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Nonpolynomial potentials with deformable topological structures

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We construct models of self-interacting scalar fields whose Bogomol'nyi-Prasad-Sommerfeld solutions exhibit kink profiles which can be continuously deformed into two kinks by varying one of the parameters of the self-interacting potential. The effective models are obtained from other models with two interacting scalar fields. The effective models are then applied in a brane-world scenario where we analyze the consequences of the thicker branes in the warped geometry and in the localization of gravity.

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I. INTRODUCTION

Nonlinear systems, particularly those that exhibit topological solutions are very important for modeling many physical systems from condensed matter to high-energy physics. One example that has been receiving considerable attention is that of topological structures in multidimensional warped space-time when one considers scalar fields coupled to gravity. In this context, one of the key points is the possibility of localizing gravitons in thin branes and reproducing effectively the four-dimensional gravity [1]. In (4, 1) dimensions, it is shown that thick branes can also localize the gravity [2]. In fact, the subject of thick branes in the context of brane worlds has received a considerable amount of attention [3]. Some years ago, it was observed that some kinds of models with two interacting scalar fields in a warped geometry can be used to describe the splitting of thick branes due to a first-order phase transition [4]. A few years ago, it was shown that Bloch branes are solutions of a model with two interacting scalar fields which can be naturally incorporated in a supersymmetric theory [5]. Later, that same model was shown to exhibit branes with richer structures [6] than those found in Ref. [5], including critical and degenerate branes. Moreover, contrary to the scenarios presented in [4,5] where the splitting of the branes is controlled by a coupling constant presented in the interaction potential, it was shown [6] that the thickness of the branes is controlled by a parameter, called *degeneracy parameter*, which is not present in the Lagrangian density of the model. Instead, it is one of the constants of integration of the orbit differential equation which relates both fields. Recently, a model with only one scalar field which incorporates thicker branes was proposed by Dutra [7]. It is characterized by a nonpolynomial interaction potential with coupling constants that control the thickness of the brane. This last property, besides the fact of exhibiting a nonpolynomial interacting potential, is shared with the *p model* introduced in Refs. [8,9]. In the later one, the changing of the parameter p in discrete jumps implies into a changing of model, and thicker branes appear only for some values of p; whereas, in the model introduced in [7], the parameter that controls the deformation of the brane is a coupling constant of the model and as such, its variation do not modify the structure of the model.

The purpose of this work is to construct nonlinear models, in classical field theory, with only one scalar field, from models with two interacting scalar fields and that exhibit solutions with *two-kink* profiles. Two-kink solutions yield thicker branes in brane-world scenarios or Bloch branes whose internal structures are somehow incorporated in the parameter that controls the thickness of the brane. The nonpolynomial effective models we construct belong to the same class of models than that proposed by Dutra [7], where the kink solutions can be continuously deformable into two-kinks by varying one of the parameters of the effective potential. Moreover, the same parameter that controls the thickness of the brane can be thought of as depending on the temperature in such a way that a first-order phase transition characterized by the brane splitting can happen. The model with two interacting scalar fields we consider here is the same whose consequences in brane-world scenarios were already studied in Refs. [5,6]. In order to construct the effective models, we apply a general orbit equation relating both fields [10] to eliminate one of the fields in favor of the other. The resulting effective model is applied to the study of the localization of gravity on thick branes in (4, 1)-dimensional warped space-time. In the next section, we introduce the kind of models with two interacting fields we are interested in and, after that, we show how the orbit equation can be used to construct models with only one scalar field. One particular example is studied in detail. In the third section, we apply one of the effective models in a brane-world scenario and analyze the consequences of thick branes over the warp factor and on the localization of gravity. In the conclusions section, we remark on possible applications of the effective models, particularly in the phenomenon of brane splitting.

II. MODELS WITH ONLY ONE SCALAR FIELD CONSTRUCTED FROM MODELS WITH TWO INTERACTING SCALAR FIELDS

The models with two interacting fields we consider here are described by the Lagrangian density

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$$\mathcal{L} = \tfrac{1}{2}\partial^\mu\phi\partial_\mu\phi + \tfrac{1}{2}\partial^\mu\chi\partial_\mu\chi - V(\phi,\chi), \qquad (1)$$

with the potential written as

$$V(\phi,\chi) = \tfrac{1}{2}(W_\phi^2 + W_\chi^2), \qquad (2)$$

where W_ϕ and W_χ are the derivatives of some function $W(\phi,\chi)$, called superpotential, with respect to the fields ϕ and χ, respectively.

It is shown that for potentials written in terms of a superpotential, as in Eq. (2), the static solutions of the first-order equations

$$\frac{d\phi}{dx} = W_\phi, \qquad \frac{d\chi}{dx} = W_\chi \qquad (3)$$

are those that minimize the energy of the system, the Bogomol'nyi-Prasad-Sommerfeld (BPS) energy [11], and are also solutions of the static equations of motions [12]. Based on the first-order differential equations, one can realize that dx is a kind of invariant and the following, in general nonlinear equation

$$\frac{d\chi}{d\phi} = \frac{W_\chi}{W_\phi} \qquad (4)$$

furnishes a relation between the classical static solutions, called the orbit equation, which can be solved analytically depending on the model under consideration [10]. For the cases in which the general orbit equation can be found, we can write the scalar field χ in terms of the scalar field ϕ, that is $\chi = f(\phi)$, and it can be used to eliminate χ in terms of ϕ in the Lagrangian density (1), that is

$$\mathcal{L} = \tfrac{1}{2}(1 + f_\phi^2)\partial^\mu\phi\partial_\mu\phi - V(\phi), \qquad (5)$$

where $f_\phi = \frac{df}{d\phi} = \frac{d\chi}{d\phi} = \frac{W_\chi}{W_\phi}$ with both, W_ϕ and W_χ, written in terms only of the field ϕ. $V(\phi)$ is the potential $V(\phi,\chi)$ with χ written in terms of ϕ. Because of the peculiar structure of the potential given in (2) and also to the differential Eq. (4), we have

$$V(\phi) = \tfrac{1}{2}(1 + f_\phi^2)W_\phi^2. \qquad (6)$$

The equation of motion for this single scalar field model is given by

$$(1 + f_\phi^2)\partial_\mu\partial^\mu\phi + f_\phi f_{\phi\phi}\partial_\mu\phi\partial^\mu\phi + \frac{dV}{d\phi} = 0, \qquad (7)$$

and the energy associated with the static classical solutions is expressed as

$$E = \int dx\left(\frac{1}{2}(1 + f_\phi^2)\phi'^2 + V(\phi)\right), \qquad (8)$$

where ϕ' stands for $d\phi/dx$. In order to find the minimum energy, we note that the energy can be rewritten as

$$E = \frac{1}{2}\int dx[((1 + f_\phi^2)^{1/2}\phi' \pm \sqrt{2V})^2 \mp 2(1 + f_\phi^2)^{1/2}\phi'\sqrt{2V}], \qquad (9)$$

and, consequently, the classical solutions with minimum energy satisfies the first-order differential equations

$$\phi' = \mp\left(\frac{2V}{1 + f_\phi^2}\right)^{1/2} = \mp W_\phi, \qquad (10)$$

where $W_\phi = dW(\phi,\chi)/d\phi$ with χ replaced with $f(\phi)$. It is easy to show that the solutions of the first-order differential equations are also solutions of the static equation of motion

$$(1 + f_\phi^2)\phi'' + f_\phi f_{\phi\phi}\phi'^2 = \frac{dV}{d\phi}, \qquad (11)$$

and the BPS energy is given by

$$E_{\rm BPS} = \int dx((1 + f_\phi^2)\phi' W_\phi(\phi, f(\phi))) = |W(\bar\phi, f(\bar\phi))(\infty) - W(\bar\phi, f(\bar\phi))(-\infty)|, \qquad (12)$$

where the classical solutions $\bar\phi(x)$ are to be taken at $\pm\infty$.

This procedure resembles the one carried out in Ref. [13] to prove the equivalence between sine-Gordon, Liouville, and other models. In that reference, the first-order differential equations obeyed by the BPS solutions of the models are used to construct a mapping between the fields of the two models whose equivalence is to be demonstrated. In fact, one deforms one known model by using the mapping function and obtains another known model. It is shown that both models possess the same BPS energy when the deformation is performed in the Lagrangian density, in contrast to what happens when the deformation is carried out in the differential equations [14]. One can see that the BPS energy found in Eq. (12) of the model given by the Lagrangian density (5) is the same of the model described by (1). Since we are not interested in proving the equivalence between those models, we would rather prefer to work with an effective Lagrangian density whose structure is of the type kinetic-potential which leads to static Euler-Lagrangian equation of the type $\phi'' = dU_{\rm eff}/d\phi$, whose classical solutions are also solutions of the Eq. (11). This procedure is more like the one of [14], and it is more convenient when one works with scalar fields in interaction with gravitation as we consider in the next section.

Particular cases

Here, we consider the same model with two interacting scalar fields applied in a *brane world* scenario in Refs. [5,6] whose superpotential is

$$W(\phi,\chi) = \phi\left[\lambda\left(\frac{\phi^2}{3} - a^2\right) + \mu\chi^2\right]. \qquad (13)$$

In a recent paper, Dutra [10] was able to show that for this case, the orbit equations relating both fields can be obtained explicitly. They can be written as

$$\rho(\chi) = \phi^2 - a^2 = c_0 \chi^{\lambda/\mu} - \frac{\mu}{\lambda - 2\mu}\chi^2, \quad \text{for } \lambda \neq 2\mu, \tag{14}$$

and

$$\rho(\chi) = \phi^2 - a^2 = \chi^2[\ln(\chi) + c_1], \quad \text{for } \lambda = 2\mu, \tag{15}$$

where c_0 and c_1 are constants of integration. In general, only the first of the above orbit equations is used to find well behaved classical solutions. One can check, for example, that the second of the above orbit equation fails to reproduce the minima of the model, namely $\phi = \pm a$ and $\chi = 0$. From now on, we consider $a > 0$.

The first orbit equation can be used to construct models with only one scalar field which exhibit the main features of this model with two interacting scalar fields. In one example, we consider the situation in which $\mu = \lambda$. In this case, the orbit equation is given by

$$\chi^2 + c_0\chi - (\phi^2 - a^2) = 0, \tag{16}$$

and the field χ can be expressed in terms of ϕ as

$$\chi = f(\phi) = -\frac{c_0}{2} \pm \frac{1}{2}\sqrt{c_0^2 + 4(\phi^2 - a^2)}. \tag{17}$$

It is important to remark that in this case the constant of integration must satisfy $c_0 < -2a$, in order to have BPS solutions (minimum energy configurations) in the model with two interacting scalar fields [10]. This domain of validity of the constant of integration is naturally incorporated in the construction of effective model as it follows below.

By substituting the expression (17) with the upper sign in the expression for $W_\phi(\phi, \chi = f(\phi))$, we obtain the following superpotential:

$$W_\phi(\phi) = 2\mu(\phi^2 + b^2 - a^2 + b\sqrt{\phi^2 + b^2 - a^2}). \tag{18}$$

From now on, we can work with a model described by the Lagrangian density

$$\mathcal{L} = \tfrac{1}{2}\partial^\mu\phi\partial_\mu\phi - U_{\text{eff}}(\phi), \tag{19}$$

with the effective potential $U_{\text{eff}}(\phi) = W_\phi^2(\phi)/2$. In this model, the first-order differential equations satisfied by $\phi(x)$ is $\phi' = \mp W_\phi$ and the solutions for the field ϕ are the same obtained in the model with two interacting scalar fields [6,10] described by the Lagrangian density (1) with $W(\phi, \chi)$ given in (13).

The choice of the lower sign in the expression (17) would result in an effective potential $U_{\text{eff}}(\phi) = W_\phi^2/2$ with only one minimum, and we are interested in effective potentials with at least two minima. The constant $b = c_0/2$ must satisfy the inequality $b < -a$, such that we have nonsingular solutions, and the effective potential presents two global minima and one local minimum for a certain range of the parameter b. On the other hand, for $b = -a$, we have an effective potential with three global minima. We show in Fig. 1 the behavior of the effective potential, in units of μ^2, as a function of ϕ, for two different values of b and $a = 1$. One can see that for $b < -a$, the effective potential tends to exhibit three minima as b gets closer to $-a$. In fact, for $b = -a$ the effective potential becomes $U_{\text{eff}}(\phi) = 2\mu^2(\phi^2 - a|\phi|)^2$ that has also a minimum at $\phi = 0$. The classical solutions of the first-order differential Eq. (10), for $b < -a$, are given by

$$\phi = \pm a \frac{\sinh(2\mu a x)}{\cosh(2\mu a x) - b/f}, \tag{20}$$

where $f = \sqrt{b^2 - a^2}$ and the upper (lower) sign stands for the kink (antikink) solution. In Fig. 2, it is shown a profile of the topological classical solution for a sufficiently large value of $|b|$, and a double kink profile, usually called two kinks, for b close to the critical value $-a$. This kind of configuration also arises as solution of the models introduced in Refs. [7,8].

Another choice of the parameter λ, namely $\lambda = 4\mu$, leads to a similar nonpolynomial effective potential. In this case, the orbit equation can be written as

$$\chi^2 = \frac{1}{4c_0}[1 \pm \sqrt{1 + 16c_0(\phi^2 - a^2)}]. \tag{21}$$

In this specific case ($\lambda = 4\mu$) the constant of integration c_0 has to be less than $1/16a^2$, for the same reason mentioned in the case that $\lambda = \mu$. Again, the domain of validity of this constant is going to be incorporated in the effective model presented right below.

By taking the upper sign in the above equation, we have the following effective superpotential:

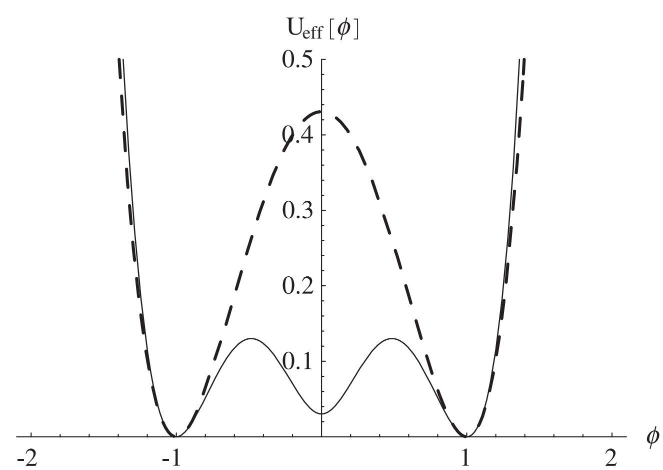

FIG. 1. The effective potential in the case $\lambda = \mu$, for $a = 1$ and $b = -1.001$ (solid line) and $b = -1.3$ (dashed line).

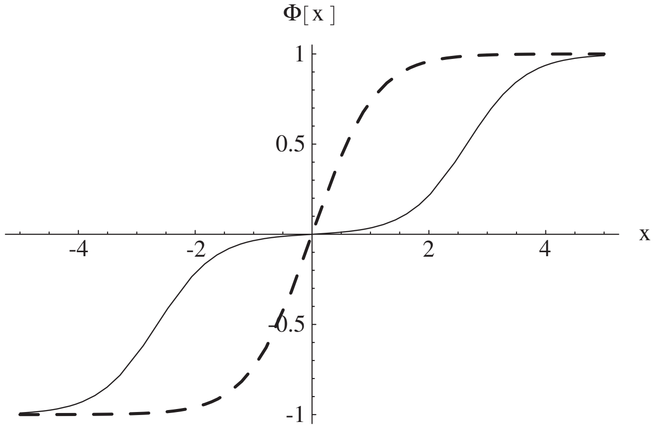

FIG. 2. Typical kink profiles, for $a = 1$ and $b = -1.001$ (solid line), corresponding to a two-kink solution and $b = -1.3$ (dashed line), corresponding to a single kink solution.

$$W_\phi = 4\mu(\phi^2 - d^2 + dc\sqrt{\phi^2 - d^2}), \tag{22}$$

where $d = a/\sqrt{c^2 - 1}$ and $c = 1/\sqrt{1 - 16c_0a^2}$. In order to have a well-defined model, the constant c_0 must satisfy the inequality $c_0 < 1/16a^2$ and the solutions are given by

$$\phi = \pm a \frac{\sinh(4\mu ax)}{\cosh(4\mu ax) + c}. \tag{23}$$

It worth mentioning that the model we have constructed in this case ($\lambda = 4\mu$) is very similar to the one proposed in Ref. [7] if we set $\mu = 1/4$ and make some identification between the parameter d with the parameter b_0 from that paper. The behavior of the effective potential as a function of ϕ is almost identical to the one presented in the previous case. Moreover, if $c_0 = 1/16a^2$ one obtains the effective potential, $U_{\rm eff}(\phi) = 8\mu^2(\phi^2 - a|\phi|)^2$, with three minima.

Effective polynomial potentials can also be obtained from the model described by the Lagrangian density (1) with $W(\phi, \chi)$ given by (13). This is done by using the orbit Eq. (14) to express the field ϕ as a function of the field χ. By conveniently rewriting the orbit equation as

$$\phi = g(\chi) = \pm\sqrt{c_0\chi^{\lambda/\mu} - \frac{\mu}{\lambda - 2\mu}\chi^2 + a^2}, \tag{24}$$

we have the effective potential given by

$$U_{\rm eff}(\chi) = \frac{1}{2}W_\chi^2 = 2\mu^2\chi^2\left(c_0\chi^{\lambda/\mu} - \frac{\mu}{\lambda - 2\mu}\chi^2 + a^2\right). \tag{25}$$

This potential has at least one minimum at $\chi = 0$, and we have to set $\lambda/\mu = n$, where n is a positive integer ($n \neq 2$), in order to have well defined potentials with, at least, two minima. This last condition leads to some constraints over the values of c_0. We note that for $\lambda/\mu = 1$, a positive definite potential with two minima, is obtained if $c_0 = \pm 2a$ and, for $\lambda/\mu = 4$, we find a positive definite potential with three minima at $\chi = 0$, $\pm 2a$ if $c_0 = 1/16a^2$. Those specific values for c_0 are the critical ones that lead to polynomials models with respect to the field ϕ which possess three minima. We look for the critical values of c_0, corresponding to different values of n, in order to have positive definite effective potentials, $U_{\rm eff}(\chi)$, with more than one minimum. First, we note that effective potentials of the type $U_{\rm eff}(\phi) = (\phi^2 - a|\phi|)^2$ has classical solutions that connect the minimum $\phi = 0$ to the minima $\phi = +a$ or $\phi = -a$ and vice versa. Moreover, we recall that those polynomial potentials, are constructed by substituting the orbit equation $\chi = f(\phi)$ into the superpotential $W_\phi = \lambda(\phi^2 - a^2) + \mu\chi^2$ and this one, by its turn, is substituted in $U_{\rm eff}(\phi) = W_\phi^2/2$, which is positive definite everywhere. If $\phi = 0$ is a minimum of this effective potential, it must correspond to $\chi = \pm\sqrt{n}a$, due to the orbit equation. By substituting one of those values of χ in the potential $U_{\rm eff}(\chi)$ and imposing that they are minima of this last, *a priori*, positive definite effective potential [$U_{\rm eff}(\chi = \pm\sqrt{n}a) = 0$], the critical value of c_0,

$$c_0^{-1} = a^{n-2}\left[\frac{n}{2} - 1\right]n^{n/2}, \tag{26}$$

is obtained. Except for the case $n = 1$ that presents two minima, one finds positive definite effective potentials with three minima only for n even. Such kinds of polynomial potentials were already discussed extensively in the literature [15]. These polynomial potentials have typical kink solutions that connect the minimum $\chi = 0$ to one of the other two minima $\chi = \pm\sqrt{n}a$ and vice versa. For $n = 4$, we have for instance

$$\chi = \mp\sqrt{2}a\frac{\cosh(\mu ax) \pm \sinh(\mu ax)}{\sqrt{\cosh(2\mu ax)}}, \tag{27}$$

which have a kink profile very similar to that of the first-order differential equation solutions for the effective models of the kind $U(\phi) = (\phi^2 - a|\phi|)^2$. These solutions that can be seen as half-torsion in a spin chain, also have similar profile to those exhibited by self-consistent solutions for inhomogeneous chiral condensates in the Nambu-Jona-Lasinio model in 1 + 1 space-time dimensions [16]. Since these solutions are not continuously deformable into two-kink solutions, they will not be applied to the brane-world scenario considered below where we are going to discuss the consequences of thicker branes in the warping of the space.

III. APPLICATION TO A BRANE WORLD SCENARIO

We now consider the scalar field coupled to gravity in (4, 1) space-time dimensions described by the action

$$S = \int d^4xdy\sqrt{|g|}\left(-\frac{1}{4}R + \frac{1}{2}g_{ab}\partial^a\phi\partial^b\phi - V(\phi)\right), \tag{28}$$

where $g \equiv \mathrm{Det}(g_{ab})$, and the metric is

$$ds^2 = g_{ab}dx^a dx^b = e^{2A(r)}\eta_{\mu\nu}dx^\mu dx^\nu - dr^2, \quad a, b = 0, \ldots, 4, \tag{29}$$

where $r = x^4$ is the extra dimension, $\eta_{\mu\nu}$ is the Minkowski metric, and $e^{2A(r)}$ is the so-called warp factor, which is supposed to depend only on the extra dimension. The Greek indices run from 0 to 3.

The static equations of motion following from the action (28) and for the case that the scalar field depends only on the extra dimension are written as

$$\frac{d^2\phi}{dr^2} + 4\frac{dA}{dr}\frac{d\phi}{dr} = \frac{dV(\phi)}{d\phi}, \tag{30}$$

$$\frac{d^2A}{dr^2} = -\frac{2}{3}\left(\frac{d\phi}{dr}\right)^2, \tag{31}$$

and

$$\left(\frac{dA}{dr}\right)^2 = \frac{1}{6}\left(\frac{d\phi}{dr}\right)^2 - \frac{1}{3}V(\phi). \tag{32}$$

We consider that the potential $V(\phi)$ can be written as [17]

$$V(\phi) = \frac{1}{2}\left(\frac{dW(\phi)}{d\phi}\right)^2 - \frac{4}{3}(W(\phi))^2. \tag{33}$$

In this case, the BPS solutions [11] of the following first-order differential equations

$$\frac{d\phi}{dr} = \pm\frac{dW(\phi)}{d\phi} \quad \text{and} \quad \frac{dA}{dr} = \mp\frac{2}{3}W(\phi) \tag{34}$$

are also solutions of the second-order differential Eqs. (30) and (31), and the Eq. (32) is identically satisfied. By taking $W_\phi(\phi)$ given by Eq. (18) together with the corresponding kink solution in (20), the superpotential is given by

$$W(\phi) = 2\mu\left[\phi\left(\frac{\phi^2}{3} + f^2 + \frac{b}{2}\sqrt{\phi^2 + f^2}\right) + \frac{bf^2}{2}\sinh^{-1}\left(\frac{\phi}{f}\right)\right], \tag{35}$$

and the warp factor is found by integrating the second of the Eqs. (34) with the classical solution (20) substituted in $W(\phi)$. We show in Fig. 3 two profiles of the warp factor ($a = 1$), corresponding to two different values of the parameter b: one close to and the other one far from the critical value $b = -1$. One can note that as far as b decreases (its modulus increases), the warp factor becomes more narrow. For values of b close to the critical value, one can observe that the warp factor becomes wide and one can see a flat region, which signalizes a Minkowskian metric inside the domain wall. Thick is the brane, wider is the warp factor. We have also analyzed the case in which $W_\phi(\phi)$ given by Eq. (22) and have verified that the behavior of the warp factor is not substantially different from that presented here.

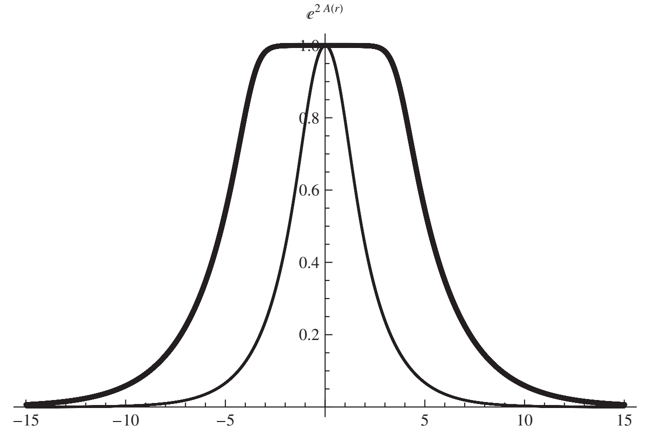

FIG. 3. Warp factor for two different values of b. One closed to the critical value (thick line) with a *meseta* shape and the other one far from the critical (thin line).

Now, we consider the stability of the system by analyzing the equations of motion of linear small fluctuations around the classical solutions. This issue is also important to realize the localization of the gravity inside the domain wall [2–17]. This is done by means of a perturbation of the metric, $ds^2 = e^{2A(r)}(\eta_{\mu\nu} + \varepsilon h_{\mu\nu})dx^\mu dx^\nu - dr^2$ and a small perturbation around the classical solution $\phi \longrightarrow \bar{\phi}(r) + \varepsilon\tilde{\phi}(r, x_\mu)$, where ε is a small number. By performing those perturbations in the Lagrangian density and by expanding it up to $\mathcal{O}(\varepsilon)$, we obtain the following equations of motion:

$$e^{-2A}\Box\tilde{\phi} - 4\frac{dA}{dr}\frac{d\tilde{\phi}}{dr} - \frac{d^2V}{d\phi^2}\tilde{\phi} = \frac{1}{2}\frac{d\phi}{dr}\eta^{\mu\nu}\frac{dh_{\mu\nu}}{dr}$$

for the fluctuation of the scalar field and

$$-\frac{1}{2}\Box h_{\mu\nu} + e^{2A}\left(\frac{1}{2}\frac{d}{dr} + 2\frac{dA}{dr}\right)\frac{dh_{\mu\nu}}{dr} - \frac{1}{2}\eta^{\alpha\beta}(\partial_\mu\partial_\nu h_{\alpha\beta} - \partial_\mu\partial_\nu h_{\beta\nu} - \partial_\nu\partial_\alpha h_{\beta\mu}) + \frac{1}{2}\eta_{\mu\nu}e^{2A}\frac{dA}{dr}\partial_r \times (\eta^{\alpha\beta}h_{\alpha\beta}) + \frac{4}{3}e^{2A}\eta_{\mu\nu}\frac{dV}{d\phi}\tilde{\phi} = 0 \tag{36}$$

for the fluctuations of the metric.

In general, it is quite difficult to take into account linear fluctuations of all components of the metric together with the quantum fluctuations of $\phi(x, t)$ in order to have a broad view of the linear stability of the whole system. This is due to fact that the above set of coupled differential equations involving the fluctuations is very intricate to be solved.

Nevertheless, it is possible to show that the transverse and traceless part of the fluctuations of the metric ($\bar{h}_{\mu\nu}$) decouple from fluctuations of $\phi(x, t)$ [17]. By constructing $\bar{h}_{\mu\nu} = P_{\mu\nu\alpha\beta}h^{\alpha\beta}$ from the projector operator $P_{\mu\nu\alpha\beta} \equiv \frac{1}{2}(\pi_{\mu\alpha}\pi_{\nu\beta} + \pi_{\mu\nu}\pi_{\nu\alpha}) - \frac{1}{3}\pi_{\mu\nu}\pi_{\alpha\beta}$ with $\pi_{\mu\nu} \equiv \eta_{\mu\nu} - \partial_\mu\partial_\nu/\Box$, we have that the Eq. (36) for the transverse and traceless part of the fluctuations of the metric simplifies to

$$\frac{d^2\bar{h}_{\mu\nu}}{dr^2} + 4\frac{dA}{dr}\frac{d\bar{h}_{\mu\nu}}{dr} - e^{-2A}\Box\bar{h}_{\mu\nu} = 0. \tag{37}$$

By using separation of variables and expressing $\bar{h}_{\mu\nu}$ conveniently as

$$\bar{h}_{\mu\nu} = e^{ik_\mu x^\mu}e^{-(3/2)A(r)}\xi_{\mu\nu}(r), \tag{38}$$

we find, by making use of the transformation of variables $z = \int e^{-A(r)}dr$, that $\xi_{\mu\nu}(z)$ satisfies the following Schrödinger-like stability equation

$$-\frac{d^2\xi_{\mu\nu}}{dz^2} + V_{\text{eff}}(z)\xi_{\mu\nu} = k^2\xi_{\mu\nu}, \tag{39}$$

where

$$V_{\text{eff}}(z) = \frac{9}{4}\left(\frac{dA}{dz}\right)^2 + \frac{3}{2}\frac{d^2A}{dz^2} \tag{40}$$

is the effective potential. We show in Fig. 4 this effective potential against the variable r for two different values of b, one close and the other one far from the critical value. One can note that the shape of the effective potential for values of b far from the critical value is similar to the shape of others presented in the literature, for example, in [2]. Therefore, the only bound-state solution is the one associated to the zero-mode which can be seen as the localization of the gravity inside the domain wall. Higher energy modes are nonlocalized states that can escape from the domain wall and propagate along the extra dimension. In any case, no matter how big the value of $|b|$ is, there is no room to have localized tachyon modes ($k^2 < 0$). The differential operator in Eq. (39) can be factorized as the product of two operators which are adjoint of each other:

$$a^\dagger a\xi_{\mu\nu} = \left(\frac{d}{dz} + \frac{3}{2}\frac{dA}{dz}\right)\left(-\frac{d}{dz} + \frac{3}{2}\frac{dA}{dz}\right)\xi_{\mu\nu} = k^2\xi_{\mu\nu}. \tag{41}$$

Then, for the n-th normalized eigenmode $|n\rangle$, we have $k_n^2 = \langle n|a^\dagger a|n\rangle = |a|n\rangle|^2 \geq 0$. Particularly, the non-normalized zero-energy eigenmode is given by $\xi_{\mu\nu}^{(0)}(z) = e^{(3/2)A(z)}$, and the corresponding transverse and traceless part of the fluctuation of the metric presents no dependence with the extra dimension.

Another aspect that the nonpolynomial models share with other models is the behavior of the matter energy density. It is given by

$$\varepsilon(r) = e^{4A(r)}\left[\frac{1}{2}\left(\frac{d\bar{\phi}}{dr}\right)^2 + V(\bar{\phi})\right], \tag{42}$$

where $V(\bar{\phi})$ is the potential in (33) evaluated at the classical solution. The behavior of the matter energy density is shown in Fig. 5 for two different values of b. The features shared with most of the models is the peak of the energy density around the thick brane, which can be observed for values of b far from the critical value, and the presence of regions outside the domain wall where the matter energy density is negative. For values of b close to the critical one, two relative small peaks of the energy density show up around each wall of the double-wall structure, this is also a common feature of the models with thicker branes [6,8]. Although the energy matter is negative, the energy functional [18]

$$F = \frac{1}{2}\int_{-\infty}^{+\infty} dr e^{4A(r)}\left[\left(\frac{d\phi}{dr}\right)^2 + 2V(\phi) - 6\left(\frac{dA}{dr}\right)^2\right] \tag{43}$$

is positive definite. This energy functional generates the

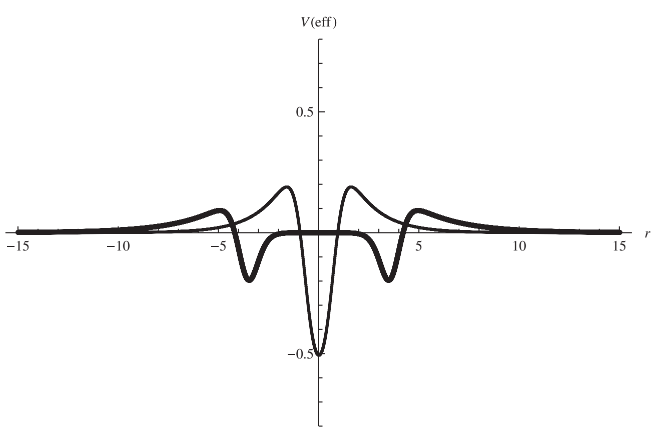

FIG. 4. The effective potential in the effective Schrödinger equation for the fluctuation of the metric for two different values b, $b = -1.0001$ (thick line), and $b = -1.2$ (thin line).

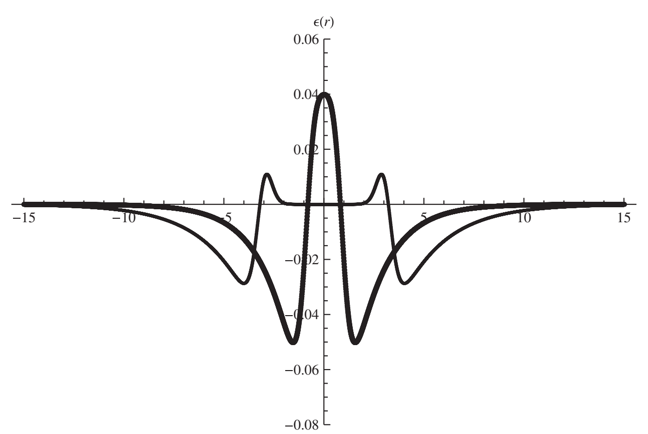

FIG. 5. The matter energy density in the case $\lambda = \mu$, for $a = 1$ and $b = -1.0001$ (thin line) and $b = -1.3$ (thick line).

Euler-Lagrange Eqs. (30)–(32) and is minimized by the solutions of the first-order differential Eqs. (34), since it can be rewritten as

$$F = \frac{1}{2}\int_{-\infty}^{+\infty} dr\bigg\{e^{4A(r)}\bigg[\bigg(\frac{d\phi}{dr} \mp W_\phi\bigg)^2 - 6\bigg(\frac{dA}{dr} \pm \frac{2}{3}W\bigg)^2\bigg] \pm \frac{d}{dr}(e^{4A(r)}W)\bigg\} = \pm e^{4A(r)}W(\bar{\phi})|_{-\infty}^{+\infty}. \tag{44}$$

Note that the $W|_{\phi(\infty)} - W|_{\phi(-\infty)}$ is bigger (less) than zero if we choose the upper (lower) sign. This energy plays the role of the topological $E_{\rm BPS}$ energy in the present scenario of scalar fields interacting with gravitation in a (4, 1) dimensional warped space-time.

IV. FINAL REMARKS

We have constructed effective models with only one scalar field which supports deformed kink solutions which has been called two-kink solutions. This has been done by means of a general orbit equation relating two scalar fields and by eliminating one of them in terms of the other one. When such a procedure is done in the Lagrangian density, the effective Lagrangian density is not given in the usual kinetic-potential form [see Eqs. (5) and (6)]. In order to have an effective canonical Lagrangian density, we adopt the elimination of one of the field in the equations of motion, instead. The model we have constructed here together with their two-kink solutions are very similar to the one proposed in [7], but, in principle, other (polynomial) models are obtained from that one with two fields we have started with. Contrary to the nonpolynomial, those polynomial potentials, as the one presented in Eq. (25), neither have BPS solutions with a two-kink profile, nor possess free parameters that can lead to continuously deformable topological structures and cannot be used to study phase transitions as mentioned below.

One of the effective nonpolynomial models constructed here is applied to a brane-world scenario where its influence in the warp factor leads to a flat geometry inside the thicker domain walls. That model can also be used to analyze universal aspects of branes splitting in a warped geometry [4]. That phenomenon can be interpreted as a phase transition due to the variation of the parameters of the potential, which in our case are the parameters b for the superpotential (18), and c for the superpotential (22). In each case, there will be a transition in the form of the potential, from one with two vacua to another one with three vacua when the parameter reaches a critical value. One can think of b, for instance, as dependent on the temperature. Far from the critical temperature, we have a single relatively thick brane and as the critical temperature is approached, the parameter b approaches $-a$ from the left. For values of b close to but less than $-a$, the thick brane splitting starts to take place (note the behavior of the solid curve in Fig. 2) and the local minimum in the effective potential $U_{\rm eff}$ presents the tendency to become a global minimum. The splitting also influences the localization of the zero modes fluctuation of the metric as can be seen by the behavior of the effective potential in Fig. 4. In that figure, one can see that for values of b far from the critical value, the effective potential has a volcano shape which leads to a narrow (more localized) zero-mode fluctuation than in the situation in which b is close to the critical value. The splitting of the brane is also manifested in the splitting of the matter energy density shown in Fig. 5. When $b = -a$, the effective potential is $U_{\rm eff}(\phi) = 2\mu^2(\phi^2 - a|\phi|)^2$ and the brane is split into two branes. We can suppose that the branes are at a distance $2L$ from each other and localized around the core of each one of the solutions $\phi_-(r) = -a/2\{1 - \tanh[\mu a(r + L)]\}$ and $\phi_+(r) = a/2\{1 + \tanh[\mu a(r - L)]$. We have not analyzed what happens to the warp factor and the fluctuation of the metric under the influence of both scalar fields simultaneously, but this might be done by following the same numerical approach adopted in Ref. [19], where it was obtained the spectrum of fermions in the background of a kink and a antikink which are far apart from each other. We think that the effective potential for the fluctuations of the metric will have the shape of two volcanos whose craters are distant $2L$ from each other. That effective potential can support a zero eigenmode and the eigenfunctions may have peaks around each brane or in the region between the branes.

The brane splitting phenomenon was originally analyzed in [4] by using a model with a complex scalar field. In that case, the solutions for the classical equations of motion for the real scalar fields are obtained numerically and it is not evident that they are BPS solutions. Here, the solutions of the first-order differential equations, the exact BPS solutions, are also solutions of the equations of motion. The model with two scalar fields described by the superpotential (13) can not be used to describe the brane splitting phenomenon and there is no appearance of metastable states, since the parameter that controls the thickness of the brane is the constant of integration c_0 that appears in the orbit Eq. (14); then, we can not think of it as a parameter that would depend on the temperature. Moreover, in that scenario one talks about Bloch branes [5] and degenerated and critical Bloch branes [6], depending on the chosen values for c_0. On the other hand, in the built nonpolynomial models, Eqs. (18) and (22), c_0 becomes a parameter of the potential and we are left with only one brane that splits when that parameter assumes a critical value.

The phenomenon of domain walls splitting in a flat geometry can also be done by using a polynomial potential, as the one given by $U(\phi) = W_\phi^2/2 = \frac{1}{2}[(1 - g)\phi^2 + g] \times (1 - \phi^2)^2$ with $g \in [0, 1]$, which also possesses kink solutions as classical configurations with minimum energy. Those kinks are also deformable into two-kinks by varying the parameter g continuously. However, in the scenario of warp geometry, the potential given in (33), in this particular case, would present severe divergences which might put in jeopardy the finiteness of the energy density (42) and the desirable smoothness of the warp factor.

The effective models presented here can also be considered in space-time with $D > 4$ dimensions and with the solutions depending only on the radial coordinate. Since nonlinear models with only scalar fields are not stable in space-time dimensions bigger than two, as has been demonstrated by Derrick's theorem [20], one should resort to a convenient bypass by introducing an explicit dependence of the interacting potential on the coordinates, as has been provided by [9]. The resulting radial solutions are very similar to those shown in Ref. [7]. Those solutions and their consequences in warped space-time with two and three extra dimensions is under study and the results will be reported elsewhere.

Finally, a myriad of models with two scalar interacting fields have been studied in Ref. [21] and from those we can, in principle, construct other effective models with only one scalar field. One of the models which is under analysis is the fourth model proposed in [21], whose potential is given by $V(\phi, \chi) = \bar{\lambda}\phi + \bar{\mu}\chi - \lambda/4(\phi^4 + \chi^4 + 6\phi^2\chi^2)$.

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