- and 2-soliton solutions for all (odd positive and negative grade) models within the hierarchy and show how they can be arranged in pairs in order to satisfy the Backlund relations. Some explicit examples are verified in section 4.

## 2. The mKdV Hierarchy

The main ingredient underlying the construction of integrable hierarchies (IH) is the Lax operator,

$$L_x = \partial_x + E^{(1)} + A_0 \quad (1)$$

where  $E^{(1)}$  and  $A_0$  are Lie algebra  $\mathcal{G}$  valued elements and carry an affine structure which classifies the IH.

The systematic construction of IH and its Lax operators of the form (1) consists in the decomposition of an affine algebra  $\hat{\mathcal{G}}$  into integer graded subspaces

$$\hat{\mathcal{G}} = \sum_a \mathcal{G}_a, \quad a \in \mathbb{Z} \quad (2)$$

induced by a choice of a grading operator  $Q$ , such that

$$[Q, \mathcal{G}_a] = a\mathcal{G}_a, \quad [\mathcal{G}_a, \mathcal{G}_b] \in \mathcal{G}_{a+b}, \quad a, b \in \mathbb{Z}. \quad (3)$$

Furthermore, the IH is determined by fixing the semi-simple grade one operator  $E^{(1)} \in \mathcal{G}_1$  such that it decomposes  $\hat{\mathcal{G}} = \mathcal{K} \oplus \mathcal{M}$  where  $\mathcal{K}$  is the Kernel of  $E^{(1)}$  and  $\mathcal{M}$  is its complement, i.e.,

$$\mathcal{K} = \{x \in \hat{\mathcal{G}}, [x, E^{(1)}] = 0\}, \quad (4)$$

such that

$$[\mathcal{K}, \mathcal{K}] \subset \mathcal{K}, \quad [\mathcal{K}, \mathcal{M}] \subset \mathcal{M}, \quad [\mathcal{M}, \mathcal{M}] \subset \mathcal{K}.$$

The equations of motion are determined by solving the zero curvature equation

$$[\partial_x + E^{(1)} + A_0, \partial_{t_N} + D^{(N)} + D^{(N-1)} + \dots + D^{(0)}] = 0, \quad (5)$$

The solution of eq. (5) may be systematically constructed by considering  $D^{(j)} \in \mathcal{G}_j$  and  $A_0 \in \mathcal{K}_0$ ,  $\mathcal{K}_0 \in \mathcal{G}_0$  and can be decomposed according to the graded structure as

$$[E, D^{(N)}] = 0 \quad (6)$$

$$\begin{aligned} [E, D^{(N-1)}] + [A_0, D^{(N)}] + \partial_x D^{(N)} &= 0 \\ \vdots &= \vdots \\ [A_0, D^{(0)}] + \partial_x D^{(0)} - \partial_{t_N} A_0 &= 0. \end{aligned} \quad (7)$$

The unknown  $D^{(j)}$ 's can be solved starting from the highest to the lowest grade projections as functionals of  $A_0$  and its  $x$ -derivatives. Notice that, in particular the highest grade equation, namely  $[E, D^{(N)}] = 0$  implies  $D^{(N)} \in \mathcal{K}$ . If we consider the fields of the theory to parametrize  $A_0 \in \mathcal{K}_0$ , the equations of motion are obtained from the zero grade component (7).

We shall now work with an explicit example of the *mkdV hierarchy* based upon the  $\hat{\mathcal{G}} = \hat{sl}(2)$  affine algebra,

$$[h^{(m)}, E_{\pm\alpha}^{(n)}] = \pm 2E_{\pm\alpha}^{(n)}, \quad [E_{\alpha}^{(m)}, E_{-\alpha}^{(n)}] = h^{(m+n)} \quad (8)$$

The grading operator is  $Q = 2\lambda \frac{d}{d\lambda} + \frac{1}{2}h$  and decomposes the affine algebra  $\hat{\mathcal{G}}$  into even and odd graded subspaces

$$\begin{aligned} \mathcal{G}_{2m} &= \{h^{(m)} = \lambda^m h\}, \\ \mathcal{G}_{2m+1} &= \{\lambda^m (E_{\alpha} + \lambda E_{-\alpha}), \lambda^m (E_{\alpha} - \lambda E_{-\alpha})\} \end{aligned} \quad (9)$$

for  $m = 0, \pm 1, \pm 2, \dots$  and  $[\mathcal{G}_a, \mathcal{G}_b] \subset \mathcal{G}_{a+b}$ . The integrable hierarchy is then specified by a choice of a semi-simple element  $E = E^{(1)}$ , where

$$E^{(2n+1)} = \lambda^n (E_{\alpha} + \lambda E_{-\alpha}) \quad (10)$$

and  $A_0 = v(x, t_n)h^{(0)}$ . The Kernel of  $E^{(1)}$  is therefore given by

$$\mathcal{K} = \mathcal{K}_{2n+1} = \{\lambda^n (E_{\alpha} + \lambda E_{-\alpha})\} \quad (11)$$

and has grade  $2n + 1$ . It thus follows from (6) that the highest grade component of  $D^{(N)}$  has grade  $N = 2n + 1$ . The component within the  $\mathcal{M}$  of the zero grade projection of (7) leads to the evolution equations according to time  $t = t_{2n+1}$ . Notice that  $D^{(0)}$  lies within the Cartan subalgebra and hence  $[A_0, D^{(0)}] = 0$ . The equations of motion are then simplified to  $\partial_{t_{2n+1}} A_0 = \partial_x D^{(0)}$ . Examples are,

$$n = 1 \quad 4\partial_{t_3} v = \partial_x (\partial_x^2 v - 2v^3) \quad mKdV \quad (12)$$

$$n = 2 \quad 16\partial_{t_5} v = \partial_x (\partial_x^4 v - 10v^2(\partial_x^2 v) - 10v(\partial_x v)^2 + 6v^5), \quad (13)$$

$$\begin{aligned} n = 3 \quad 64\partial_{t_7} v &= \partial_x (\partial_x^6 v - 70(\partial_x v)^2(\partial_x^2 v) - 42v(\partial_x^2 v)^2 - 56v(\partial_x v)(\partial_x^3 v)) \\ &- \partial_x (14v^2\partial_x^4 v - 140v^3(\partial_x v)^2 - 70v^4(\partial_x^2 v) + 20v^7) \end{aligned} \quad (14)$$

$\dots etc$

For the negative mKdV sub-hierarchy let us propose the following form for the zero curvature representation

$$[\partial_x + E^{(1)} + A_0, \partial_{t_{-N}} + D^{(-N)} + D^{(-N+1)} + \dots + D^{(-1)}] = 0. \quad (15)$$

Differently from the positive hierarchy case, the lowest grade projection now yields,

$$\partial_x D^{(-N)} + [A_0, D^{(-N)}] = 0,$$

a nonlocal equation for  $D^{(-N)}$ . Having solved for  $D^{(-N)}$ , the second lowest projection of grade  $-N + 1$ , leads to

$$\partial_x D^{(-N+1)} + [A_0, D^{(-N+1)}] + [E^{(1)}, D^{(-N)}] = 0$$

which determines  $D^{(-N+1)}$ . The process follows recursively until we reach the zero grade projection

$$\partial_{t_{-N}} A_0 - [E^{(1)}, D^{(-1)}] = 0 \quad (16)$$

which yields the evolution equation for field  $A_0$  according to time  $t = t_{-N}$ . Notice that in this case there is no condition upon  $N$ .

The simplest example is to take  $N = 1$  when the zero curvature decomposes into

$$\begin{aligned} \partial_x D^{(-1)} + [A_0, D^{(-1)}] &= 0, \\ \partial_{t_{-1}} A_0 - [E^{(1)}, D^{(-1)}] &= 0. \end{aligned} \quad (17)$$

In order to solve the first equation, we define the zero grade group element  $B = \exp(\mathcal{G}_0)$  and define

$$D^{(-1)} = B E^{(-1)} B^{-1}, \quad A_0 = -\partial_x B B^{-1}, \quad (18)$$

Under such parametrization the second eqn. (18) becomes the well known (relativistic) Leznov-Saveliev equation,

$$\partial_{t_{-1}} (\partial_x B B^{-1}) + [E^{(1)}, B E^{(-1)} B^{-1}] \quad (19)$$

which for  $\hat{sl}(2)$  with principal gradation  $Q = 2\lambda \frac{d}{d\lambda} + \frac{1}{2}h$ , yields the sinh-Gordon equation

$$\partial_{t_{-1}} \partial_x \phi = e^{2\phi} - e^{-2\phi}, \quad B = e^{-\phi h}. \quad (20)$$

where  $t_{-1} = z, x = \bar{z}, A_0 = v h \equiv \partial_x \phi h$ .

For higher values of  $N = 2, 3, 4, \dots$  we found

$$\partial_{t_{-2}} \partial_x \phi = 4e^{-2\phi} d^{-1} e^{2\phi} 4e^{2\phi} d^{-1} e^{-2\phi} \quad (21)$$

$$\partial_{t_{-3}} \partial_x \phi = 4e^{-2\phi} d^{-1} (e^{2\phi} d^{-1} (\sinh 2\phi)) + 4e^{2\phi} d^{-1} (e^{-2\phi} d^{-1} (\sinh 2\phi)) \quad (22)$$

$$\begin{aligned} \partial_{t_{-4}} \partial_x \phi &= 4e^{-2\phi} d^{-1} (e^{2\phi} d^{-1} (e^{-2\phi} d^{-1} e^{2\phi} + e^{2\phi} d^{-1} e^{-2\phi})) \\ &+ 4e^{2\phi} d^{-1} (e^{-2\phi} d^{-1} (e^{-2\phi} d^{-1} e^{2\phi} + e^{2\phi} d^{-1} e^{-2\phi})) \end{aligned} \quad (23)$$

$$\begin{aligned}
\partial_{t_{-5}} \partial_x \phi &= 8e^{-2\phi} d^{-1} \left( e^{2\phi} d^{-1} \left( e^{-2\phi} d^{-1} \left( e^{2\phi} d^{-1} (\sinh 2\phi) \right) + e^{2\phi} d^{-1} \left( e^{-2\phi} d^{-1} (\sinh 2\phi) \right) \right) \right) \\
&+ 8e^{2\phi} d^{-1} \left( e^{-2\phi} d^{-1} \left( e^{-2\phi} d^{-1} \left( e^{2\phi} d^{-1} (\sinh 2\phi) \right) + e^{2\phi} d^{-1} \left( e^{-2\phi} d^{-1} (\sinh 2\phi) \right) \right) \right),
\end{aligned} \tag{24}$$

where  $d^{-1}f = \int^x f(y)dy$ . For  $N = 2$  eqn. (21) was derived in [17] using recursion operators.

We now consider soliton solutions for the entire hierarchy. The general algebraic structure of the zero curvature representation yields a general method for constructing soliton solutions based on the fact that  $v = 0$  ( and/or  $\phi = 0$ ) is the vacuum solution for all positive and negative odd sub-hierarchies, i.e., eqns. (12)-(14) and (20), (22), (24),  $\dots$ , etc.<sup>1</sup> The idea of the dressing method is to map the *trivial vacuum* into a nontrivial configuration by gauge transformation, i.e,

$$\partial_x + A_0 = (\Theta_{\pm})^{-1} \left( \partial_x + E^{(1)} + A_0^{vac} \right) \Theta_{\pm},$$

$$\partial_{t_k} + D^{(k)} + \dots + D^{(0)} = (\Theta_{\pm})^{-1} \left( \partial_{t_k} + D_{vac}^{(k)} + \dots + D_{vac}^{(0)} \right) \Theta_{\pm}$$

where  $\Theta_{\pm}$  are group elements of the form

$$\Theta_{-}^{-1} = e^{p^{(-1)}} e^{p^{(-2)}} \dots, \quad \Theta_{+}^{-1} = e^{q^{(0)}} e^{q^{(1)}} e^{q^{(2)}} \dots,$$

$p^{(-i)}$  and  $q^{(i)}$  are linear combinations of grade  $(-i)$  and  $(i)$  generators respectively.

It thus follows that one and two soliton solutions for the mKdV hierarchy with  $A_0^{vac} = 0$  can be written as

$$\begin{aligned}
\phi_{1-sol} &= \ln \left( \frac{1 - a_1 \rho_1}{1 + a_1 \rho_1} \right) \\
\phi_{2-sol} &= \ln \left( \frac{1 - a_1 \rho_1 - a_2 \rho_2 + a_1 a_2 a_{12} \rho_1 \rho_2}{1 + a_1 \rho_1 + a_2 \rho_2 + a_1 a_2 a_{12} \rho_1 \rho_2} \right) \\
\vdots &= \vdots
\end{aligned} \tag{25}$$

$$a_{12} = \left( \frac{\kappa_1 - \kappa_2}{\kappa_1 + \kappa_2} \right)^2 \text{ and}$$

$$\rho_i(\kappa_i) = \exp 2(k_i x + t_a (k_i)^a). \tag{26}$$

These are solutions for all equations of the positive and negative odd hierarchies, i.e., eqns. (12)-(14) and (20), (22), (24),  $\dots$ , for  $a = 3, 5, 7, \dots$  and  $a = -1, -3, -5, \dots$  respectively.

It is clear that  $\phi_{vac} = \phi_0 = 0$  do not satisfy eqns. (21) or (23). It follows that the negative even cases, (21), (23), etc, do not admit zero vacuum solution, i.e.  $v_{vac} = 0$ . is not a solution. The soliton solutions (25) have to be modified accordingly to  $v_{vac} = v_0 \neq 0$ , (see [8]) and these cases shall be discussed elsewhere.

<sup>1</sup> For negative even sub-hierarchy, the vacuum solution is obtained for  $v = v_0 \neq 0$  and the dressing method works equally well generating general formulae for multisoliton solutions but with a deformation parameter  $v_0$ , see for instance [8]

### 3. Backlund Transformation

In this section we start by noticing that the zero curvature representation (5) or (15) of the form

$$[\partial_x + A_x, \partial_t + A_t] = 0 \quad (27)$$

are invariant under gauge transformations of the type

$$A_\mu(\phi, \partial_x \phi, \dots) \rightarrow \tilde{A}_\mu = U^{-1} A_\mu U + U^{-1} \partial_\mu U \quad (28)$$

If we now choose  $U(\phi_1, \phi_2)$  such that it maps one field configuration  $\phi_1$  into another field configuration  $\phi_2$  preserving the equations of motion (i.e., zero curvature (27)) see [18],

$$U A_\mu(\phi_1) = A_\mu(\phi_2) U + \partial_\mu U \quad (29)$$

If we now take  $A_\mu = A_x = E^{(1)} + A_0$  which is *common to all members* of the hierarchy, we find that

$$U = \begin{bmatrix} 1 & -\frac{\beta}{2\lambda} e^{-(\phi_1 + \phi_2)} \\ -\frac{\beta}{2} e^{(\phi_1 + \phi_2)} & 1 \end{bmatrix} \quad (30)$$

satisfies (29) provided

$$\partial_x (\phi_1 - \phi_2) = -\beta \sinh (\phi_1 + \phi_2). \quad (31)$$

For the sinh-Gordon model, the equations of motion (20) are satisfied if we further introduce the time component of the Backlund transformation,

$$\partial_{t-1} (\phi_1 + \phi_2) = \frac{1}{\beta} \sinh (\phi_2 - \phi_1). \quad (32)$$

where  $\phi_a, a = 1, 2$  satisfy the sinh-Gordon eqn,  $\partial_{t_1} \partial_x \phi_a = \frac{1}{2} \sinh \phi_a$ . The gauge transformation (30) leads to the Backlund transformation for the negative odd sub-hierarchy. Consider first the  $t = t_{-3}$  evolution equation (22) where

$$A_{t_{-3}} = D^{(-3)} + D^{(-2)} + D^{(-1)} \quad (33)$$

where

$$\begin{aligned} D^{(-3)}(\phi) &= -a \left( e^{-2\phi} E_\alpha^{(-1)} + e^{2\phi} E_{-\alpha}^{(-1)} \right), & D^{(-2)}(\phi) &= 2a I(\phi) h^{(-1)} \\ D^{(-1)}(\phi) &= -4a \left( e^{-2\phi} \int^x e^{2\phi} I(\phi) E_\alpha^{(-1)} + e^{2\phi} \int^x e^{-2\phi} I(\phi) E_{-\alpha}^{(0)} \right) \end{aligned} \quad (34)$$

where  $I(\phi_i) = \int^x \sinh(2\phi_i)$ ,  $i = 1, 2$ . Inserting the gauge transformation  $U$  (30) into (29) for  $A_\mu = A_{t_{-3}}$  given in (33) and (34) we find the Backlund transformation

$$\partial_{t_{-3}}(\phi_1 + \phi_2) = \frac{8}{\beta} e^{\phi_1 - \phi_2} \int^x e^{2\phi_2} I(\phi_2) - \frac{8}{\beta} e^{-\phi_1 + \phi_2} \int^x e^{2\phi_1} I(\phi_1), \quad (35)$$

together with the subsidiary conditions

$$\begin{aligned} I(\phi_2) - I(\phi_1) &= \beta e^{\phi_1 - \phi_2} \int^x \left( e^{-2\phi_1} I(\phi_1) + e^{2\phi_2} I(\phi_2) \right) \\ I(\phi_2) + I(\phi_1) &= \frac{2}{\beta} \sinh(\phi_2 - \phi_1) \end{aligned} \quad (36)$$

The very same argument follows for  $t = t_{-5}$  where  $A_{t_{-5}} = D^{(-5)} + D^{(-4)} + D^{(-3)} + D^{(-2)} + D^{(-1)}$ . Here (30) into (29) yields,

$$\partial_{t_{-5}}(\phi_1 + \phi_2) = \frac{16}{\beta} e^{\phi_1 - \phi_2} \int^x e^{2\phi_2} W(\phi_2) - \frac{16}{\beta} e^{-\phi_1 + \phi_2} \int^x e^{-2\phi_2} W(\phi_2) \quad (37)$$

where  $W(\phi_i) = \int^x (e^{-2\phi_i} \int^y e^{2\phi_i} I(\phi_i)) dy + \int^x (e^{2\phi_i} \int^y e^{-2\phi_i} I(\phi_i)) dy$  together with the subsidiary conditions

$$\begin{aligned} W(\phi_2) - W(\phi_1) &= \beta e^{\phi_1 - \phi_2} \left( \int^x e^{-2\phi_1} W(\phi_1) + e^{2\phi_2} W(\phi_2) \right) \\ I(\phi_2) - I(\phi_1) &= \beta e^{\phi_1 - \phi_2} \left( \int^x e^{-2\phi_1} I(\phi_1) + e^{2\phi_2} I(\phi_2) \right) \\ I(\phi_2) + I(\phi_1) &= \frac{2}{\beta} \sinh(\phi_2 - \phi_1). \end{aligned} \quad (38)$$

Other subsidiary relations are obtained from (36) and (38) by replacing  $\phi_i \rightarrow -\phi_i$ .

The key observation that allows us to extend such Backlund transformation to other positive higher grade members of the hierarchy (12 - 14), etc is to notice that the zero grade component of equation (7) is trivially solved by parametrizing  $A_0 = -\partial_x B B^{-1}$  and  $D^{(0)} = d_0 h^{(0)} = -\partial_{t_{2n+1}} B B^{-1}$ . On the other hand by solving grade by grade the zero curvature eqn. (6)-(7) we find explicit expressions (42)-(44) for  $D^{(0)}$ . We define then the multi-time evolution for the field  $v(x, t_N) = \partial_x \phi(x, t_N)$  to be

$$n = 1 \quad \partial_{t_3} \phi(x, t) \equiv d_0 = \frac{1}{4} \partial_x^3 \phi - \frac{1}{2} (\partial_x \phi)^3 \quad (39)$$

$$n = 2 \quad \partial_{t_5} \phi(x, t) \equiv d_0 = \frac{1}{16} \partial_x^5 \phi - \frac{5}{8} (\partial_x \phi)^2 \partial_x^3 \phi - \frac{5}{8} \partial_x \phi (\partial_x^2 \phi)^2 + \frac{3}{8} (\partial_x \phi)^3, \quad (40)$$

$$\begin{aligned} n = 3 \quad \partial_{t_7} \phi(x, t) \equiv d_0 &= \frac{1}{64} \partial_x^7 \phi - \frac{35}{32} (\partial_x^2 \phi)^2 (\partial_x^3 \phi) - \frac{21}{32} (\partial_x \phi) (\partial_x^3 \phi)^2 - \frac{7}{8} (\partial_x \phi) (\partial_x^2 \phi) (\partial_x^4 \phi) \\ &- \frac{7}{32} (\partial_x \phi)^2 (\partial_x^5 \phi) + \frac{35}{16} (\partial_x \phi)^3 (\partial_x^2 \phi)^2 + \frac{35}{32} (\partial_x \phi)^4 (\partial_x^3 \phi) - \frac{5}{16} \partial_x^8 \phi \end{aligned} \quad (41)$$

... etc

It follows that the Backlund transformation for the time component  $t_3$  may be derived from (40) by considering

$$4\partial_{t_3}(\phi_1 - \phi_2) = \partial_x^3 \phi_1 - \partial_x^3 \phi_2 - 2(\partial_x \phi_1)^3 + 2(\partial_x \phi_2)^3,$$

Eliminating  $\partial_x \phi_2$  from  $x$ - component of Backlund transf. for Sinh-Gordon, i.e.,  $\partial_x(\phi_1 - \phi_2) = -\beta \sinh(\phi_1 + \phi_2)$ , we find

$$\begin{aligned} 4(\partial_{t_3} \phi_2 - \partial_{t_3} \phi_1) &= \beta (\partial_x^2 \phi_1 + \partial_x^2 \phi_2) \cosh(\phi_1 + \phi_2) - \frac{\beta}{2} (\partial_x \phi_1 + \partial_x \phi_2)^2 \sinh(\phi_1 + \phi_2) \\ &- \frac{\beta^3}{2} \sinh^3(\phi_1 + \phi_2). \end{aligned} \quad (42)$$

which is in agreement with [14].

Analogously the same follows from (41) and (42) for  $t_5$  and  $t_7$  respectively, yielding

$$\begin{aligned} 16\partial_{t_5}(\phi_2 - \phi_1) &= 2\beta \partial_x^4 \phi_1 \cosh(\phi_1 + \phi_2) - 4\beta \partial_x \phi_1 (\partial_x^3 \phi_1) \sinh(\phi_1 + \phi_2) \\ &- 12\beta (\partial_x \phi_1)^2 \partial_x^2 \phi_1 \cosh(\phi_1 + \phi_2) + 2\beta (\partial_x^2 \phi_1)^2 \sinh(\phi_1 + \phi_2) \\ &+ 6\beta (\partial_x \phi_1)^4 \sinh(\phi_1 + \phi_2) - 4\beta^2 (\partial_x \phi_1)^3 + 2\beta^2 \partial_x^3 \phi_1 \\ &- 2\beta^3 (\partial_x \phi_1)^2 \sinh(\phi_1 + \phi_2) + 2\beta^3 \partial_x^2 \phi_1 \cosh(\phi_1 + \phi_2) \\ &+ 2\beta^4 \partial_x \phi_1 + \beta^5 \sinh(\phi_1 + \phi_2) \end{aligned} \quad (43)$$



and

$$\begin{aligned}
64(\partial_{t_7}\phi_2 - \partial_{t_7}\phi_1) = & -20\beta(\partial_x\phi_1)^6 \sinh(\phi_{12}) + 20\beta\partial_x(\phi_1)^2(\partial_x^2\phi_1)^2 \sinh(\phi_{12}) \\
& - 20\beta(\partial_x^2\phi_1)^3 \cosh(\phi_{12}) + 40\beta(\partial_x\phi_1)^3\partial_x^3\phi_1 \sinh(\phi_{12}) \\
& - 80\beta\partial_x\phi_1\partial_x^2\phi_1\partial_x^3\phi_1 \cosh(\phi_{12}) - 2\beta(\partial_x^3\phi_1)^2 \sinh(\phi_{12}) \\
& - 20\beta(\partial_x\phi_1)^2\partial_x^4\phi_1 \cosh(\phi_{12}) + 4\beta\partial_x^2\phi_1\partial_x^4\phi_1 \sinh(\phi_{12}) \\
& + 60\beta(\partial_x\phi_1)^4\partial_x^2\phi_1 \cosh(\phi_{12}) - 4\beta\partial_x\phi_1\partial_x^5\phi_1 \sinh(\phi_{12}) \\
& + 2\beta\partial_x^6\phi_1 \cosh(\phi_{12}) + 12\beta^2(\partial_x\phi_1)^5 - 20\beta^2\partial_x\phi_1(\partial_x^2\phi_1)^2 \\
& - 20\beta^2(\partial_x\phi_1)^2\partial_x^3\phi_1 + 2\beta^2\partial_x^5\phi_1 - 12\beta^3(\partial_x\phi_1)^4 \sinh(\phi_{12}) \\
& - 12\beta^3(\partial_x\phi_1)^2\partial_x^2\phi_1 \cosh(\phi_{12}) + 2\beta^3(\partial_x^2\phi_1)^2 \sinh(\phi_{12}) \\
& - 4\beta^3\partial_x\phi_1\partial_x^3\phi_1 \sinh(\phi_{12}) + 2\beta^3\partial_x^4\phi_1 \cosh(\phi_{12}) \\
& - 4\beta^4(\partial_x\phi_1)^3 + 2\beta^4\partial_x^3\phi_1 - 2\beta^5(\partial_x\phi_1)^2 \sinh(\phi_{12}) \\
& + 2\beta^5\partial_x^2\phi_1 \cosh(\phi_{12}) + 2\beta^6\partial_x\phi_1 + \beta^7 \sinh(\phi_{12})
\end{aligned} \tag{44}$$

where  $\phi_{12} \equiv \phi_1 + \phi_2$ .

#### 4. Examples

In this section we shall consider few solutions for the Backlund solutions for (12)-(14) and (20), (22) and (24).

##### 4.1. Vacuum - 1-soliton

Let

$$\phi_1 = \phi_{vac} = 0, \quad \phi_2 = \phi_{1-sol} = \ln \left( \frac{1 + R\rho}{1 - R\rho} \right), \quad \rho = e^{2kx + 2k^N t_N} \tag{45}$$

and  $R$  is a constant. It becomes clear that the Backlund equations (42)-(44) are satisfied by (45) for  $\beta = 2k$ .

##### 4.2. 1-soliton - 1-soliton

$$\phi_1 = \phi_{1-sol}(k_1) = \ln \left( \frac{1 + R_1\rho_1}{1 - R_1\rho_1} \right), \quad \phi_2 = \phi_{1-sol}(k_2) = \ln \left( \frac{1 + R_2\rho_2}{1 - R_2\rho_2} \right), \quad \rho_i = e^{2k_i x + 2k_i^N t_N} \tag{46}$$

and  $R_i = R_i(k_i)$ . Backlund solutions given in (46) satisfy the Backlund equations (42)-(44) for

$$k_1 = k_2 = k, \quad R_2 = \left( \frac{2k + \beta}{2k - \beta} \right) R_1. \tag{47}$$

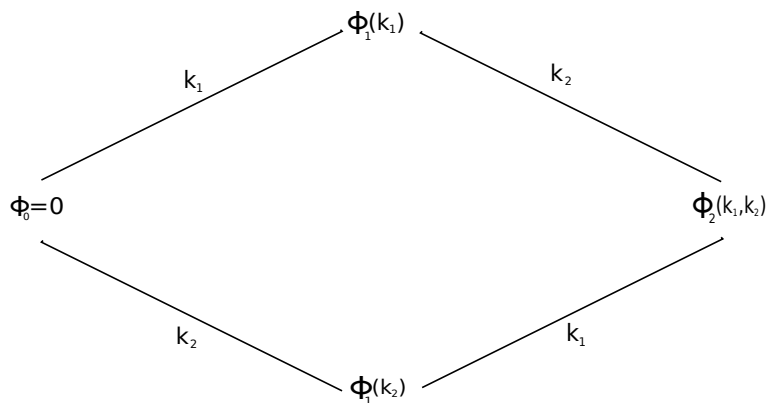
respectively for  $N = 3, 5$  and  $7$ . It was verified that the same also satisfy eqns. (20), (22) and (24) with  $N = -1, -3$  and  $-5$  respectively.

##### 4.3. 1-soliton - 2-soliton

$$\phi_1 = \phi_{1-sol} = \ln \left( \frac{1 + \rho_1}{1 - \rho_1} \right), \quad \phi_2 = \phi_{2-sol} = \ln \left( \frac{1 + \delta(\rho_1 - \rho_2) - \rho_1\rho_2}{1 - \delta(\rho_1 - \rho_2) - \rho_1\rho_2} \right), \tag{48}$$

where  $\delta = \frac{k_1 + k_2}{k_1 - k_2}$ .

We have verified that the Backlund equations for the higher members of the hierarchy, namely, (42)-(44) are satisfied by (48) for  $N = 3, 5$  and  $7$ . Also, the permutability theorem, which establishes the equality of the 2-soliton solution obtained from vacuum to 1-soliton solution for  $\beta = 2k_1$  and subsequently this 1-soliton to 2-soliton solutions and  $\beta = 2k_2$ . On the other hand the very same 2-soliton solution is obtained from the vacuum to 1-soliton solution for  $\beta = 2k_2$  and subsequently this 1-soliton to 2-soliton solutions and  $\beta = 2k_1$ . For technical reasons we were unable to verify this case for the negative odd equations (20), (22) and (24).



**Figure 1.** Permutability theorem for 2-solitons solution

## 5. Conclusions and Further Remarks

The construction of a set of non-linear integrable equations of motion were defined in terms of a zero curvature representation and an affine Lie algebra. Explicitly, we have considered the mKdV hierarchy for positive and negative odd graded time evolutions. A class of soliton solutions, labeled according to different graded time evolutions, were constructed in a universal manner to all members of the hierarchy. These were derived, by the dressing method from the trivial vacuum solution, i.e.  $v_{vac} = 0$ .

We have extended the Backlund transformation of the sinh-Gordon model to other higher and lower graded members of the mKdV hierarchy. We have shown that the spatial component of the Backlund transformation is common to all members of the hierarchy and is a direct consequence of the common Lax operator. These equations are verified to be satisfied by the same pair of solutions for the first few members of the hierarchy.

For the negative even graded equations the trivial vacuum is observed *not* to be solution of the equations of motion and a deformation of the dressing method need to be employed (see [8]).

We are extending the same arguments from the  $N = 1$  super sinh-Gordon model to the supersymmetric mKdV hierarchy [15].

So far, our arguments were based upon known spatial component of the Backlund transformation of the Sinh-Gordon model. Such relation defines the so called type I Backlund transformation. The extension to type II Backlund transformation [16], [18] is a subject for future investigation.

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