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Master Thesis

Supersymmetric S-matrix Bootstrap

Student: Matheus Augusto Fabri

Advisor: Pedro G. M. Vieira

São Paulo, February, 2019

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Master thesis *presented in the Graduate Program in Physics at Instituto de Física Teórica of UNESP as a partial requirement for obtaining the master degree in physics.*

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Resumo

Neste trabalho consideramos a matriz de espalhamento obtida por meio de otimização semidefinida em qualquer teoria quântica de campos supersimétrica bidimensional massiva dado os vínculos impostos por unitariedade e simetria de crossing. O espaço de todas as matrizes de espalhamento possíveis, com e sem pólos, compatíveis com esses vínculos foram analizadas e caracterizadas de tal maneira que conexões com clássicos modelos integráveis foram feitas, como por exemplo sine-Gordon supersimétrico.

Palavras-chave: Bootstrap de matriz S, teoria do espalhamento, supersimetria, teorias quânticas de campos bidimensionais, modelos integráveis, otimização semidefinida.

Áreas de conhecimento: Teoria quântica de campos, supersimetria e espalhamento (Física).

Abstract

We considered the S-matrix obtained by semidefinite optimization problems in any 1+1 dimensional $\mathcal{N} = 1$ supersymmetric massive quantum field theory with unitarity and crossing symmetry constraints. The set of all possible S-matrices with and without poles compatible with these conditions were analyzed and characterized such that connections with novel integrable models, like supersymmetric sine-Gordon for example, were made.

Keywords: S-matrix bootstrap, scattering theory, supersymmetry, two dimensional quantum field theory, integrable models, semidefinite optimization.

Fields of knowledge: Quantum field theory, supersymmetry and scattering (Physics).

Introduction

To motivate the S-matrix bootstrap and convince the reader of its importance we will discuss an analog and source of inspiration: conformal bootstrap. A conformal field theory (CFT) can be defined without a Lagrangian in an operator-algebraic framework, because the operator product expansion (OPE) and conformal group symmetry fully fix the spatial dependence of the correlation functions. Indeed a CFT can be defined by the set of conformal data $\{\Delta_k, C_{ijk}\}$, where Δ_k are the scaling dimensions and C_{ijk} the OPE coefficients of all primary operators in the theory. The main idea of conformal bootstrap is that the OPE imposes a semidefinite positive constraint on the OPE, named bootstrap equation, coefficients given a set of scaling dimensions, *i.e.*, given $\{\Delta_1, \cdots, \Delta_L\}$ the C_{ijk} have a maximum value. This fact implies that the space of CFTs is finite and we can non-perturbatively carve it out using this method. This yielded impressive results, for example it gave the most precise computation of the scaling dimensions for the 3d Ising model and very stringent bounds on higher dimensional superconformal models like 6d (2,0) theory of type A1 [1,2].

Using conformal bootstrap techniques the authors in ref. [3] took as an inspiration the holographic principle to carve out the space of 1+1 dimensional S-matrices of massive quantum field theories (QFTs) on flat spacetime. Basically they considered correlation functions of four equal neutral scalars lying on AdS_2 and move them to the boundary of this space that possess one dimensional conformal symmetry, the claim is that the Mellin transformation of this four point function becomes the S-matrix of a 1+1 dimensional massive QFT with a single neutral boson when the AdS_2 radius is infinite. Since the boundary has conformal symmetry the tools of conformal bootstrap can be used, where the ratios Δ_i/Δ_j and C_{ijk} become the ratio of masses and the coupling of the massive QFT in the infinite radius limit. Since the OPE coefficients attain a maximum on the conformal bootstrap this implies a maximal corresponding coupling, thus limiting the space of QFTs with a given spectrum. Which makes physical sense that a maximal value exists, because the spectrum is fixed and a higher value for the coupling would give us new bound states since the particles' interaction would be stronger. This defined the S-matrix bootstrap and established interesting connections between CFTs and QFTs, like for example that the properties of analyticity and factorization of Mellin amplitudes justified the same conditions on the S-matrix obtained in this procedure that was the sine-Gordon lightest breather-breather scattering matrix.

Much as impressive the previous results were the numerics were also impressively hard due to multiple truncations that are needed regarding conformal bootstrap and the flat space limit, so a more direct rout to derive these results was needed. The authors in ref. [4] established a much simpler approach for obtaining the same results using only analytical/physical properties of scattering in the QFT itself. What they did was to consider the most general $2 \to 2$ S-matrix of the lightest particles in a model, these being neutral bosons, with real analyticity and crossing symmetry in 1+1 dimensions, and with the presence of bound states. Then they maximized the residues with the quadratic unitarity constraint. All their results matched the ones in the holographic approach discussed before thus proving the validity of the method. However with much less computational effort and with the S-matrix bootstrap being directly translated to a semidefinite optimization without the use of conformal bootstrap[1].

The ref. [4] shed a new light on the subject and made generalizations possible. The first one was the formulation of the same problem but in dimensions higher than two, which was made in ref. [5]. This simple modification introduced multiple issues. Firstly in 1+1 dimensions there is only a single independent Mandelstam variable, however in 3+1 dimensions we have two: the scattering angle and the center-of-mass energy. The second wall to be overcome was the fact that unitarity is expressed in definite spin channels using partial waves, however crossing and analyticity are expressed in terms of Mandelstam variables, *i.e.*, the description of all constraints aren't unified which lead to complications in the numerics. Nonetheless the authors were able to impose constraints on direct experimental quantities like scattering lengths which opened the door to new explorations on the standard model in a later work [6].

With the formulation of S-matrix bootstrap problem mastered people turned into the problem of adding extra symmetries to the model. This was first done for $\mathrm{O}(N)$ symmetric models in refs. [7,8]. Here we are back to 1+1 dimensions and it was considered N scalar bosons in the vector multiplet of $\mathrm{O}(N)$. The previous ansatz of the aforementioned works was now supplied with two extra amplitudes to form the isospin channels of symmetric, antisymmetric, and singlet scatterings. Now crossing is nontrivial since unitarity is diagonalized, which happens due to the choice of channels, and the amplitudes are mixed by crossing symmetry. Again connections with novel integrable models like sine-Gordon soliton S-matrix, non-linear sigma model and Gross-Neveau model were made and some analytical results obtained just like in refs. [3,4]. Following this work in an outstanding numerical/experimental analysis in ref. [6] the authors extended the previous $\mathrm{O}(N)$ results to 3+1 dimensions and by imposing $N = 3$ they were able to explore the scalar bosonic subsector of QCD composed by mesons like pions, ρ and σ particles, for example. There they took as input some known resonances and explored how the space of scattering lengths is affected by zeros in the S-matrix that come from derivative couplings of the pions in the effective action and with it exclude some positions for these zeros.

This thesis is a next step in this chain of discussed developments. In this thesis we considered the S-matrix bootstrap of the most general 1+1 dimensional $\mathcal{N} = 1$ supersymmetric theory with a single chiral supermultiplet using semidefinite optimization. We made connections with novel integrable models like $\mathcal{N} = 1$ supersymmetric sine-Gordon and its analytic continuation in the coupling. This work can be seen as a first step into bootstrapping supersymmetric S-matrices in higher dimensions and also as an intermediate between the one boson S-matrix bootstrap and the multiparticle S-matrix bootstrap with $\mathbb{Z}_2$ symmetry worked out in [9]. This thesis is a detailed extension of part of the work available at ref. [10] in a collaboration with C. Bercini and A. Homrich and under supervision of Pedro Vieira.

The reader can ask itself what is the relation of this S-matrix bootstrap with the 60's analytic S-matrix bootstrap. This old program arose in a time where QFT wasn't well understood and myriads of resonances where appearing in the experiments due to the strong interaction, which led some physicists to look at the foundations of scattering

[1]The main distinction from conformal bootstrap is that there we find exclusion regions, however in the S-matrix bootstrap we find inclusion regions.

theory and forget QFT altogether to explain the available data. Basically they tried to derive all strong interaction physics from analytical properties of the scattering amplitudes, like Hermitian analyticity, structure of poles, dispersion relations and asymptotic behavior [11]. Despite some successes in computing scattering lengths, cross sections and predicting a new resonance for pion-nucleon scattering the program faded away due to some insurmountable obstacles in this formalism. Some of these were: their description included only bosons, no real justification for analyticity, the rise and phenomenal success of QCD in explaining the vast set of new particles and it was based on set of bootstrap equations without solutions. A more detailed explanation for why this program failed can be seen in ref. [12]. The difference to the current S-matrix bootstrap is that now the goal is much more modest and the methods reliable, inspired by conformal bootstrap the space of S–matrices that satisfies a set of physical requirements is being classified. Unlike them we do not hope to find a single theory, but instead carve out the space of possible QFTs.

The organization of this work is the following. In Chapter 1 we describe scattering process in 1+1 dimensions, analyze mathematical properties of integrable models and we finish by characterizing how $\mathcal{N} = 1$ supersymmetry acts on the S-matrix at this dimensionality. In Chapter 2 we start by defining a numerical ansatz for the optimization, and then we cast the S-matrix bootstrap as a semidefinite programming problem. With this we discuss the numerical results from the optimization with and without bound states in the ansatz and then close by elaborating some analytical remarks and possible extensions. Finally we conclude with some conceivable future paths.

"*At some point, everything's gonna go south on you... everything's going to go south and you're going to say, this is it. This is how I end. Now you can either accept that, or you can get to work. That's all it is. You just begin. You do the math. You solve one problem... and you solve the next one... and then the next. And If you solve enough problems, you get to come home. All right, questions?*"

Mark Watney - The Martian.

Chapter 1

Scattering Theory, Integrability and Supersymmetry in 1+1 Dimensional Quantum Field Theories

In this chapter we will start by briefly introducing notation and concepts in scattering theory. Then it will be described how integrability works in the context of S-matrix, what it implies in its structure, and some mathematical aspects of integrable models like quantum groups. We will also discuss some progress that has been done recently in this direction. Also it will be discussed how $\mathcal{N} = 1$ supersymmetry acts on a general $1 + 1$ dimensional S-matrix, which will be useful later to write our ansatz for numerical bootstrap.

1.1 Basics of scattering theory

Here we'll establish some notation and basic facts of S-matrix theory that will be ubiquitous in our work. We will consider massive quantum field theories in 1+1 dimensions with Poincaré symmetry or more precisely $\mathrm{ISO}(1, 1)$. The reason for considering a massive spectrum is that in this case the S-matrix can be well defined because long-range interactions are forbidden due to the mass gap and well separated non-interacting wavepackets for scattering can be constructed. Also the unitarity cuts are easily defined for us if this is the case[1] [14–16]. As is well known we define a particle as a momentum eigenstate that transforms in a irreducible representation of the little group, this scheme is known as Wigner classification. Here we encounter the first simplification, in 1+1 dimensions the little group for massive particles is SO(1). Then all particles transform as a singlet, that is, all of them have one degree of freedom on-shell. This is what will allow us to write proper ansatz for the S-matrix bootstrap for a theory with fermions since for this dimensionality they don't carry any polarizations at all. Ideas for describing particle polarizations in higher dimensions will be discussed later in the Conclusion.

Before we set up the Hilbert space it's useful to introduce the rapidity variables. In 1+1 dimensions we can parametrize a particle momenta by hyperbolic variables

$$E_a = m_a \cosh \theta_a \quad \text{and} \quad p_a = m_a \sinh \theta_a,$$

[1]Models with and without mass gap are well defined in axiomatic quantum field theory, these being described by Haag-Ruelle and Buchhholz theories, respectively [13].

where E_a, p_a and m_a are respectively the energy, spatial momentum and mass of the a-th particle. Then θ_a is the rapidity, note that for physical momenta is clear that $\theta \in \mathbb{R}$, however to study the analytical properties of the S-matrix we'll do an analytic continuation of all functions of θ_a by making simply $\theta_a \in \mathbb{C}$. We assume that this analytic continuation will be unique given that the scattering amplitudes are meromorphic functions as will be described. Note that $\theta_a \to -\theta_a$ reverses momentum but not energy and $\theta_a \to i\pi - \theta_a$ reverses the energy and leaves the momentum invariant. The former is related to parity and the latter to crossing symmetry. It can be verified that the Lorentz group act additively on these variables[2], that is a boost by a rapidity η yields $\theta_a \to \theta_a + \eta$, therefore all functions of $\theta_a - \theta_b$, denoted by θ_{ab}, will be Lorentz invariant.

Now we can define the S-matrix. Let $\mathcal{H}$ be the Hilbert space of the theory and $|p_a, a\rangle \in \mathcal{H}$ the a-th particle with momentum p_a. The idea of the S-matrix operator is to quantify the transition amplitude from a state of n-particles that are non interacting at time $t \to -\infty$ to hit each other at some finite time and transit to a m-particle state that also aren't interacting at $t \to +\infty$ in the Heisenberg picture. Let $\mathcal{H}_{in}$ and $\mathcal{H}_{out}$ be the Hilbert spaces at $t \to -\infty$ and $t \to +\infty$ respectively, then the S-matrix for $n \to m$ transition is defined as:

$$
\begin{aligned}
{}_{out}\langle (p_{b_1}, b_1), \cdots, (p_{b_m}, b_m) | (p_{a_1}, a_1), \cdots, (p_{a_n}, a_n)\rangle_{in} = \\
\langle (p_{b_1}, b_1), \cdots, (p_{b_m}, b_m) | \hat{S} | (p_{a_1}, a_1), \cdots, (p_{a_n}, a_n)\rangle .
\end{aligned}
$$

Where due to Lorentz symmetry the matrix element of $\hat{S}$ depends only on Lorentz invariant quantities build out of the momentum. Quite unlike in higher dimensions where it would have indexes that transform in irreducible representations of SO$(d-1)$ or ISO$(d-2)$ for massive and massless particles, respectively.

We have some physical requirements to establish for $\hat{S}$. First it's clear that from definition it can be regarded as the time evolution operator from the *in* state to the *out* state, then to have probability conservation it must be an unitary operator

$$\hat{S}\hat{S}^\dagger = \mathbb{1},$$

where $\mathbb{1}$ is the identity operator. Also the *in* and *out* Hilbert states are equal to the physical Hilbert space, that is $\mathcal{H}_{in} = \mathcal{H}_{out} = \mathcal{H}$ a condition known as asymptotic completeness [15]. Note that $\mathcal{H}$ here is the full Hilbert space of the theory, then it contains all "fundamental" particles as well all bound states. The reason for the quotation marks will be explained later when we link the analytical approach with the usual perturbative one. The matrix element of $\hat{S}$ from a state with n particles to one with m particles is denoted by $S_{n\to m}$. Here it's considered only theories without unstable particles, consequently decay processes like $S_{1\to 2}$ vanish on-shell.

Unitarity can be rewritten in a specific vector subspace of $\mathcal{H}$. Indeed, for example let $D \subset \mathcal{H}$ be the two-particle subspace, so:

$$\langle\psi|\hat{S}\hat{S}^\dagger|\psi\rangle = 1 \Rightarrow \sum_{a\in D} |\langle\psi|\hat{S}|a\rangle|^2 - 1 = -\sum_{a\notin D} |\langle\psi|\hat{S}|a\rangle|^2 \Rightarrow \sum_{a\in D} |\langle\psi|\hat{S}|a\rangle|^2 - 1 \le 0.$$

Then let S_D be the operator $\hat{S}$ restrict to the $D \subset \mathcal{H}$. The previous equation can be rewritten as a semidefinite condition:

$$S_D S_D^\dagger - \mathbb{1}_D \preceq 0. \tag{1.1}$$

[2]Which is expected since SO$(1,1)$ is Abelian, thus any representation of it must be Abelian.

Where $\mathbb{1}_D$ is the identity on D and $\preceq$ means that left hand side is a semidefinite negative matrix. Therefore it's clear that if there is no inelastic processes, *i.e.*, no particle production, then S_D is unitary. A linearized form of equation (1.1) is given by:

$$\begin{pmatrix} \mathbb{1}_D & S_D^\dagger \\ S_D & \mathbb{1}_D \end{pmatrix} \succeq 0. \tag{1.2}$$

Using some lemmas about determinant of block matrices it's easy to see the equivalence between the two definitions. Indeed, let A, B , C and D be square matrices of equal order and $[C, D] = 0$, then

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(AD - BC).$$

It's trivial to see that using $A = D = \mathbb{1}$, $C = S_D$ and $B = S_D^\dagger$ we get the semidefinite negative condition expressed in equation (1.1). The expression (1.2) will be central to input unitarity in the numerics in Sec. 2.1.

Have seen unitarity we can now start analyzing the analytical properties of the S-matrix. We'll assume that is satisfies the following properties:

(i). It's a function of Lorentz invariant variables. In terms of rapidity it depends on $\{\theta_{ab}\}$ and in terms of momenta is a function of generalized Mandesltam variables. Also it's defined on the entire complex plane minus the possible singularities.

(ii). It is parity, time reversal and charge conjugation invariant.

(iii). It's real analytic and satisfies crossing symmetry.

(iv). It can be a meromorphic function except at branch cuts of square root type starting at the minimal energy necessary to create n particles. With possible poles at energies corresponding to the masses of bound state particles being exchanged.

(v). It has no essential singularities at infinity.

With these axioms the poles and branch cuts are totally fixed by physical requirements, however the existence of extra structure like zeros or even periodicity of the S-matrix can also happens in some models. Now let's workout each item in detail to see the physics behind them.

For simplicity we will analyze the simplest matrix element, since unstable particles are discarded we stick with $2 \to 2$ processes. These can be denoted by S_{ab}^{cd}, where $\{a, b\}$ and $\{c, d\}$ are the incoming and outgoing particles' labels, respectively. This is a function of the Mandelstam variables as shown below in Figure 1.1. Note also that in 1+1 dimensions there is no transfer of momenta, so we can take either $p_3 = p_1$ or $p_4 = p_1$ such that either t or u vanishes. Choosing $u = 0$ we get:

$$s + t = 2(m_a^2 + m_b^2),$$

since $m_a = m_c$ and $m_b = m_d$ from momentum conservation. Due to this relation the S-matrix depends only on s or t. In terms of rapidity variables we have that s and t are given by:

$$s = m_a^2 + m_b^2 + 2m_b m_b \cosh\theta \quad \text{and} \quad t = m_a^2 + m_b^2 - 2m_b m_b \cosh\theta. \tag{1.3}$$

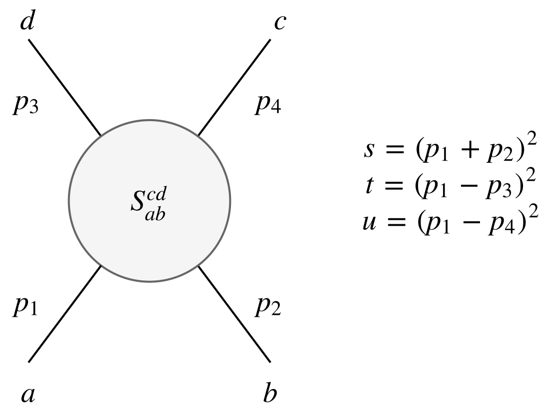


Figure 1.1: S-matrix for the $2 \to 2$ process and Mandelstam variables. Note that there are only three possible Mandesltam variables that can be formed from this scattering and they're given above.

Where $\theta = \theta_{ab}$. If the masses are equal to m these reduce to $s = 4m^2 \cosh(\theta/2)^2$ and $t = -4m^2 \sinh(\theta/2)^2$. Clearly the minimal and maximal values of s and t are $(m_a + m_b)^2$ and $(m_a - m_b)^2$ for $\theta \in \mathbb{R}$, respectively.

The condition (i) we already discussed and it comes from little group considerations and analytical continuation. So let's discuss assumption (ii). It gives us a lot of constraints, for example time reversal (T) acts by reversing the momentum of the particles and exchange *in* and *out* states. However parity (P) maintain these and exchanges the particles in each one. Lastly charge conjugation (C) just transforms all particles in their antiparticles, then:

$$\mathrm{T}: \; S^{cd}_{ab}(s) = S^{ba}_{dc}(s), \quad \mathrm{P}: \; S^{cd}_{ab}(s) = S^{dc}_{ba}(s) \quad \text{and} \quad \mathrm{C}: \; S^{\bar{c}\bar{d}}_{\bar{a}\bar{b}}(s) = S^{cd}_{ab}(s),$$

where $\bar{a}$ denotes the antiparticle of a.

Now condition (iii). First note that Hermitian analyticity can be stated as:

$$S^{ba}_{dc}(s^*) = S^{cd}_{ab}(s)^*$$

and is a condition that is directly motivated from the definition of the S-matrix as a time evolution operator between *in* and *out* states. This is a weak condition to impose, so we'll consider however a stronger one. Combining the fact that the S-matrix is time reversal invariant with Hermitian analyticity yields:

$$S^{cd}_{ab}(s^*) = S^{cd}_{ab}(s)^*.$$

This is the real analyticity condition. Two consequences of this are: in the absence of possible branch cuts the S-matrix element will be real for $s \in \mathbb{R}$ and it can be expanded in a Laurent series of real coefficients. Hermitian analyticity can't truly be derived from the Wightman axioms, however it's well justified within the usual perturbative expansion [15, 17].

Now crossing can be stated. The first caveat that should be noted is that it isn't a symmetry but it's in fact an analytic continuation of momenta where we take a particle of the *in* or *out* state to an antiparticle at the *out* or *in* state. This symmetry is immediate from the Feynman graph point of view as explicit in Figure 1.2, it can be thought as some

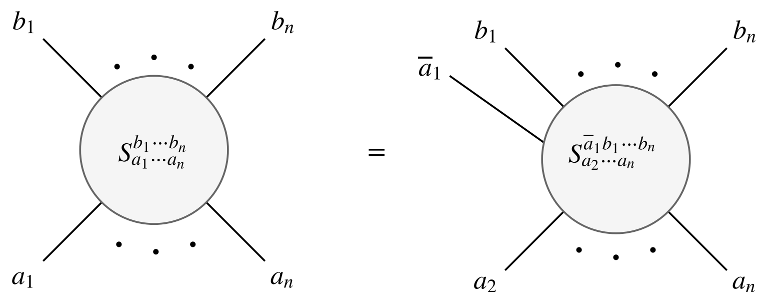


Figure 1.2: From the point of view of Feynman's diagrams crossing symmetry is natural. Indeed the left hand side amplitude is equal to the one at the right hand side if we just deform the diagram by moving a particle from the past to an antiparticle in the future. Of course we need to reverse the momentum of the crossed particle.

sort of topological invariance of the Feynman graph. Also it's a relativistic property of S-matrix, indeed in non-relativistic scattering in quantum mechanics crossing symmetry is totally missed since the dispersion relation is non-relativistic. It acts in the S-matrix as:

$$\text{Crossing}: \ S^{cd}_{ab}(t) = \alpha \ S^{da}_{bc}(s).$$

Where $\alpha \in \mathbb{C}$. More complicated realizations of crossing can be realized in exotic theories, but these will not be discussed here [18]. It can be proven that α is a phase if the S-matrix satisfies Hermitian analyticity, however it's further restrained by imposing real analyticity which sets $\alpha \in \mathbb{R}$ and then $\alpha = 1$. Since $\alpha \in \mathrm{U}(1)$ for non time reversal invariant models it can always be absorbed by redefining the scattering states [11].

Now condition (iv) has profound physical origins. The association of poles and bound states can be traced to two sources. The first is in non-relativistic scattering in quantum mechanics with delta function potentials, there is well known that the S-matrix have poles and these are associated with the bound states' energies. The second way that they appear is in the usual perturbative expansion of quantum field theories where we have simple poles that are related to exchanged particles in Feynman diagrams. Given these examples we elevated this feature to an axiom. The position of the poles are $s = M^2$ where M is the mass of a bound state. To what are the residues of these poles related? Going back to Feynman diagrams we see that tree level ones have simple poles whose residues are real. Moreover we have that the s-channel and t-channel processes, by the definition of Mandelstam variables, have residues of opposite signs. From experience with the usual quantum field theories is easy to see that if the theory is unitary, that is the action is real, then the s-channel residue is negative and the t-channel one is positive. Later we'll associate these residues with the couplings in the Lagrangian.

The existence of cuts can be derived from real analyticity and unitarity. Indeed let's separate the S-matrix in two parts, the identity and the transfer matrix $\hat{T}(s)$:

$$\hat{S} = \mathbb{1} + i\hat{T},$$

where in the identity and in front of $\hat{T}$ we have implicit delta functions for momentum conservation that we chose to omit here. Let's use $s \to s + i\epsilon$, where $\epsilon > 0$. Then the unitarity equation becomes:

$$\frac{T_{a\to a}(s+i\epsilon) - T_{a\to a}(s-i\epsilon)}{i} = \sum_b |T_{a\to b}(s+i\epsilon)|^2,$$

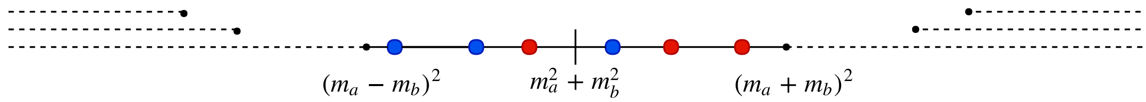


Figure 1.3: This is the analytical structure of the s-plane for $S_{ab}^{cd}(s)$. Where the red dots are the poles in the s-channel and the blue dots are the respective poles in the t-channel. Note that crossing symmetry means that the analytical structure is symmetrical upon reflection around the crossing symmetric point $m_a^2 + m_b^2$. The black dots are the branch points and the dashed lines the branch cuts for multiparticle production processes.

where $T_{a\to b}(s) = \langle a|\hat{T}|b\rangle$. Let $s^{\pm} = s \pm i\epsilon$ for $\epsilon \to 0^+$. Then:

$$\frac{T_{a\to a}(s^+) - T_{a\to a}(s^-)}{i} = \sum_b |T_{a\to b}(s^+)|^2.$$

From this equation we have two direct results. Firstly note that when the right hand side of the previous equation isn't zero the S-matrix has a discontinuity. However it isn't null only for physical energies where we can produce at least one state $|b\rangle$. Then it's clear that from the minimal energy to produce the state $|b\rangle$ to beyond we have a discontinuity. Secondly this discontinuity happens only in the imaginary part, since the right handed term is real. Then we have a series of branch cuts $[(m_{a_1} + \cdots + m_{a_p})^2, +\infty)$ starting at the threshold of p-particle production. Combining this fact with the presence of the poles we see that in order to satisfy unitarity the poles can't be on the branch cuts. Also from real analyticity:

$$T_{a\to a}(s^+) - T_{a\to a}(s^-) = 2i\mathrm{Im}\{T_{a\to a}\}(s^+).$$

From which it can be derived the usual optical theorem [14].

The immediate consequence of crossing is that every pole or cut in the s variable have an mirror pole or cut in the t variable. Let a and b the lightest particles in the model. Due to the absence of unstable particles the first cuts must be $[(m_a + m_b)^2, +\infty)$ and $(-\infty, (m_a - m_b)^2]$, thus all s and t poles are between them. The analytical structure of the S-matrix is represented in Figure 1.3.

The last analytical property to discuss is condition (v). It will become important to derive the ansatz to bootstrap in Chapter 2.1. Clearly the S-matrix satisfies:

$$S_{cd}^{ab}(s) = \oint_{\Delta(s,\epsilon)} \frac{dz}{2\pi i} \frac{S_{cd}^{ab}(z)}{z - s}, \tag{1.4}$$

where $\Delta(s,\epsilon)$ is a circle centered on s and with radius ϵ small enough such that it doesn't intersect any cut or pole. If we expand the contour to $\epsilon \to +\infty$ the integral should be the same since no pole is inside the contour, however now $S_{cd}^{ab}(s)$ must go to a constant at infinity, otherwise the integral doesn't converge. Therefore to arbitrarily enlarge this the contour for functions that doesn't have this behavior we use the subtraction method [11]:

$$S_{cd}^{ab}(s) = \oint_{\Delta(s,\epsilon)} \frac{dz}{2\pi i} \frac{S_{cd}^{ab}(z)}{z - s} \prod_{l=1}^{M} \left\{\frac{s - x_l}{z - x_l}\right\},$$

where now $\Delta(s,\epsilon)$ encompasses no singularity or cut and none of the $x_l \in \mathbb{C}$. Then for convergence reasons now $S_{cd}^{ab}(s)$ can have a singularity of order M at infinity, *i.e.*, it can

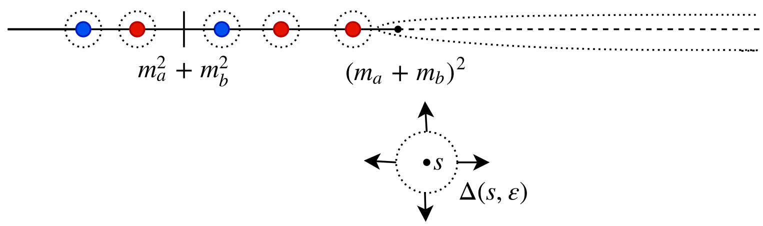


Figure 1.4: Contour of integration used in the dispersion relation of $S^{ab}_{cd}(s)$. The dots are the same as in Figure 1.2. The dotted circle around s is $\Delta(s, \epsilon)$. In the expansion the contour deforms into the dotted lines circling the branch cut and the poles. Due to crossing symmetry the same happens for the t-cut starting at $(m_a - m_b)^2$.

go at most as fast as s^M. This is what it means for the amplitude to have no essential singularity, *i.e.*, it behaves polynomially at high energies. But what is the implication of this? As shown in Figure 1.4 if we enlarge the contour the integral becomes a sum over the poles and line integrals over the cuts. Suppose for example that $S^{ab}_{cd} = S$, that is crossing symmetric and goes as $S_\infty s^M$ at infinity, where $S_\infty \in \mathbb{C}$ is a constant. We have that:

$$\begin{aligned} S(s) =& S_\infty - \sum_{l=1}^{N_{bs}} J(m_l) \left(\prod_{l=1}^{M} \left\{ \frac{s - x_l}{m_l^2 - x_l} \right\} \frac{g_l^2}{s - m_l^2} + \prod_{l=1}^{M} \left\{ \frac{(m_a + m_b)^2 - s - x_l}{(m_a + m_b)^2 - m_l^2 - x_l} \right\} \times \right. \\ & \left. \times \frac{g_l^2}{(m_a + m_b)^2 - s - m_l^2} \right) + \int_{(m_a+m_b)^2}^{+\infty} dx \prod_{l=1}^{M} \left\{ \frac{s - x_l}{x - x_l} \right\} \rho(x) \left(\frac{1}{x - s} + \right. \\ & \left. + \frac{1}{x - (m_a + m_b)^2 + s} \right), \end{aligned} \tag{1.5}$$

where $\rho(s) = 2\pi \mathrm{Im}\{S^{ab}_{cd}\}(s)$ encompasses the discontinuity and is crossing symmetric by definition, m_l is the mass of the bound state, g_l^2 is the associated residue of the T-matrix and N_{bs} is the number of bound states. The factor $J(m_l)$ is the Jacobian relating the residue of the T ans S-matrices, since the latter has some extra delta functions in comparison to the former. Equation (1.5) is the so called dispersion relation. If $m_a = m_b = m_c = m_d = m$ we have that $J(m_l)$ is given by [4]

$$J(m_j) = \frac{m^4}{2m_j\sqrt{4m^2 - m_j^2}}.$$

For non-crossing symmetric S-matrices we have to consider how $\rho(x)$ changes with crossing, which have to be worked in a case-by-case basis, an example of this behavior is given in the O(N) model where different isospin channels mix after crossing.

Now a small caveat concerning these definitions and perturbative results. First g_l^2, defined as the residue of the T-matrix, is denominated the quantum coupling or simply coupling. The relation between this and the Lagrangian coupling is clear from tree level computations. However we can do a non-relativistic approach and use Born approximation to see that g_l^2 would give the strength of interaction in the associated classical potential (Yukawa potential). Then although g_l^2 isn't exactly the coupling of the Lagrangian it can be considered a renormalized version of it. Now, the branch cut information is encoded in the $\rho(s)$ functions. An example of these in perturbation theory would be the discontinuities that appear in $2 \to 2$ one-loop S-matrix computations in QED, which means that one has a branch cut starting at the two particle threshold related to the electron mass [19].

We also can express all these analytical structures in terms of rapidity variables. Let's consider $S_{ab}^{cd}(s)$ with a and b being the lightest particles in the theory. Using equation (1.3) we have that the cuts $[(m_a+m_b)^2, +\infty)$ and $(-\infty, (m_a-m_b)^2]$ are mapped to the lines $\mathrm{Im}\{\theta\} = 0$ and $\mathrm{Im}\{\theta\} = i\pi$ in the θ-plane, respectively. Where s^+ and s^- in the former cut are mapped to $\theta > 0$ and $\theta < 0$ and the for the latter these are mapped to $\mathrm{Re}\{\theta\} > 0$ and $\mathrm{Re}\{\theta\} < 0$. Other cuts are mapped to lines on $\mathrm{Im}\{\theta\} = i\pi N$, where $N \in \mathbb{Z}$ as can be seen on ref. [20].

The poles are mapped to points in the imaginary axis with $\mathrm{Im}\{\theta\} \in [0, \pi]$. In these variables crossing is given by $\theta \to i\pi - \theta$ as it can easily be checked, which is a reflection around the crossing symmetric point $\theta = i\pi/2$. Note also that $\cosh(\theta)$ has $2\pi i$ periodicity, then its inverse function must be limited to some sheet of the Riemann surface determined by the inverse. It can be shown then that the entire physics is encoded within $\mathrm{Im}\{\theta\} \in [0, \pi]$ which is called the physical strip. However conjugation of the s variable clearly corresponds to reflection around the imaginary axis, *i.e.*, $\mathrm{Re}\{\theta\} \to -\mathrm{Re}\{\theta\}$. Since we can work entirely on the physical strip all cuts disappear in this parametrization and we have that the S-matrix elements are nice meromorphic functions on the physical strip without any cuts. If we don't scatter the lightest particles it's necessary to use extended unitarity relations, which was work out in ref. [10].

In these variables crossing and unitarity become

$$\text{Unitarity: } S_D(\theta) S_D(-\theta)^T - \mathbb{1}_D \preccurlyeq 0 \quad \text{and} \quad \text{Crossing: } S_{ab}^{cd}(\theta) = S_{bc}^{da}(i\pi - \theta),$$

with crossing phase already absorbed and unitarity being evaluated at the s cut, so in the θ-plane it's valid for $\mathrm{Im}\{\theta\} = 0$. However we can analytically extended this for entire $\mathbb{C}$. This extension implies that for every pole at θ_0 there is a corresponding zero at $-\theta_0$. Also if the amplitude is self-crossing, it is easy to see that the S-matrix is periodic on the imaginary axis:

$$S_{cd}^{ab}(\theta) = S_{cd}^{ab}(\theta + 2\pi i).$$

These analytical properties of the S-matrix on the θ-plane are explicit on Figure 1.5.

1.2 Yang-Baxter Equation and Quantum Groups

Beyond significant simplifications in kinematics, why is $1+1$ dimensions so special? One reason is that there exist very particular quantum field theories called integrable models that are completely solvable and nontrivial. Note that they're fully interacting quantum field theories with weak/strong coupling limits and interesting non-perturbative behaviour. These yield toy models to analyze the longstanding problem of non-perturbative effects on quantum field theories.

The question is how these integrable models arise. As usual in physics if they are solvable it means that some symmetry is in action. This notion is clear in the context of classical integrability. A classical system evolving in a $2n$-dimensional phase space is integrable in the Liouville sense if there are n conserved and linearly independent charges. Then one can use these and construct the action angle variables and solve the system [21]. For us a integrable 1+1 dimensional quantum field theory is a model with an with an

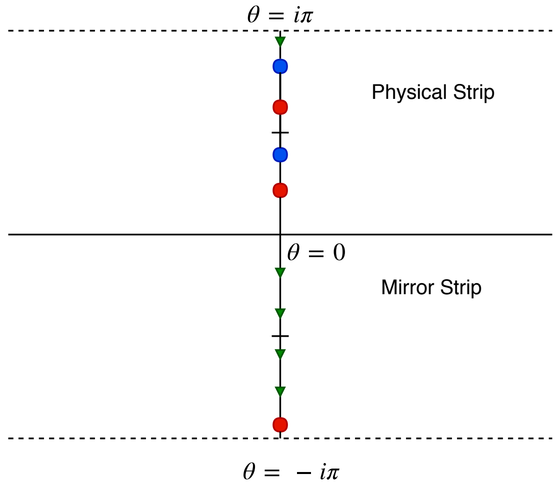


Figure 1.5: Schematics of the analytical structure of the S-matrix in the θ-plane. Where the red points, blue points and green triangles are the s poles, the corresponding t poles and the zeros of the S-matrix, respectively. The set with $\text{Im}\{\theta\} \in [-\pi, 0]$ is denominated mirror strip.

infinite set of conserved charges that in our case depend on momentum or equivalently on the rapidity. Then these charges by Noether's theorem would define an symmetry action that restrict the S-matrix and have dramatic consequences. The keen reader should note that these charges don't commute with the Lorentz generators, thus violating the Coleman-Mandula theorem and the corresponding S-matrix should be trivial. However this is not the case as we'll see later. The discussion here will be briefly since these facts are well known and aren't the focus of the work. All the following arguments are more detailed in refs. [22–24].

First let's establish the notation for the states in terms of rapidity variables. The one particle state is given by $|A_a(\theta)\rangle \in \mathcal{H}$, where A_a denotes the particle A with index a coming from some internal symmetry, for example in a $O(N)$ symmetric theory it means the isospin. We want to describe scattering states, then the multiparticle state must represent well separated wavepackets. The following conventions are established:

$$|A_{a_1}(\theta_1)\cdots A_{a_n}(\theta_n)\rangle \text{ is an } in \text{ state if } \theta_1 > \cdots > \theta_n$$
$$|A_{a_1}(\theta_1)\cdots A_{a_n}(\theta_n)\rangle \text{ is an } out \text{ state if } \theta_1 < \cdots < \theta_n.$$

The S-matrix is defined just as before, however in 1+1 dimensions two conventions can be used. For $2 \to 2$ scattering these are:

$$|A_a(\theta_1)A_b(\theta_2)\rangle_{in} = S_{ab}^{cd}(\theta)|A_c(\theta_1)A_d(\theta_2)\rangle_{out} \quad \text{and}$$
$$|A_a(\theta_1)A_b(\theta_2)\rangle_{in} = \hat{S}_{ab}^{cd}(\theta)|A_c(\theta_2)A_d(\theta_1)\rangle_{out}.$$

The only difference between the two appear in the presence of fermions where the second one gain a minus sign with respect to the first when two fermions are exchanged. We'll workout with the first convention and in the end introduce the correct minus signs. Note that only the second convention yields a C, P and T invariant S-matrix as will be detailed in the next sections. The conventions established are useful to represent the particles diagrammatically as lines in the plane that carries a parameter θ_i and in each crossing of worldlines it's defined a S-matrix.

Then let's describe the conserved charges in 1+1 dimensions. We consider those charges coming from local integration of currents of spin s and denote them by Q_s. Since their spin is s it's easy to see that if $\Lambda(\theta)$ is a boost with rapidity θ then

$$\Lambda(\theta)^{-1}Q_s\Lambda(\theta) = e^{-s\theta}Q_s,$$

which implies the following action on one-particle states if they're eigenstates:

$$Q_s|A_a(\theta)\rangle = q_a^{(s)}e^{s\theta}|A_a(\theta)\rangle.$$

Where $q_a^{(s)}$ is the corresponding eigenvalue in the rest frame. The parity conjugated charge is denoted by $\bar{Q}_s$ and it acts as

$$\bar{Q}_s|A_a(\theta)\rangle = \bar{q}_a^{(s)}e^{-s\theta}|A_a(\theta)\rangle,$$

since we assume parity invariant theories these charges always exists. The locality argument enters as the fact that the charges act additively on asymptotic states, *i.e.*, let Δ_n be the coproduct map that takes a charge $Q_s \in \mathrm{End}(\mathcal{H})$ and yields its representative on $\mathrm{End}(\otimes_{j=1}^{n}\mathcal{H})$ then[3]

$$\Delta_n(Q_s) = Q_s \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1} + \mathbb{1} \otimes Q_s \otimes \cdots \otimes \mathbb{1} + \cdots + \mathbb{1} \otimes \cdots \otimes Q_s.$$

Consequently their action is additively in the multiparticle states

$$\Delta_n(Q_s)|A_{a_1}(\theta_1)\cdots A_{a_n}(\theta_n)\rangle = \left(\sum_{j=1}^{n} q_{a_j}^{(s)}e^{s\theta_j}\right)|A_{a_1}(\theta_1)\cdots A_{a_n}(\theta_n)\rangle.$$

An example of a charge is the momentum in lightcone coordinates where $s = 1$ and $q_a^{(1)} = m_a$ as it can easily be checked.

We consider all the infinite charges to be local and in involution, *id est*, they commute. Therefore the multiparticle states will be eigenstates of all Q_s, since they're simultaneously diagonalizable. Since these are conserved charges we have that in a $n \to m$ scattering:

$$\sum_{j=1}^{n} q_{a_j}^{(s)}e^{s\theta_j} = \sum_{k=1}^{m} q_{a_k}^{(s)}e^{s\theta_k}.$$

However these equations must be true for an infinite set of spins, thus the only solution is $n = m$ with charges and rapidities permuted. Due to momentum conservation the set of masses in the initial and final states are equal, although the particles themselves can be different if there is some mass degeneracy, like in supersymmetric models or $\mathrm{O}(N)$ theories for example. Therefore integrable models have no particle production and consequently unitarity is saturated. A nice question is the reverse statement:

"If some quantum field theory in 1+1 dimensions has no particle production it must be integrable?"

[3]The coproduct defined here must preserve the Lie algebra, indeed for other kinds of algebra the map is different and consequently the charges or generators don't act additively. Also let V be some vector space then $\mathrm{End}(V)$ denotes the set of endomorphisms of V.

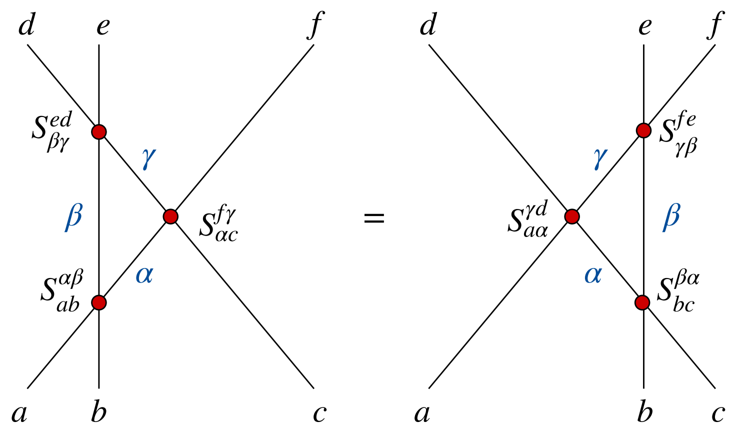


Figure 1.6: Factorizability of the S-matrix also called Yang-Baxter relation. Where the worldlines a, b and c carry rapidities θ_1, θ_2 and θ_3, respectively. The blue indexes represent internal states and we must sum over all possible ones with the care that a fermion loop carries an associated minus sign.

In this work we partially answered this question as will be discussed in the next chapter. However this was solved at least in a perturbative level. Indeed, in ref. [25] the authors proved that if we impose no particle production at tree-level for models with bosonic spectrum in $2d$ the only theories compatible with this are Toda field theories and non-linear sigma models, which are well-known integrable models. Also in ref. [26] the authors proved that with the same assumptions plus supersymmetry (for a single supermultiplet and only chiral superpotentials) we only find supersymmetric sine-Gordon. Then, at least in tree-level, the only models without particle production are integrable.

Now it looks like that to completely solve an integrable we would have to find all the infinite conserved charges and see the effect of all these in the S-matrix. However there is a much simpler approach that follows directly from their existence. First note that the action of the symmetry operator $\exp(i\alpha Q_s)$ at the wave packets shifts position by a momentum dependent function. Since the charges commute with the S-matrix then the scattering with the wavepackets shifted is equal to the one with them unshifted. Therefore in a $n \to n$ event with all the wavepackets prepared to collide at the same point the existence of these charges implies that the particles' worldlines can be moved around such that they hit each other in $2 \to 2$ processes and don't violate microcausality[4]. This is the factorizability of the S-matrix. Thus for $3 \to 3$ scattering there is only two possibilities as shown in Figure 1.6. The equality in this picture is the famous Yang-Baxter equation.

Let's formulate this more precisely. Let $S_{ij}(\theta_{ij})$ with $i, j \in \{1, 2, 3\}$ be an operator in $\mathrm{End}(\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H})$ defined as the $2 \to 2$ S-matrix acting on the i and j entries only and the identity operator in the remaining state. Therefore the Yang-Baxter equation is an identity on $\mathrm{End}(\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H})$ given by:

$$S_{23}(\theta_{23})S_{13}(\theta_{13})S_{12}(\theta_{12}) = S_{12}(\theta_{12})S_{13}(\theta_{13})S_{23}(\theta_{23}).$$

Following the conventions established in Figure 1.6 we can write the Yang-Baxter relation

[4]That means basically that particles of the *in* state can't collide with the ones at the *out* state before these collide between themselves.

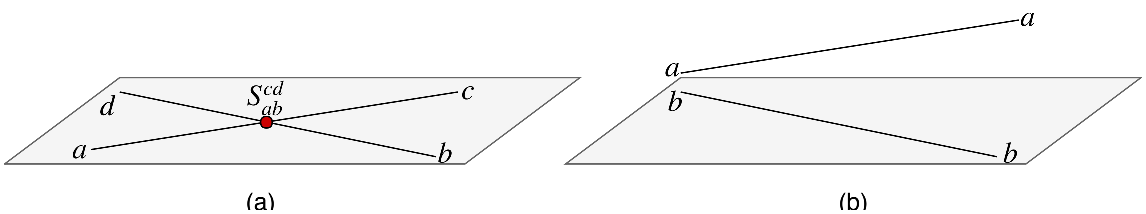


Figure 1.7: Loophole of the Coleman-Mandula theorem. In figure (a) we have the plane defined by the worldlines of a and b and their scattering at a point. In figure (b) we have a translation in the orthogonal direction of the worldline of a generated by some conserved charge Q_s, then there is no scattering anymore and the S-matrix is trivial since the two scenarios must match.

in components, which is

$$\sum_{\alpha,\beta,\gamma} S^{\alpha\beta}_{ab}(\theta_{12}) S^{f\gamma}_{\alpha c}(\theta_{13}) S^{ed}_{\beta\gamma}(\theta_{23}) = \sum_{\alpha,\beta,\gamma} S^{\beta\alpha}_{bc}(\theta_{23}) S^{\gamma d}_{a\alpha}(\theta_{13}) S^{fe}_{\gamma\beta}(\theta_{12}).$$

It's clear that if a model is integrable, it must satisfy Yang-Baxter equations, thus we'll take this as the definition of an integrable model. Later we'll use this criterion as a test for the integrability of the models obtained in the numerics. In these type of theories the $S_{2\to 2}$ completely fixes the entire S-matrix. Also note that there is at most $\dim(\mathcal{H})^2$ unknown $2 \to 2$ processes and Yang-Baxter implies at most $\dim(\mathcal{H})^3$ equations, therefore we can use the latter to find the former up to an ambiguity that comes from multiplying by a function of θ all amplitudes in $S_{2\to 2}$. The multiplicative ambiguity can be partially tamed if we impose unitarity. However we still have functions that have unit norm and some level of ambiguity remains, these are fixed by physical requirements specific to each model and comparisons with the perturbative expansion.

Let's come back now to the paradox of obtaining non trivial integrable theories even though they appear to violate the Coleman-Mandula theorem. The charges considered before violate the assumptions of the theorem since they depend explicitly on the momentum, therefore the group that they generate isn't in a direct product with $\mathrm{SO}(1,1)$. Note that these charges generates momentum dependent phase shifts that translate wavepackets and yield us factorization equations. However in 1+1 dimensions if two worldlines intersect at a single point then every translation of them also intersect. Thus the theory can never be trivial, unlike in higher dimensions where this translation can occur in some orthogonal direction to the plane they originally defined, then rendering a trivial theory as shown in Figure 1.7. Consequently this is a "loophole" of the Coleman-Mandula theorem and nontrivial integrable models can be constructed.

Note that the Yang-Baxter relation is a very constrained cubic equation. It would be interesting to classify all solutions of it similar to Cartan's classification of Lie algebras. Although there is no general framework we have some partial results concerning quasi-classical R-matrices[5] due to A. A. Belavin and V. G. Drinfeld [27]. A quasi-classical R-matrix is an operator that satisfies Yang-Baxter and that depends on some continuous parameter $\hbar$ and a spectral parameter[6] $z \in \mathbb{C}$, then it can be written as $R_\hbar(z)$. Let V be

[5] Let V be some vector space. In statistical mechanics, a R-matrix is an operator in $\mathrm{End}(V \otimes V)$ that satisfies Yang-Baxter relation. To be a proper S-matrix in a quantum field theory it must be supplemented with Lorentz invariance, unitarity and crossing symmetry.

[6] More specifically the R-matrix depends on the spectral parameters of the two crossing lines let's say z_i and z_j, however we'll assume that it depends only $z = z_i - z_j$.

some vector space and $R_\hbar(z) \in \mathrm{End}(V \otimes V)$. Then we can power expand the R-matrix as a series in $\hbar$ around $\hbar = 0$, *i.e.*, $\hbar$ works like a loop counting parameter in perturbative quantum field theory. Therefore

$$R_\hbar(z) = \mathbb{1}_{V\otimes V} + \sum_{n=1}^{+\infty} \frac{\hbar^n}{n!} r^{(n)}(z),$$

where $\mathbb{1}_{V\otimes V}$ is the identity on $\mathrm{End}(V \otimes V)$ and $r^{(1)}(z) = r(z)$. Note that the multiplicative ambiguity in the Yang-Baxter relation can be used to fix the leading term to be the identity operator. Just as before $r_{ij} \in \mathrm{End}(V \otimes V \otimes V)$ denotes the same as $S_{ij}(\theta_{ij})$. Thus the Yang-Baxter relation up to order $\mathcal{O}(\hbar^2)$ is

$$[r_{23}(z_{23}), r_{12}(z_{12}) + r_{13}(z_{13})] + [r_{13}(z_{13}), r_{12}(z_{12})] = 0.$$

This is the classical Yang-Baxter equation and is a central object in the theory of classical integrable systems [21].

Suppose that r takes values on some semi-simple Lie algebra $\mathfrak{g}$ and that it's non-degenerated, *i.e.*,

$$r(z) = \sum_{a,b} r_{ab}(z)(t_a \otimes t_b) \text{ with and } \det(r_{ab}(z)) \neq 0,$$

where $t_a \in \mathfrak{g}$. Then with these assumptions, plus some general analytical remarks that don't concern us here, the authors in ref. [27] proved that exist only three classes of quasi-classical R-matrices: rational, trigonometric and elliptic. And each one is associated Hopf algebras these being the Yangian, quantum affine algebra and elliptic algebras, respectively.

Recently there is a new interesting and radically different approach to the Yang-Baxter equation. In 1988 M. Atiyah proposed that the factorization equations could be derived from some higher dimensional topological field theory [28]. Now in recent years K. Costello, E. Witten, and M. Yamazaki in a series of papers developed this idea in detail at refs. [29–33]. We'll briefly describe the main idea here. Consider a perturbative gauge theory with action given by

$$I[A] = \frac{1}{2\pi} \int_{\Sigma\times C} \omega \wedge \mathrm{CS}(A),$$

where Σ is a two dimensional differentiable manifold, C is a holomorphic manifold with complex dimension one, ω is a fixed 1-form at C, A a gauge field and $\mathrm{CS}(A)$ the usual Chern-Simons 3-form. Also the gauge group G has a semi-simple Lie algebra $\mathfrak{g}$. Note that the action defined has explicit diffeomorphism invariance for transformations in Σ and is topological[7]. Then the correlation functions don't depend on position, since we have a topological freedom to move objects around. This theory doesn't have holomorphic invariance because ω is a fixed form. It's argued that to have a perturbative theory there is only three possible choices for ω and C:

$$\begin{aligned}&(i) : C = \mathbb{C} \text{ and } \omega = dz,\\ &(ii) : C = \mathbb{C}/\mathbb{Z} \text{ and } \omega = dz/z,\\ &(iii) : C = \mathbb{C}/\ (\mathbb{Z} + \tau\mathbb{Z}) \text{ and } \omega = dz.\end{aligned}$$

[7]Technically it's a topological field theory of Schwarz type since it doesn't depend explicitly on the metric.

Where $\tau \in \mathbb{C}$. Each one of these choices is linked to the three classes of quasi-classical R-matrices discussed previously.

But before we discuss this relation, how the R-matrix and Yang-Baxter arise in this context? Basically they considered an object analogous to Wilson lines given by

$$W_K(z) = \mathcal{P}\exp\left(\int_K A(x,y,z)\right),$$

where (x,y) and z are, respectively, coordinates on Σ and C and $K \subset \Sigma$ is some curve[8]. So this "Wilson line" is interpreted as a wordline of a single particle if K is a straight line and the correlation function of two of these operators become the R-matrix. The matrix structure of the R-matrix is obviously present and it assumes values in $\mathfrak{g} \otimes \mathfrak{g}$ by construction, and it's given by a series in $\hbar$. Then it's possible to identify these operators with the quasi-classical R–matrices from before. Also Yang-Baxter follows trivially from topological invariance, indeed given three of these Wilson lines with distinct spectral parameters that meet in the same point of Σ we can move them around since they don't actually cross due to the distinct spectral parameters, *i.e.*, they lie at different points at C. This yields the factorization equations.

The relation with the three classes of quase-classical R-matrices is given by the three choices of manifolds. The case (i) one denotes rational R-matrices and they depend only on differences of spectral parameters. Now for (ii) is clear that $C = \mathbb{C}/\mathbb{Z}$ is a cylinder, thus all functions in it are periodic in a single direction and depend on ratios z_i/z_j. Therefore this choice yields trigonometric R-matrices. Lastly in the case (iii) we identify $\mathbb{C}$ in both directions, consequently C is a torus. Then all functions here are periodic in the real and imaginary axis, consequently we have elliptic R-matrices. Thus it's recovered all classes of quasi-classical R-matrices based on semi-simple Lie algebras. With this way of deriving R-matrices and Yang-Baxter it's hoped that new light will be shed on these issues, which in turn could provide some classification framework for all solutions of Yang-Baxter equation.

To end this section let's see how the set of conserved charges is linked with a specific model. Consider a $2 \to 2$ scattering with a single bound state being exchanged. By locality and unitarity its residue is the product of two off-shell three point amplitudes. Indeed, let the scattering $S^{cd}_{ab}(\theta)$ with particle A_e being exchanged and the corresponding pole at $\theta = iU^e_{ab}$, where $U^e_{ab} \in [0,\pi]$. Consequently the three-point process S^e_{ab} can be interpreted as the particles A_a and A_b approaching each other and fuse to form A_e. Which conforms with the notion of bound state and U^e_{ab} is named the fusing angle which is the minimal (complex) energy for this process to occur. A bound state is thus defined as:

$$\lim_{\epsilon\to 0} |A_a(\theta+\epsilon+iU^e_{ab}/2)A_b(\theta - iU^e_{ab}/2)\rangle = |A_e(\theta)\rangle$$

This has an interesting consequence. Due to integrability the worldlines of the particles can be moved around and the scattering process remains invariant, thus we could scatter say A_a and A_b at minimal energy to form A_e and then hit the latter (which is at rest) with some other particle A_c and this would be equal to scatter A_c with particles A_a and A_b before the fusing took place. This equality is graphically represented in Figure 1.8. In terms of the S-matrix the fusion identity is:

$$S^{ce}_{ce}(\theta) = S^{ac}_{ac}(\theta - iU^e_{ab}/2)S^{bc}_{bc}(\theta + iU^e_{ab}/2).$$

[8]Although the operator doesn't look gauge invariant since the trace is missing it's invariant due to the infrared free nature of the model [31].

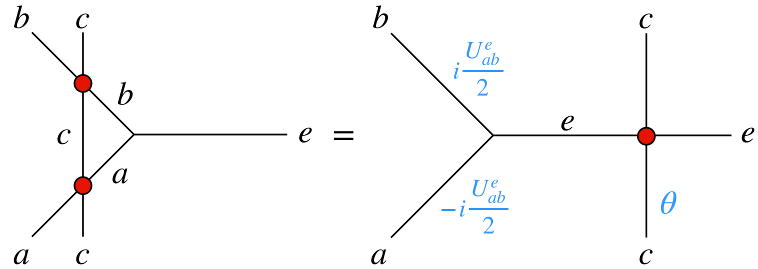


Figure 1.8: S-matrix diagram of the fusion rule. Where the blue labels are the rapidities of the corresponding worldlines. Note that strictly saying this process isn't physical since it occurs for complex momenta, therefore it should be taken as an axiom for integrable models based on physical grounds.

Then we'll consider as definition of integrable model that it must satisfy both Yang-Baxter and the fusion identity. If the latter is not realized then there is some new particles in the spectrum which must be consistently added in a way that closes the fusion identity. Now if we regard the fusion of A_a and A_b as a physical process it must satisfy charge conservation, which in turn implies

$$q_a^{(s)} + q_b^{(s)} e^{-isU_{ab}^c} + q_c^{(s)} e^{-is(U_{ab}^c + U_{cb}^a)} = 0.$$

This equation can be interpreted geometrically in terms of a generalized cosine law with the fusing angles and the charges being the external angles and sides of the corresponding triangle, respectively. This puts severe restrictions in the the conserved charges' set of spins if we know the charges of the particles. Indeed, consider that we have only a single particle A_a in the spectrum, therefore $U_{aa}^a = 2\pi/3$ and

$$1 + e^{-is2\pi/3} + e^{-is4\pi/3} = 0,$$

whose solution is $s = (1,5) \mod 6$, that is $s \in \{1, 5, 7, 11, 13, 17, \cdots\}$. Consequently the spectrum of spins is directly linked with the particle content of a model, thus forming some sort of fingerprint of the integrable model.

Let us make a final remark suggested by this picture of bound states. A principle that drives these analytical S-matrix results we described is the *Nuclear Democracy* or *Bootstrap Principle*: "*All particles in a model are in the same footing*". Then for us bound states are as fundamental as the excitations described by a Lagrangian for example.

1.3 Supersymmetric S-matrices

In this section we'll apply supersymmetry to a 1+1 dimensional S-matrix in a theory with a single boson and fermion. This servers to define the S-matrix used in the numerical bootstrap. First it's needed to see how to realize supersymmetry in $\mathcal{H}$. In 1+1 dimensions we can use lightcone coordinates to describe the supercharges, thus the superalgebra becomes [34]:

$$\{Q_-, Q_+\} = 0,\ Q_+^2 = P_+ \text{ and } Q_-^2 = P_-.$$

Where $Q_\pm$ are the supercharges and $P_\pm$ the lightcone momenta. We can have a central charge in the superalgebra that would yield a BPS bound for the masses in the spectrum which appears for example in the presence of solitons. Clearly the supercharges have spin

1/2, however more convoluted realizations of the superalgebra can have lower fractional spin charges. For instance we could have $Q^n = H$ with $n \in \mathbb{N}$, Q is the supercharge and H the Hamiltonian, thus Q have spin $1/n$. This is the phenomenon of fractional supersymmetry and is exclusive to 1+1 dimensions [35].

We consider in this work a 1+1 dimensional quantum field theory with one boson (ϕ) and one fermion (ψ) both with mass m and forming a supersymmetric multiplet. Also ϕ is a neutral scalar which implies that ψ is Majorana fermion, since they belong to the same supermultiplet. Our Hilbert space is $\mathcal{H} = \mathbb{CP}^1$ with the basis being $\{|\phi(\theta)\rangle, |\psi(\theta)\rangle\}$. These are the lightest particles in the theory, therefore we consider no central charge in the superalgebra. Here we delineate the supersymmetric S-matrix description given in refs. [24, 36–38]. A representation of the superalgebra in $\mathcal{H}$ is given by

$$Q_+ = \sqrt{m}\, e^{\theta/2} \begin{pmatrix} 0 & e^{-i\pi/4} \\ e^{i\pi/4} & 0 \end{pmatrix} \text{ and } Q_- = \sqrt{m}\, e^{-\theta/2} \begin{pmatrix} 0 & e^{i\pi/4} \\ e^{-i\pi/4} & 0 \end{pmatrix},$$

where θ is the rapidity of the state they act on[9]. To apply supersymmetry to the $2 \to 2$ S-matrix we need its action on $\mathcal{H} \otimes \mathcal{H}$. Note that every time a supercharge pass through a fermion we gain an extra minus sign. Then it follows

$$\Delta_2(Q_\pm) = Q_\pm \otimes \mathbb{1}_2 + \sigma_3 \otimes Q_\pm,$$

where σ_3 yields the aforementioned minus sign and $\Delta_2 : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ is the coproduct that takes an algebra $\mathcal{A}$ to its tensor product.

Now the most general $2 \to 2$ S-matrix with only fermion number conservation is:

$$S(\theta) = \begin{pmatrix} S^{\phi\phi}_{\phi\phi}(\theta) & 0 & 0 & S^{\psi\psi}_{\phi\phi}(\theta) \\ 0 & S^{\phi\psi}_{\phi\psi}(\theta) & S^{\psi\phi}_{\phi\psi}(\theta) & 0 \\ 0 & S^{\phi\psi}_{\psi\phi}(\theta) & S^{\psi\phi}_{\psi\phi}(\theta) & 0 \\ S^{\phi\phi}_{\psi\psi}(\theta) & 0 & 0 & S^{\psi\psi}_{\psi\psi}(\theta) \end{pmatrix}. \tag{1.6}$$

Here the eight amplitudes are independent, however imposing C, P and T this number can be decreased, indeed

$$S^{\psi\psi}_{\phi\phi}(\theta) = S^{\phi\phi}_{\psi\psi}(\theta), \quad S^{\phi\psi}_{\phi\psi}(\theta) = S^{\psi\phi}_{\psi\phi}(\theta) \text{ and } S^{\psi\phi}_{\phi\psi}(\theta) = S^{\phi\psi}_{\psi\phi}(\theta).$$

These constraints applied on eq. (1.6) yields five independent amplitudes. Also crossing symmetry acts as

$$\begin{aligned} S^{\phi\phi}_{\phi\phi}(i\pi - \theta) &= S^{\phi\phi}_{\phi\phi}(\theta), \quad S^{\psi\psi}_{\psi\psi}(i\pi - \theta) = S^{\psi\psi}_{\psi\psi}(\theta), \\ S^{\psi\psi}_{\phi\phi}(i\pi - \theta) &= S^{\psi\phi}_{\phi\psi}(\theta), \quad S^{\phi\psi}_{\phi\psi}(i\pi - \theta) = S^{\phi\psi}_{\phi\psi}(\theta). \end{aligned}$$

With all these constraints we reduce the number of independent amplitudes from eight to only four.

Now let's apply supersymmetry to the $2 \to 2$ S-matrix satisfying C, P and T. It must commute with the supercharges, so

$$[\Delta_2(Q_+), S(\theta)] = 0 \quad \text{and} \quad [\Delta_2(Q_-), S(\theta)] = 0,$$

[9]We chose this realization for the supercharges instead of Pauli matrices because it yields a crossing phase equal to unity and thus a real analytic S-matrix. If you start with the Pauli matrices a change of basis will be needed to have real analyticity as seen in refs. [36, 37].

in terms of the matrix components these yield the following linear system

$$\begin{aligned}
&S^{\phi\phi}_{\phi\phi}(\theta) + 2i \ \mathrm{csch}(\theta/2) \ S^{\phi\phi}_{\psi\psi}(\theta) = S^{\psi\psi}_{\psi\psi}(\theta),\\
&S^{\psi\psi}_{\phi\phi}(\theta) + S^{\phi\phi}_{\psi\psi}(\theta) = 0,\\
&S^{\phi\psi}_{\phi\psi}(\theta) + i \ \mathrm{csch}(\theta/2) \ S^{\phi\phi}_{\psi\psi}(\theta) = S^{\psi\psi}_{\psi\psi}(\theta),\\
&S^{\psi\phi}_{\phi\psi}(\theta) + i \ \mathrm{coth}(\theta/2) \ S^{\phi\phi}_{\psi\psi}(\theta) = 0,\\
&S^{\phi\psi}_{\psi\phi}(\theta) + i \ \mathrm{coth}(\theta/2) \ S^{\phi\phi}_{\psi\psi}(\theta) = 0,\\
&S^{\psi\phi}_{\psi\phi}(\theta) + i \ \mathrm{csch}(\theta/2) \ S^{\phi\phi}_{\psi\psi}(\theta) = S^{\psi\psi}_{\psi\psi}(\theta).
\end{aligned}$$

Solving it in terms of the crossing symmetric amplitudes $S^{\phi\phi}_{\phi\phi} = S^{(1)}$ and $S^{\phi\psi}_{\phi\psi} = S^{(2)}$ we get

$$S(\theta) = \begin{pmatrix} S^{(1)}(\theta) & 0 & 0 & -i\sinh(\theta/2)f(\theta) \\ 0 & S^{(2)}(\theta) & \cosh(\theta/2)f(\theta) & 0 \\ 0 & \cosh(\theta/2)f(\theta) & S^{(2)}(\theta) & 0 \\ i\sinh(\theta/2)f(\theta) & 0 & 0 & 2S^{(2)}(\theta) - S^{(1)}(\theta) \end{pmatrix}, \tag{1.7}$$

where $f(\theta) = S^{(1)}(\theta) - S^{(2)}(\theta)$. The resulting S-matrix is real analytic, however it violates the C, P and T constraints since $S^{\psi\psi}_{\phi\phi} \neq S^{\psi\psi}_{\phi\phi}$. This isn't too alarming if we note that the $S(\theta)$ defined before isn't the physical S-matrix ($\hat{S}$) since it doesn't possess a minus sign obtained in the exchanging of two fermions. The relation between S and $\hat{S}$ is given by the sign matrix P via

$$\hat{S}(\theta) = P \cdot S(\theta), \ \text{where } P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Then it's clear that

$$\hat{S}(\theta) = \begin{pmatrix} S^{(1)}(\theta) & 0 & 0 & -i\sinh(\theta/2)f(\theta) \\ 0 & S^{(2)}(\theta) & \cosh(\theta/2)f(\theta) & 0 \\ 0 & \cosh(\theta/2)f(\theta) & S^{(2)}(\theta) & 0 \\ -i\sinh(\theta/2)f(\theta) & 0 & 0 & -2S^{(2)}(\theta) + S^{(1)}(\theta) \end{pmatrix}, \tag{1.8}$$

which is real analytic and C, P and T invariant. Note that we can use either S or $\hat{S}$ for the numerical bootstrap since the ansatz enters the procedure through the unitarity equation and there all these phase redefinitions are reabsorbed, thus yielding the same relations.

Equation (1.8) is the most general S-matrix compatible with C, P, T and supersymmetry. In terms of the Mandelstam variables we get

$$\hat{S}(s) = \begin{pmatrix} S^{(1)}(s) & 0 & 0 & -i\sqrt{s-4}f(s)/2 \\ 0 & S^{(2)}(s) & \sqrt{s}f(s)/2 & 0 \\ 0 & \sqrt{s}f(s)/2 & S^{(2)}(s) & 0 \\ -i\sqrt{s-4}f(s)/2 & 0 & 0 & -2S^{(2)}(s) + S^{(1)}(s) \end{pmatrix}, \tag{1.9}$$

this will be the S-matrix used in the numerical bootstrap and it contains two unknown amplitudes $S^{(1)}$ and $S^{(2)}$. Note also that all bound states come in pairs since they must form a supermultiplet, for instance $S^{\psi\psi}_{\phi\phi}$ and $S^{\phi\psi}_{\phi\psi}$ have poles in the same position and they correspond to fermionic and bosonic bound states, respectively.

Given this realization of supersymmetry a natural question that comes up can be answered. What is the distinction between this model with a supermultiplet to another with two bosons of equal mass, however one of them has $\mathbb{Z}_2$ symmetry? Note that in the former $S^{\phi\psi}_{\phi\psi}$ and $S^{\psi\phi}_{\phi\psi}$ exchange the same bound state and factorize in the same three-point amplitudes, due to unitariry and locality [39]. However in the latter the exchanged bound state could be bosons with or without $\mathbb{Z}_2$ symmetry, therefore they don't necessarily factorize equally. Thus for the supersymmetric model with a pole at $s = m_j^2$ we have

$$\mathrm{Res}(S^{\phi\psi}_{\phi\psi}, m_j^2) = \mathrm{Res}(S^{\psi\phi}_{\phi\psi}, m_j^2).$$

Where $\mathrm{Res}(f, z)$ denotes the residue of the function $f(w)$ at $w = z$. Consequently

$$\mathrm{Res}(S^{(1)}, m_j^2) = \left(1 + \frac{2}{m_j}\right) \mathrm{Res}(S^{(2)}, m_j^2), \tag{1.10}$$

therefore in a supersymmetric model the residues ratio of the independent amplitudes is fixed unlike the $\mathbb{Z}_2$ model, this result is supported by the numerical bootstrap in ref. [9]. Also it can be seen that supersymmetry supplements the fusion rules with extra relations upon action with the supercharges which further restricts the spectrum [36].

A nice feature of supersymmetric integrable models is that they factorize as a tensor product of the S-matrices from the non-supersymmetric model and another that commutes with supercharges called Bose-Fermi (BF) S-matrix. Indeed, as an example consider the supersymmetric non-linear sigma model which satisfies

$$S(\theta) = S_{\mathrm{NLSM}}(\theta) \otimes S_{\mathrm{BF}}(\theta),$$

where $S_{\mathrm{NLSM}}(\theta)$ is the S-matrix of the O(N) model and $S_{\mathrm{BF}}(\theta)$ the Bose-Fermi term that mixes bosons and fermions [24]. Indeed $S_{\mathrm{NLSM}}(\theta)$ have three independent amplitudes (singlet, antisymmetric and symmetric channels) and $S_{\mathrm{BF}}(\theta)$ possess only two, which yields six independent amplitudes for the supersymmetric $O(N)$ model. This sort of factorization will be further explored in the next chapter (Section 2.2) when we discuss $\mathcal{N} = 1$ supersymmetric sine-Gordon.

Chapter 2

Numerical S-Matrix Bootstrap

In this chapter we will explore the numerical bootstrap of supersymmetric 1+1 dimensional S-matrices in two dimensions. In the first section we set up the numerical problem as a semidefinite optimization programming problem. Then in the second section we detail the analytic construction of $\mathcal{N} = 1$ supersymmetric S-matrices like supersymmetric sine-Gordon and Bose-Fermi scattering for example. In the third and fourth sections we detail the numerics obtained for S-matrices without and with bound states.

2.1 Spectrahedron and Semidefinite Programming

The main objective of this work is to explore the space of supersymmetric massive quantum field theories in 1+1 dimensions. But how is this space configured and how to probe it? Note that we impose C, P, T, crossing symmetry, supersymmetry, and unitarity. The last constraint tells us that the space of unitary S-matrices is convex due to equation (1.2). Indeed, let's prove that positive semidefinite matrices form a convex space. Let A_1 and A_2 be positive semidefinite matrices and let $B = tA_1 + (1-t)A_2$ with $t \in [0, 1]$. Then this space is convex if $B \succeq 0$. Let V be some complex vector space and $A_1, A_2 \in \mathrm{End}(V)$. Therefore $A_1 \succeq 0$ and $A_2 \succeq 0$ if, and only if, they define a positive inner product:

$$v^\dagger A_1 v \geq 0 \text{ and } v^\dagger A_2 v \geq 0 \text{ for any } v \in V.$$

Clearly, given B we have

$$v^\dagger B v \geq 0.$$

Thus $B \succeq 0$ and the space of positive semidefinite matrices is convex. Therefore by equation (1.2) the space of unitary S-matrices is convex.

Now to explore a convex space is well known that you can maximize any linear functional of the variables in the problem that you will end up in the boundary of this space [8, 40, 41]. Then we consider the maximization of linear functionals with the linearized unitarity constraint of eq. (1.2) given the most general ansatz of a $\mathcal{N} = 1$ supersymmetric S-matrix in eq. (1.9). This is a well posed semidefinite programming (SDP) problem that is standard in the literature. Indeed it can be cast in the form:

$$\text{Maximize } f(x_1, \cdots, x_n) = \sum_{j=1}^{n} b_j x_j \text{ with } A_0 + \sum_{j=1}^{n} A_j x_j \succeq 0,$$

where A_j and b_j are known hermitian matrices and real coefficients, respectively, and $x_j \in \mathbb{R}$. The set of possible solutions

$$\mathcal{S} = \left\{ (x_1, \cdots, x_n) \in \mathbb{R}^n \middle| A_0 + \sum_{j=1}^{n} A_j x_j \succeq 0 \right\},$$

is called the spectrahedron in the literature [40]. Since the semidefinite condition can be expressed in terms of minors by the Sylvester's criterion or with the eigenvalues then the spectrahedron is basically a semi-algebraic set[1]. This leads to interesting interactions between algebraic geometry, convex geometry, and optimization theory that created a field called convex algebraic geometry [41].

From what we described before it is clear that any linear functional being maximized will return a S-matrix that saturates unitarity, *i.e.*, the boundary of the spectrahedron defined by the unitarity relation. Thus any integrable model that can be found must lay at the boundary. A nice question is that if the entire boundary is given by integrable models which is a problem that we will probe later. Note that if the answer turns out to be negative we would find models that have no particle production, yet that are not integrable, which is quite unexpected and would be interesting to see if these possible models are physically realizable or have any interesting dynamics on their own.

As discussed in ref. [8] the boundary of the spectrahedron has an interesting structure. We expect the presence of cusps that corresponds to integrable models. Indeed, suppose we have an integrable model, then we presume that any generic deformation of it to spoil the Yang-Baxter equation. Therefore it lies at a cusp of the boundary since any small change of it would not be integrable anymore. However there is a flaw on the previous argument. There are integrable models that depend on continuous parameters, thus they admit small deformations and would remain integrable. Examples of such are the supersymmetric sine-Gordon without poles, O(2) models and elliptic deformation of sine-Gordon [9,36,42]. These theories do not lie at a cusp, they are on an edge of the boundary since we can deform on one direction. To summarize: integrable models with continuous parameters lie at edges of the spectrahedron and the ones without at a extremal point of it. The framework described here is represented schematically on Figure 2.1.

Before we state the maximization problem, we need the numerical ansatz for the S-matrix. Record from Chapter 1 that we solved the supersymmetry constraints in terms of two crossing symmetric amplitudes $S^{(1)}$ and $S^{(2)}$. To find the ansatz for them we use the same procedure of ref. [4]. Consider the dispersion relations of the amplitude $S^{(i)}$ with N_{bs} bound states and that goes to a constant $S^{(i)}_\infty$ at infinity. It follows from eq. (1.5) that

$$S^{(i)}(s) = S^{(i)}_\infty - \sum_{j=1}^{N_{bs}} J(m_j) \left(\frac{g^2_{i,j}}{s - m_j^2} + \frac{g^2_{i,j}}{4m^2 - s - m_j^2} \right) + \int_{4m^2}^{\infty} dx \left(\frac{\rho^{(i)}(x)}{x - s} + \frac{\rho^{(i)}(x)}{x - 4m^2 + s} \right), \tag{2.1}$$

we assumed this behavior at infinity because it is compatible with the asymptotic form of supersymmetric sine-Gordon as will be discussed at next section[2]. Without bound states we simply get

$$S^{(i)}(s) = S^{(i)}_\infty + \int_{4m^2}^{\infty} dx \left(\frac{\rho^{(i)}(x)}{x - s} + \frac{\rho^{(i)}(x)}{x - 4m^2 + s} \right). \tag{2.2}$$

[1] A semi-algebraic set is defined by a group of algebraic equalities and inequalities.

[2] Sometimes this has to be modified. In ref. [7] the authors needed to use subtractions since the S-matrix of the O(N) model goes like $1/\log(s)$ which is a consequence of asymptotic freedom.

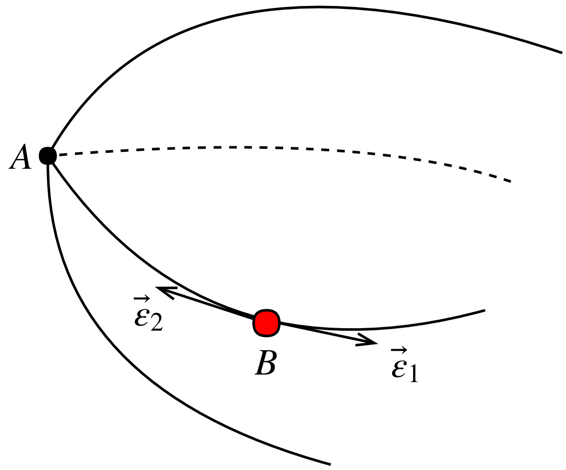


Figure 2.1: Schematics of the spectrahedron. The point A would be a cusp and we expect it to be a integrable model without deformations. In this geometric interpretation a small change would be given by the tangent vector in some direction, but for point A there is no tangent vector at any direction. However point B would be a candidate for an integrable deformable model since it can be moved in some directions let's say $\vec{\varepsilon}_1$ and $\vec{\varepsilon}_2$, but not all directions. Therefore it lives in an edge.

From real analyticity it follows that $S_\infty^{(i)}$ is real and from unitarity that $S_\infty^{(i)} \in [-1, 1]$. Remember that by definition $\rho^{(i)}(x) \in \mathbb{R}$ and it is just the discontinuity of the imaginary part of the S-matrix.

For convenience let's use units where $m = 1$, *i.e.*, all dimensionful constants are set in units of m. To parametrize $\rho^{(i)}(x)$ we consider a linear splitting. That is, suppose a grid $\{x_0, x_1, \cdots, x_M\}$ on the branch cut with $\rho^{(i)}(x_a) = \rho_a^{(i)}$ and $x_0 = 4$. We approximate $\rho^{(i)}$ by lines joining x_a and x_{a+1} for $a \in \{0, 1, \cdots, M-1\}$. Then

$$\rho^{(i)}(s) = \rho_{a+1}^{(i)} \frac{s - x_a}{x_{a+1} - x_a} + \rho_a^{(i)} \frac{x_{a+1} - s}{x_{a+1} - x_a} \text{ for } s \in [x_a, x_{a+1}].$$

Clearly the more refined the grid becomes the greater is the precision of this approximation as shown in Figure 2.2. The amplitude must go to a constant at infinity, then for $s > x_M$ it is assumed

$$\rho^{(i)}(s) = \rho_M^{(i)} \frac{x_M}{s} \text{ for } s > x_M,$$

which is reasonable since introduces the desired behavior for $S^{(i)}(s)$. Also it is considered $\rho^{(i)}(x_0) = 0$, that is, the discontinuity at the branch point vanishes. Using

$$\int_{x_a}^{x_{a+1}} dx\, \frac{\rho^{(i)}(x)}{x - s} = \rho_{a+1}^{(i)} \left(1 + \frac{s - x_a}{x_{a+1} - x_a} \log\left(\frac{x_{a+1} - s}{x_a - s}\right)\right)$$
$$- \rho_a^{(i)} \left(1 - \frac{x_{a+1} - s}{x_{a+1} - x_a} \log\left(\frac{x_{a+1} - s}{x_a - s}\right)\right) \text{ for } a \in \{0, \cdots, M-1\},$$

and

$$\int_{x_M}^{+\infty} dx\, \frac{\rho^{(i)}(x)}{x - s} = \frac{\rho_M^{(i)} x_M}{s} \log\left(\frac{x_M}{x_M - s}\right)$$

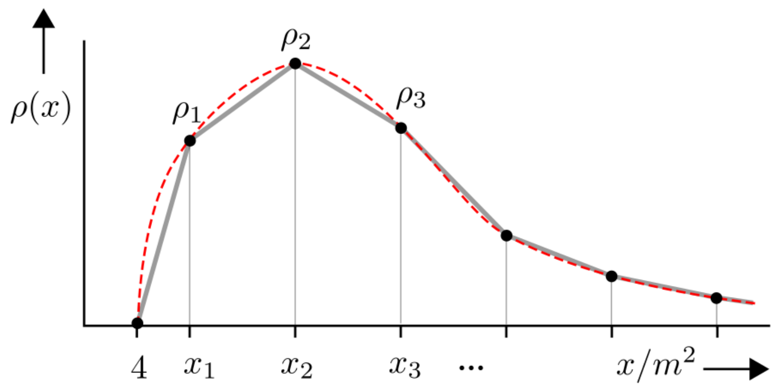


Figure 2.2: Linear splitting of $\rho(x)$. As said the function $\rho(x)$ between two grid points is approximated by a line and after x_M it is assumed a $1/x$ decay. Figure taken from ref. [4].

we can rewrite eq. (2.1) as

$$S^{(i)}(s) = S^{(i)}_\infty - \sum_{j=1}^{N_{bs}} J(m_j)\left(\frac{g^2_{i,j}}{s-m_j^2} + \frac{g^2_{i,j}}{4m^2-s-m_j^2}\right) + \sum_{a=1}^{M} \rho_a^{(i)} K_a(s). \tag{2.3}$$

Also eq. (2.2) for cases without bound state just becomes

$$S^{(i)}(s) = S^{(i)}_\infty + \sum_{a=1}^{M} \rho_a^{(i)} K_a(s). \tag{2.4}$$

Where the function K_a is given by

$$\begin{aligned}
K_a(s) = &\frac{x_{a+1}-s}{x_{a+1}-x_a}\log\left(x_{a+1}-s\right) + \frac{x_{a-1}-s}{x_a-x_{a-1}}\log\left(x_{a-1}-s\right) \\
&-\log\left(x_a-s\right)\frac{(x_{a+1}-x_{a-1})(x_a-s)}{(x_a-x_{a-1})(x_{a+1}-x_a)} + \frac{x_{a+1}-4+s}{x_{a+1}-x_a}\log\left(x_{a+1}-4+s\right) \\
&+\frac{x_{a-1}-4+s}{x_a-x_{a-1}}\log\left(x_{a-1}-4+s\right) - \log\left(x_a-4+s\right)\frac{(x_{a+1}-x_{a-1})(x_a-4+s)}{(x_a-x_{a-1})(x_{a+1}-x_a)},
\end{aligned}$$

for $a \in \{1, \cdots, M-1\}$ and for the last grid point it is

$$\begin{aligned}
K_M(s) = &\frac{x_M}{s}\log(x_M) + \frac{(x_M-s)\left(x_{M-1}-x_M-s\right)}{s(x_M-x_{M-1})}\log\left(x_M-s\right) \\
&-\frac{s-x_{M-1}}{x_M-x_{M-1}}\log\left(x_{M-1}-s\right) + \frac{x_M}{4-s}\log(x_M) \\
&\frac{(x_M-4+s)\left(x_{M-1}-x_M-4+s\right)}{(4-s)(x_M-x_{M-1})}\log\left(x_M-4+s\right) \\
&-\frac{4-s-x_{M-1}}{x_M-x_{M-1}}\log\left(x_{M-1}-4+s\right).
\end{aligned}$$

Note that $K_a(s)$ is explicitly crossing symmetric and it has the desired branch cuts, indeed the logarithms in their definition yields us this characteristic. By construction, unitarity is a s-dependent constraint then we will evaluate it at the grid points chosen before.

For the case with poles there are two residues for each one of the bound states. However using equation (1.10) it is clear that

$$g_{1,j}^2 = \left(1+\frac{2}{m_j}\right) g_j^2 \quad \text{where} \quad g_{2,j}^2 = g_j^2.$$

Therefore there is only a single coupling for each supersymmetric multiplet of bound states. Consequently the variables in the optimization problem are the quantum couplings $\{g_1^2, \cdots, g_{N_{bs}}^2\}$ and the discontinuities of each amplitude $\{\rho_1^{(1)}, \cdots, \rho_M^{(1)}, \rho_1^{(2)}, \cdots, \rho_M^{(2)}\}$. Thus for N_{bs} bound states and M grid points we have $2M + N_{bs}$ real variables in the problem.

Now we formulate the optimization problem. In the presence of bound states it is clear what have to be done. Following the approach of refs. [4,7] we maximize the couplings or a linear combination of it. Given a fixed spectrum it must have a maximum value for the couplings, otherwise new bound states would be added to the model thus violating the initial assumption. Note that linear functionals of g_j^2 are maximized not of g_j since the former are the variables in the problem. Thus in the presence of N_{bs} bound states the SDP problem is

$$\boxed{\begin{array}{l}\text{Maximize } f_{\phi_1,\cdots,\phi_{N_{bs}}}(g_1^2,\cdots,g_{N_{bs}}^2) = \displaystyle\sum_{j=1}^{N_{bs}} \left\{ \left(\prod_{l=1}^{j-1} \sin(\phi_l)\right) \cos(\phi_j)\ g_j^2 \right\} \text{given} \\ A + B(x_k)\left(S_\infty^{(1)} + \displaystyle\sum_{j=1}^{M} K_j(x_k)\rho_j^{(1)}\right) + C(x_k)\left(S_\infty^{(2)} + \displaystyle\sum_{j=1}^{M} K_j(x_k)\rho_j^{(2)}\right) + \displaystyle\sum_{j=1}^{N_{bs}} D_j(x_k)\ g_j^2 \succeq 0.\end{array}}$$

Where $x_k \in \{x_1, \cdots, x_M\}$, $\phi_1, \cdots, \phi_{N-1} \in [0, \pi]$ and $\phi_N \in [0, 2\pi)$, that is, we used a parametrization of the N_{bs}-dimensional hypersphere for the linear functionals. The coefficients A, B, C and D_j are obtained directly from the linear unitarity relation (1.2). Indeed, these are

$$A = \begin{pmatrix} \mathbb{1}_D & 0 \\ 0 & \mathbb{1}_D \end{pmatrix}, \ B(x_k) = \begin{pmatrix} 0 & \mathcal{M}_1(x_k) \\ \mathcal{M}_1(x_k) & 0 \end{pmatrix}, \ C(x_k) = \begin{pmatrix} 0 & \mathcal{M}_2(x_k) \\ \mathcal{M}_2(x_k) & 0 \end{pmatrix},$$

and

$$D_j(x_k) = -J(m_j)\left(\frac{1}{x_k - m_j^2} + \frac{1}{4 - x_k - m_j^2}\right)\left(B(x_k) + \left(1 + \frac{2}{m_j}\right)C(x_k)\right).$$

Where

$$\mathcal{M}_1(x_k) = \begin{pmatrix} 1 & 0 & 0 & \sqrt{x_k - 4}/2 \\ 0 & 0 & \sqrt{x_k}/2 & 0 \\ 0 & \sqrt{x_k}/2 & 0 & 0 \\ \sqrt{x_k - 4}/2 & 0 & 0 & -1 \end{pmatrix}$$
$$\mathcal{M}_2(x_k) = \begin{pmatrix} 0 & 0 & 0 & -\sqrt{x_k - 4}/2 \\ 0 & 1 & -\sqrt{x_k}/2 & 0 \\ 0 & -\sqrt{x_k}/2 & 1 & 0 \\ -\sqrt{x_k - 4}/2 & 0 & 0 & 2 \end{pmatrix}.$$

Also $\mathcal{M}_j(x_k)$ is hermitian, thus A, B, C and D_j are also hermitian and the SDP problem is well posed. Note that C, P and T implied a hermitian S-matrix, which is essential for the semidefinite programming. However if in another problem one of these assumptions must be violated we can simply write a bigger unitarity relation in block matrix form until it becomes Hermitian, although it would make the numerics harder.

For the case without bound states we consider a functional similar to one used in ref. [8]. Then the optimization put in a SDP form is

$$\boxed{\begin{array}{l}\text{Maximize } f_\alpha(S^{(1)}(s_0), S^{(2)}(s_0)) = \cos(\alpha) S^{(1)}(s_0) + \sin(\alpha) S^{(2)}(s_0) \text{ given} \\ A + B(x_k)\left(S_\infty^{(1)} + \sum_{j=1}^{M} K_j(x_k)\rho_j^{(1)}\right) + C(x_k)\left(S_\infty^{(2)} + \sum_{j=1}^{M} K_j(x_k)\rho_j^{(2)}\right) + \sum_{j=1}^{N_{bs}} D_j(x_k)\ g_j^2 \succeq 0.\end{array}}$$

Where $\alpha \in [0, 2\pi)$ and $s_0 \in (0, 4)$. Therefore we consider the linear functionals of $S^{(1)}(s_0)$ and $S^{(2)}(s_0)$.

As said before at a vertex could lie an interesting integrable model, however note also that the spectrahedron is a semi-algebraic set. Thus we could use algebraic geometry tools to find these points. Indeed these methods exist, however the algorithms in `Mathematica` were numerically unstable when we use arbitrary real numbers in the polynomial equations given before, thus we could not tame them[3]. It would be interesting to dominate these tools and with them maybe find new integrable models that lie on $\mathcal{S}$.

However to explore the spectrahedron we have a small caveat: it is an infinite dimensional space. Indeed, note that $\mathcal{S}$ is formed by the unitarity relation at each grid point, thus as we increase precision we also increase the number of grid points and variables of the problem and since the dimension of the spectrahedron is directly linked with the latter it must become infinite at arbitrarily high precision. Another way to see this is start directly from the dispersion relation, there the S-matrix depends on the residues and the space of all discontinuities $\rho^{(i)}(x)$, thus the spectrahedron is clearly infinite dimensional. Therefore to visualize it we will look at two dimensional cross sections since the ansatz has two independent amplitudes. These will be the planes formed by tuples $(S^{(1)}(s_0), S^{(2)}(s_0))$ with $s_0 \in (0, 4)$ being fixed and chosen like this for the tuples to be real. We name this cross-section as $\mathcal{S}(s_0)$ and we will see how these vary with s_0. It would be interesting to see in future works if other cross-sections of the full spectrahedron would yield new information. Note that it does not mean that integrable models should only lie at the boundary $\mathcal{S}(s_0)$, indeed they can also lie at their interior as we will see later.

Now it is needed a numerical algorithm to attack these problems. We took two routes. First we considered the non-linear unitarity relation (1.1) and did the same approach as in refs. [3,4]. These problems are simple enough to be solved using the function `FindMaximum` in `Mathematica`. However it took quite a lot of computational time even for low precision numerics. Since we could cast the S-matrix bootstrap as SDP problems we used `SDPB` to get a higher precision and efficiency, because it was developed to attack exactly this kind of problem [43]. Then we compared the low precision numerics of `Mathematica` with the ones from `SDPB`, and they matched nicely. For completeness let's end the section by briefly describing `SDPB`. Basically it is an open-source high efficiency SDP solver with arbitrary numerical precision enabled with parallel computing. It uses the primal-dual interior point method for SDP specialized for polynomial matrix programming, which is the kind of problem we are dealing with. The source code is written in `C++` and uses several libraries, like for example OpenMP for parallelization, GNU multiprecision

[3]Possible softwares for this purpose could be `cdd` and `cdd+` developed by Komei Fukuda and available in `https://www.inf.ethz.ch/personal/fukudak/cdd_home/index.html`.

library and MPACK for arbitrary precision machine and linear algebra, respectively. Its high efficiency allowed the most precise computation of the scaling dimensions of the 3D Ising field theory [43].

2.2 $\mathcal{N} = 1$ Supersymmetric Sine-Gordon S-matrix

In the boundary of the spectrahedron we have models that saturate unitarity, then integrable theories with the considered spectrum will lie there. Now we will briefly describe one integrable theory that will be ubiquitous in the discussion of the numerics: $\mathcal{N} = 1$ supersymmetric sine-Gordon (SSG). To derive the S-matrix for the lightest particles of this model we will start with the ansatz (1.7) and apply Yang-Baxter directly to it to obtain a S-matrix without bound states. Then we justify why SSG has poles and how to correctly introduce them by considering some physical aspects of the theory. Then we end the section by considering the analytic continuation of SSG, the so called $\mathcal{N} = 1$ supersymmetric sinh-Gordon theory (SShG) that, spoiler alert, also appears in the numerics.

The SSG model is the simplest supersymmetric extension of the sine-Gordon (SG) lagrangian, where the latter SG model is

$$\mathcal{L} = \frac{(\partial\phi)^2}{2} + \frac{m^2}{\beta^2}\cos(\beta\phi),$$

with ϕ being a real scalar, m the mass of the scalar field and β the coupling constant. As is well known this model is integrable which has been worked out in numerous papers, the most prominent one being the Alexander B. Zamolodchikov and Alexey B. Zamolodchikov classical work [42]. Now the $\mathcal{N} = 1$ supersymmetric sine-Gordon model introduces a Majorana field ψ which together with ϕ builds a chiral supermultiplet of mass m. As states these are the particles $|\phi(\theta)\rangle$ and $|\psi(\theta)\rangle$ being the fermion and boson with rapidity θ, respectively. The $\mathcal{N} = 1$ SSG lagrangian is

$$\mathcal{L} = \frac{(\partial\phi)^2}{2} + \frac{i}{2}\bar{\psi}\gamma^\mu\partial_\mu\psi + \frac{m^2}{4\beta^2}\cos(\beta\phi)^2 - \frac{m}{2\beta^2}\cos(\beta\phi)\bar{\psi}\psi,$$

where γ^μ are the usual Dirac matrices in 1+1 dimensions. This model possess $\mathcal{N} = 1$ supersymmetry on-shell and it is integrable since it can be seen as the Toda field theory based on $\mathrm{C}^{(2)}(2)$ twisted super affine Lie algebra and possess an infinite set of conserved charges [37, 44, 45].

Now let's find the amplitudes of the lightest particles in the model using the Yang-Baxter equations. Consider the factorization relation shown in Figure 2.3 for the amplitude $\phi(\theta_1)\phi(\theta_2)\phi(\theta_3) \to \phi(\theta_1)\psi(\theta_2)\psi(\theta_3)$, then we get:

$$\begin{aligned} S^{\phi\phi}_{\phi\phi}(\theta_{12})S^{\phi\phi}_{\phi\phi}(\theta_{13})S^{\psi\psi}_{\phi\phi}(\theta_{23}) - S^{\psi\psi}_{\phi\phi}(\theta_{12})S^{\psi\phi}_{\phi\psi}(\theta_{13})S^{\psi\psi}_{\psi\psi}(\theta_{23}) = \\ S^{\phi\phi}_{\phi\phi}(\theta_{23})S^{\psi\psi}_{\phi\phi}(\theta_{13})S^{\psi\phi}_{\phi\psi}(\theta_{12}) + S^{\psi\psi}_{\phi\phi}(\theta_{23})S^{\phi\psi}_{\phi\psi}(\theta_{13})S^{\phi\psi}_{\phi\psi}(\theta_{12}). \end{aligned} \tag{2.5}$$

Considering $S^{(2)}(\theta) \neq 0$, it's possible to define

$$\theta_{12} = \theta,\ \theta_{23} = \theta' \text{ and } \frac{S^{(1)}(\theta)}{S^{(2)}(\theta)} = 1 + f(\theta),$$

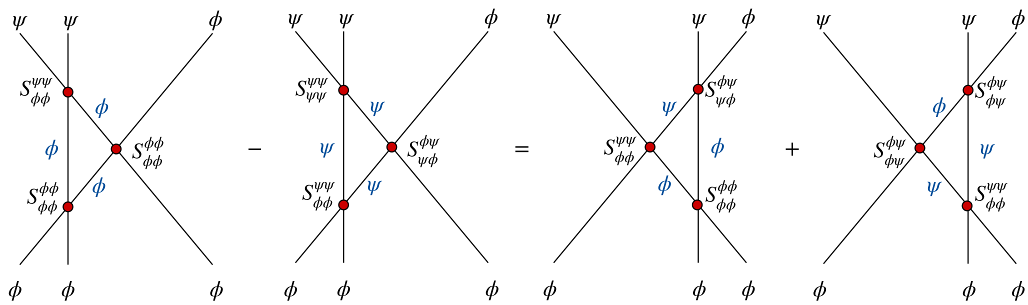


Figure 2.3: The Yang-Baxter equation for SSG that yields expression (2.5). We are using the same terminology of Figure 1.6. The minus sign in the second term is due to the fermion loop on the diagram.

where $f(\theta)$ is some arbitrary function. With these conventions and the supersymmetric S-matrix expressed in (1.7), equation (2.5) can be written as

$$f(\theta')\sinh\left(\frac{\theta'}{2}\right)\left(f(\theta+\theta')+f(\theta)\right)=f(\theta)f(\theta+\theta')\sinh\left(\theta+\frac{\theta'}{2}\right). \tag{2.6}$$

It can be seen that the solution of expression (2.6) is

$$f\left(\theta\right)=2i\alpha\ \mathrm{csch}(\theta),$$

with in principle $\alpha\in\mathbb{C}$, however real analyticity implies $\alpha\in\mathbb{R}$. Therefore the full S-matrix can be written as

$$\hat{S}(\theta)=S^{(2)}(\theta)\begin{pmatrix}1+2i\alpha\ \mathrm{csch}(\theta) & 0 & 0 & \alpha\ \mathrm{sech}\left(\theta/2\right)\\ 0 & 1 & i\alpha\ \mathrm{csch}\left(\theta/2\right) & 0\\ 0 & i\alpha\ \mathrm{csch}\left(\theta/2\right) & 1 & 0\\ \alpha\ \mathrm{sech}\left(\theta/2\right) & 0 & 0 & -1+2i\alpha\,\mathrm{csch}(\theta)\end{pmatrix}. \tag{2.7}$$

Note that this solution has a real free parameter α and no poles between the thresholds, only on the branch points. Therefore a physical pole needs to be inserted by hand if the model has a bound state and we will do this later. Also this solution has explicit C, P, T invariance, real analyticity, unit crossing phase and satisfies all others factorization equations as can be checked.

To find the expression for $S^{(2)}(\theta)$ we use saturation of unitarity and crossing symmetry. Therefore it must satisfy

$$S^{(2)}(\theta)=S^{(2)}(i\pi-\theta)\ \ \text{and}\ \ S^{(2)}(\theta)S^{(2)}(-\theta)=\frac{\sinh(\theta/2)^2}{\sinh(\theta/2)^2+\alpha^2}, \tag{2.8}$$

where the unitarity relation is valid for $\theta\geq 0$. There are two ways of writing the solution for these equations: an infinite product and an integral formula. For the first one we start by simply considering the guess

$$S^{(2)}(\theta)=h(\theta),\ \text{where}\ h(\theta)=\frac{\sinh(\theta/2)}{\sinh(\theta/2)+i\alpha}.$$

This satisfies the unitarity relation and is real analytic, but fails crossing then we fix it by making

$$S^{(2)}(\theta)=h(\theta)h(i\pi-\theta),$$

which now is crossing symmetric, however it fails unitarity thus we fix it by doing

$$S^{(2)}(\theta) = \frac{h(\theta)h(i\pi-\theta)}{h(i\pi+\theta)}.$$

But this now misses crossing symmetry, consequently we fix it again and in the end it is recursively obtained

$$S^{(2)}(\theta) = h(\theta)\prod_{n=1}^{\infty}\left\{\frac{h(ni\pi+(-1)^n\theta)}{h(ni\pi-(-1)^n\theta)}\right\}.$$

Which is a solution of the system (2.8) by construction, where α is a free constant, thus we can start with $+\alpha$ or $-\alpha$ for the initial guess. However this product can be written in terms of the gamma function, which is more common in the old literature as seen in refs. [24,36,37,42]. Indeed, consider the Euler's reflection formula for the gamma function:

$$\sinh(x) = \frac{\pi}{i}\frac{1}{\Gamma\left(ix/\pi\right)\Gamma\left(1-ix/\pi\right)} \text{ for } x \neq in\pi, \text{ with } n \in \mathbb{Z}.$$

Using it we get the usual product of gamma functions

$$S^{(2)}(\theta) = \frac{1}{1-\frac{\alpha}{\pi}\,\Gamma\left(\frac{i\theta}{2\pi}\right)\Gamma\left(1-\frac{i\theta}{2\pi}\right)}\prod_{n=1}^{\infty}\left\{\frac{1-\frac{\alpha}{\pi}\,\Gamma\left(-\frac{\pi n+i(-1)^n\theta}{2\pi}\right)\Gamma\left(1+\frac{n\pi+i(-1)^n\theta}{2\pi}\right)}{1-\frac{\alpha}{\pi}\,\Gamma\left(\frac{i(-1)^n\theta-n\pi}{2\pi}\right)\Gamma\left(1+\frac{n\pi-i(-1)^n\theta}{2\pi}\right)}\right\}.$$

Since we have an infinite product of gamma functions this formula is not viable for numerical evaluation.

Now let's derive the integral solution which is better for computations. Consider a positive function $p(\theta)$ for $\theta \in \mathbb{R}$ and some crossing symmetric function $g(\theta)$ such that

$$g(\theta)g(-\theta) = p(\theta).$$

In refs. [4, 38] is given an integral solution for g in terms of p. The expression is

$$g(\theta) = \exp\left(\int_{-\infty}^{\infty}\frac{dx}{2\pi i}\frac{\log\left(p(x)\right)}{\sinh(x-\theta^+)}\right),$$

where $\theta^+ = \theta + i0$. This shift is important since when $\theta \in \mathbb{R}$ it allows us to compute the integral using the Cauchy's principal value formula. Indeed we have

$$g(\theta) = \exp\left(\mathrm{PV}\left[\int_{-\infty}^{\infty}\frac{dx}{2\pi i}\frac{\log\left(p(x)\right)}{\sinh(x-\theta)}\right] + \frac{\log(p(\theta))}{2}\right), \text{ for } \theta \in \mathbb{R}.$$

Where PV denotes the Cauchy's principal value. For $\theta \notin \mathbb{R}$ the integral is straightforward since there are no poles at the real axis and it can be computed numerically. There is however a small caveat pointed out in ref. [38]. The integral exists only if p has no zeros on the real axis, otherwise it introduces an essential singularity in the integrand due to the logarithm. The trick used to fix this minor issue was to redefine g in a way consistent with crossing that removes this singularity. Since for $S^{(2)}(\theta)$ the respective function $p(\theta)$ has a zero we consider

$$g(\theta) = \frac{iS^2(\theta)}{\sinh(\theta)}, \text{ thus } p(\theta) = \frac{\sinh(\theta/2)^2}{\sinh(\theta)^2(\sinh(\theta/2)^2+\alpha^2)},$$

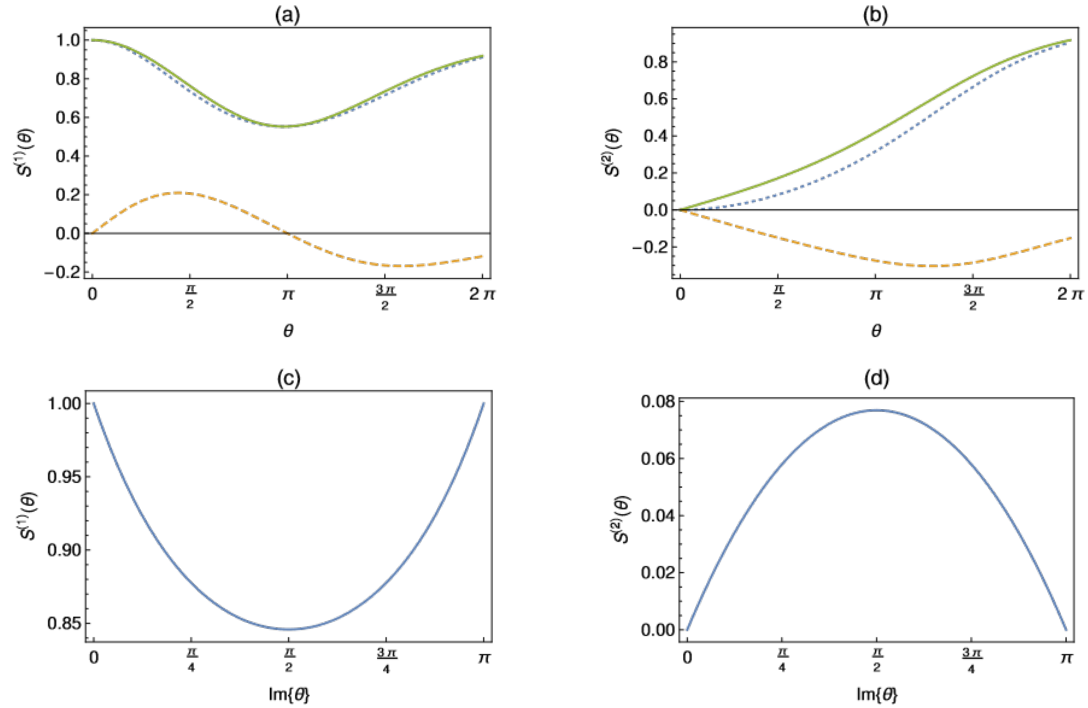


Figure 2.4: Figures (a) and (b) are the plots of the amplitudes $S^{(1)}(\theta)$ and $S^{(2)}(\theta)$ for $\theta \in \mathbb{R}$, respectively. The dotted, dashed and continuous line on (a) and (b) are the real part, imaginary part and norm of the amplitude, respectively. Now Figures (c) and (d) corresponds the same respectively plots, but with θ purely imaginary and in the physical strip. From them it is clear that the Bose-Fermi S-matrix has no poles. All the plots here were done with $\alpha = 5$ and at high energies the S-matrix goes to a constant, thus no subtractions are needed in the dispersion relations.

which maintains crossing symmetry and removes the undesired zero. We can choose i instead of $-i$ for $g(\theta)$, however the $\pm i$ must be there to ensure real analyticity. Consequently the integral solution is

$$S^{(2)}(\theta) = -i\sinh(\theta)\exp\left(\int\limits_{-\infty}^{\infty}\frac{dx}{2\pi i}\frac{1}{\sinh(x-\theta^{+})}\log\left(\frac{\sinh(x/2)^2}{\sinh(x)^2(\sinh(x/2)^2+\alpha^2)}\right)\right). \tag{2.9}$$

Then equations (2.7) and (2.9) yield us the minimal solution[4] for the Yang-Baxter equations and since it commutes with supersymmetry this is the aforementioned Bose-Fermi S-matrix. In Figure 2.4 we have examples of the S-matrix described by these equations for a given value of α. Since this S-matrix has a free parameter and is integrable we expect it to lie at the boundary of the spectrahedron and at some edge of it, indeed it plays even a more fundamental role in $\mathcal{S}$ as will be detailed in the next section. Now we will detail the analysis of the S-matrix without bound states for general α and then later we consider the S-matrix of SSG.

[4]This solution still have an ambiguity where we can multiply $S^{(2)}(\theta)$ by any function of θ that is crossing symmetric and saturate unitarity, that is why it is called minimal. However as discussed in ref. [4] the class of functions we can add either introduces new poles, zeros or resonances (like the CDD factors) or yield essential singularities at infinity (like gravitational dreesing factors).

For generic α there is not anything very distinct from the behavior shown in Figure 2.4. However we are interested in two regimes: $|\alpha| \to \infty$ and $\alpha \to 0$. The latter is the easiest to analyze, from it follows

$$S^{(1)}(\theta) = S^{(2)}(\theta) = 1,$$

therefore for $\alpha = 0$ we get the free theory S-matrix. Despite its simplicity on this point the S-matrix is not differentiable on $\alpha = 0$. Indeed consider the variation on α

$$\frac{\partial S^{(2)}(\theta)}{\partial \alpha} = -\left(\int_{-\infty}^{\infty} \frac{dx}{i} \frac{\alpha}{\sinh(x-\theta^+)(\sinh(x/2)^2 + \alpha^2)} \right) S^{(2)}(\theta). \tag{2.10}$$

For the limit $\alpha \to 0$ we can use the Lorentzian delta function representation, thus

$$\lim_{\alpha\to 0^\pm} \left\{ \frac{\partial S^{(2)}(\theta)}{\partial \alpha} \right\} = \pm \frac{2}{i \sinh(\theta)} \text{ then } \lim_{\alpha\to 0^+} \left\{ \frac{\partial S^{(2)}(\theta)}{\partial \alpha} \right\} - \lim_{\alpha\to 0^-} \left\{ \frac{\partial S^{(2)}(\theta)}{\partial \alpha} \right\} = \frac{4}{i \sinh(\theta)}.$$

Thus in $\alpha = 0$ the derivative is discontinuous, which as we will see in the discussion of the numerics later, implies that a cusp exist on the $\mathcal{S}(s_0)$.

Now the $|\alpha| \to \infty$ regime is more delicate to analyze. It can be checked numerically that the amplitude $S^{(2)}(\theta)$ goes as $1/|\alpha|$, then it vanishes at infinity and $S^{(1)}(\theta)$ goes to some non-zero function due to the term α at the numerator in equation (2.7). Therefore a nice way to derive the expression for $\alpha \to \pm\infty$ is to start with equation (1.8) and set $S^{(2)}(\theta) = 0$, thus we get

$$\hat{S}(\theta) = S^{(1)}(\theta) \begin{pmatrix} 1 & 0 & 0 & -i\sinh(\theta/2) \\ 0 & 0 & \cosh(\theta/2) & 0 \\ 0 & \cosh(\theta/2) & 0 & 0 \\ -i\sinh(\theta/2) & 0 & 0 & 1 \end{pmatrix}. \tag{2.11}$$

Consequently the amplitude $S^{(1)}(\theta)$ must satisfy

$$S^{(1)}(\theta) = S^{(1)}(i\pi - \theta) \quad \text{and} \quad S^{(1)}(\theta) S^{(1)}(-\theta) = \operatorname{sech}(\theta/2)^2.$$

Then we can write the full solution for the S-matrix with $\alpha \to \pm\infty$ as

$$S^{(1)}(\theta) = \operatorname{sgn}(\alpha) \exp\left(\int_{-\infty}^{\infty} \frac{dx}{2\pi i} \frac{\log\left(\operatorname{sech}(x/2)^2\right)}{\sinh(x-\theta^+)} \right), \tag{2.12}$$

where $\operatorname{sgn}(\alpha)$ gives the sign of α for the limit being taken. Equations (2.11) and (2.12) indeed match the result for asymptotic α, which can be checked numerically as shown in Figure 2.5. Note that this S-matrix satisfies Yang-Baxter automatically since it is a limit of the Bose-Fermi result.

We said that the S-matrix of SSG must have a pole, let's see why and how to introduce it. As aforementioned in Section 1.3, it is expected that the S-matrix of a supersymmetric theory factorizes as a tensor product of a term that commutes with supersymmetry and another one that is a non-supersymmetric S-matrix. Indeed as described in ref. [37] the SSG S-matrix possess the following factorization:

$$S_{\text{SSG}}(\theta) = S^{(2)}_{\text{RSG}}(\theta) \otimes S_{\text{SG}}(\theta), \tag{2.13}$$

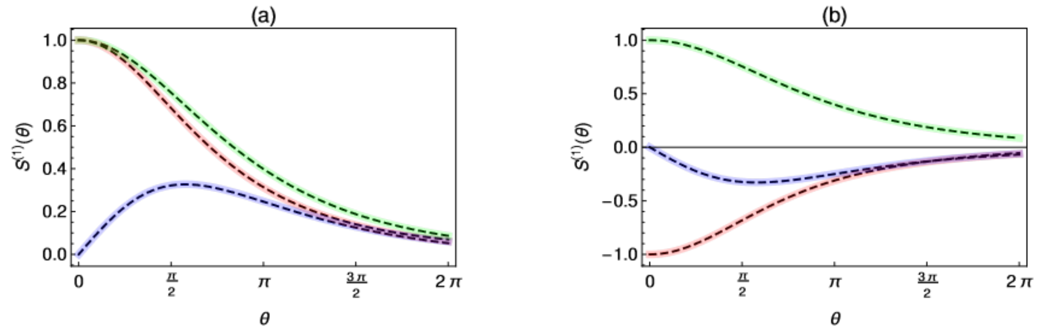


Figure 2.5: In these plots we have the behavior of $S^{(1)}(\theta)$ for $\alpha \to +\infty$ and $\alpha \to -\infty$ in (a) and (b), respectively. The red, blue and green solid lines are the real part, imaginary part and norm of the expression (2.12), respectively. The black dashed lines in (a) are the same curves now computed using the Bose-Fermi S-matrix with $\alpha = 10^3$ and in plot (b) these are BF S-matrix with opposite sign and $\alpha = -10^3$. This confirms the validity of equation (2.12).

where $S^{(2)}_{\text{RSG}}(\theta)$ and $S_{\text{SG}}(\theta)$ are the S-matrices of restricted sine-Gordon (RSG) and sine-Gordon models, respectively. Before we describe what the former is, a quick dive in quantum affine algebras, is needed.

Let's describe the quantum deformation of the $\mathfrak{sl}(2,\mathbb{R})$ algebra. The undeformed algebra is

$$[J_+, J_-] = h \text{ and } [h, J_\pm] = 2J_\pm,$$

where $\{J_-, J_+, h\}$ are the generators. By definition this is a Lie algebra with the usual coproduct map. We can deform it using a parameter $q \in \mathbb{C}$ in such a way that preserves some properties like associativity. This deformation is denoted as $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$ and it is known as a quantum group, more specifically a quantum affine algebra [46]. This deformed algebra is

$$[J_+, J_-] = \frac{q^h - q^{-h}}{q - q^{-1}} \text{ and } [h, J_\pm] = 2J_\pm, \text{ where } q^h = \exp(h \log q)\,.$$

In the limit $q \to 1$ we recover the $\mathfrak{sl}(2,\mathbb{R})$ algebra. Clearly $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$ is not a Lie algebra since the Lie bracket is non-linear. Also the coproduct map is unusual and expressed as

$$\Delta_2(h) = \mathbb{1} \otimes h + h \otimes \mathbb{1} \text{ and } \Delta_2(J_\pm) = q^{h/2} \otimes J_\pm + J_\pm \otimes q^{-h/2}.$$

This preserves the algebra and as expected in the limit $q \to 1$ we recover the usual coproduct map of Lie algebras.

The link with the sine-Gordon theory is the following. It is claimed that SG has $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$ as a symmetry group and then its states must be in some representation of it [47]. The representation theory of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$ can be worked out just as the one of the usual Lie algebra $\mathfrak{sl}(2,\mathbb{R})$, however note that the normalization of the states is distinct. Indeed, let $|m, j\rangle$ be an eigenstate of h with eigenvalue m from the spin-j multiplet[5], we have for example that

$$\langle j-1, j | j-1, j\rangle = \frac{q^{j-1} - q^{1-j}}{q - q^{-1}},$$

[5]Due to the factor of 2 in the algebra the spin is always integer, as an example the spin-1/2 representation have spins $\{-1, 1\}$.

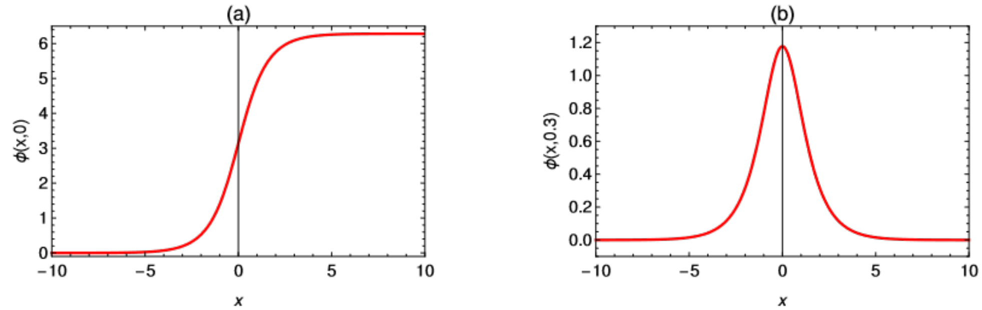


Figure 2.6: Classical solutions of the equations of motion for the sine-Gordon theory. Plot (a) is the kink solution for velocity $v = 0.1$ and at time $t = 0$ and plot (b) is the breather solution for velocity $v = 0.1$ and at time $t = 0.3$. Note the kink solution has non-zero topological charge as expected due to the asymptotic conditions, unlike the breather solution that is a even function thus its topological charge vanishes.

where $|j, j\rangle$ is normalized to unit. Then the Clebsch–Gordan coefficients have corrections due to q and these are named q-CG. The interesting fact is that when q is root of unity the q-CG coefficients are ill defined unless the Hilbert space is truncated, *id est*, there is a maximum spin allowed. These values of q imposes serious constraints on the coupling of SSG. Indeed the relation between q and the coupling of SG is

$$\gamma(\beta) = \frac{4\beta^2}{1-\beta^2/4\pi}, \text{ then } q = \exp\left(-\frac{8\pi^2 i}{\gamma(\beta)}\right).$$

At the values

$$\gamma(\beta) = 8\pi(L+2) \text{ where } q = \exp\left(-\frac{8\pi i}{L+2}\right),$$

we have the truncation of the Hilbert space and the theory at this value of coupling is named restricted sine-Gordon at level $L \in \mathbb{Z}$. The maximum allowed spin is related to L by $j_{\max} = (L+2)/2 - 1$. So for $L = 2$ we have only the states of spin $\{0, 1/2, 1\}$.

The spins of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$ have an interesting physical interpretation. In the SG model there is a topological conserved current given by

$$J^{\mu\nu}(x) = \frac{\epsilon^{\mu\nu}\partial_\nu\phi(x)}{2\pi} \text{ whose charge is } Q_0 = \frac{\phi(+\infty,t)-\phi(-\infty,t)}{2\pi}.$$

Being a conserved charge it must be time (t) independent. It can be seen that with this normalization we have $Q_0 \in \mathbb{Z}$ for the classical solutions. This integer counts between how many vacua the solution interpolates and it is the classical realization of the spins of the aforementioned quantum affine algebra. There are two types of solutions to the classical equations of motion that are distinguishable by the topological charge: the kink/antikink and breathers. The former have $Q_0 = 1$ and $Q_0 = -1$ and fits in the spin 1/2-multiplet of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$, however the latter have $Q_0 = 0$ and are in the singlet representation of the quantum group. Higher spin states are interpreted as multiparticle states of kink/antikink. Examples of the classical solutions can be seen in Figure 2.6 and make evident these charges.

We will not describe the full kink/antikink S-matrix of sine-Gordon, however suffices to say that it has the following set of poles (θ_p) and zeros (θ_z) (without considering their crossing pairs):

$$\theta_p = i\left(\frac{m\gamma}{8} + 2n\pi\right) \text{ and } \theta_z = i\left(\frac{m\gamma}{8} + (2n+1)\pi\right), \text{ where } m, n \in \mathbb{Z} \text{ and } n \geq 1.$$

These poles correspond to bound states of kink/antikink and are the previously mentioned breathers. By definition they carry no topological charge and are parity eigenstates then

$$\mathrm{P}|B_n(\theta)\rangle = (-1)^n|B_n(\theta)\rangle, \; Q_0|B_n(\theta)\rangle = 0, \text{ with masses } m_n = 2M\sin\left(\frac{n\gamma}{16}\right),$$

where M is the kink/antikink mass and $|B_n\rangle$ the breather state. Note that for the restricted sine-Gordon the poles and zeros cancel each other. Then the S-matrix of RSG does not have breathers which is an evidence that $S^{(2)}_{\mathrm{RSG}}(\theta)$ must match the previously obtained Bose-Fermi S-matrix. Using the fusion identities one could derive the kink/antikink-breather and breather-breather S-matrices as done in ref. [42]. The conclusion is that the poles of the former are kink/antikink and of the latter are breather, which follows from charge conservation. The supersymmetry generalization is straightforward, basically the number of states are doubled to form supermultiplets and we consider the factorization in expression (2.13) to write the full S-matrix of SSG, consequently it must have poles introduced by $S_{\mathrm{SG}}(\theta)$. Note that the scattering states are tensor products of RSG at level two and SG as shown in ref. [37].

Now we can finally introduce the S-matrix of SSG. In refs. [37, 42] it is given that for the lightest breather-breather scattering $S^{(2)}_{\mathrm{RSG}}$ is the Bose-Fermi S-matrix described by equations (2.7) and (2.9). Also S_{SG} is the sine-Gordon lightest breather-breather S-matrix, with pole at $s = m_{bs}^2$ given by:

$$[m_{bs}](\theta) = \frac{\sinh(\theta) + m_{bs}\sqrt{(4 - m_{bs}^2)/4}}{\sinh(\theta) - m_{bs}\sqrt{(4 - m_{bs}^2)/4}}. \tag{2.14}$$

Since the pole is given by a simple CDD factor it is by definition crossing symmetric, saturates unitarity and multiplies all channels making supersymmetry manifest. The parameter α is free in the Bose-Fermi S-matrix, however in the presence of the pole it must be fixed. Indeed, the amplitudes $S^{\phi\psi}_{\phi\psi}$ and $S^{\psi\phi}_{\phi\psi}$ factorize in the same three point amplitudes and therefore must have equal residue, which implies that

$$\alpha = \sqrt{\frac{4 - m_{bs}^2}{4}}. \tag{2.15}$$

Thus joining equations (2.7), (2.9), (2.14) and (2.15) we get the supersymmetric sine-Gordon S-matrix. It can be expressed as

$$\hat{S}(\theta) = Y(\theta)[m_{bs}](\theta)\begin{pmatrix} 1 + \frac{i\sqrt{4-m_{bs}^2}}{\sinh(\theta)} & 0 & 0 & \frac{1}{\cosh(\theta/2)}\sqrt{\frac{4-m_{bs}^2}{4}} \\ 0 & 1 & \frac{i}{\sinh(\theta/2)}\sqrt{\frac{4-m_{bs}^2}{4}} & 0 \\ 0 & \frac{i}{\sinh(\theta/2)}\sqrt{\frac{4-m_{bs}^2}{4}} & 1 & 0 \\ \frac{1}{\cosh(\theta/2)}\sqrt{\frac{4-m_{bs}^2}{4}} & 0 & 0 & -1 + \frac{i\sqrt{4-m_{bs}^2}}{\sinh(\theta)} \end{pmatrix}, \tag{2.16}$$

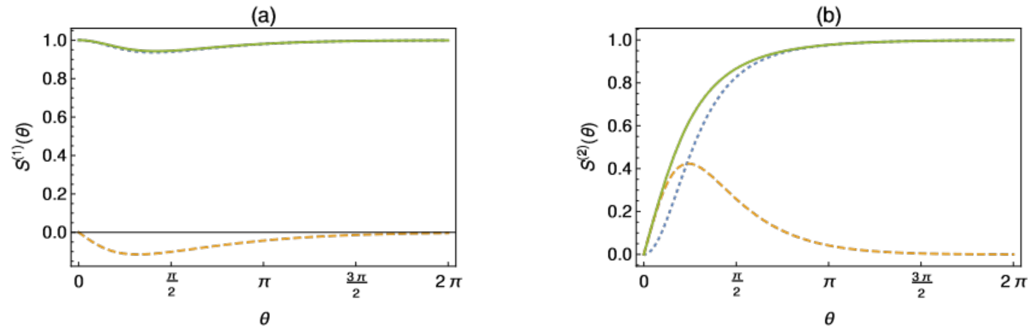


Figure 2.7: Figures (a) and (b) are the plots of the amplitudes $S^{(1)}(\theta)$ and $S^{(2)}(\theta)$ for SSG S-matrix for $m_{bs} = \sqrt{3}$. The blue, orange and green line on (a) and (b) are the real part, imaginary part and norm of the amplitude, respectively.

which explicitly have all the desired properties and where

$$Y(\theta) = -i\sinh(\theta)\exp\left(\int_{-\infty}^{\infty}\frac{dx}{2\pi i}\frac{1}{\sinh(x-\theta^{+})}\log\left(\frac{4\sinh(x/2)^2}{\sinh(x)^2(4\sinh(x/2)^2+4-m_{bs}^2)}\right)\right).$$

Note that it has no free parameters except the mass of the bound state which is fixed in the optimization, thus this model should lie on a vertex of the spectrahedron. It can also be checked that the SSG S-matrix goes to a constant at infinity, more specifically that $S^{(1)}_{\infty} = S^{(2)}_{\infty} = 1$, thus no subtractions are needed in the dispersion relations. The behavior of some S-matrix elements are shown in Figure 2.7.

In the SDP problem with bound states we maximize a linear functional of the couplings. We define the quantum couplings by the residue of the SSG S-matrix at the bound state pole $s = m_{bs}^2$. The computation is direct and it yields

$$\operatorname{Res}\left(S, m_{bs}^2\right) = \operatorname{sgn}(m_{bs}^2-2)\frac{2\,(4-m_{bs}^2)}{1-2/m_{bs}^2}Y(2i\cos^{-1}(m_{bs}/2))\begin{pmatrix} 1+\frac{2}{m_{bs}} & 0 & 0 & -\sqrt{\frac{4-m_{bs}^2}{m_{bs}^2}} \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ -\sqrt{\frac{4-m_{bs}^2}{m_{bs}^2}} & 0 & 0 & -1+\frac{2}{m_{bs}} \end{pmatrix}, \tag{2.17}$$

where $\operatorname{sgn}(m_{bs}^2 - 2)$ is here to fix the sign of the residue and to obtain the associated coupling in the T matrix we simply divide by $J(m_{bs})$.

Another theory that could appear in the numerics is the analytic continuation of SSG. Indeed consider $\beta = i\hat{\beta}$ with $\hat{\beta} \in \mathbb{R}$, then now we have only a single vacuum and thus all bound states disappear. The obtained model is the supersymmetric sinh-Gordon (SShG). It can be seen that in terms of the S-matrix this implies that the mass becomes negative and $-m_{bs} \in [0, 2]$, which turns the CDD pole into a CDD zero such that the bound state is now missing as expected. So the S-matrix of SShG is the same as SSG but with negative mass parameter.

2.3 Optimization Problem I: No Bound States

In this section we will explain the numerical results of the SDP problem without bound states. For the numerical computations we have to fix a set of grid points. To

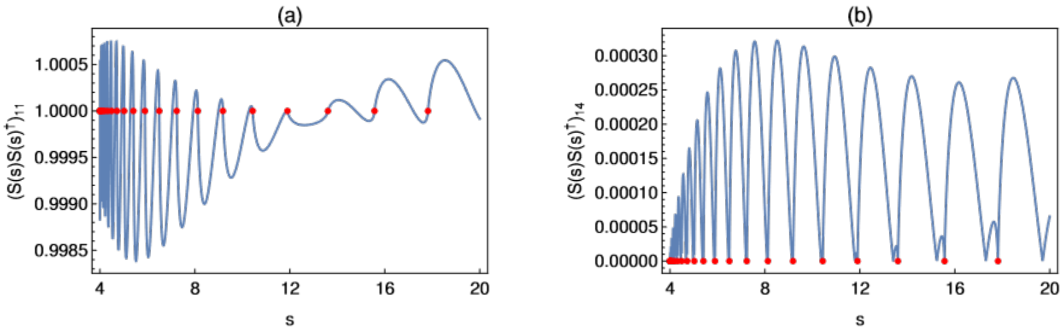


Figure 2.8: Numerical saturation of unitarity for the S-matrix obtained at the optimization in the case without bound states. In (a) we have the plot of $(S(s)S(s)^{\dagger})_{11}$ and in (b) it's shown the element $(S(s)S(s)^{\dagger})_{14}$, therefore they should be unity and zero, respectively. Note that indeed at the grid points (red dots) where unitarity is enforced, `SDPB` forces the former and the latter to be one and zero, however outside these points the S-matrix is free to vary. As seen in the plots the error is of order 10^{-4} which is acceptable and it can be decreased with more computational power.

better modeling the behavior of the S-matrix we chose one with the points concentrated at low energy, since at high energies it goes to a constant. Therefore a good choice is

$$x_n = 4 + \left(\frac{n}{N_p}\right)^4 x_{\max},$$

where $x_{\max}$ is the maximal grid point and N_p is the number of grid points. It can be seen that this concentrates most points around the branch point $x = 4$. In our optimization in `SDPB` we chose $x_{\max} = 40$, $N_p = 30$ and used 350 digits of precision. As described before the S-matrices obtained in the optimization must saturate unitarity, which they do within numerical errors as shown in Figure 2.8.

There are two ways of obtaining a cross section of the spectrahedron. The first was the described before, just maximize all linear functionals of the independent amplitudes at $s = s_0$, where we chose $s_0 = 2$. From this SDP problem we obtained only Bose-Fermi S-matrices, as shown in Figure 2.9. The keen reader noted that these S-matrices have the opposite sign for the real part of the ones exhibited in Figure 2.4. This discrepancy is allowed since they do not have to match exactly, only up to an arbitrary phase. For the fit with the BF S-matrix we use eq. (2.7) and $s = 2$ to fix α.

The second way to obtain the spectrahedron is the following: set an amplitude $S^{(i)}(2)$ at a specific value in the interval $[-1, 1]$ and maximize the other amplitude $S^{(j)}(2)$. Doing this for a chosen grid of points in the aforementioned interval yields the same $\mathcal{S}(2)$ as in the SDP problem described previously. So to build this cross-section of the spectrahedron we can use the latter method if we do not known the set of functionals to cover it, which is a very useful fact in the SDP problem with bound states. In Figure 2.10 we have the plot of $\mathcal{S}(2)$ and there we see that the boundary of this set is entirely described by the Bose-Fermi S-matrix with the SShG lying inside of it. The choice of s_0 in $\mathcal{S}(s_0)$ does not affect much this set, for all the values tested it is described perfectly by the BF S-matrix.

There are two small details concerning the case without bound states that will be useful later: derivatives and Yang-Baxter equations. Previously it was proved for the

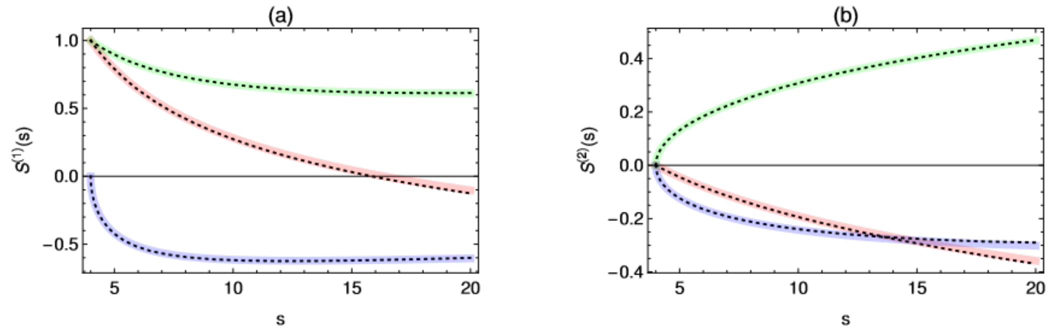


Figure 2.9: In these plots we have the match of the BF S-matrix with $\alpha \approxeq -3.7772737$ and the numerics for $S^{(1)}(s)$ and $S^{(2)}(s)$ in plots (a) and (b), respectively. The red, blue and green solid lines are the real part, imaginary part and norm of the analytical expression for $S^{(1)}(s)$ and $S^{(2)}(s)$, respectively. The black dashed lines are the corresponding numerical data.

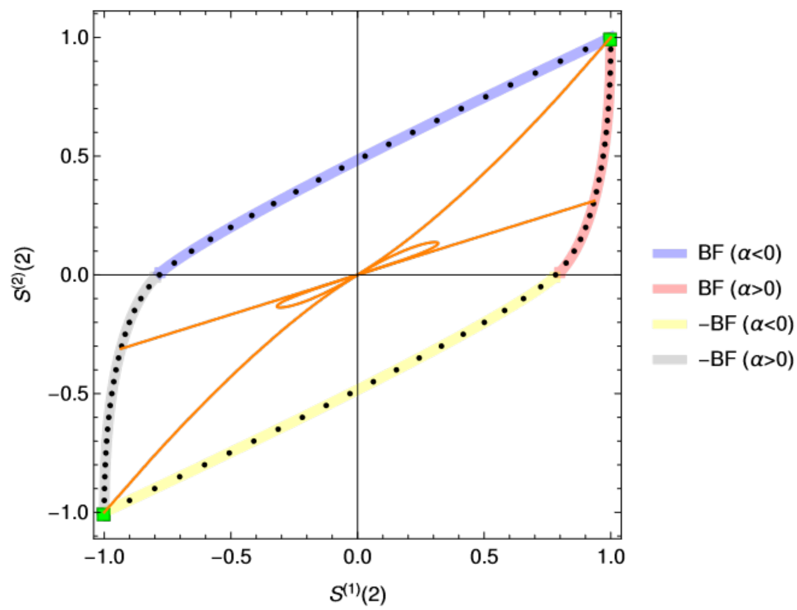


Figure 2.10: Unitarity bounds on the cross-section $\mathcal{S}(2)$, with the interior containing all allowed S-matrices like for example the SShG described by the orange curve. The solid curve is the BF S-matrix and the dots are the numerics. We used 82 points in the discretization of the boundary. Each point in the boundary is identified with $\pm$BF for a fixed value of α. The green squares are the free theory limits, that is $\alpha \to 0$. Note that we have $S^{(2)}(2) \to 0$ at $|\alpha| \to \infty$ as discussed before, also the upper half-plane contains +BS and the lower half-plane $-$BF. The values where SShG and BF S-matrix match are in the limits where the zeros of the former are in the branch points.

Bose-Fermi S-matrix that the derivative of $S^{(2)}(2)$ with respect to α is discontinuous at $\alpha = 0$. Indeed the only discontinuity occur at this point, as shown in Figure 2.11 where it nicely matches the analytical formula. Thus our algorithm to compute the cusps works and we use it in the next section. Now concerning the Yang-Baxter we have that the entire boundary of this cross section of the spectrahedron is described by $\pm$BF S-matrix

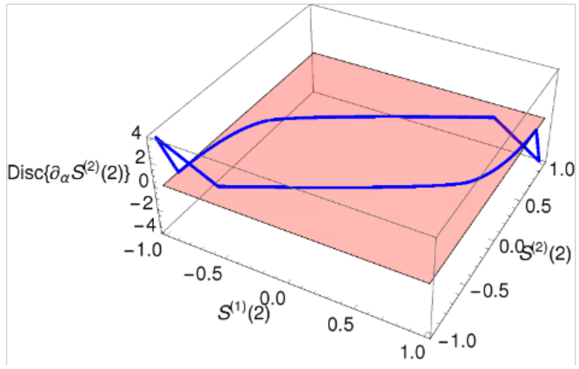


Figure 2.11: Numerical test for the discontinuity of the Bose-Fermi S-matrix derivative using the analytical expression. The blue line is the analytical discontinuity and the red plane marks the value of zero. Note that the jumps occur only at $\alpha = 0$ in this cross-section of the spectrahedron and at the cusps it yields the same values as the analytical formula.

therefore all theories in the boundary are integrable. We could use it to define a numerical function that measures if some theory satisfies the factorization equations or not. There are various measures that one could create, however here is what we did: using a numerical S-matrix you compute all the 32 Yang-Baxter relations, then calculate the absolute value of the left hand side of a Yang-Baxter equation minus its right handed side and finally take the maximal value of all the factorization equations which yields a number μ. Now to find the error ε we considered all the 82 grid points of the boundary of $\mathcal{S}(2)$ that was obtained in this section and took the maximal μ, the value found was $\varepsilon \approxeq 0.0023149629$. If for some S-matrix $\mu \leq \varepsilon$ then we say that it satisfies Yang-Baxter, otherwise it is not integrable. Note that to numerically evaluate the factorization equations we have to choose two rapidities, for convenience it was taken two grid points to do this.

2.4 Optimization Problem II: One Bound State

Now in this section we explore the spectrahedron for the case with bound states. The choice of grid points and precision for the numerical computations are the same as in the last section. Note however that with a single bound state the associated SDP problem has a simple functional: it is just the coupling and it must have a maximal value to not violate the spectrum taken as a input. We found by maximizing the residue that the associated S-matrix is the supersymmetric sine-Gordon as shown in Figure 2.12. We will not show here but the precision of the numerics fall in the same 10^{-4} range as the case without bound states. For a single bound state we denote $m_1 = m$ and $g_1 = g$ for the bound state mass and coupling, respectively.

As said before, to apply the SDP problem computationally we need to fix the mass of the bound state, then given a range of masses we obtain the maximal value of the coupling for each choice of m. The numerical result nicely matches the residue of SSG as shown in Figure 2.13. The divergence in the plot at $m = \sqrt{2}$ is due to the cancellation of poles, that is when the mass is in the crossing symmetric point the s and t poles cancel out and there is no bound on the residue and it can be arbitrarily high.

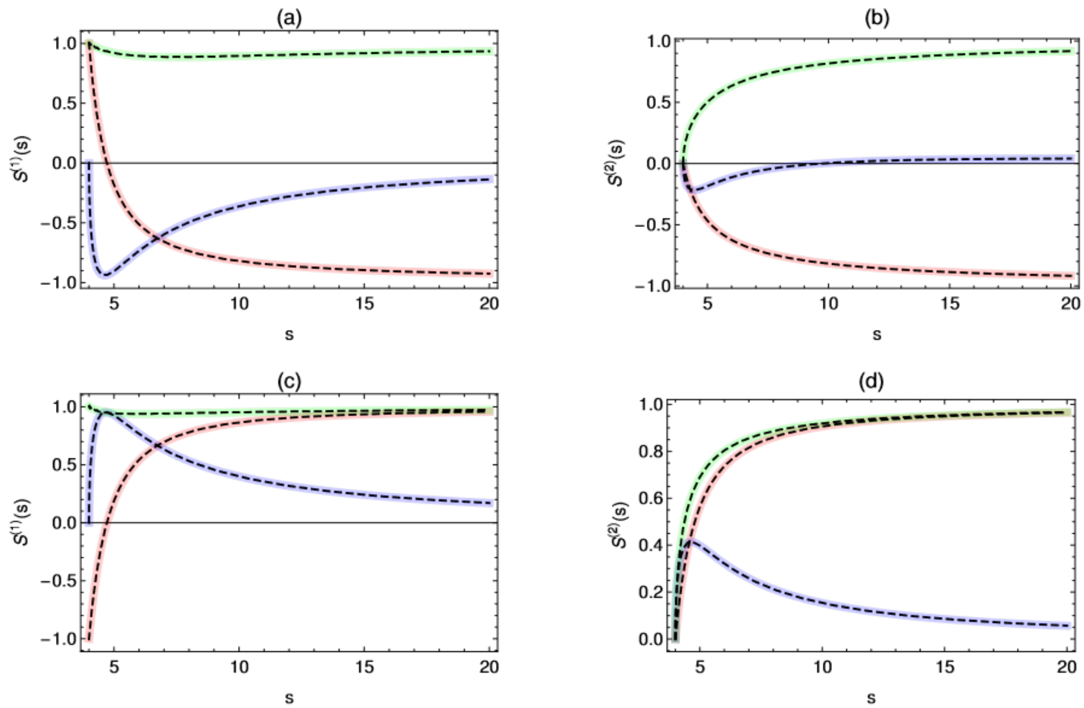


Figure 2.12: Here we have the match of the SSG S-matrix and the numerics for $S^{(1)}(s)$ and $S^{(2)}(s)$. The red, blue and green solid lines are the real part, imaginary part and norm of the analytical expression for $S^{(1)}(s)$ and $S^{(2)}(s)$, respectively. The black dashed lines are the corresponding numerical data. Plots (a) and (b) are for mass $m = 1$, however (c) and (d) correspond to mass $m = 1.7$. Note that for $m < \sqrt{2}$ we have to match -SSG instead of +SSG.

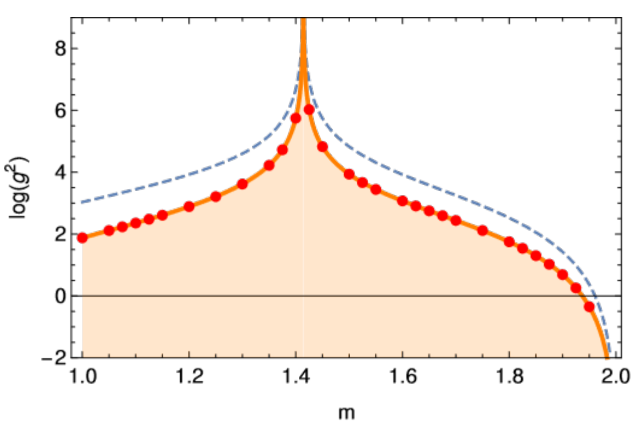
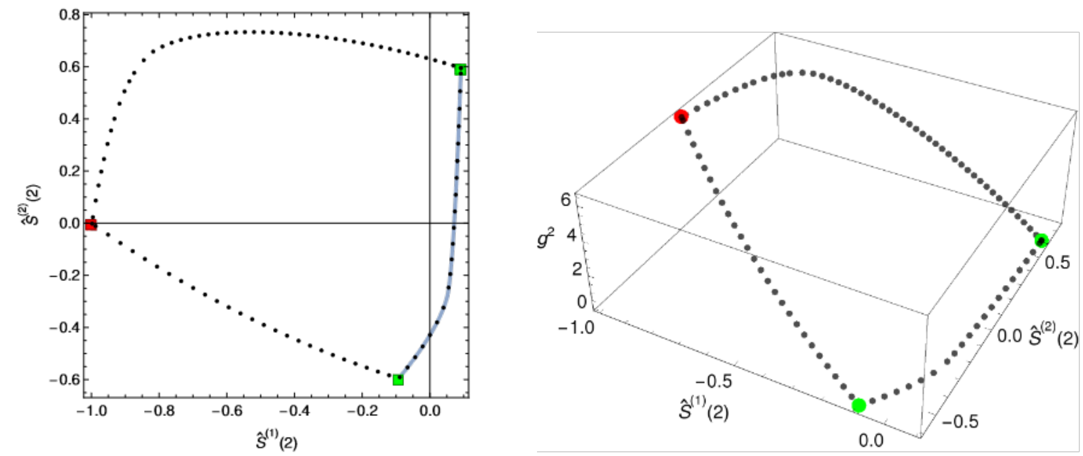


Figure 2.13: Plot of the coupling g^2 for SSG, for the lightest breather-breather scattering of SG, and the numerical results, each one being the orange line, blue dashed line and the red dots, respectively. Note the agreement between the numerics and SSG residue. Also we can see that the SG has bigger residue than SSG, *id est*, the space of supersymmetric QFTs and given spectrum seems smaller than that of non-supersymmetric QFTs, as expected. Another thing to note is that $g^2 \to 0$ as $m \to 2$, *i.e.*, as the bound state mass approach the threshold the coupling enters the perturbative regime, as expected.



(a) (b)

Figure 2.14: In plot (a) we have $\mathcal{S}(2)$, where $\left(\hat{S}^{(1)}(2), \hat{S}^{(2)}(2)\right)$ denote a rotated and rescaled axes such that the point corresponding to SSG, red marker, is on $(-1,0)$. The black dots, blue curve, up and lower green markers correspond to the numerical data, BF S-matrix, plus and minus free theories, respectively. Here we considered $m=1$ and 98 grid points. Since part of $\mathcal{S}(2)$ is perfectly modeled by Bose-Fermi it should have vanishing residue, which is confirmed in (b). There it is clear that SSG have the highest residue and the BF region have vanishing coupling.

Knowing that the coupling maximization yields supersymmetric sine-Gordon we can explore $\mathcal{S}(2)$. The procedure is the same as in the case without bound states, however with very distinct results. In Figure 2.14 (a) we have $\mathcal{S}(2)$ in the presence of a single bound state. Using the derivative test we found three cusps corresponding to the free theories and SSG, thus corroborating the idea that integrable models without free parameters should live at a cusp in the boundary. Also an entire region modeled by the BF S-matrix with vanishing residue appeared. Using the aforementioned measure of integrability we conclude that only the Bose-Fermi region and the point corresponding to SSG satisfy Yang-Baxter equations. Also from Figure 2.14 (b) it is clear that the model with maximal residue is indeed minimal supersymmetric sine-Gordon.

Since the Yang-Baxter test failed for the majority of $\mathcal{S}(2)$ and all models obtained saturate unitarity we have an answer to the conundrum presented in Chapter 1. Here we obtained **non integrable models without particle production**. The question is what kind of deformations of SSG these models are. To analyze this we choose $S^{(2)}(s)$ because there are some interesting dynamics for this amplitude. First note that it has a zero on the branch point $s=4$ and no zeros for physical energies. However as we move from SSG to the plus free theory in plot (a) of Figure 2.14 the zero at the branch point moves in the branch cut and goes to $s\to+\infty$ as seen in Fig. 2.15. We will not show here but if we go to the bottom part of $\mathcal{S}(2)$ and move from minus identity to SSG the values of $S^{(2)}(s)$ at the branch point goes from one to zero as we hit the integrable model.

It would be interesting to see if there is a way of obtaining the analytical form of these models. We could try is to start with S-matrix that saturates unitarity and from it immediately follows that

$$\frac{S^{(1)}(\theta)}{S^{(2)}(\theta)} = 1 + ib(\theta),$$

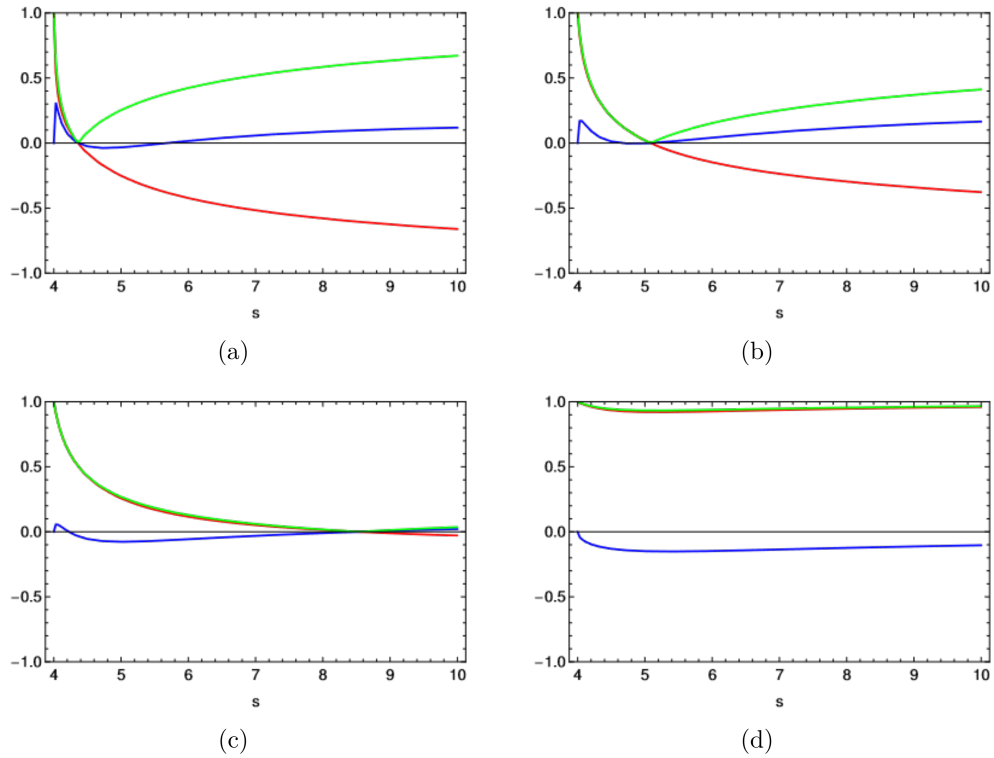


Figure 2.15: Non integrable models that saturate unitarity obtained in the numerics, where we plotted $S^{(2)}(s)$ with the red, blue and green curves being the real part, imaginary part and norm of this amplitude. Note that plot (a) is a S-matrix near the SSG point in $\mathcal{S}(2)$ and as we move away from it in direction of the plus identity in plots (b) and (c) the zero on the branch cut goes to higher and higher energies until the model becomes the identity as seen in plot (d), which is a point near it.

where $b(\theta)$ is a function that must satisfy a series of properties that follow from the independent amplitudes. Let $\lambda \in [0, \pi]$ be the position of the pole in rapidity variables, then:

$$b(i\pi - \theta) = b(\theta), \quad b(\theta)b(-\theta) = \frac{1}{1 + \cosh(\theta/2)^2 b(\theta)}, \quad b(-\theta) = -b(\theta) \;\text{ and }\; ib(i\lambda) \in \mathbb{R}.$$

The first two follow from crossing and unitarity and the remaining one from real analyticity. Also $b(\theta)$ cannot have poles on the physical strip, because $S^{(1)}(s)$ and $S^{(2)}(s)$ poles are in the same position and thus are canceled. Also at high energies $b(\theta)$ should vanish as indicated by the numerics. Note that the unknown function could depend on some continuous parameter such that this model is a homotopic deformation of SSG. Since Yang-Baxter implies $b(\theta) \propto \text{cosech}(\theta)$ the reader would be tempted to guess that $b(\theta)$ is a Jacobi function, more specifically, the ns function that reduces to the cosech(θ) formula at the limit of vanishing modular parameter. However this idea is rather naive since the former function would imply periodic S-matrices for real energies, which is contradicted by the numerics. At the moment an expression for $b(\theta)$ still elusive for us[6].

[6]After the publication of this thesis we found the analytical expression for this S-matrix as it is detailed in ref. [10].

Conclusion

In this work we considered the S-matrix bootstrap of $2 \to 2$ amplitudes of the lightest particles in a supersymmetric theory with a single supermultiplet containing a real scalar boson (ϕ) and a Majorana fermion (ψ) in two dimensions. Also the S-matrix bootstrap was casted as a semidefinite programming problem and we considered various functionals to maximize depending if we consider a model with or without bound states. To summarize we achieved:

- For S-matrices without bound states we obtained, from functional maximization described in Section 2.1, only S-matrices described entirely by the Bose-Fermi term of SSG. It has the physical interpretation of restricted sine-Gordon at level two. Also we encountered SShG in the numerics since, which is expected since this model does not have bound states.

- Considering bound states it was found that the S-matrix with maximum residue is $\mathcal{N} = 1$ supersymmetric sine-Gordon.

- In $\mathcal{S}(2)$ with bound states we found a class of non integrable models without particle production. Also these are deformations of SSG such that zeros move until hit their optimal position in a way that maximizes the residue.

Also in ref. [9] it was observed that the supersymmetric cases discussed here are special models that appear when we consider particles of distinct mass that possess $\mathbb{Z}_2$ symmetry.

Despite these achievements there are two unanswered questions that appeared. Firstly we would like to known the analytical description of the boundary of $\mathcal{S}(2)$ in the case with bound states, *i.e.*, what are the expressions for the deformations of SSG. Knowing these we could analyze exactly the motion of zeros described earlier. Also this would maybe give hints to solve the second problem: prove that SSG is the only model with a single bound state that maximizes the residue as indicated by the numerical results. A possible strategy for this is to consider the Maximum Modulus Principle which worked in the bootstrap of 1+1 dimensional models containing only bosons as it was done in refs. [4, 5]. However it is not clear how it would work here and at the moment we have no answer to this. These problems are important since solving them would give clues why the maximization procedure works.

Note that besides solving these conundrums, future works may be done in very interesting directions. There are basically two ways to go: extra symmetries in 1+1 dimensions or supersymmetry in 3+1 dimensions. The first is immediate and not very interesting. For example we could consider $\mathrm{O}(N)$ symmetry just like in ref. [7] and this would yield us six independent amplitudes, thus making numerics complicated but not impossible, and it is expected that in the boundary of the spectrahedron theories we have supersymmetric non-linear sigma models first discussed in ref. [24]. Also we could include a central charge in the superalgebra (just like in ref. [36]) which yields a topological charge in the spectrum and it is expected to recover the supersymmetric kink/antikink S-matrix. Another direct extension is $\mathcal{N} \geq 2$ supersymmetry, which could yield $\mathcal{N} = 2$ supersymmetric sine-Gordon described in ref. [48] for example.

A much more promising and hard problem to tame would be supersymmetric models in $3 + 1$ dimensions. The first complication is that now the Majorana fermions have two polarizations and we must add a spin structure into the S-matrix. Though applying supersymmetry would be the same as described in this work, we just need to find the action of it on the two particle subspace. However if one could described the spin structure of the S-matrix and then impose crossing symmetry and unitarity in the same fashion as ref. [5] it would be possible to describe the fermionic sector of the Standard Model. Also after controlling supersymmetry we could maybe even impose restrictions on Minimal Supersymmetric Standard Model, which is a much harder direction but it is very exciting. These are exciting directions to go and we hope to dwell in these matters soon.

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