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FACULDADE DE ENGENHARIA
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**ATTITUDE CONTROL OF INVERTED PENDULUMS USING REACTION
WHEELS AND VARIABLE SPEED CONTROL MOMENT GYROSCOPE**

ILHA SOLTEIRA

2020

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WHEELS AND VARIABLE SPEED CONTROL MOMENT GYROSCOPE**

A thesis submitted to the Faculdade de Engenharia de Ilha Solteira - UNESP in partial fulfillment of the requirements for obtaining the Doctorate in Mechanical Engineering.

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*"We cannot solve our problems with the same
thinking we used when we created them."*

Albert Einstein (1879 - 1955)

Abstract

The control of inverted pendulums has attracted attention and motivated the research development of the control and robotics fields for many decades. These systems can approximate the dynamics of engineering and real-world applications. Additionally, different angular momentum exchange devices can be used as actuators to control inverted pendulums. This thesis investigates new pendulum configurations that employ reaction wheels or a variable speed control moment gyroscope to drive and to control a rigid pendulum in the inverted position. A pendulum configuration using two reaction wheels is introduced, and linear and nonlinear controllers are designed. Their performances are verified experimentally using a self-manufactured plant and low-cost hardware to employ the real-time controllers. Moreover, the use of a variable speed control moment gyroscope combining the angular momentum variation in direction and amplitude is carried out to control two different pendulums. Firstly, a classical rigid pendulum is studied where the model and a nonlinear controller derived from the Lyapunov theory are presented. Furthermore, for a better evaluation, the pendulum configurations proposed are compared to the classical reaction wheel pendulum and the results of the proposed pendulum configurations highlight the performance improvement. Afterwards, the model and controller of a generic 3D pendulum with a variable speed control moment gyroscope are derived. Under appropriate assumptions, this new model yields a spherical pendulum that is also controlled in the inverted position.

Keywords: Inverted pendulum. Reaction wheels. Nonlinear dynamics. Nonlinear control. Low-cost hardware. Variable speed control moment gyroscope.

Resumo

O controle de pêndulos invertidos têm atraído atenção e motivado o desenvolvimento de pesquisa nas áreas de controle e robótica por muitas décadas. Esses sistemas podem aproximar a dinâmica de aplicações de engenharia e do mundo real. Além disso, diferentes dispositivos de troca de momento angular podem ser utilizados como atuadores para o controle de pêndulos invertidos. Essa tese de doutorado investiga novas configurações de pêndulo que empregam rodas de reação ou controle por momento giroscópico com velocidade variável para levar e controlar um pêndulo rígido na posição invertida. Uma configuração utilizando duas rodas de reação é introduzida e controladores lineares e não lineares são projetados. O desempenho deles é verificado experimentalmente em uma planta construída pelo autor e que emprega uma *hardware* de baixo custo para os controladores em tempo real. Além do mais, a utilização do controle por momento giroscópico com velocidade variável combinando a variação do momento angular em amplitude e direção é realizada para o controle de dois pêndulos invertidos diferentes. Primeiramente, um pêndulo rígido planar clássico é estudado onde são apresentados o modelo e um controlador não linear obtido a partir da teoria de Lyapunov. Ademais, para uma melhor avaliação, as configurações de pêndulo propostas são comparadas com o conhecido pêndulo com roda de reação e os resultados das configurações propostas mostram o aperfeiçoamento no desempenho. Posteriormente, o modelo e controlador para um pêndulo 3D genérico com um dispositivo de controle por momento giroscópico com velocidade variável são apresentados. A partir de hipóteses apropriadas, esse novo modelo resulta em um pêndulo esférico que também é controlado na posição invertida.

Palavras-chave: Pêndulo invertido. Rodas de reação. Dinâmica não linear. Controle não linear. Hardware de baixo custo. Controle por momento giroscópico com velocidade variável.

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List of Acronyms

2-RWP	- Two reaction wheels pendulum
CMG	- Control moment gyroscope
IAE	- Integral of the absolute magnitude of the error
IDA-PBC	- Interconnection and damping assignment passivity-based control
ISE	- Integral of the square of the error
ITAE	- Integral of time multiplied by the absolute error
ITSE	- Integral of time multiplied by the square error
LMIs	- Linear matrix inequalities
LPRS	- Locus of a perturbed relay system
LQR	- Linear quadratic regulator
MRPs	- Modified Rodrigues parameters
PID	- Proportional integral and derivative
PWM	- Pulse width modulation
RMP	- Reaction mass pendulum
RW	- Reaction wheel
RWP	- Reaction wheel pendulum
SMC	- Sliding mode control
VSC	- Variable structure control
VSCMG	- Variable speed control moment gyroscope

List of Symbols

$\{\hat{b}_1, \hat{b}_2, \hat{b}_3\}$	- Body frame unit vectors
B	- Body frame
$[BG]$	- Rotation matrix between body moving frame B and gimbal moving frame G
$[BN]$	- Rotation matrix between the inertial frame and moving frame B
C_c	- Coefficient of the Coulomb friction
C_v	- Coefficient of viscous friction
D_{ist}	- Disturbance
e	- Error
$\hat{e}$	- Principal axis unit vector
$\{\hat{g}_t, \hat{g}_g, \hat{g}_s\}$	- Gimbal frame unit vectors
g	- Gravity
G	- Gimbal moving frame
h	- Nominal angular momentum value
$\mathbf{H}$	- Angular momentum
$\dot{\mathbf{H}}$	- Rate of the angular momentum
$\mathbf{H}$	- Total angular momentum
$\mathbf{H}_B$	- Angular momentum of the body
$\mathbf{H}_G$	- Angular momentum of the gimbal
$\mathbf{H}_W$	- Angular momentum of the reaction wheel
$\{\hat{i}, \hat{j}, \hat{k}\}$	- RWP and 2-RWP inertial frame unit vectors
$\{\hat{i}_1, \hat{j}_1, \hat{k}_1\}$	- RWP and 2-RWP body frame unit vectors
$\{\hat{i}_2, \hat{j}_2, \hat{k}_2\}$	- RWP and 2-RWP reaction wheel frame unit vectors
$\{\hat{i}_3, \hat{j}_3, \hat{k}_3\}$	- 2-RWP reaction wheel 2 frame unit vectors
I_a	- Armature current
$[I]$	- Total inertia matrix of 3D pendulum with VSCMG
$[I_{3 \times 3}]$	- 3x3 identity matrix
$[I_g]$	- Gimbal inertia matrix
$[I_p]$	- VSCMG pendulum inertia matrix

$[I_s]$	- 3D pendulum inertia matrix
$[I_W]$	- Reaction wheel inertia matrix
I_{zeq}^O	- Moment of inertia of the pendulum and the reaction wheel
I_{zw}^A	- Moment of inertia of the reaction wheel relative to the fixed point A
I_{zw1}^A	- Moment of inertia of the reaction wheel 1 relative to fixed point A
I_{zw2}^B	- Moment of inertia of the reaction wheel 2 relative to fixed point B
J	- Moment of inertia
$[J]$	- Combined inertia matrix of the gimbal and reaction wheel
k_d	- Derivative gain
k_i	- Integral gain
k_p	- Proportional gain
K	- Stationary gain
K_t	- Motor torque constant
K_v	- Back electromotive force constant
K_γ	- Gimbal subservo gain
K_ψ	- Reaction wheel subservo gain
K_Ω	- Reaction wheel subservo gain
ℓ	- Length of the pendulum
ℓ_2	- Length of counterweight rod
L	- Inductance
$\mathbf{L}$	- External torque sum
$\mathbf{L}_r$	- Required torque
m_1	- Mass of the first reaction wheel of 2-RWP
m_2	- Mass of the second reaction wheel of 2-RWP
m	- Mass of the pendulum
m_{cw}	- Mass of the counterweight
m_{dcm}	- Mass of the DC motor
m_{h2}	- Mass of the counterweight rod
m_p	- Mass of the pendulum
M	- Mass of the reaction wheel
$\{\hat{n}_1, \hat{n}_2, \hat{n}_3\}$	- Inertial frame unit vectors
N	- Inertial frame
$[Q]$	- Attitude matrix
r_1	- Radius of the first reaction wheel of 2-RWP
r_2	- Radius of the second reaction wheel of 2-RWP
r	- Distance from fixed point O to the center of mass

r_{cw}	- Radius of the counterweight
R	- Radius of the reaction wheel
R_a	- Armature resistance
sat	- Saturation function
sgn	- Sign function
T	- Torque provided by the reaction wheel
T_1	- Torque provided by the reaction wheel 1
T_2	- Torque provided by the reaction wheel 2
T_{dcn}	- motor torque
T_{Gfr}	- Generic friction torque
$T_{Fr_{RWP}}$	- Friction torque of the RWP
T_{fr}	- Friction torque of the 2-RWP
$\mathbf{T}_O$	- Torque sum
u	- Control law using sliding mode control
u_{PD}	- Proportional-derivative control law
u_g	- Gimbal motor torque
u_s	- Reaction wheel motor torque
$V(t)$	- Voltage applied to the DC motor
$V(s)$	- Voltage applied to the DC motor in Laplace domain
V_α	- Voltage applied to the DC motor coupled to reaction wheel 1
V_β	- Voltage applied to the DC motor coupled to reaction wheel 2
$\mathcal{V}$	- Lyapunov function
$\dot{\mathcal{V}}$	- Derivative of the Lyapunov function
W	- Wheel moving frame
$[W]$	- Weights matrix
W_w	- Reaction wheel mode weight
W_g	- Gimbal mode weight

Greek Letters

α	- Angle that rotates solidarity to the first reaction wheel of 2-RWP
$\dot{\alpha}$	- Angular velocity of the first reaction wheel of 2-RWP
$\ddot{\alpha}$	- Angular acceleration of the first reaction wheel of 2-RWP
β	- Angle that rotates solidarity to the second reaction wheel of 2-RWP
$\dot{\beta}$	- Angular velocity of the second reaction wheel of 2-RWP
$\ddot{\beta}$	- Angular acceleration of the second reaction wheel of 2-RWP
γ	- Gimbal angle
$\dot{\gamma}$	- Gimbal angular velocity
$\ddot{\gamma}$	- Gimbal angular acceleration
Δ	- Width of the saturation function
$\Delta\dot{\gamma}$	- Gimbal rate tracking error
$\Delta\Omega$	- Reaction wheel velocity tracking error
δ	- Non dimensional scalar factor
$\delta\omega$	- Body angular velocity error
ζ	- Positive constant related to closed loop bandwidth
$\boldsymbol{\eta}$	- State vector
$\boldsymbol{\eta}_{2-RWP}$	- State vector for the 2-RWP
$\boldsymbol{\eta}_{sw}$	- State vector for the swing-up strategy
θ_d	- Desired reference set for pendulum angular position
θ	- Angle that rotates solidarity to the pendulum
$\dot{\theta}$	- Angular velocity of the pendulum
$\ddot{\theta}$	- Angular acceleration of the pendulum
κ	- Proximity of singularity factor
ξ	- Positive switching gain
$\sigma(e, t)$	- Sliding function
$\boldsymbol{\sigma}$	- MRP vector
$\boldsymbol{\sigma}^S$	- MRP shadow set vector
$\boldsymbol{\sigma}_e$	- Relative attitude vector

τ	-	Time constant
Φ	-	Principal angle rotation
ψ	-	Angular position of the reaction wheel of the RWP
$\dot{\psi}$	-	Angular velocity of the reaction wheel
$\ddot{\psi}$	-	Angular acceleration of the reaction wheel
Ω	-	Angular velocity
$\omega(t)$	-	Angular velocity of the DC motor
$\dot{\omega}(t)$	-	Angular acceleration of the DC motor
$\omega(s)$	-	Angular velocity in Laplace domain
Ω	-	Reaction wheel angular velocity
$\dot{\Omega}$	-	Reaction wheel angular acceleration
$\omega_{B/N}$	-	Body angular velocity relative to inertial frame
$\omega_{G/B}$	-	Gimbal angular velocity relative to body frame
$\omega_{W/G}$	-	Wheel angular velocity relative to gimbal frame

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1 INTRODUCTION

This chapter presents the motivation, the main objectives, the main contributions, and the outline of this thesis.

1.1 MOTIVATION

The control and stabilization of inverted pendulums have been studied for decades. Although these systems seem simple, they have many attractive characteristics that motivate research and are often used as benchmarks to test new controllers and new arrangements of actuators. A few examples of these systems are: the reaction wheel pendulum (SPONG; CORKE; LOZANO, 2001), the wheeled inverted pendulum (GUO; XU; LEE, 2014), the pendulum on a cart system (OZANA; SCHLEGEL, 2018), the reaction mass pendulum (SANYAL; GOSWAMI, 2014), the reaction wheel unicycle (NEVES; ANGÉLICO; AGULHARI, 2019), the variable speed control moment gyroscope pendulum (TRENTIN; SILVA; SCHAUB, 2019b), the pendulum with two reaction wheels (TRENTIN *et al.*, 2020a), among others. Due to each of the system's particularities, which can be unstable, highly nonlinear, underactuated, and nonminimum phase, the controllers must present many different features.

Other inverted pendulum-like systems have been studied recently. Owczarkowski, Horla and Zietkiewicz (2019) have investigated the control of a real bicycle robot which has its balance controlled by a reaction wheel pendulum, where a combination of feedback-linearization-based techniques and robust linear quadratic regulator (LQR) control laws was carried out. Moreover, the studies of unicycles were also conducted, where Jin *et al.* (2016) use the gyroscopic precession of two gyroscopes to provide lateral balance to the system. Neves, Angélico and Agulhari (2019) have designed a robust controller that minimized the $\mathcal{H}_2$ system norm considering parametric uncertainties in the polytopic domain for a reaction wheel unicycle, and the controller performance was verified experimentally.

A ball-bot system which is a pendulum balanced on a ball, has also attracted attention in the past few years (CAI; LU; LI, 2019; LIMA *et al.*, 2019; SEKHAR *et al.*, 2019). This system consists of an inverted pendulum rolling on a sphere and can be controlled by its Omni-wheels (LIMA *et al.*, 2019) or using reaction wheels (SEKHAR *et al.*, 2019). Sekhar *et al.* (2019) have derived a mathematical model using Lagrangian mechanics, where it included reaction wheels to control the ball-bot system. Proportional-Integral-Derivative (PID) and Linear Quadratic Regulator (LQR) controllers were designed to control the system with or without the reaction wheels to compare the performances. Furthermore, Lima *et al.* (2019) have verified experimentally a Linear Matrix Inequalities (LMI) based robust $\mathcal{H}_2$ controller for such a system.

Therefore, most of these inverted pendulum-like systems are nonlinear and under-actuated, which makes their control and stabilization more complicated (LIU; YU, 2013). Thereby, the attitude control of devices has the challenges mentioned earlier, besides taking into account uncertainties and disturbances during operation. The problem of balancing inverted pendulum systems is of paramount importance for control and robotic communities.

Reaction wheels can be applied to many fields and in particular in the control and stabilization of mechanisms due to its simple configuration and its reliability to aerospace industry applications (RUI; KOLMANOVSKY; MCCLAMROCH, 2000; NUDEHI *et al.*, 2008; JEPSEN *et al.*, 2009; ISMAIL; VARATHARAJOO, 2010). This actuator class is very common in systems that require attitude change, and it can store angular momentum, which may aid in stabilization and rejection of external disturbances torques. The most usual way to use a reaction wheel to produce torque is from the angular velocity variation causing an angular momentum variation in amplitude. In this scenario, from specific angular velocities, it is possible to control the oscillation and direction rate of rigid bodies in space through the application of torques in the reaction wheel. The fundamental physical principle consists of the validity of the conservation of the angular momentum and the equilibrium of its rate of variation with the involved torques (SPERRY, 1913; DEIMEL, 1950; SCARBOROUGH, 1958; CANNON, 1967; WHITTAKER; MCCRAE, 1988; SCHAUB; JUNKINS, 2014; GOLDSTEIN; POOLE; SAFKO, 2011).

Furthermore, the use of the torque amplification feature from control moment gyroscopes (CMGs) combined to the reaction wheels' reliability brings forward unique results for the control of inverted pendulum systems. This can be done when using variable speed control moment gyroscope (VSCMG) to control inverted pendulums. Such a device is able to use the properties of the CMGs and of the reaction wheels favorably.

1.2 OBJECTIVES

The main objective of this doctorate is to investigate new pendulum configurations changing the dynamics of known systems. Some of the new configurations proposed are verified experimentally using low-cost hardware to employ linear and nonlinear controllers. The objectives can be summarized as:

- To investigate the new pendulum configuration, the two reaction wheels pendulum (2-RWP), evaluating the difference between using one and two actuators;
- To evaluate the use of a Variable Speed Control Moment Gyroscope mechanism coupled to an inverted pendulum.

1.3 MAIN CONTRIBUTIONS

The main contributions of this thesis to the literature are:

- Experimental comparison of the control of an inverted pendulum using one and two actuators to drive the pendulum from the downward position to the unstable equilibrium point in the inverted position (TRENTIN *et al.*, 2020a);
- Design and real-time implementation of a sliding mode controller and of a non-linear proportional-derivative controller for a pendulum with two reaction wheels improving the results of a simple and well-know PID controller (TRENTIN *et al.*, 2020b);
- Design of a swing-up control strategy that uses a reaction wheel at a time and takes into account the plant limitations, also using the known features of sliding mode control favorably;
- The proposition of a new inverted pendulum system controlled by a Variable Speed Control Moment Gyroscope. The model and controller are described and the comparison of the performance to a well-established pendulum configuration is carried out (TRENTIN; SILVA; SCHAUB, 2019b).
- Derivation of a nonsingular model for a 3D pendulum configuration with a VSCMG that under appropriate assumptions yield a spherical pendulum. The control design for such configuration assures global asymptotic stabilization.

1.4 OUTLINE

This thesis is structured into the following chapters:

- **Chapter 1 - INTRODUCTION:** covers the motivation, objectives, main contributions, and the scope of the thesis.
- **Chapter 2 - A BRIEF STATE-OF-THE-ART REVIEW:** illustrates the main contributions from the literature to the control of inverted pendulum systems that employ reaction wheels. It also presents some crucial works on inverted pendulum-like systems.
- **Chapter 3 - PENDULUMS WITH REACTION WHEELS:** exhibits the mathematical model of the pendulums, the identification of the friction of the experimental devices which are included into the model, the experimental description of the devices, and the comparison of using one or two actuators employing a PID controller.
- **Chapter 4 - NONLINEAR CONTROL OF AN INVERTED PENDULUM WITH TWO REACTION WHEELS:** carries out the design of nonlinear controllers, the design of a swing-up control strategy for the pendulum with reaction wheels and verifies the performance experimentally;
- **Chapter 5 - VARIABLE SPEED CONTROL MOMENT GYROSCOPE USED TO CONTROL INVERTED PENDULUMS:** brings forward the model and design of a nonlinear controller of a new planar pendulum configuration controlled by a VSCMG and compares its performance to the reaction wheel pendulum. It also shows the derivation of the model and controller for the 3D pendulum with VSCMG.
- **Chapter 6 - FINAL REMARKS:** comprises the conclusions, the contributions to the literature, and the suggestions for future work.

2 A BRIEF STATE-OF-THE-ART REVIEW

This chapter reviews the state of the art of pendulums controlled using reaction wheels and interesting inverted pendulum-like systems that motivated and helped to develop this thesis.

2.1 REACTION WHEEL PENDULUMS AND THEIR CONTROL

Control and stabilization of mechanisms using reaction wheels are common due to their simple configuration and reliability for applications in the aerospace industry (RUI; KOLMANOVSKY; MCCLAMROCH, 2000; JEPSEN *et al.*, 2009). This kind of actuator is commonly used in systems that require an attitude change, such as, robots, (WALKER; WEE, 1991; DUBOWSKY; PAPADOPOULOS, 1993) satellites (NUDEHI *et al.*, 2008; ISMAIL; VARATHARAJOO, 2010), and spacecraft (URAKUBO; TSUCHIYA; TSUJITA, 2004). Reaction wheels can store angular momentum, which may help to stabilize and reject external disturbance torques.

The most common way of using a reaction wheel to produce torque is from the angular velocity variation, which causes an amplitude variation of angular momentum. In this scenario, specific angular velocities can control the oscillation and direction rates of change of rigid bodies in space by applying controlled torques in the reaction wheel. The fundamental physical principle consists of the conservation of the angular momentum and the equilibrium of its rate of variation being equal to the involved torques (SPERRY, 1913; DEIMEL, 1950; SCARBOROUGH, 1958; CANNON, 1967; ARDEMA, 2005; GOLDSTEIN; POOLE; SAFKO, 2011).

To illustrate such concept, the reaction wheel pendulum can be used. It is relatively well known in the literature and it has a wide range of studies about it. A reaction wheel pendulum (RWP) consists of a rigid pendulum with a free spinning disk attached

to its free end. The free spinning disk is the reaction wheel, which is the actuator of the system. RWP is an underactuated nonlinear system and in this kind of system, the number of actuators is lower than the degrees of freedom of the system, which makes the system's attitude control more difficult. The RWP is first deeply discussed by Spong, Corke and Lozano (2001), where feedback linearization and passivity-based control were used for swing up and control it in the inverted position. Ortega *et al.* (2002) have proposed the stabilization of a class of underactuated systems through interconnection and damping-assignment using a new passivity-based formulation; this new controller was tested in the RWP. Hernández (2003) has designed a control strategy that combined sliding modes and generalized PI control for swinging up and controlling the RWP around its unstable equilibrium point. Bapiraju *et al.* (2004) have presented some balancing control strategies for the RWP that included three kinds of linearization and a fuzzy logic controller. Santibanez, Kelly and Sandoval (2005) carries out the control of RWP applying limited torques and following also a formulation based on interconnection and damping assignment passivity-based control (IDA-PBC) verifying the possibility of steering the pendulum to the inverted position with the actuation within prescribed limits.

Block, Astrom and Spong (2007) present in detail all the aspects related to the RWP including its history and background, the mathematical model encompassing the sensors and actuators for feedback control use, the stabilization of the pendulum in the inverted position using linear control as well as the nonlinear control problem of the swing-up control. Furthermore, important practical issues are addressed discussing the friction modeling and compensation, quantization of sensor signals, and saturation. Ye, Wang and Wang (2007) perform the control of an RWP using saturation techniques where the its dynamic model is transformed into a chain of integrators subjected to nonlinear perturbations. Aguilar-Ibanez, F. and Castanon (2008) consider the RWP strongly damped and propose a controller also using saturation techniques transforming the system into a chain of integrators making all the states converge to zero. Srinivas and Behera (2008) propose some other control strategies to take the RWP to the inverted position also using interconnection and damping assignment passivity-based control (IDA-PBC). Jepsen *et al.* (2009) have introduced a hybrid switching controller that to swing up the pendulum used a simple bang-bang controller and to control it in the inverted position an observer-based feedback controller was designed. Bobtsov, Pyrkin and Kolyubin (2009) design a hybrid controller using an algorithm with an adaptive adjustment of the controller's parameters. Lopez-Martinez, Acosta and Cano (2010) have proposed using a nonlinear sliding surface to control underactuated systems, among them the RWP; they

have also considered uncertainties in physical parameters.

Andrievsky (2011) has designed an energy-based controller that uses the speed gradient to swing up the pendulum and a variable structure controller with forced sliding modes to balance the RWP in the inverted position. Pereira and Weber (2011) places the RWP in the horizontal plane and performed the position control using a PID controller adjusting its gains with the Ziegler-Nichols method. Aguilar *et al.* (2012) have designed a two-relay controller combined with feedback linearization to generate oscillations with the desired amplitude and frequency to control RWP in the inverted position. Instead of using approximated solutions, they proposed two exact ones based on the first return of the Poincaré map and the Locus of a Perturbed Relay System (LPRS). Iriarte, Aguilar and Fridman (2013) present a second-order sliding mode controller for the swing up and control of the RWP. Olivares and Albertos (2014) have performed the linear control of RWP assuming a linearization in the unstable equilibrium point (inverted position) using, at first, a PID controller to stabilize the system. In the sequence, the PID controller is replaced by an observer-based state feedback control. Sun, Fang and Chen (2015) introduce a new version of a sliding mode controller wherein the RWP model was included parametric uncertainties to illustrate the robustness of the proposed controller. Sowman, Laila and Longo (2015) have demonstrated that a predictive controller that uses a nonlinear model for prediction presented very interesting results to control RWP in the inverted position.

Most recently, Rosól, Pilat and Tumaú (2016) solve an optimization problem using a control algorithm in an open loop, and a real-time experiment was performed to evaluate the proposed algorithm. Zhang *et al.* (2016) introduce a control strategy based on coordinates transformation, and in a three-step time-reverse strategy. Moreno-Valenzuela *et al.* (2017) have proposed two controllers, one based on an adaptive neural network, and the other one based on adaptive regressors to control RWP in the inverted position. Aguilar-Avelar *et al.* (2017) are the first ones to discuss the effects of friction in the model and control of RWP, and what could be done to compensate for this effect. After the inclusion of Coulomb friction in the model, a feedback-linearization controller was designed, and the results with and without friction were compared with a real-time experiment. Gritli and Belghith (2017) have proposed a robust controller based on Linear Matrix Inequalities (LMIs) to control RWP in the inverted position. Haddad, Chemori and Belghith (2017) present a robustness improvement related to disturbances combining interconnection and damping assignment passivity-based control (IDA-PBC) with an adaptive controller to swing up and control RWP in the inverted position. Kant

and Mukherjee (2018) study the swing-up problem for the reaction wheel pendulum using a sequence of impulsive inputs to simplify the dynamics and to allow the design of optimal sequences under general assumptions. Ramírez-Neria *et al.* (2019) use a flatness based approach for the linear active disturbance rejection control to stabilize the RWP around its inverted position. Franco (2019) has investigated a new control strategy based on interconnection and damping assignment passivity-based control (IDA-PBC) including a term for the disturbance compensation and the effectiveness of this new strategy was verified in the reaction wheel pendulum. Montoya and Gil-González (2019) carries out a nonlinear analysis and control based on a Lyapunov approach, comparing classical methodologies to swing-up and control the RWP in the inverted position. Guo and Liu (2019) have studied the stabilization problem around the unstable equilibrium point using the classic sliding mode method and used two sliding mode observers to obtain the unmeasurable states. This literature review has shown so far that the reaction wheel pendulum has been used as a benchmark for testing and developing new control strategies. However, this is not the goal of this thesis.

There is a large variety of control techniques to be applied to stabilize inverted pendulums as mentioned above. The present thesis also investigates the use of two nonlinear control techniques, one Lyapunov-based and the other based on variable structure control (VSC), to be applied in the same configuration proposed in Trentin *et al.* (2020a). The Lyapunov direct method can provide a helpful tool to analyze the stability of dynamical systems. From this method, it is possible to propose energy-like functions that must be positive definite with its derivative being negative semi-definite or definite (IBÁÑEZ; FRIAS; CASTAÑÓN, 2005; IBÁÑEZ; AZUELA, 2007; QUEIROZ *et al.*, 2012; SCHAUB; JUNKINS, 2014). Under these conditions, this function is a Lyapunov function and can be used to conclude about the system stability.

The variable structure control (VSC) is an advanced nonlinear control strategy that has provided adequate means to design robust state feedback controllers. The aim is to obtain the desired plant behavior, changing its structure automatically on reaching a switching surface. It is an efficient control strategy used to a broad class of both nonlinear and linear systems subject to external disturbances and modeling uncertainty (SLOTINE; LI, 1991; LOPEZ-MARTINEZ; ACOSTA; CANO, 2010; LIU; WANG, 2012). The controller structure is changed by using a high-frequency switching feedback control. The switching feedback control law drives the system's trajectory onto a specified and user-chosen surface, which is termed the sliding surface or the switching surface and maintains the trajectory on this sliding surface for all subsequent time (UTKIN, 1978; SLOTINE; LI,

1991; LIU; WANG, 2012). The primary mode of variable structure control operation is sliding mode control, which includes excellent properties, such as low sensitivity to plant parameter uncertainty, rejection of a particular class of external disturbances, reduced-order modeling of plant dynamics, and finite-time convergence (UTKIN, 1978; SLOITINE; LI, 1991; LIU; WANG, 2012; ZHAO *et al.*, 2017; LOPEZ-MARTINEZ; ACOSTA; CANO, 2010).

VSC theory's basic principle is to design a switching surface, which has the desired behavior for the states, and a control law that brings the states to that surface in finite time. The advantage is that once the system's states reach the switching surface, the system operation mode depends only on the reduced-order dynamic equation underlying of the switching surface, regardless of the system's parameters, since it can be written as a matched disturbance term (SLOITINE; LI, 1991; LIU; WANG, 2012). Therefore, the VSC theory make the overall closed-loop system invariant to matched disturbances or model uncertainties. It can reduce the order of the system and the algorithm is simple and easy to realize (DECARLO; ZAK; MATTHEWS, 1988).

Although the theory of variable structure systems has undergone extensive and detailed studies in the control community (DECARLO; ZAK; MATTHEWS, 1988; GARCIA *et al.*, 2005; REN; HU, 2017), in recent years, researches on VSC theory have been successfully applied to a wide variety of practical systems, such as satellite and spacecraft attitude control (WEI; LIU; RADICE, 2015; HU, 2010; YEH, 2010; ZHANG *et al.*, 2017; SONG *et al.*, 2018), power model (MONDAI; RAHMAN, 2017), cranes (NGO; HONG, 2012; TUAN; LEE, 2013), quad-rotors (IZAGUIRRE-ESPINOSA *et al.*, 2016; SILVA; SANTOS, 2020) and robot control (ARDAKANI *et al.*, 2017; HUANG *et al.*, 2010; RUBAGOTTI; VEDOVA; FERRARA, 2011). The variable structure control technique also has wide application in advanced microprocessor technology of the on-line control of dynamic systems, such as robotic manipulators, heat processes, and electro-pneumatic system (GARCIA *et al.*, 2005). Additionally, for inverted pendulum systems, Riachy *et al.* (2008) have designed a second-order sliding mode controller for such a device under uncertainty conditions. Furthermore, Wang (2012) carries out the stabilization and tracking control using sliding modes of an X-Z inverted pendulum that can move combining vertical and horizontal forces. Adhikary and Mahanta (2013) propose an integral back-stepping sliding mode controller for the swing-up and stabilization of the pendulum on a cart system. Guo, Xu and Lee (2014) have carried out the design and implementation of a sliding mode controller for the wheeled inverted pendulum.

2.2 INVERTED PENDULUM-LIKE SYSTEMS

Different types of pendulums can be tested and controlled using reaction wheels as actuators as aforementioned. Additionally, the use of inverted pendulums to approximate the dynamics of robots and real-world mechanisms has been extensively studied (KAJITA *et al.*, 2001; SUGIHARA; NAKAMURA; INOUE, 2002; GRASSER *et al.*, 2002; FURUTA, 2003; KAJITA *et al.*, 2003; LEE; GOSWAMI, 2007; KAJITA *et al.*, 2010; TSAI *et al.*, 2010; SANYAL; GOSWAMI, 2014; YANG *et al.*, 2014; VASUDEVAN; DOLLAR; MORRELL, 2015; JEONG; HAYASHI, 2018; LIU; NING; CHEN, 2018; AN *et al.*, 2018; HE *et al.*, 2018).

Kajita *et al.* (2001) have concerned with a better understanding of the biped locomotion's theoretical background. They analyzed the 3D walking control of a biped robot using a three-dimensional linear inverted pendulum model resulting in simpler dynamics. Grasser *et al.* (2002) have discussed a very interesting pendulum configuration, a mobile inverted pendulum. Nowadays, this configuration is known as the wheeled inverted pendulum and it can be found as a toy, very similar to a Segway (GUO; XU; LEE, 2014). Grasser *et al.* (2002) have used two decoupled state-space controllers to command the DC motors to keep the system balanced. Figure 1 depicts the system studied by them.

Figure 1 – Wheeled inverted pendulum, Joe.



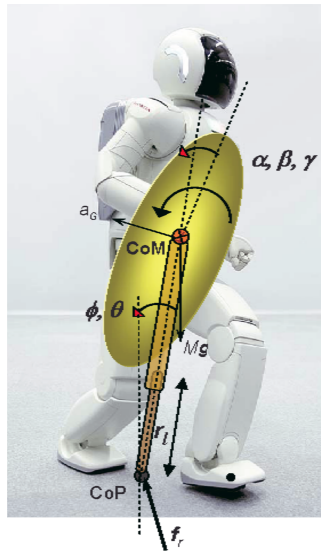
Source: Grasser *et al.* (2002).

Furuta (2003) has illustrated the control of some pendulum configurations such as the simple pendulum and the Furuta pendulum, presenting a new approach to analyze and to design a nonlinear controller based on the fractal. Moreover, Kajita *et al.* (2003) have introduced a method that linearly derived the total momentum of a robot based on

its physical parameters and considering the robot's constraints. This method provides a framework that generates various maneuvers for humanoid robots.

For instance, the introduction of the Reaction Mass Pendulum (RMP) in which a 3D multi-body pendulum has represented well the model of a humanoid robot with variable body inertia (LEE; GOSWAMI, 2007; SANYAL; GOSWAMI, 2014). This interesting system is illustrated in Fig. 2. The RMP is also a generalization of the well-known reaction wheel pendulum, and it shows how the generic geometry and dynamics of a robot can be mapped into the RMP.

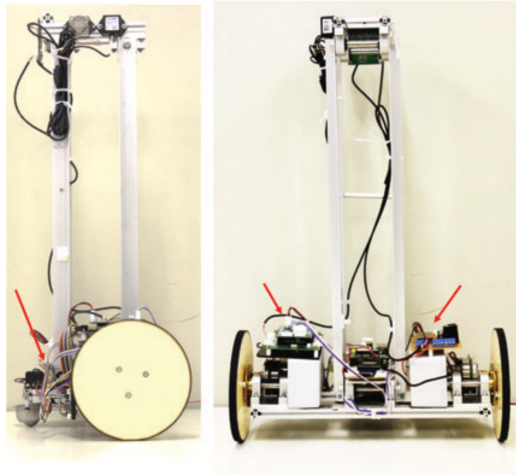
Figure 2 – Reaction mass pendulum.



Source: Lee and Goswami (2007).

Tsai *et al.* (2010) have carried out a physics-based approach that used an inverted pendulum model to adjust the desired motion trajectory of a 3D biped character animation that reacted to dynamical environments. Yang *et al.* (2014) have investigated the wheeled inverted pendulum models where it was decomposed into a second-order subsystem. The radial basis neural network was employed for approximating uncertain dynamics and a neural network-based motion control was also developed. Vasudevan, Dollar and Morrell (2015) have studied different design aspects for the control of wheeled inverted pendulum platforms so that the design process would be the least iterative. Jeong and Hayashi (2018) have developed a variation of the wheeled inverted pendulum with a four-bar parallel mechanism and verified experimentally two application tasks, object delivery and image taking, which were successfully performed. This system is shown in Fig. 3, illustrating that nowadays, the variation of proposed inverted pendulum-like systems attract research attention for different applications.

Figure 3 – Wheeled inverted pendulum with a four-bar parallel mechanism.



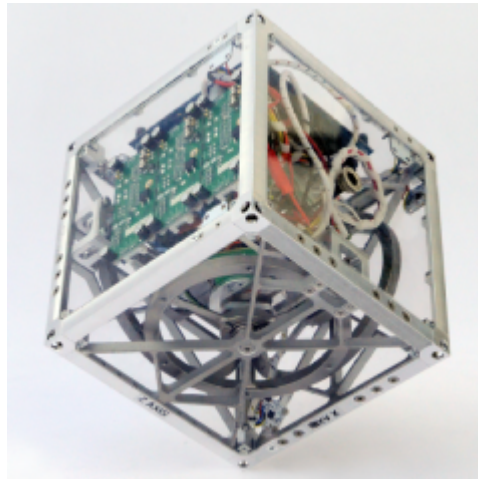
Source: Jeong and Hayashi (2018).

Another important inverted pendulum-like configuration is the 3D pendulum that is introduced by (SHEN *et al.*, 2004). They analyze a new pendulum configuration that consists of a rigid body fixed to a pivot and allows three rotational degrees of freedom. The gravity acts on this rigid body, and three control torques are considered, one for each direction. Moreover, they demonstrate that symmetry assumptions can lead this 3D pendulum configuration to the planar 1D pendulum and 2D spherical pendulum. Chaturvedi *et al.* (2005) presents several different classifications for the 3D pendulum. Besides, two different strategies for the stabilization of the 3D pendulum are proposed. Chaturvedi and McClamroch (2007) have demonstrated the asymptotic stabilization of the hanging equilibrium manifold of the 3D pendulum, and Chaturvedi, McClamroch and Bernstein (2008) have shown the stabilization in the case where the 3D pendulum is axially symmetric, and the center of mass lies on the axis of symmetry. The results show the control in the hanging and inverted position, although very large control torques are needed to be applied to control the system. More recently, Yao (2020) presents the robust adaptive finite-time attitude tracking control of the 3D pendulum. Real-time experiments are conducted, and external disturbances are applied to the system to verify the performance under this condition. All these works let open different ways of actuating the 3D pendulum once the majority of them assume three control torques directly applied to each of the rotational degrees of freedom.

Based on the 3D inverted pendulums model, Gajamohan *et al.* (2012) have introduced the Cubli, shown in Fig. 4 which is a cube that can jump up and balance using reaction wheels that are mounted on three faces of the cube. Muehlebach and Andrea (2017) carry out the nonlinear analysis and control of the Cubli detailing this system's main features,

i. e., the ability to balance on its edge and to jump from up and down. Analyzing the equations of motion, the authors have found a reduced description of the dynamics that is used for the control design. The design of two nonlinear control strategies is presented and the results were verified experimentally.

Figure 4 – Cubli.



Source: Muehlebach and Andrea (2017).

Thus, the control moment gyroscope is a widely known form of control that has many applications, such as the attitude control of aerospace devices (ROSER; SGHEDONI, 1997; BAILEY, 2000a; BAILEY, 2000b; HAN; PECHEV, 2007; MACKUNIS *et al.*, 2008; OMAGARI; FUJIHASHI; MATUNAGA, 2008; CLARK; WORRALL; YAVUZOĞLU, ; NOUMI; TAKAHASHI, 2013), and robots (SENDA *et al.*, 1995). There are also other uses of a CMG, such as a personal balance assistance (BERRY *et al.*, 2016; LEMUS; FRANKENHUYZEN; VALLERY, 2017).

A simple CMG consists of a rotating disk at a constant rate and in a gimbal mechanism that can change its direction generating a gyroscopic torque by the direction variation of angular momentum (SCHAUB *et al.*, 1998; SCHAUB; JUNKINS, 2000). CMGs have the advantage of a small angular variation in the gimbal being able to produce a high magnitude torque (SCHAUB *et al.*, 1998). However, CMGs with a constant speed in the reaction wheel can present singularities in some specific angular positions and then they may not be so effective. Therefore, to avoid such singularities, Schaub *et al.* (1998) discuss variable speed control moment gyroscope in which the favorable aspects of using reaction wheels are combined with CMGs. In this case, the reaction wheel angular velocity can be adjusted and controlled for situations in which singularities can occur and the gyroscopic torque can also be applied (SCHAUB *et al.*, 1998; SCHAUB; JUNKINS, 2000; MCMAHON; SCHAUB, 2009; STEVENSON; SCHAUB, 2012). For this reason, the

VSCMG is a suitable method to control an inverted pendulum with a reaction wheel.

2.3 SCOPE OF THE THESIS

In the present thesis, different reaction wheel pendulum configurations are proposed, involving two reaction wheels and a variable speed control moment gyroscope. In many aerospace systems, it is necessary to use more than one actuator to perform the attitude control. Thus, it is essential to study pendulums' control where two actuators are used, such as the pendulum with two reaction wheels (2-RWP) proposed by Trentin *et al.* (2020a). This new configuration aims to evaluate how the control of the pendulum could be performed using two actuators. This system may attract attention for many reasons, ranging from academic and pedagogic purposes to developing new control techniques. It is also interesting that one can use one reaction wheel or both to drive the pendulum to the inverted position. For this new system, at a first step, a PID controller has been designed and evaluated using simulations as well as a real-time experiment using a low-cost platform.

The present work also describes a different approach to the well-known RWP showing that it is possible to control the RWP in the inverted position using a simple PID controller and low-cost hardware to apply this classical and straightforward controller for the experimental tests. In the developed controller, RWP is not swung up to its inverted position. Thus, the reaction wheel is accelerated by the DC motor generating torque, and thereby, the pendulum leaves the downward position and goes to the upward position where it is balanced around its unstable equilibrium point. Another interesting point here is the use of a counterweight to balance the torque caused by the weight of the DC motor. However, the counterweight does not entirely balance the weight of the reaction wheel. In this stage, the main idea is to elaborate a position controller for both pendulum configurations (RWP and 2-RWP) through the variation of angular momentum of the reaction wheels. The angular momentum variation is caused by the acceleration of a DC motor coupled to the reaction wheel. The DC motor's angular velocity must be controlled so that the angular acceleration of the reaction wheel is also controlled because this acceleration is what provides torque to the system. The torque produced by the reaction wheels enables the control of RWP and 2-RWP in the inverted position.

One of the main contributions of this thesis is to describe new control realizations regarding the pendulum with two reaction wheels (2-RWP) to improve the performance compared with our previous paper (TRENTIN *et al.*, 2020a). This thesis illustrates the

design of two nonlinear controllers for the two reaction wheels pendulum (2-RWP). The first one designed is a classical nonlinear proportional-derivative controller derived from the Lyapunov stability theory. The proof of stability is provided using the the Lyapunov function candidate's higher-order derivatives, where the asymptotic stability is proven. Additionally, considering that many plant parameters are known with limited accuracy, a sliding mode controller is also designed for this novel pendulum configuration. The results obtained experimentally using low-cost hardware are compared to established ones. Moreover, the way that the plant was manufactured using DC motors without encoders makes it impossible to know many of the reaction wheels' states. This was taken into account in the design of the controllers, which have to be able to overcome this practical challenge.

Many of these different pendulum configurations have encouraged the development of the control of the humanoids robots and wheeled inverted pendulum systems. These applications usually have their motion approximated by the dynamics of inverted pendulums. However, to the best of our knowledge, an inverted pendulum has not been controlled using the variable speed control moment gyroscope yet. The other two main contributions of this thesis are the development of a pendulum configuration controlled by a VSCMG (TRENTIN; SILVA; SCHAUB, 2019b) where the pendulum is kept in the plane and the derivation of the 3D pendulum model with a VSCMG to yield a spherical pendulum. This is detailed in Chapter 5.

3 PENDULUMS WITH REACTION WHEELS

Reaction wheels have been extensively used to control and stabilize a wide range of plants due to their angular momentum exchange capability. Additionally, the attitude control using reaction wheels as actuators has been one of the most popular ways to stabilize and reject external disturbances in aerospace devices (ISMAIL; VARATHARAJOO, 2010). From the controlled change of the angular momentum rate of change using reaction wheels, it is possible to stabilize and guide rigid bodies in space. Therefore, this chapter deeply investigates two pendulum configurations controlled using reaction wheels.

Primarily, section 3.1 describes the model of the well-known reaction wheel pendulum (RWP) and proposes a pendulum equipped with two reaction wheels (2-RWP). Section 3.2 presents the experimental device used for the real-time tests and introduces the low-cost hardware. In addition, the DC motor's characterization and how the DC motors models are coupled to the dynamic models are illustrated. Nonlinear friction models are identified for the pendulum configurations studied in this chapter. Section 3.3 exhibits one of the main contributions of this thesis: the comparison of using one or two actuators to drive and control and inverted pendulum using a simple PID controller. Finally, section 3.4 comprises the concluding remarks of this chapter.

3.1 DYNAMIC MODELING

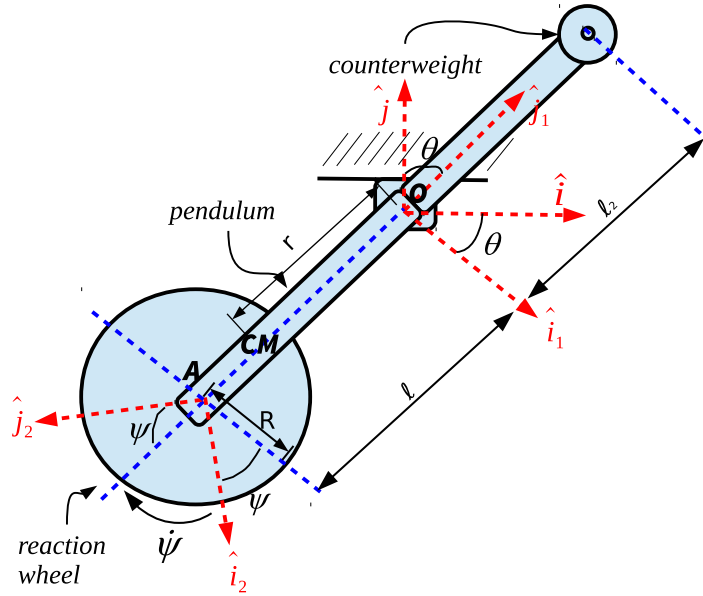
This section presents the dynamic modeling of the classical and well-known reaction wheel pendulum, and of a new and unusual pendulum with two reaction wheels proposed by this thesis.

3.1.1 Mathematical model of the reaction wheel pendulum

The Newton-Euler dynamic modeling of the reaction wheel pendulum is presented in this section. Figure 5 shows a rigid pendulum with mass m and length ℓ fixed at O .

The inertial frame is described by the orthonormal basis $\{\hat{i}, \hat{j}, \hat{k}\}$. The moving frame B , or body frame, is described by the orthonormal basis $\{\hat{i}_1, \hat{j}_1, \hat{k}_1\}$ and rotates solidarity to the pendulum with the angle θ . A reaction wheel with mass M and radius R is fixed at A . The moving frame W is described by the orthonormal basis $\{\hat{i}_2, \hat{j}_2, \hat{k}_2\}$ and rotates with the angle ψ solidarity to the reaction wheel. The directions $\hat{k}$, $\hat{k}_1$ and $\hat{k}_2$ are parallel and are pointing outside the plan.

Figure 5 – Reaction wheel pendulum.



Source: Trentin *et al.* (2020a).

The angular momentum of the rigid pendulum represented in the moving frame B is computed by:

$${}^B \mathbf{H} = (I_{zeq}^O \dot{\theta} + I_{zw}^A \dot{\psi}) \hat{k}_1 \quad (1)$$

where $I_{zeq}^O = \frac{m\ell^2}{3} + m\ell^2 + \frac{MR^2}{2} + M\ell^2 + \frac{m_{h2}\ell_2^2}{3} + m_{h2}\ell_2^2 + m_{cw}r_{cw}^2 + m_{cw}\ell_2^2$ is the moment of inertia of the pendulum and the reaction wheel computed relative to the fixed point O using Steiner's Theorem where ℓ_2 is the length of counterweight rod, m_{h2} is the mass of this rod, m_{cw} is the mass of the counterweight and r_{cw} is the radius of the counterweight. $I_{zw}^A = \frac{1}{2}MR^2$ is the moment of inertia of the reaction wheel relative to the fixed point A , $\dot{\theta}$ is the angular velocity of the pendulum and $\dot{\psi}$ is the angular velocity of the reaction wheel. The rate of the angular momentum, ${}^B \dot{\mathbf{H}}$ is equal to the torque sum relative to the

fixed point O :

$${}^B\dot{\mathbf{H}} = \frac{{}^B d}{dt}(\mathbf{H}) + \underbrace{{}^B\boldsymbol{\Omega} \times {}^B\mathbf{H}}_{=0} \equiv {}^B\mathbf{T}_O \quad (2)$$

where ${}^B\mathbf{T}_O$ is the torque sum relative to the fixed point O , $\frac{{}^B d}{dt}(\mathbf{H})$ is the amplitude rate of change of the angular momentum and ${}^B\boldsymbol{\Omega} \times {}^B\mathbf{H}$ is the direction rate of change of the angular momentum, also known as gyroscopic torque. The reaction wheel pendulum in the configuration shown in Fig. 5 does not present gyroscopic torque because the angular velocity ${}^B\boldsymbol{\Omega}$ is parallel to the direction of the angular momentum ${}^B\mathbf{H}$. Thereby, the rate of change of angular momentum must be equal to the torque sum relative to the fixed point O :

$$I_{zeq}^O \ddot{\theta} + I_{zw}^A \ddot{\psi} = r(m + M + m_{dcm})g \sin \theta - \ell_2(m_{cw} + m_{h2})g \sin \theta \quad (3)$$

where $\ddot{\theta}$ is the angular acceleration of the pendulum, $\ddot{\psi}$ is the angular acceleration of the reaction wheel, r is the distance from the fixed point O to the center of mass of the device and m_{dcm} is the mass of the DC motor. An analysis for the reaction wheel is carried out to find the reaction wheel motor torque equation, where the rate of change of the angular momentum must be equal to the sum of torques applied on it, and thus, for the reaction wheel:

$$T = I_{zw}^A \ddot{\theta} + I_{zw}^A \ddot{\psi} \quad (4)$$

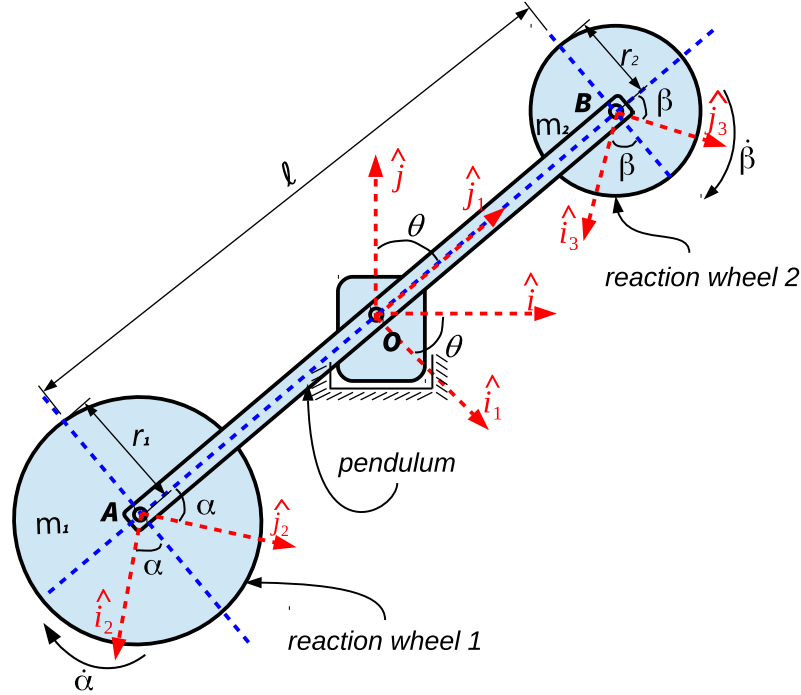
where T is the reaction wheel motor torque. Thus, eq. (3) and eq. (4) describe the reaction wheel pendulum motion. These equations can be integrated by Runge-Kutta method considering $m = 0.11$ kg, $M = 0.21$ kg, $m_{h2} = 0.09$ kg, $m_{dcm} = 0.22$ kg, $m_{cw} = 0.324$ kg, $\ell = 0.3$ m, $\ell_2 = 0.23$ kg, $R = 0.1$ m $r = 0.27$ m and sampling rate of 3 milliseconds. These values are the physical parameters based on the experimental set-up, as shown in Fig. 7. The torque T generated by spinning the reaction wheel is done by coupling a DC motor to the reaction wheel.

3.1.2 Mathematical model of the two reaction wheels pendulum

In this section, the complete modeling of the two reaction wheels pendulum (2-RWP) is presented. Figure 6 shows the 2-RWP with mass m_p , length ℓ and fixed at O . For modeling, an inertial frame described by the orthonormal basis $\{\hat{i}, \hat{j}, \hat{k}\}$ and three moving

frames were adopted. The first moving frame (B), or body frame, is described by the orthonormal basis $\{\hat{i}_1, \hat{j}_1, \hat{k}_1\}$ and rotates solidarity to the pendulum with the angle θ . The second moving frame (W_1) is described by the orthonormal basis $\{\hat{i}_2, \hat{j}_2, \hat{k}_2\}$ rotating with angle α solidarity to reaction wheel 1. This reaction wheel has a mass m_1 , radius r_1 and it is fixed at A . The third moving frame (W_2) is described by the orthonormal basis $\{\hat{i}_3, \hat{j}_3, \hat{k}_3\}$ rotating solidarity to the second reaction wheel with angle β , with a mass m_2 and radius r_2 .

Figure 6 – Two reaction wheels pendulum.



Source: Trentin *et al.* (2020a).

The angular momentum of the rigid pendulum described in moving frame B is:

$${}^B \mathbf{H} = (I_{zeq}^O \dot{\theta} + I_{zw1}^A \dot{\alpha} + I_{zw2}^B \dot{\beta}) \hat{k}_1 \quad (5)$$

where $I_{zeq}^O = \frac{1}{12}m_p \ell^2 + \frac{1}{2}(m_1 r_1^2 + m_2 r_2^2) + \frac{1}{4}(m_1 \ell^2 + m_2 \ell^2)$ is the moment of inertia of the pendulum and both reaction wheels computed relative to the fixed-point O using Steiner's Theorem (WHITTAKER, 1952). $I_{zw1}^A = \frac{1}{2}m_1 r_1^2$, $I_{zw2}^B = \frac{1}{2}m_2 r_2^2$ are the moment of inertia of the reaction wheels computed relative to the fixed-points A and B , respectively. $\dot{\theta}$ is the angular velocity of the pendulum, $\dot{\alpha}$ is the angular velocity of reaction wheel 1 and $\dot{\beta}$ is the angular velocity of reaction wheel 2. The rate of the angular momentum, $\dot{\mathbf{H}}$ is equal to the torque sum relative to the fixed point O as it is shown in eq. (2). The

mathematical model of 2-RWP obtained is:

$$I_{zeq}^O \ddot{\theta} + I_{zw1}^A \ddot{\alpha} + I_{zw2}^B \ddot{\beta} = (m_1 - m_2)g \frac{\ell}{2} \sin \theta \quad (6)$$

The same analysis for both reaction wheels was also carried out. The rate of change of the angular momentum must be equal to the torques sum applied to it, and thus, for both reaction wheels:

$$T_1 = I_{zw1}^A \ddot{\theta} + I_{zw1}^A \ddot{\alpha} \quad (7)$$

$$T_2 = I_{zw2}^B \ddot{\theta} + I_{zw2}^B \ddot{\beta} \quad (8)$$

where T_1 and T_2 are the motor torques of the reaction wheels 1 and 2, respectively. For the simulations, the parameters used were $m_1 = 0.21$ kg, $r_1 = 0.11$ m, $m_2 = 0.13$ kg, $r_2 = 0.1$ m, $m_p = 0.16$ kg and $\ell = 0.5$ m. These parameters were extracted from the experimental device exhibited in Fig. 9.

3.2 DESCRIPTION OF THE EXPERIMENTAL SETUP

This section presents the details of the experimental apparatus and the low-cost hardware used to perform the real-time controllers. Furthermore, the characterization of the DC motor and the identification of the pendulums friction are carried out.

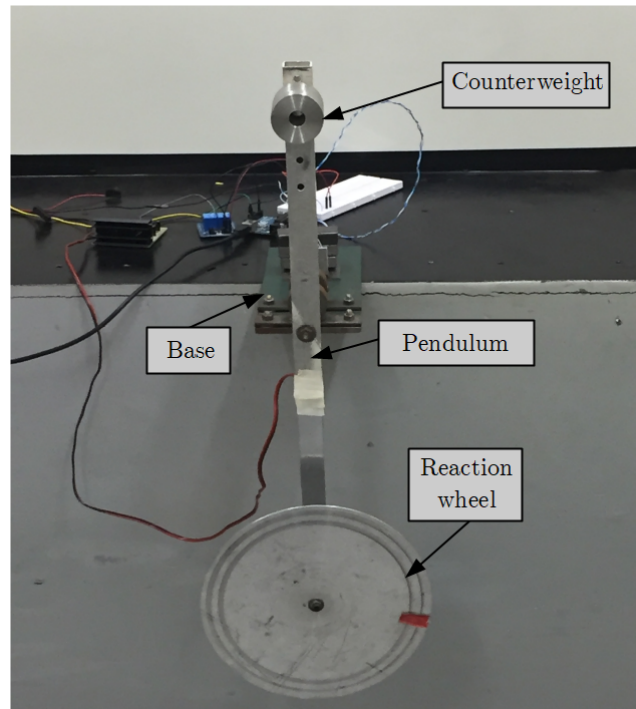
Figure 7 shows the reaction wheel pendulum that was built using an aluminum bar and a disk. Figure 8 illustrates other details of the experimental device, such as the position of the precision potentiometer, the DC motor coupled to the reaction wheel and the bearings that hold the pendulum. The counterweight used to balance the DC motor weight can also be seen.

The DC motor is an Akiyama model AK555/390ML 12S18200C, which operates with a nominal voltage of 12 V and can reach up to 15700 RPM. A Vishay Spectrol multi-turn potentiometer model 533 with a resistance range from 50 Ω to 20 $k\Omega$ was used. This potentiometer was chosen due to its low-cost, as well as its availability in the laboratory.

Figure 9 shows the two reaction wheels pendulum constructed for experimental tests. This pendulum was also manufactured using an aluminum bar and two disks connected to each of its ends. The same base is used for both pendulum configurations.

Figure 10 shows the low-cost hardware used for the real-time controller for the reaction wheel pendulum, including the Arduino Uno prototype platform with a microcontroller,

Figure 7 – Reaction wheel pendulum built for the experimental tests.

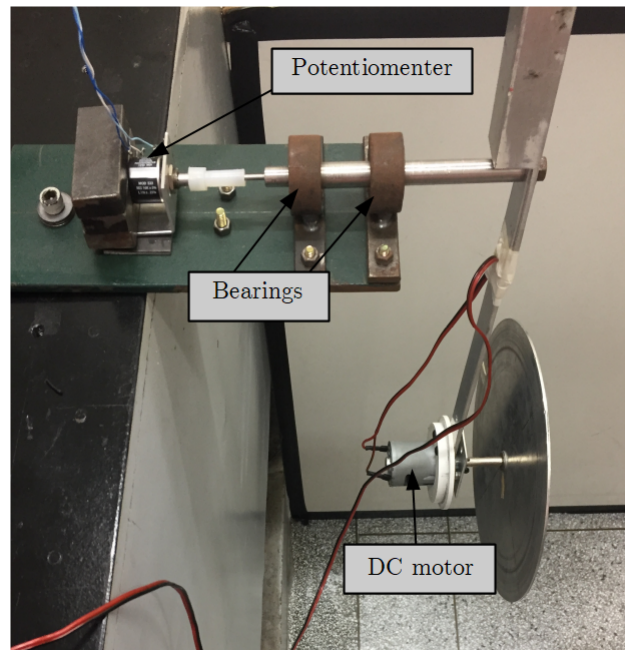


Source: Trentin *et al.* (2020a).

a two-relay module from Shun Yu model BTE13-003 and a PWM board that controls the DC motor. The angular position of the RWP is read using the multi-turn potentiometer. This signal is sent to the Arduino Uno microcontroller, where a script is developed with the PID controller. The relay module is used to evaluate which way the DC motor should rotate.

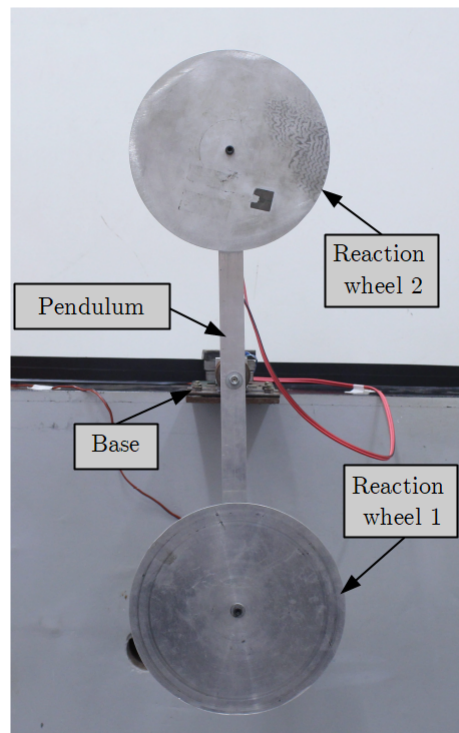
Figure 11 displays the low-cost hardware employed for the experimental tests with the 2-RWP. In this situation, the attachment of a PWM board can be viewed, since now there are two DC motors to be controlled. Each board is responsible for controlling each motor separately.

Figure 8 – Details of the reaction wheel pendulum built for the experimental tests.



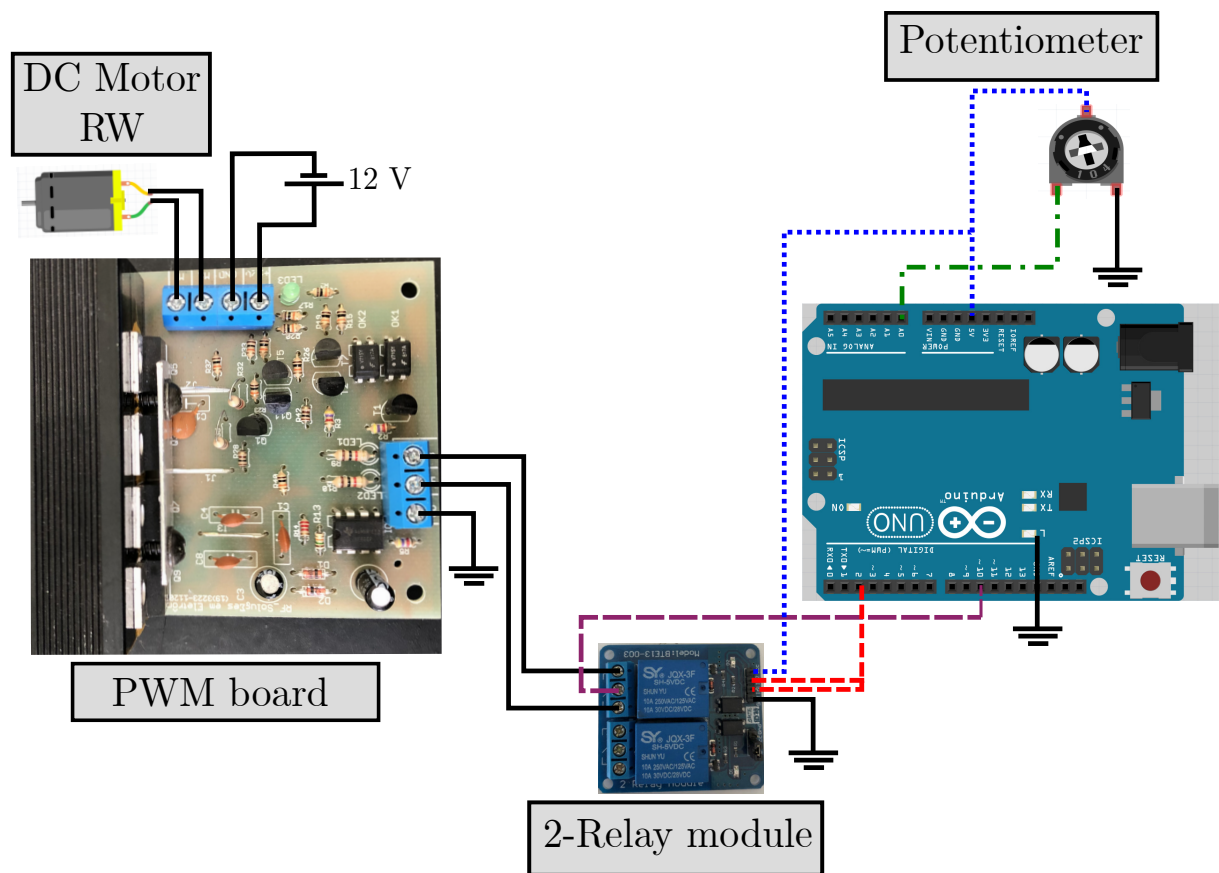
Source: Trentin *et al.* (2020a).

Figure 9 – Two reaction wheels pendulum built for the experimental tests.



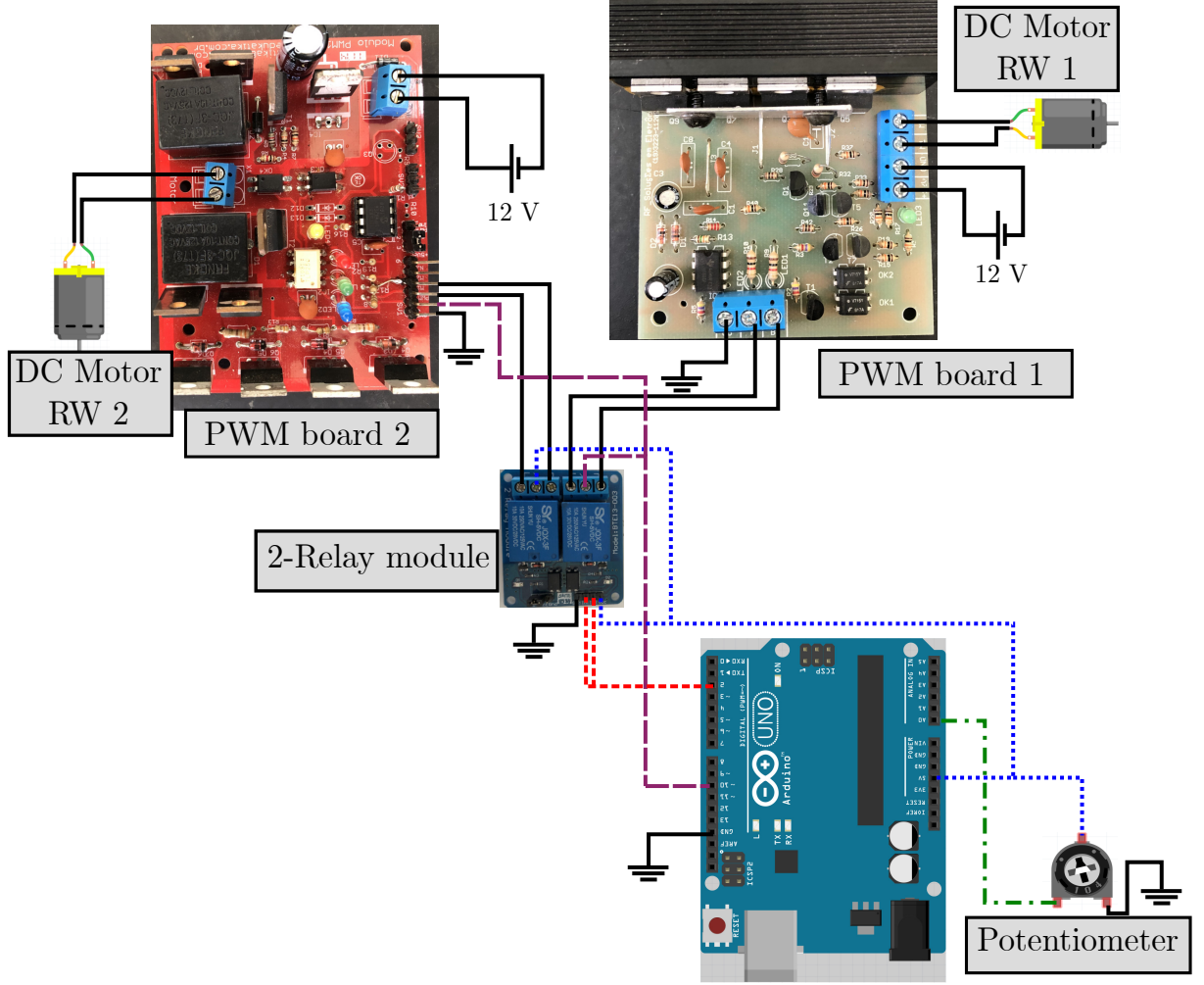
Source: Trentin *et al.* (2020a).

Figure 10 – Low-cost hardware used for the experimental tests of the RWP.



Source: Trentin *et al.* (2020a).

Figure 11 – Low-cost hardware used for the experimental tests of the 2-RWP.



Source: Trentin *et al.* (2020a).

3.2.1 Experimental characterization of the DC motor parameters

This section discusses how the experimental characterization of the parameters of the DC motor was done. These parameters are used for both pendulum configurations studied in this chapter.

For a generic DC motor, its equation of motion is given by:

$$T_{dcm}(t) = J\dot{\omega}(t) + B\omega(t) \quad (9)$$

where $T_{dcm}(t)$ is the motor torque, J is the moment of inertia of the system, $\dot{\omega}(t)$ is the angular acceleration, B is viscous friction coefficient and $\omega(t)$ is the angular velocity. The

torque provided by the DC motor linear relating the flux and armature current is:

$$T_{dcn}(t) = K_t I_a(t) \quad (10)$$

where K_t is the motor torque constant and $I_a(t)$ is the armature current. The armature circuit is:

$$V(t) = R_a I_a(t) + K_v \omega(t) + L \frac{d}{dt}(I_a(t)) \quad (11)$$

where $V(t)$ is the voltage applied to the DC motor, R_a is the armature resistance, K_v is the back electromotive force constant and L is the inductance that will not be considered in the modeling. Solving Eq. (11) for the current:

$$I_a(t) = \frac{V(t)}{R_a} - \frac{K_v}{R_a} \omega(t) \quad (12)$$

To obtain a relation between the voltage applied to the DC motor and the angular velocity, the expression for the current shown in Eq. (12) can be substituted into the motor torque equation presented in Eq. (10):

$$K_t I_a(t) = J \dot{\omega}(t) + B \omega(t) \quad (13)$$

$$K_v \left(\frac{V(t)}{R_a} - \frac{K_v}{R_a} \omega(t) \right) = J \dot{\omega}(t) + B \omega(t) \quad (14)$$

Applying Laplace transform with null initial conditions, it is easy to obtain the following expression:

$$\frac{\omega(s)}{V(s)} = \frac{K}{\tau s + 1} \quad (15)$$

where

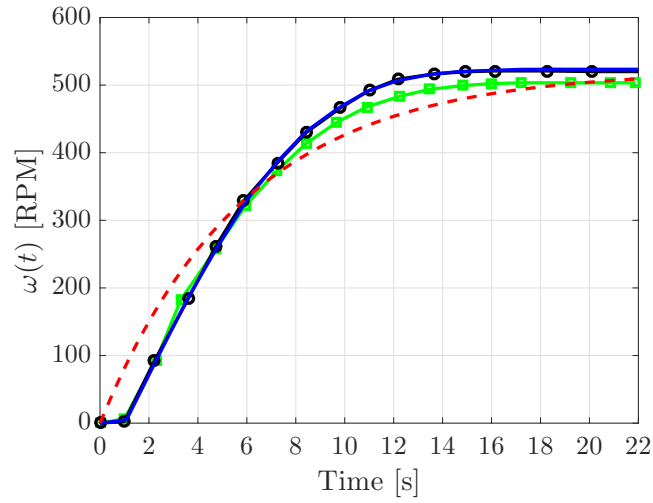
$$K = \frac{K_v}{R_a B + K_t K_v} \quad (16)$$

$$\tau = \frac{R_a J}{R_a B + K_t K_v} \quad (17)$$

The time constant τ and stationary gain K were obtained from an experimental characterization of a DC motor Akiyama model AK555/390ML 12S18200C. A step input voltage signal was applied to the DC motor, and the angular velocity of the reaction wheel was measured to characterize the DC motor. This result is charted in Fig. 12, where the values represent three experimental measurements of time and angular velocity of the DC motor coupled to the reaction wheel. With the angular velocity of the reaction and

the time of the experiment, it was possible to evaluate the response of the DC motor. As expected, the response of a first order system is observed, and the stationary gain is determined to be the value of regime: $K = 521.6$ V/RPM. The time constant is found when the time is equal to $0.632K$, which results in $\tau = 5.88$ s. After obtaining these parameters, the transfer functions was estimated, shown by Eq. (15) to compare with the experimental ones.

Figure 12 – Experimental characterization of the DC motor where $- \circ -$ indicates experiment 1, $-$ indicates experiment 2, $- \square -$ indicates experiment 3 and $- -$ indicates the calculated transfer function.



Source: Trentin *et al.* (2020a).

The torques generated by spinning the reaction wheels are done by coupling DC motors to the reaction wheels. Hence, the DC motor model must be coupled to the pendulums models. For the reaction wheel pendulum, the motor torque is:

$$T(t) = V(t) \frac{K_t}{R_a} - \frac{K_t K_v}{R_a} \dot{\psi}(t) \quad (18)$$

For the 2-RWP, the torques T_1 and T_2 are also provided by DC motors coupled to the reaction wheels, whose equations are given by:

$$T_1(t) = V_\alpha(t) \frac{K_t}{R_a} - \frac{K_t K_v}{R_a} \dot{\alpha}(t) \quad (19)$$

$$T_2(t) = V_\beta(t) \frac{K_t}{R_a} - \frac{K_t K_v}{R_a} \dot{\beta}(t) \quad (20)$$

And equations (18), (19), and (20) can be rearranged in such manner that they will

provide the necessary voltage for each DC motor based on the motor torques:

$$V(t) = T(t) \frac{R_a}{K_t} + K_v \dot{\psi}(t) \quad (21)$$

$$V_\alpha(t) = T_1(t) \frac{R_a}{K_t} + K_v \dot{\alpha}(t) \quad (22)$$

$$V_\beta(t) = T_2(t) \frac{R_a}{K_t} + K_v \dot{\beta}(t) \quad (23)$$

From experimental characterization of the DC motor already presented, and considering the values obtained for the stationary gain and for the time constant: $K = 54.62$ V.s/rad and $\tau = 5.88$ s, and the moment of inertia of each reaction wheel, it is possible to find that $R_a = 1.06 \, \Omega$ and $K_t = 0.0063$ Vs/rad and $K_v = 0.0063$ Nm/A.

3.2.2 Experimental characterization of the pendulum friction

It is essential to highlight that the pendulum built for the experimental tests has some construction imperfections and presents a nonlinear behavior regarding its friction. The pendulum angular motion has small velocities for which is difficult to model the effect of friction (KELLY; LLAMAS, 1999). Therefore, it is imperative to include such a phenomenon in the model of the pendulum configurations that were built. The model must consider the contributions of Coulomb and viscous friction to be able to represent this phenomenon, and it is described by (KELLY; LLAMAS, 1999):

$$T_{Gfr} = \text{sgn}(\dot{\theta}) (C_v |\dot{\theta}| + C_c) \quad (24)$$

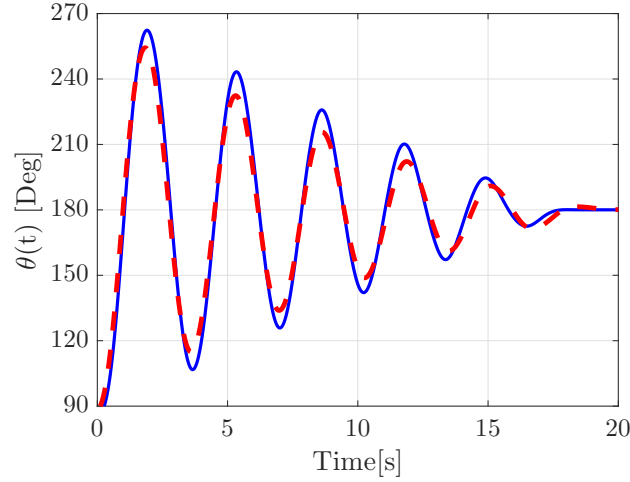
where T_{Gfr} is a generic friction torque, C_v is the coefficient of viscous friction, and C_c the Coulomb friction value.

The parameters of this friction model were identified for both pendulum configurations, and the model was included in the equations of motion. Regarding the identification of the model, the pendulum was released from $\theta = 90$ degrees so that the friction behavior could be observed until the pendulum stopped in the stable equilibrium point, the downward position. Figures 13 and 14 present the experimental result of releasing the pendulum and waiting for it to stop and compares these results to the equations of motions. For the RWP, the equation of motion considering the friction model is:

$$I_{zeq}^O \ddot{\theta} + I_{zw}^A \ddot{\psi} = r (m + M + m_{dcn}) g \sin \theta - \ell_2 (m_{cw} + m_{h2}) g \sin \theta - T_{Fr_{RWP}} \quad (25)$$

where T_{FrRWP} is the friction torque for the RWP, and the coefficients of the friction model in this case are: $C_v = 0.011$ Nms/rad and $C_c = 0.016$ Nm. Figure 13 depicts the comparison between the experimental result and the model for the RWP where the coefficients aforementioned were used in the friction model. The model still does not fit perfectly the experimental behavior. However, it is close and it can be considered in the simulations for the RWP to try to approximate the experimental set-up.

Figure 13 – Identification of the friction of the RWP. - - - indicates numerical result and - the experimental result.



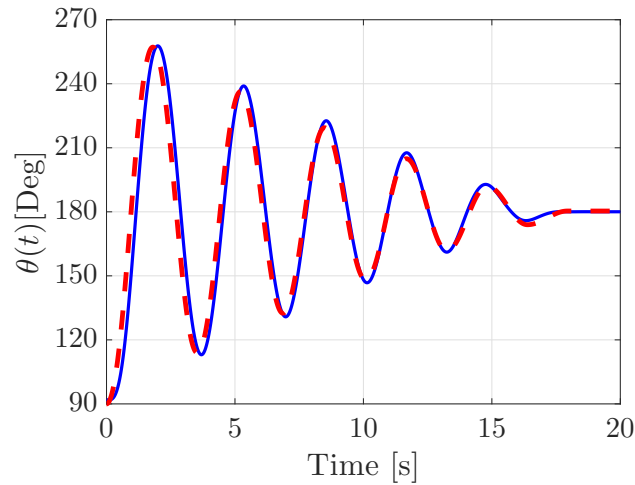
Source: Prepared by the author.

For the 2-RWP the equation of motion with the friction model included given by:

$$I_{zeq}^O \ddot{\theta} + I_{zw1}^A \ddot{\alpha} + I_{zw2}^B \ddot{\beta} = (m_1 - m_2)g \frac{\ell}{2} \sin \theta - T_{fr} \quad (26)$$

where T_{fr} is the friction torque for the 2-RWP. This model has a very close fit to the experimental result presented in Fig. 14, where $C_v = 0.0031$ Nms/rad and $C_c = 0.011$ Nm. Also, the friction model used here, composed by viscous and Coulomb friction contribution, is trendy in the literature (KELLY; LLAMAS, 1999). Moreover, this has not been considered in our previous paper (TRENTIN *et al.*, 2020a), and the friction contribution is significant to be identified and included in the design of the controllers once it has a strong nonlinear behavior and this could influence in the performance of the controllers.

Figure 14 – Identification of the friction of the 2-RWP. - - - indicates numerical result and - the experimental result.



Source: Prepared by the author.

It is important to state that, during the identification process of the friction torque, we use different signals for updating the friction torque model as well as to validate it. The signals presented in figures 13 and 14 are not the signals used to tune the coefficients for the friction torque model.

3.3 COMPARISON OF USING ONE OR TWO ACTUATORS EMPLOYING A PID CONTROLLER

This section puts forward one of the main contributions of the thesis. Firstly, the real-time controller is explained in detail, then, the comparison of using one or two reaction wheels to drive and control an inverted pendulum is carried out. Additionally, a quantitatively comparison of the pendulum configurations is performed to highlight the differences.

3.3.1 Operational details of the real-time PID controller

The steps to perform the real-time controller in both pendulums are illustrated in Fig. 15. These steps are described as:

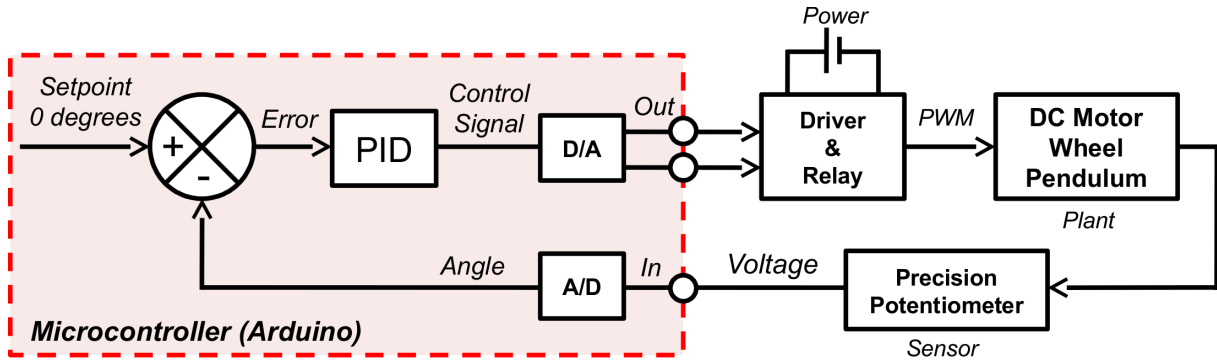
1. The algorithm executed by the prototype platform Arduino, which has a microcontroller, is responsible to keep the controller running in real-time;
2. The microcontroller reads an analog signal of the common pin of the precision

potentiometer coupled to the pendulum, thus reading a voltage value. The algorithm converts that voltage value to an angle, which is the angular position of the pendulum at that instant;

3. The algorithm then calculates the control signal with the PID library and uses the value of the angular position as input;
4. The library performs all necessary calculations and outputs a digital value ranging from -255 to 255;
5. This digital value is sent to the relay module that will evaluate which way the motor should spin. Then the relay module will send the digital value to the PWM control board;
6. The board then reads the digital value sent by the relay module and performs an analog PWM, sending the DC motor the appropriate voltage signal, so that the pendulum reaches the required position. Now the real-time controller returns to the second step and these steps are restarted controlling the system.

Additionally, the PMW board used has a power input and can be powered up to 21 V. However, in the tests, it was powered up to only 12 V to avoid overheating the power transistors. The board also has an output responsible for feeding the load that controls the DC motor and a control input. After the control signal provided by the Arduino flows through the relay module that evaluates which direction the DC motor must spin, it triggers a digital value to this port. The printed circuit board is responsible for converting this signal into a Pulse Width Modulation (PWM) signal. The use of PWM can be very efficient for speed control of a DC motor, usually with voltages between 6 and 24 V.

Figure 15 – Representative scheme for the real-time controller for both systems using PID controller.



Source: Trentin *et al.* (2020a).

Thus, a PWM circuit generates a continuous square wave that can vary from its highest amplitude to the lowest from a connected to a switched-off position.

The PWM performs the DC motor speed control by the square wave pulse width. Briefly, the printed circuit board allows the Arduino to control the DC motor's angular acceleration, and consequently the angular position of the pendulum by generating the appropriate angular acceleration that provides the necessary torque to take the pendulum to the inverted position.

A nonvariable source of power was used, with sufficient output current to drive the DC motor for occasions that require quick accelerations in which the DC motor can need up to 6 A. Since it was decided to use a low-cost hardware to perform the real-time controller, a 200 W computer source with some adaptations was used as the power source. The PID gains used for the simulations and for both real-time controllers designed can be seen in Tab. 1. The gains were tuned using the experimental set-ups and the ones with the best performances are presented in Tab. 1.

Table 1 – Gains used for both real-time PID controllers.

Configuration	k_p	k_i	k_d
RWP	8	18	8
2-RWP	20	15	7.5

Source: Trentin *et al.* (2020a).

To control the 2-RWP, some extra hardware was necessary. One PWM control board could not simultaneously manage both DC motors; thus, a second PWM control board was added. The real-time controller for the position control of 2-RWP is very similar to

the one designed for RWP. It follows the same steps shown previously and its operation is the same as the one shown in the scheme presented in Fig. 15. For the 2-RWP, as done in the simulated results, the control signal used is the same for both reaction wheels. Thus, the control signal generated by the PID library in the Arduino goes through the relay module and, afterward, the signal goes to the PWM control boards. Therefore, two PWM control boards were used, one for each DC motor. These boards are responsible for reproducing the desired control signal for the reaction wheel so that the reaction wheels can provide the necessary torque to take the 2-RWP to the inverted position. Here, the distinction between the hardware of the RWP and 2-RWP was that new connections had to be made to supplement the second PWM control board and communicate it with the Arduino and to the relay module.

Simulated PID controllers were designed to compare them to the experimental controllers designed. As the sampling time used in the real-time experiment is three milliseconds, it is enough to perform a continuous simulated PID controller to emulate the experimental discrete one. The PID controller implemented has the following equation:

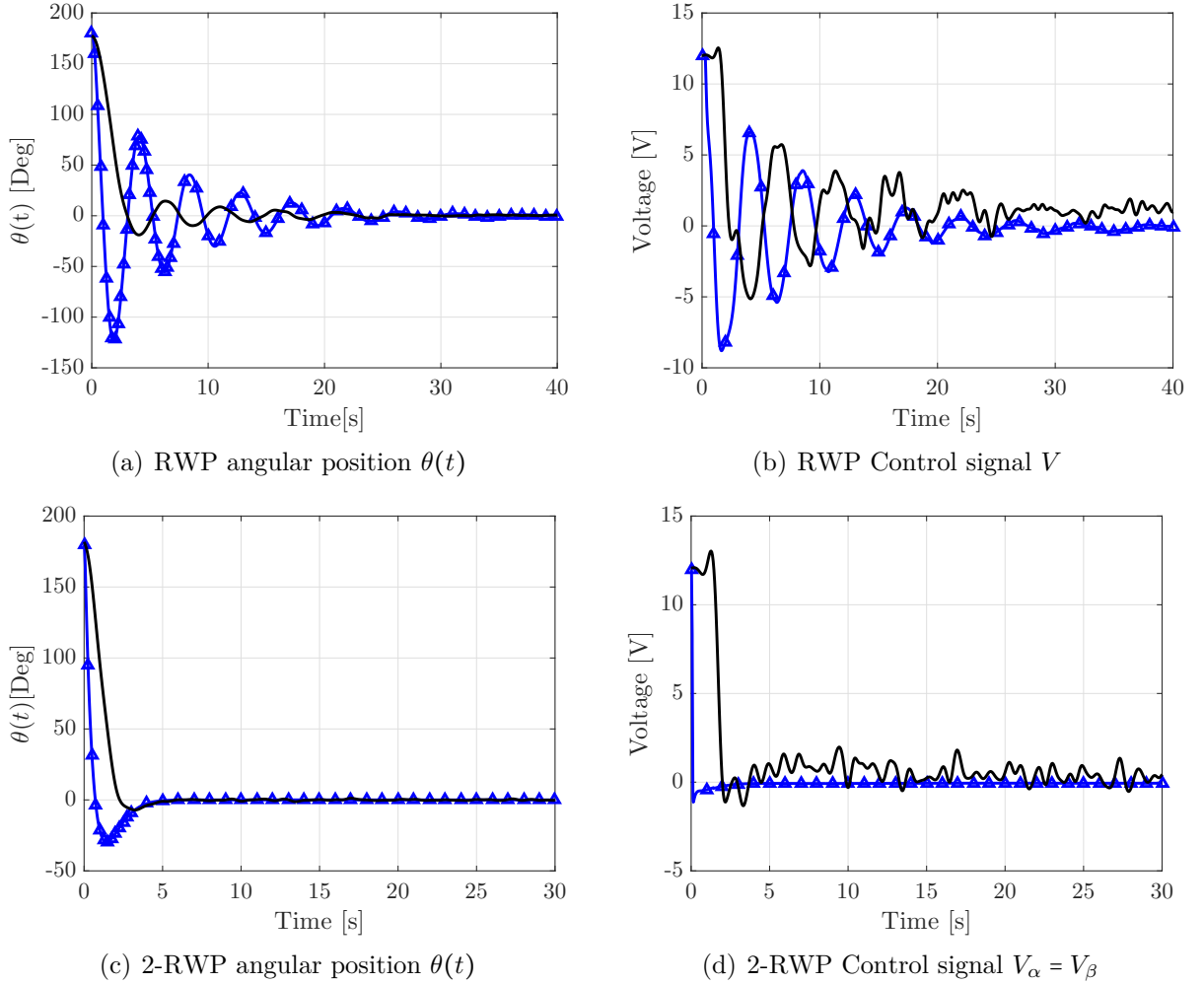
$$C(s) = k_p + k_i \frac{1}{s} + k_d s \quad (27)$$

where k_p is the proportional gain, k_i is the integral gain and k_d is the derivative gain. The same gains used for the experimental tests were used in the simulations. The operation of the designed PID controller starts with comparing the desired reference set θ_d and the effective angular position of the pendulum θ . From this comparison, the PID controller generates a voltage control signal that will be applied to the DC motor. In the RWP, this control signal will be applied to the single system's reaction wheel, but for the 2-RWP the voltage control signal generated will be applied equally to both DC motors coupled to each reaction wheel. After that, for both systems, the angular position of the pendulum θ is fed back, so that the comparison is restarted. This process will continue for the simulation time set. Both pendulums are controlled by the acceleration variation applied to the reaction wheels, so that an angular momentum variation will generate a torque. This torque is used as a control torque and from this torque it is possible to move RWP and 2-RWP to the inverted position.

3.3.2 Results using PID control to compare the configurations

A comparison between the numerical and experimental results is presented in Fig. 16. Figure 16(a) shows the experimental and numerical controlled angular positions for RWP. The control signal used, that is, the voltage applied to the DC motor, is shown in Fig. 16(b). The results for controlling 2-RWP in the inverted position can be seen in Fig. 16(c) where the controlled angular position for 2-RWP is presented. Figure 16(d) illustrates the control signal used for 2-RWP. These results indicate that the designed PID controllers could control both configurations of pendulums studied in this thesis.

Figure 16 – Results obtained for controlling RWP and 2-RWP in the inverted position where line $-\triangle-$ indicates the numerical PID control and line $-$ indicates the real-time PID controller.



Source: Trentin *et al.* (2020a).

Both real-time controllers designed could control both pendulum configurations studied by this work. However, some differences compared to the simulated results can be seen. Taking the RWP to the inverted position without swinging it up, but only

by accelerating the reaction wheel to produce a torque, may be more difficult. This also makes it harder to stabilize the experimental RWP using the real-time controller, where some oscillations could be observed. The model also did not consider, the damping coefficients from the bearings and the DC motor. And, as a potentiometer is used to read the angular position, it introduces some problems for the real-time controller due to the measurement of the potentiometer range as well as noise, which may cause the real-time controller to produce a different control signal. In addition, the flatness of the reaction wheels can lead to different results from the simulated ones, since the plant is self-manufactured and it has some construction imperfections that are hard to be modeled.

For both systems studied in this chapter, the controllers were designed to have similar behavior so that the difference between using one and two actuators could be compared. The use of two actuators allows a faster and easier control of the pendulum. The simulated and experimental results to control 2-RWP are much closer than the ones from RWP. The use of the same control signal for both reaction wheels for the 2-RWP was also shown to be effective. Using two reaction wheels leads to much fewer oscillations than using just one reaction wheel to control a pendulum in the inverted position. Furthermore, the difference between the configurations' settling time found that the 2-RWP is stabilized at 5 seconds while RWP takes 25 seconds to be stabilized.

3.3.3 Quantitatively analysis of performance

To quantitatively measure the performance for both pendulum configurations, four performance indices were calculated to evaluate the controlled angular position $\theta(t)$. The indices are suitable for comparing the performance between two forms of actuating an inverted pendulum using reaction wheels. Usually, these indices are minimized during the design of the controller to guarantee the best performance, but they can also be used to analyze the resulting output of systems (DORF; BISHOP, 2011). The configuration that presents the smaller indices is the one with the best performance.

The integral of time multiplied by the absolute error (ITAE) is a well-known performance index which reduces the contribution at the beginning of the signal and emphasizes the steady state error (DORF; BISHOP, 2011). The ITAE is given by:

$$\text{ITAE} = \int_0^T t|e(t)|dt \quad (28)$$

where t is time, $|e(t)|$ is the absolute error, and T is the total time. Other performance

index commonly used is the integral of the square of the error (ISE) given by:

$$\text{ISE} = \int_0^T e(t)^2 dt \quad (29)$$

where $e(t)$ is the error. This index discriminates an over damped system for an under damped system. The quadratic error is welcome even for analytical reasons than numerical ones (DORF; BISHOP, 2011). Another performance index is the integral of the absolute magnitude of the error (IAE). IAE is usually used in computational simulations and it is described by:

$$\text{IAE} = \int_0^T |e(t)| dt \quad (30)$$

The forth index that was calculated to evaluate the performance was the integral of time multiplied by the square error (ITSE):

$$\text{ITSE} = \int_0^T te(t)^2 dt \quad (31)$$

Thus, the performance indices were calculated for both pendulums for the experimental results. Table 2 presents the experimental case values for all four performance indices. The use of more than one index was applied to verify if, with other metrics, the same result for the analysis of the performance is obtained.

Table 2 – Performance indices for the experimental results of PID controller comparing RWP and 2-RWP.

Index	RWP	2-RWP
ITAE	1711.71	307.38
ISE	39111.09	25593.01
IAE	408.36	217.56
ITSE	36114.12	13235.51

Source: Trentin *et al.* (2020a).

Analyzing these values, it is evident that the 2-RWP has a better result because its performance indices have lower values. Each performance index has a different way to analyze the result of the controlled angular position $\theta(t)$ for both pendulum configurations, but it is clear that all indices state that the 2-RWP has a much better performance.

Two videos are available online that demonstrates the performance of the real-time controllers implemented:

- Real time controller for RWP:

<https://www.youtube.com/watch?v=awZPw5eWAZM>

- Real time controller for 2-RWP:

<https://youtu.be/EnZR9gfhOTE>

3.4 FINAL REMARKS

The dynamic modeling of two pendulum configurations, a classical and well-known one (RWP) and a new one (2-RWP), is carried out in this chapter. Additionally, the experimental set-ups built for the real-time experiments are described, also presenting the low-cost hardware used for the real-time tests. This hardware cost around USD 150,00. Moreover, the characterization of the DC motor and the friction of the experimental devices are conducted. The first main contribution of this thesis is the comparison of using one or two actuators to drive and control an inverted pendulum applying experimentally a simple PID controller. The difference between the settling time and the performance indices evidence that using two actuators could be an easier way to control an inverted pendulum.

4 NONLINEAR CONTROL OF AN INVERTED PENDULUM WITH TWO REACTION WHEELS

This chapter covers three control strategies for the pendulum with two reaction wheels (2-RWP). Nonlinear control techniques intend to find out if these controllers can result in better performance than the previous PID controller used. Firstly, section 4.1 carries out a nonlinear proportional-derivative controller design deriving it from the Lyapunov theory which ensures asymptotic stability of the closed-loop system. Section 4.2 describes the design of a sliding mode controller and section 4.3 presents a swing-up control strategy where only a reaction wheel is used at a time, combining the sliding mode control and the swing-up. All these controllers were verified experimentally. The results and performance comparison are presented in section 4.5. Lastly, section 4.6 presents the concluding remarks of this chapter.

4.1 DESIGN OF A LYAPUNOV BASED NONLINEAR PROPORTIONAL-DERIVATIVE CONTROLLER

This section presents the design of a proportional-derivative controller using the Lyapunov stability theory and also considering the nonlinear friction model of the 2-RWP. To design a feedback control law for the 2-RWP using the Lyapunov stability theory, it is necessary to select an energy-like Lyapunov candidate function:

$$\mathcal{V}(\delta\theta, \delta\dot{\theta}) = I_{zeq}^O \frac{\delta\dot{\theta}^2}{2} + k_p \frac{\delta\theta^2}{2} \quad (32)$$

where $\delta\dot{\theta} = \dot{\theta} - \dot{\theta}_d$, $\delta\theta = \theta - \theta_d$, k_p is a positive feedback gain, and I_{zeq}^O is the moment of inertia of the 2-RWP relative to point O . For the system to be stable, the time derivative of the Lyapunov candidate function with respect to time must be negative semi-definite.

The time derivative of the Lyapunov candidate function is given by:

$$\dot{\mathcal{V}}(\delta\theta, \delta\dot{\theta}) = \delta\dot{\theta} (I_{zeq}^O \delta\ddot{\theta} + k_p \delta\theta) \quad (33)$$

Therefore, $\dot{\mathcal{V}}(\delta\theta, \delta\dot{\theta})$ is set to be equal to a negative semi-definite function, in this case:

$$\dot{\mathcal{V}}(\delta\theta, \delta\dot{\theta}) \triangleq \delta\dot{\theta} (I_{zeq}^O \delta\ddot{\theta} + k_p \delta\theta) = -k_d \delta\dot{\theta}^2 \quad (34)$$

Developing the derivative of the Lyapunov candidate function, it is possible to obtain the following stable closed-loop dynamical system described as:

$$I_{zeq}^O \ddot{\theta} - I_{zeq}^O \ddot{\theta}_d + k_d \delta\dot{\theta} + k_p \delta\theta = 0 \quad (35)$$

To find the control law, it is necessary to substitute the equation of motion of the system, Eq. (26), into the closed-loop system presented in Eq. (35), yielding:

$$-I_{zw1}^A \ddot{\alpha} - I_{zw2}^B \ddot{\beta} + (m_1 - m_2)g \frac{\ell}{2} \sin \theta - T_{fr} - I_{zeq}^O \ddot{\theta}_d + k_p \delta\theta + k_d \delta\dot{\theta} = 0 \quad (36)$$

and rearranging, we obtain the control law for the 2-RWP:

$$u_{PD} \triangleq \begin{bmatrix} I_{zw1}^A & I_{zw2}^B \end{bmatrix} \begin{Bmatrix} \ddot{\alpha} \\ \ddot{\beta} \end{Bmatrix} = (m_1 - m_2)g \frac{\ell}{2} \sin \theta - T_{fr} - I_{zeq}^O \ddot{\theta}_d + k_p \delta\theta + k_d \delta\dot{\theta} \quad (37)$$

To guarantee the asymptotic stability of the equilibrium point, the candidate for Lyapunov function $\mathcal{V}(\delta\theta, \delta\dot{\theta})$ must be positive definite, whereas its derivative $\dot{\mathcal{V}}(\delta\theta, \delta\dot{\theta})$ must be negative definite. If the Lyapunov function derivative is negative semi-definite, it is still possible for the equilibrium point to be asymptotically stable (SCHAUB; JUNKINS, 2014). This can be typically concluded by using the Barbalat's Lemma. The theorem enunciated below, introduced by Mukherjee and Chen (1993), provides the sufficient conditions for proving the asymptotic stability when the Lyapunov function derivative is only semi-definite, i.e., $\dot{\mathcal{V}}(\delta\theta, \delta\dot{\theta}) \leq 0$, by investigating higher-order derivatives of the Lyapunov functions (MUKHERJEE; CHEN, 1993; MUKHERJEE; JUNKINS, 1993; JUNKINS; KIM, 1993).

Theorem (Mukherjee and Chen (1993)): Assume that there exists a Lyapunov function $\mathcal{V}(\mathbf{x})$ for the dynamical system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$. Let Ω be a nonempty set of state-space vectors such that $\dot{\mathcal{V}}(\mathbf{x}) = 0$. If the first $k - 1$ derivatives of $\mathcal{V}(\mathbf{x})$, evaluated on the set Ω ,

are zero

$$\frac{d^i \mathcal{V}(\mathbf{x})}{dt^i} = 0, \quad \forall \mathbf{x} \in \Omega, \quad i = 1, 2, \dots, k-1,$$

and the k -th derivative is negative definite on the set Ω

$$\frac{d^k \mathcal{V}(\mathbf{x})}{dt^k} < 0, \quad \forall \mathbf{x} \in \Omega,$$

then the system $\mathbf{x}$ is asymptotically stable if k is an odd number.

The proof of the theorem above can be found in Mukherjee and Chen (1993). Thereby, to find out which kind of stability the proposed control law, Eq. (37), guarantees, the higher-order derivatives of the candidate to the Lyapunov function are analyzed. Evaluating the second derivative of the Lyapunov function, presented in Eq.(38), none information is obtained.

$$\ddot{\mathcal{V}}(\delta\theta, \delta\dot{\theta}) = -2k_d \delta\dot{\theta} \delta\ddot{\theta}, \quad \ddot{\mathcal{V}}(\delta\theta, \delta\dot{\theta} = 0) = 0 \quad (38)$$

Thus, it is necessary to analyze the third derivative of $\mathcal{V}(\delta\theta, \delta\dot{\theta})$:

$$\ddot{\mathcal{V}}(\delta\theta, \delta\dot{\theta}) = -2k_d \delta\ddot{\theta}^2 - 2k_d \delta\dot{\theta} \delta\ddot{\ddot{\theta}}, \quad \ddot{\mathcal{V}}(\delta\theta, \delta\dot{\theta} = 0) = -2k_d \delta\ddot{\theta}^2 \quad (39)$$

and from the closed-loop system when $\delta\dot{\theta} = 0$, we can obtain:

$$\delta\ddot{\theta} = -\frac{k_p}{I_{zeq}^O} \delta\theta \quad (40)$$

Substituting the result presented in Eq. (40) into Eq. (39):

$$\ddot{\mathcal{V}}(\delta\theta, \delta\dot{\theta} = 0) \triangleq -2k_d \left(-\frac{k_p}{I_{zeq}^O} \delta\theta \right)^2 = -2k_d \left(\frac{k_p}{I_{zeq}^O} \right)^2 \delta\theta^2 < 0 \quad (41)$$

The third derivative of $\mathcal{V}(\delta\theta, \delta\dot{\theta})$ is negative definite in terms of $\delta\dot{\theta}$, and because the first nonzero $\mathcal{V}$ derivative is of odd order, the equilibrium point of the closed-loop system described by (26) and (37) is asymptotically stable.

4.2 DESIGN OF A SLIDING MODE CONTROLLER

This section presents the design of a sliding mode controller for the 2-RWP to evaluate how the results of a simple approach to robust control would perform when compared

with other control techniques. It is crucial to specify a sliding surface or to choose a sliding function to design a controller using sliding modes (LIU; WANG, 2012). The dynamic regime of this surface is chosen in such a way that all the trajectories within this surface converge to the desired value (SLOTINE; LI, 1991; LIU; WANG, 2012; LOPEZ-MARTINEZ; ACOSTA; CANO, 2010). However, in the design of a sliding mode controller, this surface must be chosen in a way where the dynamic of the system become stable and, therefore, a control law must be defined where all trajectories are driven to the sliding surface.

For the design of the sliding mode controller, we used a sliding function instead of a surface, since the majority of the states of the experimental 2-RWP are unknown. Thereby, the controller was designed using the known states. The tracking error, defined as $e = \theta - \theta_d$, is associated to the desired trajectory and the sliding function is defined by:

$$\sigma(e, t) = \dot{e} + \zeta e \quad (42)$$

where ζ is a positive constant. Equation (42) represents a straight line of sliding in the error configuration space. The defined sliding function must converge to zero in finite time. From the sliding variable presented in Eq. (42), its derivative yields:

$$\dot{\sigma} = \ddot{e} + \zeta \dot{e} \equiv \ddot{\theta} - \ddot{\theta}_d + \zeta \dot{e} \quad (43)$$

Here, it is necessary to select a Lyapunov function $\mathcal{V}(\sigma)$ being positive definite and that has a negative definite derivative, where a possible one is:

$$\mathcal{V}(\sigma) = \frac{1}{2} \sigma^2 \quad (44)$$

With the presented Lyapunov candidate function, the sliding mode can be shown to exist, and the system will be stable if $\dot{\mathcal{V}}(\sigma)$ is negative definite. The time derivative of the Lyapunov candidate function is shown below:

$$\dot{\mathcal{V}}(\sigma) = \sigma \dot{\sigma}, \quad \sigma \neq 0 \quad (45)$$

The motion equation of the 2-RWP including the friction torque, Eq. (26), can be rearranged as:

$$\ddot{\theta} = \frac{(m_1 - m_2)g \frac{\ell}{2} \sin \theta}{I_{zeq}^O} - \frac{I_{zw1}^A}{I_{zeq}^O} \ddot{\alpha} - \frac{I_{zw2}^B}{I_{zeq}^O} \ddot{\beta} - \frac{T_{fr}}{I_{zeq}^O} \quad (46)$$

The motion equation of the 2-RWP presented in Eq. (46) can be substituted into the

derivative of the sliding function shown in Eq. (43):

$$\dot{\sigma} = \frac{(m_1 - m_2)g_2^\ell \sin \theta}{I_{zeq}^O} - \frac{I_{zw1}^A}{I_{zeq}^O} \ddot{\alpha} - \frac{I_{zw2}^B}{I_{zeq}^O} \ddot{\beta} - \frac{T_{fr}}{I_{zeq}^O} - \ddot{\theta}_d + \zeta \dot{e} \quad (47)$$

which leads to:

$$\dot{\sigma} = \frac{(m_1 - m_2)g_2^\ell \sin \theta}{I_{zeq}^O} - \underbrace{\left[\frac{I_{zw1}^A}{I_{zeq}^O} \quad \frac{I_{zw2}^B}{I_{zeq}^O} \right]}_u \begin{Bmatrix} \ddot{\alpha} \\ \ddot{\beta} \end{Bmatrix} - \frac{T_{fr}}{I_{zeq}^O} - \ddot{\theta}_d + \zeta \dot{e} \quad (48)$$

The following control law u can be adopted:

$$u = \left[\frac{I_{zw1}^A}{I_{zeq}^O} \quad \frac{I_{zw2}^B}{I_{zeq}^O} \right] \begin{Bmatrix} \ddot{\alpha} \\ \ddot{\beta} \end{Bmatrix} = \frac{(m_1 - m_2)g_2^\ell \sin \theta}{I_{zeq}^O} - \frac{T_{fr}}{I_{zeq}^O} - \ddot{\theta}_d + \zeta \dot{e} + \xi \operatorname{sgn}(\sigma) \quad (49)$$

The control law proposed in Eq. (49) can be substituted into the derivative of the sliding function shown in Eq.(48):

$$\dot{\sigma} = \frac{(m_1 - m_2)g_2^\ell \sin \theta}{I_{zeq}^O} - \frac{T_{fr}}{I_{zeq}^O} - \left(\frac{(m_1 - m_2)g_2^\ell \sin \theta}{I_{zeq}^O} - \frac{T_{fr}}{I_{zeq}^O} - \ddot{\theta}_d + \zeta \dot{e} + \xi \operatorname{sgn}(\sigma) \right) - \ddot{\theta}_d + \zeta \dot{e} \quad (50)$$

yielding:

$$\dot{\sigma} = -\xi \operatorname{sgn}(\sigma) \quad (51)$$

This result is substituted into the derivative of the Lyapunov candidate function presented in Eq. (45) to prove that the selected function is a Lyapunov function where its derivative is negative definite:

$$\dot{\mathcal{V}}(\sigma) = \sigma (-\xi \operatorname{sgn}(\sigma)) = -\xi |\sigma| < 0, \quad \sigma \neq 0 \quad (52)$$

where ξ is an arbitrary positive constant and $\operatorname{sgn}(\sigma)$ is a sign function defined as:

$$\operatorname{sgn}(\sigma) = \begin{cases} 1 & \text{if } \sigma > 0 \\ 0 & \text{if } \sigma = 0 \\ -1 & \text{if } \sigma < 0 \end{cases} \quad (53)$$

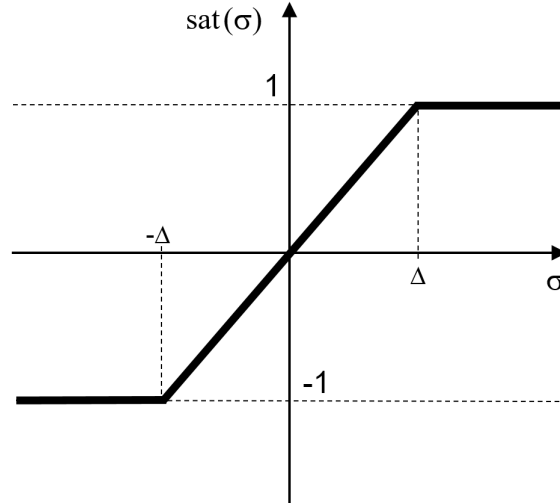
From a practical point of view, a sliding mode cannot exist because this would require that the controller switches at an infinite frequency. Considering the sampling time, the

discontinuity in controller feedback produces a particular behavior known as chattering (SLOTINE; LI, 1991; LIU; WANG, 2012). Slotine and Li (1991) proposed to smooth the sign function used in the control law to avoid chattering. Thus, a boundary layer around the sliding function is established. This work follows a similar approach to restrain the chattering phenomenon, thus, the following saturation function $\text{sat}(\sigma)$ is adopted:

$$\text{sat}(\sigma) = \begin{cases} 1 & \text{if } \sigma > \Delta \\ k\sigma & \text{if } |\sigma| \leq \Delta, \quad k = \frac{1}{\Delta} \\ -1 & \text{if } \sigma < -\Delta \end{cases} \quad (54)$$

Figure 17 illustrates the saturation function used.

Figure 17 – Saturation function



Source: Prepared by the author.

Thus, employing the saturation function in the adopted control law that was presented in eq. (49) and rewriting it:

$$u = \begin{bmatrix} \frac{I_{zw1}^A}{I_{zeq}^O} & \frac{I_{zw2}^B}{I_{zeq}^O} \end{bmatrix} \begin{Bmatrix} \ddot{\alpha} \\ \ddot{\beta} \end{Bmatrix} = \frac{(m_1 - m_2)g \frac{\ell}{2} \sin \theta}{I_{zeq}^O} - \frac{T_{fr}}{I_{zeq}^O} - \ddot{\theta}_d + \zeta \dot{e} + \xi \text{sat}(\sigma) \quad (55)$$

Since the sliding mode control law can be separated into two main components, the equivalent and the switching parts, equation (55) can be written as:

$$u = u_{eq} + u^\pm \quad (56)$$

where

$$u_{eq} = \frac{(m_1 - m_2)g\frac{\ell}{2}\sin\theta}{I_{zeq}^O} - \frac{T_{fr}}{I_{zeq}^O} - \ddot{\theta}_d + \zeta\dot{e} \quad (57)$$

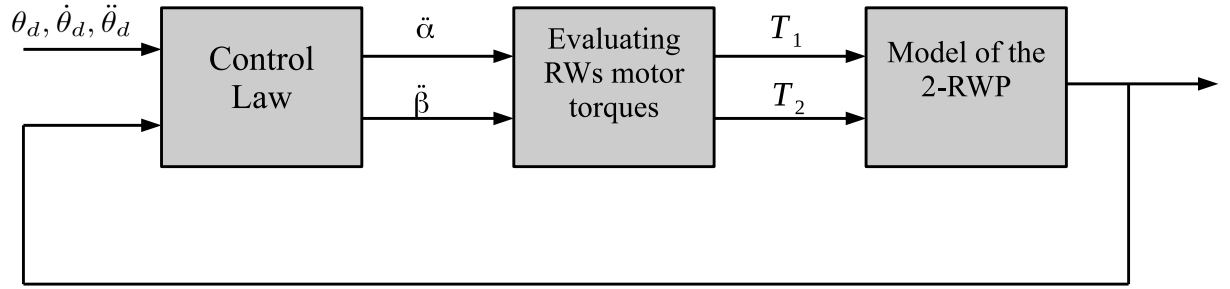
and

$$u^\pm = \xi \text{sat}(\sigma) \quad (58)$$

The switching component of the control law ($u^\pm$) takes the system to the sliding function, and the equivalent component (u_{eq}) is important when the system is sliding, since this component is responsible for driving the states to the equilibrium point.

Figure 18 exhibits the following block diagram which demonstrates how both controllers work for the numerical simulations.

Figure 18 – Blocks diagram for both controllers.



Source: Prepared by the author.

The desired and actual states are compared and the control laws are evaluated. With the desired rates evaluated by the control laws, the reaction wheel motor torques are calculated since they are necessary for the complete integration of the mathematical model of the 2-RWP. The resulting rates of the integration process are fed back and the process is restarted.

4.3 DESIGN OF A SWING-UP CONTROL STRATEGY

There are many different swing-up control strategies already presented in the literature. Some of them consider an energy approach looking at the potential energy of the pendulum. Here, the strategy is straightforward and combines an on-off controller with the sliding mode controller designed in subsection 4.2.

This sliding mode control law was designed to provide the desired angular acceleration for the reaction wheels. Moreover, we chose to swing up the pendulum using a reaction wheel at a time. To accomplish this task, the control law needs to be rearranged to turn off one of the control modes, i. e., to one of the reaction wheels, an angular acceleration equals to zero is commanded.

Thereby, the sliding mode control law presented in Eq. (55) for the 2-RWP can be rewritten such that each control mode can be allocated separately. The 2×1 state vector $\boldsymbol{\eta}_{2-RWP}$ is introduced to simplify the notation:

$$\boldsymbol{\eta}_{2-RWP} = \begin{Bmatrix} \ddot{\alpha} \\ \ddot{\beta} \end{Bmatrix} \quad (59)$$

And the 1×2 matrix $[Q_{2-RWP}]$:

$$[Q_{2-RWP}] = \begin{bmatrix} \frac{I_{zw1}^A}{I_{zeq}^O} & \frac{I_{zw2}^B}{I_{zeq}^O} \end{bmatrix} \quad (60)$$

Thus, the control law presented in Eq. (55) can be rewritten as:

$$[Q_{2-RWP}]\boldsymbol{\eta}_{2-RWP} = L_{2-RWP} \quad (61)$$

where $L_{2-RWP} = u_{eq} + u^\pm$. The inverse of the $[Q_{2-RWP}]$ can be evaluated using a weighted pseudo-inverse as done in Schaub *et al.* (1998):

$$\boldsymbol{\eta}_{2-RWP} = [W_{2-RWP}][Q_{2-RWP}]^T ([Q_{2-RWP}][W_{2-RWP}][Q_{2-RWP}]^T)^{-1} L_{2-RWP} \quad (62)$$

where for the case of the 2-RWP, $[W_{2-RWP}]$ is a 2×2 diagonal matrix:

$$[W_{2-RWP}] = \begin{bmatrix} W_1 & 0 \\ 0 & W_2 \end{bmatrix} \quad (63)$$

where W_1 and W_2 are the weights related to each reaction wheel respectively. Thus, when it is desired to use only reaction wheel 1: $W_1 = 1$ and $W_2 = 0$, and when using reaction wheel 2: $W_1 = 0$ and $W_2 = 1$.

The swing-up control law can be summarized as:

$$\boldsymbol{\eta}_{sw} = \begin{cases} \boldsymbol{\eta}_{2-RWP} & \text{if } \text{sgn}(\dot{\theta}) = 1 \\ \boldsymbol{\eta}_{2-RWP} & \text{if } \text{sgn}(\dot{\theta}) = 0 \\ 0 & \text{if } \text{sgn}(\dot{\theta}) = -1 \text{ and } |\theta| > 20^\circ \\ \boldsymbol{\eta}_{2-RWP} & \text{if } \text{sgn}(\dot{\theta}) = -1 \text{ and } |\theta| < 20^\circ \end{cases} \quad (64)$$

where $\boldsymbol{\eta}_{sw}$ is the resulting desired rates for the reaction wheels' angular acceleration, taking into account which reaction wheel is actuating and employing the sliding mode control law designed, and $\dot{\theta}$ is the pendulum's angular velocity.

This swing-up control strategy was developed to deal with the limitations of the experimental set-up. Moreover, the controller is set to turn-off when the DC motor starts to be unable to provide the necessary torque, i.e., the pendulum's angular position starts to repeat, and the pendulum angular velocity changes its sign. The controller is turned back on when the pendulum reaches another point where the angular position will be repeated, and the angular velocity sign will change again.

In this strategy, the swing-up always seeks for the sliding function, and it does not switch between controllers. Once the controller is on, the sliding mode control law is responsible for providing adequate angular acceleration for the reaction wheel that is being used. Thus, the motor torque is evaluated, and the voltage is commanded to the DC motor. Additionally, turning off the controller when the saturation of the DC motor is reached uses the energy of swinging the pendulum to help take it to the unstable equilibrium point.

One of the hypotheses made based on experimental observation was that the swing-up control law proposed is only turned off when the pendulum's angular position is in a region different than $|\theta| < 20^\circ$, otherwise the sliding mode control actuates. Thus, the process of swinging up the pendulum will drive the pendulum to the inverted position using only a reaction wheel at a time, and the designed sliding mode control law will balance the pendulum in the inverted position.

4.4 EXPERIMENTAL PARTICULARITIES FOR THE REAL-TIME NONLINEAR CONTROLLERS

Some differences in the real-time implementation for the nonlinear controller are illustrated in this section. The low-cost hardware used for these experiments is the same as the one used for the PID controller.

The real-time control algorithm developed in the Arduino Uno works simply for both nonlinear controllers designed. The angular position of the 2-RWP is read using the multi-turn potentiometer. The value read by the potentiometer is converted into an angle value, and also its derivatives are calculated in the script developed in the Arduino. In this way, the angular position, the angular velocity, and the angular acceleration of the

pendulum are known for each iteration since they are necessary to evaluate the nonlinear control laws. The reaction wheels angular acceleration are the control signals provided by the control laws, and considering the pendulum angular acceleration, the desired motor torques can be evaluated as shown by equations (7) and (8).

It is essential to highlight some difficulties encountered in the plant. There are no encoders coupled to the DC motors. This means that the states related to the reaction wheels are unknown, and only the desired rates are commanded. Additionally, considering that $\frac{R_a}{K_t}$ is much greater than K_v and that the reaction wheels angular velocities are unknown, for the practical application, equations (22) and (23) were considered as:

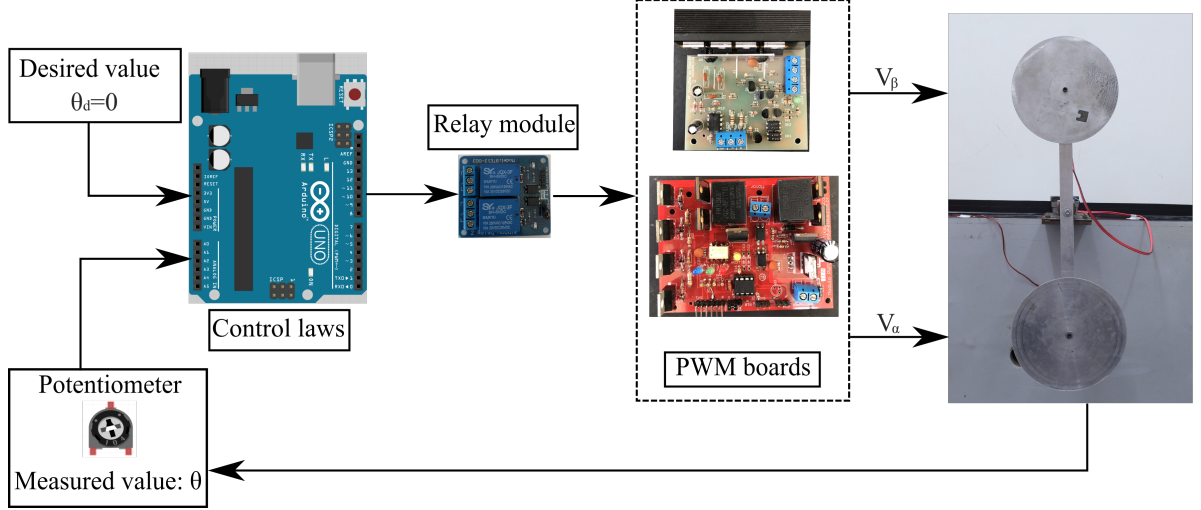
$$V_\alpha(t) = T_1 \frac{R_a}{K_t} \quad (65)$$

$$V_\beta(t) = T_2 \frac{R_a}{K_t} \quad (66)$$

After calculating the desired voltage values for each DC motor, as shown by equations (65) and (66), these values are converted into PWM values ranging from -255 to 255 . The relay module verifies which way the DC motors should rotate evaluating if these values are positive or negative indicating one way or the other. Each PWM board is fed with an external power source, and it receives the controlled PWM signal from the Arduino. Thus, they can provide the correct and controlled voltage signal to the DC motors. After this point, the process is restarted. Moreover, a saturation limit of 12 V is imposed on the voltage applied to both DC motors for both controllers designed. This limit does not affect the performance of the controllers, and it is imposed to protect the component

The flowchart depicted in Fig. 19 exemplifies how both controllers work. The desired and actual rates are compared, and the control laws are evaluated. With the desired rates evaluated by the control laws, the reaction wheel motor torques are calculated so that the voltage of each reaction wheel can be evaluated. The voltages are converted into PWM values, and after the relay analysis which way the wheels must spin, they are commanded to the PWM boards. These boards combine the controlled signal with the external power supply to provide the correct voltage to the DC motors coupled to the reaction wheels. Subsequently, the angle value measured by the potentiometer is fed back, and the process is restarted.

Figure 19 – Flowchart of the real-time application.



Source: (TRENTIN *et al.*, 2020b).

4.5 RESULTS FOR THE NONLINEAR CONTROLLERS FOR THE 2-RWP

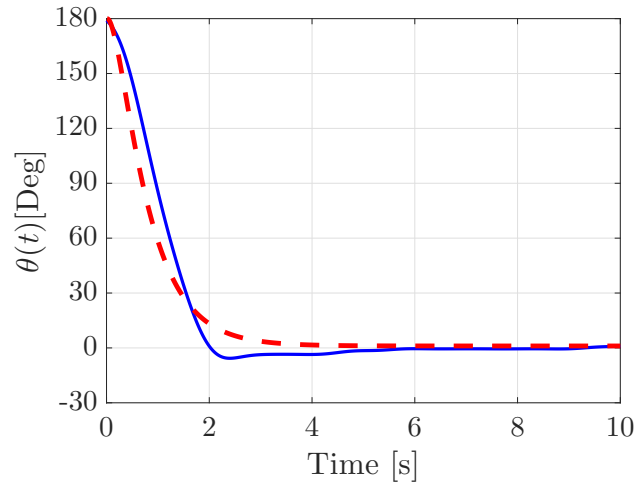
This section presents the numerical and experimental results for both nonlinear controllers designed. The goal is to drive the two reaction wheels pendulum to the inverted position. This is carried out at first using both reaction wheels where subsection 4.5.1 shows the results using the nonlinear proportional-derivative controller and subsection 4.5.2 exhibits the results using the sliding modes controller. Furthermore, a swing-up control strategy was designed to use only one reaction at a time. Thus, subsection 4.3 depicts the results to swing-up and control the 2-RWP using only a reaction wheel combining an on-off controller and the sliding mode controller designed. Subsection 4.5.4 displays a comparison between the experimental results for both nonlinear controllers and previous results using a PID controller.

4.5.1 Results for the nonlinear proportional-derivative controller

Using the proposed control law presented by equation (37), the numerical and experimental pendulum angular position are charted in Fig. 20 being close to each other. In the experimental result, a small overshoot can be noticed while the numerical one does not present such behavior. But, the equilibrium position is achieved in a very close

instant near 6 seconds where the controller demonstrates to be efficient enough to drive and control the pendulum in the inverted position. The gains used for the simulation were $K = 6$ and $P = 7$ and for the experimental application were $K = 60$ and $P = 15$.

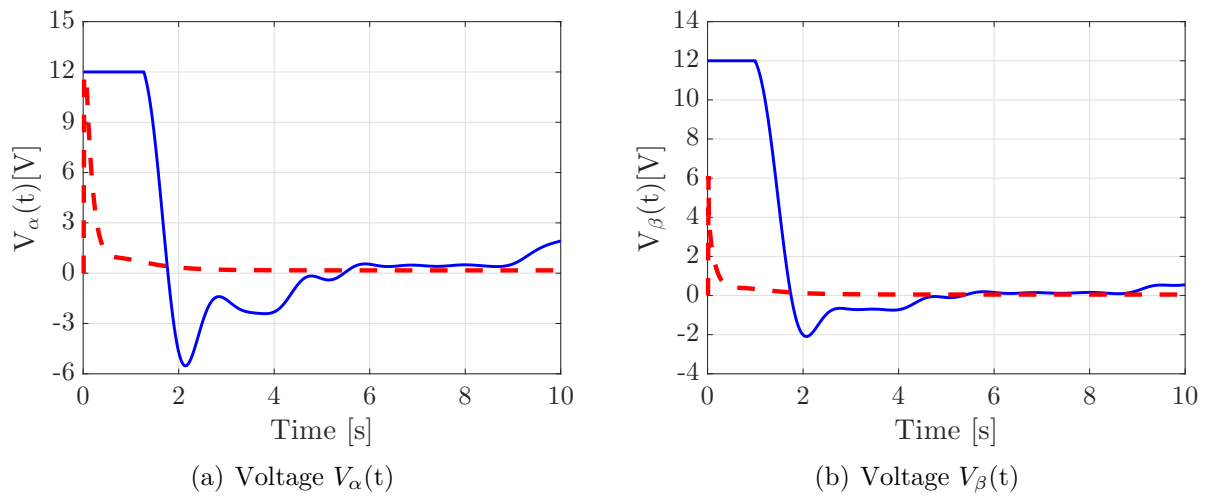
Figure 20 – Pendulum angular position to drive and control the 2-RWP in the inverted position ($\theta = 0^\circ$) using nonlinear PD controller. - - - indicates numerical result and - the experimental result.



Source: Prepared by the author.

Figure 21 exhibits the voltage applied to each DC motor. The numerical and experimental results are close only after the pendulum is controlled in the inverted position. The big difference in the first seconds can be due to unmodeled dynamics from the DC motor.

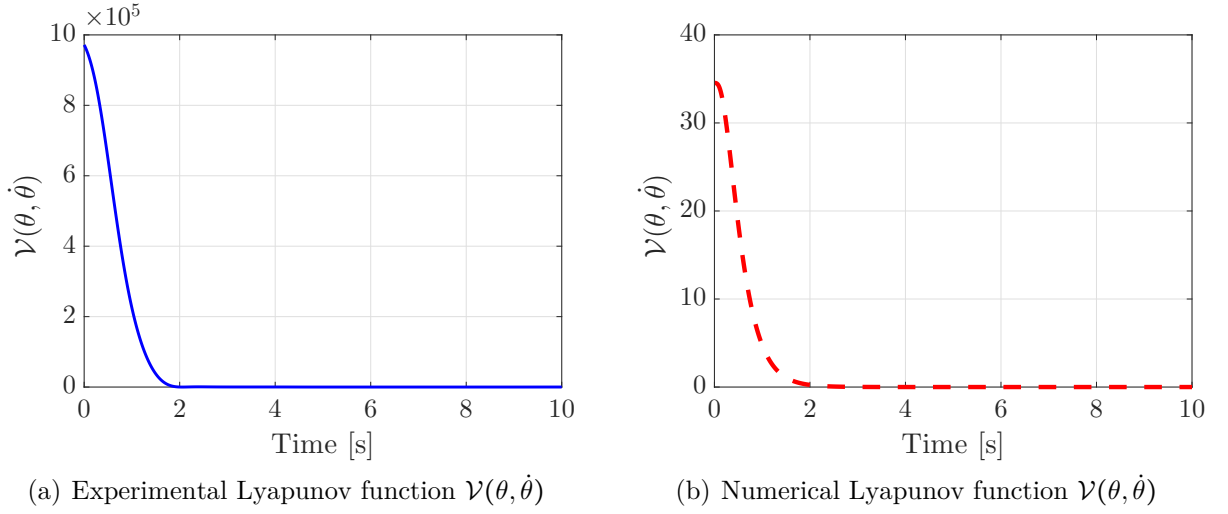
Figure 21 – Voltage values applied to the DC motors to control the 2-RWP in the inverted position ($\theta = 0^\circ$). - - - indicates numerical result and - the experimental result.



Source: Prepared by the author.

It is important to analyze the behavior of the proposed Lyapunov function for the nonlinear PD controller, presented in equation (32), to verify if the analytical proven asymptotically stability is maintained in the numerical and experimental application. Figure 22 illustrates such a result, and due to the different gains used in the experimental and numerical application these results had to be charted in two different plots so that the Lyapunov function behavior could be verified.

Figure 22 – Lyapunov function to control the 2-RWP in the inverted position ($\theta = 0^\circ$). - - indicates numerical result and --- the experimental result.

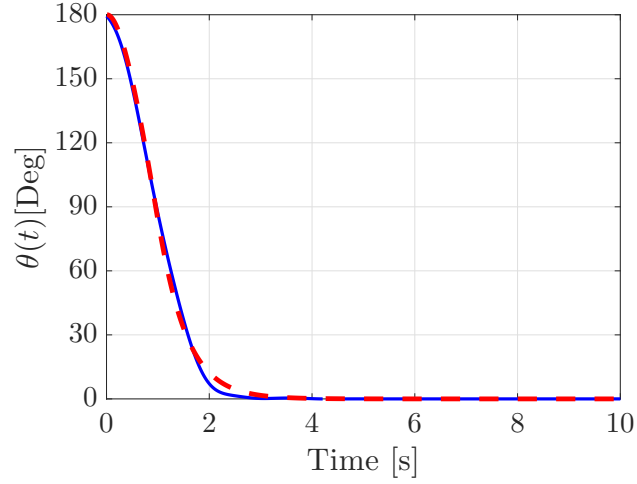


Source: Prepared by the author.

4.5.2 Results for the sliding mode controller

This subsection shows the results of using a sliding mode controller to control a pendulum with two reaction wheels. Figure 23 depicts the numerical and experimental results for the pendulum angular position.

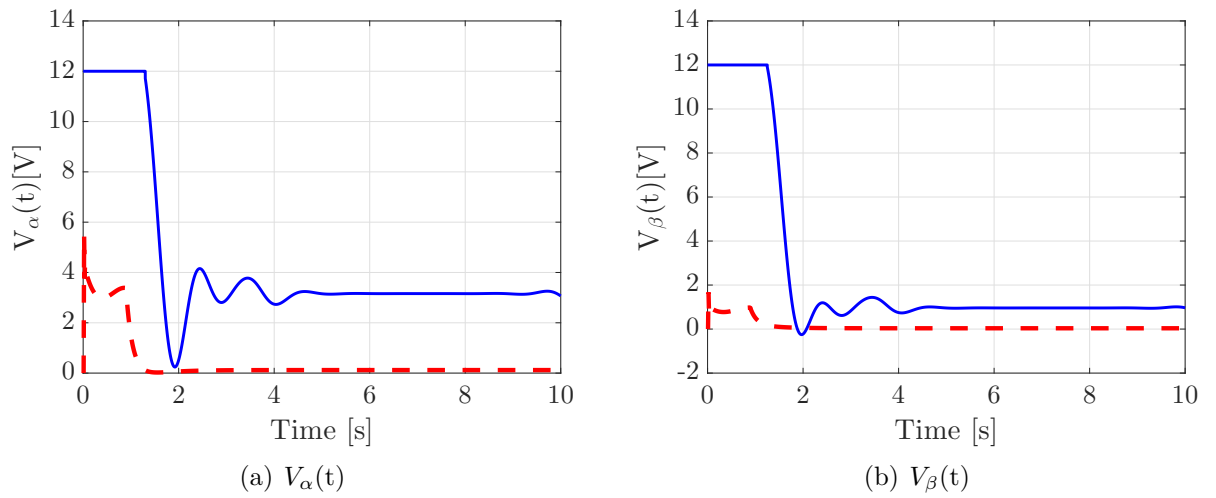
Figure 23 – Pendulum angular position to drive and control the 2-RWP in the inverted position ($\theta = 0^\circ$) using SMC. - - - indicates numerical result and - the experimental result.



Source: Prepared by the author.

In this case, the numerical and experimental results are very close. This is due to the same slope of the sliding function being used for both results ($\zeta = 2$). However, the gain ξ for numerical simulation was $\xi = 10$, and for the real-time experiment $\xi = 100$. For all instances using the sliding mode controller, the parameter Δ is adopted as $\Delta = 1$, including the swing-up strategy. The voltage values for this case are presented in Fig. 24. Note that a big difference is observed when comparing the numerical and experimental results.

Figure 24 – Voltage values applied to the DC motors to control the 2-RWP in the inverted position ($\theta = 0^\circ$) using SMC. - - - indicates numerical result and - the experimental result.

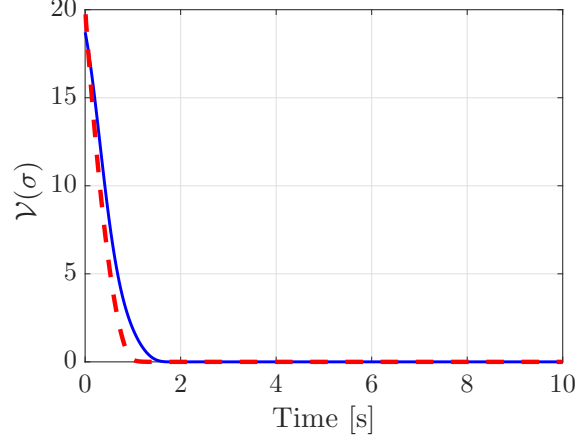


Source: Prepared by the author.

In this case, the numerical and experimental Lyapunov functions are very close, as

shown in Fig. 25, and also the analytical proof of stability could be verified.

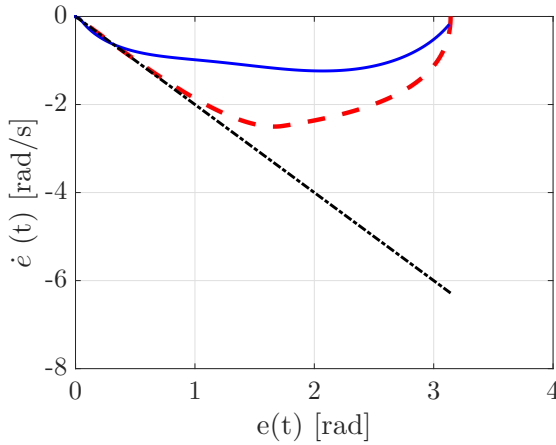
Figure 25 – Lyapunov function ($\mathcal{V}(\sigma)$) to control the 2-RWP in the inverted position ($\theta = 0^\circ$) using SMC. - - - indicates numerical result and - the experimental result.



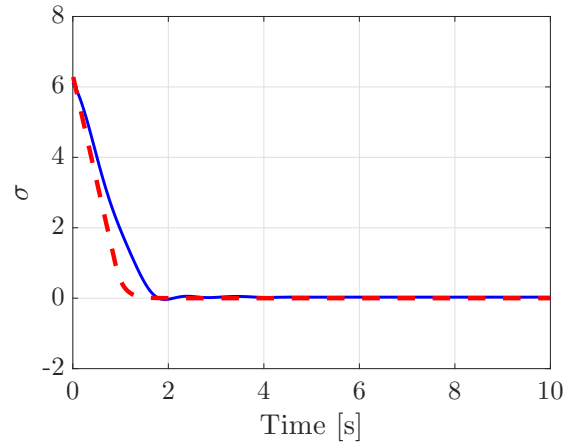
Source: Prepared by the author.

Figure 26 presents specific results of the sliding mode controller. Figure 26(a) shows the sliding function designed in the error configuration space where the numerical and experimental results for the controller can reach the function and slide to the equilibrium position.

Figure 26 – Error configuration space and sliding function to control the 2-RWP in the inverted position ($\theta = 0^\circ$). - - - indicates numerical result, - the experimental result and - • - the sliding function



(a) Error configuration space



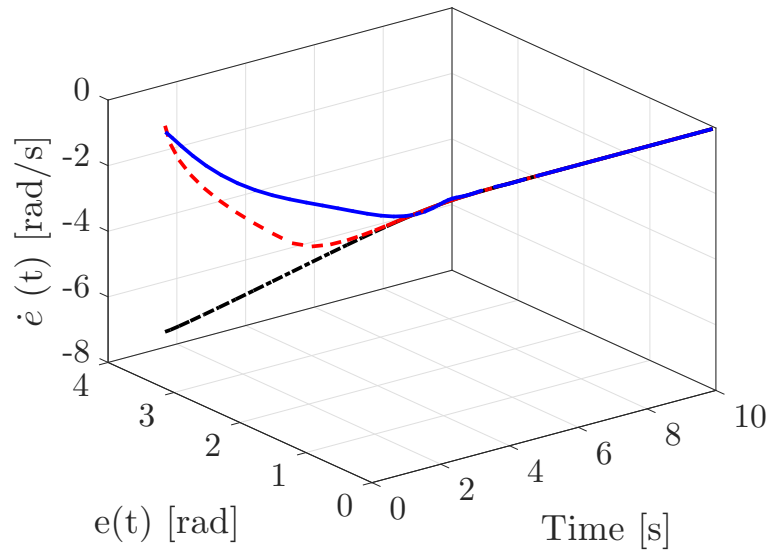
(b) Sliding function

Source: (TRENTIN *et al.*, 2020b).

Fig. 26(b) depicts the behavior of the sliding function ($\sigma(e, t)$). When it is zero, it means that the system has reached the sliding function designed in equation (42). If the function remains at zero, it shows that the system is sliding. The system will thereby

slide to the equilibrium point, describing the property of attractiveness, which is also one of the features of the sliding mode control. The reachability and the attractiveness properties are proven by analyzing the numerical and experimental results presented in Fig. 26. Additionally, it is very interesting to observe how the configuration space evolves with time which is present in Fig. 27. The control law proposed can take the system to the sliding function and to keep it there controlled.

Figure 27 – Sliding function varying with time to control the 2-RWP in the inverted position ($\theta = 0^\circ$). - - - indicates numerical result, - the experimental result and - • - the sliding function.



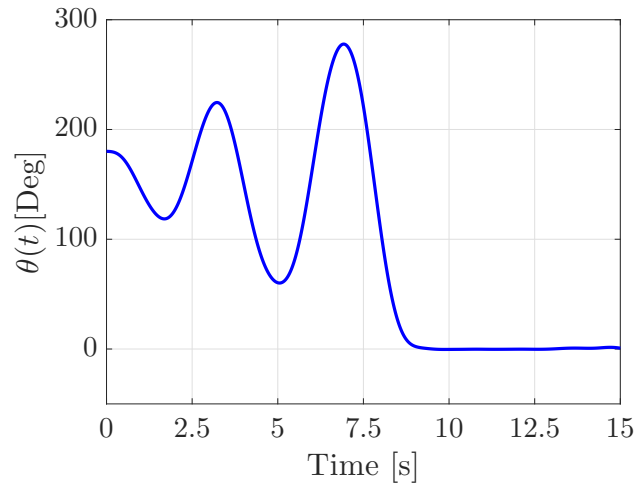
Source: Prepared by the author.

It is normal in any control design to find discrepancies between the model and the real-time experiment because of unmodelled dynamics, parameter uncertainty, and the presence of external disturbances (LIU; YU, 2013; LIU; WANG, 2012). Besides, the following link allows the reader to see the performance of the real-time controller implemented: <https://youtu.be/HbPgJYSusmc>.

4.5.3 Results for the swing-up control

The results for the swing-up and control of the 2-RWP are presented in this subsection. These results were implemented only in real-time. Thus, using only reaction wheel 1, the pendulum angular position of the 2-RWP is charted in Fig. 28 where the controller can drive the pendulum to the inverted position after swinging it up and the pendulum remains stable in the inverted position using the sliding mode controller. For this case, the parameters for the sliding mode controller were $\zeta = 2.7$ and $\xi = 100$.

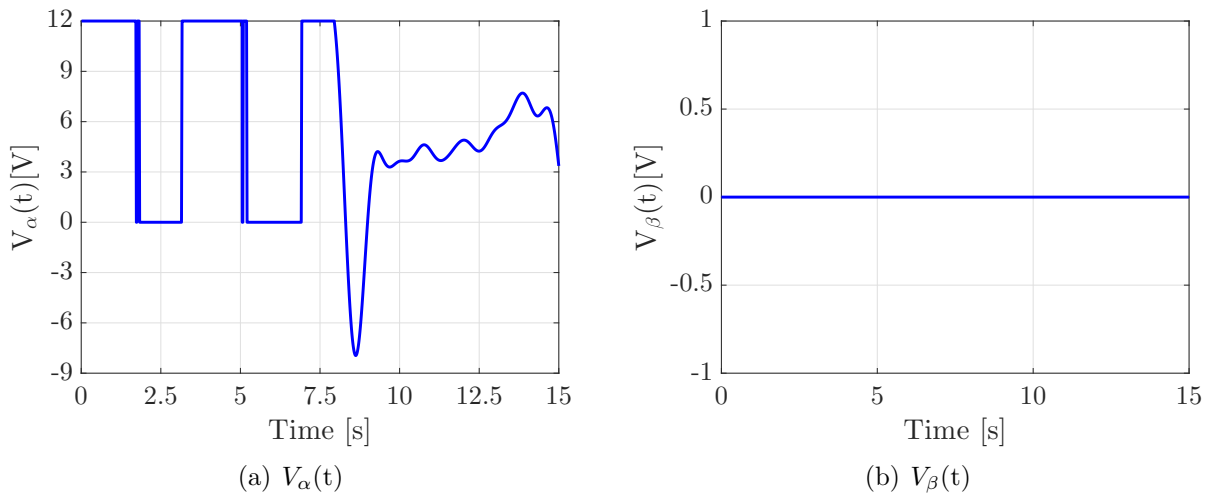
Figure 28 – Pendulum angular position to swing-up and control the 2-RWP in the inverted position ($\theta = 0^\circ$) using only RW1. - indicates the experimental result.



Source: Prepared by the author.

Figure 29 presents the voltage values applied to the reaction wheels. Actuating only reaction wheel 1, the voltage applied to DC motor of reaction 2 must be zero for the entire experiment as observed.

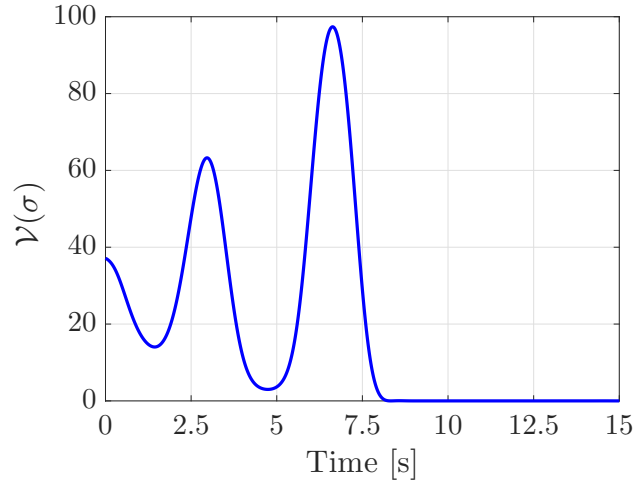
Figure 29 – Voltage values applied to the DC motors to swing-up and control the 2-RWP in the inverted position ($\theta = 0^\circ$) using only RW1. - indicates the experimental result.



Source: Prepared by the author.

Figure 30 depicts the Lyapunov function for the sliding mode controller for the entire experiment. When the controller is being turned on and off the stability is not reached. But in the attempt that the pendulum reaches the sliding surface, the positive definite function behavior can be observed.

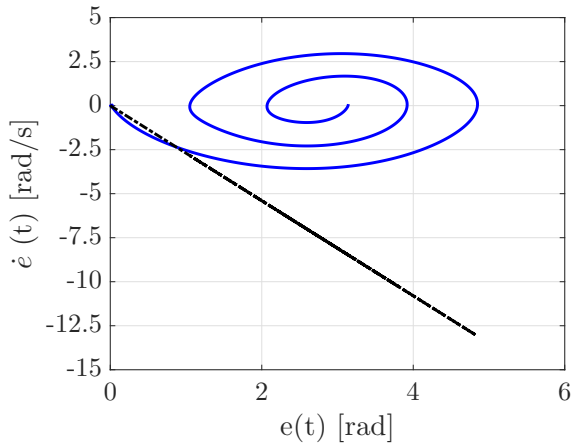
Figure 30 – Lyapunov function ($\mathcal{V}(\sigma)$) to swing-up and control the 2-RWP in the inverted position ($\theta = 0^\circ$) using only RW1. - indicates the experimental result.



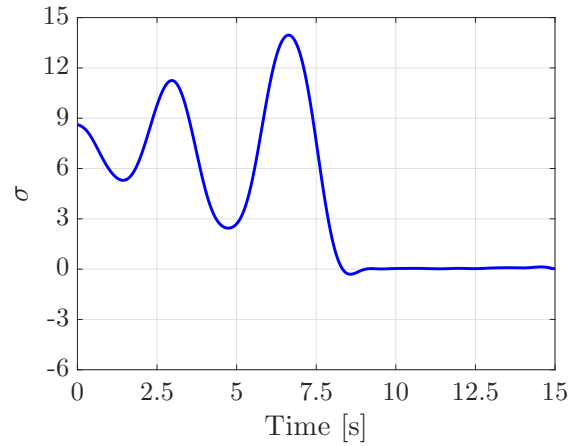
Source: Prepared by the author.

Figure 31 exhibits the error configuration space where the sliding mode controller is designed and the behavior of the sliding function. Figure 32 shows the configuration space evolving with time where the swing-up strategy and the process of reaching the sliding function using it can be observed.

Figure 31 – Error configuration space and sliding function to swing-up and control the 2-RWP in the inverted position ($\theta = 0^\circ$) using only RW1. - indicates the experimental result and -•- the sliding function.



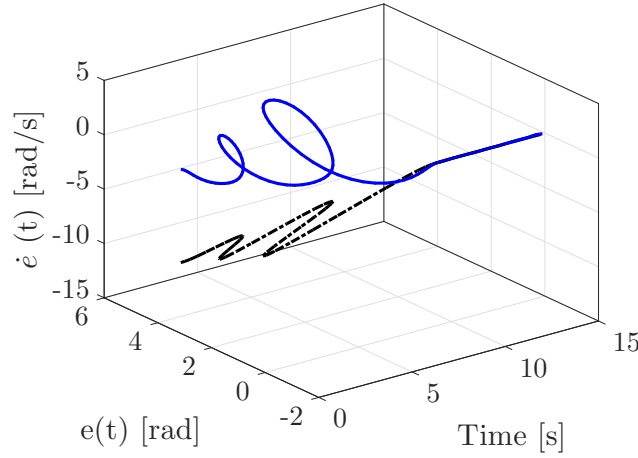
(a) Error configuration space



(b) Sliding function

Source: Prepared by the author.

Figure 32 – Sliding function varying with time to swing-up and control the 2-RWP in the inverted position ($\theta = 0^\circ$) using only RW1. - indicates the experimental result and - • - the sliding function.

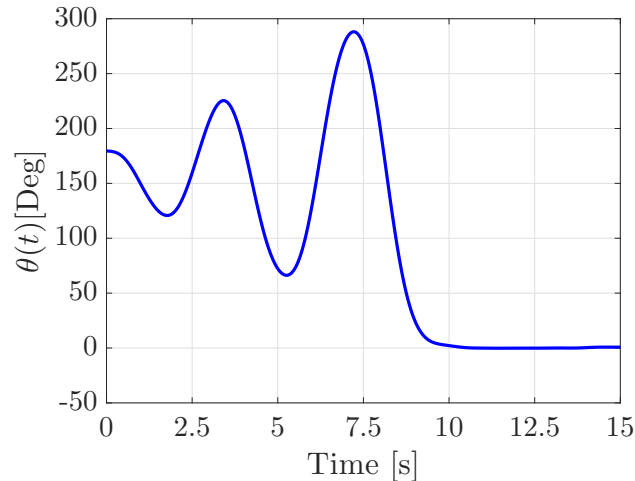


Source: Prepared by the author.

Here, it is also possible to observe that the proposed strategy of combining on-off and sliding modes works very well and that in a certain point when the pendulum angular position is closer to the desired position, the proposed control law using sliding modes is able to reach the sliding function and to slide to the equilibrium position.

Figure 33 presents the pendulum angular position to swing-up and control the 2-RWP using only reaction wheel 2. Here, the process took more time to swing-up and control the 2-RWP in the inverted position when compared to the use of reaction wheel 1 presented in Fig. 28, whereas the 2-RWP reaches the inverted position closer to 10 seconds. For this case, the gains selected for the experimental application were $\zeta = 2.4$ and $\xi = 100$.

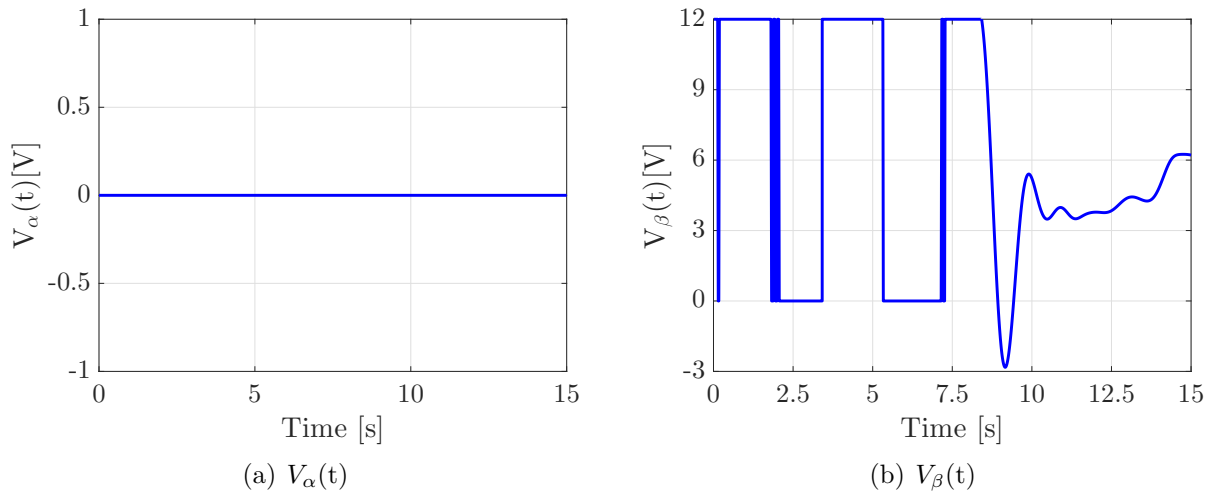
Figure 33 – Pendulum angular position to swing-up and control the 2-RWP in the inverted position ($\theta = 0^\circ$) using only RW2. - indicates the experimental result.



Source: Prepared by the author.

Figure 34 depicts the voltages applied to the DC motors. As expected, the voltage applied to the reaction wheel 1 is zero since this wheel mode is turned off.

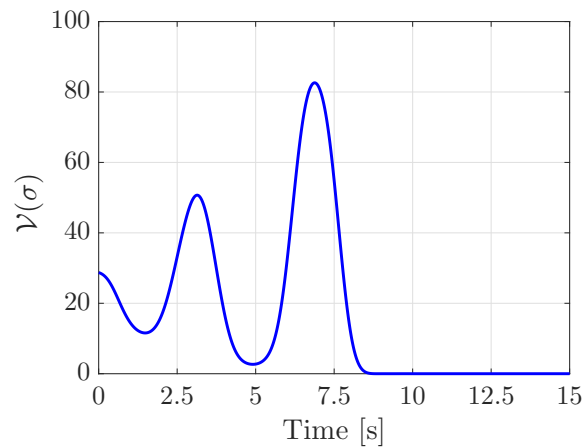
Figure 34 – Voltage values applied to the DC motors to swing-up and control the 2-RWP in the inverted position ($\theta = 0^\circ$) using only RW2. - indicates the experimental result.



Source: Prepared by the author.

Figure 35 exhibits the Lyapunov function's behavior when using only reaction wheel 2. This behavior is the same as the one observed when using only reaction wheel 1.

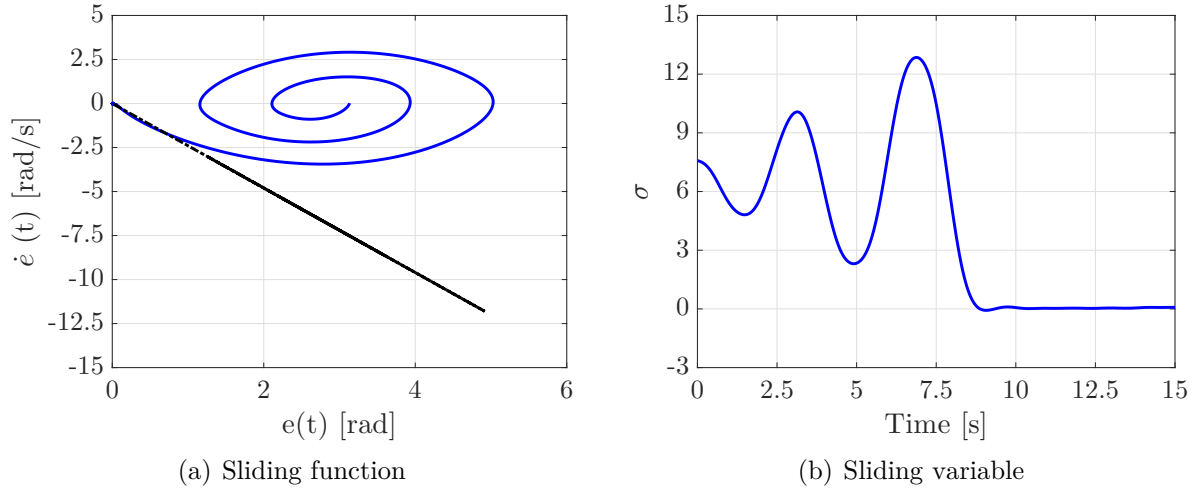
Figure 35 – Lyapunov function ($\mathcal{V}(\sigma)$) to swing-up and control the 2-RWP in the inverted position ($\theta = 0^\circ$) using only RW2. - indicates the experimental result.



Source: Prepared by the author.

Furthermore, Fig. 36 shows the error configuration space and the sliding variable when using only reaction wheel 2 to swing up and control the 2-RWP.

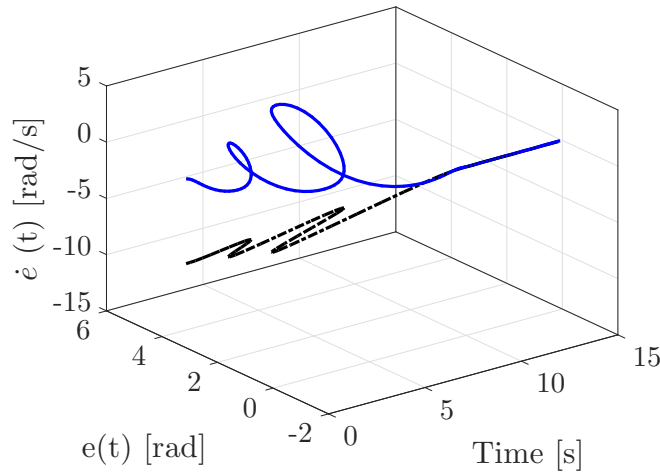
Figure 36 – Error configuration space and sliding function to swing-up and control the 2-RWP in the inverted position ($\theta = 0^\circ$) using only RW2. - indicates the experimental result and -•- the sliding function.



Source: Prepared by the author.

For this experiment, the configuration space evolving with time is charted in Fig. 37.

Figure 37 – Sliding function varying with time to swing-up and control the 2-RWP in the inverted position ($\theta = 0^\circ$) using only RW2. - indicates the experimental result and -•- the sliding function.



Source: Prepared by the author.

Discussion on the results for the swing-up strategy

After 10 seconds, both cases can control the pendulum in the inverted position. The swing-up process can be considered slow. However, the hardware used, the self-manufactured pendulum, and its big dimensions have to be considered. This series of limitations that the plant imposes on this control problem is responsible for these slower

results. Although they emphasize the strength of the control law used, which, despite all the difficulties encountered, manages to swing up and control the pendulum in the inverted position with the adaptations made.

Additionally, the difference in the performance between using reaction wheels 1 and 2 can be explained due to the inertia of each one of them. Reaction wheel 2 is smaller and has a lower mass, which implicates in a lower moment of inertia. When considering equations (7) and (8) to calculate the motor torque for the DC motors, it could explain the difference found in the results. The smaller reaction wheel is easier accelerated. However, when comparing the results depicted in Figures 28 and 33, this does not turn out to be an advantage, since when actuating reaction wheel 1, the pendulum reaches the inverted position faster.

Since only one reaction wheel is actuated at a time, the voltage applied reaches the DC motor's saturation limit of $|12|$ V in the swing-up process. This means that at those instants, the DC motor employs its maximum acceleration capacity. Furthermore, the noise presented in the measurement of the pendulum's angular position by the potentiometer causes some imperfections to the controller during the transitions between turning it off and on. This happens because the controller is based on the pendulum angular velocity to turn off and on. This rate is obtained by deriving the angular position, which amplifies the noise. Although a moving average is used to diminish the potentiometer measurements' noise, when the pendulum's angular position is derived to obtain the pendulum's angular velocity the remaining noise is still amplified.

Moreover, when the pendulum approaches the desired set-point ($\theta = 0^\circ$), the voltages, either using reaction wheel 1 or 2, are not close to the saturation limit and they are almost constant. However, small oscillations can be observed. This happens because we are not altogether actuating the system. There are some contributions due to constructions imperfections that were not possible to be modeled once this pendulum was self-manufactured. There is also the nonlinear behavior of the friction phenomenon.

Figures 31(a) and 36(a) have illustrated the error configuration space in which the sliding function was designed. The swing-up process generates the spirals where we can interpret that one of the control modes was on, and it was not possible to reach the sliding function. The system then goes to a farther away position, and it uses the pendulum energy to help in the swing-up process. Thus, a new attempt is made to try to reach the designed sliding function. In the attempt that the pendulum reaches the sliding function, it passes the function. This also can be verified in Figs. 31(b) and 36(b), where we show

the behavior of the sliding function over time.

Analyzing the sliding function's behavior over time, we can verify exactly the moment when the control is turned off due to the DC motor's limitations, comparing the time when this occurs to Figs. 29 and 34 in which the voltages applied to the DC motors are presented.

In the controller designed, if the pendulum's angular position is repeated and the pendulum's angular velocity changes its signal, the controller is turned off. Thus, the controller will only be turned back on when this happens again, i. e., the pendulum reached another point where its angular velocity was changed. This idea is due to the limitations of the DC motors employed in the plant. The DC motor used for the experimental tests reaches a high angular velocity. This means that low torque is achieved, which is not very interesting for this kind of control strategy. It would be more important than the DC motor provided a higher torque than achieving high angular velocity. From the experimental characterization of the DC motor shown in (TRENTIN *et al.*, 2020a), we can also see that the DC motors used are slow, they have a high time constant $\tau = 5.88$ s. This is also an issue because even if 12 V is commanded to the DC motors, it takes some time for them to provide the acceleration needed.

Additionally, this controller does not look into an energy approach. We combine a simple swing-up control strategy that deals with the experimental limitations of the plant. We know that the DC motor used does not provide high torques. Therefore, we have chosen to turn off the controller when the torque provided is not enough to reach either the sliding function or the inverted position. The disadvantage is that it does not drive the pendulum energy and disk velocity to zero, once the voltages applied to the DC motors for both cases studied are almost constant after the pendulum is controlled in the inverted position.

Furthermore, some features make the swing-up and control of the configuration studied in this paper easier than the classical reaction wheel pendulum. In the equation of motion presented in (26), we need to consider the accompanying term of $\sin(\theta)$, that is $(m_1 - m_2)g\frac{\ell}{2}$. To model this different configuration, the masses of each DC motors balance themselves. Thus, the term mentioned depends on the reaction wheels' masses difference. However, this configuration remains an underactuated mechanical system, but this mass difference is smaller than when considering the classical reaction wheel pendulum. Besides, if this new system has both reaction wheels with the same mass, this system is no longer a pendulum, it is a linear system if we do not consider the pendulum friction, and it will

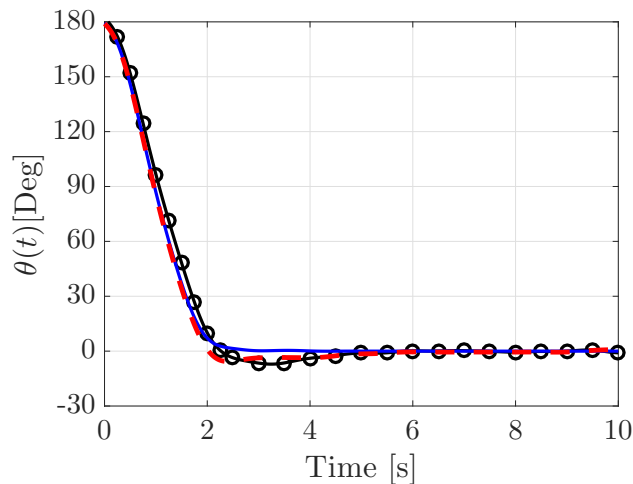
not oscillate like a pendulum.

It is essential to state that the strategy presented was adapted to work within the experimental device when actuating only a reaction wheel at a time. When both reaction wheels are actuated for such configuration, it is unnecessary to swing up the pendulum. The real-time implementation of swing-up control strategies in practice is not very simple to be carried out. In this case, an extra challenge is to do so using low-cost hardware costing around USD 150 (TRENTIN *et al.*, 2020a). The reader can examine the performance of the controller in a video in <https://youtu.be/evk3qk9j9Qs>.

4.5.4 Comparison between controllers

This section compares the experimental results to drive and control an inverted pendulum using two reaction wheels. Trentin *et al.* (2020a) proposed a simple PID controller and this result is compared with the two nonlinear control techniques proposed in this thesis. Figure 38 presents the controlled angular position for the 2-RWP for the three controllers where the sliding mode control technique exhibits a better performance reaching the equilibrium position first than the other controllers and presenting no overshoot. The SMC reached the inverted position in less than 3 seconds while the other two controllers took around 5 seconds to drive and stabilize the pendulum in the inverted position. It is interesting to note that the process of swinging-up using only a reaction wheel takes more time to drive and control the pendulum in the inverted position than using both reaction wheels.

Figure 38 – Comparison of the experimental pendulum angular position to drive and control the 2-RWP in the inverted position ($\theta = 0^\circ$). --- indicates nonlinear PD controller, - sliding mode controller, and -o- PID controller.



Source: (TRENTIN *et al.*, 2020b).

Nevertheless, the experimental results are close, and a quantitative measure of performance can be helpful to evaluate them. Section 3.3.3 discusses four different performance indices. They are also used here to compare the performance of each of the control laws designed for the 2-RWP.

The values of the indices are presented in Tab. 3. Analyzing the values in Tab. 3, the nonlinear controllers had smaller indices than the PID, attesting better results of the nonlinear controllers as seen in Fig. 38. Moreover, the sliding mode controller's indices are the smaller ones, indicating that it has the best performance among all the controllers designed when looking at the pendulum's angular position. Due to the swing-up process when using only a reaction wheel at a time, the performance indices were much higher than when both reaction wheels are used.

Table 3 – Performance indices for the experimental results to analyze pendulum angular position.

Controller	ISE	IAE	ITAE	ITSE
Sliding modes	6.8202	3.2083	2.1256	3.2102
Nonlinear PD	6.9212	3.3379	2.9042	3.2498
PID	7.7952	3.6902	3.2878	4.0235
Swing-up RW1	74.2889	23.4838	101.4637	337.2547
Swing-up RW2	83.3009	25.6012	117.5165	404.5634

Source: Prepared by the author.

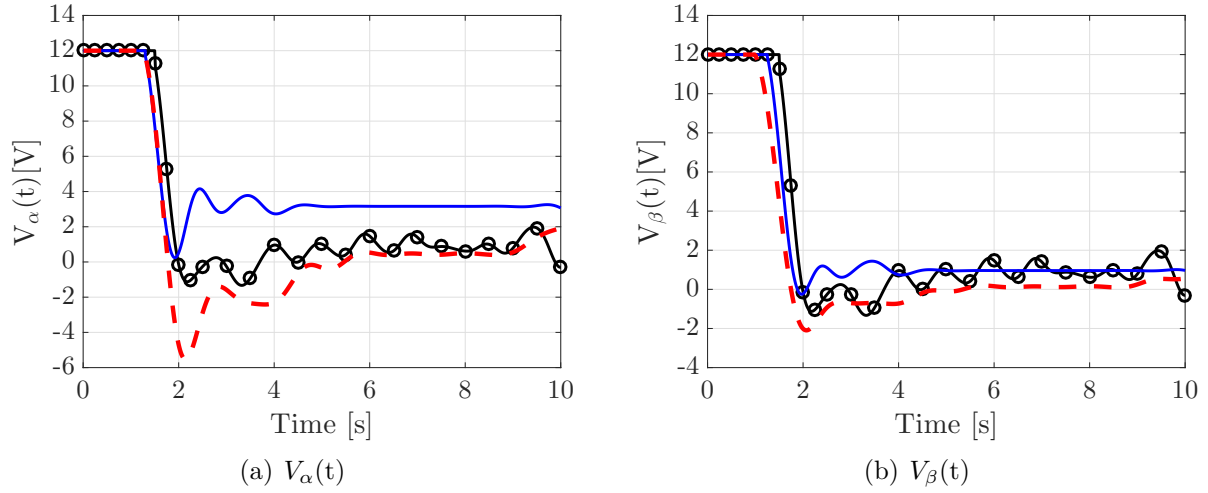
Nonetheless, taking into account that the sliding mode controller does not present overshoot and the pendulum reaches the inverted position faster than the nonlinear PD controller, the sliding mode controller outperforms the other controllers regarding the angular pendulum position.

Figure 39 compares the voltages applied to the DC motors by each controller. By looking at the voltage applied to the DC motor of reaction wheel 1, Fig. 39(a), the sliding mode controller presents a higher value of voltage, while in steady-state, the voltage applied by the nonlinear proportional derivative controller and PID are very close. However, when analyzing the voltage applied to the DC motor of reaction wheel 2, charted in Fig. 39(b), the values for all three controllers are very close

Thus, to a better evaluation of these results, the root mean square value of the voltages were calculated and for a generic signal (x_{rms}) is given by:

$$x_{rms} = \sqrt{\frac{1}{n} (x_1^2 + x_2^2 + \dots + x_n^2)} \quad (67)$$

Figure 39 – Comparison of the voltage values applied to the DC motors to control the 2-RWP in the inverted position ($\theta = 0^\circ$). - - - indicates nonlinear PD controller, - sliding mode controller, and -o- PID controller.



Source: (TRENTIN *et al.*, 2020b).

The RMS values for the voltage control signal for all the controller are presented in Tab. 4.

Table 4 – Root mean square values of the voltages applied to the DC motors.

Controller	RMS ($V_\alpha(t)$)	RMS ($V_\beta(t)$)
Sliding modes	5.4235	4.6609
Nonlinear PD	4.8650	4.3569
PID	4.9391	4.9391
Swing-up RW1	7.6991	0
Swing-up RW2	0	7.5573

Source: Prepared by the author.

The nonlinear proportional-derivative controller's RMS values are the smaller ones, which indicates that this controller has smaller energy consumption when both reaction wheels are actuated simultaneously.

Analyzing the RMS values of reaction wheel 1, we note that the sliding mode controller had the highest value. When we evaluate the RMS values for the voltage applied to DC motor 2, the PID controller had the highest value. The swing-up control strategy uses a higher value in the reaction wheel mode that is on. On the other hand, the other mode does not spend any energy. In this way, to swing-up and control the pendulum in the inverted position using only a reaction wheel had a smaller energy consumption but it takes more time to drive and stabilize the pendulum in the inverted position. The RMS

values of the voltage control signals were calculated for 10 seconds when both reaction wheels are actuated and 15 seconds in the swing-up cases.

For the better performance of the sliding mode controller, it is necessary to spend more energy. Nonetheless, it is essential to observe that the SMC control signal is much more steady, and it outperforms the other controllers tested. This is possible due to the SMC low sensitivity to the noise presented in the real-time measurements, whereas the other two controllers present much more oscillatory control voltage signals. When the sliding function is reached (as seen in Fig. 26 approximately at 2 seconds), one of the main features of sliding mode control is to maintain the system's behavior under the minimum influence of parametric uncertainties and external disturbances. When this occurs, the system behaves with the designed dynamic characteristics given by equation (42), with $\sigma(e, t) = 0$, providing robustness to the system and a much more steady control signal.

However, one reason for the high voltage values can be the nonlinear behavior of the friction encountered in the self-manufactured plant. After the pendulum is controlled in the inverted position, its angular velocity is almost zero. The Coulomb friction component has a more significant contribution and the controller has to deal with this. The voltages values should be closer to zero or around zero, but a higher voltage value is commanded for reaction wheel 1 when using the sliding mode control due to the friction. It would be interesting to develop a control strategy that can reduce the voltage after the pendulum is controlled in the inverted position or a better manufactured experimental device could be built where the friction encountered can be minimized.

Both nonlinear controllers designed can be applied for any initial conditions. The controllers can also track the desired trajectory. The function for the desired trajectory (θ_d) and its rates have to be provided as input to the controllers. Thereby, it is also possible for the controllers designed to track the desired trajectory. It is recommended that the pendulum with two reaction wheels be controlled in the inverted position or track the desired trajectory. If a set-point that is not an equilibrium point is desired, the torque caused by gravity is not zero, and it is also added. The controller will have to deal with the friction and the torque caused by gravity. Thus, a higher voltage signal than the one employed to control the pendulum in the inverted position will have to be applied to the DC motors, and in steady-state, this value tends to increase until the saturation limit is reached. After some time, the DC motors' saturation limits can be reached, or the reaction wheels will reach very high angular velocities, which can cause a catastrophic failure to the system. Either way, this system will no longer be able to be controlled in that position.

4.6 FINAL REMARKS

This chapter has presented two of the principal contributions of this thesis. Firstly, the application of two nonlinear control techniques to drive and control a pendulum using two reaction wheels where the design and proof of the stability of each technique were carried out. A novel approach to swing up and control an inverted pendulum with two reaction wheels combining on-off and sliding mode control has also been performed. This different configuration of an inverted pendulum can be further explored to test new controllers, for example, to track the desired trajectory instead of controlling the pendulum in the inverted position. Additionally, the use of low-cost hardware to reproduce these controllers in a real-time application has proven to be very efficient despite the difficulties encountered.

The nonlinear control strategies that actuate both reaction wheels simultaneously presented adequate experimental results, which were much better than the previous one using a simple PID controller. The performance indices quantitatively measured this improvement. The sliding mode control had a more reliable performance when compared to the nonlinear proportional-derivative, confirming that the use of a technique that deals with model uncertainty provides a better outcome. When analyzing the voltage applied to the DC motors to control the inverted pendulum, we found that the values were higher than the expected. These values for a perfect inverted pendulum system controlled in an equilibrium point should be zero. In real-time experiments, it is expected that these values be close to zero or around zero. One possible reason might be due to the presence of the Coulomb friction encountered when identifying the friction of the experimental device. For future work, a control strategy to decrease the voltage values and bring them as close as possible to zero will be carried out.

Regarding the swing-up control strategy, note that the properties achieved using SMC are essential to deal with many of the plant's limitations, which include the absence of sensors in the DC motors, the low-cost hardware, and the nonlinear behavior of the friction. Additionally, the controller presented is first designed to actuate the system entirely and it is adapted to work using only a reaction wheel. The strategy presents excellent experimental results and can swing up the pendulum and control it in the inverted position using a single controller.

5 VARIABLE SPEED CONTROL MOMENT GYROSCOPE USED TO CONTROL INVERTED PENDULUMS

The use of variable speed control moment gyroscope (VSCMG) is an effective way to stabilize aerospace devices. It is possible to control the oscillation and direction rate of change in rigid bodies in space due to the controlled change of angular momentum rate. Thus, this chapter proposes two novel pendulum configurations controlled by a VSCMG actuator. The VSCMG pendulum idea is to use both the angular momentum variation in amplitude and direction to implement the control. The controllers are designed using the Lyapunov theory to stabilize the pendulum in the inverted position.

Section 5.1 presents the first pendulum configuration studied that is controlled with a VSCMG. In this case, the pendulum motion is restricted to the plane. The results illustrate the control using a VSCMG in an inverted pendulum showing how the inverted pendulum stabilization is performed using two control actions. Also, the proposed pendulum's comparison with a classical configuration, the reaction wheel pendulum, is presented. Part of the results of this chapter was published in (TRENTIN; SILVA; SCHAUB, 2019a; TRENTIN; SILVA; SCHAUB, 2019b).

Furthermore, section 5.2 introduces the model and control of the 3D pendulum with a VSCMG. Under appropriate assumptions, the model of the 3D pendulum is used to yield the model of a spherical pendulum. To the best of the author's knowledge, it is the first time that a spherical pendulum is controlled using a VSCMG.

5.1 VARIABLE SPEED CONTROL MOMENT GYROSCOPE IN AN PLANAR INVERTED PENDULUM

This chapter's first main contribution is to propose a new configuration of a pendulum with a VSCMG mechanism coupled to it. The control torque is produced by the reaction wheel and by the gyroscopic effect when the reaction wheel is gimbaled, driving the pendulum from the downward position to the inverted position. This section is organized as follows: section 5.1.1 presents the dynamic modeling of the new pendulum configuration proposed. Section 5.1.2 shows the controller's design using the Lyapunov control theory, and section 5.1.3 shows the numerical results for driving the pendulum to the inverted position and also when it is subject to an external disturbance. Additionally, the comparison with a well-known pendulum configuration, the reaction wheel pendulum, is carried out. Finally, section 5.3 shows the concluding remarks of the chapter and future work.

5.1.1 Dynamic modeling of the VSCMG pendulum

Figure 40 illustrates three frames used for modeling the Variable Speed Control Moment Gyroscope coupled to a pendulum. The inertial frame is described by axes (x, y, z) and orthonormal basis $\{\hat{n}_1, \hat{n}_2, \hat{n}_3\}$. The first moving frame (B), or body frame, is described by axes (x_1, y_1, z_1) and the orthonormal basis $\{\hat{b}_1, \hat{b}_2, \hat{b}_3\}$ is rotating solidarity to the pendulum with the angle θ , assuming a negative rotation. The second moving frame (G), or gimbal frame, is described by axes (x_2, y_2, z_2) and orthonormal basis $\{\hat{g}_t, \hat{g}_g, \hat{g}_s\}$. This frame is responsible for rotating the gimbal with angle γ , which makes the reaction wheel move outside of the plane; and for rotating the gimbal, a positive rotation is considered. The rotation matrix between the inertial frame and moving frame B is described by:

$$\begin{Bmatrix} \hat{b}_1 \\ \hat{b}_2 \\ \hat{b}_3 \end{Bmatrix} = \underbrace{\begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{[BN]} \begin{Bmatrix} \hat{n}_1 \\ \hat{n}_2 \\ \hat{n}_3 \end{Bmatrix} \quad (68)$$

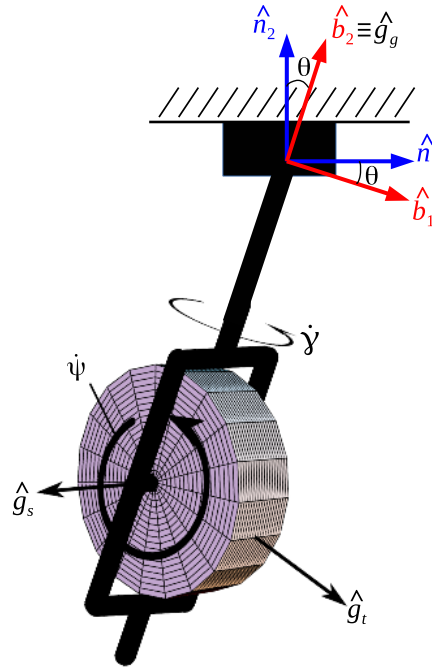
The rotation matrix between the body moving frame B and the gimbal moving frame

G is given by:

$$\begin{Bmatrix} \hat{g}_t \\ \hat{g}_g \\ \hat{g}_s \end{Bmatrix} = \underbrace{\begin{bmatrix} \cos \gamma & 0 & -\sin \gamma \\ 0 & 1 & 0 \\ \sin \gamma & 0 & \cos \gamma \end{bmatrix}}_{[BG]^T} \begin{Bmatrix} \hat{b}_1 \\ \hat{b}_2 \\ \hat{b}_3 \end{Bmatrix} \quad (69)$$

In this new system there are three different angular velocities to be evaluated: the pendulum angular velocity ${}^B\boldsymbol{\omega}_{B/N} = \dot{\theta} \hat{b}_3$, the gimbal angular velocity ${}^B\boldsymbol{\omega}_{G/B} = \dot{\gamma} \hat{b}_2$ and the reaction wheel angular velocity ${}^B\boldsymbol{\omega}_{W/G} = \dot{\psi} \sin \gamma \hat{b}_1 + \dot{\psi} \cos \gamma \hat{b}_3$. In this way, the inertial angular wheel velocity is ${}^B\boldsymbol{\omega}_{W/N} = \dot{\psi} \cos \gamma \hat{b}_1 + \dot{\gamma} \hat{b}_2 + (\dot{\theta} + \dot{\psi} \cos \gamma) \hat{b}_3$.

Figure 40 – Variable Speed Control Moment Gyroscope Pendulum.



Source: Trentin, Silva and Schaub (2019b).

Before calculating the angular momentum, it is necessary to compute the inertia matrix. This is done for the pendulum, the gimbal, and the reaction wheel. The principal inertia directions of the gimbal and the reaction wheel are in a different frame. It is necessary to rotate these inertia matrices to the body frame B , where the VSCMG pendulum is adequately modeled. The pendulum inertia matrix in the body moving

frame B is:

$${}^B[I_p] = \begin{bmatrix} I_{p_x} & 0 & 0 \\ 0 & I_{p_y} & 0 \\ 0 & 0 & I_{p_z} \end{bmatrix} = \begin{bmatrix} I_p & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_p \end{bmatrix} \quad (70)$$

where $I_{p_x} = I_{p_z} = \frac{ml^2}{3} = I_p$ and $I_{p_y} = 0$, since for the pendulum the inertia of a rod was considered. For the gimbal inertia matrix, it is necessary to rotate it to the body frame B . The inertia matrix of the gimbal in the gimbal frame is given by:

$${}^G[I_g] = \begin{bmatrix} I_{gt} & 0 & 0 \\ 0 & I_{gg} & 0 \\ 0 & 0 & I_{gs} \end{bmatrix} \quad (71)$$

moreover, the inertia matrix of the gimbal in the body frame B is:

$$\begin{aligned} {}^B[I_g] &= [BG][I_g][BG]^T = \\ &= \begin{bmatrix} I_{gs} \sin^2 \gamma + I_{gt} \cos^2 \gamma & 0 & (I_{gs} - I_{gt}) \cos \gamma \sin \gamma \\ 0 & I_{gg} & 0 \\ (I_{gs} - I_{gt}) \cos \gamma \sin \gamma & 0 & I_{gt} \sin^2 \gamma + I_{gs} \cos^2 \gamma \end{bmatrix} \end{aligned} \quad (72)$$

where $[BG]$ is the rotation matrix between the body and gimbal frame. It is worth noting that the reaction wheel is symmetric and the principal inertia directions of the gimbal and the reaction wheel are equivalent. The inertia matrix of the reaction wheel can be written in the gimbal moving frame G :

$${}^G[I_W] = \begin{bmatrix} I_{wt} & 0 & 0 \\ 0 & I_{wt} & 0 \\ 0 & 0 & I_{ws} \end{bmatrix} \quad (73)$$

Finally, the inertia matrix of the reaction wheel in the body frame is:

$$\begin{aligned} {}^B[I_W] &= [BG][I_W][BG]^T = \\ &= \begin{bmatrix} I_{ws} \sin^2 \gamma + I_{wt} \cos^2 \gamma & 0 & (I_{ws} - I_{wt}) \cos \gamma \sin \gamma \\ 0 & I_{wt} & 0 \\ (I_{ws} - I_{wt}) \cos \gamma \sin \gamma & 0 & I_{wt} \sin^2 \gamma + I_{ws} \cos^2 \gamma \end{bmatrix} \end{aligned} \quad (74)$$

The total angular momentum of the system (${}^B\mathbf{H}$) is the sum of the angular momentum

of the pendulum (${}^B\mathbf{H}_B$), of the gimbal (${}^B\mathbf{H}_G$) and the reaction wheel (${}^B\mathbf{H}_W$), given by:

$${}^B\mathbf{H} = {}^B\mathbf{H}_B + {}^B\mathbf{H}_G + {}^B\mathbf{H}_W \quad (75)$$

that can be rewritten as:

$${}^B\mathbf{H} = {}^B[I_p]{}^B\boldsymbol{\omega}_{B/N} + {}^B[I_g]{}^B\boldsymbol{\omega}_{G/N} + {}^B[I_W]{}^B\boldsymbol{\omega}_{W/N} \quad (76)$$

where ${}^B\boldsymbol{\omega}_{G/N} = {}^B\boldsymbol{\omega}_{B/N} + {}^B\boldsymbol{\omega}_{G/B}$. The time derivative of the angular momentum in frame B taken relative to the inertial frame N is:

$$\dot{\mathbf{H}} = \frac{{}^N d}{dt}\mathbf{H} = \frac{{}^B d}{dt}({}^B\mathbf{H}) + {}^B\boldsymbol{\omega}_{B/N} \times {}^B\mathbf{H} = {}^B\mathbf{L} \quad (77)$$

Finally, the motion equation is obtained by:

$$\mathcal{A}\ddot{\theta} + \mathcal{B}\ddot{\psi} + \mathcal{C}\dot{\gamma} + \mathcal{D} = 0 \quad (78)$$

where

$$\mathcal{A} = I_{wt} \sin^2 \gamma + I_{ws} \cos^2 \gamma + I_p + I_{gt} \sin^2 \gamma + I_{gs} \cos^2 \gamma \quad (79)$$

$$\mathcal{B} = I_{ws} \cos \gamma \quad (80)$$

$$\mathcal{C} = \dot{\theta} \cos \gamma \sin \gamma (2I_{wt} + 2I_{gt} - 2I_{ws} - 2I_{gs}) - \dot{\psi} I_{ws} \sin \gamma \quad (81)$$

$$\mathcal{D} = -rmg \sin \theta \quad (82)$$

The motor torque equation for the gimbal is obtained by analyzing it separately from the whole system. This can be done in the body frame or the gimbal frame. However, if it is carried out in the gimbal frame, it is not required to rotate the gimbal matrix, which leads to simpler equations. Thus, the angular momentum used to obtain the motor torque equation for the gimbal is:

$${}^G\mathbf{H}_{GW} = {}^G[I_g]{}^G\boldsymbol{\omega}_{G/N} + {}^G[I_W]{}^G\boldsymbol{\omega}_{W/N} \quad (83)$$

The motor torque equation of the gimbal can be obtained by performing the following derivation of the angular momentum of the gimbal and the reaction wheel in the gimbal frame G relative to the inertial frame N :

$$\dot{\mathbf{H}}_{GW} = \frac{{}^N d}{dt}\mathbf{H}_{GW} = \frac{{}^G d}{dt}({}^G\mathbf{H}_{GW}) + {}^G\boldsymbol{\omega}_{G/N} \times {}^G\mathbf{H}_{GW} = {}^G\mathbf{L}_{GW} \quad (84)$$

Thus, equation (85) is the motor torque equation for the gimbal, where u_g is the control torque produced by the gimbal:

$$u_g = (I_{gg} + I_{wt})\ddot{\gamma} + I_{ws}\dot{\psi}\dot{\theta}\sin\gamma \quad (85)$$

It is also essential to analyze the reaction wheel separately to obtain its motor torque equation. As the direction in which this torque is produced is $\hat{g}_s$, it is more convenient to perform the analysis in the gimbal frame. The angular momentum for the reaction wheel is:

$${}^G\mathbf{H}_W = {}^G[I_W]{}^G\boldsymbol{\omega}_{W/N} \quad (86)$$

By performing the derivation of the angular momentum of the reaction wheel in the gimbal frame G relative to the inertial frame N :

$$\dot{\mathbf{H}}_W = \frac{{}^N d}{{}^G dt}\mathbf{H}_W = \frac{{}^G d}{{}^G dt}({}^G\mathbf{H}_W) + {}^G\boldsymbol{\omega}_{G/N} \times {}^G\mathbf{H}_W = {}^G\mathbf{L}_W \quad (87)$$

the following reaction wheel motor torque equation is obtained:

$$u_s = I_{ws}\ddot{\theta}\cos\gamma - I_{ws}\dot{\theta}\dot{\gamma}\sin\gamma + I_{ws}\ddot{\psi} \quad (88)$$

where u_s is the control torque provided by the reaction wheel. The simultaneous numerical integration of equations (78), (85) and (88) provides the motion of the VSCMG pendulum.

5.1.2 Nonlinear control design for the VSCMG pendulum

First, it is necessary to select a candidate to be a Lyapunov function to design a feedback control law using the Lyapunov control theory:

$$\mathcal{V}(\delta\theta, \delta\dot{\theta}) = \frac{\delta\dot{\theta}^2}{2} + K\frac{\delta\theta^2}{2} \quad (89)$$

where $\delta\dot{\theta} = \dot{\theta} - \dot{\theta}_r$, $\delta\theta = \theta - \theta_r$ and K is a scalar attitude feedback gain. To ensure stability, the derivative of the Lyapunov candidate function should be at least negative semidefinite, as:

$$\dot{\mathcal{V}}(\delta\theta, \delta\dot{\theta}) = \delta\dot{\theta}(\delta\ddot{\theta} + K\delta\theta) \quad (90)$$

Therefore, $\dot{\mathcal{V}}(\delta\theta, \delta\dot{\theta})$ is set to be equal to a negative definite function, in this case:

$$\dot{\mathcal{V}}(\delta\theta, \delta\dot{\theta}) = \delta\dot{\theta} (\delta\ddot{\theta} + K\delta\theta) = -P\delta\dot{\theta}^2 \quad (91)$$

where P is a positive gain. By carrying out the derivative of the Lyapunov candidate function, it is possible to obtain the following stable closed-loop dynamical system:

$$\ddot{\theta} - \ddot{\theta}_r + P\delta\dot{\theta} + K\delta\theta = 0 \quad (92)$$

To find the control law, it is necessary to substitute the equation of motion of the system into the closed-loop system. Thus, it is necessary to analyze the higher order derivatives of the Lyapunov candidate function in order to find out which kind of stability it ensures (MUKHERJEE; CHEN, 1993):

$$\ddot{\mathcal{V}}(\delta\theta, \delta\dot{\theta}) = -2P\delta\dot{\theta}\delta\ddot{\theta}; \quad \ddot{\mathcal{V}}(\delta\theta, \delta\dot{\theta} = 0) = 0 \quad (93)$$

So, it is essential to analyze the third derivative of $\mathcal{V}(\delta\theta, \delta\dot{\theta})$:

$$\ddot{\mathcal{V}}(\delta\theta, \delta\dot{\theta}) = -2P\delta\ddot{\theta}^2 - 2P\delta\dot{\theta}\delta\ddot{\theta} \quad (94)$$

$$\ddot{\mathcal{V}}(\delta\theta, \delta\dot{\theta} = 0) = -2P\delta\ddot{\theta}^2 \quad (95)$$

And from the close-loop system when $\delta\dot{\theta} = 0$:

$$\delta\ddot{\theta} = -K\delta\theta \quad (96)$$

Substituting the result presented in Eq. (96) into Eq. (95):

$$\ddot{\mathcal{V}}(\delta\theta, \delta\dot{\theta} = 0) = -2P(-K\delta\theta)^2 = -2PK^2\delta\theta^2 < 0 \quad (97)$$

The third derivative of $\mathcal{V}(\delta\theta, \delta\dot{\theta})$ is negative definite, and because the first nonzero $\mathcal{V}$ derivative is of odd order, the control law is asymptotically stable.

Control Law

By substituting the motion equation shown in Eq. (78) into the closed-loop dynamics presented in Eq. (92) and solving for the desired control vector:

$$[\mathcal{B} \quad \mathcal{C}] \begin{Bmatrix} \ddot{\psi} \\ \dot{\gamma} \end{Bmatrix} = -\mathcal{D} - \mathcal{A}\ddot{\theta}_r + P\mathcal{A}\delta\dot{\theta} + K\mathcal{A}\delta\theta = \mathbf{L}_r \quad (98)$$

The control law given by Eq. (98) can avoid many of the conventional CMG singularities once that the RW rotor speed is time-varying (SCHAUB; JUNKINS, 2014). The state vector $\boldsymbol{\eta} = \{\ddot{\psi} \quad \dot{\gamma}\}^T$ and the matrix $[Q] = [\mathcal{B} \quad \mathcal{C}]$ are used to simplify the notation. Thus, the steering law presented in Eq. (98) can be rewritten as:

$$[Q]\boldsymbol{\eta} = \mathbf{L}_r \quad (99)$$

The inverse of $[Q]$ matrix is computed to find the rates for the reaction wheel and the gimbal. This can be done using the standard Moore-Penrose inverse to obtain a minimum norm solution. However, assuming that a VSCMG should act as a classical CMG away from singular configurations, a weighted pseudo-inverse is used (JUNKINS, 1978). For the case of the VSCMG pendulum $[W]$ is a diagonal matrix:

$$[W] = \begin{bmatrix} W_w & 0 \\ 0 & W_g \end{bmatrix} \quad (100)$$

where W_w and W_g are the weights related to the reaction wheel and gimbal modes, respectively; these weights represent how active each mode is. The desired $\boldsymbol{\eta}$ can be found as done in Schaub *et al.* (1998):

$$\boldsymbol{\eta} = \begin{Bmatrix} \ddot{\psi} \\ \dot{\gamma} \end{Bmatrix} = [W][Q]^T ([Q][W][Q]^T)^{-1} \mathbf{L}_r \quad (101)$$

To obtain the required performance from the VSCMG, the weights are estimated to see how close to a single-gimbal singularity the VSCMG is. By computing the nondimensional scalar factor κ the proximity of the singularity can be evaluated:

$$\kappa = \frac{1}{h^2} ([C][C]^T) \quad (102)$$

where h is the RW nominal angular momentum. If the parameter κ goes to zero, this

indicates a singular configuration for the CMG. The weight W_w can be defined as:

$$W_w = W_w^o e^{-\mu\kappa} \quad (103)$$

The parameters W_w^o and μ are positive scalars to be chosen by the designer. When the VSCMG is operating away from singularities, it should behave as a CMG, i.e., the weights on the reaction wheel mode are smaller than the one from the CMG mode. The weight of W_g is held constant.

Torque level control law

It is important to obtain a torque level control command for the system that is being studied since the steering law presented in Eq. (98) provides only kinematic directive. It is necessary to investigate a subservo control law for the VSCMG pendulum steering law, i.e., develop the control for $\ddot{\psi}$ and $\dot{\gamma}$.

To evaluate the motor torque u_g , the gimbal angular acceleration $\ddot{\gamma}$ is needed. So, it is necessary to develop a feedback control to obtain an expression for $\ddot{\gamma}$ to yield a stable servo dynamics. To develop the gimbal rate servo, it is essential to determine $\ddot{\gamma}$ such that the actual gimbal rate $\dot{\gamma}$ approaches the desired gimbal $\dot{\gamma}_d$ that is determined through the VSCMG steering law presented in Eq. (98). The servo tracking error is defined as:

$$\Delta\dot{\gamma} = \dot{\gamma} - \dot{\gamma}_d \quad (104)$$

and the Lyapunov function selected is:

$$\mathcal{V}(\Delta\dot{\gamma}) = \frac{1}{2}\Delta\dot{\gamma}^2 \quad (105)$$

where this function is a positive definite measure of the servo tracking error. Evaluating the time derivative of the Lyapunov function and forcing it to be definite negative leads to:

$$\dot{\mathcal{V}}(\Delta\dot{\gamma}) = \Delta\dot{\gamma}\Delta\ddot{\gamma} \equiv -K_\gamma\Delta\dot{\gamma}^2 \quad (106)$$

with $K_\gamma > 0$. Solving Eq. (106) for the desired gimbal acceleration yields:

$$\ddot{\gamma} = \ddot{\gamma}_d - K_\gamma\Delta\dot{\gamma} \quad (107)$$

The feedforward term $\ddot{\gamma}_d$ may be evaluated by numerically differentiating the desired gimbal steering law rates $\dot{\gamma}_d$. Substituting Eq. (107) into Eq. (85) leads to the gimbal-rate

servo control torque:

$$u_g = (I_{gg} + I_{wt}) (\ddot{\gamma}_d - K_\gamma \Delta \dot{\gamma}) + I_{ws} \dot{\psi} \dot{\theta} \sin \gamma \quad (108)$$

A servo for the reaction wheel acceleration control is also developed similarly. Let $\ddot{\psi}_d$ be the desired reaction wheel acceleration that results from the steering law presented in Eq. (98). The speed tracking error for the reaction wheel is:

$$\Delta \dot{\psi} = \dot{\psi} - \dot{\psi}_d \quad (109)$$

As done before for the gimbal, the following Lyapunov function is chosen:

$$\mathcal{V}(\Delta \dot{\psi}) = \frac{1}{2} \Delta \dot{\psi}^2 \quad (110)$$

This function is also a definite positive measure of the servo tracking error. The time derivative of the Lyapunov function for the reaction wheel feedback control is evaluated, and it is forced to be definite negative, leading to:

$$\dot{\mathcal{V}}(\Delta \dot{\psi}) = \Delta \dot{\psi} \Delta \ddot{\psi} \equiv -K_\psi \Delta \dot{\psi}^2 \quad (111)$$

with $K_\psi > 0$, the following is obtained:

$$\ddot{\psi} = \ddot{\psi}_d - K_\psi \Delta \dot{\psi} \quad (112)$$

such that the new reaction wheel torque control is given by:

$$u_s = I_{ws} (\ddot{\psi}_d - K_\psi \Delta \dot{\psi}) + I_{ws} \ddot{\theta} \cos \gamma - I_{ws} \dot{\theta} \dot{\gamma} \sin \gamma \quad (113)$$

5.1.3 Numerical results for the VSCMG pendulum

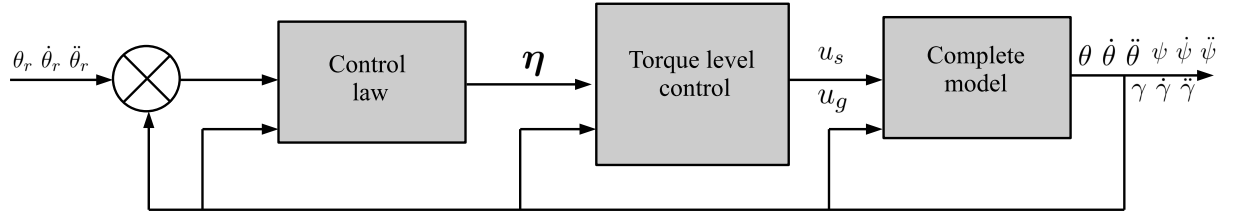
This section shows the results obtained to control the VSCMG pendulum in the inverted position. How this new system and its controller react when the pendulum is subjected to external disturbances after controlled in the inverted position is also presented.

For the VSCMG pendulum, a reaction wheel with radius $R = 0.11$ m, mass $m_w = 0.2151$ kg, a pendulum length of $\ell = 0.35$ m with a mass of $m_p = 0.1$ kg are considered. The gimbal structure has a mass equal to $m_g = 0.082$ kg, and its inertia was obtained using *SolidWorks*[®], where $I_{gt} = 0.0020441$ kg.m², $I_{gg} = 0.0020466$ kg.m² and $I_{gs} = 0.0000054$

kg.m². The radius of the center of mass is $r = (m_g \ell / 2 + m_g \ell + m_w \ell) / (m_p + m_g + m_w)$. The following initial conditions for all the simulations were assumed to be: $\theta(0) = 180^\circ$, $\dot{\theta}(0) = 0$, $\psi(0) = 0$, $\dot{\psi}(0) = 100$ RPM, $\gamma(0) = 90^\circ$ and $\dot{\gamma}(0) = 0$.

For the controller designed, the gains used in the simulations were $K = 5$, $P = 5$, $K_\gamma = 100$ and $K_\psi = 100$. The weights for the controller were $W_w = 2$, $W_g = 1$ and $\mu = 10^{-9}$. In the control algorithm, first, the initial conditions are set, i.e., the angular position where it is intended to control the pendulum ($\theta = 0^\circ$) and the controller parameters. Afterwards, $\mathbf{L}_r$ is obtained, such that the desired rates for the gimbal and the reaction wheel can be obtained. With these desired rates, the motor torques of the gimbal (u_g) and the reaction wheel (u_s) can be evaluated. Then, the integration of the complete model is performed, and the new states are obtained. Figure 41 shows the block diagram that demonstrates the controller operation.

Figure 41 – Block diagram for the control of the VSCMGP.



Source: Prepared by the author.

Furthermore, the proposed inverted pendulum configuration with a VSCMG is compared with the classical and well-known reaction wheel pendulum (RWP). The model of the RWP has been described in many papers (SPONG; CORKE; LOZANO, 2001; HERNÁNDEZ, 2003; BLOCK; ASTROM; SPONG, 2007; BAPIRAJU *et al.*, 2004; SRINIVAS; BEHERA, 2008; ANDRIEVSKY, 2011). One can notice that if we lock the gimbal at $\gamma = 0^\circ$ the VSCMG pendulum (VSCMGP) becomes the classical RWP. This can also be addressed in the equations that describe the motion of the system. Considering $\gamma = 0^\circ$, the equation of motion of VSCMG pendulum, presented in Eq. (78), becomes:

$$(I_{ws} + I_p + I_{gs})\ddot{\theta} + (I_{ws})\ddot{\psi} + \mathcal{D} = 0 \quad (114)$$

The motor torque of the gimbal yields zero since the gimbal has to be locked in the RWP. Moreover, the motor torque of the reaction wheel becomes:

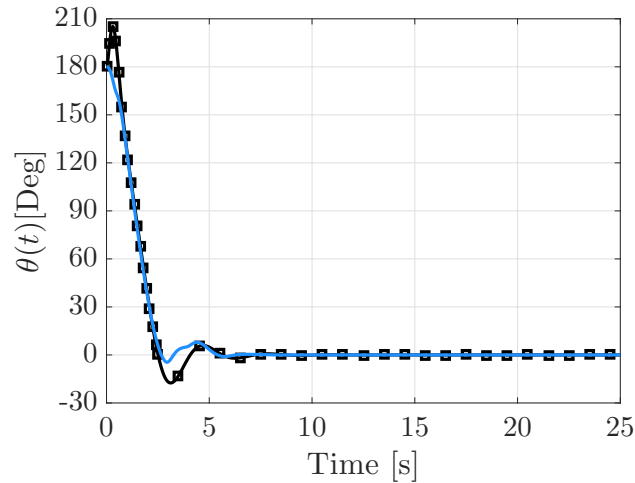
$$u_s = I_{ws}\ddot{\theta} + I_{ws}\ddot{\psi} \quad (115)$$

Equations (114) and (115) are the exact same equations of the RWP that were proposed by Spong, Corke and Lozano (2001) and describe its motion. Therefore, we have shown from the VSCMGP equation of motion that we can obtain the RWP equation of motion as well as the motor torque equation for the reaction wheel. For the numerical simulations of the RWP, the same parameters presented were used; the only difference is that the weight of the gimbal mode W_g in the controller is set to be zero, since for the RWP the reaction wheel is not gimballed. Additionally, the equation of motion of the RWP presented in chapter 3 in Eq. (25) can be reduced to Eq. (114) if the friction model and the counter weight are not considered, obtaining the classical RWP model. And the motor torque for the RWP presented in Eq. (4) in chapter 3 is the same than the one presented in Eq. (115).

Numerical results for controlling the pendulums in the inverted position

Figure 42 presents the results for driving the VSCMG pendulum and RWP from the downward position ($\theta = 180^\circ$) to the upward position ($\theta = 0^\circ$). The pendulum angular position for both configurations is close but the VSCMG seems to have a better outcome.

Figure 42 – Pendulum angular position $\theta(t)$, where --- indicates the result for VSCMG pendulum and --□-- indicates the result for RWP.



Source: Trentin, Silva and Schaub (2019b).

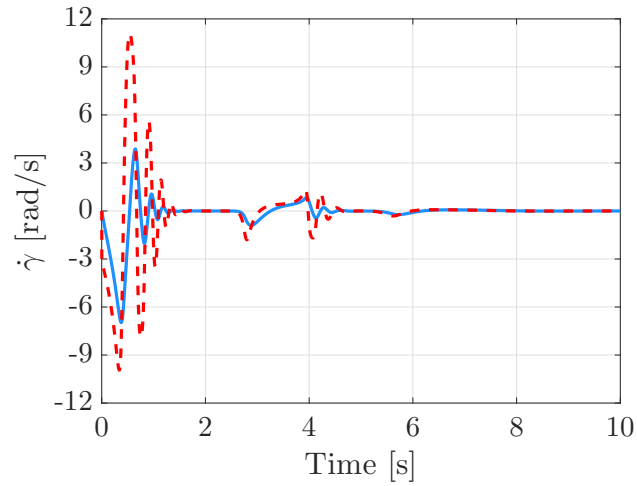
Figure 43 depicts the gimbal rates for the VSCMGP. For the RWP, the gimbal rate must be zero for all the simulation, this it is not displayed in the figure. Analyzing the VSCMG pendulum, the gyroscopic torque is more used in the beginning of the simulation to start moving the pendulum, and after some seconds the torque is mainly produced by

the reaction wheel. Some differences between the desired gimbal rate and the effective gimbal rate for the VSCMG can be noticed because for the CMG torque u_g , a saturation limit of 2.5 N.m has been added, which makes the control more realistic.

For both pendulum configurations, the reaction wheel angular velocities are shown in Fig. 44, where after the pendulum reaches the equilibrium in the inverted position the wheel speed is constant, for all cases.

Additionally, the angular velocity of the reaction wheel of the RWP reaches a higher value in the beginning of the simulation; this occurs because in the RWP the torque is provided only by the reaction wheel mode and it does not have the contribution of the CMG mode.

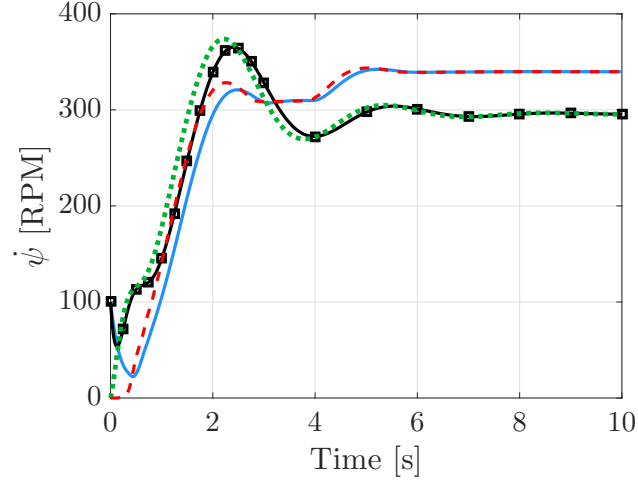
Figure 43 – Gimbal rate $\dot{\gamma}(t)$. --- indicates the effective gimbal rate and - - - indicates the desired gimbal rate for the VSCMG pendulum.



Source: Trentin, Silva and Schaub (2019b).

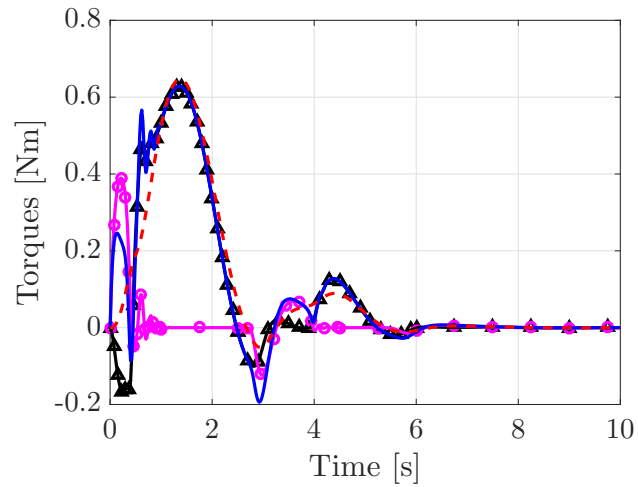
Figure 45 shows the contribution of each control action for the VSCMG pendulum and the external torque caused by gravity. In addition, the angular momentum variation can also be observed. Here, the gyroscopic torque has a higher contribution in the start of the simulation, and after some time the control is performed mainly by the reaction wheel mode.

Figure 44 – RW angular velocity $\dot{\psi}(t)$. --- indicates the effective reaction wheel angular velocity and - - - indicates the desired reaction wheel angular velocity for the VSCMG pendulum. --□-- indicates the effective reaction wheel angular velocity and indicates the desired reaction wheel angular velocity for the RWP.



Source: Trentin, Silva and Schaub (2019b).

Figure 45 – VSCMGP control actions and torques. - - - indicates the external torque caused by gravity ($\mathcal{D}$), - indicates the composition of the angular momentum variation ($\dot{H}_3 = \mathcal{A}\ddot{\theta} + \mathcal{B}\ddot{\psi} + \mathcal{C}\dot{\gamma}$), -△- indicates the reaction wheel control action ($\mathcal{B}\ddot{\psi}$) and -○- indicates the CMG control action ($\mathcal{C}\dot{\gamma}$).

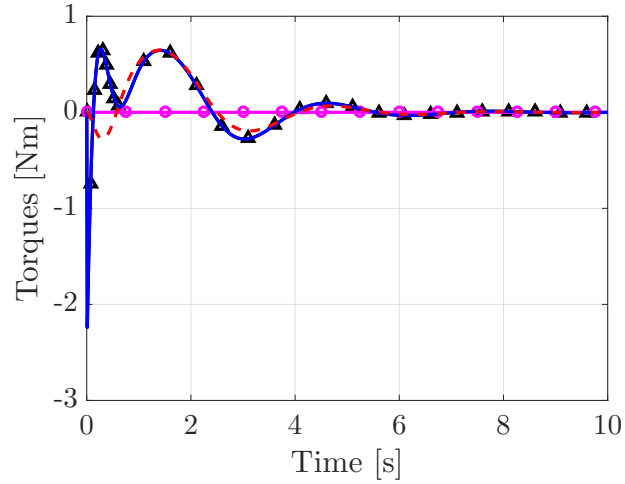


Source: Trentin, Silva and Schaub (2019b).

Figure 46 shows the control actions, the external torque caused by gravity and the variation of angular momentum for the RWP. The gimbal control action is kept at zero, as expected, since the gimbal is not actuated for this configuration. Thus, the control is performed only by the reaction wheel control action.

In this way, a higher action from the reaction wheel is necessary in the beginning of the simulation. When analyzing the VSCMG pendulum in Fig. 45 this is not observed because the actions are distributed between the gimbal and reaction wheel. The RMS values of the control actions of VSCMG pendulum and RWP were calculated and they are presented in Tab. 6 in subsection 5.1.3, where the comparison between pendulum configurations is discussed.

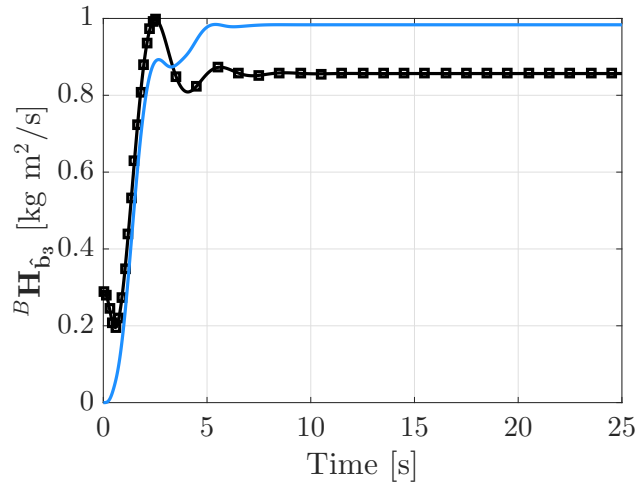
Figure 46 – RWP control actions and torques. - - - indicates the external torque caused by gravity ($\mathcal{D}$), --- indicates the composition of the angular momentum variation ($\dot{H}_3 = \mathcal{A}\ddot{\theta} + \mathcal{B}\ddot{\psi} + \mathcal{C}\dot{\gamma}$), -Δ- indicates the reaction wheel control action ($\mathcal{B}\ddot{\psi}$) and -○- indicates the CMG control action ($\mathcal{C}\dot{\gamma}$).



Source: Trentin, Silva and Schaub (2019b).

Figure 47 shows the angular momentum component around $\hat{b}_3$ for both pendulums, where after the pendulum is controlled in the inverted position, it becomes a constant value, as expected, because when the pendulum is balanced in the inverted position gravity does not cause a torque, illustrating the conservation of the angular momentum. This result is observed for both pendulum configurations.

Figure 47 – Angular momentum ${}^B\mathbf{H}$ around $\hat{b}_3$, where --- indicates the result for VSCMG pendulum and --□-- indicates the result for RWP.

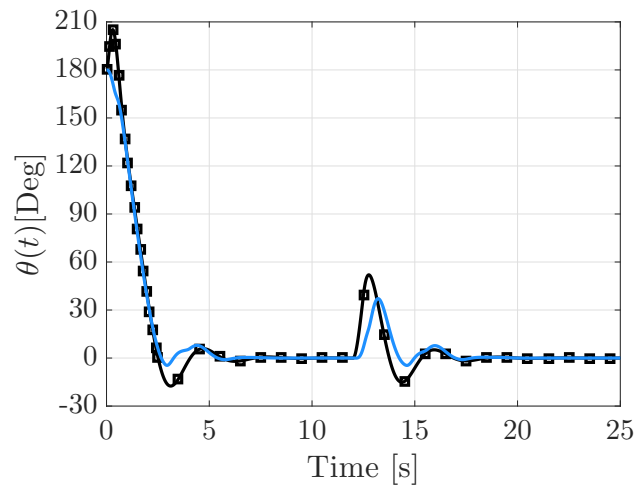


Source: Trentin, Silva and Schaub (2019b).

Numerical results for controlling the pendulums in the inverted position with an external disturbance

Simulations were performed when an external disturbance is applied after the pendulum is controlled in the inverted position. Also, we investigate how the RWP would react when subjected to a disturbance so that this situation can also be compared between the pendulum configurations. For all the simulations in this subsection, the disturbance was applied at 12 seconds for 0.5 s with an amplitude of $D_{ist} = 0.3$ N.m. The pendulums angular positions are presented in Fig. 48.

Figure 48 – Pendulum angular position $\theta(t)$ when subject to external disturbance, where --- indicates the result for VSCMG pendulum and --□-- indicates the result for RWP.

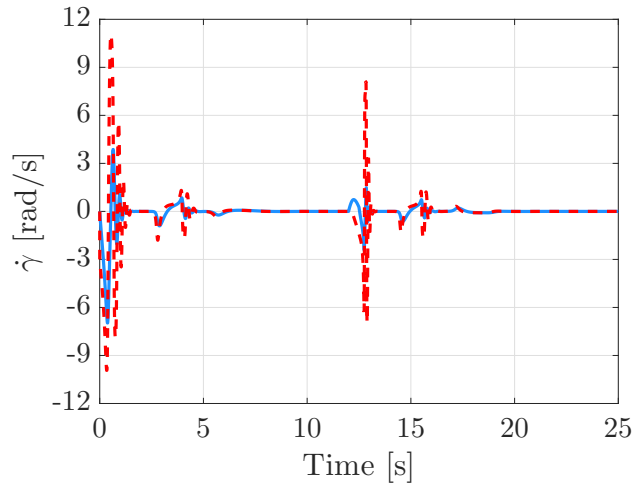


Source: Trentin, Silva and Schaub (2019b).

It is also interesting to note that the controller designed can bring both pendulums back to the desired position ($\theta = 0^\circ$) after these systems are subjected to an external disturbance. Comparing the results between the VSCMG pendulum and the RWP, the angular displacement of the VSCMG is slightly smaller, which means that the system that employs two control actions reacts better when subjected to a disturbance, whereas the RWP angular displacement reaches a higher value. Since these results are still close to each other, the performance indices were calculated for this situation, and they are presented in Tab. 5 in subsection 5.1.3 showing that the VSCMG pendulum presented a better performance than the RWP.

Figure 49 shows the gimbal rates ($\dot{\gamma}$) for the VSCMG, where the gimbal is used in the beginning of the simulation and also when the disturbance is applied. Again, there is a difference between the desired rate and the actual gimbal rate due to the saturation limit imposed on the CMG torque (u_g). For the RWP, the desired and effective gimbal rate is zero since for this configuration the gimbal must be locked and the torques must be generated only by the reaction wheel.

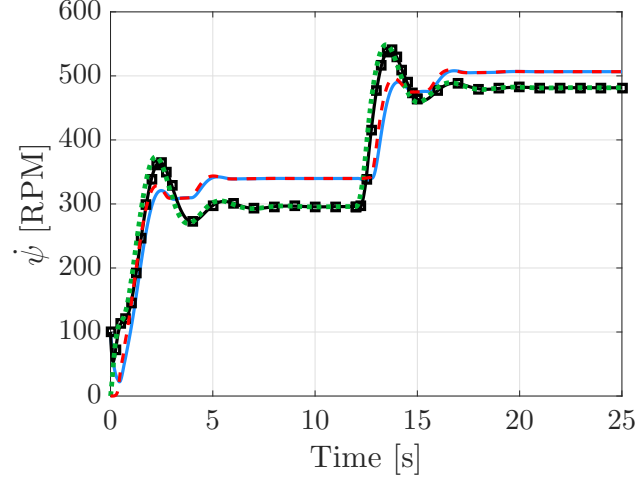
Figure 49 – Gimbal rate $\dot{\gamma}(t)$ when subject to an external disturbance. --- indicates the effective gimbal rate and - - - indicates the desired gimbal rate for the VSCMG pendulum.



Source: Trentin, Silva and Schaub (2019b).

Figure 50 shows the angular velocities of the reaction wheels for both systems. After the application of the disturbances, the controller needs to accelerate the reaction wheel to produce torque to compensate for the disturbance applied to the pendulums. Since for the RWP the torque is produced only by the reaction wheel, it demands a higher acceleration for the reaction wheel to be able to deal with the disturbance applied, whereas for the VSCMG pendulum this acceleration is smaller, due to the fact that the gyroscopic contribution is also added.

Figure 50 – RW angular velocity $\dot{\psi}(t)$ when subject to external disturbance. --- indicates the effective reaction wheel angular velocity and - - - indicates the desired reaction wheel angular velocity for the VSCMG pendulum. --□-- indicates the effective reaction wheel angular velocity and indicates the desired reaction wheel angular velocity for the RWP.



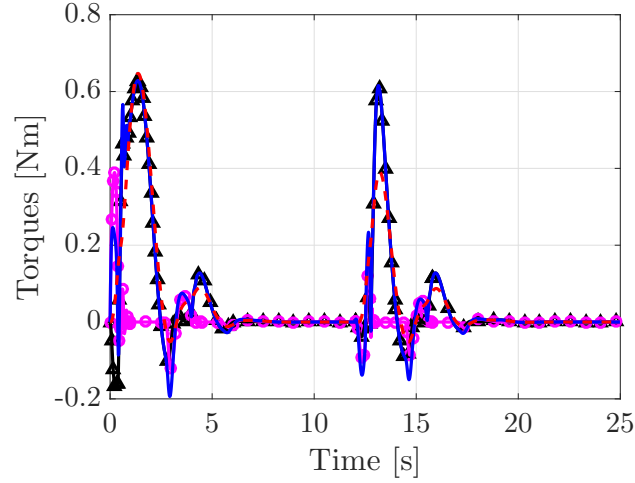
Source: Trentin, Silva and Schaub (2019b).

Analyzing the contribution of each control action presented in Fig. 51 for the VSCMG pendulum subjected to a disturbance after being controlled in the inverted position, both modes are used to deal with the disturbance applied to the system even if a higher value for the weight of the wheel mode (W_w) is considered compared to the one related to the gimbal mode (W_g). However, both control actions help to control the pendulum by employing the gyroscopic torque and the torque produced by the reaction wheel.

Figure 52 presents the control actions for the RWP. Here, just the reaction wheel control action is used to control the inverted pendulum, and there is no contribution from the gyroscopic torque since the gimbal is locked for this case.

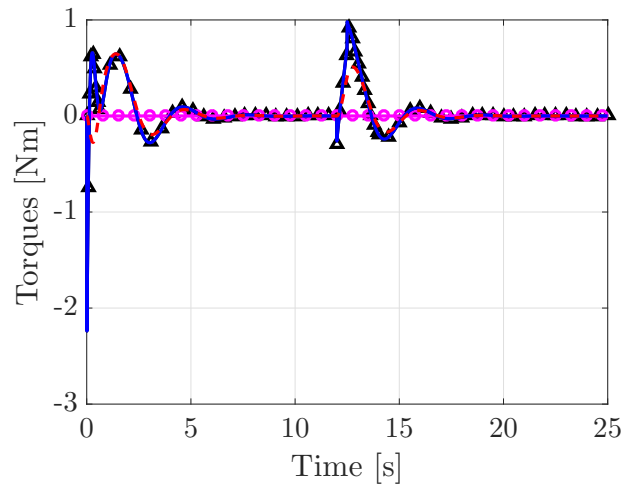
Since only one control action is used, the values reached by this action are higher than the ones obtained for the VSCMG pendulum presented in Fig. 51. The RMS values for the control actions for the RWP were calculated, and they are presented in Tab. 6 to be compared with the ones for the VSCMG pendulum, when these systems are subjected to an external disturbance.

Figure 51 – VSCMGP control actions and torques when subject to external disturbance. - - - indicates the external torque caused by gravity ($\mathcal{D}$), - indicates the composition of the angular momentum variation ($\dot{H}_3 = \mathcal{A}\ddot{\theta} + \mathcal{B}\ddot{\psi} + \mathcal{C}\dot{\gamma}$), $-\Delta-$ indicates the reaction wheel control action ($\mathcal{B}\ddot{\psi}$) and $-\circ-$ indicates the CMG control action ($\mathcal{C}\dot{\gamma}$).



Source: Trentin, Silva and Schaub (2019b).

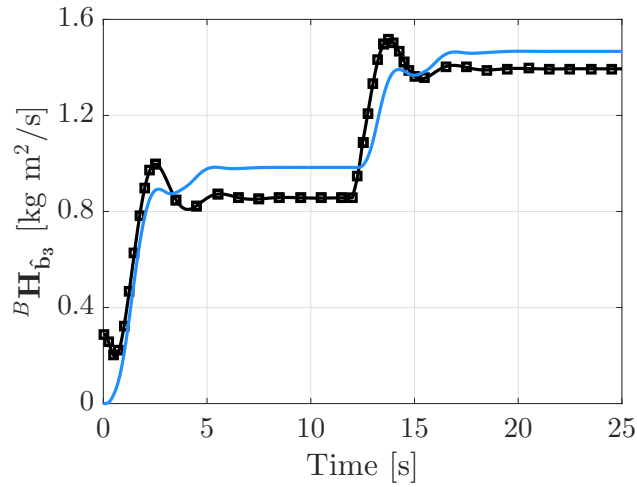
Figure 52 – RWP control actions and torques when subject to external disturbance. - - - indicates the external torque caused by gravity ($\mathcal{D}$), - indicates the composition of the angular momentum variation ($\dot{H}_3 = \mathcal{A}\ddot{\theta} + \mathcal{B}\ddot{\psi} + \mathcal{C}\dot{\gamma}$), $-\Delta-$ indicates the reaction wheel control action ($\mathcal{B}\ddot{\psi}$) and $-\circ-$ indicates the CMG control action ($\mathcal{C}\dot{\gamma}$).



Source: Trentin, Silva and Schaub (2019b).

Figure 53 presents the angular momentum around $\hat{b}_3$ for both systems, illustrating the change in the angular momentum after the disturbance is applied. The conservation of the angular momentum can also be seen since the pendulum was controlled in the inverted position; then, a disturbance was applied; at last, the angular pendulum position was controlled again, and the angular momentum was conserved once again.

Figure 53 – Angular momentum ${}^B\mathbf{H}$ around $\hat{b}_3$ when subject to external disturbance, where --- indicates the result for VSCMG pendulum and --□-- indicates the result for RWP.



Source: Trentin, Silva and Schaub (2019b).

Comparison between performances

This section presents a detailed analysis of the performance of the two pendulum systems analyzed. This comparison is conducted to evaluate how the new configuration proposed performs when confronted to an existing one.

The results presented in Figures 42 and 48 for the angular pendulum position considering the two situations of both pendulum configurations analyzed in this chapter were close. Therefore, four performance indices were calculated to evaluate the results for controlling the angular pendulum position of the systems studied. The indices were introduced in subsection 3.3.3 and they are a suitable measure of performance.

The values of the indices are presented in Tab. 5. For both cases studied in this chapter (that is, in the inverted position and subjected to a disturbance) with the parameters chosen, the VSCMG pendulum outperforms the RWP configuration, presenting lower values for all performances indices.

Table 5 – Performance indices of the pendulums angular position calculated to compare VSCMG pendulum and RWP.

Inverted			Disturbance	
Index	VSCMG	RWP	VSCMG	RWP
ISE	10.09	12.01	10.42	12.65
IAE	4.57	5.10	5.44	6.34
ITAE	4.60	5.57	16.66	22.49
ITSE	6.57	7.58	10.92	15.92

Source: Trentin, Silva and Schaub (2019b).

Moreover, the RMS value of the control actions presented in Figures 45, 46, 51 and 52 is evaluated. Thereby, the RMS of the reaction wheel control action and the gimbal control action were calculated for the two situations analyzed in this chapter for both pendulum configurations. The RMS values obtained are presented in Tab. 6.

Table 6 – RMS of the control actions of VSCMG pendulum and RWP.

Inverted			Disturbance	
Action	VSCMG	RWP	VSCMG	RWP
$\mathcal{B}\ddot{\psi}$	0.1345	0.1706	0.1631	0.2230
$\mathcal{C}\dot{\gamma}$	0.0418	0	0.0459	0

Source: Trentin, Silva and Schaub (2019b).

Analyzing the RMS values obtained, it is hard to conclude that one configuration is more energy efficient than the other one. Note that the control actions of the VSCMG pendulum are divided into two. Therefore, they must be added to yield the total. Thus, for the case where the pendulums are controlled in the inverted position the RWP has lower values for the control actions whereas when the disturbance is applied to the pendulums, the VSCMG pendulum has lower RMS values accounting both control actions.

5.2 DYNAMIC MODELLING AND CONTROL OF A 3D PENDULUM WITH A VSCMG

The main objective of this approach is to obtain a nonsingular model of a spherical pendulum that is controlled in the inverted position using a VSCMG. Thus, some assumptions and remarks are carried out to properly obtain the model of an spherical pendulum from the 3D pendulum. Furthermore, to the best of the author's knowledge,

such pendulum configuration controlled using a VSCMG has not been investigated before.

The results presented in this chapter are adapted from Schaub and Junkins (2014) that carries out the complete derivation of the model and control of a spacecraft controlled using VSCMG. A similar approach is used here to obtain the dynamic model of a 3D pendulum controlled using a VSCMG.

If this pendulum configuration is modeled using a sequence of Euler angles rotation, it will return a singular model in the point of interest of this study, that is, the inverted position. This is the main reason why in this part of the work the model is not derived as done previously. Thus, subsection 5.2.1 presents the complete derivation of the model of the 3D pendulum with a VSCMG, subsection 5.2.2 comprises the design of the control strategy that assures global asymptotic stabilization of the system, and subsection 5.2.3 shows the numerical results obtained.

5.2.1 Dynamic modeling of the 3D pendulum with a VSCMG

This subsection presents the assumptions, definitions and the complete derivation of the model of a 3D pendulum with a VSCMG. The Modified Rodrigues Parameters (MRPs) are used to describe the attitude of this new pendulum configuration to yield the nonsingular motion. The shadow set property of the MRPs allows a complete three-dimensional nonsingular attitude description.

Frames definitions and kinematics

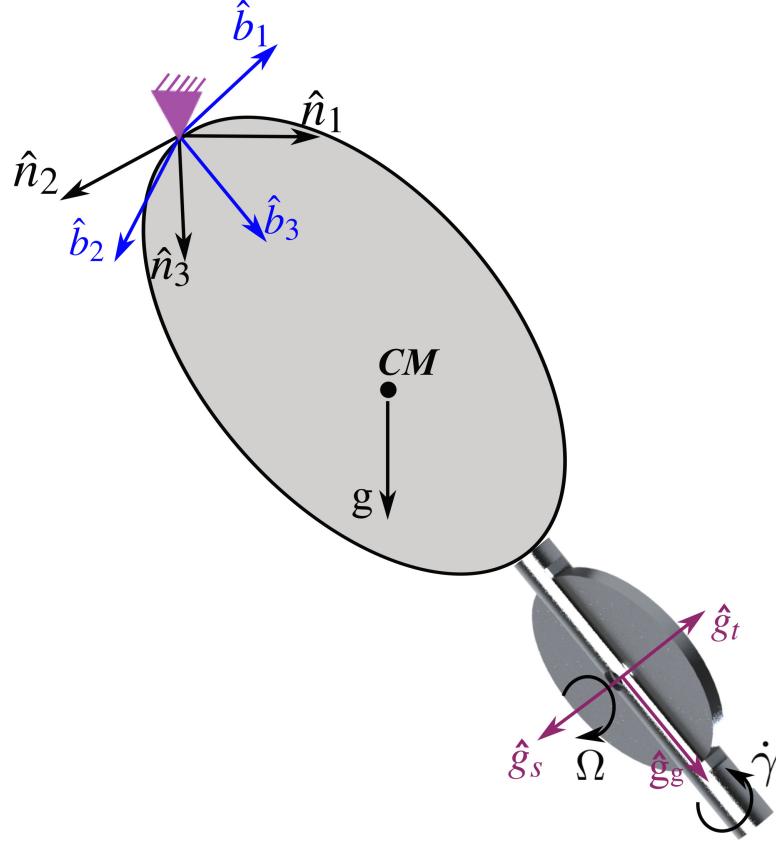
The 3D pendulum is a rigid body with three rotational degrees of freedom as illustrated in Fig. 54. It is fixed at a pivot point and gravity acts on it (SHEN *et al.*, 2004; CHATURVEDI; MCCLAMROCH; BERNSTEIN, 2008; YAO, 2020). This work couples a VSCMG mechanism to the 3D pendulum to obtain a generic model.

To do so, firstly consider two coordinate frames, the inertial defined as $N : \{\hat{n}_1, \hat{n}_2, \hat{n}_3\}$ and the body fixed frame $B : \{\hat{b}_1, \hat{b}_2, \hat{b}_3\}$. It is understood that in all instances where the frame in which the vector is taken is not defined, it denotes that such vector or matrix is in the body frame B . For this problem, the Modified Rodrigues Parameters (MRPs) are used to describe the attitude of the 3D pendulum. They are defined as

$$\boldsymbol{\sigma} = \tan \frac{\Phi}{4} \hat{\mathbf{e}} \quad (116)$$

where $\boldsymbol{\sigma} \in \mathbb{R}^3$, $\Phi \in (-2\pi, 2\pi)$ is the principal angle rotation, $\hat{\mathbf{e}}$ is the principal axis unit

Figure 54 – 3D pendulum with a VSCMG



Source: Prepared by the author.

vector referring to Euler's principal rotation theorem (SCHAUB; JUNKINS, 2014). The MRPs are singular if a $\pm 2\pi$ rotation is carried out. However, a very important feature that the MRPs provide is the shadow MRP set. This means that this singularity can be avoided if the MRP set is switched to the shadow set (SCHAUB; JUNKINS, 2014). This leads to a complete nonsingular attitude description, although being discontinuous. Therefore, when $\sigma^T \sigma > 1$ the MRP set is mapped to the shadow set as:

$$\sigma^S = -\frac{\sigma}{\sigma^T \sigma} \quad (117)$$

This allows a nonsingular attitude description in three-dimensional space.

It is also important to define a frame for the desired reference, denoted by $R : \{\hat{r}_1, \hat{r}_2, \hat{r}_3\}$. The following relations between frames are defined considering the short-notation for triad of unit vector $\{\hat{\mathbf{b}}\} = \{\hat{b}_1, \hat{b}_2, \hat{b}_3\}$, $\{\hat{\mathbf{n}}\} = \{\hat{n}_1, \hat{n}_2, \hat{n}_3\}$ and $\{\hat{\mathbf{r}}\} = \{\hat{r}_1, \hat{r}_2, \hat{r}_3\}$:

$$\{\hat{\mathbf{b}}\} = [BN(\sigma_{B/N})] \{\hat{\mathbf{n}}\} \quad (118)$$

$$\{\hat{\mathbf{b}}\} = [BR(\sigma_{B/R})] \{\hat{\mathbf{r}}\} \quad (119)$$

where $[BN(\boldsymbol{\sigma}_{B/N})]$ is the rotation matrix between the body and inertial frames defined in terms of the MRPs:

$$[BN(\boldsymbol{\sigma}_{B/N})] = [I_{3 \times 3}] + \frac{8[\tilde{\boldsymbol{\sigma}}_{B/N}][\tilde{\boldsymbol{\sigma}}_{B/N}] - 4(1 - \boldsymbol{\sigma}_{B/N}^T \boldsymbol{\sigma}_{B/N})[\tilde{\boldsymbol{\sigma}}_{B/N}]}{(1 + \boldsymbol{\sigma}_{B/N}^T \boldsymbol{\sigma}_{B/N})^2} \quad (120)$$

and, analogously, $[BR(\boldsymbol{\sigma}_{B/R})]$ the rotation matrix between the reference and body frame:

$$[BR(\boldsymbol{\sigma}_{B/R})] = [I_{3 \times 3}] + \frac{8[\tilde{\boldsymbol{\sigma}}_{B/R}][\tilde{\boldsymbol{\sigma}}_{B/R}] - 4(1 - \boldsymbol{\sigma}_{B/R}^T \boldsymbol{\sigma}_{B/R})[\tilde{\boldsymbol{\sigma}}_{B/R}]}{(1 + \boldsymbol{\sigma}_{B/R}^T \boldsymbol{\sigma}_{B/R})^2} \quad (121)$$

where $[I_{3 \times 3}]$ is an identity 3x3 matrix, $[\tilde{\boldsymbol{\sigma}}_{B/N}]$ and $[\tilde{\boldsymbol{\sigma}}_{B/R}]$ are skew-symmetric matrices defined as:

$$[\tilde{\boldsymbol{\sigma}}_{B/N}] = \begin{bmatrix} 0 & -\sigma_3 & \sigma_2 \\ \sigma_3 & 0 & -\sigma_1 \\ -\sigma_2 & \sigma_1 & 0 \end{bmatrix}; \quad [\tilde{\boldsymbol{\sigma}}_{B/R}] = \begin{bmatrix} 0 & -\sigma_{e3} & \sigma_{e2} \\ \sigma_{e3} & 0 & -\sigma_{e1} \\ -\sigma_{e2} & \sigma_{e1} & 0 \end{bmatrix} \quad (122)$$

Let $\boldsymbol{\sigma}_{R/N}$ and $\boldsymbol{\omega}_r$ denote the reference attitude and angular velocities, respectively. The relative attitude vector $\boldsymbol{\sigma}_{B/R}$ in terms of $\boldsymbol{\sigma}_{B/N}$ and $\boldsymbol{\sigma}_{R/N}$ is expressed as (SCHAUB; JUNKINS, 2014):

$$\boldsymbol{\sigma}_{B/R} = \frac{(1 - \boldsymbol{\sigma}_{R/N}^T \boldsymbol{\sigma}_{R/N})\boldsymbol{\sigma}_{B/N} - (1 - \boldsymbol{\sigma}_{B/N}^T \boldsymbol{\sigma}_{B/N})\boldsymbol{\sigma}_{R/N} - 2[\tilde{\boldsymbol{\sigma}}_{R/N}]\boldsymbol{\sigma}_{B/N}}{1 + \boldsymbol{\sigma}_{R/N}^T \boldsymbol{\sigma}_{R/N} \boldsymbol{\sigma}_{B/N}^T \boldsymbol{\sigma}_{B/N} + 2\boldsymbol{\sigma}_{R/N}^T \boldsymbol{\sigma}_{B/N}} \quad (123)$$

The kinematic differential equation of MRPs is:

$$\dot{\boldsymbol{\sigma}}_{B/N} = \frac{1}{4}[B(\boldsymbol{\sigma}_{B/N})]\boldsymbol{\omega}_{B/N} \quad (124)$$

where $\boldsymbol{\omega}_{B/N}$ is the body angular velocity in the body frame with respect to the inertial frame and the $[B(\boldsymbol{\sigma}_{B/N})]$ matrix is given by:

$$[B(\boldsymbol{\sigma}_{B/N})] = \begin{bmatrix} 1 - \sigma^2 + 2\sigma_1^2 & 2(\sigma_1\sigma_2 - \sigma_3) & 2(\sigma_1\sigma_3 + \sigma_2) \\ 2(\sigma_2\sigma_1 + \sigma_3) & 1 - \sigma^2 + 2\sigma_2^2 & 2(\sigma_2\sigma_3 - \sigma_1) \\ 2(\sigma_3\sigma_1 - \sigma_2) & 2(\sigma_3\sigma_2 + \sigma_1) & 1 - \sigma^2 + 2\sigma_3^2 \end{bmatrix} \quad (125)$$

that can also be written in vector form as:

$$[B(\boldsymbol{\sigma}_{B/N})] = \frac{1}{4} \left[(1 - \boldsymbol{\sigma}_{B/N}^T \boldsymbol{\sigma}_{B/N})[I_{3 \times 3}] + 2[\tilde{\boldsymbol{\sigma}}_{B/N}] + 2\boldsymbol{\sigma}_{B/N} \boldsymbol{\sigma}_{B/N}^T \right] \quad (126)$$

Besides, it is also important to define the body angular velocity error $\delta\boldsymbol{\omega}$:

$$\delta\boldsymbol{\omega} = \boldsymbol{\omega}_{B/N} - \boldsymbol{\omega}_r \quad (127)$$

where $\boldsymbol{\omega}_r = [BR(\boldsymbol{\sigma}_{B/R})]^R \boldsymbol{\omega}_r$. The angular velocity error and the MRP rate are related through:

$$\dot{\boldsymbol{\sigma}}_{B/R} = [B(\boldsymbol{\sigma}_{B/R})]\delta\boldsymbol{\omega} \quad (128)$$

To simplify the notation for the reminder of this work, it is assumed that $\boldsymbol{\sigma} = \boldsymbol{\sigma}_{B/N}$, $\boldsymbol{\sigma}_e = \boldsymbol{\sigma}_{B/R}$.

The orientation of the gimbal frame G is given by the triad of unit vectors $G : \{\hat{\mathbf{g}}_s, \hat{\mathbf{g}}_t, \hat{\mathbf{g}}_g\}$. Note that $\hat{\mathbf{g}}_g$ is aligned to $\hat{\mathbf{b}}_3$ as shown in Fig. 54. As seen by the B frame, $\hat{\mathbf{g}}_s$ and $\hat{\mathbf{g}}_t$ are time varying given by:

$$\hat{\mathbf{g}}_s(t) = \cos(\gamma(t) - \gamma_0)\hat{\mathbf{g}}_s(t_0) + \sin(\gamma(t) - \gamma_0)\hat{\mathbf{g}}_t(t_0) \quad (129)$$

$$\hat{\mathbf{g}}_t(t) = -\sin(\gamma(t) - \gamma_0)\hat{\mathbf{g}}_s(t_0) + \cos(\gamma(t) - \gamma_0)\hat{\mathbf{g}}_t(t_0) \quad (130)$$

The rotation matrix $[BG]$ relates the gimbal and body frame:

$$\{\hat{\mathbf{b}}\} = [BG]\{\hat{\mathbf{g}}\} \quad (131)$$

where $\{\hat{\mathbf{g}}\} = \{\hat{\mathbf{g}}_s, \hat{\mathbf{g}}_t, \hat{\mathbf{g}}_g\}$ and $[BG] = [\hat{\mathbf{g}}_s \ \hat{\mathbf{g}}_t \ \hat{\mathbf{g}}_g]$. The components of this rotation matrix maps a vector taken in the G frame into one in the B frame.

Besides, regarding the VSCMG, we need to draw attention to two angular velocities. The angular velocity of the gimbal, i.e., the angular velocity of the G frame relative to B is given by:

$${}^G\boldsymbol{\omega}_{G/B} = \dot{\gamma}\hat{\mathbf{g}}_g \quad (132)$$

And the reaction wheel angular velocity, i.e., the angular velocity vector of the reaction wheel frame W relative to the gimbal frame G is:

$${}^G\boldsymbol{\omega}_{W/G} = \Omega\hat{\mathbf{g}}_s \quad (133)$$

Inertia matrices

Note that three inertia matrices have to be defined to model the 3D pendulum with a VSCMG. The inertia matrix of the 3D pendulum $[I_s]$ contains the inertias of the body and of the VSCMG calculated relative to the overall center of mass. It is given in B frame and it is constant as seen from the B frame.

The inertia matrix of the gimbal ${}^G[I_G]$ is defined in the gimbal frame as:

$${}^G[I_G] = \begin{bmatrix} I_{Gs} & 0 & 0 \\ 0 & I_{Gt} & 0 \\ 0 & 0 & I_{Gg} \end{bmatrix} \quad (134)$$

Here, the inertia matrix of the reaction wheel ${}^G[I_W]$ is already presented in the gimbal frame due to the disk symmetry:

$${}^G[I_W] = \begin{bmatrix} I_{ws} & 0 & 0 \\ 0 & I_{wt} & 0 \\ 0 & 0 & I_{wg} \end{bmatrix} \quad (135)$$

These two matrices need to be rotated to the body frame where they are time-varying given by:

$${}^B[I_G] = [BG]{}^G[I_G][BG]^T = I_{Gs}\hat{\mathbf{g}}_s\hat{\mathbf{g}}_s^T + I_{Gt}\hat{\mathbf{g}}_t\hat{\mathbf{g}}_t^T + I_{Gg}\hat{\mathbf{g}}_g\hat{\mathbf{g}}_g^T \quad (136)$$

$${}^B[I_W] = [BG]{}^G[I_W][BG]^T = I_{ws}\hat{\mathbf{g}}_s\hat{\mathbf{g}}_s^T + I_{wt}\hat{\mathbf{g}}_t\hat{\mathbf{g}}_t^T + I_{wg}\hat{\mathbf{g}}_g\hat{\mathbf{g}}_g^T \quad (137)$$

In further development might be convenient sometimes to express the inertia matrices of the gimbal and of the reaction wheel as a single inertia matrix of the VSCMG $[J]$ denoted by:

$${}^G[J] = {}^G[I_G] + {}^G[I_W] = \begin{bmatrix} J_s & 0 & 0 \\ 0 & J_t & 0 \\ 0 & 0 & J_g \end{bmatrix} \quad (138)$$

Moreover, the total inertia matrix $[I]$ in the body frame can be expressed as:

$$[I] = [I_s] + J_s\hat{\mathbf{g}}_s\hat{\mathbf{g}}_s^T + J_t\hat{\mathbf{g}}_t\hat{\mathbf{g}}_t^T + J_g\hat{\mathbf{g}}_g\hat{\mathbf{g}}_g^T \quad (139)$$

Equations of motion

To obtain the equations of motion of the 3D pendulum with the VSCMG, the angular momentum of the 3D pendulum $\mathbf{H}_B$, of the gimbal $\mathbf{H}_G$ and of the reaction wheel must be evaluated to yield the total angular momentum of the complete system around the center of mass:

$$\mathbf{H} = \mathbf{H}_B + \mathbf{H}_G + \mathbf{H}_W \quad (140)$$

and the angular momentum of the body is given by:

$$\mathbf{H}_B = [\mathbf{I}_s] \boldsymbol{\omega}_{B/N} \quad (141)$$

The gimbal frame angular momentum is:

$$\mathbf{H}_G = [\mathbf{I}_G] \boldsymbol{\omega}_{G/N} \quad (142)$$

where $\boldsymbol{\omega}_{G/N} = \boldsymbol{\omega}_{G/B} + \boldsymbol{\omega}_{B/N}$. Using the definitions for the inertia matrix and for the angular velocities, the angular momentum of the gimbal frame can be rewritten as:

$$\mathbf{H}_G = \left(I_{Gs} \hat{\mathbf{g}}_s \hat{\mathbf{g}}_s^T + I_{Gt} \hat{\mathbf{g}}_t \hat{\mathbf{g}}_t^T + I_{Gg} \hat{\mathbf{g}}_g \hat{\mathbf{g}}_g^T \right) \boldsymbol{\omega}_{B/N} + I_{Gg} \dot{\gamma} \hat{\mathbf{g}}_g \quad (143)$$

Here, it is introduced the projection of $\boldsymbol{\omega}_{B/N}$ onto G frame unit axes

$$\omega_s = \hat{\mathbf{g}}_s^T \boldsymbol{\omega}_{B/N}; \quad \omega_t = \hat{\mathbf{g}}_t^T \boldsymbol{\omega}_{B/N}; \quad \omega_g = \hat{\mathbf{g}}_g^T \boldsymbol{\omega}_{B/N} \quad (144)$$

These relations help to simplify many steps of the derivation of this model and they are also used in the design of the controller. They allow the angular momentum of the gimbal to be written as:

$$\mathbf{H}_G = I_{Gs} \omega_s \hat{\mathbf{g}}_s + I_{Gt} \omega_t \hat{\mathbf{g}}_t + I_{Gg} (\omega_g + \dot{\gamma}) \hat{\mathbf{g}}_g \quad (145)$$

The angular momentum of the reaction wheel is given by:

$$\mathbf{H}_W = [\mathbf{I}_W] \boldsymbol{\omega}_{W/N} \quad (146)$$

where $\boldsymbol{\omega}_{W/N} = \boldsymbol{\omega}_{W/G} + \boldsymbol{\omega}_{G/B} + \boldsymbol{\omega}_{B/N}$. Analogously for what was done for the gimbal angular momentum, the reaction wheel angular momentum can be written as:

$$\mathbf{H}_W = I_{ws} (\omega_s + \Omega) \hat{\mathbf{g}}_s + I_{wt} \omega_t \hat{\mathbf{g}}_t + I_{wg} (\omega_g + \dot{\gamma}) \hat{\mathbf{g}}_g \quad (147)$$

The Euler's equation is used to derive the equations of motion of the 3D pendulum with a VSCMG. The inertial derivative of the total angular momentum is calculated and equaled to the external torques acting on the system:

$$\dot{\mathbf{H}} = \mathbf{L} \quad (148)$$

where $\mathbf{L} = mg\boldsymbol{\rho} \times [BN]^T \hat{n}_3$ denotes the external torque that gravity exerts onto the system, where m is the mass of the system, g the acceleration of gravity, $\boldsymbol{\rho}$ the center of mass vector, and $\hat{n}_3$ the inertial direction in which gravity acts.

Differently from the first part of this chapter, due to the higher complexity of the system, it is recommended to evaluate the inertial derivatives of the gimbal frame's angular momentum and the reaction wheel first. With these results they are combined with the inertial derivative of the body angular momentum to yield the total inertial derivative of $\mathbf{H}$. Thus, they are added together and equaled to the external torques acting onto the system to yield the equations of motion. To simplify the notation, from this point onwards $\boldsymbol{\omega} = \boldsymbol{\omega}_{B/N}$.

Note that the vectors $\{\hat{\mathbf{g}}_s, \hat{\mathbf{g}}_t, \hat{\mathbf{g}}_g\}$ appear in the angular momentum of the gimbal and of the reaction wheel. Therefore, the inertial derivatives of these vector must be evaluated:

$$\dot{\hat{\mathbf{g}}}_s = \frac{{}^B d}{dt}(\hat{\mathbf{g}}_s) + \boldsymbol{\omega} \times \hat{\mathbf{g}}_s = (\dot{\gamma} + \omega_g)\hat{\mathbf{g}}_t - \omega_t\hat{\mathbf{g}}_g \quad (149)$$

$$\dot{\hat{\mathbf{g}}}_t = \frac{{}^B d}{dt}(\hat{\mathbf{g}}_t) + \boldsymbol{\omega} \times \hat{\mathbf{g}}_t = -(\dot{\gamma} + \omega_g)\hat{\mathbf{g}}_s - \omega_s\hat{\mathbf{g}}_g \quad (150)$$

$$\dot{\hat{\mathbf{g}}}_g = \frac{{}^B d}{dt}(\hat{\mathbf{g}}_g) + \boldsymbol{\omega} \times \hat{\mathbf{g}}_g = \omega_t\hat{\mathbf{g}}_s - \omega_s\hat{\mathbf{g}}_t \quad (151)$$

once the B frame derivatives are given by:

$$\frac{{}^B d}{dt}(\hat{\mathbf{g}}_s) = \dot{\gamma}\hat{\mathbf{g}}_t ; \quad \frac{{}^B d}{dt}(\hat{\mathbf{g}}_t) = -\dot{\gamma}\hat{\mathbf{g}}_s ; \quad \frac{{}^B d}{dt}(\hat{\mathbf{g}}_g) = 0 \quad (152)$$

Additionally, it is important to evaluate the inertial derivatives of the G frame body angular velocities:

$$\dot{\boldsymbol{\omega}}_s = \dot{\hat{\mathbf{g}}}_s^T \boldsymbol{\omega} + \hat{\mathbf{g}}_s^T \dot{\boldsymbol{\omega}} = \dot{\gamma}\omega_t + \hat{\mathbf{g}}_s^T \dot{\boldsymbol{\omega}} \quad (153)$$

$$\dot{\boldsymbol{\omega}}_t = \dot{\hat{\mathbf{g}}}_t^T \boldsymbol{\omega} + \hat{\mathbf{g}}_t^T \dot{\boldsymbol{\omega}} = -\dot{\gamma}\omega_s + \hat{\mathbf{g}}_t^T \dot{\boldsymbol{\omega}} \quad (154)$$

$$\dot{\boldsymbol{\omega}}_g = \dot{\hat{\mathbf{g}}}_g^T \boldsymbol{\omega} + \hat{\mathbf{g}}_g^T \dot{\boldsymbol{\omega}} = \hat{\mathbf{g}}_g^T \dot{\boldsymbol{\omega}} \quad (155)$$

All these expressions developed here are used to derive the inertial derivatives of $\mathbf{H}_W$ and $\mathbf{H}_G$. Thus, the inertial derivative of the $\mathbf{H}_W$ is given by:

$$\begin{aligned}\dot{\mathbf{H}}_W &= \hat{\mathbf{g}}_s (I_{ws}(\dot{\Omega} + \dot{\gamma}\omega_t + \hat{\mathbf{g}}_s^T \dot{\boldsymbol{\omega}})) + \hat{\mathbf{g}}_t ([I_{ws}(\dot{\gamma}(\omega_s + \Omega) + \Omega\omega_g) + I_{wt}\hat{\mathbf{g}}_t^T \dot{\boldsymbol{\omega}} + (I_{ws} - I_{wt})\omega_s\omega_g) + \\ &- \hat{\mathbf{g}}_t(2I_{wt}\omega_s\dot{\gamma}) + \hat{\mathbf{g}}_g (I_{wt}(\hat{\mathbf{g}}_g^T \dot{\boldsymbol{\omega}} + \ddot{\gamma}) + (I_{wt} - I_{ws})\omega_s\omega_t - I_{ws}\Omega\omega_t)\end{aligned}\quad (156)$$

Analyzing this result in the format of Euler's equation $\dot{\mathbf{H}}_W = \mathbf{L}_W$ where $\mathbf{L}_W$ is the torque that the gimbal frame exerts on the reaction wheel, note that the torque component around $\hat{\mathbf{g}}_s$ has to be produced by the motor that spins the reaction wheel. Thus, the reaction wheel motor torque is expressed as:

$$u_s = I_{ws}(\dot{\Omega} + \hat{\mathbf{g}}_s^T \dot{\boldsymbol{\omega}} + \dot{\gamma}\omega_t) \quad (157)$$

Using the expressions defined from Eq.(149) to (155), the inertial derivative of the gimbal frame is obtained as:

$$\begin{aligned}\dot{\mathbf{H}}_G &= \hat{\mathbf{g}}_s ((I_{Gs} - I_{Gt} + I_{Gg})\dot{\gamma}\omega_t + I_{Gs}\hat{\mathbf{g}}_s^T \dot{\boldsymbol{\omega}} + (I_{Gg} - I_{Gt})\omega_t\omega_g) + \\ &+ \hat{\mathbf{g}}_t ((I_{Gs} - I_{Gt} - I_{Gg})\dot{\gamma}\omega_s + I_{Gt}\hat{\mathbf{g}}_t^T \dot{\boldsymbol{\omega}} + (I_{Gs} - I_{Gg})\omega_s\omega_g) + \\ &+ \hat{\mathbf{g}}_g (I_{Gg}(\hat{\mathbf{g}}_g^T \dot{\boldsymbol{\omega}} + \ddot{\gamma}) + (I_{Gt} - I_{Gs})\omega_s\omega_t)\end{aligned}\quad (158)$$

Now we need to analyze the gimbal and the reaction wheel combined, i.e., let's evaluate Euler's equation as:

$$\dot{\mathbf{H}}_G + \dot{\mathbf{H}}_W = \mathbf{L}_G \quad (159)$$

Considering that $\mathbf{L}_G$ is the torque vector that the 3D pendulum causes onto the VSCMG and noting that the component around $\hat{\mathbf{g}}_g$ has to be provided by the motor responsible for gimbaling the CMG. The motor torque u_g of the gimbal is given by:

$$u_g = J_g(\hat{\mathbf{g}}_g^T \dot{\boldsymbol{\omega}} + \ddot{\gamma}) - (J_s - J_t)\omega_s\omega_t - I_{ws}\Omega\omega_t \quad (160)$$

Finally, the inertial derivative of the body angular momentum is given by:

$$\dot{\mathbf{H}}_B = [I_s]\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times [I_s]\boldsymbol{\omega} \quad (161)$$

Therefore, considering Eq. (148), and combining the inertial derivatives of $\mathbf{H}_B$, $\mathbf{H}_G$

and $\mathbf{H}_W$, the equations of motion of the 3D pendulum with a VSCMG are given by:

$$\begin{aligned} [I]\dot{\boldsymbol{\omega}} &= -\boldsymbol{\omega} \times [I]\boldsymbol{\omega} - \hat{\mathbf{g}}_s (J_s \dot{\gamma} \boldsymbol{\omega}_t + I_{ws} \dot{\Omega} - (J_t - J_g) \boldsymbol{\omega}_t \dot{\gamma}) + \\ &- \hat{\mathbf{g}}_t ((J_s \boldsymbol{\omega}_s + I_{ws} \Omega) \dot{\gamma} - (J_t + J_g) \boldsymbol{\omega}_s \dot{\gamma} + I_{ws} \Omega \boldsymbol{\omega}_g) - \hat{\mathbf{g}}_g (J_g \ddot{\gamma} - I_{ws} \Omega \boldsymbol{\omega}_t) + \mathbf{L} \end{aligned} \quad (162)$$

In this format, the inertia matrices were combined as presented in Eq. (139). The expression presented in Eq. (162) can be more simplified if we consider $J_s \approx I_{ws}$ once the gimbal frame inertia I_{G_s} is commonly very small and can be neglected when compared to I_{W_s} . Therefore, the equations of motions of the 3D pendulum with a single VSCMG are given by:

$$\begin{aligned} [I]\dot{\boldsymbol{\omega}} &= -\boldsymbol{\omega} \times [I]\boldsymbol{\omega} - \hat{\mathbf{g}}_s (J_s (\dot{\Omega} + \dot{\gamma} \boldsymbol{\omega}_t) - (J_t - J_g) \boldsymbol{\omega}_t \dot{\gamma}) + \\ &- \hat{\mathbf{g}}_t (J_s (\boldsymbol{\omega}_s + \Omega) \dot{\gamma} - (J_t + J_g) \boldsymbol{\omega}_s \dot{\gamma} + J_s \Omega \boldsymbol{\omega}_g) - \hat{\mathbf{g}}_g (J_g \ddot{\gamma} - J_s \Omega \boldsymbol{\omega}_t) + \mathbf{L} \end{aligned} \quad (163)$$

The complete model of the 3D pendulum with VSCMG has to be solved together. This encompasses the kinematic differential equations, the equations of motions and the motor torque equation. Thereby, the complete model can be written as:

$$\begin{bmatrix} [I_{3x3}] & [0_{3x3}] & [0_{3x1}] & [0_{3x1}] & [0_{3x1}] \\ [0_{3x3}] & [I] & [0_{3x1}] & J_g \hat{\mathbf{g}}_g & J_s \hat{\mathbf{g}}_s \\ [0_{1x3}] & [0_{1x3}] & 1 & 0 & 0 \\ [0_{1x3}] & J_g \hat{\mathbf{g}}_s^T & 0 & J_g & 0 \\ [0_{1x3}] & J_s \hat{\mathbf{g}}_s^T & 0 & 0 & J_s \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{\sigma}} \\ \dot{\boldsymbol{\omega}} \\ \dot{\gamma} \\ \ddot{\gamma} \\ \dot{\Omega} \end{bmatrix} = \begin{bmatrix} \frac{1}{4} [B(\boldsymbol{\sigma})] \boldsymbol{\omega} \\ \mathbf{f}_\omega \\ \dot{\gamma} \\ f_g \\ f_s \end{bmatrix} \quad (164)$$

where

$$\begin{aligned} \mathbf{f}_\omega &= -\boldsymbol{\omega} \times [I]\boldsymbol{\omega} - \hat{\mathbf{g}}_s (J_s \dot{\gamma} \boldsymbol{\omega}_t - (J_t - J_g) \boldsymbol{\omega}_t \dot{\gamma}) - \hat{\mathbf{g}}_t (J_s (\boldsymbol{\omega}_s + \Omega) \dot{\gamma} - (J_t + J_g) \boldsymbol{\omega}_s \dot{\gamma} + J_s \Omega \boldsymbol{\omega}_g) + \\ &+ \hat{\mathbf{g}}_g (J_s \Omega \boldsymbol{\omega}_t) + \mathbf{L} \end{aligned} \quad (165)$$

$$f_g = u_g + (J_s - J_t) \boldsymbol{\omega}_s \boldsymbol{\omega}_t + J_s \Omega \boldsymbol{\omega}_t \quad (166)$$

$$f_s = u_s - J_s \dot{\gamma} \boldsymbol{\omega}_t \quad (167)$$

Dynamic analysis

Shen *et al.* (2004) and Chaturvedi, McClamroch and Bernstein (2008) have shown that the 3D pendulum presents two equilibrium manifolds. The hanging equilibrium is when the center of mass of the pendulum is below the pivot. The inverted equilibrium is when the center of mass is directly above the pivot. Shen *et al.* (2004) have proven that

the hanging equilibrium is stable in the sense of Lyapunov, and the inverted is unstable.

Assumptions to yield the spherical pendulum

As aforementioned, the objective is to control a spherical pendulum with a VSCMG. The alternative found to model this pendulum configuration was to derive a general model for a 3D pendulum and based on the assumptions made in Shen *et al.* (2004), Chaturvedi *et al.* (2005), Chaturvedi, McClamroch and Bernstein (2008) to obtain a spherical pendulum.

The assumptions considered to yield the spherical pendulum are: the rigid pendulum is axis symmetric; the pivot is located in the axis of symmetry; and the body has only two possible rotations. Therefore, ω_3 which is the angular velocity of the body around $\hat{b}_3$ is assumed to be zero. This also leads to $\dot{\omega}_3 = 0$.

5.2.2 Controller design for the 3d pendulum with VSCMG

The goal of this section is to design a feedback control law for the 3D pendulum being controlled using a VSCMG. An unconstrained control law that assures global asymptotic stabilization is presented. To design the control strategy, the following candidate for Lyapunov function is selected:

$$\mathcal{V}(\delta\omega, \sigma_e) = \frac{1}{2}\delta\omega^T[I]\delta\omega + 2K\ln(1 + \sigma_e^T\sigma_e) \quad (168)$$

which is positive definite once K is a positive attitude feedback gain and the inertia matrix $[I]$ is a positive definite matrix. The time derivative of the candidate for Lyapunov function is given by:

$$\dot{\mathcal{V}}(\delta\omega, \sigma_e) = \delta\omega^T \left([I] \frac{B}{dt}(\delta\omega) + \frac{1}{2} \frac{B}{dt}([I])\delta\omega + K\sigma_e \right) \quad (169)$$

It is important to note that for this case the inertia matrix $[I]$ is time-varying. This happens because of the CMG gimabilling. It is important to recall that the Lyapunov function is a scalar quantity, thus the derivative involves only the scalar components. Therefore, the derivatives of $[I]$ and $\delta\omega$ are evaluated as seen by the B frame. The derivative of the inertia matrix in the B frame is given by:

$$\frac{B}{dt}([I]) = \dot{\gamma}(J_s - J_t)(\hat{g}_s\hat{g}_t^T + \hat{g}_t\hat{g}_s^T) \quad (170)$$

To yield this result, the gimbal frame unit vectors derivatives are used as presented in Eq. (152).

The same approach as presented before in this thesis is used here to guarantee the stability of the closed-loop system. The rate of the Lyapunov function is forced to be negative semi-definite as:

$$\dot{\mathcal{V}} = -\delta\boldsymbol{\omega}^T[P]\delta\boldsymbol{\omega} \quad (171)$$

where $[P]$ is a positive definite gain matrix. This leads to the following stability constraint:

$$\delta\boldsymbol{\omega}^T \left([I] \frac{Bd}{dt}(\delta\boldsymbol{\omega}) + \frac{1}{2} \frac{Bd}{dt}([I])\delta\boldsymbol{\omega} + K\boldsymbol{\sigma}_e \right) = -\delta\boldsymbol{\omega}^T[P]\delta\boldsymbol{\omega} \quad (172)$$

$$[I] \frac{Bd}{dt}(\delta\boldsymbol{\omega}) = -K\boldsymbol{\sigma}_e - [P]\delta\boldsymbol{\omega} - \frac{1}{2} \frac{Bd}{dt}([I])\delta\boldsymbol{\omega} \quad (173)$$

It is necessary to substitute the equations of motions of the 3D pendulum with a VSCMG, presented in Eq. (163), the derivative of the inertia matrix, given by Eq. (170), into (173). After algebraic manipulations, the stability constraint can be written as:

$$\begin{aligned} & J_s \dot{\Omega} \hat{\mathbf{g}}_s + J_g \ddot{\gamma} \hat{\mathbf{g}}_g + \dot{\gamma} \left(J_s \Omega \hat{\mathbf{g}}_t + \frac{1}{2} (J_s - J_t) (\omega_t \hat{\mathbf{g}}_s + \omega_s \hat{\mathbf{g}}_t) + J_g (\omega_t \hat{\mathbf{g}}_s - \omega_s \hat{\mathbf{g}}_t) \right) + \\ & + \dot{\gamma} \left(\frac{1}{2} (J_s - J_t) (\hat{\mathbf{g}}_s \hat{\mathbf{g}}_t^T \boldsymbol{\omega}_r + \hat{\mathbf{g}}_t \hat{\mathbf{g}}_s^T \boldsymbol{\omega}_r) \right) = K\boldsymbol{\sigma}_e + [P]\delta\boldsymbol{\omega} + \mathbf{L} - [\tilde{\omega}][I]\boldsymbol{\omega} - [I](\dot{\boldsymbol{\omega}}_r - [\tilde{\omega}]\boldsymbol{\omega}) + \\ & - J_s (\Omega \omega_g \hat{\mathbf{g}}_t - \Omega \omega_t \hat{\mathbf{g}}_g) \end{aligned} \quad (174)$$

For a better and simpler representation, the following matrices are defined:

$$[D_0] = [J_s \hat{\mathbf{g}}_s] \quad (175)$$

$$[D_1] = \left[J_s \left((\Omega + \frac{1}{2} \omega_s) \hat{\mathbf{g}}_t + \frac{1}{2} \omega_t \hat{\mathbf{g}}_s \right) \right] \quad (176)$$

$$[D_2] = \left[\frac{1}{2} J_t (\omega_t \hat{\mathbf{g}}_s + \omega_s \hat{\mathbf{g}}_t) \right] \quad (177)$$

$$[D_3] = [J_g (\omega_t \hat{\mathbf{g}}_s - \omega_s \hat{\mathbf{g}}_t)] \quad (178)$$

$$[D_4] = \left[\frac{1}{2} (J_s - J_t) (\hat{\mathbf{g}}_s \hat{\mathbf{g}}_t^T + \hat{\mathbf{g}}_t \hat{\mathbf{g}}_s^T) \boldsymbol{\omega}_r \right] \quad (179)$$

$$[B] = [J_g \hat{\mathbf{g}}_g] \quad (180)$$

Thus, the stability constraint in a compact form as done in Schaub and Junkins (2014) is:

$$[D_0]\dot{\Omega} + [B]\ddot{\gamma} + [D]\dot{\gamma} = \mathbf{L}_r \quad (181)$$

where $[D] = [D_1] - [D_2] + [D_3] + [D_4]$ and the required torque vector $\mathbf{L}_r$ is given by:

$$\mathbf{L}_r = K\boldsymbol{\sigma}_e + [P]\delta\boldsymbol{\omega} + \mathbf{L} - [\tilde{\omega}][I]\boldsymbol{\omega} - [I](\dot{\boldsymbol{\omega}}_r - [\tilde{\omega}]\boldsymbol{\omega}) - J_s(\Omega\omega_g\hat{\mathbf{g}}_t - \Omega\omega_t\hat{\mathbf{g}}_g) \quad (182)$$

The proof of stability of this control law is also analogous to the one presented in (SCHAUB; JUNKINS, 2014). Note that the Lyapunov function rate was set to be only negative semi-definite, thereby, from the stability constraint, at this stage, we can infer that the proposed control scheme only assures global stability in the sense of Lyapunov. To assure that the control law proposed is global asymptotic stabilizing the system, the higher order derivatives of the Lyapunov function are investigated as done previously using the theorem provided by (MUKHERJEE; CHEN, 1993). The theorem was presented in Chapter 4, section 4.1, where it is also used to prove the stability of the controller.

If $\delta\boldsymbol{\omega}$ is zero, then, $\dot{\mathcal{V}}$ is also zero. Evaluating the second order derivative $\ddot{\mathcal{V}}$ yields:

$$\ddot{\mathcal{V}} = -2\delta\boldsymbol{\omega}^T[P]\delta\dot{\boldsymbol{\omega}} \quad (183)$$

Analyzing Eq. (183) when $\delta\boldsymbol{\omega} = 0$, it yields $\ddot{\mathcal{V}} = 0$ and nothing can be concluded regarding the stability. Thus, the third derivative is evaluated:

$$\ddot{\mathcal{V}} = -2\delta\boldsymbol{\omega}^T[P]\delta\ddot{\boldsymbol{\omega}} - 2\delta\dot{\boldsymbol{\omega}}^T[P]\delta\dot{\boldsymbol{\omega}} \quad (184)$$

For the sake of simplicity, $\delta\dot{\boldsymbol{\omega}}$ is $\frac{B}{dt}(\delta\boldsymbol{\omega})$ and from Eq. (173):

$$\delta\dot{\boldsymbol{\omega}} = -[I]^{-1}K\boldsymbol{\sigma}_e \quad (185)$$

and substituting this relation into Eq. (184) when $\delta\boldsymbol{\omega} = 0$ yields:

$$\ddot{\mathcal{V}}(\sigma_e, \delta\boldsymbol{\omega} = 0) = -2K^2\boldsymbol{\sigma}_e^T[I]^{-1}[P][I]^{-1}\boldsymbol{\sigma}_e \quad (186)$$

which is negative definite once $[I]$ and $[P]$ are positive definite matrices. Thus, the global asymptotic stability is guaranteed by (MUKHERJEE; CHEN, 1993).

However, the main goal is to control the a spherical pendulum with a VSCMG in the

inverted position, i.e., an stationary attitude. Thus, the reference for the angular velocity is always zero. This introduces some simplifications to the stability constraint that can be rewritten in compact form as:

$$[D_0]\dot{\Omega} + [B]\ddot{\gamma} + [D_1]\dot{\gamma} = \mathbf{L}_r \quad (187)$$

where $[D] = [D_1] - [D_2]$. $[D_0]$ and $[B]$ are the same ones shown before. The matrices $[D_1]$ and $[D_2]$ are simplified to:

$$[D_1] = [J_s(\Omega + \omega_s)\hat{\mathbf{g}}_t] \quad (188)$$

$$[D_2] = [J_t\omega_s\hat{\mathbf{g}}_t] \quad (189)$$

The proof of stability also holds for this case.

Control law

The motor torques of the reaction wheel and of the gimbal do not appear directly on the stability constraint. Only the rate and acceleration of the gimbal, and reaction wheel acceleration appear. To include the motor torques in the stability constraint would yield equivalent control laws. However, some undesirable terms would show up that provides excessive gimbal rates. Besides, CMGs usually requires the gimbal rate as input (SCHAUB; JUNKINS, 2014). The objective is to use favorably the torque produced by the reaction wheel and the torque amplification feature that the gyroscopic torque produces when the reaction wheel is gimbaled.

To do so, the term $[B]\ddot{\gamma}$ is dropped to avoid torque being generated mainly through this term, since that in the $[B]$ term, the inertia J_g is generally smaller than the ones in the spin direction J_s . This leads to higher gimbal angular accelerations $\ddot{\gamma}$, which would make the CMG works as a reaction wheel and the torque amplification effect that the CMG provides would not be deeply exploited. Thereby, the stability constraint can be written as:

$$[D_0]\dot{\Omega} + [D_1]\dot{\gamma} = \mathbf{L}_r \quad (190)$$

Again, to simplify the notation, the state vector $\boldsymbol{\eta}$ is introduced:

$$\boldsymbol{\eta} = \begin{Bmatrix} \Omega \\ \gamma \end{Bmatrix} \quad (191)$$

and also the matrix $[Q] = [D_0 \ D_1]$. Thus, Eq. (190) can be written as:

$$[Q]\dot{\boldsymbol{\eta}} = \mathbf{L}_r \quad (192)$$

Equation (192) is the control law that yields the kinematic directive for the VSCMG and some of the traditional CMG singularities can be avoided when the reaction wheel angular velocity is changed. Once more, the weighted pseudo-inverse is recommended to be used as done in (SCHAUB *et al.*, 1998; SCHAUB; JUNKINS, 2014):

$$\dot{\boldsymbol{\eta}} = [W][Q]^T ([Q][W][Q]^T)^{-1} \mathbf{L}_r \quad (193)$$

where

$$[W] = \begin{bmatrix} W_s & 0 \\ 0 & W_g \end{bmatrix} \quad (194)$$

where W_s is the weight of the reaction wheel mode and W_g is the weight of the gimbal mode. None specific criterion is used here, the designer should choose these weights, thus, allowing to specify how active each control is. For instance, the control mode of the reaction wheel can be turned off and the reaction wheel speed held constant. Therefore, the VSCMG should act as a traditional CMG, and it would be used to control the 3D pendulum.

The control law in the way shown here does not use any mechanism to avoid CMG singularities or to verify if the CMG configuration is close to a singularity. The only feature exploited is that if the CMG can not exert any torque to the pendulum, the reaction wheel can produce torque since the spin up or down of it is allowed in the VSCMG mechanism.

Torque level control

As stated before, the motor torques of the gimbal and of the reaction wheel do not appear in the control law designed. Therefore, the steering law's sub servo control laws must be investigated to yield a torque-level control command once the steering law provides only kinematic directive. Thereby, the servo control for the CMG and for the reaction is developed.

Consider the motor torque equation of the gimbal presented in Eq. (160). A simplification is made to the motor torque equation of the gimbal, since $\dot{\boldsymbol{\omega}}$ is usually much smaller than the gimbal rates, this term is dropped, reducing the complexity of the

servo control once $\dot{\omega}$ and $\ddot{\gamma}$ would have to be solved together to provide the gimbal motor torque. Thus, the gimbal motor torque equation is simplified to:

$$u_g \approx J_g \ddot{\gamma} - (J_s - J_t) \omega_s \omega_t - I_{Ws} \Omega \omega_t \quad (195)$$

To evaluate u_g , the gimbal angular acceleration is needed. The steering law provides the desired gimbal rate $\dot{\gamma}_d$ that when numerically differentiated yields $\ddot{\gamma}_d$. Thus, stable servo dynamics is achieved developing a feedback control to generate an expression for $\ddot{\gamma}$. So, the servo control for the gimbal have to determine $\ddot{\gamma}$ while $\dot{\gamma}$ approaches $\dot{\gamma}_d$. The following Lyapunov function is used:

$$\mathcal{V}(\Delta\dot{\gamma}) = \frac{1}{2} \Delta\dot{\gamma}^2 \quad (196)$$

is a positive definite and the tracking error is $\Delta\dot{\gamma} = \dot{\gamma} - \dot{\gamma}_d$. As done before, the derivative of the Lyapunov function is evaluated and forced to be negative definite yields:

$$\dot{\mathcal{V}} = \Delta\dot{\gamma} \delta\ddot{\gamma} = -K_\gamma \Delta\dot{\gamma}^2 \quad (197)$$

where K_γ is a positive gain. Thus, the expression for the gimbal angular velocity is given by:

$$\ddot{\gamma} = \ddot{\gamma}_d - K_\gamma \Delta\dot{\gamma} \quad (198)$$

This expression is then substituted into Eq.(195) leading to the gimbal rate servo control torque:

$$u_g = J_g (\ddot{\gamma}_d - K_\gamma \Delta\dot{\gamma}) - (J_s - J_t) \omega_s \omega_t - I_{Ws} \Omega \omega_t \quad (199)$$

The servo control for the reaction wheel is developed analogously. Consider the reaction wheel motor equation shown in Eq.(157). The same arguments are used to simplify the motor torque equation of the reaction wheel to:

$$u_s = I_{ws} (\dot{\Omega} + \dot{\gamma} \omega_t) \quad (200)$$

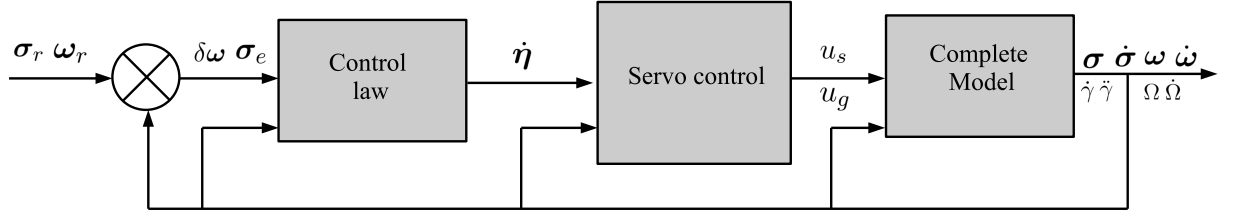
Carrying out the same procedure, the following reaction wheel torque control is obtained:

$$u_s = I_{ws} (\dot{\Omega}_d - K_\Omega \Delta\Omega + \dot{\gamma} \omega_t) \quad (201)$$

where K_Ω is a positive gain and the velocity tracking error is $\Delta\Omega = \Omega - \Omega_d$.

An important remark is that the servo control needs to be faster than the steering control law, i.e., it needs to be updated at a higher frequency. Figure 55 illustrates the block diagram of the complete control set-up. With the reference and actual rates, the relative attitude and the body angular velocity error are evaluated. These two quantities and the actual states of the 3D pendulum with the VSCMG are used as inputs for the steering law. At this stage, the designer has already decided which are the weights for the gimbal and for the reaction wheel. Thus, the control law calculates the desired gimbal rate $\dot{\gamma}_d$ and $\dot{\Omega}_d$ that constitutes $\dot{\boldsymbol{\eta}}$. Then, the servo control law computes the motor torques for the gimbal and for the reaction which are necessary for the complete model integration. Afterwards, the states are fed back and the process is restarted.

Figure 55 – Flowchart of the control loop of the spherical pendulum with a VSCMG



Source: Prepared by the author.

5.2.3 Results for the control fo the 3d pendulum with VSCMG

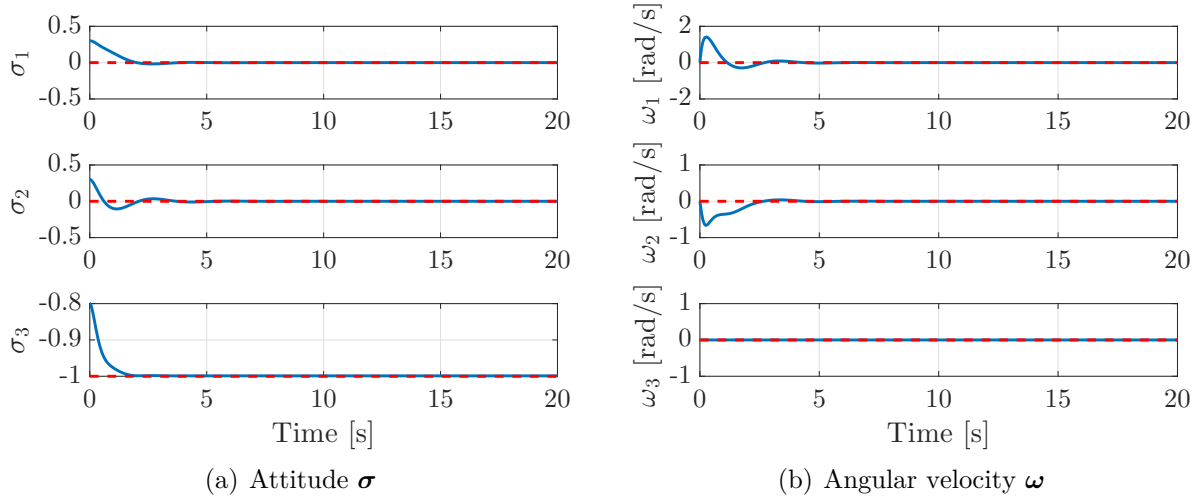
This section presents the numerical results obtained when controlling the spherical pendulum with a VSCMG. The following initial conditions were used $\boldsymbol{\sigma}(0) = (0.3, 0.3, -0.8)^T$, $\boldsymbol{\omega}(0) = (0, 0, 0)^T$, $\gamma(0) = 30^\circ$, $\dot{\gamma}(0) = 0$, and $\Omega(0) = 72$ rad/s. The mass of the pendulum is $m = 30.5$ kg and the inertia matrices of the pendulum and of the VSCMG are:

$$[I_s] = \begin{bmatrix} 5.3 & 0 & 0 \\ 0 & 5.3 & 0 \\ 0 & 0 & 14.5 \end{bmatrix} \quad {}^G[J] = \begin{bmatrix} 0.13 & 0 & 0 \\ 0 & 0.04 & 0 \\ 0 & 0 & 0.03 \end{bmatrix} \quad (202)$$

The gains used for the controller are $K = 400$, $[P] = \text{diag}(45)$, $K_\gamma = 2$, and $K_\Omega = 1$. To avoid non physical motions, i.e., motions that might be impossible to reproduce in practical experiments, we chosen to release the pendulum close to the inverted position, taking into account the possible limitations of a future experimental set-up. The interest of studying such pendulum configuration is related to the control of humanoid robots. Thus, it is more appropriate to chose the initial conditions as done here.

Figure 56 depicts the attitude and the angular velocity of the spherical pendulum with a VSCMG. The desired reference for the attitude is $\sigma_{R/N} = \{0, 0, -1\}^T$ which denotes the inverted position, and for the body angular velocity $\omega_r = 0, 0, 0^T$. Note that the controller designed can take the pendulum to the desired position in a few seconds.

Figure 56 – Attitude and angular velocity to control the spherical pendulum with VSCMG in the inverted position. --- actual rates and - - - reference.

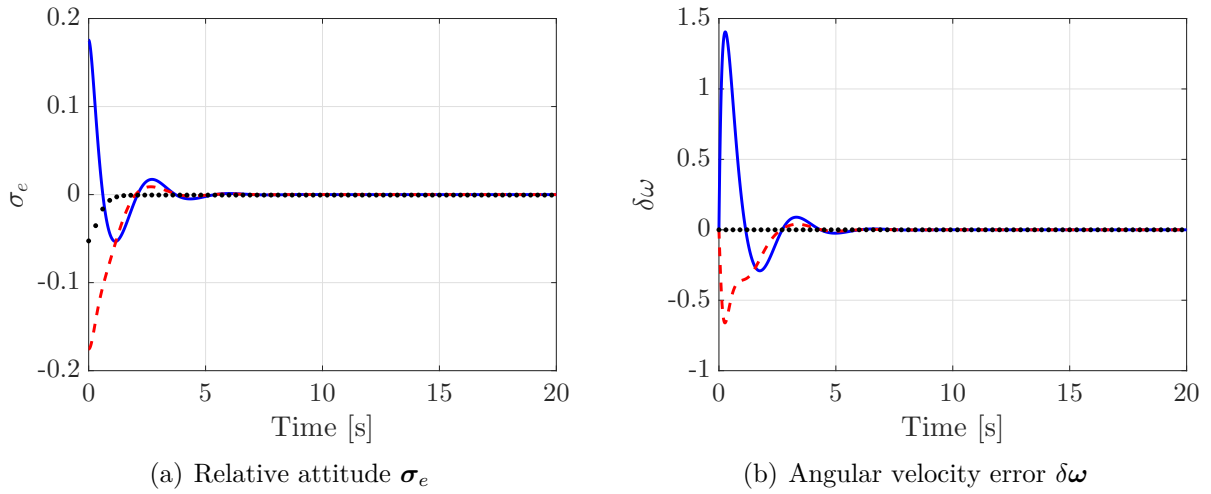


Source: Prepared by the author.

Figure 57 shows the relative attitude and the body angular velocity error. The controller goal is to take both quantities to zero. Note that this task is accomplished a little bit after 5 seconds. The controller designed demonstrates to have a very good performance.

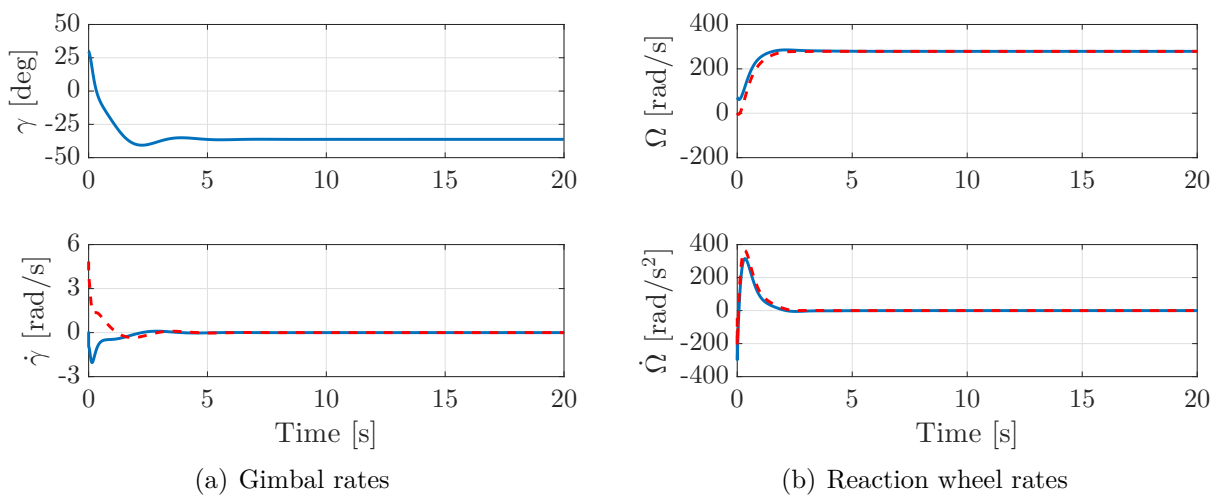
Moreover, figure 58 exhibits the rates for the gimbal and for the reaction wheel. Note that the gimbal motion is smooth and the reaction wheel is accelerated while the gimbal is also actuated. As aforementioned, the goal is to use the gyroscopic torque combined with the reaction wheel torque. This figure helps us to visualize such phenomena. Besides, the reaction wheel works within reasonable rates, i.e., these rates can be reproduced in a practical experiment.

Figure 57 – Relative attitude and angular velocity error to control the spherical pendulum with VSCMG in the inverted position. --- denotes σ_{e1} and $\delta\omega_1$, - - - reference denotes σ_{e2} and $\delta\omega_2$ and ... denotes σ_{e3} and $\delta\omega_3$.



Source: Prepared by the author.

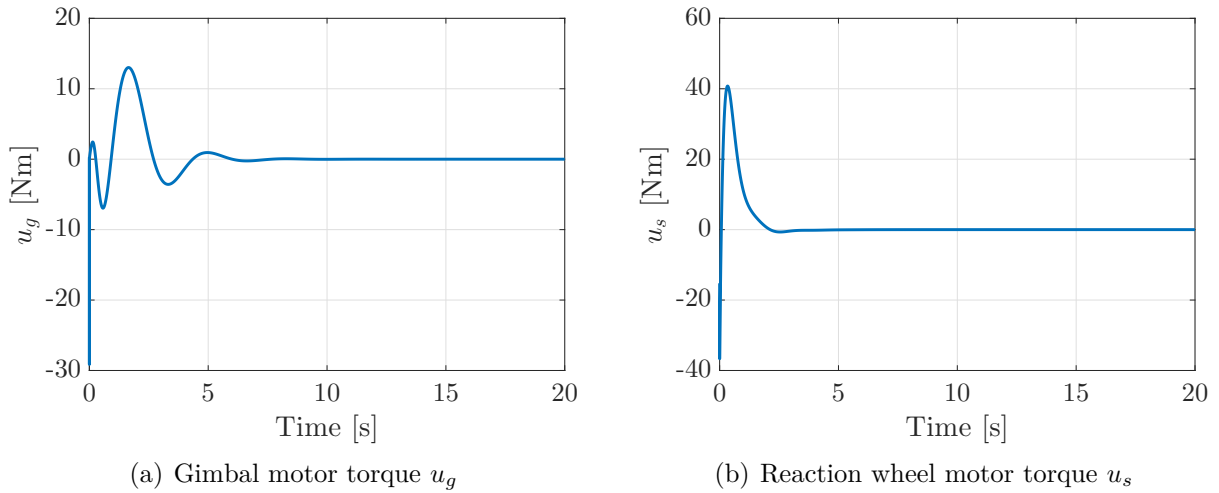
Figure 58 – Gimbal and reaction wheel rates to control the spherical pendulum with VSCMG in the inverted position. --- actual rates and - - - desired rates $\dot{\gamma}_d$, Ω_d , $\dot{\Omega}_d$.



Source: Prepared by the author.

Figure 59 illustrates the motor torques obtained for the gimbal and the reaction wheel. Although large torques are demanded at the beginning of the simulation, it is imperative to take into account that the body, i.e, the pendulum has a large mass and a large inertia. The controller also has to deal with the torque caused by gravity. Thus, the motor torques of the gimbal and reaction wheel also yield reasonable values.

Figure 59 – Motor torques of the gimbal and of the reaction wheel to control the spherical pendulum with VSCMG in the inverted position.



Source: Prepared by the author.

5.3 FINAL REMARKS

This chapter presented a new configuration of a pendulum with a VSCMG mechanism stabilized in the inverted position. A nonlinear controller using the Lyapunov control theory was designed to perform the position control of this new configuration. Thus, the use of two control actions, gyroscopic torque and the reaction wheel's torque, was beneficial to drive the pendulum from the downward position to its upward position, thus controlling it in the unstable equilibrium point. Additionally, when this new system was subjected to an external disturbance, the controller designed was able to return the pendulum to the desired position by employing both control actions. Also, the reaction wheel mode is more used than the CMG mode because the pendulum motion is restricted to the plane. Moreover, the comparison between the VSCMG pendulum with a well-established configuration in the literature, the reaction wheel pendulum, was carried out. The comparison results have shown that for the cases studied in this chapter, the proposed pendulum configuration had a better performance.

Furthermore, this chapter has also introduced the complete modeling of a 3D pendulum with a VSCMG. The objective was to obtain a nonsingular model for a spherical pendulum controlled using a VSCMG. This was done after common assumptions were used. The main assumptions are to provide an axis-symmetric inertia matrix for the 3D pendulum and to allow no rotation around $\hat{\mathbf{b}}_3$, i.e., $\omega_3 = 0$ which leads to $\dot{\omega}_3 = 0$. A classical nonlinear controller was also developed. The control law designed assures global asymptotic stability. Furthermore, the numerical results presented reasonable rates and they are unique, since a pendulum configuration like this one has never been studied before to the best of the author's knowledge. The pendulum configuration proposed needs a deeper investigation considering that the 3D pendulum configuration already presented by the literature let some points open such as how to actuate differently these systems and to consider them underactuated. Since this is the first time such pendulum configuration is controlled using a VSCMG, a classical approach was used. Future work includes the design of other control techniques.

6 FINAL REMARKS

This chapter presents the concluding remarks of this thesis, the contributions to the literature and the propositions for future work.

6.1 CONCLUSIONS

This thesis has proposed three new pendulum configurations; the pendulum with two reaction wheels, the variable speed control moment gyroscope pendulum; and the spherical pendulum with a variable speed control moment gyroscope.

The first new pendulum configuration studied was the pendulum with two reaction wheels (2-RWP). After the models and the experimental considerations were carried out, the comparison of using one or two actuators to drive and control an inverted pendulum using a simple PID controller was performed, depicting the first main contribution of the thesis. The new configuration proposed was compared to a classical one, the reaction wheel pendulum (RWP). This result illustrated that two actuators become much easier to drive and control the pendulum in the inverted position.

Furthermore, still regarding the pendulum with two reaction wheels, this thesis reported the design and implementation of nonlinear controllers for it using low-cost hardware. A nonlinear proportional-derivative controller was derived from Lyapunov theory, and the asymptotic stability of the proposed control law was verified using the higher-order derivative of the candidate to Lyapunov function. Also, a sliding mode controller based on a sliding function was designed. These controllers had much better performance than the one designed previously. The sliding mode controller that can deal with parametric uncertainties had the best result considering that many parameters of the plant are known with limited accuracy. This was confirmed by analyzing performance indices in which the SMC obtained the smaller ones. Besides, the use of low-cost hardware in plants that were self-manufactured has presented to be efficient. Additionally, the design of a swing-up control strategy that considers the plant limitations and uses the

well-known properties of the sliding mode control favorably was conducted and presented good results.

Moreover, the variable speed control moment gyroscope pendulum was introduced by this thesis. This pendulum configuration is a variation of the classical reaction wheel pendulum. In this case, the reaction wheel can be gimballed, producing torque from the angular momentum's amplitude and direction variation. The model and the nonlinear control of this new pendulum configuration were proposed. Additionally, this configuration was compared to the classical reaction wheel pendulum. For the situations investigated by this thesis, the results have shown that the VSCMG pendulum had a better outcome verified by the indices of performance.

Besides, the model of the 3D pendulum with a VSCMG is derived. The modified Rodrigues parameters are used to describe this pendulum configuration's attitude to yield nonsingular dynamics. The shadow set property allows this attitude description to provide a three-dimensional nonsingular description. The assumptions used to yield a spherical pendulum from the 3D pendulum model are also presented. Additionally, the controller design's is carried out, and the results demonstrated adequate values for the control of the new pendulum configuration in the inverted position.

6.2 CONTRIBUTIONS TO THE LITERATURE

The main contribution of this thesis to the literature was the proposition of new pendulum configurations. Linear and nonlinear controllers were designed for the configurations studied and implemented in real-time using low-cost hardware. Some of the results were published in the following journal articles:

- Trentin *et al.* (2020a): " Attitude control of inverted pendulums using reaction wheels: Comparison between using one and two actuators".
- Trentin, Silva and Schaub (2019b): "Variable Speed Control Moment Gyroscope in an Inverted Pendulum."
- Trentin *et al.* (2020b) "Inverted Pendulum Nonlinear Controllers Using two Reaction Wheels: Design and Implementation".

And in the following conferences:

- Trentin, Silva and Abreu (2016): "Controle de posição de um Pêndulo com Roda de Reação";
- Trentin, Abreu and Silva (2016): "Controle Não Linear de um Pêndulo com Roda de Reação";
- Trentin *et al.* (2017): "Attitude Control of a Pendulum using Two Reaction Wheels";
- Cenale, Trentin and Silva (2017): "Evitando Saturação no Controle de Posição de Pêndulo com Duas Rodas de Reação";
- Trentin, Cenale and Silva (2018): "Real-time Control of a Reaction Wheel Pendulum using a Low-cost Hardware";
- Trentin, Silva and Schaub (2018): "Nonlinear Control of an Inverted Pendulum using Two Reaction Wheels";
- Trentin, Silva and Schaub (2019a): "Inverted Pendulum Control via Variable Speed Control Moment Gyroscope".

6.3 FUTURE WORK

The next steps of the research are:

- To design and implement a fuzzy-LMI controller for the 2-RWP considering uncertainties and to compare it to the sliding mode controller. To do so, the experimental device needs to be rearranged where new DC motors will be bought with encoders. In this way, all the states would be known;
- To finish the construction of the Variable Speed Control Moment Gyroscope mechanism after assembling precisely the best hardware to use and to see if it is possible to build such a complex system using a low-cost platform;
- To design a swing-up control strategy for the VSCMG pendulum combining it with sliding mode control;
- To study more deeply the new 3D pendulum with VSCMG comparing it to the situation where it is used only the CMG, and carry out the tracking control for this configuration.

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