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The issue of time in Quantum Mechanics

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Abstract

Time is everywhere. A concept so ubiquitous and natural for us, human beings, that we rarely wonder about its nature. For science, there is a clear need of understanding more and more about the fundamental structure of the world around us, so that an understanding about the concept of time surely is a noble target for physics. In this essay, we explore the concept of time from the perspective of two conflicting paradigms: the relativist and the quantum one. If on the one hand General Relativity allows to characterize time as that measured by *honest clocks* — pointwise apparatus that neither delay or anticipate irrespective of its past history —, on the other the laws of Quantum Mechanics forbid its existence in nature. Our proposal is to furnish a precise understanding about the reason for such prohibition, as well as a solid perspective about the role of time in quantum theory as a measurable quantity.

Key-words: Time, Quantum Mechanics, Relativity

Field of knowledge: Quantum Fundamentals

Resumo

O tempo está em todo lugar. Um conceito tão onipresente e natural para nós, seres humanos, que raramente indagamos sobre a sua natureza. Para a ciência, existe a necessidade clara de se entender cada vez mais sobre a estrutura fundamental do mundo a nossa volta, de modo que um entendimento completo sobre o tempo certamente é um objetivo nobre para a Física. Nesta dissertação, investigamos o conceito de tempo a partir da perspectiva de dois paradigmas fundamentalmente conflitantes: o relativístico e o quântico. Se por um lado a Relatividade Geral permite caracterizar tempo como aquilo medido por *relógios honestos* — aparatos pontuais que não atrasam e nem adiantam independendemente de suas histórias passadas —, por outro, as leis da Mecânica Quântica proibem a existência dos mesmos na natureza. Nossa proposta é fornecer um entendimento preciso da razão de tal proibição, bem como uma perspectiva sólida do papel do tempo na teoria quântica enquanto uma quantidade passível de medição.

Palavras-chaves: Tempo, Mecânica Quântica, Relatividade

Áreas do conhecimento: Fundamentos da teoria quântica

Contents

List of Figures

Summary of notations and conventions

In this work, we use natural units, setting $c = \hbar = G = 1$.

The spacetime metric has signature $(-,+,+,+)$

- $(\mathcal{M}, g)$: spacetime with metric tensor g
- $\mathcal{T}_p\mathcal{M}$: tangent space at the point p.
- Einstein summation convention is used, with greek indices running from 0 to 3 and latin ones from 1 to 3. Examples:
  - . $T_{\mu\nu}v^{\nu} = \sum_{\nu=0}^{4} T_{\mu\nu}v^{\nu}$
  - . $x_i x^i = \sum_{i=1}^{3} x_i x^i$
  - . $R_{\mu\nu}R^{\mu\nu} = \sum_{\mu=0}^{3}\sum_{\nu=0}^{3} R_{\mu\nu}R^{\mu\nu}$
- $\mathcal{H}$: complex Hilbert space
- $\mathcal{L}(\mathcal{H})$: set of all linear operators acting on $\mathcal{H}$. We distinguish operators by using a "hat": $\hat{A}$, $\hat{B}$, etc.
- $\mathcal{L}(\mathcal{H})^+$: the subset of $\mathcal{L}(\mathcal{H})$ containing only the positive operators.
- $\hat{A}^\dagger$: adjoint of the operator $\hat{A}$

- $L^2([a,b])$: Hilbert space of the square-integrable functions defined over the interval $[a,b]$.

Introduction

The difference between past, present and future is a mere illusion.

Time is everywhere. No matter who you are or where you go, it is always with you, a silent and unceasing flow. A concept so ubiquitous and natural for human beings that we rarely wonder about it, except when we face it wickedness on continuing to push us forward to our unavoidable destiny. However, when we are asked about *what is time*, surely no trivial and short answer readily comes out. As well grasped by Saint Augustine [1]: "*What then is time? If no one asks me, I know; if I want to explain it to a questioner, I do not know*".

Perhaps our perception of time is inseparable from that of change. Systems in nature are in constant change, and those events that happen in a periodic manner — like the sunrise — can be used to mark the passage of time. In this scenario, one could think about time as a human abstraction created to describe succession of *events*, like the successive positions assumed by the hand of a clock.

When it come to physics, time is present in almost every description of natural phenomena.[2] Before the advent of Einstein's theory of General Relativity [3], the physical theories were shaped within the *absolute time* paradigm coined by Isaac Newton (1643 - 1727), absolutely befitting with our human intuition. The beginning of the XX century brought the rupture with this paradigm; the concepts of space and time were unified within a more fun-

[1]Saint Augustine (354 - 430) was a remarkable philosopher for the Christianity. Among his writings, there are reflections about the nature of time.

[2]If you are a physicist, I bet you to think about what was the largest *time* you worked without using the concept of time.

[3]Albert Einstein (1879-1955).

damental underlying structure: the *spacetime*. This new abstract structure was posed as the "background" where all physical theories should be defined.

In general lines, the message of General Relativity is that "*different observers extract different space and time contents from the same spacetime*."[4] The effect of gravity is to change the spacetime geometrical features, defining in an univocal manner how different *events* (spacetime points without any temporal or spatial extension) relate to each other and, thence, the values of the physical quantities that different observers will measure when performing certain experiments.

Within the classical physics paradigm — which considers General Relativity — it is possible to supply an objective definition for the concept of time: *Time is what honest clocks measure*. At first sight, such a definition may seem circular. However, since physics is a experimental science, it does not make sense to define an observable without telling how to measure it; in this sense, there is no problem to outscore the definition of time for that of an honest clock, provided we give a clear prescription of such apparatus using the variables of the theory.

Honest clocks are pointwise apparatus that irrespective of its past histories attribute the same real number to each pair of events arbitrarily close they visit on spacetime. As we shall argue, there is no problem with this definition within the scenario of General Relativity.

The issues come out when one realizes the honest clocks, while apparatus that supposedly exist in nature, must ultimately be subject to the laws of Quantum Mechanics. Thus, we have the following questions: How Quantum Mechanics impacts on the physical reliability of honest clocks? If, somehow, Quantum Mechanics forbids the existence of honest clocks, how one can use the theory to predict the time interval between two events? Furthermore, it is important to realize that insofar as these questions explore the concept of time in the context of both quantum and relativity theories, they lead us to pursue an even more fundamental scientific problem: what would be time in a (real) quantum theory of gravity?

[4] I must admit that this statement, which is a kind of catchphrase of my advisor, is perhaps the most meaningful way of talking about Relativity in just one line.

This essay is organized as follows: Chapter 1 is devoted to the definition of time according the the Classical Physics paradigm. After a brief review about some fundamental aspects of General Relativity, we define the concept of an honest clock and, then, formulate the main issue of the essay: *How Quantum Mechanics impacts on the definition of honest clocks*? For the reader who is not familiar with General Relativity and is only interested in Quantum Mechanics, we stress that this first chapter can be safely skipped without any loss to the understanding of the main topic.

Chapter 2 contains a review on the subject of Quantum Mechanics, focusing on the extended formulation of observables as described by Positive Operator Value Measures (POVMs).

In chapter 3, we begin our analysis on the issue of time in Quantum Mechanics through a discussion about formulations and interpretations of the time-energy uncertainty relation.

Chapter 4 is concerned with a celebrated no-go theorem due to Pauli that points out the impossibility of constructing a self-adjoint operator canonically conjugated to the Hamiltonian of any physical system. However, given the interpretation of observables as described most generally by POVMs, we show how to use such formalism to construct a time observable in the quantum realm for a particular — but rather interesting — case. Finally, we discuss a fundamental result due to Wald and Unruh that *rules out the realizability of honest clocks in Quantum Mechanics.*

In the last chapter, we address the issue of using particular quantum systems as quantum clocks. The conclusion then follows.

1

Time in classical physics

This first chapter is concerned with a precise definition for the concept of time within the classical physics paradigm. By "classical physics", we mean all theories that do not take Quantum Mechanics into account. Among these theories, General Relativity surely is the one that — until now — best describes the fundamental notions of space and time. Thus, we begin by giving an overview about General Relativity, focusing on the conceptual features that will be crucial for defining time itself.

Since the present work focuses on issues arising from Quantum Mechanics, the discussion about General Relativity will be brief. For those willing to read more about the subject of General Relativity and its underlying mathematical structure, we recommend references [1] and [2].

1.1 The spacetime of General Relativity

The spacetime is the background where all the physical theories are to be defined. Before the advent of General Relativity, all classical physics can be seen as taking place in the *Galileo spacetime*, where time is an absolute entity. Different observers could disagree about space measurements — what is really intuitively based on our own daily experience — but never about time measurements.

This paradigm is changed when Albert Einstein (1879 - 1955) introduces the special theory of Relativity in the year of 1905, thereby showing that the *Minkowski spacetime* is the "correct" background to describe all physical phenomena in the absence of gravity. Later on 1915, General Relativity would teach us how gravity can be understood as the own geometric structure of spacetime. In this new scenario, *different observers extract different space and time contents from the same spacetime.* Both space and time measurements outcomes are dependent on the observer who is asking the experimental questions. As a consequence, notions like simultaneity, past and future — which seem to be absolute based on our everyday experiences — also need to be reformulated in an observer-dependent way.

Formally, the spacetime of General Relativity is a four-dimensional pseudo-Riemannian manifold $\mathcal{M}$ equipped with a non-degenerated bi-linear form g, the so-called *metric tensor*. The set $\mathcal{M}$, which has the usual structure of a differentiable manifold,[1] is the collection of the spacetime *events*, whereas the mathematical object g is what allows one to ascribe a real number— the *distance* — between each pair of events, being understood as the ultimate description of gravity itself. The adjective "pseudo-Riemannian" comes from the fact that the metric g is not positive-definitive — in which case we would have a Riemaniann manifold —, thus meaning that distances between different events of $\mathcal{M}$ can be null or even negative.

Since distances between events on a spacetime cannot be interpreted as the usual "positive distances" of our everyday life Euclidian geometry, it is necessary to give a proper physical interpretation to the new possibilities unraveled when one uses all the tools of pseudo-Riemanninan geometry to describe the spacetime. Indeed, vectors in a spacetime can be divided into three classes, according to the value of its length: given a vector $v \in \mathcal{T}_p\mathcal{M}$, we say that

- v is *timelike* if $g(v, v) < 0$
- v is *spacelike* if $g(v, v) > 0$

[1] Intuitively, a differentiable manifold of dimension n is a space that locally behaves like $\mathbb{R}^n$, being the proper generalization to the concept of surfaces. If you want to review the mathematical definition of a manifold, see chapter 2 of reference [2].

- v is *null* if $g(v, v) = 0$

This same nomenclature applies to vector fields defined in an open set of $\mathcal{M}$. In particular, if v is the tangent vector field to some curve $\gamma : I \subset \mathbb{R} \to \mathcal{M}$, γ is to be named after the classification of v. Physically, such a classification is important to determine the causal structure of the spacetime: two events that can be joined by a timelike curve are causally connected. That is: if A, $B \in \mathcal{M}$ can be joined by a timelike curve and some observer says that A occurred before B, then *all* observes will agree on this causal relation. In this way, timelike curves are the *worldlines* of massive particles. On the other hand, events joined by a spacelike curve have no causal relation, so that no object — massive or not — can follow a spacelike trajectory on the spacetime. As for the null curves, they define the trajectories of the massless particles that travel at the speed of light.

In principle, it should be possible to discuss all relativity issues without using coordinates. After all, physical phenomena do not care about which coordinates one chooses to describe them. Nonetheless, one cannot deny that without coordinates the physicist task would be much harder[2] or even impossible. For this reason, we follow the common trend and introduce coordinates in order to discuss a few more issues in a clearer way.

Given a generic spacetime $(\mathcal{M}, g)$, one can always introduce a system of coordinates (x^μ) covering some open set $\mathcal{U} \subset \mathcal{M}$. Thus, the description of the metric tensor becomes

$$g = g_{\mu\nu} dx^\mu \otimes dx^\nu. \tag{1.1}$$

The components $g_{\mu\nu}$ are to be understood as real functions of the coordinates describing each point of $\mathcal{U}$, whereas the objects dx^μ, when considered at some point $p \in \mathcal{U}$, are the one-form basis for the cotangent space $T_p^*\mathcal{M}$. The corresponding coordinate basis for the tangent space $T_p\mathcal{M}$ is denoted by $\dfrac{\partial}{\partial x^\mu}$ (the evaluation point is omitted by the sake of simplicity), being fully characterized by the relation

[2] It is already hard enough!

$$dx^\mu \left(\frac{\partial}{\partial x^\nu} \right) = \delta^\mu_\nu. \tag{1.2}$$

When there is no room for ambiguities regarding the coordinate system being used, the coordinante basis vectors are denoted by ∂_μ.

In a more general fashion, a tensor field T defined over $\mathcal{U}$ can be written as

$$T = T^{\alpha_1 \dots \alpha_r}{}_{\beta_1 \dots \beta_s} \frac{\partial}{\partial x^{\alpha_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{\alpha_r}} \otimes dx^{\beta_1} \otimes \dots \otimes dx^{\beta_s}. \tag{1.3}$$

The actual description of the tensor T is then read from its components. In particular, the action of the metric over two vector fields $v = v^\mu \partial_\mu$ and $w = w^\mu \partial_\mu$ can be written as

$$g(v, w) = g_{\mu\nu} v^\mu w^\nu. \tag{1.4}$$

Within this formulation, the distance between arbitrarily close events is given by

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu. \tag{1.5}$$

This is the so-called *line element*. Usually, this object is referred as the metric itself. Of course, such abuse of nomenclature is no harmful: given the relation (1.5), we can extract all physical features from spacetime.

The actual dynamics of free particles — here understood as particles subject solely to the action of gravity — in the spacetime is described by *geodesics*, trajectories that minimize the line element (1.5). Let $\gamma(\lambda) = (x^\mu(\lambda))$ be a curve with affine parameter λ and tangent field

$$u^\mu = \frac{dx^\mu}{d\lambda}.$$

If γ represents a valid trajectory for a real particle on spacetime, it needs to be either timelike or null. In the first case we have $u^\mu u_\mu = -1$, whereas for null curves we have $u^\mu u_\mu = 0$. On both cases, u^μ is called the *four-velocity* of the particle.

One way to impose γ to be a geodesic is to demand that its *intrinsic acceleration* must be zero. By intrinsic acceleration, we mean the *covariant derivative* of the four-velocity $u = u^\mu \partial_\mu$ along the curve γ, namely

$$\nabla_u u = 0 \implies \frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu{}_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} = 0. \tag{1.6}$$

Here, the quantities $\Gamma^\mu{}_{\alpha\beta}$ are the Levi-Civita connection symbols and ∇ is the connection itself (see reference [2] if you want to review the subject of connections and covariant derivatives). In the usual formulation of General Relativity, the connection — an abstract rule that allows one to take derivatives of tensor fields on manifolds — chosen is the Levi-Civita's, the only that is symmetric and compatible with the metric tensor g. This is a derivative rule that is rather natural if one assumes that the manifold can be immersed in a large Euclidean space. In terms of the metric components, we have (with $\nabla_{\partial_\mu} \equiv \nabla_\mu$)

$$\nabla_\alpha \partial_\beta = \Gamma^\mu{}_{\alpha\beta} \partial_\mu \implies \Gamma^\mu{}_{\alpha\beta} = \frac{g^{\mu\sigma}}{2} \left(\partial_\alpha g_{\sigma\beta} + \partial_\beta g_{\alpha\sigma} - \partial_\sigma g_{\alpha\beta} \right). \tag{1.7}$$

Now, let us turn our attention to more fundamental issues regarding how an observer can extract space and time contents from spacetime.

Consider an observer characterized by the four-velocity u^μ. Any vector field $X = X^\mu \partial_\mu$ admits a decomposition of the form

$$X^\mu = \underbrace{\Upsilon^\mu{}_\nu X^\nu}_{\text{timelike vector}} + \underbrace{\Pi^\mu{}_\nu X^\mu}_{\text{spacelike vector}} , \tag{1.8}$$

where Υ and Π are the projectors onto the timelike and spacelike subspaces with respect to the observer u^μ. That is: we are decomposing the vector X into a portion that lies on what the observer calls *space* and another portion over what he calls *time*. Since such a construction is observer dependent, we can take it as the mathematical realization of the statement that different observers extract different space and time contents from the same spacetime.

The expression for the time projector Υ is

$$\Upsilon^{\mu}{}_{\nu} = -u^{\mu} u_{\nu}, \tag{1.9}$$

so that the spacelike part of X according to the observer with four-velocity u can be calculated through the relation

$$\Pi^{\mu}{}_{\nu} X^{\nu} = X^{\mu} + (u_{\nu} X^{\nu}) u^{\mu}. \tag{1.10}$$

A concept that is central to Relativity is that of *proper time*. Given two events A and B joined by a timelike geodesic on $\mathcal{M}$ whose coordinate description is $\gamma(\lambda) = (x^{\mu}(\lambda))$, with $A = \gamma(\lambda_1)$ and $B = \gamma(\lambda_2)$, the *proper time* between the events can be calculated as

$$\Delta\tau_{AB} = \int_{\lambda_1}^{\lambda_2} d\lambda \, \sqrt{-g_{\mu\nu}(\lambda) u^{\mu} u^{\nu}}, \tag{1.11}$$

where $u^{\mu} = \dfrac{dx^{\mu}}{d\lambda}$.

Notice that if λ is taken to be an affine parameter τ along γ, then the previous expression reduces to

$$\Delta\tau_{AB} = \tau_2 - \tau_1. \tag{1.12}$$

Proper time has a straightforward interpretation: if a free-falling observer carrying a clock follows the trajectory γ, the quantity $\Delta\tau_{AB}$ would be the time, *as told by his clock*, between the events A and B.

Nonetheless, any observer following a distinct trajectory on the spacetime would say that the time elapsed between events A and B is different. Indeed, an observer with four-velocity v^{μ} says that the time elapsed between the events A and B is

$$\Delta t_{AB} = \int_{\lambda_1}^{\lambda_2} d\lambda\ \sqrt{-g_{\mu\nu}(\Upsilon^{\mu}{}_{\alpha}v^{\alpha})(\Upsilon^{\nu}{}_{\beta}v^{\beta})} = \int_{\lambda_1}^{\lambda_2} d\lambda\ (v^{\alpha}u_{\alpha})\sqrt{-g_{\mu\nu}u^{\mu}u^{\nu}}. \tag{1.13}$$

For spacelike-separated events, one can always find some observer $\mathcal{O}$ that says the events are simultaneous. Mathematically, this amounts to choosing a coordinate description naturally associated with the rest frame of this observer, where both events lie on the same *spacelike surface* determined by fixing the value of x^0. In this case, those events can be used to define the *proper length* of some object. Indeed: given two events A and B linked by a spacelike geodesic $\gamma(\lambda) = (x^0, x^i(\lambda))$, with $A = \gamma(\lambda_1)$ and $B = \gamma(\lambda_2)$, the proper distance between them reads

$$\Delta L_{AB} = \int_{\lambda_1}^{\lambda_2} d\lambda\ \sqrt{g_{ij}(\lambda)\frac{dx^i}{d\lambda}\frac{dx^j}{d\lambda}}. \tag{1.14}$$

From the point of view of another observer, the events A and B may not be simultaneous. Indeed, let $w = \left(0, \frac{dx^i}{d\lambda}\right)$ and consider an observer $\mathcal{P}$ following a timelike worldline with four-velocity $v = v^{\mu}\partial_{\mu}$ such that

$$v^{\mu}w_{\mu} \neq 0.$$

In this case, the observer $\mathcal{K}$ believes that the events A and B are separated in time by an amount

$$\Delta t^{(\mathcal{K})}_{AB} = \int_{\lambda_1}^{\lambda_2} d\lambda\ (v_{\alpha}w^{\alpha})\sqrt{-g_{\mu\nu}v^{\mu}v^{\nu}}. \tag{1.15}$$

The simplest spacetime model is the Mikowski spacetime, whose line element in Cartesian coordinates reads

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2. \tag{1.16}$$

Both null and timelike geodesics in this spacetime are straight-lines. Curved lines represent non-geodesic (accelerated) observers. Below, we display a diagram representing some basic features of the Minkowski spacetime.

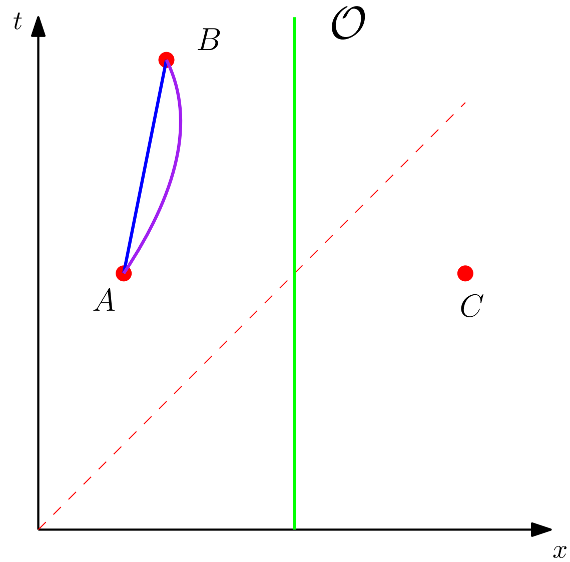

Figure 1.1: A representative diagram of the Minkowski spacetime with one spatial dimension, displayed using Cartesian coordinates where the observer $\mathcal{O}$ — green line — is at rest. The red dotted-line stands for the trajectory of a light-ray, which determines the causal structure of this spacetime. The blue and purple lines are timelike curves connecting events A and B, the former representing the trajectory of an inertial observer and the latter of an accelerated observer. The events A and C are spacelike-separated: they can happen simultaneously from the point of view of a specific inertial observer, but cannot have any absolute causal relation.

Such a spacetime is *flat*, meaning that it posses no intrinsic curvature. That is: if one computes all the components of the Riemman curvature tensor, one finds that they all vanish:

$$R_{\mu\nu\alpha\beta} = 0.$$

Physically, this means that this is a spacetime without gravity. Including gravity, the spacetime in the vicinity of a spherical massive object is well described by the *Schwarzschild metric*:

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2(\sin^2\theta d\theta^2 + d\phi^2) \tag{1.17}$$

Studying the timelike and null geodesics of this spacetime, it is possible to predict two important phenomena that are important tests for the theory: the perihelion precession of Mercury and the deflection of light by the sun.

Contrary to (1.16), the Schwarzschild metric describes a *curved spacetime*. Mathematically, this is represented by a non-vanish Riemann tensor, as outlined above.

Both Minkowski and Schwarzschild metrics are solutions of the Einstein's equations:

$$R_{\mu\nu} - \frac{R}{2} g_{\mu\nu} = 8\pi T_{\mu\nu}. \tag{1.18}$$

Also known as the field equations of General Relativity, the equations above relate the matter-energy content of the spacetime — described by the energy-momentum tensor $T_{\mu\nu}$ — with its intrinsic geometric strucuture. Recall that the Ricci tensor is the only independent contraction of the Riemman tensor; in components notation, we have

$$R_{\mu\nu} = R^{\alpha}{}_{\mu\alpha\nu}. \tag{1.19}$$

A fundamental point that is worth mentioning here is the *Equivalence Principle*, the General Relativity's conceptual cornerstone. Essentially, this principle states that the physics in a freely-falling frame is the same of special relativity. Formally, this principle has its mathematical counterpart in the existence of the so-called *Riemann normal coordinates* (see chapter 5 of reference [2] for more details). Indeed, around each point p of a generic spacetime $(\mathcal{M}, g)$, one can introduce a system of coordinates (n^{μ}) where, up to second order in the curvature, the metric components takes the form

$$g_{\mu\nu}(n^{\sigma}) = \eta_{\mu\nu}(p) + \frac{1}{3} R_{\mu\alpha\beta\nu}(p) n^{\alpha} n^{\beta} + \mathcal{O}(R^2), \tag{1.20}$$

where $\eta_{\mu\nu} = diag(-,+,+,+)$ is the Minkowski metric. That is: it is always possible describe the spacetime metric at a *particular point* as the Minkowski metric. Of course, locally a curved spacetime still remains distinguishable from a spacetime without gravity through experiments sensible to $R_{\alpha\beta\mu\nu}$.

Now, it is time to state our main issue. First of all, we claim that if somehow one has access to the value of the proper time (1.11) between every pair of events belonging to a generic spacetime $(\mathcal{M}, g)$, then one has a complete description of the geometric structure of the spacetime. That is: measuring the proper time between two events can, in principle, distinguish a particular spacetime from all other. This statement will be sustained on the next section, where we show how clocks can be used to also measure spacelike distances as well.

Next, it is important to recall that in physics, observables have no meaning at all if it is not possible to measure them. Then, concerning theoretical fundamentals, we need to introduce an apparatus that allows one to measure the proper time between two events belonging to a given spacetime. In doing so, notice that such an apparatus will allows one to *test the spacetime itself*, insofar as the recording of proper time measures will reveal the geometric structure of spacetime itself.

Before introducing the aforementioned apparatus, we need to convince the reader that there is a "hidden" asymmetry between space and time concepts on a classical spacetime, making time measurements more fundamental — in a sense to be specified afterwards — than spatial ones.

1.2 Using clocks to measure spatial distances

There is no doubt that General Relativity treats space and time in a unified picture. Nonetheless, this does not mean that they have the same nature. Indeed, the fact all geometrical features are determined by a pseudo-Riemannian metric with signature $(-,+,+,+)$

already indicates that all the four spacetime dimensions cannot be interpreted as sharing the same features. The own distinction between timelike and spacelike distances also made clear the difference regarding space and time.

Now, we are going to argue that everything we need to extract space and time measurements from the spacetime is *a clock*. At this point, the shall call a clock any pointwise apparatus that allows one to measure the proper time along some timelike trajectory. Notice that the hypothesis of being pointwise is necessary in order to assure that the clock follows this same trajectory. Later on, we shall make this notion more precise introduction the concept of an *honest clock*.

In the Minkowski spacetime, consider the following experiment: two observers, Jonas and Martha, are at rest in the same inertial frame and located a distance d apart. Consider a system of Cartesian coordinates (x, t) such that $x = 0$ is Jonas's position and $x = d$ is Martha's. At $t = 0$ (event A), Jonas sends a clock to Martha with constant velocity v_1. Martha grabs the clock at the event B and, immediately, sends back a clock moving with constant velocity v_2, which reaches Jonas at event C. During all this process, Jonas has a clock with him that measures a proper time τ between the events A and C. As for the traveling clocks, the first one attributes a proper time of τ_1 between the events A and B, whereas the clock sent by Martha reads τ_2 between B and C. This situation is described on the spacetime diagram of figure 1.2 below.

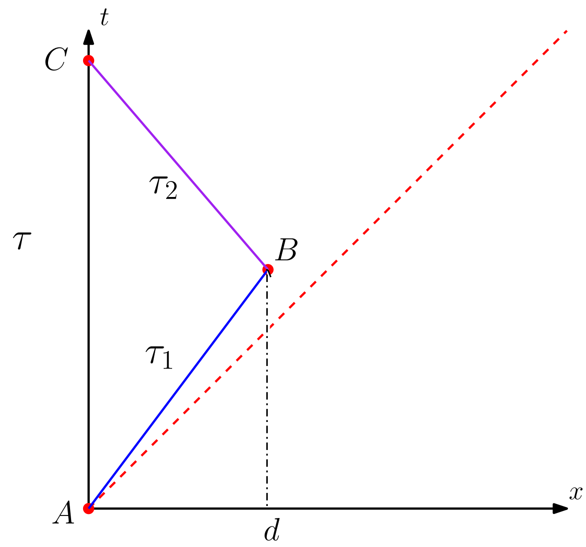

Figure 1.2: Minkowski diagram describing the experiment of Martha and Jonas. They want to measure the distance d based on the readings of three clocks.

Our claim is that the distance d can be expressed exclusively in terms of the proper times τ, τ_1 and τ_2. Let us prove this.

The first thing to do is to recall how proper time is related to the coordinate time t. From the Minkowski metric in one spatial dimension, we have

$$d\tau^2 = -ds^2 = -dt^2 + dx^2 \implies d\tau = \sqrt{1-v^2}dt, \tag{1.21}$$

where $v = \dfrac{dx}{dt}$ is the velocity as measured in the rest frame of Jonas and Martha.

Thus, if t_B denotes the coordinate time of the event B, we can write

$$\tau_1 = t_B\sqrt{1-v_1^2} \tag{1.22}$$

and

$$\tau_2 = (\tau - t_B)\sqrt{1-v_2^2}. \tag{1.23}$$

Notice that $v_1 = \dfrac{D}{t_B}$ and $v_2 = \dfrac{D}{\tau - t_B}$. Then, using relation (1.23), we get

$$t_B = \tau - \sqrt{\tau_2^2 + D^2}.$$

Substituting this on equation (1.22) and performing further algebraic manipulations, we obtain our result:

$$d = \frac{1}{2\tau}\sqrt{(\tau_1^2 + \tau_2^2 - \tau^2)^2 - 2(\tau_1\tau_2)^2}. \tag{1.24}$$

Therefore, we see that it is possible to design a measurement scheme where spatial distances are measured only with clocks. For a generic spacetime, this same conclusion holds provided one works locally. Thus, using solely clocks, only can, in principle, determine the geometric structure of the spacetime.

At this point, one can ask: why not try to use rules to measure time intervals? After some thinking, you should convince yourself that there is no natural way of doing that without resorting to some other object (like a light-ray). So far, we do not know any example in the specialized literature showing that this is even possible. Thus, we shall work under the hypothesis that *clocks are the most fundamental measuring tools that one can introduce on spacetime*.

Recall that we have introduced a clock as being just an apparatus that can be used to measure the proper time between two timelike-separated events. Moreover, we have shown that clocks can also be used to measure spacelike distances. Now, it remains to discuss more deeply what we mean by a clock. Additional restrictions on this apparatus will led us to introduce the concept of *honest clock*.

1.3 Honest clocks

Honest clocks are pointwise apparatus that attribute the same real number to each pair of *arbitrarily close* events they visit *irrespective of its past history*. Equivalently, they can be characterized as periodic dynamical systems whose evolution does not depend upon the trajectory described by the clock on the spacetime. Furthermore, in order to avoid ambiguities, we always take this trajectory to be a *timelike geodesic*.

More precisely, let $\gamma : I \subset \mathbb{R} \rightarrow \mathcal{M}$ be a timelike geodesic followed by a clock. Without loss of generality, we can always assume that γ can be parameterized by the proper time τ. Moreover, consider that the trajectory of the clock in its phase space is described by the map $\Gamma(\lambda) = (q_i(\lambda), p_i(\lambda))$, where q_i and p_i are canonically conjugated dynamical variables, with $i = 1, \ldots, n$. Then, for such a clock to be called *honest* there must exist some function $f(q_i(\lambda), p_i(\lambda), \lambda)$ linear with respect to the real parameter λ whose output is the *time as told by the clock*. That is: we are assuming that it is possible to construct an one-to-one association between the parameter λ that keeps track of the clock's internal evolution and the real number $\Delta\tau_{AB} \equiv \tau_B - \tau_A$ that the clock ascribes for the events $A = \gamma(\tau_A)$ and $B = \gamma(\tau_B)$.

Hence the internal dynamics of the clock does not depend upon the timelike trajectory, thus fulfilling the condition that the clock readings $\Delta\tau_{AB}$ cannot depend on the clock's past history. Additionally, we also assume that the classical Hamiltonian governing the evolution of the clock in its phase space does not depend explicitly on the parameter λ. Physically, such a condition ensures that the clock will not suffer deteriorations, so that its recordings will always advance at the same pace.

Now, we need to describe a protocol to decide whether a given clock is honest or not. In fact, from the experimental point of view it is not possible to decide if an isolated clock is honest. The best we can do is to decide whether *a clock is honest compared to another honest clock*. At first, this might seem a kind of tautology; however, you shall see in a moment that what we are really constructing here is an *equivalency class* of honest clocks. Before describing the protocol in detail, it is worth making a small digression about such a mathematical concept.

Let X be generic a non-empty set. An *equivalence relation* on X is a binary relation $\sim$ satisfying the following conditions:

1. $a \sim a, \forall\ a \in X$

2. If $a \sim b$ then $b \sim a$, $\forall\ a, b \in X$

3. If $a \sim b$ and $b \sim c$ then $a \sim c$, $\forall\ a, b, c \in X$

The *equivalence class* of an element $a \in X$, denoted by $[a]$, is defined as the set of all other elements of X that are $\sim$ related with a:

$$[a] = \{x \in X : x \sim a\}.$$

Now comes the protocol: suppose you are given a collection of clocks (the set X). As a fundamental hypothesis, we assume that X contain *at least two honest clocks*. Indeed, there is no reason to believe that classical General Relativity forbids the existence of honest clocks. Also, the existence of only one honest clock in the whole universe does not make sense: if there exists one, you can just reply it to construct an infinite amount of honest clocks!

Next, we separate those clocks whose readings advance at the same pace, obtaining a new set of clocks $A \subset X$. It is important to realize that one cannot guarantee that all clocks belonging to the set A are honest: some clocks could be devised to exhibit an *equally dishonest* bevahior. So, the proper procedure is to submit all those clocks to different conditions like, e.g., moving them, turning them off for arbitrary periods, and so on. The idea is to force all the clocks to follow quite distinct worldlines on the spacetime they are placed in. After this, the clocks that still advance at the same pace surely are the honest ones, belonging to the same equivalence class.

1.4 What is time?

Throughout this introductory chapter, we have briefly reviewed some fundamental aspects of the theory of General Relativity, focusing on how one can extract spatial and time measurements from a given spacetime. Now, we turn back to the promise made at the beginning of the chapter and give a precise definition of time within the paradigm of classical physics: *time is what honest clocks measure.*

At first, one may be disappointed (or not?) with this definition. Nonetheless, from the point of view of physics as an experimental science, we defend once more that observables are meaningless if one cannot state clearly how to measure them. Henceforth, the definition of a measurable quantity cannot be dissociated from the measuring apparatus itself. After all, even though we have outsourced the definition of time to that of honest clocks, we were able to give a precise description of such apparatus using classical physics.

Another point that is worth thinking about is that questions regarding the "true" nature of some concept often lie outside the scope of physics. At the end of the day, all that we have are models that predict the values of quantities to be observed experimentally. For instance, one cannot be sure that gravity "really" is a "manifestation" of the spacetime curvature; all we can be sure is that by using the model of gravity proposed by the theory of General Relativity we can make predictions that are observed in nature.

Therefore, we can safely state that we do have a precise characterization for the concept of time within the paradigm of classical physics.

However, we cannot forget that although we assume that an honest clock is an object that exists in nature, its ultimate description must be given by Quantum Mechanics. Thus, it is unavoidable to ask: how quantum theory impacts on our definition of a honest clock?

In the next chapters, we shall approach this question by analyzing different views about the role of time in Quantum Mechanics and how one can predict time measurements within the quantum formalism.

2

Essential aspects of Quantum theory

The purpose of this chapter is to review some aspects of Quantum Mechanics, focusing on what will be used to discuss the issue of time. After presenting topics that are more traditional, which can be found in more detail on references like [3, 4, 5, 6, 7], we discuss the concept of positive operator-valued measures (POVMs), a notion that is often used on quantum information theory [8]. This will help us to construct a generalized view of quantum observables [10, 13, 14], which will be important when we address some attempts of defining an operator for time in Quantum Mechanics.

2.1 Basic concepts of probability theory

Since Quantum Mechanics is a theory where all predictions are made within the framework of probability, we begin this review chapter by presenting the basic language of probability theory. In the present exposition we define some concepts aiming to outline the core of the probabilistic interpretation of Quantum Mechanics. For more details regarding probability theory, see e.g reference [9]. A more complete discussion about the probability concepts used in quantum theory may be found in reference [10].

In the same way that Linear Algebra happens in vector spaces, the background for probability theory is the so-called *probability space*. To construct such a space, one needs three ingredients:

- Sample space
- σ - algebra
- Probability measure

A *sample space* is simply any set $\Omega \neq \emptyset$ that contains all the events of our probabilistic theory. To give it a "soul", we define a σ- algebra $\mathcal{A}$ on it, which is a set of subsets of Ω satisfying the following conditions:

1. $\Omega \in \mathcal{A}$ and $\emptyset \in \mathcal{A}$;
2. If $A_1, A_2 \in \mathcal{A}$ then $A_1 \cup A_2, A_1 \cap A_2$, and $A_1 - A_2 \in \mathcal{A}$;
3. If $A_n \in \mathcal{A}\ \forall n \in \mathbb{N}$, then $\displaystyle\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$.

The structure $(\Omega, \mathcal{A})$ is called a *measurable space*. The mathematical theory of these spaces is the *Measure Theory*, whose treatment is completely out of the scope of this essay. If a subset $A \subset \Omega$ is in $\mathcal{A}$, we shall call it an *event*.

Given a measurable space $(\Omega, \mathcal{A})$ we can turn it into a probability space by defining a *probability measure on it*, which is a map $P : \mathcal{A} \to \mathbb{R}$ that satisfies:

1. $0 \leq P(A) \leq 1, \qquad \forall A \in \mathcal{A}$;
2. $P(\Omega) = 1$;
3. For $A_1, A_2, \ldots, A_n, \ldots \in \mathcal{A}$, with $A_i \cap A_j = \emptyset$ whenever $i \neq j$, one has

$$P\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} P(A_n).$$

Notice that these conditions (also known as the *Kolmogorov axioms*) just formalize our intuitive understanding about probability: (1) the probability of any event is a number between 0 and 1; (2) the probability of all the sample space ("anything happens") is 1; (3) the probability of the union of disjoint events is the sum of the probability of each event (the "or rule").

A basic tool in the framework of probability is the concept of *random variable*, which can be defined as a map $X : \Omega \to \mathbb{R}$ that associates real numbers to "elementary events" (points in the sample space). The key feature of random variables is that they allow us to work directly on the real line $\mathbb{R}$, thus being an important quantitative tool. At this point, it is important to mention that the set $\mathbb{R}$ itself can be made a measurable space with the aid of the σ-algebra of *Borel sets* $\mathcal{B}$, defined as the smallest σ-algebra of $\mathbb{R}$ that contains all subsets of the form $(-\infty, x]$. Thus, for X to be a well-defined random variable, the following condition must holds:

$$B \in \mathcal{B} \implies X^{-1}(B) \in \mathcal{A}.$$

Assuming this condition we can define a probability distribution for the random variable X as

$$P_X(B) = P(X^{-1}(B)). \tag{2.1}$$

Let us work a little bit more with definitions to construct the probability density function of a random variable, a central quantitative tool for Quantum Mechanics. Consider the set

$$A_x = \{\omega \in \Omega : X(\omega) \leq x\}.$$

Notice that A_x is the pre-image of the Borel set $(-\infty, x]$. The *cumulative distribution function* of X is then defined by

$$F_X(x) = P(A_x) \equiv P(X \leq x) \tag{2.2}$$

Finally, we say that X has a *probability density function* p_X if we can write

$$F_X(x) = \int_{-\infty}^{x} d\lambda \; p_X(\lambda). \tag{2.3}$$

Given this definition, one can show that the following properties hold:

- $F_X(x_1) \leq F_X(x_2)$ whenever $x_1 < x_2$.
- $\lim_{\epsilon \to 0^+} F_X(x + \epsilon) = F_X(x)$ (continuity from the right)
- $\lim_{x \to \infty} F_X(x) = 1$
- $\lim_{x \to -\infty} F_X(x) = 0$

Conversely, any function satisfying these four properties is a cumulative distribution function.

Notice that definition (2.3) implies on

$$p_X(x) = \frac{dF_X(x)}{dx}. \tag{2.4}$$

Moreover, the relation with the probability distribution of X is straightforward:

$$P_X(B) = \int_B dx \; p_X(x). \tag{2.5}$$

2.2 The probabilistic interpretation of Quantum Mechanics

In the framework of Quantum Mechanics, the degrees of freedom of each physical system are represented by a complex Hilbert space $\mathcal{H}$. Pure states of the system are described by normalized vectors $|\psi\rangle \in \mathcal{H}$. For most physical situations of interest, one assumes that $\mathcal{H}$

is *separable*, meaning that it is possible to find an *countable orthonormal* basis $\{|\phi_n\rangle\}$ which allows us to write any vector in the form

$$|\psi\rangle = \sum_n \langle\phi_n|\psi\rangle\,|\phi_n\rangle\,. \tag{2.6}$$

Usually, one assumes that measurable quantities (i.e., the *observables* of the theory) are represented by *self-adjoint operators* acting on $\mathcal{H}$. For now, we restrict our discussion within this view. Later on, we shall present some mathematical tools that allow for a more generic approach towards the issue of measurement in Quantum Mechanics.

For a self-adjoint operator $\hat{A} : \mathcal{H} \to \mathcal{H}$ with discrete spectrum $\{\lambda_n\}$, the *spectral theorem* allows us to write

$$\hat{A} = \sum_n \lambda_n \hat{\Pi}_n, \tag{2.7}$$

where $\hat{\Pi}_n$ is the projector onto the eigenspace belonging to the eigenvalue λ_n, i.e., the projector onto the space $\mathrm{Ker}(\hat{A} - \lambda_n \hat{\mathbb{I}})$[1]. The family of projectors $\{\hat{\Pi}_n\}$ is orthogonal, in the sense that

$$\hat{\Pi}_i \hat{\Pi}_j = \delta_{ij} \hat{\Pi}_i. \tag{2.8}$$

This means that eigenvectors belonging to different eigenvalues are orthogonal.

From the mathematical point of view, a proper generalization of (2.7) to the case where $\hat{A}$ has a continuous spectrum needs some tools from functional analysis. Since such a subject is out of our scope, we will just give a formal motivation for the generalization, aiming to understand how to operationally associate a random variable to a self-adjoint operator in Quantum Mechanics.

Still considering the operator (2.7), let us define

[1]The *kernel* of a linear operator $\hat{A}$ is defined as the set $\mathrm{Ker}(\hat{A}) = \{|\psi\rangle \in \mathcal{H} : \hat{A}\,|\psi\rangle = 0\}$.

$$\hat{E}(\lambda) = \sum_{\lambda_n \leq \lambda} \hat{\Pi}_n. \tag{2.9}$$

The collection of operators $\{\hat{E}(\lambda)\}_{\lambda\in\mathbb{R}}$ is called the *spectral family* of the operator $\hat{A}$. Notice that

- $\lim_{\epsilon\to 0^+} \hat{E}(\lambda+\epsilon) = \hat{E}(\lambda)$
- $\lim_{\lambda\to -\infty} \hat{E}(\lambda) = 0$ (null-operator)
- $\lim_{\lambda\to \infty} \hat{E}(\lambda) = \hat{\mathbb{I}}$ (identity operator)
- $[\hat{E}(\lambda), \hat{A}] = 0, \qquad \forall \lambda \in \mathbb{R}$
- $\langle\psi|\hat{E}(\alpha)|\psi\rangle \leq \langle\psi|\hat{E}(\beta)|\psi\rangle$ whenever $\alpha \leq \beta$, for all $|\psi\rangle \in \mathcal{H}$.

Moreover,

$$\Delta\hat{E}(\lambda_n) \equiv \hat{E}(\lambda_n) - \hat{E}(\lambda_{n-1}) = \hat{\Pi}_n. \tag{2.10}$$

So, the self-adjoint operator $\hat{A}$ can be written as

$$\hat{A} = \sum_n \lambda_n \Delta\hat{E}(\lambda_n). \tag{2.11}$$

It turns out that this is the expression that should be used for a generalization to the continuous case. Indeed, the general form of the spectral theorem reads

$$\hat{A} = \int_{-\infty}^{\infty} \lambda \, d\hat{E}(\lambda). \tag{2.12}$$

Here, the spectral family satisfies the same conditions outlined previously. For our purposes, it will not be necessary to give a rigorous definition for the "spectral measure" $d\hat{E}(\lambda)$. (for more details regarding technical issues of functional analysis, see reference [11]). Let us treat it operationally, keeping in mind that it is the generalization of (2.10) for the continuous case.

As an example consider $\mathcal{H} = \mathbb{L}^2(\mathbb{R})$, the Hilbert space of the square-integrable functions on the real line. Let $\hat{X}$ be the operator that takes a function $f \in \mathcal{H}$ and returns the function $g : x \mapsto xf(x)$. The spectral family of this operator can be defined as

$$\hat{E}(\lambda)(f)(x) = \Theta(x - \lambda)f(x), \tag{2.13}$$

where $\Theta(x)$ is the Heaviside step function. In this case, the spectral measure $d\hat{E}(\lambda)$ acts on each function f to generate a measure on the real line, so that the integral (2.12) can be computed by the usual rules of calculus. Formally,

$$d\hat{E}(\lambda)(f)(x) = \delta(\lambda - x)f(x)d\lambda. \tag{2.14}$$

Thus, we see that

$$\hat{X}(f)(x) = \int d\lambda \ \lambda\delta(\lambda - x)f(x) = xf(x). \tag{2.15}$$

Returning to the generic case, it remains to address the question regarding the eigenvalues of the operator $\hat{A}$ represented by (2.12). It turns out that one can give a rigorous definition for the spectrum of a self-adjoint operator solely by the properties of its spectral family. First of all, notice that in the discrete case the eigenvalues of $\hat{A}$ correspond to "jumps" in the operators of the spectral family. For the sake of illustration, suppose that the spectrum of $\hat{A}$ is $Spec(A) = \{\lambda_0, \lambda_1, \ldots \lambda_n\}$. In this case we have

$$\hat{E}(\lambda) = \hat{\Pi}_0 \qquad \lambda_0 \leq \lambda < \lambda_1$$

$$\hat{E}(\lambda) = \hat{\Pi}_0 + \hat{\Pi}_1 \qquad \lambda_1 \leq \lambda < \lambda_2$$

$$.$$

.

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$$\hat{E}(\lambda) = \hat{\mathbb{I}} \qquad \lambda \geq \lambda_n.$$

The key concept here is that the presence of eigenvalues is related to changes in the spectral family. When the spectral family is constant in some neighborhood of a point $\lambda \in \mathbb{R}$, we say that this point is a *stationary* one. Precisely: $\lambda \in \mathbb{R}$ is a stationary point of the spectral family $\{\hat{E}(\lambda)\}$ if there is some $\epsilon > 0$ such that

$$\hat{E}(\alpha) = \hat{E}(\beta) \qquad \forall \alpha, \beta \in (\lambda - \epsilon, \lambda + \epsilon). \tag{2.16}$$

From the definition (2.12), one sees that for the interval $J = (\lambda - \epsilon, \lambda + \epsilon)$ the expression

$$\int_J \lambda \; d\hat{E}(\lambda)$$

represents a null operator. Thus, it does not make sense to include the point λ in the spectrum of $\hat{A}$. In this way, the spectrum may be defined as

$$Spec(\hat{A}) = \{\lambda \in \mathbb{R} : \lambda \text{ is a non-stationary point of the spectral family} \{\hat{E(\lambda)}\}\}.$$

In a moment, we shall interpret the spectrum of a self-adjoint operator as the *collection of all the possible outcomes that can be observed in a measurement of the physical quantity described by the operator*. But first, we must figure out some way to connect all these mathematical definitions with the probability tools we have considered previously.

At first, consider a physical system completely characterized by a pure state $|\psi\rangle \in \mathcal{H}$ (a normalized vector in a complex Hilbert space). It turns out that each self-adjoint operator $\hat{A}$ gives raise to a real-valued random variable A whose cumulative distribution function is

$$F_A(\lambda) = \langle\psi|\hat{E}(\lambda)|\psi\rangle. \tag{2.17}$$

The proof that F_A is indeed a cumulative distribution function follows directly from the properties of the spectral family $\hat{E}(\lambda)$.

Now, assume that $\lambda \notin Spec(\hat{A})$. Then, λ is a stationary point of the spectral family $\{\hat{E}(\lambda)\}$, i.e, there is some $\epsilon > 0$ such that $\hat{E}(x)$ does not change for $x \in (\lambda - \epsilon, \lambda + \epsilon)$. So, the cumulative distribution F_A is constant in this same interval, which means that the probability associated with $I = (\lambda - \epsilon, \lambda + \epsilon)$ is zero. Thus, not only λ, but all the real values pertaining to the interval I are not valid outcomes for a measurement of $\hat{A}$. On the other hand, if $\lambda \in Spec(\hat{A})$, the function F_A will exhibit changes in every neighborhood of λ, so that there is always a non-zero probability of observing values sufficiently close to this point (of course, the probability of a specific point is always zero). Therefore, within such probabilistic paradigm we see that the spectrum of a self-adjoint operator can indeed be associated to the possible outcomes for the measurement of some physical quantity described by $\hat{A}$.

Furthermore, given the cumulative distribution function (2.17) the probability that the outcomes of a measurement of $\hat{A}$ will lie in some interval $J \subset \mathbb{R}$ is

$$P_A(J) = \int_J dF_A(\lambda) = \int_J \langle\psi|d\hat{E}(\lambda)|\psi\rangle. \tag{2.18}$$

From the experimental point of view, the quantity that deserves more attention is the *mean value* of $\hat{A}$ over the state $|\psi\rangle$, to be denoted by $\langle\hat{A}\rangle_\psi$. Indeed, the real number

$$\langle\hat{A}\rangle_\psi$$

is what Quantum Mechanics predicts the experimentalist will obtain if he performs infinite measurements of the physical property described by $\hat{A}$ in a system prepared in the state $|\psi\rangle$. Of course, since no laboratory in the real world can deal with infinite repetitions of the same experiments, one needs to establish some error criteria to claim that the measurements match or not with the previsions of theory. Concerning Quantum Mechanics, today there is no doubt that it is a theory that describe very well all natural phenomena apart the ones that involves gravity.

Now, we are in position of stating the principal result of this section: the formula that encodes the core of the probabilistic interpretation of Quantum Mechanics. Recall that the mean value of a random variable X is given by

$$\langle X\rangle = \int_{-\infty}^{\infty} dx\ xp_X(x) = \int_{-\infty}^{\infty} xdF_X(x). \tag{2.19}$$

So, using the random variable A associated with the self-adjoint operator $\hat{A}$, we can write

$$\langle\hat{A}\rangle_\psi \equiv \langle A\rangle = \int_{-\infty}^{\infty} \lambda dF_A(\lambda) = \int_{-\infty}^{\infty} \lambda d\langle\psi|\hat{E}(\lambda)|\psi\rangle = \langle\psi|\hat{A}|\psi\rangle. \tag{2.20}$$

As already mentioned, this equation is the core of the probabilistic interpretation of Quantum Mechanics, and will be the basis for further generalizations for the measurement concept as represented by an operator. This will be the subject of section 2.5.

Another statistical quantity that deserves mentioning is the *variance* (or *dispersion*) of a self-adjoint operator over a state $|\psi\rangle$. Using the random variable A naturally associated with the operator $\hat{A}$ again, such a quantity is defined by

$$\Delta_\psi\hat{A} \equiv \sqrt{\langle A^2 - \langle A\rangle^2\rangle} = \sqrt{\langle\psi|\hat{A}^2|\psi\rangle - \langle\psi|\hat{A}|\psi\rangle^2}. \tag{2.21}$$

Before finishing the present issue, we shall discuss how to extend the relation (2.20) to the case where it is not possible to describe the system under study by a vector in the Hilbert space. As is well known from the standard formalism of Quantum Mechanics (see [3], for instance), in this case one need to resort to the concept of *density matrix*. Indeed: the most general characterization of a physical system in Quantum Mechanics is given by an operator $\hat{\rho} : \mathcal{H} \to \mathcal{H}$ that satisfies to the following conditions:

- $\hat{\rho} = \hat{\rho}^\dagger$
- $\langle\psi|\hat{\rho}|\psi\rangle \geq 0$, for all $|\psi\rangle \in \mathcal{H}$
- $\mathrm{Tr}(\hat{\rho}) = 1$

One way to motivate the introduction of such a mathematical object is by considering a statistical mixture of quantum states. For instance, consider a case where there is ignorance about which state the system was prepared in. Suppose that everything we know is that there is a probability p_i that the state of the system is $|\psi_i\rangle$, for $i = 1, \ldots, N$. Then, in a measurement of $\hat{A}$, the mean value would be

$$\langle\hat{A}\rangle = \sum_{n=1}^{N} p_i \langle\psi_i|\hat{A}|\psi_i\rangle. \tag{2.22}$$

Introducing the operator

$$\hat{\rho} = \sum_{n=1}^{N} p_i |\psi_i\rangle\langle\psi_i|, \tag{2.23}$$

the quantity $\langle\hat{A}\rangle$ can be cast into

$$\langle\hat{A}\rangle = \mathrm{Tr}\left[\hat{\rho}\hat{A}\right]. \tag{2.24}$$

Notice that from the condition $\hat{\rho} = \hat{\rho}^\dagger$, every density matrix in a finite-dimensional Hilbert space can always be diagonalized, so that expression (2.23) is quite general. For an infinite dimensional Hilbert space there are more mathematical subtleties to be taken into account, but from a pragmatic point of view things work in the same way. Moreover, the equation (2.24) must be taken as the proper extension of (2.20) to a state described by a density matrix $\hat{\rho}$. Finally, if all the coefficients p_i vanish but one — say, $p_k = 1$ — we will have

$$\hat{\rho} = |\psi_k\rangle\langle\psi_k| \Longleftrightarrow |\psi_k\rangle\,,$$

so that the usual description of a pure state is recovered.

2.3 Quantum dynamics

The dynamics of quantum systems is generated by the *Schrödinger equation*:

$$i\frac{d}{dt}\left|\psi(t)\right\rangle = \hat{H}(t)\left|\psi(t)\right\rangle . \tag{2.25}$$

In this equation, the self-adjoint operator $\hat{H} : \mathcal{H} \to \mathcal{H}$ is the system's *Hamiltonian*, whose eigenstates are the states with defined energy value. This equation is completely deterministic: if we know the state of the system at some particular time $t = t_0$, equation (2.25) determines it for later all later times $t > t_0$. It is important to mention that here *time is just a parameter* that keeps track of the evolution of the system under study. Of course, it could be interpreted as the time as told by an external (classical) clock that guides the experimentalist to formulate statements like "at time $t = t_0$ the system was prepared in state $|\psi_0\rangle$" or "the measurement was carried at instant t".

The evolution of states can be given in terms of the *temporal evolution operator* $\hat{U}(t, t_0)$, which takes the state of the system at $t = t_0$ and evolves it to some other instant t according to:

$$\hat{U}(t, t_0)\left|\psi(t_0)\right\rangle = \left|\psi(t)\right\rangle \tag{2.26}$$

In the case where the Hamiltonian does not depend upon the evolution parameter t, such an operator has a simple closed expression, namely

$$\hat{U}(t, t_0) = e^{-i(t-t_0)\hat{H}} \tag{2.27}$$

More generally, it can be described by a *time ordered exponential*:

$$\hat{U}(t, t_0) = \mathcal{T}\exp\left(-i\int_{t_0}^{t} ds\ \hat{H}(s)\right) \tag{2.28}$$

The time ordering symbol $\mathcal{T}$ is an operation that can be defined as follows: given a product of time-dependent operators $\hat{A}(t_1)\hat{A}(t_2)...\hat{A}(t_n)$, the operation $\mathcal{T}$ reorganizes the order of the factors so that

$$\mathcal{T}[\hat{A}(t_1)\hat{A}(t_2)...\hat{A}(t_n)] = \hat{A}(s_1)\hat{A}(s_2)...\hat{A}(s_n), \qquad s_1 > s_2 > \ldots > s_n. \tag{2.29}$$

For the case where the state of our system is a mixture, the evolution of the density matrix reads

$$\hat{\rho}(t) = \hat{U}(t,t_0)\hat{\rho}(t_0)\hat{U}^\dagger(t,t_0). \tag{2.30}$$

All these results were presented within the *Schrödinger picture*, a description where the operators are fixed and the states evolve with time. *In the Heisenberg picture*, we have a description where the roles are reversed: states are fixed and operators evolve in time. We also have the *interaction picture*, whose limiting cases are just the two pictures already mentioned. For the sake of completeness — since we will eventually use the Heisenberg picture — we review these pictures on appendix A. Certainly, the physical predictions of Quantum Mechanics do not depend upon the picture we choose to perform our mathematical description.

2.4 Symmetries

In the context of Quantum Mechanics, symmetries are understood as operations realized over a system that do not change the physical predictions of the theory. In this sense, a symmetry operation must be implemented on the Hilbert space of the system by operators that preserve probabilities. Mathematically speaking: let $|\psi\rangle$ and $|\phi\rangle$ be two allowable quantum states of some physical system. Assume that the outlined operation can be represented by the action of an operator $\hat{S} : \mathcal{H} \to \mathcal{H}$. Define the transformed states

$$|\Psi\rangle = \hat{S}\,|\psi\rangle \tag{2.31}$$

and

$$|\Phi\rangle = \hat{S}\,|\phi\rangle\,. \tag{2.32}$$

Thus, for the operator $\hat{S}$ to be regarded as the proper implementation of a symmetry on the Hilbert space, the following condition must holds:

$$|\langle\psi|\phi\rangle|^2 = |\langle\Psi|\Phi\rangle|^2. \tag{2.33}$$

The *Wigner's theorem* states that in order to an operator to satisfy relation (2.33), it must be either *unitary* or *anti-unitary.* Recall that an operator $\hat{A}$ is said to be anti-unitary if for any vectors $|\psi\rangle\,, |\phi\rangle \in \mathcal{H}$ and scalars $\alpha, \beta \in \mathbb{C}$ the following conditions holds:

$$\langle\psi|\hat{A}^\dagger\hat{A}|\phi\rangle = \langle\psi|\phi\rangle^*$$

and

$$\hat{A}(\alpha\,|\psi\rangle + \beta\,|\phi\rangle) = \alpha^*\hat{A}\,|\psi\rangle + \beta^*\hat{A}\,|\phi\rangle\,.$$

The main example of a symmetry represented by an anti-unitary operator is the *time reversal*.

Now, we restrict the discussion to unitary operators: to each element g belonging to some symmetry group $\mathcal{G}$ we associate an unitary operator $\hat{U}(g) : \mathcal{H} \to \mathcal{H}$. That is: formally speaking, the implementation of a symmetry in Quantum Mechanics means to construct a *representation* of the group $\mathcal{G}$ that encapsulates the symmetry operations on the physical space.

Given any self-adjoint operator $\hat{A} \in \mathcal{L}(\mathcal{H})$, its transformation under the action of the symmetry described by $g \in \mathcal{G}$ is given by the relation

$$\hat{A}' = \hat{U}(g)\hat{A}\hat{U}^{\dagger}(g). \tag{2.34}$$

The representation of $\mathcal{G}$ on the Hilbert space is called *projective* if for any $g_1, g_2 \in \mathcal{G}$ one can write

$$\hat{U}(g_1)\hat{U}(g_2) = e^{i\phi(g_1,g_2)}\hat{U}(g_1 g_2), \tag{2.35}$$

for some real function ϕ. When it is possible to choose this phase to be zero, we say that we have a *complete representation.*

Example 1: *Galileo invariance.*

The invariance under Galileo transformations is a symmetry of non-relativistic Quantum Mechanics. Its representation on the Hilbert space of a spinless particle moving along one dimension, e.g., is the most notable case of a projective representation. Indeed, if $\hat{p}$ and $\hat{x}$ denote the usual momentum and position operators, the action of a Galileo boost with velocity v over the system's state at some instant t is described by the operator

$$\hat{G}(v) = e^{\frac{imv^2 t}{2}} e^{imv\hat{x}} e^{-ivt\hat{p}}. \tag{2.36}$$

When we combine two boosts with velocities v_1 and v_2 we obtain:

$$\hat{G}(v_1 + v_2) = e^{2imv_1 v_2 t}\hat{G}(v_1)\hat{G}(v_2). \tag{2.37}$$

Example 2: *Continuous symmetries.*

In physics, continuous symmetries are described by the so-called Lie groups, which essentially are differentiable manifolds with a group structure [12].

Let $\mathcal{G}$ be a connected Lie Group. Consider each element $g \in \mathcal{G}$ to be described by n real variables $x^a \equiv (x^1, \ldots, x^n)$: $g = g(x^a)$. Certainly, the identity element $I_{\mathcal{G}}$ corresponds to $x^a = 0$.

Suppose the composition law of $\mathcal{G}$ is

$$g(x^a)g(\bar{x}^b) = g(f(x^a, \bar{x}^b)),$$

where $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a differentiable function. We shall denote its component functions by $f^a : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$. For the sake of consistency, notice that $f^a(x^j, 0) = f^a(0, x^j) = x^a$.

As a quite natural ansatz, the representation of each $g(x^a)$ on the Hilbert space will be described by the operator

$$\hat{U}(g(x^a)) = \hat{I} + ix^a\hat{M}_a + \frac{1}{2}x^a x^b \hat{M}_{ab} + \dots . \tag{2.38}$$

The operators $\hat{M}_a$, $\hat{M}_{ab}$, etc., are self-adjoint operators that do not depend on the real coordinates x^a. Now, assuming a complete representation we have

$$\hat{U}(g(x^a))\hat{U}(g(\bar{x}^b)) = \hat{U}(g(f(x^a, \bar{x}^b))). \tag{2.39}$$

Consider then the Taylor expansion of the component functions f^a:

$$\begin{aligned} f^a(x^i, \bar{x}^j) &= f^a(x^i, 0) + f^a(0, \bar{x}^j) + f^a{}_{ij}x^i\bar{x}^j + \dots \\ &= x^a + \bar{x}^a + f^a{}_{ij}x^i\bar{x}^j + \dots \end{aligned} \tag{2.40}$$

Applying this last relation on (2.39) and then using the expansion outlined in equation (2.38), we obtain the *Lie algebra* for the symmetry generators on the Hilbert space, namely

$$[\hat{M}_b, \hat{M}_c] = iC^a{}_{bc}\hat{M}_a. \tag{2.41}$$

Here, $C^a{}_{bc} = f^a{}_{cb} - f^a{}_{bc}$ are the structure constants of $\mathcal{G}$ [12]. Thus, we see that the commutation relations of the symmetry generators on the Hilbert space are determined by the structure constants of the Lie group that describe the symmetry operations on the physical space. Let us see two examples.

Exemple 2.1: *Linear momentum.*

The linear momentum concept arises naturally on the context of spatial translations. Thus, its associated symmetry group can be taken as $\mathcal{G} = (\mathbb{R}, +)$, the additive group of real numbers. In this case, all the structure constants vanish.

As is well known, the generators of the translation symmetry in the Hilbert space of Quantum Mechanics are the momentum operators $\hat{p}_a$, $a = 1, 2, 3$, which satisfy to the commutation relations

$$[\hat{p}_a, \hat{p}_b] = 0. \tag{2.42}$$

Finite translations by a vector $\vec{a} \in \mathbb{R}^3$ are implemented by the unitary operator

$$\hat{U}(\vec{a}) = e^{-i\vec{a}\cdot\vec{\hat{p}}}. \tag{2.43}$$

Exemple 2.2: *Angular momentum.*

Since the concept of angular momentum is closely related to that of rotations, the symmetry group that gives rise to the angular momentum algebra is $\mathcal{G} = SO(3)$, the multiplicative group of 3×3 orthogonal real matrices. In this case, the commutation relation of the generators — the angular momentum operators $\hat{J}_a$ — is given by

$$[\hat{J}_a, \hat{J}_b] = i\epsilon_{abc}\hat{J}^c. \tag{2.44}$$

In order to keep the notation's elegance (in the context of Einstein's summation convention), we have defined $\hat{J}_a \equiv \hat{J}^a$.

The rotation by an angle $\phi \in [0, 2\pi]$ around the axis defined by the unit vector $\vec{n} \in \mathbb{R}^3$ — on the counter-clockwise sense and adopting the passive view — is implemented by the unitary operator

$$\hat{U}(\vec{n}, \phi) = e^{-i\phi\vec{n}\cdot\vec{\hat{J}}}. \tag{2.45}$$

2.5 Generalized observables

In this section, we use the concepts outlined previously to discuss how one can describe quantum measurements in a more general way, showing how to predict probabilities for outcomes in experiments where the measured quantity cannot be represented by a self-adjoint operator. For the sake of completeness, a generic system will always be assumed to be described by a density matrix $\hat{\rho}$.

Adapting the approach of references like [8, 10, 14], we generalize the concept of obserbables so that they now are represented by *positive operator-valued measures* (POVMs). A general definition of a POVM goes as follows: let $(\Omega, \mathcal{A})$ be a measurable space, as defined in section 2.1. A POVM over this space is a map $E : \mathcal{A} \to \mathcal{L}(\mathcal{H})^+$ such that

1. $0 \leq \langle\psi|\hat{E}(X)|\psi\rangle \leq 1 \qquad \forall\ X \in \mathcal{A}, \forall\ |\psi\rangle \in \mathcal{H}$
2. $\hat{E}(\cup_i X_i) = \sum_i \hat{E}(X_i)$, for any countable collection $\{X_i\}$ of disjoint sets belonging to the σ-algebra $\mathcal{A}$
3. $\hat{E}(\Omega) = \hat{\mathbb{I}}$

If the state of the system is represented by a density matrix $\hat{\rho}$, then the POVM E is supposed to describe a measurement process whose outcomes can be associated with the sets $X \in \mathcal{A}$. Furthermore, the probability distribution is

$$P(X) = \mathrm{Tr}\left[\hat{\rho}\hat{E}(X)\right] \tag{2.46}$$

The key point here is that properties 1-3 above ensure that this function is a truly probability distribution. Indeed, we can always find an orthonormal basis $\{|\phi_n\rangle\}$ of $\mathcal{H}$ such that the density matrix assumes a diagonal form

$$\hat{\rho} = \sum_n w_n |\phi_n\rangle\langle\phi_n|, \tag{2.47}$$

where $w_n \geq 0$ for each n and $\sum_n w_n = 1$. Using this decomposition, relation (2.46) can be cast into

$$P(X) = \sum_n \sum_j w_j \langle\phi_n|\phi_j\rangle \langle\phi_j|\hat{E}(X)|\phi_n\rangle = \sum_n w_n \langle\phi_n|\hat{E}(X)|\phi_n\rangle \tag{2.48}$$

From property 1, we have $0 \leq \langle\phi_n|\hat{E}(X)|\phi_n\rangle \leq 1$ for each n. Thus,

$$0 \leq P(X) \leq \sum_n w_n = 1.$$

Property 2 ensures the usual addition law for the probability of union of disjoint sets, whereas the condition $\hat{E}(\Omega) = \hat{\mathbb{I}}$ implies on $P(\Omega) = 1$.

In practice, we usually are concerned with the case where $\Omega = \mathbb{R}$ and $\mathcal{A} = \mathcal{B}(\mathbb{R})$. What we really do here is to associate each open set $J \subset \mathbb{R}$ to a positive operator $\hat{E}(J)$ whose properties ensure the probabilistic interpretation of (2.46). Then, the quantity $P(J)$ is interpreted as the probability that the measurement outcome lies in the set J. For the case where possible measurement outcomes can be labeled by a set of discrete indexes $\mathcal{J}$, we proceed in the same way by associating an positive operator $\hat{E}_i$ to each $i \in \mathcal{J}$. In this case,

$$\sum_i \hat{E}_i = \hat{\mathbb{I}}, \tag{2.49}$$

and the probability of obtaining the outcome labeled by i is to be given by

$$P(i) = \mathrm{Tr}\left[\hat{\rho}\hat{E}_i\right]. \tag{2.50}$$

In the remainder of this section, we shall assume the discrete case for the sake of simplicity.

A *projective measurement*, i.e., a measurement described by a self-adjoint operator $\hat{A}$, surely is a particular case of a POVM. Indeed, considering again the spectral decomposition

$$\hat{A} = \sum_i \lambda_i \hat{\Pi}_i,$$

it suffices to take $\hat{E}_i = \hat{\Pi}_i$.

Essentially, a POVM can be viewed as a particular case of the general measurement formalism where one is not concerned with the state of the system after the measurement but solely with its statistical description [8], since the probability distribution of the measurement's outcomes is still well defined. At the same time, it provides us with a theoretical framework that is more general than the one of projective measurements, thus allowing us to describe experimental situations that are not covered by the latter. The point here is that a projective measurement presupposes repeatability: if the measurement of the observable $\hat{A}$ yields the result i, then when $\hat{A}$ is measured again in the same system the result i will be observed with probability 1. It turns out that not all experimental measurements schemes meet this condition; e.g, when one uses a silvered screen to measure the position of a photon, the particle is destroyed.

Example[2]: suppose we were given a qubit prepared in one of the states

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

or

$$|\psi_2\rangle = |0\rangle \tag{2.51}$$

since these states are not orthogonal, there is no measurement that can safely distinguish between them (for a rigorous proof of this claim, see [8]). However, one can perform a POVM measurement to avoid misidentifications. Indeed, consider the operators

$$\hat{E}_1 = \frac{\sqrt{2}}{1+\sqrt{2}}|1\rangle\langle 1|,$$

$$\hat{E}_2 = \frac{\sqrt{2}}{2(1+\sqrt{2})}(|0\rangle\langle 0| - |0\rangle\langle 1| - |1\rangle\langle 0| + |1\rangle\langle 1|),$$

[2] This example was taken from reference [8].

and

$$\hat{E}_3 = \hat{\mathbb{I}} - \hat{E}_1 - \hat{E}_2.$$

One can easily verify that $\mathbb{M} \equiv \{\hat{E}_1, \hat{E}_2, \hat{E}_3\}$ is a POVM measurement.

Now, suppose that the state of the system is $|\psi\rangle = |\psi_2\rangle$. Because

$$\hat{E}_1 |\psi_2\rangle = 0,$$

if the outcome of the measurement $\mathbb{M}$ is $i = 1$, then we can be sure that the qubit that was given to us *is not* in the state $|\psi_2\rangle$. In the same token, since

$$\hat{E}_2 |\psi_1\rangle = 0$$

if the resultant outcome is $i = 2$ we have certainty that $|\psi\rangle \neq |\psi_1\rangle$.

Therefore, the measurement scheme described by $\mathbb{M}$ allow us to distinguish between the states $|\psi_1\rangle$ and $|\psi_2\rangle$ whenever the outcome is $i = 1$ or $i = 2$. Of course, if the outcome is $i = 3$ nothing can be said, but at least we have ruled out the possibility of misidentifications.

Notwithstanding, one could object against the understanding of POVMs as associated to distinct outcomes of the physical observable to be measured in the real experiment due to the lack of orthogonality. Indeed, for the example just exposed, one can verify that in general

$$\hat{E}_i \hat{E}_j \neq \delta_{ij} \hat{E}_i,$$

as opposed to what happens to the projectors that can be used to construct the self-adjoint operator in relation (2.12).

However, it turns out that there is a way to circumvent such a "problem" by enlarging the Hilbert space of the system to include the measuring apparatus. Indeed, suppose now that the Hilbert space $\mathcal{H}$ of the system can be written as

$$\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_A,$$

where $\mathcal{H}_S$ is the Hilbert space of the system of interest — i.e., the one where the measurements are to be carried on — and $\mathcal{H}_A$ describes the degrees of freedom of the measuring apparatus. Furthermore, assume that $\dim \mathcal{H}_A = N$ and let $\{|a_1\rangle, \ldots, |a_N\rangle\}$ be a basis of the same space such that each $|a_i\rangle$ is associated with a possible outcome of the measurement we want to perform in $\mathcal{H}_S$. Define the projectors

$$\hat{\Pi}_i = |a_i\rangle\langle a_i|, \qquad i = 1, \ldots, N. \tag{2.52}$$

Initially, suppose the global state of the extended system is describe by separable density matrix

$$\hat{\rho} = \hat{\rho}_S \otimes \hat{\rho}_A. \tag{2.53}$$

Notice that such a description stands for a particular instant before the experiment starts, where it is reasonable to suppose that there are no correlations between the system S and the apparatus A.

The measurement process will correspond to a interaction between the quantum subsystems S and A. Indeed, in such a scenario, one generally works with a Hamiltonian of the form

$$\hat{H}(t) = \hat{H}_S \otimes \hat{\mathbb{I}}_A + \hat{\mathbb{I}}_S \otimes \hat{H}_A + \hat{H}_I(t). \tag{2.54}$$

Due to the presence of the interaction Hamiltonian $\hat{H}_I(t)$, the temporal evolution as determined by the total Hamiltonian will create correlations between S and A. The study of how such correlations impact on local measurements on the system of interest — in this case, S — is very important, for instance, in *decoherence models*.

Hence, if $\hat{U}(t)$ is the time evolution operator arising from the complete Hamiltonian $\hat{H}(t)$, the probability for the outcome i in the measurement scheme under consideration is given by the formula

$$P(i) = \mathrm{Tr}\Big[\hat{\rho}(t)(\hat{\mathbb{I}}_S \otimes \hat{\Pi}_i)\Big] = \mathrm{Tr}\Big[\hat{U}(t)\hat{\rho}\hat{U}^\dagger(t)(\hat{\mathbb{I}}_S \otimes \hat{\Pi}_i)\Big]. \tag{2.55}$$

Using expression (2.53), this last result can be cast into

$$P(i) = \mathrm{Tr}_S[\hat{\rho}_S \hat{E}_i], \tag{2.56}$$

where Tr_S denotes the partial trace over $\mathcal{H}_S$ and the operator $\hat{E}_i$ is defined by

$$\hat{E}_i = \mathrm{Tr}_A[\hat{\rho}_A \hat{U}^\dagger(t)\hat{\Pi}_i\hat{U}(t)]. \tag{2.57}$$

It turns out that the collection $\{\hat{E}_i\}$ represents a POVM. Indeed, it is quite clear that each operator $\hat{E}_i$ is positive. Moreover,

$$\sum_{i=1}^{N} \hat{E}_i = \mathrm{Tr}_A\left[\hat{\rho}_A \hat{U}^\dagger(t)\left(\sum_{i=1}^{N}\hat{\Pi}_i\right)\hat{U}(t)\right] = \mathrm{Tr}_A[\hat{\rho}_A \hat{U}^\dagger(t)\hat{\mathbb{I}}_A\hat{U}(t)] = \mathrm{Tr}_A[\hat{\rho}_A] = 1. \tag{2.58}$$

Therefore, a set of orthogonal projectors acting in the Hilbert space of the measuring apparatus induce a POVM in $\mathcal{H}_S$.

As a concluding remark, we mention here that POVMs constitute the most general definition of an observable that is compatible with the probabilistic interpretation of Quantum Mechanics. Since a mathematically rigorous statement of this result is out of our scope, we just refer the reader to reference [13]. The chapter 2 of the book [14] also deals with this same issue.

3

The time-energy uncertainty relation

The time-energy uncertainty relation

$$\Delta E \Delta t \geq \frac{\hbar}{2}, \tag{3.1}$$

has been a matter of controversies since its proposal at the early days of quantum theory. The main reason for this is that there is no direct way of interpreting (3.1) in the same fashion as one interprets the uncertainty relation between position and momentum.

Indeed, a general result from linear algebra states that for any operators $\hat{A}, \hat{B} \in \mathcal{H}$ one has the inequality

$$\Delta_\psi \hat{A} \Delta_\psi \hat{B} \geq \frac{1}{2} |\langle [\hat{A}, \hat{B}] \rangle_\psi|, \tag{3.2}$$

Here, the quantity $\Delta \hat{A}_\psi$ is the dispersion of the operator $\hat{A}$ over the state $|\psi\rangle$, as defined previously on equation (2.21). Choosing $\hat{A} = \hat{x}$ and $\hat{B} = \hat{p}$ we have (for any state $|\psi\rangle$):

$$\Delta_\psi \hat{x} \Delta_\psi \hat{p} \equiv \Delta x \Delta p \geq \frac{\hbar}{2}. \tag{3.3}$$

Surely, there is no doubt about the meaning of the quantity ΔE on relation (3.1): it is just the dispersion of the Hamiltonian $\hat{H}$ when it is measured over the state $|\psi\rangle$ under study, namely

$$\Delta E = \sqrt{\langle\psi|\hat{H}^2|\psi\rangle - \langle\psi|\hat{H}|\psi\rangle^2}. \tag{3.4}$$

Nonetheless, in principle it is not clear how to define the quantity Δt.

In this chapter, we shall discuss some formulations and interpretations of the time-energy uncertainty relation.

3.1 The Mandelstamm-Tamm relation

We begin our discussion regarding the time-energy uncertainty relation by revisiting a formal derivation due to Mandelstamm and Tamm [20] (see also [19]).

At first, consider a classical clock equipped with a dynamical variable $C = C(t)$ from which the time measurements are determined. E.g., C may represent the angular position of a pointer in an analog clock. Let ΔC be the uncertainty in a measurement of C. Then, it is reasonable to suppose that the corresponding uncertainty in a time measurement, Δt, is related with the first one by

$$\Delta C = \left|\frac{dC}{dt}\right| \Delta t. \tag{3.5}$$

The fundamental hypothesis of the derivation considered here is based on the extrapolation of the relation (3.5) for the quantum case, where the variable C must be replaced by the observable $\hat{C}$. In doing so, we use the *Correspondence Principle* to replace the variable $C(t)$ in (3.5) by the mean value of $\hat{C}$ over the state of the system. The clock must now be understood as a quantum system whose time evolution (followed by the parameter t) is determined by some Hamiltonian $\hat{H}_c$. By simplicity, we suppose that such Hamiltonian does not depend on t. So, in the quantum case one can write

$$\Delta\hat{C} = \left|\frac{d\langle\hat{C}\rangle}{dt}\right| \Delta t(\hat{C}). \tag{3.6}$$

In the previous expression, the quantity $\Delta\hat{C}$ must be read as the dispersion of the operator $\hat{C}$ over the state of the system, as defined on (2.21). The change of notation $\Delta t \to \Delta t(\hat{C})$ was made to stress out that equation (3.6) must be understood as a *definition* of the quantity $\Delta t(\hat{C})$ that, a priori, depends on the dynamical behavior of the observable represented by the operator $\hat{C}$. A clear way to interpret such a quantity is to say that it represents a measure of the characteristic time scale during which a significant change of the expectation value of $\hat{C}$ — that is, a change by an amount of the order of the dispersion $\Delta\hat{C}$ — takes place on a specified system's state.

Thus, in this scenario the equation (3.6) defines what we understand by "time uncertainty" in Quantum Mechanics. Further assuming that the operator $\hat{C}$ does not depend explicitly on the parameter t, we can use Ehrenfest's theorem [3, 4] to write

$$i\hbar \frac{d\langle\hat{C}\rangle}{dt} = \langle[\hat{C}, \hat{H}]\rangle. \tag{3.7}$$

In this way, we have

$$\Delta\hat{C} = \frac{1}{\hbar}|\langle[\hat{C}, \hat{H}]\rangle|\Delta t(\hat{C}). \tag{3.8}$$

Then, applying the result (3.2) with $\hat{A} = \hat{H}$ and $\hat{B} = \hat{C}$, we get

$$\Delta E \Delta t(\hat{C}) \geq \frac{\hbar}{2}. \tag{3.9}$$

It is crucial to realize that the explicit dependence upon the observable $\hat{C}$ in this relation does not alter its universal validity: no matter which operator $\hat{C}$ we use to construct the quantity $\Delta t(\hat{C})$, the uncertainty relation will hold.

Example: *Free particle described by a Gaussian wave packet.*

Let us see what definition (3.6) tell us in the simplest particular case possible: a free particle. Here, we work in the Heisenberg picture (see appendix A) with $\hat{C}(t) = \hat{x}(t)$. To simplify the notation, we write Δt instead of $\Delta t(\hat{x})$.

The Hamiltonian of the system is

$$\hat{H} = \frac{\hat{p}^2}{2m},$$

so that $\hat{p}$ is a constant of motion. Thus, we write $\hat{p}(t) = \hat{p}(0) \equiv \hat{p}_0$. Using the Heisenberg equation (A.9), we can have

$$\hat{x}(t) = \frac{\hat{p}_0 t}{m} + \hat{x}(0) \equiv \frac{\hat{p}_0 t}{m} + \hat{x}_0. \tag{3.10}$$

Moreover,

$$\Delta\hat{x}(t) = \sqrt{\frac{t^2}{m^2}(\Delta\hat{p}_0)^2 + (\Delta\hat{x}_0)^2 + \frac{t}{m}(\langle\hat{x}_0\hat{p}_0 + \hat{p}_0\hat{x}_0\rangle - \langle\hat{p}_0\rangle\langle\hat{x}_0\rangle)}. \tag{3.11}$$

Now, assume that the state of our particle is described by a Gaussian wave packet centered at $x_0 = 0$ and with width σ in position space. Let p_0 be the mean value of the momentum operator over this sate. In this way, the normalized wave function for $t = 0$ can be written as

$$\langle x|\psi_0\rangle \equiv \psi_0(x) = \left(\frac{1}{2\pi\sigma^2}\right)^{1/4} e^{-\left(\frac{x}{2\sigma}\right)^2} e^{ip_0 x}. \tag{3.12}$$

As for the wave function in the momentum space (that is, the Fourier transform) we have

$$\langle p|\psi_0\rangle \equiv \tilde{\psi}_0(p) = \left(\frac{2\sigma^2}{\pi}\right)^{1/4} e^{-\sigma^2(p-p_0)^2}. \tag{3.13}$$

Switching back to the Schrödinger picture, we understand the mean and dispersion values of the operators $\hat{x}_0$ and $\hat{p}_0$ in equation (3.11) as the corresponding values for the Schrödinger picture operators $\hat{x}$ and $\hat{p}$ computed over the state $|\psi_0\rangle$. Thus, we have

$$\Delta\hat{x}_0 = \langle\psi_0|\hat{x}|\psi_0\rangle = \sigma,$$

$$\Delta\hat{p}_0 = \langle\psi_0|\hat{p}|\psi_0\rangle = \frac{1}{2\sigma},$$

$$\langle\hat{x}_0\rangle = \langle\psi_0|\hat{x}|\psi_0\rangle = 0,$$

and

$$\langle\hat{p}_0\rangle = \langle\psi_0|\hat{p}|\psi_0\rangle = p_0$$

For the sake of completeness, we outline the computation of the last quantity

$$\langle\hat{x}_0\hat{p}_0 + \hat{p}_0\hat{x}_0\rangle.$$

First of all, notice that

$$\langle\hat{x}_0\hat{p}_0 + \hat{p}_0\hat{x}_0\rangle = \langle\psi_0|\hat{x}\hat{p} + \hat{p}\hat{x}|\psi_0\rangle = 2\langle\psi_0|\hat{p}\hat{x}|\psi_0\rangle + i\,\langle\psi_0|\psi_0\rangle = 2\langle\psi_0|\hat{p}\hat{x}|\psi_0\rangle + i.$$

Inserting the identity operator (in position representation) into the first term of this last expression, it follows that

$$\langle\psi_0|\hat{p}\hat{x}|\psi_0\rangle = \int dx\ \langle\psi_0|\hat{p}|x\rangle\langle x|\hat{x}|\psi_0\rangle = \int dx\ \langle\psi_0|\hat{p}|x\rangle x\psi_0(x) \equiv \int dx\ J(x)x\psi_0(x).$$

It turns out that the function $J(x)$ is basically the derivative of $\psi_0^*(x)$. Indeed,

$$\begin{aligned}
J(x) &= \int dp\ \langle\psi_0|\hat{p}|p\rangle\langle p|x\rangle \\
&= \frac{1}{\sqrt{2\pi}}\int dp\ pe^{-ipx}\tilde{\psi}_0(p) \\
&= i\frac{\partial}{\partial x}\left[\frac{1}{\sqrt{2\pi}}\int dp\ e^{-ipx}\tilde{\psi}_0(p)\right] \\
&= i\frac{\partial\psi_0^*(x)}{\partial x} \\
&= \left(p_0 - i\frac{x}{2\sigma^2}\right)\psi_0^*(x).
\end{aligned}$$

So,

$$\langle\psi_0|\hat{p}\hat{x}|\psi_0\rangle = \int dx\ x\left(p_0 - i\frac{x}{2\sigma^2}\right)|\psi_0(x)|^2 = \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} dx\ xe^{-\frac{x^2}{2\sigma^2}}\left(p_0 - i\frac{x}{2\sigma^2}\right)$$

$$\Longrightarrow \langle\psi_0|\hat{p}\hat{x}|\psi_0\rangle = -\frac{i}{2}.$$

This means that

$$\langle\hat{x}_0\hat{p}_0 + \hat{p}_0\hat{x}_0\rangle = 0.$$

Thus, for a free particle initially prepared in a state described by the Gaussian wave packet (3.12), the dispersion of the position operator $\hat{x}$ at any time t reads

$$\Delta\hat{x}(t) = \sigma\sqrt{1 + \left(\frac{t}{2m\sigma^2}\right)^2}. \tag{3.14}$$

The picture here is the following: as the center of the wave packet moves like it were a classical particle with constant velocity, the packet spreads out and the uncertainty about its position increases with the parameter t.

Finally, using this result on the relation (3.6) with $\dfrac{d\langle\hat{x}\rangle}{dt} = \dfrac{p_0}{m}$, the definition of the time uncertainty Δt reads

$$\Delta t = \frac{m\sigma}{p_0}\sqrt{1 + \left(\frac{\hbar t}{2m\sigma^2}\right)^2}. \tag{3.15}$$

In order to give a proper interpretation to this quantity, we have restored the fundamental constant $\hbar$.

The first point to notice here is that the factor

$$\frac{m\sigma}{p_0}$$

is the ratio between the uncertainty about the initial position of the particle, $\Delta x_0 = \sigma$, and the group velocity $v_0 = \frac{p_0}{m}$, which is to be interpreted as the (constant) velocity of the free classical particle associated to our quantum particle. So, in the classical limit $\hbar \to 0$, Δt could be read as the time of flight through a region with spatial extent Δx_0, just as expected. In the quantum case, the presence of the term

$$\sqrt{1 + \left(\frac{\hbar t}{2m\sigma^2}\right)^2}$$

allows us to interpret Δt as the time interval where the probability of finding the particle within the spatial range defined by (3.14) is appreciable. That is: there is an appreciable change of detecting the particle at position $\dfrac{p_0 t}{m} + \Delta\hat{x}(t)$ on the time interval $t + \Delta t$.

3.2 Lifetime of a property

Following [15], we present a derivation of a slightly different type of the time-energy uncertainty relation concerning the lifetime of a *property* (see also [17, 20]). By property, we mean a particular instance of some physical observable that can be represented by a projector operator $\hat{\Pi}$. E.g., the physical observable represented by the self-adjoint operator

$$\hat{A} = \sum_{n=1}^{N} \lambda_n \hat{\Pi}_n,$$

has N properties described by each one of the projectors onto its eigenspaces.

Let $\hat{\Pi}$ be a projector operator representing some property P. We say that a state $|\psi\rangle$ has property P if

$$\hat{\Pi}\,|\psi\rangle = |\psi\rangle\,. \tag{3.16}$$

Once more, we work within the Heisenberg picture. Consider the differential equation that gives the evolution of the projector $\hat{\Pi}(t)$:

$$\frac{d\hat{\Pi}(t)}{dt} = i[\hat{H}, \hat{\Pi}(t)].$$

Now, the state $|\psi\rangle$ of the system is fixed. Define the function

$$p(t) = \langle\psi|\hat{\Pi}(t)|\psi\rangle. \tag{3.17}$$

Since $\hat{\Pi}$ is a self-adjoint operator, $p(t)$ is a real function of the time parameter t. Furthermore, we assume that the property P holds at the initial time instant $t = 0$, so that $p(0) = 1$. Given that, we can define the *lifetime* τ_P of property P to be the first value of $t > 0$ such that

$$p(\tau_P) = \frac{1}{2}. \tag{3.18}$$

Taking the time derivative of $p(t)$ and using relation (3.2), it follows that

$$\frac{dp(t)}{dt} = \frac{1}{\hbar}|\langle[\hat{\Pi}(t), \hat{H}]\rangle| \geq \frac{2}{\hbar}\Delta\hat{H}\Delta\hat{\Pi}. \tag{3.19}$$

In the expression above, the quantities $\Delta\hat{H} \equiv \Delta E$ and $\Delta\hat{\Pi}$ are to be calculated in the Heisenberg picture state $|\psi\rangle$. Since $\Delta\hat{\Pi} = \sqrt{p(t)(1 - p(t))}$, we can write

$$\frac{dp}{dt} \geq \frac{2}{\hbar}\Delta E\sqrt{p(t)(1 - p(t))}. \tag{3.20}$$

Recalling that $p(0) = 1$, the integration of (3.20) yields:

$$p(t) \geq \cos^2\left(\frac{\Delta E t}{\hbar}\right), \text{ for } 0 \leq t \leq \frac{\pi\hbar}{2\Delta E}.$$

Therefore, we must conclude that

$$\tau_P \Delta E \geq \frac{\hbar\pi}{4} > \frac{\hbar}{2} \tag{3.21}$$

3.3 The Aharonov - Bohm point of view

In the paper [21], Aharonov and Bohm criticize the following statement:"*In a measurement of energy carried out in a time interval* Δt*, there must be a minimum uncertainty* ΔE *in the transfer of energy to the observed system*". In doing so, they refute this particular interpretation of the time-energy uncertainty relation. Next, we shall discuss a model for measurement of energy that illustrates the core of such criticism.

The model is the following: a bipartite system $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ composed by two interacting particles, one representing the object of interest (the "observed system") and the other – the probe — playing the role of a "measurement apparatus". Let $\hat{x}$, $\hat{p}$ and m be the position, momentum and mass of the first particle (A), whereas $\hat{X}$, $\hat{P}$ and M stands for the corresponding quantities of the probe (B). The total Hamiltonian of the system is

$$\hat{H}(t) = \frac{\hat{p}^2}{2m} \otimes \hat{\mathbb{I}}_B + \hat{\mathbb{I}}_A \otimes \frac{\hat{P}^2}{2M} + g(t)\hat{p} \otimes \hat{X}, \tag{3.22}$$

where

$$g(t) = \begin{cases} g_0, & if\ 0 \leq t \leq \Delta t \\ 0, & otherwise \end{cases} \tag{3.23}$$

On the following, we omit the tensor product symbol for local operators (the first two terms of the total Hamiltonian).

Notice that the presence of the interaction term,

$$g(t)\hat{p} \otimes \hat{X},$$

enables us to interpret this model as representing a measurement scheme of the particle's momentum ($\hat{p}$) by means of a coupling with the probe's position ($\hat{X}$). Such a coupling has a constant strength g_0 and is "turned on" during a time Δt. Since measuring the momentum of a free particle is equivalent to determining the value of its energy, the model indeed represents a measurement scheme for the particle's energy that lasts for a time interval Δt.

Now, in order to see how this time can be made as short as one pleases without disturbing the measurement's outcome, we analyze the dynamics of the relevant operators using the Heisenberg picture. Applying the Heisenberg equation (A.9) to each one of the four operators that compose the total Hamiltonian $\hat{H}(t)$, we get

$$\frac{d\hat{x}(t)}{dt} = \frac{\hat{p}(t)}{m} + g(t)\hat{X}(t), \tag{3.24}$$

$$\frac{d\hat{X}(t)}{dt} = \frac{\hat{P}(t)}{M}, \tag{3.25}$$

$$\frac{d\hat{P}(t)}{dt} = -g(t)\hat{p}(t), \tag{3.26}$$

and

$$\frac{d\hat{p}(t)}{dt} = 0. \tag{3.27}$$

From the last equation, we see the the particle's momentum $\hat{p}$ is a constant of motion. So, we write

$$\hat{p}(t) = \hat{p}(0) \equiv \hat{p}_0.$$

Restricting the time parameter t in the range $0 \leq t \leq \Delta t$, so that $g(t) = g_0$ in equation (3.26), the time evolution of the probe's momentum $\hat{P}(t)$ is described by the relation

$$\hat{P}(t) = \hat{P}(0) - g_0 t\hat{p}_0. \tag{3.28}$$

Using these results in equations (3.24) and (3.25), we have (still for the limited time range $0 \leq t \leq \Delta t$)

$$\hat{X}(t) = \hat{X}(0) + \frac{\hat{P}(0)}{M} - \frac{g_0 t \hat{p}_0}{M}, \tag{3.29}$$

$$\hat{x}(t) = \frac{\hat{p}_0}{m} + g_0 \hat{X}(0) + \frac{g_0 \hat{P}(0)}{M} - \frac{g_0^2 t \hat{p}_0}{M}. \tag{3.30}$$

Now, let us focus on the measurement of the particle's energy. According to what we know so far, during the time interval Δt the energy of the particle is constant, being described by the Hamiltonian

$$\hat{H}_0 = \frac{\hat{p}_0^2}{2m}. \tag{3.31}$$

So, all we need is to find a way to measure the observable $\hat{p}_0$. Notice that it can be achieved by a measurement of the change on the probe's momentum in the course of the time interval Δt. Indeed, if one considers an ensemble of identically prepared systems ("particle" + "probe"), equation (3.28) tells us that the outcomes of measurements in both particle and probe's momenta will have mean values related by

$$\langle \hat{P}(\Delta t) - \hat{P}(0) \rangle = -g_0 \Delta t \langle \hat{p}_0 \rangle. \tag{3.32}$$

Without loss of generality, from now on we assume that $\hat{P}(0) = 0$.

In the same token, the dispersion of these quantities in the same measurement process satisfies

$$\Delta \hat{P}(\Delta t) = -g_0 \Delta t \Delta \hat{p}_0. \tag{3.33}$$

This result is the mathematical core of Aharonov - Bohm's argument: for a given resolution $\Delta \hat{P}(\Delta t)$ on the measurement of the probe's momentum, the value of the product

$$\Delta t \Delta \hat{p}_0$$

can be made as small as one pleases just by adjusting the coupling strength g_0. Therefore, in the present scenario one can measure the particle's energy $\hat{H}_0$ with *any desired accuracy in an arbitrarily small time interval*.

It is important to recognize that the Aharonov-Bohm's energy measurement scheme only refutes a particular interpretation of the time-energy uncertainty relation, according to which there is an intrinsic "*uncertainty relation between the duration of the measurement and the energy transfer to the observed system*" [21]. However, as the authors of [21] point out, "*it is commonly realized, of course, that the "inner" times of the observed system do obey an uncertainty relation*". In this statement, they endorse that there is no doubt on the physical validity of the time-energy uncertainty relation as formulated in the same fashion we did on section 3.1.

Therefore, the argument presented by Aharonov and Bohm shows that there is no scope for a generic time-energy uncertainty relation concerning time intervals that are defined by objects *external* to the system where the energy measurement is carried on. Surely, this is not the case of the formulation due to Maldestamm and Tamm, where the time uncertainty is defined by means of some inner observable of the system (see equation (3.6)).

3.4 A derivation using the fact that "energy weighs"

In the last section we have seen that there is no scope for a time-energy uncertainty when one speaks of measuring the energy of a system during a time interval defined by a clock external to the system. Nonetheless, this situation changes when one considers energy measurements that are performed from within the system itself. In the present section, we discuss a *gadanken* experiment proposed on [22] that uses the fact that energy weighs — a fundamental conclusion of Relativity — to derive a uncertainty relation between the measured energy and the duration of the measurement as defined by a clock internal to the system.

In general lines, the ideia is to weigh the total mass of a closed system from within the system itself. In doing so, it is possible to argue that the time τ an internal observer takes to measure the system's energy with an accuracy ΔE is bounded by the relation

$$\Delta E \tau \geq \frac{\hbar}{2}. \tag{3.34}$$

The proposed gedanken experiment consists of measuring the mass M of a spherical shell with radius R by means of tracking the time it takes for an ejected spherical shell (*test shell*), with mass $m \ll M$, to reach a certain height $z \ll R$, to be specified below. The point here is that this time must be measured from a clock that is inside the test shell. According to General Relativity, the time as told by such clock will be affected by gravity. So, the peculiarities of the mass ejection event will alter the way the clock ticks inside the shell. Indeed: working within the post-Newtonian approximation, a clock located at a height z in a gravitational field with potential $\phi(z)$ will have its time measurements dilated according to the relation

$$\tau(z) = t\left(1 + \frac{\phi(z)}{c^2}\right). \tag{3.35}$$

Next, assume that the internal clock is located somewhere inside the test shell. Setting the ground level of the potential at the surface of the original spherical shell, we have $z = r - R$, where r is the radius of the test shell. The situation is represented in the figure below.

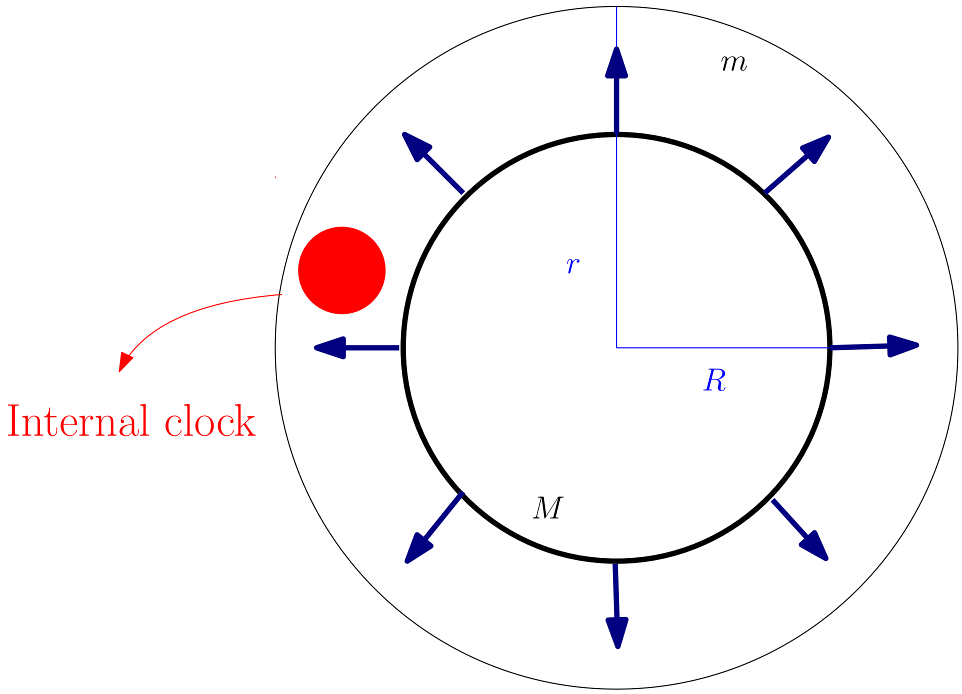


Figure 3.1: Illustrative scheme of the gedanken experiment. The size of the internal clock is exaggerated for the sake of clearness; the clock is located inside the spherical shell that is being ejected.

Recalling the assumption $z \ll R$, we can write the change on the gravitational potential felt by the clock when the shell moves from $z = 0$ to $z = r - R$ as

$$\Phi \equiv \phi(z) - \phi(0) = \frac{Gmz}{R^2}. \tag{3.36}$$

Now, we turn to Quantum Mechanics and state that the uncertainty in the height determination is Δz. Of course, this amounts to a uncertainty in the radial momentum Δp_z such that

$$\Delta z \Delta p_z \geq \frac{\hbar}{2}. \tag{3.37}$$

It turns out that we can relate the quantity Δz to the uncertainty $\Delta\tau$ in the duration τ

of the experiment. Since τ is the tine recorded by the internal clock between the events "test shell with radius R" and "test shell with radius $r > R$", equation (3.35) allows us to write

$$\tau = t\left(1 + \frac{\Phi}{c^2}\right) = t\left(1 + \frac{Gmz}{c^2 R^2}\right). \tag{3.38}$$

Thus,

$$\Delta\tau = \frac{Gm\Delta z}{c^2 R^2} t. \tag{3.39}$$

Furthermore, assuming that $\frac{\Phi}{c^2} \ll 1$, one can make the following approximation:

$$\frac{\Delta\tau}{\tau} \approx \frac{Gm\Delta z}{c^2 R^2} \implies \Delta z \approx \frac{c^2 R^2 \Delta\tau}{\tau G m}. \tag{3.40}$$

As for Δp_z, we can give an upper bound to it by means of the following reasoning: if the mass of the original shell is to be measured within an accuracy ΔM, the change in the test shell's momentum due to the impulse of the gravitational force $F = mg = \frac{mGM}{R^2}$ is well approximated by

$$\delta p_z \approx \frac{Gm\Delta M\tau}{R^2}. \tag{3.41}$$

So, we take the above quantity as an upper bound to the uncertainty about the determination of p_z: $\Delta p_z \leq \delta p_z$.

Therefore, in the context of the outlined assumptions, a direct application of the relation (3.37) yields

$$c^2 \Delta M \Delta\tau \geq \frac{\hbar}{2}. \tag{3.42}$$

Finally, using that the energy of the spherical shell relates to its mass according to the celebrated relation $E = Mc^2$, as well as the requirement $\tau \geq \Delta\tau$, we get the claimed result:

$$\Delta E \tau \geq \frac{\hbar}{2}.$$

3.5 Concluding remarks

At first, the absence of a general self-adjoint operator for time makes it harder to formulate an universal uncertainty relation regarding simultaneous measurement of energy and time. Nonetheless, apart from the content of the last section — which tends to be more "heuristic" due to the incorporation of gravitational effects in quantum systems — the discussion presented in this chapter supports the view that the Mandelstamm-Tamm formulation is — to the best of our knowledge — the most precise formulation of the time-energy uncertainty relation.

According to such a formulation, any attempt to keep track of the dynamics of some quantum system's observable to define elapsed time will be subject to the limitation (3.9). This fact *per se* already leads one to suspect that objects like honest clocks are forbidden by quantum theory. In the next chapter we go deeper into this discussion, arguing that any quantum system is indeed forbidden to play the role of an honest clock due to more fundamental issues.

4

Time observables in Quantum Mechanics and the nonexistence of honest clocks

In standard Quantum Mechanics textbooks it is usually said that time cannot be represented by a self-adjoint operator (like energy, position and momentum do); instead, it must be regarded simply as a parameter that labels the dynamics of states (or operators). We begin this chapter by presenting a no-go theorem — *Pauli's theorem* — that supports such a statement.

Nonetheless, since the requirement that only self-adjoint operators can represent observables in the quantum realm is not true at all — as supported by our exposition of POVMs on section 2.5 —, Pauli's theorem by itself is not sufficient to rule out the possibility of a consistent definition for quantum measurements of time. Indeed, as we shall present in this chapter, there is an approach that leads to the construction of a time observable in a quantum system, namely, a POVM that allows one to extract predictions about time of arrival measurements performed over a free particle.

After discussing this construction, we finish the chapter by presenting a fundamental result due to Unruh and Wald that completely rules out the realization of honest clocks in the quantum realm.

4.1 Pauli's theorem

Wolfgang Pauli quotes in [24] a celebrated argument that rules out the possibility of constructing a self adjoint operator canonically conjugated to the Hamiltonian of a *system that admits ground state*. In this section, we state and proof Pauli's theorem.

Theorem: Consider a physical system whose Hamiltonian $\hat{H} \in \mathcal{L}(\mathcal{H})$ is bounded from below. Then, it is not possible to construct a self adjoint operator $\hat{T} \in \mathcal{L}(\mathcal{H})$ such that

$$[\hat{H}, \hat{T}] = i\hat{\mathbb{I}}. \tag{4.1}$$

Proof: First of all, the hypothesis over $\hat{H}$ means that there exists an energy eigenstate $|E_0\rangle$ with energy E_0 such that $E \geq E_0$ for any other value of energy E allowed in this system. It is important to recognize that such assumption is quite natural, since in nature we do not observer systems without an energy ground state.

Now, suppose by contradiction that there is a self-adjoint operator $\hat{T}$ that satisfies equation (4.1). For $\alpha \in \mathbb{R}$, consider the following unitary operator:

$$\hat{U}_\alpha = e^{i\alpha\hat{T}}. \tag{4.2}$$

Notice that unitariness of $\hat{U}_\alpha$ follows from the assumption of $\hat{T}$ being self-adjoint, so that $\hat{T} = \hat{T}^\dagger$.

For the sake of completeness, we recall the following identity:

$$e^{\lambda\hat{A}}\hat{B}e^{-\lambda\hat{A}} = \hat{B} + \lambda[\hat{A}, \hat{B}] + \frac{\lambda^2}{2}[\hat{A}, [\hat{A}, \hat{B}]] + \mathcal{O}(\lambda^3). \tag{4.3}$$

In the previous equation, $\hat{A}$ and $\hat{B}$ are any linear operators, and $\lambda \in \mathbb{C}$. Adapting such equation to our context, we have

$$\hat{U}_\alpha \hat{H} \hat{U}_\alpha^\dagger = \hat{H} + \alpha \hat{\mathbb{I}}. \tag{4.4}$$

Since the parameter α can assume any real value, the spectrum of the operator

$$\hat{H} + \alpha \hat{\mathbb{I}}$$

is the whole real line. Moreover, the unitariness of $\hat{U}_\alpha$ ensures that the spectrum of $\hat{H}$ and $\hat{U}_\alpha \hat{H} \hat{U}_\alpha^\dagger$ are the same. Indeed, one can show that their characteristic polynomials are equal by reasoning as follows:

$$\det\Big(\hat{U}_\alpha \hat{H} \hat{U}_\alpha^\dagger - \lambda \hat{\mathbb{I}}\Big) = \det\Big(\hat{U}_\alpha (\hat{H} - \lambda \hat{\mathbb{I}}) \hat{U}_\alpha^\dagger\Big) = \det\Big(\hat{H} - \lambda \hat{\mathbb{I}}\Big).$$

Therefore, we must conclude that the spectrum of the Hamiltonian $\hat{H}$ is the whole set $\mathbb{R}$, what gives the contradiction that completes the proof of Pauli's theorem.

■

4.2 Time as a POVM in Quantum Mechanics

Following the approach of [26] (see also [27, 28, 31, 32, 33]), we discuss the possibility to define a time observable for a particular quantum system — the free particle — within the formalism of POVMs (refer to section 2.5).

Let us begin by defining the concept of *time POVM* for a generic quantum system.

Consider $\mathcal{G}$ to be a group representing some symmetry in the physical space (spatial translations, rotations, etc.). For each $g \in \mathcal{G}$, we denote by $\hat{U}(g) \in \mathcal{L}(\mathcal{H})$ the unitary representation of the group. We say that a POVM $\mathbb{P} : (\Omega, \mathcal{A}) \to \mathcal{L}^+(\mathcal{H})$ is *covariant* with respect to this unitary representation if

$$\hat{U}(g)\hat{\mathbb{P}}(X)\hat{U}^{\dagger}(g) = \hat{\mathbb{P}}(X_g), \qquad \forall\; X \in \mathcal{A}, \tag{4.5}$$

where X_g denotes the set of $\mathcal{A}$ obtained through the action of the group element g on X. For instance, if $X = [a, b)$ and $\mathcal{G}$ is the translation group, with $\hat{U}(g) = e^{-ig\hat{p}}$, then $X_g = [a + g, b + g)$.

Thereby, we define a time POVM simply as a POVM that is covariant with respect to the group of time translations. Explicitly: $\mathbb{T}$ is a time POVM if, and only if

$$e^{i\tau\hat{H}}\hat{\mathbb{T}}(X)e^{-i\tau\hat{H}} = \hat{\mathbb{T}}(X + \tau), \tag{4.6}$$

where $X + \tau = \{x + \tau : x \in X\}$.

Now, consider that $\mathcal{H}$ is the Hilbert space of a particle in one dimension. Moreover, suppose the particle is free and has mass m, so that the Hamiltonian is

$$\hat{H} = \frac{\hat{p}^2}{2m}. \tag{4.7}$$

If we were dealing with a classical particle with velocity v whose location at the instant $t = t_0 = 0$ is $x = x_0 \neq 0$, then it would arrive at the origin $x = 0$ when

$$t = -\frac{x_0}{v}. \tag{4.8}$$

This is the *time of arrival* at the origin. Notice that it can be negative, meaning that the particle was at the origin before the instant $t = 0$.

A quite natural extension of (4.8) to the quantum case is

$$\hat{T} = -\frac{m}{2}(\hat{x}\hat{p}^{-1} + \hat{p}^{-1}\hat{x}). \tag{4.9}$$

Some authors make reference to this operator as the “Arahonov-Bohm time operator”, since it was first proposed — with opposed signal — as a time operator in the paper [21].

Notice that $\hat{T}$ as defined in this way is canonically conjugated to the free Hamiltonian:

$$[\hat{H}, \hat{T}] = i\hat{\mathbb{I}}. \tag{4.10}$$

Of course, this relation must be understood in the domain $\mathcal{D}(\hat{T})$ of $\hat{T}$. For more details concerning the nature of $\mathcal{D}(\hat{T})$, as well as related mathematical issues, see appendix C.

Nonetheless, due to Pauli's theorem one can be sure that such an operator cannot be self-adjoint. The idea, then, is to construct a POVM in $\mathbb{R}$ from which we can compute probabilities regarding time measurements.

A natural way of beginning such a task is to look for the eigenstates of the operator $\hat{T}$. Formally, we shall employ the notation

$$\hat{T} \left| t \right\rangle = t \left| t \right\rangle, \tag{4.11}$$

with $t \in \mathbb{R}$.

In momentum representation, we have

$$\langle p|\hat{T}|\psi\rangle = \frac{im}{2}\left(\frac{\psi(p)}{p^2} - \frac{2}{p}\frac{d\psi(p)}{dp}\right), \tag{4.12}$$

where, as usual, $\psi(p) = \langle p|\psi\rangle$.

Introducing the notation $\phi_t(p) = \langle p|t\rangle$, relation (4.11) yields the following differential equation:

$$\frac{d\phi_t(p)}{dp} = \left(\frac{1}{2p} + \frac{ipt}{m}\right)\phi_t(p). \tag{4.13}$$

Due to the divergence at $p = 0$, this equation has two families of eigenfunctions: one for the branch $p > 0$ and other for $p < 0$. Indeed, for $\alpha = \pm 1$, we can write these eigenfunctions in the form

$$\phi_t(p;\alpha) = \frac{1}{\sqrt{2\pi m}}\Theta(\alpha p)\sqrt{|p|}e^{\frac{ip^2 t}{2m}}, \tag{4.14}$$

where $\Theta(x)$ is the Heaviside step function, which takes the value 1 when $x > 0$ and vanishes otherwise. The multiplicative constant $\frac{1}{\sqrt{2\pi m}}$ is included just for the sake of convenience.

Realize that the eigenfunctions above do not belong to the Hilbert space $L^2(\mathbb{R})$, since they are not normalizable. However, just as happens in the case of plane waves — momentum eigenfunctions in the free space — this does not prevent us to associate them with eigenstates $|t,\alpha\rangle$ in the abstract Hilbert space $\mathcal{H}$. That is: here, we shall treat the state $|t,\alpha\rangle$ just like one treats the states $|x\rangle$ and $|p\rangle$ in the case where the associated wave functions are non-normalizable. Thus, it makes sense to write

$$\langle p|t,\alpha\rangle = \frac{1}{\sqrt{2\pi m}}\Theta(\alpha p)\sqrt{|p|}e^{\frac{ip^2 t}{2m}}. \tag{4.15}$$

Concerning the analogy between the time-of-arrival eigenstates and the usual position (or momentum) eigenstates, a crucial difference must be stressed out: while these eigenstates are orthogonal, in the sense that they satisfy the relations

$$\langle x|x'\rangle = \delta(x-x') \text{ and } \langle p|p'\rangle = \delta(p-p'),$$

the same does not happens with the time-of-arrival eigenstates.

Indeed, for $\alpha = 1$ and $\alpha' = -1$ one has

$$\langle t,\alpha|t',\alpha'\rangle = \frac{1}{2\pi m}\int dp\ \Theta(p)\Theta(-p)|p|e^{\frac{ip^2(t'-t)}{2m}} = 0. \tag{4.16}$$

Certainly, this happens whenever $\alpha \neq \alpha'$. On the other hand, if $\alpha' = \alpha$ (we take $\alpha = 1$ without loss of generality) then

$$\langle t,\alpha|t',\alpha\rangle = \frac{1}{2\pi m}\int dp\ \Theta(p)|p|e^{\frac{ip^2(t-t')}{2m}} = \frac{1}{2\pi m}\int_0^\infty dp\ pe^{\frac{ip^2(t-t')}{2m}}. \tag{4.17}$$

Performing the change of variable $p \to u = \dfrac{p^2}{2m}$ allows one to identify this last integral as the Fourier transform of the Heaviside step function (apart from a multiplicative factor of $(2\pi)^{-1/2}$). Thus, we have

$$\langle t, \alpha | t', \alpha \rangle = \frac{1}{2}\delta(t' - t) + \frac{i}{2\pi(t' - t)} \tag{4.18}$$

Therefore, the eigenstates $|t, \alpha\rangle$ do not form an orthogonal set in $\mathcal{H}$. Everything is consistent so far: we know beforehand that $\hat{T}$ is not a self-adjoint operator, meaning that its eigenstates belonging to different eigenvalues t does not have to be always orthogonal.

At this point, one may raise objections against the physical interpretation of the eigenstates $|t, \alpha\rangle$. Indeed, since they do not form an orthogonal set, time of arrival measurements carried on a system prepared on the state $|t, \alpha\rangle$ will not necessarily have an outcome sharply peaked around t. Nonetheless, one cannot forget that, rigorously speaking, these states do not belong to the Hilbert space $\mathcal{H}$ because they are not normalizable. Thus, a proper physical meaning of the eigenstates of the operator $\hat{T}$ can only be achieved after one uses them to construct normalizable states, just like the case with position and momentum eigenstates. We postpone this discussion to the next subsection, where we construct a Gaussian wave packet from the eigenstates $|t, \alpha\rangle$.

Returning to the main issue, it turns out that the eigenstates of $\hat{T}$ do satisfy a completeness relation. As will become clear next, it is precise this property — along with the commutation relation (4.10) — that allows one to construct a consistent time-of-arrival POVM from the eigenstates $|t, \alpha\rangle$.

The completeness relation for the states $|t, \alpha\rangle$ is a consequence of the following expression:

$$\sum_{\alpha \in \{-1,1\}} \int dt \;\; \langle p' | t, \alpha \rangle \, \langle t, \alpha | p \rangle = \sum_{\alpha} \frac{1}{2\pi m} \int dt \; \sqrt{|p|}\sqrt{|p'|}\Theta(\alpha p)\theta(\alpha p') e^{\frac{it(p^2 - (p')^2)}{2m}}. \tag{4.19}$$

Indeed, recalling the widely used integral representation of the Dirac's delta function,

$$\delta(x-a) = \frac{1}{2\pi}\int dt\; e^{it(x-a)}, \tag{4.20}$$

we can write

$$\frac{1}{2\pi m}\int dt\; e^{\frac{it(p^2-(p')^2)}{2m}} = \frac{1}{m}\delta\left(\frac{p^2}{2m} - \frac{(p')^2}{2m}\right). \tag{4.21}$$

For the sake of completeness, we also recall that if $g(x)$ is a function such that $g(x_i) = 0$ for $i = 1, \ldots, n$, then

$$\delta(g(x)) = \sum_{i=1}^{n} \frac{\delta(x-x_i)}{|g'(x_i)|}. \tag{4.22}$$

Thus, we have

$$\frac{1}{2\pi m}\int dt\; e^{\frac{it(p^2-(p')^2)}{2m}} = \frac{1}{|p'|}\left(\delta(p-|p'|) + \delta(p+|p'|)\right). \tag{4.23}$$

Using this in relation (4.19) and paying attention to the step functions, we get the expected result:

$$\sum_{\alpha}\int dt\;\; \langle p'|t,\alpha\rangle\,\langle t,\alpha|p\rangle = \delta(p'-p). \tag{4.24}$$

Since $\langle p'|p\rangle = \delta(p'-p)$, we are led to conclude that

$$\sum_{\alpha}\int dt\; |t,\alpha\rangle\langle t,\alpha| = \hat{\mathbb{I}}. \tag{4.25}$$

Finally, we are in position to present the time POVM for the free particle: for each set $X = (a,b)$, it is defined by the expression

$$\hat{\mathbb{T}}(X) = \sum_{\alpha}\int_a^b dt\; |t,\alpha\rangle\langle t,\alpha|. \tag{4.26}$$

It is straightforward to verify that this expression really defines a POVM; the only tricky property, $\hat{\mathbb{T}}((-\infty,\infty)) = \hat{\mathbb{I}}$, is ensured by the completeness relation (4.24). Moreover, this POVM is covariant with respect to the group of time translations. Indeed, notice that since $[\hat{H}, \hat{T}] = i\hat{\mathbb{I}}$ we have

$$e^{i\tau\hat{H}} |t,\alpha\rangle = |t+\tau,\alpha\rangle \,. \tag{4.27}$$

Thus, taking $\hat{U}(\tau) = e^{i\tau\hat{H}}$ we can reproduce the relation (4.5):[1]

$$\begin{aligned} \hat{U}(\tau)\hat{\mathbb{T}}(a,b)\hat{U}^{\dagger}(\tau) &= \sum_{\alpha}\int_a^b dt\ |t+\tau,\alpha\rangle\langle t+\tau,\alpha| \\ &= \sum_{\alpha}\int_{a+\tau}^{b+\tau} dt\ |t,\alpha\rangle\langle t,\alpha| \\ &= \hat{\mathbb{T}}(a+\tau,b+\tau). \end{aligned} \tag{4.28}$$

For the sake of simplifying the notation, we use $\hat{\mathbb{T}}(a,b)$ in the place of $\hat{\mathbb{T}}((a,b))$.

Physically, this result means that the probability for arriving at time t is equal to the probability for arriving at $t+\tau$ when the temporal evolution of the original state is tracked backwards by a time τ.

In the context of the formalism under consideration, the physical predictions that really are relevant from the experimental point of view are given by expressions like (2.46). For instance, consider the following experimental setup: a particle detector is placed at the position $x = 0$. The free quantum particle is prepared in some state $|\psi\rangle$. Let t be the time at which the detector fires, i.e., the time at which the particle arrives in the origin $x = 0$. Then, the probability that t lies in some interval $J = (t_1, t_2) \subset \mathbb{R}$ is given by

$$p(J) = \langle\psi|\hat{\mathbb{T}}(J)|\psi\rangle = \sum_{\alpha}\int_J dt\ \langle\psi|t,\alpha\rangle\,\langle t,\alpha|\psi\rangle = \sum_{\alpha}\int_{t_1}^{t_2} dt\ |\langle t,\alpha|\psi\rangle|^2. \tag{4.29}$$

Notice that the probability density is simply

[1] Warning: this is the *time translation operator*, and not the time evolution operator.

$$\mathcal{T}(t) = \sum_{\alpha} |\langle t, \alpha|\psi\rangle|^2. \tag{4.30}$$

In terms of the particle's wave function in momentum representation, this same result reads

$$\mathcal{T}(t) = \sum_{\alpha} \frac{1}{2\pi m} \int dpdp' \; \Theta(\alpha p)\Theta(\alpha p')\psi^*(p)\psi(p')\sqrt{|pp'|}e^{\frac{it(p^2-(p')^2)}{2m}}. \tag{4.31}$$

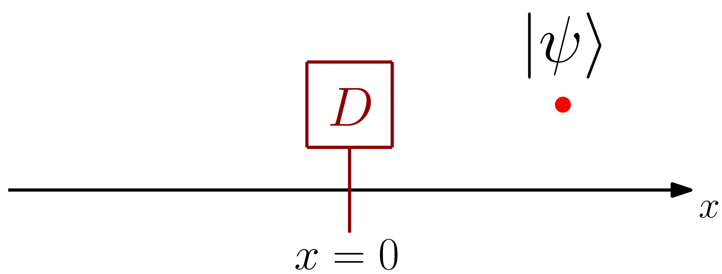


Figure 4.1: Illustrative diagram of an time-of-arrival experiment. Given a particle prepared at a state $|\psi\rangle$, all we want to know is the probability that the detector D fires at a time t.

The probability distribution (4.31) is also known as *Kijowski's distribution*, since the work of J. Kijowski [30] seems to be the first where this result appears. It is important to mention that Kijowski's derivation is completely different from the one presented here, consisting on an axiomatic approach based off an analogy with time of arrival distributions for classical ensembles.

As a final remark, notice that the probability distribution $\mathcal{T}(t)$ is defined for all $t \in \mathbb{R}$. However, the state $|\psi\rangle$ we use to compute (4.30) is to be interpreted as a Schrödinger picture state prepared at some fixed instant $t = \lambda$. In this case, only the values $t > \lambda$ can be registered by the detector, since it makes no sense to detect the particle before its state was prepared. For instance: if $\lambda = 0$, how one can interpret negative values for the time of arrival?

Here, we circumvent this problem adopting the point of view of Grot *et all* (reference

[31]): we suppose that the system was prepared in the far past ($\lambda \to -\infty$) and have evolved without any perturbation to the state $|\psi(t=0)\rangle$ we use to make the predictions. Of course, on the laboratory the experimentalist will only be able to measure a positive time of arrival, but the point here is this picture allows one to give a proper interpretation to the negative values predicted by the theory.

4.2.1 Interpreting the eigenstates of the Aharonov-Bohm time operator

As mentioned before, the eigenstates of the time operator (4.9) are not always orthogonal, which can difficult the task of ascribing a physical meaning to the abstract objects $|t,\alpha\rangle$. However, it is still possible to construct physical states — wave packets — from these eigenstates in the same fashion one does for position or momentum eigenstates. Moreover, as we shall see such wave packets can be taken as peaked distributions around the time of arrival t.

Let us see how this construction can be made. See also reference [32], which obtain similar results.

For the sake of definiteness, we shall construct wave packets using combinations of the states $|t,+\rangle$. Consider

$$\Phi(p) = N \int dt\; e^{-\delta^2(t-t_0)^2} \langle p|t,+\rangle = \frac{N\Theta(p)\sqrt{p}}{\sqrt{2\pi m}} \int dt\; e^{-\delta^2(t-t_0)^2} e^{\frac{ip^2 t}{2m}}, \tag{4.32}$$

where $N \in \mathbb{R}$ is a constant to ensure the normalization of $\Phi(p)$. The parameter $\delta > 0$ is inversely proportional to the width of the wave packet, and $t = t_0$ is the central value.

Performing the calculation, we get

$$\Phi(p) = \frac{N\Theta(p)}{\delta}\sqrt{\frac{p}{2m}} \exp\left(-\frac{p^4}{16m^2\delta^2} + \frac{ip^2 t_0}{2m}\right). \tag{4.33}$$

In order to give a proper interpretation to this state, we must compute the time of arrival distribution (4.31). The "temporal" representation of the state $|\Phi\rangle$ reads

$$\langle t,+|\Phi\rangle = \frac{N}{2m\delta\sqrt{\pi}}\int_0^{\infty} dp\; p\exp\left(-\frac{p^4}{16m^2\delta^2} - \frac{ip^2(t-t_0)}{2m}\right). \tag{4.34}$$

Solving this, we obtain

$$\langle t,+|\Phi\rangle = \frac{N}{2}e^{-\delta^2(t-t_0)^2}\left[1 - i\frac{2}{\sqrt{\pi}}F(\delta(t-t_0))\right], \tag{4.35}$$

where we have defined the function

$$F(z) = \int_0^{\delta(t-t_0)} du\; e^{u^2}. \tag{4.36}$$

Notice that $\langle t,-|\Phi\rangle$ vanishes, since the state $|\Phi\rangle$ contains only states with $\alpha = 1$ (see relation (4.16)). Thus, the time of arrival probability distribution is given by

$$\mathcal{T}(t) = |\langle t,+|\Phi\rangle|^2 = \frac{N^2}{4}e^{-2\delta^2(t-t_0)^2}\left[1 + \frac{4}{\pi}F(\delta(t-t_0))^2\right]. \tag{4.37}$$

As this result suggests, the probability distribution $\mathcal{T}(t)$ is centered at $t = t_0$, becoming more and more peaked as one increases the parameter δ, which is inversely proportional to the width of the wave packet (4.32). Below, we show some representative plots of $\mathcal{T}$ — ignoring the normalizing constant $\frac{N^2}{4}$ — for $t_0 = 1$ and different values of δ.

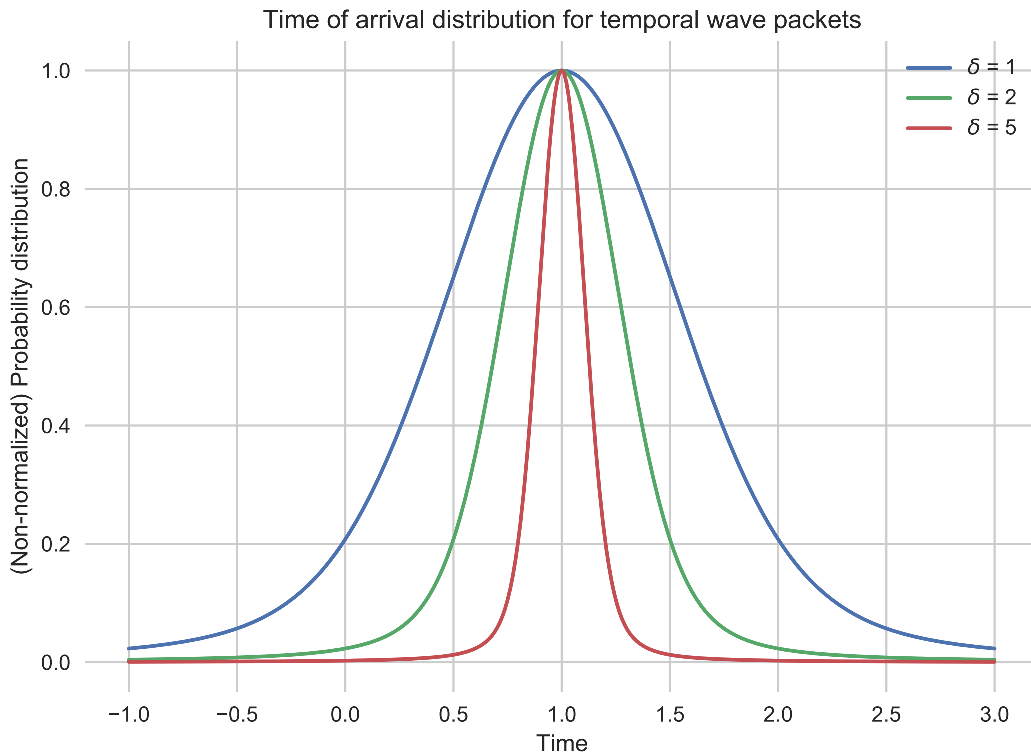


Figure 4.2: Time of arrival distribution for temporal wave packets centered at $t = t_0$. The parameter δ is inversely proportional to the width of the wave packet. The probability distributions are not normalized (obviously, this does not change the qualitative features that one can grasp by staring at these results).

Therefore, the normalized state $|\Phi\rangle$ behaves as expected: it represents a physical state that when subjected to a time of arrival measurement gives outcomes that can be made as sharp as one pleases around some specific value.

4.2.2 Example: time of arrival probability distribution for a Gaussian wave packet

Consider a free particle with mass m described by a Gaussian wave packet centered at some position x_0 and with characteristic momentum p_0. We are going to use the formalism just presented above to predict the probability distribution $\mathcal{T}(t)$ for this particle to arrive at position $x = 0$ in the time t.

For a Gaussian wave packet at $t = 0$, we have

$$\psi(x) = \left(\frac{1}{2\pi\sigma^2}\right)^{1/4} e^{-\left(\frac{x-x_0}{2\sigma}\right)^2} e^{ip_0 x}. \tag{4.38}$$

The constant $\sigma > 0$ represents the spatial width of the wave packet. Indeed, recall that the standard deviation of the position operator over this state is

$$\Delta\hat{x} = \sigma.$$

The same state in momentum representation reads

$$\psi(p) = \left(\frac{2\sigma^2}{\pi}\right)^{1/4} e^{-i(p-p_0)x_0} e^{-\sigma^2(p-p_0)^2}. \tag{4.39}$$

Using this relation, we can calculate the functions $\langle t, \alpha|\psi\rangle$ as

$$\langle t, \alpha|\psi\rangle = \int dp\ \langle t, \alpha|p\rangle\langle p|\psi\rangle = \frac{1}{\sqrt{2\pi m}}\left(\frac{2\sigma^2}{\pi}\right)^{1/4} \int dp\ \Theta(\alpha p)\sqrt{|p|}e^{-i(p-p_0)x_0} e^{-\sigma^2(p-p_0)^2} e^{-\frac{ip^2 t}{2m}}. \tag{4.40}$$

Thus, the probability density is

$$\mathcal{T}(t) \equiv \sum_\alpha |\langle t, \alpha|\psi\rangle|^2 = \frac{\sigma}{m\pi\sqrt{2\pi}} \sum_\alpha \left| \int dp\ \Theta(\alpha p)\sqrt{|p|}e^{-i(p-p_0)x_0} e^{-\sigma^2(p-p_0)^2} e^{-\frac{ip^2 t}{2m}} \right|^2. \tag{4.41}$$

Using the software Mathematica [48], we have solved this analytically. The result can be written as

$$\mathcal{T}(t) = \frac{g(t)}{h(t)}\left(|f_1(t)|^2 + |f_2(t)|^2\right), \tag{4.42}$$

where we have made the following definitions:

$$g(t)=\frac{\sigma}{16m^3}\sqrt{\frac{\pi}{2}}\exp\left[-\sigma^2\left(\frac{m^2x_0^2+4m^2p_0^2\sigma^4+2p_0t(p_0t+mx_0)}{4m^2\sigma^4+t^2}\right)\right], \tag{4.43}$$

$$h(t)=\left(4\sigma^4+\frac{t^2}{m^2}\right)^{3/2}\sqrt{4p_0^2\sigma^4+x_0^2}, \tag{4.44}$$

$$z=\frac{m(2p_0\sigma^2-ix_0)^2}{8m^2+4it}, \tag{4.45}$$

$$f_1(t)=m(2p_0\sigma^2-ix_0)^2\left[I_{\frac{3}{4}}(z)+I_{-\frac{1}{4}}(z)-I_{\frac{5}{4}}(z)-I_{\frac{1}{4}}(z)\right]+[2i(mx_0p_0\sigma^2-t)-4m\sigma^2]I_{\frac{1}{4}}(z), \tag{4.46}$$

$$f_2(t)=m(2p_0\sigma^2-ix_0)^2\left[I_{\frac{3}{4}}(z)+I_{-\frac{1}{4}}(z)+I_{\frac{5}{4}}(z)+I_{\frac{1}{4}}(z)\right]+[4m\sigma^2+2i(t-mx_0p_0\sigma^2)]I_{\frac{1}{4}}(z). \tag{4.47}$$

Here, $I_\alpha(x)$ is the modified Bessel function of first kind (see appendix D).

The plot of this distribution for $m=x_0=1$, $p_0=-1$ and some values of σ is displayed next.

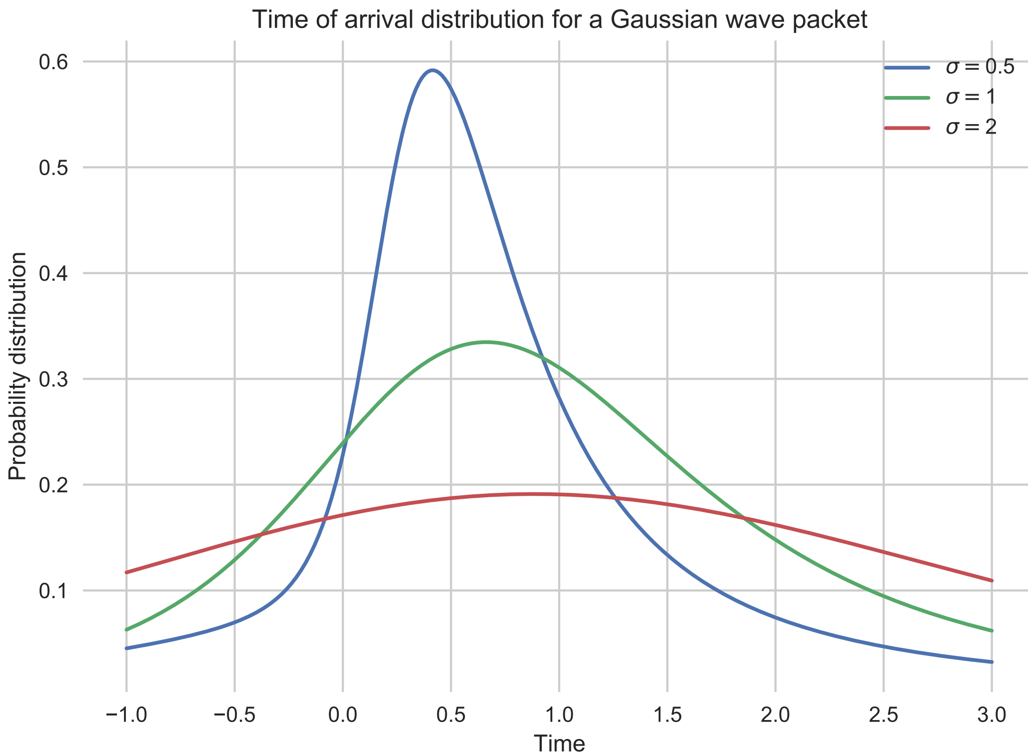


Figure 4.3: Time of arrival distribution for a particle with mass $m = 1$ described by a Gaussian wave packet whose center has position $x_0 = 1$ and momentum $p_0 = -1$. The parameter σ represents the width of the packet in position representation.

As we can see from the plots displayed on figure 4.3, for fixed values of p_0 and x_0 the probability distribution is highly sensitive with respect to the parameter σ. Moreover, at first it is not clear what is the relation of the distributions with classical time of arrival $t_c = -\dfrac{mx_0}{p_0} = 1$.

In order to get a better understanding about the meaning of the result (4.42), we consider the limit of "quasi-classical states", where the *relative dispersions* of position and momentum, namely

$$\left|\frac{\Delta x}{x_0}\right| = \frac{\sigma}{|x_0|}, \qquad \left|\frac{\Delta p}{p_0}\right| = \frac{1}{2\sigma|p_0|} \tag{4.48}$$

are sufficiently small. That is: we are going to focus on the case where the Gaussian wave packet approximates a state localized both in space and momentum.

Let us still fix the classical time of arrival to be 1, with $m = \sigma = 1$. Then, the outlined relative dispersions can be made sufficiently small by increasing x_0 and p_0 in the same proportion. The following plot illustrates this point.

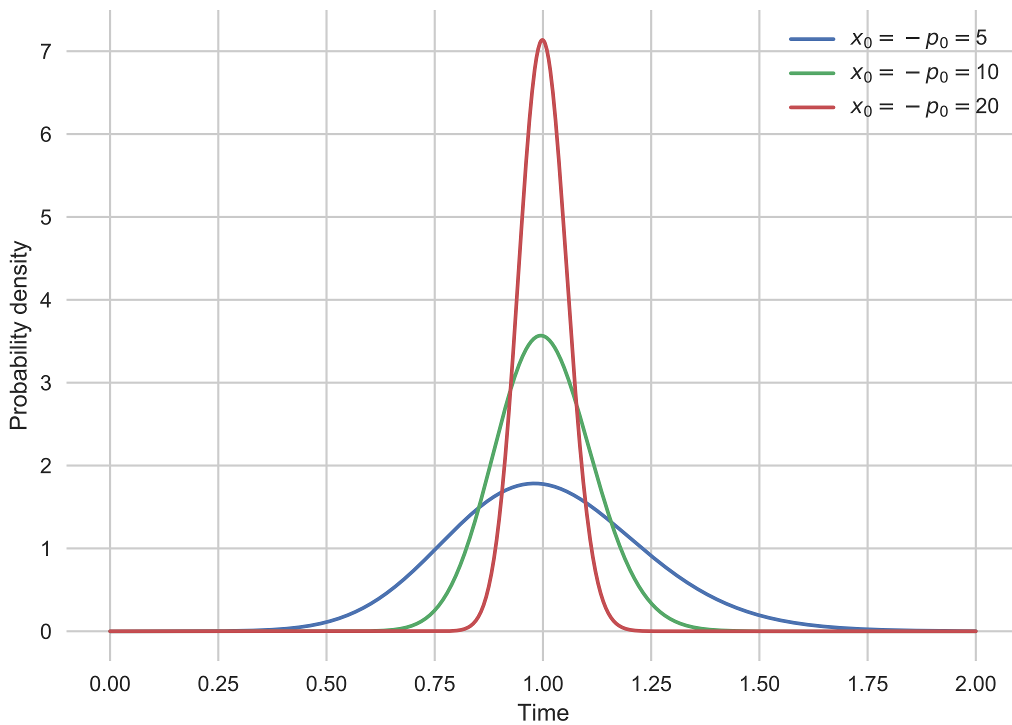


Figure 4.4: Time of arrival distribution for Gaussian wavepackets approaching the limit of a classical state

Notice that for $\left|\dfrac{\Delta x}{x_0}\right| = \dfrac{1}{20}$ and $\left|\dfrac{\Delta p}{p_0}\right| = \dfrac{1}{40}$, the distribution is already highly peaked around the classical time of arrival $t_c = 1$, thus reproducing what is expected for the classical limit.

The case corresponding to $x_0 = p_0 = 0$ needs to be discussed separately. Classically, it makes no sense to define a time of arrival, since the particle holds still at the origin. In the quantum case, however, we obtain a symmetric distribution around $t = 0$. Indeed, setting

$x_0 = p_0$ one can solve equation (4.41) analytically, getting as result the following expression:

$$\mathcal{T}(t) = \frac{\Gamma(3/4)^2}{2m\sigma^2\pi\sqrt{2\pi}} \left[1 + \left(\frac{t}{2m\sigma^2}\right)^2\right]^{-3/4}, \tag{4.49}$$

where $\Gamma(z)$ is the Euler gamma function (defined on appendix D).

Next, we include a plot of this distribution for $m = 1$. Again, notice the sensible dependence with the parameter σ.

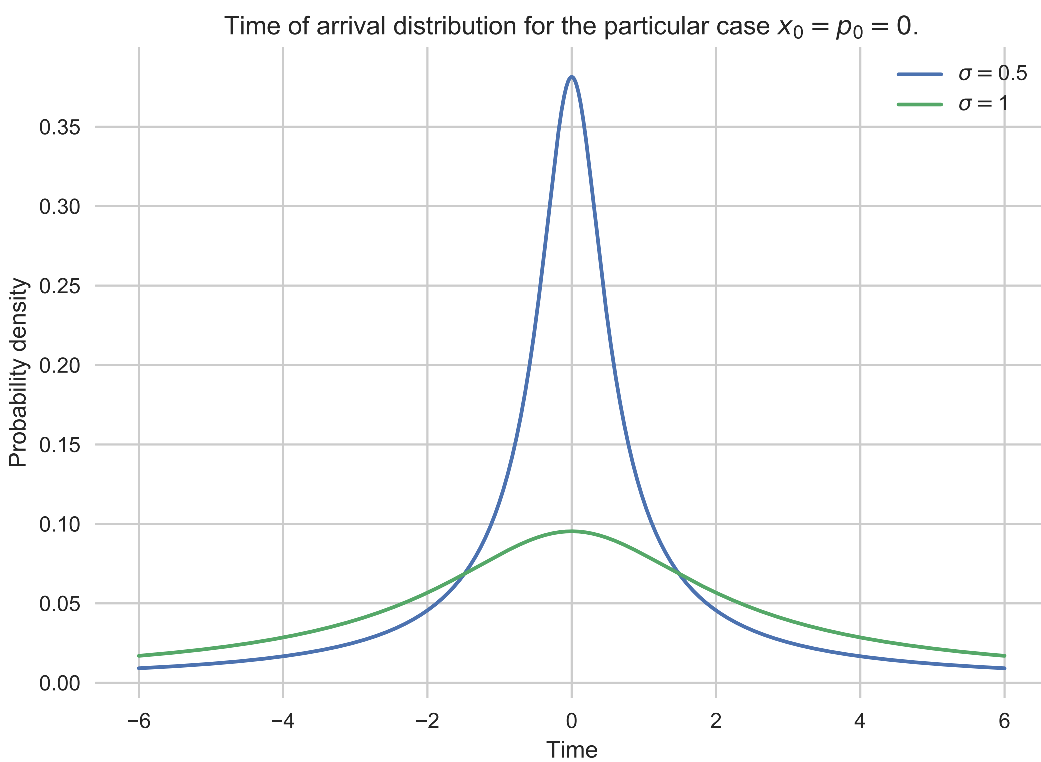


Figure 4.5: Time of arrival distribution for a Gaussian wave packet with $x_0 = p_0 = 0$. The particle's mass was set equal to 1.

4.3 Ruling out the existence of honest clocks in Quantum Mechanics

In this section, we review the *no perfect clock theorem*, a result due to Unruh and Wald (see reference [?]) that points out to the impossibility of construct a time operator for quantum systems that possess an energy ground state from a more fundamental perspective than that from Pauli's theorem, since it does not assume that such a time operator must be self-adjoint. After giving a precise statement and a proof of the result, we discuss its implications for the *nonexistence of honest clocks in Quantum Mechanics*.

Consider a quantum system whose Hamiltonian $\hat{H} : \mathcal{H} \to \mathcal{H}$ is bounded from below, in the same sense discussed in section 4.1. We would like to construct an operator $\hat{T}$ that allows one to codify the time measurements that can be realized on the system under study. So far, the only thing we require of $\hat{T}$ is that it must posses a set of eigenstates $\{|T_\alpha\rangle\}_{\alpha\in\Lambda}$ that can be indexed by a complete ordered set Λ.

In order to fix ideas, one can regard the parameter α as being directly related to the proper time along the worldline of the observer that defines the operator $\hat{T}$. That is: if an observer following a timelike trajectory on spacetime whose affine parameter is α, the measurements he or she extracts from some hypothetical quantum device associated with the operator $\hat{T}$ are described by the eigenvalue equation

$$\hat{T}\,|T_\alpha\rangle = T_\alpha\,|T_\alpha\rangle\,, \tag{4.50}$$

with $T_\alpha \in \mathbb{R}$ for each $\alpha \in \Lambda$.

As opposed to the statement of Pauli's theorem, the operator $\hat{T}$ *does not need to be considered self-adjoint*. Indeed, the only additional conditions Wald and Unruh assume it must satisfies are:

(i) $T_\alpha > T_\beta$ for each $\alpha, \beta \in \Lambda$ with $\alpha > \beta$.

(ii) For each $\alpha \in \Lambda$ there exists $\beta > \alpha$ and $t > 0$ such that $\langle T_\beta | \hat{U}(t) | T_\alpha \rangle \neq 0$. Here, t must be understood as the time evolution parameter that appears in the Schrödinger equation (2.25) and $\hat{U}(t)$ is the time evolution operator arising from the Hamiltonian $\hat{H}$.

(iii) Given $\alpha \in \Lambda$, for all $\beta < \alpha$ and $t > 0$ we have $\langle T_\beta | \hat{U}(t) | T_\alpha \rangle = 0$.

Physically, condition (ii) means that for any time interval t elapsed in the evolution of the system, the probability amplitude of its time measurements to advance forwards is non zero. In the same token, the assumption (iii) presupposes that at any instant t the probability for time measurements to run backwards is zero. If we imagine that the quantum system under study realizes a *honest clock*, all that we are stating is that the construction of an observable ($\hat{T}$) that allows one to extract time measurements from such apparatus must obey the specifications about how an honest clock works: whereas we always must have a non zero probability for the clock to run forward and its reading display a certain value, the clock is forbidden to have any probability of "going back in time".

Now comes the core of the *no perfect clock theorem*:

Theorem: It is not possible to construct a linear operator with real spectrum whose eigenstates $\{|T_\alpha\rangle\}$ satisfy the outlined features $1-3$.

Proof:

The proof to be presented here is based on the approach of [**?**].

First of all, we suppose that our Hamiltonian $\hat{H}$ does not depend on the time parameter t, so that the time evolution operator $\hat{U}(t)$ is given simply by

$$\hat{U}(t) = e^{-it\hat{H}}. \tag{4.51}$$

Recall that if the system associated whose evolution is dictated by $\hat{H}$ is to represent a honest clock, then such assumption is obligatory.

Next, consider the function

$$f(z) = \langle T_\beta | e^{-iz\hat{H}} | T_\alpha \rangle \,, \tag{4.52}$$

with $\beta < \alpha$ and $z \in \mathbb{C}$.

Without loss of generality, suppose that the spectrum of $\hat{H}$ is discrete: $\hat{H} |E_n\rangle = E_n |E_n\rangle$, where $E_i < E_j$ for $i < j$ and $n \geq 0$. Evidently, this is in concord with the fundamental assumption of $\hat{H}$ being bounded from below. Thus, we can cast the function $f(z)$ as

$$f(z) = \sum_{n=0}^{\infty} e^{-izE_n} \langle T_\beta | E_n \rangle \langle E_n | T_\alpha \rangle \,. \tag{4.53}$$

Setting $z = x + iy$, it is clear the function f is holomorphic[2] in the complex half-plane defined by $y \leq 0$. That is because the only possibility of "bad behavior" in such a case would be the presence of a divergent real exponential e^{yE_n} for $y \leq 0$. But, since we have a ground state no energy level E_n extends to $-\infty$ and, so, such behavior is ruled out.

According to the hypothesis (iii), $f(z) = 0$ for $y = 0$ and $x \geq 0$. However, adapting the mathematical result exposed in appendix B to our case, we see that if a holomorphic function vanishes in some interval of the real semi-axis $x \geq 0$, then it vanishes in the whole semi-plane $y \leq 0$. In particular, this means that the function f must vanish for $y = 0$ and $x < 0$.

With these considerations at hand, it turns out that for $t \in \mathbb{R}$, $t > 0$, we have

$$\langle T_\alpha | e^{-it\hat{H}} | T_\beta \rangle = f^*(-t) = 0, \tag{4.54}$$

Notice that such a result contradicts the hypothesis (ii). Therefore, we conclude that for a quantum system that admits a state of lowest energy it is not possible to construct a time observable satisfying simultaneously the hypotheses $(i) - (iii)$.

■

Therefore, if one assumes that that the measurements of any clock subject to the laws

[2] A complex function $f : \Omega \subset \mathbb{C} \to \mathbb{C}$ is said to be holomorphic if for each $p \in \Omega$ there exists a neighborhood of p where f is complex differentiable.

of Quantum Mechanics is to be represented by the hypothetical operator $\hat{T}$ considered here, then there is always a probability that the readings go "backwards in time", meaning that the time as told by the clock will not always evolve according to the classical proper time, as defined along the worldline of the observer that carries the clock.

Since there is no doubts concerning the physical reasonableness of the assumptions made about the time operator $\hat{T}$, we understand this result as *ruling out the existence of honest clocks*.

A related result points out to the impossibility of using quantum states to perfectly keep track of the external time parameter t. More precisely: given a system prepared as the Schrödinger-picture quantum state $|\psi(t_0)\rangle$ that is not an eigenstate of the Hamiltonian, the values of $t > t_0$ for which $|\psi(t)\rangle$ is completely distinguishable from the initial state (i.e., $\langle\psi(t_0)|\psi(t)\rangle = 0$) cannot form an interval of the real line.

In order to prove this, we consider the function (and take $t_0 = 0$ without loss of generality)

$$F(t) = \langle\psi(0)|\psi(t)\rangle = \langle\psi(0)|e^{-it\hat{H}}|\psi(0)\rangle. \tag{4.55}$$

Let

$$A = \{t \in \mathbb{R}_+ : F(t) = 0\}.$$

Notice that if A were an open interval of $\mathbb{R}_+$, then using essentially the same argument we employed just after equation (4.53) we would conclude that $F(t) = 0$ for all $t \geq 0$, which surely is a contradiction for $t = 0$.

That is: it is not possible to have a continuous succession of time evolved orthogonal states. Nonetheless, as will be discussed in the next chapter, it is still possible to use the evolution of a system through a succession of orthogonal states (that are not eigenstates of the Hamiltonian) to define a quantum clock with *discrete* readings.

4.4 Concluding remarks

Although Pauli's theorem reveals that it is not possible to construct a self-adjoint time operator canonically conjugated to the Hamiltonian of a physical system — admitting that all physical systems in nature must possess a ground state —, we have seen that it is still possible to define time measurements on the quantum realm resorting to the concept of a POVM. Within this scenario, we have Kijowski's distribution as a quite reasonable answer to the experimental question about the time of arrival of a free particle. Indeed, other references ([27, 33, 35, 36]) derive the same result using distinct approaches.

Now, regarding the main issue of this essay ("how quantum theory impacts on our definition of honest clocks?"), we have seen that a result due to Unruh and Wald rules out the possibility of using quantum systems as honest clocks. However, this does not prevent us to give operational definitions for the elapsed time between two events using a quantum system that realizes a particular clock. This is the subject of the last chapter.

5

Quantum Clocks

The last chapter of the present essay is concerned about three particular formulations of quantum systems as clocks. First, we review the model of Salacker and Wigner (later "revitalized" by Peres), which is the most general quantum clock build from a system with a finite energy spectrum. Then, we discuss a toy model that uses the effect of Larmor precession to extract an intrinsic time from a concrete experimental situtation. Finally, we end the chapter presenting a model of a quantum clock based on *phase observables* — to be defined with the aid of POVMs — measured over coherent states of the quantum harmonic oscillator.

5.1 The Salecker-Wigner-Peres quantum clock

The work of Salecker and Wigner [41] may be considered as the first historical landmark on the issue of quantum clocks. In that work they introduce the notion of a quantum clock as a system with a finite number of energy levels, the "tic-tac" being characterized as the evolution through successive orthogonal states. Later on, Asher Peres [42] revisited this model studying some applications and drawing a considerable attention to the issue again. For this reason, the quantum clock model we are going to discuss in this section is most time referred in the literature as the *Salecker-Wigner-Peres* (SWP) quantum clock.

According to Asher Peres, "*a clock is a dynamical system which passes through a succession of states at constant time intervals*" [42]. Going to the quantum realm, such idea can be realized by a system prepared in a state whose dynamical evolution imitates the periodic circular motion of a (classical) clock hand over the dial. Let us see how this can be done.

Consider any quantum system described by N energy levels. For $N = 2$, e.g., one can imagine simply a nonrelativistic spin-1/2 particle placed in a magnetic field. The Hamiltonian of this system is

$$\hat{H} = \sum_{n=0}^{N-1} E_n |n\rangle\langle n|, \tag{5.1}$$

so that $|n\rangle$ is an energy eigenstate corresponding to the energy eigenvalue E_n. Moreover, the collection $\{|n\rangle\}$ constitutes an orthonormal basis for the Hilbert space of the system, namely $\mathcal{H} \cong \mathbb{C}^N$. Here, for the sake of simplicity, we consider the energy levels to be equally separated: $E_n = n\omega$.

The initial state of our quantum clock will be taken as

$$|\psi(0)\rangle = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} |n\rangle \, . \tag{5.2}$$

Working in the Schrödinger picuture, the dynamical evolution of this state reads

$$|\psi(t)\rangle = \hat{U}(t)\, |\psi(0)\rangle = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} e^{-in\omega t} \, |n\rangle \, . \tag{5.3}$$

The key point here is that the dynamics of the state $|\psi(t)\rangle$ is such that it passes through a succession of orthogonal states at constant coordinate time intervals. Indeed, one has

$$\langle\psi(0)|\psi(t)\rangle = 0 \iff \sum_{n=0}^{N-1} e^{-in\omega t} = \frac{e^{-iN\omega t} - 1}{e^{-i\omega t} - 1} = 0. \tag{5.4}$$

Thus, $|\psi(t)\rangle$ will be orthogonal to the initial state at each instant $t_n = n\tau$, $n \geq 1$, where we have defined

$$\tau = \frac{2\pi}{\omega N}. \tag{5.5}$$

Notice that such a quantity may be understood as the *clock precision*. We shall return to this point later.

It turns out that the N states defined by

$$|\xi_n\rangle = |\psi(n\tau)\rangle\,, \qquad n = 0, 1\ldots, N-1 \tag{5.6}$$

constitute an orthonormal basis for $\mathcal{H}$. These states shall be referred as the *clock eigenstates*, in the sense that the transition between them marks what one may call the "tic-tack" of the clock. Precisely, they are eigenstates of the operator

$$\hat{T}_c = \tau \sum_{n=0}^{N-1} n|\xi_n\rangle\langle\xi_n|, \tag{5.7}$$

which gives the discrete clock readings t_n modulo $N\tau$. Here, it is important to recognize that $\hat{T}_c$ cannot be regarded as a time operator in the sense considered in the previous chapter, since it is not canonically conjugate to the Hamiltonian. Moreover, it respects the no-go theorem of Wald and Unruh (refer to section 4.3): for $\delta t > 0$, $t_1 < t_2$, we have

$$\langle\psi(t_1)|\hat{U}(\delta t)|\psi(t_2)\rangle = \frac{1}{N}\left(\frac{e^{-iN\omega(\delta t + t_2 - t_1)} - 1}{e^{-i\omega(\delta t + t_2 - t_1)}}\right), \tag{5.8}$$

which is nonzero in general. That is: through all the evolution of the system, there is always a nonzero probability of the clock readings — when regarded as continuous from the theoretical point of view — to go backwards in time. A possible way around this is to regard the clock readings as being consisted only by the discrete readings t_n. From the experimental point of view, we know that it is always possible to perform measurements that allows one to completely distinguish between two orthogonal states. Thus, the picture of the clock is the following: every time we known for sure that the system has made a transition from $|\xi_n\rangle$ to

$|\xi_{n+1}\rangle$, some device will add τ to the clock's display. Of course, this procedure comes at the cost that one cannot measure time with precision better than τ.

This discussion may be supported by looking at the mean value and dispersion of $\hat{T}_c$ over the clock state $|\psi(t)\rangle$ for arbitrary t. A straightforward calculation shows that

$$\langle \hat{T}_c \rangle(t) = \frac{\tau}{N^2} \sum_{n=0}^{N-1} n \left[\frac{\sin(N\omega(t - n\tau)/2)}{\sin(\omega(t - n\tau)/2)} \right]^2 \tag{5.9}$$

and

$$\Delta \hat{T}_c(t) = \left\{ \frac{\tau^2}{N^2} \sum_{n=0}^{N-1} n^2 \left[\frac{\sin(N\omega(t - n\tau)/2)}{\sin(\omega(t - n\tau)/2)} \right]^2 - (\langle \hat{T}_c \rangle(t))^2 \right\}^{1/2} . \tag{5.10}$$

As expected, the dispersion vanishes at every instant $t_n = n\tau$, $n = 0, 1, \ldots, N-1$, when the mean value of $\hat{T}_c$ is exactly t_n. However, it is still possible to *define* the clock reading at "classical time" t as $\langle \hat{T}_c \rangle(t)$. Of course, such a definition comes at the cost that the confidence interval of this statement is $\langle \hat{T}_c \rangle \pm \Delta \hat{T}_c$.

The next plot shows how the mean value of $\hat{T}_c$ compares with the readings of an hypothetical classical clock whose readings coincide with the evolution parameter t. For the SWP quantum clock, we choose $N = 6$ and $\omega = \frac{\pi}{3}$, so that $\tau = 1$.

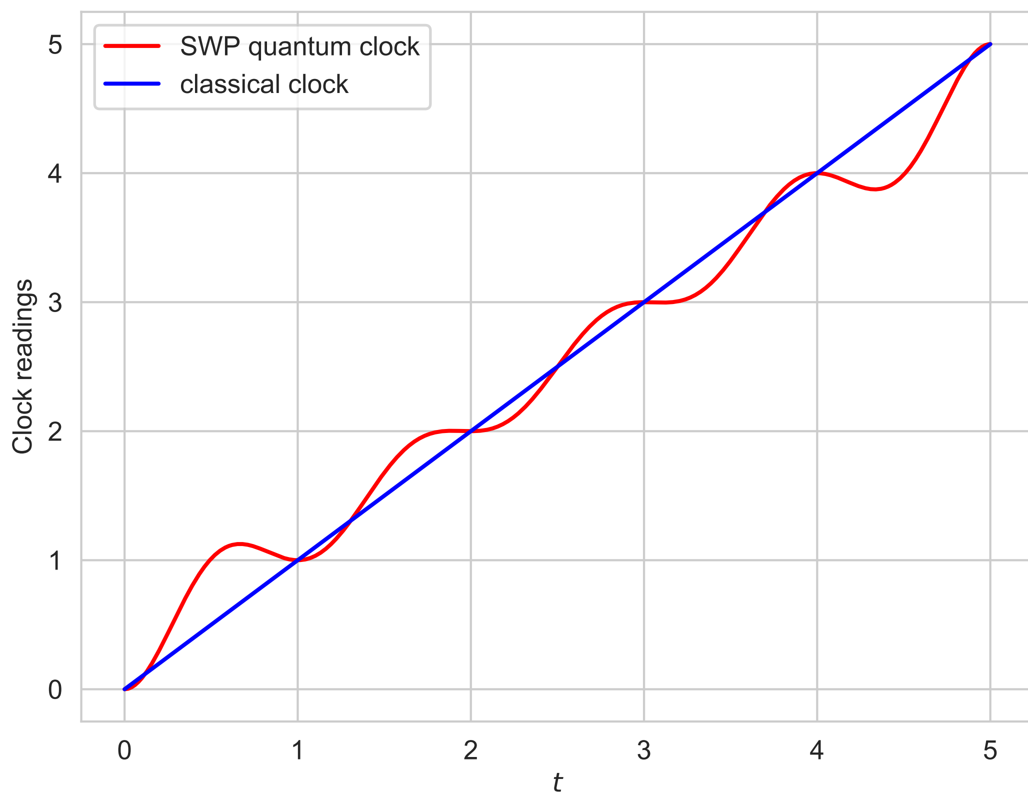


Figure 5.1: Comparing the readings of the SWP quantum clock with those of a classical clock that measures the evolution parameter t.

As this plot illustrates, if the time as told by the quantum clock is considered to be the mean value of the operator $\hat{T}_c$, then we will observe the clock readings going "backwards" in time when compared to the evolution parameter t. For the next plot, we concentrate only on the quantum clock readings, showing it along with the confidence interval due to the spread $\Delta\hat{T}_c$.

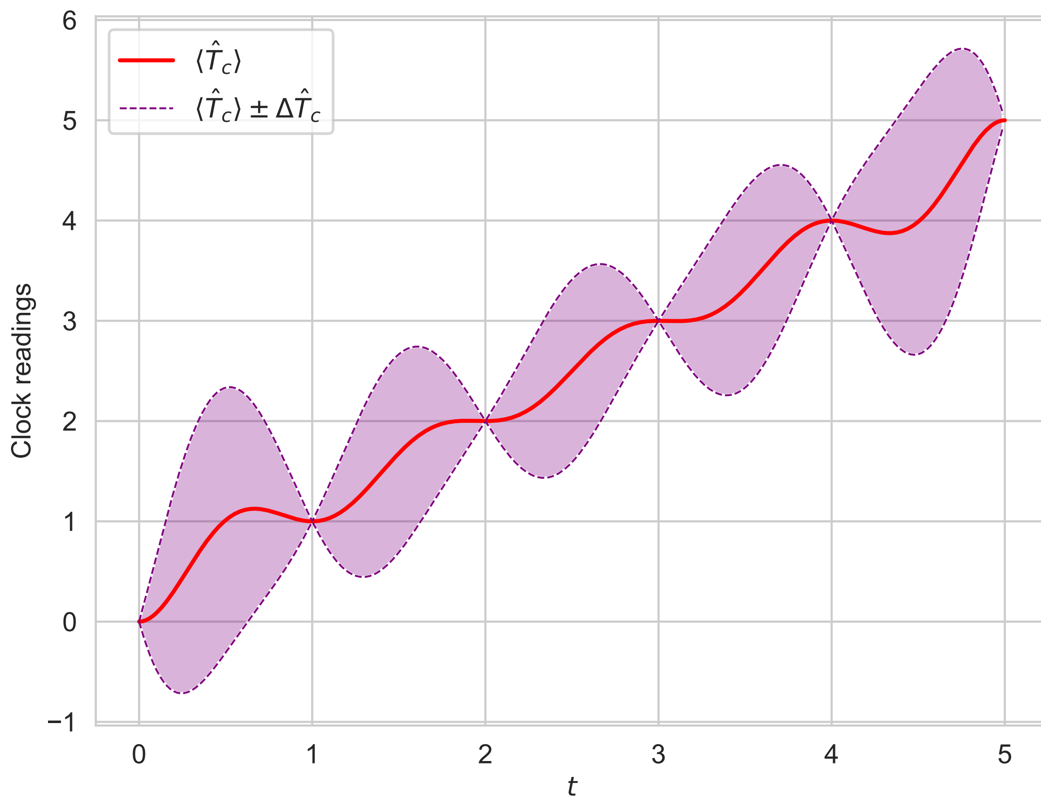


Figure 5.2: Quantum clock readings and its confidence interval.

Given all these considerations, we *define* the SWP quantum clock readings as the integer multiples of the quantity τ (5.5), thus regarding τ as the clock precision. It is important to realize that given any experimental procedure that determines when the evolution of our system passes exactly through the orthogonal states $|\xi_n\rangle$, then the quantities $n\tau$ can be measured without the necessity of any classical clock. Of course, the SWP clock cannot perform better than the former, since the laws of Quantum Mechanics — when applied to the particular system under study — prohibits time to be sharply defined at scales smaller than τ.

At this point, one may wonder: what forbids us to take the limit $\tau \to 0$? From relation (5.5), one sees that the precision of the SWP quantum clock can be improved by increasing the number of energy levels N. However, changing N has a direct impact on the energy dispersion over the state $|\psi(t)\rangle$. Indeed, since

$$\hat{H}\,|\psi(t)\rangle = \frac{\omega}{\sqrt{N}} \sum_{n=0}^{N-1} n e^{-in\omega t}\,|n\rangle\,, \tag{5.11}$$

the mean value of the energy at each instant t reads.

$$\langle \hat{H} \rangle = \frac{\omega}{N} \sum_{n=0}^{N-1} n = \frac{\omega(N-1)}{2}. \tag{5.12}$$

In the same token,

$$\langle \hat{H}^2 \rangle = \frac{\omega^2}{N} \sum_{n=0}^{N-1} n^2 = \frac{\omega^2}{6}(2N-1)(N-1), \tag{5.13}$$

so that

$$\Delta \hat{H} = \omega \sqrt{\frac{N^2-1}{12}}. \tag{5.14}$$

This means that a quantum clock with arbitrary precision ($\tau \to 0$) would require measurements to be carried on a state with an arbitrarily high energy dispersion. More than that: for $N \gg 1$, the largest energy available to the system, ωN, becomes arbitrarily high, so that introducing such an apparatus on the spacetime could lead to the formation of a black hole. Surely, we want to avoid this kind of aberration on conceptual grounds!

Now, it remains to understand how the model just presented can be applied to measure time in physical situations. Following Peres [42], we shall discuss the simplest case possible: the measurement of the time a particle spent on some region of space.

In the model proposed by Peres, one has a bipartite system $\mathcal{H} = \mathcal{H}_p \otimes \mathcal{H}_c$ composed by a free particle moving in one dimension and a clock. The total Hamiltonian of the system is,

$$\hat{H} = \frac{\hat{p}^2}{2m} \otimes \hat{\mathbb{I}}_c + \Theta(x)\Theta(d-x)\hat{\mathbb{I}}_p \otimes \hat{H}_c \tag{5.15}$$

where $\hat{H}_c$ stands for the clock Hamiltonian as defined on equation (5.1). The presence of the term $\Theta(x)\Theta(d-x)$ guarantees that the clock couples with the particle solely on the region $0 < x < d$, which is precisely the region one wants to know the time that the particle takes to transverse it.

We assume that the global state of the system before the particle enters in the region $0 < x < d$ is described by

$$|\Phi_i\rangle = |p\rangle \otimes |\xi_0\rangle \equiv |p, \xi_0\rangle = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} |p, n\rangle \tag{5.16}$$

Here, $|p\rangle$ denotes a momentum eigenstate, which is a plane wave in the position representation:

$$\langle x|p\rangle = \frac{1}{\sqrt{2\pi}} e^{ipx} \tag{5.17}$$

Notice that the clock is in its initial state ("off" or "ready" state) $|\xi_0\rangle$. After it interacts with the particle on the measurement process, the global state of the system will no longer be separable, and the correlations created will be used to determine how the clock accuses the time the particle takes to transverse the region. Thus, we write the state of the system particle + clock after the experiment as

$$|\Phi_f\rangle = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} |\phi_n, n\rangle \, . \tag{5.18}$$

In order to determine the states $|\phi_n\rangle$ in terms of the relevant quantities of the problem, it is useful to work with the representation

$$\langle x, \psi(t)|\Phi_f\rangle = \frac{1}{N} \sum_{n=0}^{N-1} \phi_n(x) e^{in\omega t}, \tag{5.19}$$

where $|\psi(t)\rangle$ is the clock state (5.3).

Now, imposing $|\Phi_f\rangle$ to be a stationary state with energy $E = \dfrac{p^2}{2m}$ it follows that

$$\hat{H}\left|\Phi_f\right\rangle = E\left|\Phi_f\right\rangle \implies \sum_{n=0}^{N-1}\left[-\frac{1}{2m}\frac{d^2\phi_n(x)}{dx^2} + \omega n\Theta(x)\Theta(d-x)\phi_n(x)\right]e^{in\omega t} = E\sum_{n=0}^{N-1}\phi_n(x)e^{in\omega t}. \tag{5.20}$$

So, each function ϕ_n satisfies to the differential equation

$$-\frac{1}{2m}\frac{d^2\phi_n(x)}{dx^2} + \omega n\Theta(x)\Theta(d-x)\phi_n(x) = E\phi_n(x). \tag{5.21}$$

Note that this problem is equivalent to the scattering of a free particle by the potential barrier

$$V_n(x) = \begin{cases} \omega n, & 0 < d < x \\ 0, & \text{otherwise} \end{cases} \tag{5.22}$$

This problem can be solved in the traditional way: in each region, the proposed solution is

$$\phi_n(x) = e^{ipx} + R_n e^{ipx}, \qquad x < 0 \tag{5.23}$$

$$\phi_n(x) = A_n e^{ik_n x} + B_n e^{-ik_n x}, \qquad 0 < x < d \tag{5.24}$$

$$\phi_n(x) = T_n e^{ipx}, \qquad x > d \tag{5.25}$$

where $k_n = \sqrt{2m(E - n\omega)}$. Here, we are assuming that the kinetic energy of the particle is bigger than the largest energy available to the quantum clock, i.e, $E > \omega(N-1)$. In this way, $k_n \in \mathbb{R}$ for all n.

Imposing the boundary conditions — continuity of the function and its derivative — at the points $x = 0$ and $x = d$, one can compute the constants R_n, A_n, B_n and T_n. Since we are interested on the description of the state $\left|\Phi_f\right\rangle$ for $x > d$, when the clock is "turned off", we just give the expression for the last one:

$$T_n = \frac{4p^2k_n^2 e^{-ipd}[\cos(k_n d) + \frac{i(p^2+k_n^2)}{2pk_n}\sin(k_n d)]}{4p^2k_n^2\cos^2(k_n d) + (p^2+k_n^2)^2\sin^2(k_n d)}. \tag{5.26}$$

Therefore, the global state of the system just after the particle leaves the region $0 < x < d$ can be written as

$$|\Phi_f\rangle = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} |\phi_n, n\rangle = |p\rangle \otimes \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} T_n \, |n\rangle \, . \tag{5.27}$$

That is: the particle is still described by the momentum eigenstate $|p\rangle$, whereas the clock state $|\xi_0\rangle$ changes due to the couplings T_n given by expression (5.26). In order to understand how these quantities can be associated to the clock readings, let us consider the limit of a "quasi-classical" particle, where the kinetic energy E is much larger than all the energy levels of the clock, i.e., $E \gg n\omega$ for all n. In this case, one can show that

$$T_n \approx e^{id(k_n - p)} \approx e^{-in\omega \frac{md}{p}} . \tag{5.28}$$

Then, in this limit the clock state after the measurement reads

$$\frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} e^{-in\omega \frac{md}{p}} .$$

Comparing this with the generic clock state (5.3), we see that the quantity

$$t_c \equiv \frac{md}{p}$$

can be related with the clock reading. More precisely: assuming that there exists some $n \in \mathbb{N}$ such that

$$d = \frac{2\pi np}{N\omega m},$$

then within the approximations considered the SWP quantum clock will accuse that the particle has spent a time t_c in the region with spatial extent d. Notice that this result reproduces exactly what is expected in the classical limit.

5.2 Larmor clock

The Larmor precession is a well known quantum phenomenon where the mean value of the spin of a particle performs a rotational motion when placed in a magnetic field. It seems that Baz [43] was the first author to use such a phenomenon to propose a measurement scheme for the duration of scattering events. Later on, Rybachenko [44] adapted the idea to determine the time that particles spent in a tunneling barrier. The works [45, 46] deal with this same issue as well.

The so-called Larmor clock is a quantum clock based on the monitoring of the spin rotation that is caused by a constant magnetic field within a limited region, usually a pontential barrier. By considering the weak field limit, it is possible to read directly a characteristic time from the spin precession around the axis determined by the field. Let us see how this can be done.

Consider a spin-1/2 particle in the presence of potential barrier of length L. The motion of the particle is restricted to one dimension and we choose coordinates such that the barrier is located at $0 < y < L$. Inside this barrier, we turn on a constant magnetic field $\vec{B} = B\hat{z}$. Making a explicit separation between the spin degree of freedom and the kinetic one ($\mathcal{H} = \mathcal{H}_p \otimes \mathcal{H}_S$), the Hamiltonian of the system can be written as

$$\hat{H} = \frac{\hat{p}^2}{2m} \otimes \hat{\mathbb{I}}_S + \chi(y) \left(V_0 \hat{\mathbb{I}} + \frac{\mu B}{2} \hat{\mathbb{I}}_p \otimes \hat{\sigma}_z \right), \tag{5.29}$$

where μ is the particle's magnetic moment and

$$\chi(y) = \begin{cases} 1, & 0 < y < L \\ 0, & \text{otherwise} \end{cases} \tag{5.30}$$

Here, it is convenient to define the *Larmor frequency*:

$$\omega_L = \frac{\mu B}{2}. \tag{5.31}$$

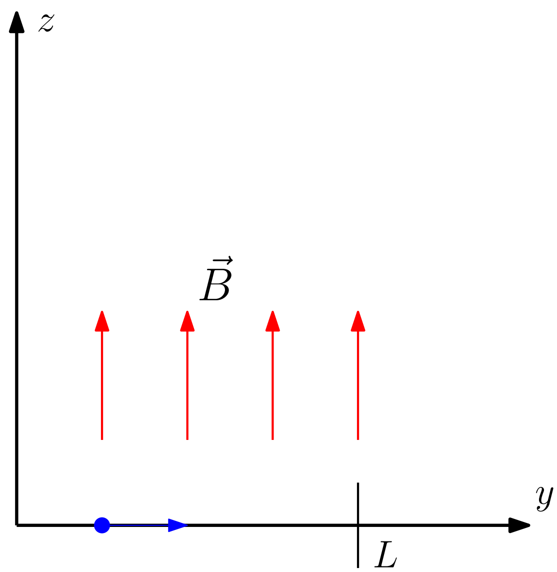


Figure 5.3: Simplified representation of the physical situation corresponding to the Larmor clock.

Notice that we are making the simplifying assumption that the motion of the particle is restricted to one dimension even in the presence of a magnetic field, thus ignoring any deflections that would appear in a full realistic situation. Since at the end of the calculations we will be interested in the weak field limit, such assumption is quite reasonable and does not invalidates the idea behind the quantum clock model under consideration.

For the spin degree of freedom, we work in the basis $\{|+\rangle, |-\rangle$ of the eigenstates of the operator $\hat{\sigma}_z$, with

$$\hat{\sigma}_z |\pm\rangle = \pm |\pm\rangle . \tag{5.32}$$

The components of the spin operator are

$$\hat{S}_z = \frac{1}{2}\hat{\sigma}_z = \frac{1}{2}(|+\rangle\langle+|-|-\rangle\langle-|), \tag{5.33}$$

$$\hat{S}_y = \frac{1}{2}\hat{\sigma}_y = \frac{i}{2}(|-\rangle\langle+|-|+\rangle\langle-|), \tag{5.34}$$

and

$$\hat{S}_x = \frac{1}{2}\hat{\sigma}_x = \frac{1}{2}(|+\rangle\langle-|+|-\rangle\langle+|). \tag{5.35}$$

In the matricial representation of the choosen basis we have the usual Pauli matrices:

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \qquad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{5.36}$$

The initial state of the system is taken to be a free particle with energy $E = \dfrac{p^2}{2m}$ and spin polarized in the x direction, namely

$$|\Phi(0)\rangle = |\psi_0\rangle \otimes \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle), \tag{5.37}$$

where

$$\langle y|\psi_0\rangle = \frac{1}{\sqrt{2\pi}} e^{ipy}. \tag{5.38}$$

After the particle enters into the potential barrier, the state $|\Phi(0)\rangle$ evolves to a state of the form

$$|\Phi\rangle = |\psi_+\rangle \otimes |+\rangle + |\psi_-\rangle \otimes |-\rangle\,, \tag{5.39}$$

since the overall potential will be felt differently by each possible spin configuration. Employing the notation $\psi^{(\pm)}(y) = \langle y|\psi_\pm\rangle$, we can treat this problem in the same fashion of that one considered in the previous section, so that

$$\psi^{(\pm)}(y) = e^{ipy} + R^{(\pm)} e^{-ipy}, \qquad y < 0 \tag{5.40}$$

$$\psi^{(\pm)}(y) = B_1^{(\pm)} e^{\lambda_\pm y} + B_2^{(\pm)} e^{-\lambda_\pm y}, \qquad 0 < y < L \tag{5.41}$$

$$\psi^{(\pm)}(y) = A^{(\pm)} e^{ipy}, \qquad y > L \tag{5.42}$$

The main difference here is that now we distinguish between spin configurations, with the quantities $\lambda_\pm$ being defined as

$$\lambda_\pm = \sqrt{2m(V_0 - E) \mp m\omega_L}. \tag{5.43}$$

Furthermore, since we are assuming a tunneling barrier — with real exponentials for the wave function inside the potential barrier — we have $V_0 > E$. Recalling once more that our objective is to consider the weak field limit (which in this case amounts to work with small enough ω_L), we always consider $\lambda_\pm \in \mathbb{R}$.

Thus, we see that the situation is equivalent to a usual tunneling problem where the potential V_0 is replaced by

$$V_0 \to V_0 \mp \frac{\omega_L}{2},$$

according to the spin of the particle. Adapting the result (5.26) for the present case, we get

$$A^{(\pm)} = \frac{4p^2\lambda_\pm^2 e^{-ipL}[\cosh\lambda_\pm L + i\frac{(p^2-\lambda_\pm^2)}{2p\lambda_\pm}\sinh\lambda_\pm L]}{4p^2\lambda_\pm^2\cosh^2\lambda_\pm L + (p^2-\lambda_\pm^2)^2\sinh^2\lambda_\pm L}. \tag{5.44}$$

In order to read out the results with more clarity, we cast this expression into

$$A^{(\pm)} = |A^{(\pm)}| e^{-ipL} e^{i\phi_\pm} \equiv \sqrt{T_\pm} e^{-ipL} e^{i\phi_\pm}, \tag{5.45}$$

where the phases $\phi_\pm$ satisfy

$$\tan\phi_\pm = \left(\frac{p^2 - \lambda_\pm^2}{2p\lambda_\pm}\right)\tanh\lambda_\pm L. \tag{5.46}$$

As for the probabilities $T_\pm$, they can be simplified as

$$T_{\pm} = \frac{2p\lambda_{\pm}}{\sqrt{4p^2\lambda_{\pm}^2 + (p^2+\lambda_{\pm}^2)^2 \sinh^2 \lambda_{\pm} L}} \tag{5.47}$$

Therefore, the state of the system corresponding to the particles that have tunneled through the barrier can be written in the following form (incorporating the multiplicative factor $(2\pi)^{-1/2}$ as a global phase):

$$|\Phi_T\rangle = |\psi_0\rangle \otimes \frac{1}{\sqrt{(T_+ + T_-)}} \left(A^{(+)} |+\rangle + A^{(-)} |-\rangle \right). \tag{5.48}$$

Now, it is straightforward to compute the mean value of each spin component over this state. The results are:

$$\langle \Phi_T | \hat{S}_z | \Phi_T \rangle = \frac{1}{2} \left(\frac{T_+ - T_-}{T_+ + T_-} \right), \tag{5.49}$$

$$\langle \Phi_T | \hat{S}_x | \Phi_T \rangle = \cos(\phi_+ - \phi_-) \frac{\sqrt{T_+ T_-}}{T_+ + T_-}, \tag{5.50}$$

$$\langle \Phi_T | \hat{S}_y | \Phi_T \rangle = -\sin(\phi_+ - \phi_-) \frac{\sqrt{T_+ T_-}}{T_+ + T_-}. \tag{5.51}$$

The spin precession around the axis determined by the magnetic field — the z axis, in this case — can be read from the mean values of $\hat{S}_x$ and $\hat{S}_y$. In order to relate these quantities with time measurements, we consider the weak field limit, i.e., $\omega_L \ll 1$. Up to second order in the Larmor frequency, we have

$$\lambda_{\pm} = \lambda \mp \frac{m\omega_L}{2\lambda} + \mathcal{O}(\omega_L^2), \tag{5.52}$$

where $\lambda = \sqrt{2m(V_0 - E)}$.

In order to express the approximate results (5.49)-(5.51) up to first order on ω_L, we regard the quantities $T_{\pm}$ as functions of $\lambda_{\pm}$,

$$T_\pm \to T(\lambda_\pm)$$

so that it makes sense to work with the expansion

$$T(\lambda_\pm) = T(\lambda) + \frac{\partial T(\lambda)}{\partial \lambda}(\lambda_\pm - \lambda) + \mathcal{O}((\lambda_\pm - \lambda)^2) = T(\lambda) \mp \frac{\partial T(\lambda)}{\partial \lambda}\frac{m\omega_L}{2\lambda} + \mathcal{O}(\omega_L^2). \quad (5.53)$$

Thus,

$$T(\lambda_+) - T(\lambda_-) = -\frac{m\omega_L}{\lambda}\frac{\partial T(\lambda)}{\partial \lambda} + \mathcal{O}(\omega_L^2), \quad (5.54)$$

$$T(\lambda_+) + T(\lambda_-) = 2T(\lambda) + \mathcal{O}(\omega_L^2), \quad (5.55)$$

and

$$T(\lambda_+)T(\lambda_-) = T(\lambda)^2 + \mathcal{O}(\omega_L^2). \quad (5.56)$$

Therefore, up to second order in the magnetic field we can write

$$\langle \Phi | \hat{S}_z | \Phi \rangle = -\frac{m\omega_L}{4\lambda T(\lambda)}\frac{\partial T(\lambda)}{\partial \lambda} + \mathcal{O}(\omega_L^2), \quad (5.57)$$

$$\langle \Phi | \hat{S}_x | \Phi \rangle = \frac{1}{2} + \mathcal{O}(\omega_L^2), \quad (5.58)$$

$$\langle \Phi | \hat{S}_y | \Phi \rangle = -\frac{m\omega_L}{2\lambda}\frac{\partial \phi(\lambda)}{\partial \lambda} + \mathcal{O}(\omega_L^2). \quad (5.59)$$

In this limit, we see that the mean value of $\hat{S}_y$ defines a characteristic time for the Larmor precession, namely

$$\tau_y = -\frac{m}{\lambda}\frac{\partial\phi}{\partial\lambda} = \frac{mp}{\lambda}\frac{2L\lambda(\lambda^2 - p^2) + (p^2 + \lambda^2)\sinh(2\lambda L)}{4p^2\lambda^2\cosh^2(\lambda L) + (p^2 - \lambda)^2\sinh^2(\lambda L)}. \tag{5.60}$$

Therefore, within the approximations considered, successive repetitions of the experiment — spin measurement along the direction of motion of the particles' beam — allows one to ascribe an intrinsic time for the effect and, thus, use it to define a quantum clock.

5.3 Harmonic oscillator as a quantum clock

Anyone who has ever seen a pendulum clock has appreciated the quite natural idea of associate oscillation phenomena with time measurements. In fact, the classical harmonic oscillator is perhaps the most natural proposal for the realization of an honest clock.

In the quantum realm, we already know that the situation is different: quantum systems cannot be used to realize perfectly honest quantum clocks. Even so, one may wonder if it is possible to construct a quantum clock that in some limit regime allows us to retrieve the behavior of an honest clock. Within that old philosophy that teach us that simpler answers are probably the right ones, it is quite natural to look at one of the most (if not the most) favorite toys of a theoretical physicist: the quantum harmonic oscillator.

The Hamiltonian of the quantum harmonic oscillator is defined by

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2\hat{x}^2}{2}. \tag{5.61}$$

As is well known from standard Quantum Mechanics, the analysis of this system is eased by the introduction of the *annihilation* and *creation* operators, namely:

$$\hat{a} = \sqrt{\frac{m\omega}{2}}\left(\hat{x} + i\frac{\hat{p}}{m\omega}\right), \qquad \hat{a}^\dagger = \sqrt{\frac{m\omega}{2}}\left(\hat{x} - i\frac{\hat{p}}{m\omega}\right) \tag{5.62}$$

In terms of these operators, the Hamiltonian (5.61) can be cast into

$$\hat{H} = \omega \left(\hat{a}^\dagger \hat{a} + \frac{1}{2}\hat{\mathbb{I}} \right). \tag{5.63}$$

Using the properties of $\hat{a}$ and $\hat{a}^\dagger$, one can show that $\hat{H}$ admits a discrete basis of eigenstates $\{|n\rangle\}_{n\in\mathbb{N}}$, which are eigenstates of the operator $\hat{N} = \hat{a}^\dagger \hat{a}$ as well. This latter is the so-called *number operator* which counts how many "quanta of oscillations" are present in each energy eigenstate of the harmonic oscillator. Indeed, one has $\hat{N}|n\rangle = n|n\rangle$, so that each state $|n\rangle$ is an energy eigenstate corresponding to the eigenvalue $E_n = \omega(n + 1/2)$. Finally, the action of the annihilation and creating operators over each state $|n\rangle$ is characterized by the following relations:

$$\hat{a}|n\rangle = \sqrt{n}\,|n-1\rangle, \qquad \hat{a}^\dagger |n\rangle = \sqrt{n+1}\,|n+1\rangle. \tag{5.64}$$

Of course, for $n = 0$ (the *vacuum state*) we have $\hat{a}|0\rangle = 0$.

Recall the construction of the time POVM for the free particle: we started by quantizing the classical expression for the time of arrival, obtaining an operator whose set of complete but nonorthogonal states were used to built a legitimate POVM fulfilling the covariance condition (4.5) with respect to the shifts generated by the Hamiltonian.

Once more, our motivation starts on the corresponding classical system. Given the initial data $x_0 = x(0)$ and $p_0 = p(0)$, the dynamics of the classical harmonic oscillator in its phase space (x, p) is described by

$$x(t) = \frac{x_0}{\cos\varphi_0}\cos(\omega t + \varphi_0), \qquad p(t) = -\frac{m\omega x_0}{\cos\varphi_0}\sin(\omega t + \varphi_0), \tag{5.65}$$

where the phase φ_0 reads

$$\varphi_0 = -\arctan\left(\frac{p_0}{m\omega x_0}\right). \tag{5.66}$$

In this case, we have a quite natural variable that allows us to use this system as a clock:

the phase

$$\varphi(t) = \omega t + \varphi_0. \tag{5.67}$$

Thereby, the approach to be described here relies on finding the proper quantum version of a *phase observable* for the quantum harmonic oscillator. As starting point, consider the classical analogue of the annihilation operator, namely

$$a(t) = \sqrt{\frac{m\omega}{2}} \left(x(t) + i\frac{p(t)}{m\omega} \right) = \sqrt{\frac{m\omega}{2}} \frac{x_0}{\cos\varphi_0} (\cos\varphi(t) - i\sin\varphi(t)). \tag{5.68}$$

Rewriting this variable as

$$a(t) = |a|e^{-i\varphi(t)}$$

suggests that a natural candidate for a quantum phase operator would arise in the quantization of this last expression. Indeed, since in the quantum realm the amplitude $|a|$ can be described by the well defined operator $\sqrt{\hat{N}}$ [1], we are led to consider the following decomposition of the annihilation operator:

$$\hat{a} = \sqrt{\hat{N}}\hat{V} \tag{5.69}$$

Working in the basis of energy eigenstates, it is easy to figure out a closed expression for the operator $\hat{V}$, which reads

$$\hat{V} = \sum_{n=0}^{\infty} |n\rangle\langle n+1|. \tag{5.70}$$

Following the approach of Paul Busch [29] (see also references [37, 38]), we will show how the eigenstates of this operator can be used to construct a *phase* POVM for the quantum harmonic oscillator, i.e., a POVM that is covariant with respect to the shifts generated by the number operator $\hat{N}$. Of course, given the linear relationship between phase and time,

[1] Any analytical function of a self-adjoint operator can be straightforwardly defined through its Taylor series.

such a POVM will allow us to speak about time measurements over states of the quantum harmonic oscillator.

It turns out that the eigenstates of $\hat{V}$ can be labeled by an angular variable $\theta \in [\varphi_0, \varphi_0 + 2\pi]$, being described by

$$|\theta\rangle = \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} e^{in\theta} |n\rangle \, . \tag{5.71}$$

Then, for any open set $X = (\theta_a, \theta_b) \subset [\theta_0, \theta_0 + 2\pi)$ the proposed phase POVM takes the form

$$\hat{\mathbb{F}}(X) = \int_{\theta_a}^{\theta_b} d\theta |\theta\rangle\langle\theta| = \frac{1}{2\pi} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \int_{\theta_a}^{\theta_b} d\theta \ e^{i(n-m)\theta} |n\rangle\langle m|. \tag{5.72}$$

It is straightforward to check that this expression defines a POVM: $\hat{\mathbb{F}}(X)$ is a positive operator for each set X and the evaluation over the "whole" space $[\theta_0, \theta_0 + 2\pi]$ gives the identity operator, as is verified below:

$$\hat{\mathbb{F}}([\theta_0, \theta_0 + 2\pi)) = \frac{1}{2\pi} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \underbrace{\int_{\theta_0}^{2\pi+\theta_0} d\theta \ e^{i(n-m)\theta}}_{2\pi\delta_{nm}} |n\rangle\langle n| = \sum_{n=0}^{\infty} |n\rangle\langle n| = \hat{\mathbb{I}}. \tag{5.73}$$

Furthermore,

$$e^{i\varphi\hat{N}} \hat{\mathbb{F}}(X) e^{-i\varphi\hat{N}} = \hat{\mathbb{F}}(X + \varphi), \tag{5.74}$$

where the translated set $X + \varphi$ is defined modulo 2π. This result confirms our claim that (5.73) is a phase POVM.

Henceforth, we shall work with $\theta_0 = 0$.

Now that the phase POVM is defined, we know how to compute the main observable: for an oscillator prepared on a state $|\psi\rangle$, the measurement of the phase will yield a result belonging to the set $X = (\theta_a, \theta_b) \subset [0, 2\pi)$ with probability

$$p(X) = \langle\psi|\hat{\mathbb{F}}(X)|\psi\rangle = \int_{\theta_a}^{\theta_b} d\theta\ f(\theta), \tag{5.75}$$

where we have defined the *probability density*

$$f(\theta) = |\langle\theta|\psi\rangle|^2. \tag{5.76}$$

Using this, the mean value of the phase over an arbitrary state reads

$$\int_0^{2\pi} d\theta\ \theta f(\theta) \equiv \langle\psi|\hat{F}|\psi\rangle. \tag{5.77}$$

On this last expression, we have defined the phase operator

$$\hat{F} = \int_0^{2\pi} d\theta\ \theta|\theta\rangle\langle\theta|, \tag{5.78}$$

which may be considered as the "analogue" of the Aharonov-Bohm time operator (3.3) for the present case. It is important to realize that $\hat{F}$ is not the θ-multiplicative operator, since relation (5.71) makes it clear that the states $|\theta\rangle$ do not form an orthogonal set. Thereby, the proper way to compute the dispersion is through the probability density (5.76):

$$(\Delta\hat{F})_\psi \equiv \left[\int_0^{2\pi} d\theta f(\theta)(\theta - \langle\psi|\hat{F}|\psi\rangle)^2\right]^{1/2}. \tag{5.79}$$

In order to build an intuition about the meaning of the analytical results at hand, let us evaluate them over "special" states of the quantum harmonic oscillator: coherent states. Such states deserve a careful attention because they can approximate the behavior of classical systems in an appropriate limit. For the sake of completeness, we brief recall their definition and main properties.

Coherent states of the harmonic oscillator can be labeled by a complex number $\alpha \in \mathbb{C}$, being defined as the eigenstates of the annihilation operator $\hat{a}$. In terms of energy eigenstates, they read

$$|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \,. \tag{5.80}$$

The interesting point about these states becomes clear when one computes the mean value of the position and momentum operators: writing $\alpha = |\alpha|e^{-i\varphi}$, we have

$$\langle\alpha|\hat{x}|\alpha\rangle = \sqrt{\frac{2\hbar}{m\omega}}|\alpha|\cos\varphi, \tag{5.81}$$

$$\langle\alpha|\hat{p}|\alpha\rangle = -\sqrt{2m\omega\hbar}|\alpha|\sin\varphi. \tag{5.82}$$

These results resemble the classical expressions (5.65). The fundamental constant $\hbar$ was restored for the sake of interpretive clearness; indeed, these relations suggest that the classical behavior should emerge on the limit $|\alpha|\to\infty$. In this limit, the quantity α plays the role of the (classical) amplitude of motion, whereas the phase φ becomes just the classical variable (5.67). Such a point of view is further strengthened when one also considers the measurement of the Hamiltonian over coherent states. In fact, since

$$\Delta\hat{H}_\alpha = \omega|\alpha| \tag{5.83}$$

and

$$\langle\alpha|\hat{H}|\alpha\rangle = \omega\left(|\alpha|^2 + \frac{1}{2}\right), \tag{5.84}$$

the relative dispersion of the mean energy vanishes in the limit of large enough $|\alpha|$:

$$\lim_{\alpha\to\infty} \frac{\Delta\hat{H}_\alpha}{\langle\alpha|\hat{H}|\alpha\rangle} = 0. \tag{5.85}$$

As a side remark, it is important to notice that coherent states are also states that saturate the position-momentum uncertainty. A straightforward calculation results in

$$\Delta\hat{x}_\alpha = \sqrt{\frac{\hbar}{2m\omega}}, \tag{5.86}$$

$$\Delta\hat{p}_\alpha = \sqrt{\frac{m\omega\hbar}{2}}, \tag{5.87}$$

so that

$$\Delta\hat{x}_\alpha\Delta\hat{p}_\alpha = \frac{\hbar}{2}. \tag{5.88}$$

Now that we already have reviewed the definition and properties of coherent states, it is natural to expect that the phase distribution (5.76) to be peaked around the phase φ of the complex number $\alpha = |\alpha|e^{-i\varphi}$ that characterizes a coherent state. As a matter of fact, using the state (5.80) to compute this distribution we obtain

$$f_\alpha(\theta) = |\langle\theta|\alpha\rangle|^2 = \frac{e^{-|\alpha|^2}}{2\pi}\left|\sum_{n=0}^{\infty}\frac{|\alpha|^n}{\sqrt{n!}}e^{in(\varphi-\theta)}\right|^2. \tag{5.89}$$

A numerical plot of this distribution is shown below for $\varphi = \pi$ and different values of $|\alpha|$. Just as expected, the curve becomes more peaked around φ as one increases the modulus of α.

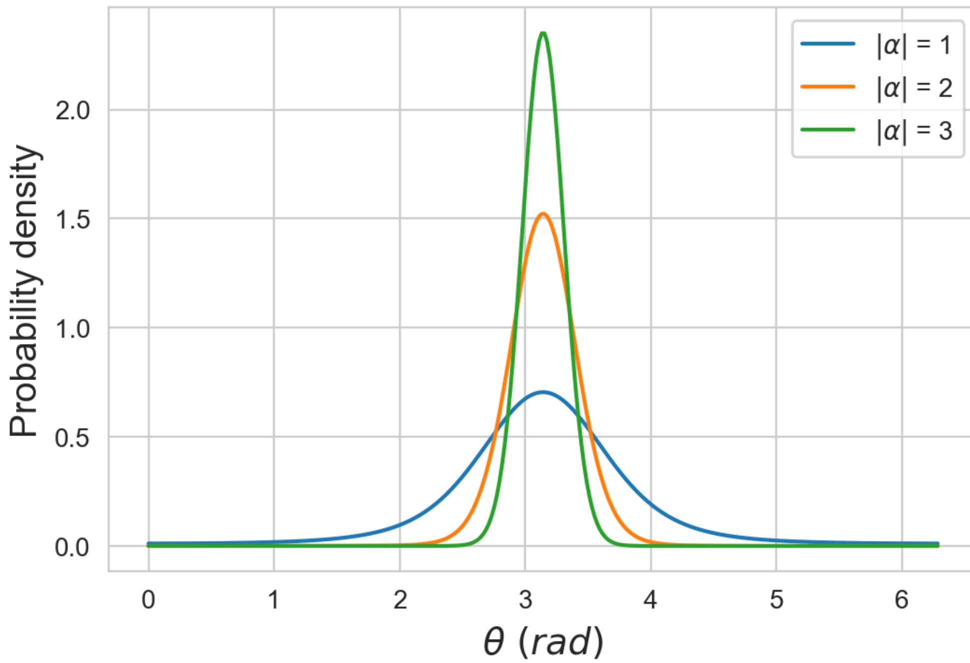


Figure 5.4: Phase distribution for coherent states of the harmonic oscillator.

Furthermore, it turns out that in the limit $|\alpha| \to \infty$ we have precisely the analytical result

$$\lim_{|\alpha|\to\infty} f_\alpha(\theta) = \delta(\theta - \varphi) \tag{5.90}$$

A possible way to prove this is to consider the asymptotic behavior of the complex function

$$g(\lambda) = \sum_{n=0}^{\infty} \frac{\lambda^n}{\sqrt{n!}} \tag{5.91}$$

As shown in reference [37], for large enough $|\lambda| \equiv r$ such a function is well approximated by

$$g(re^{i\beta}) \sim (2\pi)^{1/4}\sqrt{2r}\exp\left[\frac{r^2}{2} - r^2\beta^2 + i\beta\left(r^2 - \frac{1}{2}\right)\right]. \tag{5.92}$$

Thus, since $f_\alpha(\theta) = \dfrac{e^{-|\alpha|^2}}{2\pi}\left|g(|\alpha|e^{i(\varphi-\theta)})\right|^2$ it follows that

$$
\begin{aligned}
\lim_{|\alpha|\to\infty} f_\alpha(\theta) &= \lim_{|\alpha|\to\infty} \frac{2e^{-|\alpha|^2}|\alpha|}{\sqrt{2\pi}} \left|\exp\left[\frac{|\alpha|^2}{2} - |\alpha|^2(\theta-\varphi)^2 + i(\varphi-\theta)\left(|\alpha|^2 - \frac{1}{2}\right)\right]\right|^2 \\
&= \lim_{|\alpha|\to\infty} \sqrt{\frac{2}{\pi}}|\alpha|e^{-2|\alpha|^2(\theta-\varphi)^2} \\
&= \delta(\theta-\varphi).
\end{aligned}
\tag{5.93}
$$

Consequently,

$$\lim_{|\alpha|\to\infty} \langle\alpha|\hat{F}|\alpha\rangle = \varphi. \tag{5.94}$$

An analogous calculation shows that

$$\lim_{|\alpha|\to\infty} (\Delta\hat{F})_\alpha = 0. \tag{5.95}$$

Therefore, in the classical limit coherent states have a well defined phase. Since phase measurements allows one to directly infer elapsed times, one can regard coherent states of the harmonic oscillator as *approximations for honest clocks*.

Now, for the general case where $|\alpha| < \infty$, we can operationally define a quantum clock as follows: in a generic spacetime, let A and B be two events joined by a timelike curve and labeled by the parameters τ_A and τ_B (with $\tau_B > \tau_A$), respectively. The time interval between these events as told by the quantum clock will be defined as

$$\Delta\tau_{AB} \equiv \frac{1}{\omega}(\langle\alpha(\tau_B)|\hat{F}|\alpha(\tau_B)\rangle - \langle\alpha(\tau_A)|\hat{F}|\alpha(\tau_A)\rangle). \tag{5.96}$$

For the sake of consistency, notice that the temporal evolution of a coherent state labeled by the complex number $\alpha = |\alpha|e^{-i\varphi}$ amounts to a shift in the phase of α by a quantity related to the elapsed time:

$$|\alpha(t)\rangle = e^{-it\hat{H}}\,|\alpha(t=0)\rangle = e^{-i\omega t/2}e^{-\frac{|\alpha|^2}{2}}\sum_{n=0}^{\infty}\frac{\alpha^n e^{-i\omega t}}{\sqrt{n!}}\,|n\rangle \sim \left|\alpha e^{-i\omega t}\right\rangle. \tag{5.97}$$

Thus,

$$\lim_{|\alpha|\to\infty}\Delta\tau_{AB} = \tau_B - \tau_A. \tag{5.98}$$

Of course, we cannot forget that relation (5.96) has an associated dispersion, which is non-vanishing along all the temporal evolution of the coherent state (see the figure below). Thus, even though such dispersion can be made sufficiently small by allowing $|\alpha|$ to increase, from a fundamental point of view the situation here is no better than the one with the SWP quantum clock: it is not possible to define precisely the elapsed time between all the events along the worldline of the observer who carries such a quantum clock.

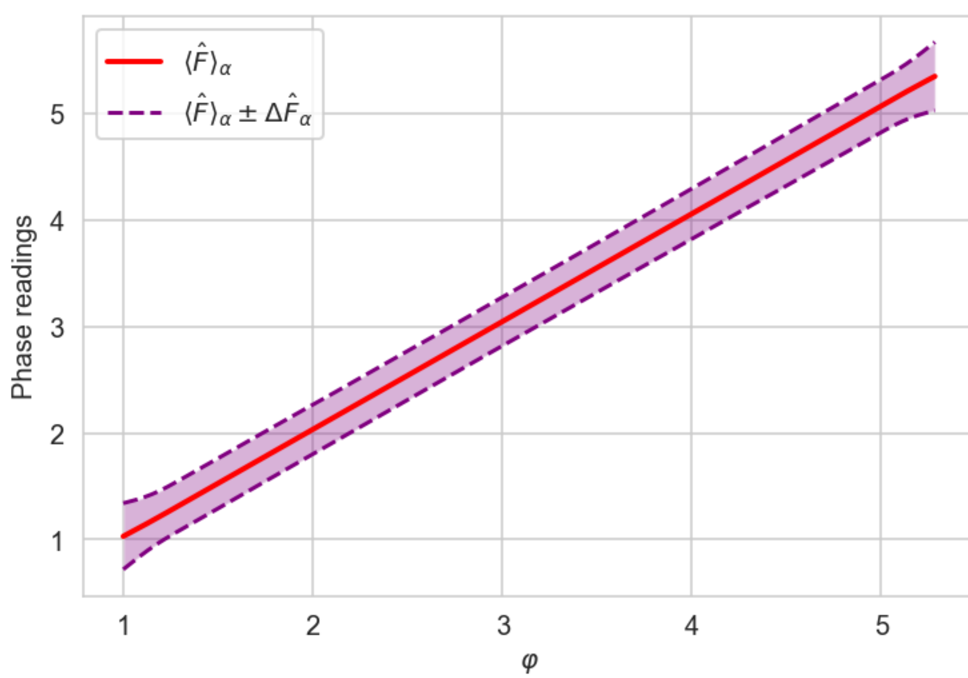


Figure 5.5: Phase readings and corresponding uncertainties for a coherent state with $|\alpha|= 2$.

6

Conclusion

Time is everywhere. We already know that. But an important lesson we can take from this essay is that there is still more to learn about this concept within the scientific framework. Although it is possible to give a precise meaning for time in the context of classical physics paradigm, the same is no longer true when Quantum Mechanics comes into play. As a matter of fact, we have learned that the laws of quantum realm forbid physical systems to be used as honest clocks.

At first, we have said that time is what honest clock measures. Nonetheless, if one accepts that physical systems cannot posses arbitrarily negative energy values — an assumption that we understand as so fundamental that makes no sense to consider otherwise — then the laws of nature show that there is no such thing like an honest clock. Then, it is perfectly honest to ask once more: *What is time*? We do not know. Surely, the objective of the present work never was to give a straight answer to this question that is compatible both with General Relativity and Quantum Mechanics. Instead, we have learned a few points that at least can guide a path for a future answer.

Regarding the role of time in Quantum Mechanics, our exposition has proven that makes no sense to stick with the traditional view that time is just a sharply defined parameter used to describe the dynamics of quantum states or operators. Indeed, once one accepts that observables can be represented by POMVs, it is possible to construct a time observable on the quantum realm, as we have seen for the free particle and harmonic oscillator cases. Moreover, the intrinsic dynamics of quantum systems may be sufficient for an operational definition of quantum clocks. To the best of our knowledge, it seems that measurements of a phase observable over coherent states of the harmonic oscillator is the closest we can get to an "honest clock" in quantum theory.

At this point, one may wonder: since it is possible to use quantum theory to build clocks (but not honest ones), why not define time just as the quantity measured by *any* clock? In short terms, the answer is: *causality*. In physics, you can challenge many notions, but a cornerstone like causality is accepted — until now — as a fundamental principle underlying all physical phenomena. As we have supported on chapter 4, the no-go theorem of Unruh and Wald implies that any physical system used as clock is subject to a lack of causality on its readings.

Thereby, under the assumption that honest clocks must be understood as apparatus that allows one to test spacetime itself, Quantum Mechanics revels the impossibility of having a consistent causal structure at all scales. Any theory of quantum gravity should account for that. Regarding the question about the nature of time, a possibility opened by the present investigation is that perhaps time is an emergent concept — yet to be fully understood — and it makes no sense to try to define it for all scales.

We hope that this work can be used as a basic guide for further investigations on the subject.

Appendix A

The pictures of Quantum Mechanics

Since the Heisenberg and Schrödinger pictures can be obtained as particular cases of the so-called interaction picture, we begin this appendix by considering the latter. In this picture, one supposes that it is possible to write the Hamiltonian of the quantum system under study in the following way:

$$\hat{H}(t) = \hat{H}_0 + \hat{H}_I(t). \tag{A.1}$$

The above equation is to be understood in the usual Schrödinger picture. The operator $\hat{H}_0$ is the "free" or "unperturbed" Hamiltonian, and is assumed to be time independent. As for $\hat{H}_I(t)$, we shall call it the "interaction" Hamiltonian, which in general may depend on the time parameter t.

Let $\hat{U}_0(t_0, t)$ be the time evolution operator arising from the Hamiltonian $\hat{H}_0$. Since such Hamiltonian is time independent, we can write

$$\hat{U}_0(t_0, t) = \exp\left(-i(t - t_0)\hat{H}_0\right). \tag{A.2}$$

The time evolution operator arising from the total Hamiltonian $\hat{H}(t)$ will be denoted by $\hat{U}(t_0, t)$.

Now, consider that the system is described by a density matrix $\hat{\rho}(t)$ (in the Schrödinger picture). Then, the mean value of a general observable $\hat{A}$ measured in such state is given by

$$\langle\hat{A}\rangle(t) = \mathrm{Tr}\Big[\hat{A}\hat{\rho}(t)\Big] = \mathrm{Tr}\Big[\hat{A}\hat{U}(t,t_0)\hat{\rho}(t_0)\hat{U}^\dagger(t,t_0)\Big]. \tag{A.3}$$

Recall that the crucial point when switching between pictures is to ensure that the mean values of observables remain unchanged. That is: we may change the description of states and operators, but the physical predictions one makes with all this stuff cannot depend upon any choice of picture. Hereupon, notice that one can cast the right hand side of the previous expression into

$$\begin{aligned}\langle\hat{A}\rangle(t) &= \mathrm{Tr}\Big[\hat{A}\hat{U}_0(t,t_0)\hat{U}_0^\dagger(t,t_0)\hat{U}(t,t_0)\hat{\rho}(t_0)\hat{U}^\dagger(t,t_0)\hat{U}_0(t,t_0)\hat{U}_0^\dagger(t,t_0)\Big]\\ &= \mathrm{Tr}\Big[\hat{U}_0^\dagger(t,t_0)\hat{A}\hat{U}_0(t,t_0)\hat{U}_0^\dagger(t,t_0)\hat{U}(t,t_0)\hat{\rho}(t_0)\hat{U}^\dagger(t,t_0)\hat{U}_0(t,t_0)\Big]\\ &= \mathrm{Tr}\Big[\hat{A}^{(I)}(t)\hat{\rho}^{(I)}(t)\Big],\end{aligned} \tag{A.4}$$

where we have defined

$$\hat{A}^{(I)}(t) \equiv \hat{U}_0^\dagger(t,t_0)\hat{A}\hat{U}_0(t,t_0) \tag{A.5}$$

and

$$\hat{\rho}^{(I)}(t) \equiv \hat{U}_0^\dagger(t,t_0)\hat{U}(t,t_0)\hat{\rho}(t_0)\hat{U}^\dagger(t,t_0)\hat{U}_0(t,t_0) = \hat{U}_0^\dagger(t,t_0)\hat{\rho}(t)\hat{U}_0(t,t_0). \tag{A.6}$$

The equations (A.5) and (A.6) completely characterizes the switch from the Schrödinger picture to the interaction picture. Furthermore, the differential equation for the density matrix of the system can be written as

$$\frac{d\hat{\rho}^{(I)}(t)}{dt} = -i[\hat{H}_I^{(I)}(t), \hat{\rho}^{(I)}(t)]. \tag{A.7}$$

Therefore, the time evolution of a pure state $|\psi\rangle$ in the interaction picture is given by

$$|\psi(t)\rangle = \hat{U}_I(t,t_0)\,|\psi(t_0)\rangle\,, \tag{A.8}$$

where $\hat{U}_I(t, t_0)$ is the time evolution operator that arises from the interaction Hamiltonian in the interaction picture, $\hat{H}_I^{(I)}(t)$.

Finally, notice that the particular case $\hat{H}_0 \equiv 0$ corresponds to the Schrödinger picture. Indeed, in this case equation (A.5) implies that all operators are fixed in time, whereas the evolution of pure states — or, more generally, mixed states — is given by the full time operator $\hat{U}_I(t, t_0) = \hat{U}(t, t_0)$. On the other hand, the case $\hat{H}_I(t) \equiv 0$ corresponds to the so-called *Heisenberg picture*, where states — pure or mixed — do not depend on the evolution parameter t and operators evolve in time according to relation (A.5). That is: if $\hat{A}^{(H)}(t)$ denotes an operator on the Heisenberg picture, one has $\hat{A}^{(H)}(t) = \hat{A}^{(I)}(t)$. Moreover, in this same particular case the differential equation that matters is the *Heisenberg equation*:

$$\frac{d}{dt}\hat{A}^{(H)}(t) = i[\hat{H}_0, \hat{A}^{(H)}(t)]. \tag{A.9}$$

Usually, the free Hamiltonian $\hat{H}_0$ is chosen to be time independent, as we have done here. Nonetheless, nothing prohibits one to work in the Heisenberg picture using solely a total Hamiltonian which may depend on the parameter t.

As a final remark, we emphasize that no matter what picture one chooses to work with, the physical predictions of the theory remain the same, since all pictures agree on the mean values of operators.

Appendix B

A result from complex analysis

For the sake of completeness, in this appendix we state and prove the result from complex analysis that was used in the formulation of the *no perfect clock theorem* (see section 4.3). Our approach is based on the material presented on reference [49]. See also reference [51] for more details about the subject of complex analysis.

On the following, we employ the notation: $\mathbb{H}_+ = \{z \in \mathbb{C} : Im(z) \geq 0\}$ and $\mathbb{H}_- = \{z \in \mathbb{C} : Im(z) \leq 0\}$, where $Im(z)$ is the imaginary part of $z \in \mathbb{C}$.

Consider the holomorphic functions $f_j : \Omega_j \subset \mathbb{C} \to \mathbb{C}$ $(j = 1, 2)$, where Ω_1 is an open set in $\mathbb{H}_+$ and Ω_2 is open in $\mathbb{H}_-$. Suppose that the boundary of both sets Ω_j contains the interval $(a, b) \subset \mathbb{R}$. Moreover, we shall assume that the limits

$$f_1(x) = \lim_{y \to 0^+} f_1(x + iy) \tag{B.1}$$

and

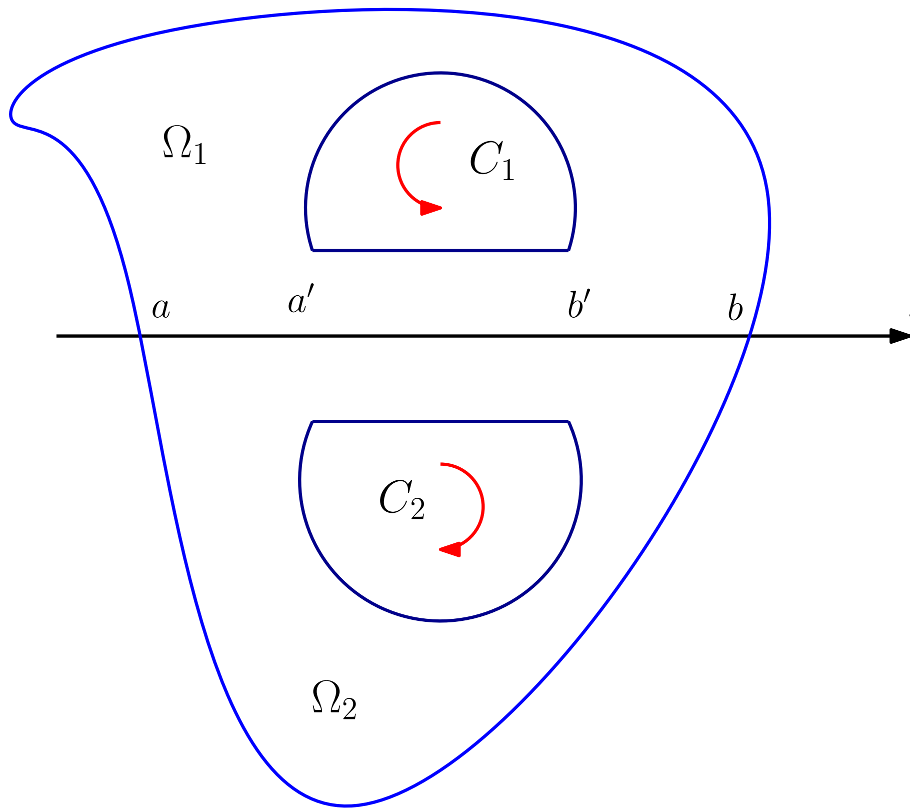
$$f_2(x) = \lim_{y \to 0^+} f_2(x - iy) \tag{B.2}$$

exist uniformly for $a < x < b$. Given all these considerations, we have the following result.

<u>Theorem</u>: Let f_1 and f_2 be the functions defined above. If the limits (B.1) and (B.2) are such that $f_1(x) = f_2(x)$, then f_1 and f_2 are the same holomorphic function, i.e., they have the same analytical extension.

Proof:

Consider two closed paths C_1 and C_2 entirely contained in the sets Ω_1 and Ω_2, respectively. This situation is represented in the figure below, where $a < a' < b' < b$.



Using the Cauchy's theorem, it follows that

$$\frac{1}{2\pi i}\oint_{C_1}\frac{f_1(\xi)}{\xi - z}d\xi = \begin{cases} f_1(z), & z \in Int(C_1) \\ 0, & z \in Int(C_2) \end{cases} \tag{B.3}$$

and

$$\frac{1}{2\pi i}\oint_{C_2}\frac{f_2(\xi)}{\xi - z}d\xi = \begin{cases} f_2(z), & z \in Int(C_2) \\ 0, & z \in Int(C_1) \end{cases} \tag{B.4}$$

In the previous integral, $Int(C_j)$ denotes the region delimited by the closed path C_j.

Now, using the hyphotesis of the uniform existence of the limits (B.1) and (B.2), we can write

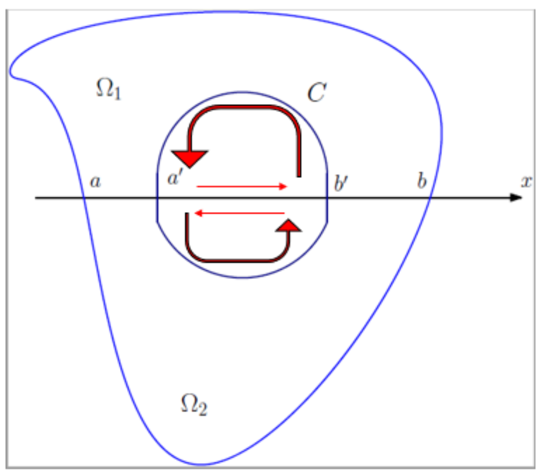
$$\lim_{\epsilon \to 0^+} \int_{a'}^{b'} \frac{f_j(\xi \pm i\epsilon)}{\xi - z \pm i\epsilon} d\xi = \int_{a'}^{b'} \frac{f_j(x)}{x - z} dx, \tag{B.5}$$

for $j = 1, 2$.

Consider then the function $G : \Omega_1 \cup \Omega_2 \to \mathbb{C}$ defined by the expression

$$G(z) = \frac{1}{2\pi i} \oint_{C_1} \frac{f_1(\xi)}{\xi - z} d\xi + \frac{1}{2\pi i} \oint_{C_2} \frac{f_2(\xi)}{\xi - z} d\xi. \tag{B.6}$$

Let C be the path constructed from the concatenation of the paths C_1, C_2 and the line segment $[a', b']$, to be covered as follows: beginning at b', we go along the path C_1 in counter-clockwise sense, (see figure below); after returning to b', we follow to the path C_2 also in coonter-clockwise sense.



According to the result (B.5), the contributions along the segment $[a', b']$ cancel each other. So, we can replace the sum of integrals in the definition of G above by just on integral performed along the close path Γ obtained from the path C through the exclusion of the real line segment. Thus, we can write

$$G(z) = \frac{1}{2\pi i} \oint_{\Gamma} \frac{f(\xi)}{\xi - z} d\xi, \tag{B.7}$$

where the function f is defined by

$$f(z) = \begin{cases} f_1(z), & z \in \Omega_1 \\ f_2(z), & z \in \Omega_2 \end{cases} \tag{B.8}$$

So, using the Cauchy's theorem once more and recalling that both functions f_1 and f_2 are holomorphic in its respective domains, we conclude that the function G is, at the same time, the analytical continuation of f_1 to the domain Ω_2 and the of f_2 to the domain Ω_1. Since the analytical continuation of a complex function is unique, we must conclude that the function f_1 and f_2 are, in the functional sense, the same holomorphic function.

$\blacksquare$

Appendix C

Mathematical properties of the Aharonov-Bohm time operator

In section 4.2 we have worked with the time operator

$$\hat{T} = -\frac{m}{2}(\hat{x}\hat{p}^{-1} + \hat{p}^{-1}\hat{x}), \tag{C.1}$$

focusing on the physical questions that can be answered when we use it to build a lawful POVM. Now, we shall be concerned with some mathematical subtleties concerning this same operator. (See also [50])

Let us work in the Hilbert space of a free particle in momentum representation. For the sake of mathematical clarity, we shall employ the following notation

$$\langle p|\hat{T}|\psi\rangle \equiv \hat{T}_p(\psi).$$

Then, for each function ψ in the domain of $\hat{T}_p$ we have

$$\hat{T}_p(\psi) = \frac{im}{2}\left(\frac{1}{p^2} - \frac{2}{p}\frac{d}{dp}\right)\psi(p). \tag{C.2}$$

What we are going to do next is to identify a special condition that the functions belonging to $\mathcal{D}(\hat{T}_p)$ must satisfy. Such a condition will be crucial to show that the Aharonov-Bohm operator is Hermitian, but not self-adjoint.

First, we must compute the norm of $\hat{T}_p(\psi)$. Due to the divergence at $p = 0$, we have (see [50] for more details)

$$\begin{aligned}\langle \hat{T}_p(\psi), \hat{T}_p(\psi)\rangle = \frac{m^2}{4} \lim_{\epsilon\to 0^+} \Bigg[\int_{-\infty}^{-\epsilon} dp \, \left(\frac{\psi(p)}{p^2} - \frac{2}{p}\frac{d\psi(p)}{dp}\right)\left(\frac{\psi^*(p)}{p^2} - \frac{2}{p}\frac{d\psi^*(p)}{dp}\right) \\ + \int_{\epsilon}^{\infty} dp \, \left(\frac{\psi(p)}{p^2} - \frac{2}{p}\frac{d\psi(p)}{dp}\right)\left(\frac{\psi^*(p)}{p^2} - \frac{2}{p}\frac{d\psi^*(p)}{dp}\right)\Bigg].\end{aligned} \tag{C.3}$$

Manipulating this expression with integration by parts, one can show that

$$\begin{aligned}\langle \hat{T}_p(\psi), \hat{T}_p(\psi)\rangle = \lim_{\epsilon\to 0^+} \Bigg[\int_{-\infty}^{-\epsilon} dp \, \left(\frac{4}{p^2}\frac{d\psi(p)}{dp}\frac{d\psi^*(p)}{dp} - 5\frac{|\psi(p)|^2}{p^4}\right) \\ + \int_{\epsilon}^{\infty} dp \, \left(\frac{4}{p}^2\frac{d\psi(p)}{dp}\frac{d\psi^*(p)}{dp} - 5\frac{|\psi(p)|^2}{p^4}\right) + 2\left(\frac{|\psi(\epsilon)|^2 + |\psi(-\epsilon)|^2}{\epsilon^3}\right)\Bigg].\end{aligned} \tag{C.4}$$

On deriving such a result, we have assumed that $\lim_{p\to\pm\infty} \psi(p) = 0$. Notice that this is a necessary condition for $\psi \in \mathbb{L}^2(\mathbb{R})$.

Since the function ψ will belong to the domain of $\hat{T}_p$ only if (C.4) is finite, we surely must impose the limit

$$\lim_{\epsilon\to 0^+} \left(\frac{|\psi(\epsilon)|^2 + |\psi(-\epsilon)|^2}{\epsilon^3}\right)$$

to be finite. In fact, since $\psi(p)$ must vanish as $p \to \pm\infty$, it cannot be a polynomial. So, for the limit above to be finite, it must be equal to zero. Thus, the functions belonging to the domain of $\hat{T}_p$ need to satisfy the following condition:

$$\lim_{p\to 0} \frac{\psi(p)}{p^{3/2}} = 0. \tag{C.5}$$

Now, we are in position to argue that $\hat{T}_p$ is a Hermitian operator. Indeed, for $\psi, \phi \in \mathcal{D}(\hat{T}_p)$ we have

$$
\begin{aligned}
&\langle \hat{T}_p(\phi), \psi \rangle \\
&= \frac{im}{2} \lim_{\epsilon \to 0^+} \left[\int_{-\infty}^{-\epsilon} dp\ \psi(p) \left(\frac{\phi^*(p)}{p} - \frac{2}{p} \frac{d\phi^*(p)}{dp} \right) + \int_{\epsilon}^{\infty} dp\ \psi(p) \left(\frac{\phi^*(p)}{p} - \frac{2}{p} \frac{d\phi^*(p)}{dp} \right) \right],
\end{aligned} \tag{C.6}
$$

and

$$
\langle \phi, \hat{T}_p(\psi) \rangle = \frac{im}{2} \lim_{\epsilon \to 0^+} \left[\int_{-\infty}^{-\epsilon} dp\ \phi^*(p) \left(\frac{\psi(p)}{p} - \frac{2}{p} \frac{d\psi(p)}{dp} \right) + \int_{\epsilon}^{\infty} dp\ \phi^*(p) \left(\frac{\psi(p)}{p} - \frac{2}{p} \frac{d\psi(p)}{dp} \right) \right]. \tag{C.7}
$$

Integrating by parts this last relation we get

$$
\langle \phi, \hat{T}_p(\psi) \rangle = \langle \hat{T}_p(\phi), \psi \rangle + im \lim_{\epsilon \to 0^+} \left[\frac{\phi^*(\epsilon)\psi(\epsilon) + \phi^*(-\epsilon)\psi(-\epsilon)}{\epsilon} \right]. \tag{C.8}
$$

Finally, since both functions ψ and ϕ belong to $\mathcal{D}(\hat{T}_p)$, they need to satisfy the condition (C.5). Then, we readily see that the limit in the previous equation vanishes, so that

$$
\langle \phi, \hat{T}_p(\psi) \rangle = \langle \hat{T}_p(\phi), \psi \rangle. \tag{C.9}
$$

Therefore, $\hat{T}_p$ is Hermitian. Since this conclusion does not depend on the particular representation we are working with – the momentum representation –, we can state that the time operator $\hat{T}$ itself is Hermitian.

Nonetheless, $\hat{T}$ cannot be self-adjoint. Indeed, resorting to the definition of the adjoint operator $\hat{T}^\dagger$, we have that for each $\psi \in \mathcal{D}(\hat{T}_p)$ and $\phi \in \mathcal{D}(\hat{T}_p^\dagger)$ the following relation holds:

$$
\langle \hat{T}_p^\dagger(\phi), \psi \rangle = \langle \phi, \hat{T}_p(\psi) \rangle. \tag{C.10}
$$

It turns out that $\mathcal{D}(\hat{T}_p)$ is a proper subset of $\mathcal{D}(\hat{T}_p^\dagger)$. For instance, consider

$$
\psi(p) = p^2 e^{-p^2}, \qquad \phi(p) = \frac{1}{p}.
$$

Notice that the function ϕ does not belong to $\mathcal{D}(\hat{T}_p)$, because it does not satisfy condition (C.5). However, for this particular choice we still have

$$\lim_{\epsilon \to 0^+} \left[\frac{\phi^*(\epsilon)\psi(\epsilon) + \phi^*(-\epsilon)\psi(-\epsilon)}{\epsilon} \right] = 0,$$

so that no problem will arise at all when one evaluates the right hand side of (C.10). Thus, ϕ as defined above is a function belonging to $\mathcal{D}(\hat{T}_p^\dagger)$ that does not belong to $\mathcal{D}(\hat{T}_p)$. Therefore, the Aharonov-Bohm time operator is not self-adjoint.

Appendix D

Special functions used

Gamma function:

$$\Gamma(z) = \int_0^\infty x^{z-1} e^{-z}, \qquad z \in \mathbb{C}, Re(z) > 0. \tag{D.1}$$

Modified Bessel function of the first kind:

$$I_\alpha(z) = \sum_{n=0}^{\infty} \frac{1}{n!\,\Gamma(n+\alpha+1)} \left(\frac{z}{2}\right)^{2n+\alpha}, \qquad z \in \mathbb{C}. \tag{D.2}$$

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