Julio Marny Hoff da Silva

Aspectos Físicos e Algébricos de Espinores Escuros

Guaratinguetá
2016

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Texto Sistemático Crítico Apresentado à Faculdade de Engenharia da Universidade Estadual Paulista "Júlio de Mesquita Filho"- Câmpus de Guaratinguetá para o Concurso de Livre-Docência.

Guaratinguetá, setembro de 2016
2016

Hoff da Silva, Julio Marny

Aspectos Físicos e Algébricos de Espinores Escuros

XXXXX páginas

Texto Sistemático Crítico - Faculdade de Engenharia - Campus de Guaratinguetá - Universidade Estadual Paulista "Júlio de Mesquita Filho". Departamento de Física e Química.

I. Universidade Estadual Paulista "Júlio de Mesquita Filho". Faculdade de Engenharia - Campus de Guaratinguetá. Departamento de Física e Química.

Dedicatória

Para Ane, pois estaremos sempre sob a mesma lua...

Agradecimentos

Gostaria de expressar minha sincera gratidão

- aos colegas e funcionários do Departamento de Física e Química, cuja colaboração e aprendizado são de grande valia;
- em particular aos colaboradores do grupo de partículas e campos, por toda a possibilidade de aprendizado que me proporcionam;
- ao amigo Roldão da Rocha pelo exemplo de resiliência;
- ao professor Dharam Vir Ahluwalia;
- ao CNPq;
- aos diversos estudantes (de graduação, pós-graduação, e orientados) cujos questionamentos e vontade de aprender nos impelem sempre em frente;
- aos contribuintes do Estado de São Paulo.

Julio Marny Hoff da Silva

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Apresentado à Faculdade de engenharia da Universidade Estadual Paulista - Campus de Guaratinguetá para o Concurso de Livre-Docência

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.

Texto Sistemático Sobre Nossas Contribuições aos "Aspectos Físicos e Algébricos de Espinores Escuros"

Elaborado por

Julio Marny Hoff da Silva

APRESENTAÇÃO

Optei pela apresentação de uma análise crítica de alguns trabalhos levados a termo após o ingresso como Docente do Departamento de Física e Química da Faculdade de Engenharia de Guaratinguetá - UNESP, em subsituição à tradicional tese de Livre-Docência.

Essa opção na sistemática de apresentação se coaduna com um duplo aspecto: apresentar criticamente as contribuições realizadas na área e consolidar perspectivas de continuidade do trabalho exposto.

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Capítulo 1

Introdução

Nada menos propositado, talvez, do que uma citação no intróito de um texto formal. Havendo ainda o agravo do texto pretender conter uma parcela de crítica, a boa norma de uso do cálamo sempre nos remete a uma certa dose de parcimônia. Ainda assim, dada a relevância do autor da citação, comecemos nossa introdução apreciando a opinião de Sir Michael Atiyah no que concerne aos objetos centrais dessa tese:

> *No one fully understands spinors. Their algebra is formally understood but their general significance is mysterious. In some sense they describe the "square root" of geometry and, just as understanding the square root of -1 took centuries, the same might be true of spinors.*

Em certo sentido, abordaremos ao longo dos trabalhos explicitados nesse texto diversas facetas do entendimento corrente de espinores do ponto de vista matemático e de sua relação com a física, especialmente no que tange ao estudo de candidatos à matéria escura. A fim de encerrar nessa introdução um contexto explicativo do que se segue, recorremos novamente à frase supracitada, desta vez ponto a ponto.

No one fully understand spinors.

O começo desalentador da afirmação guarda também ampla perspectiva de trabalho. Em física, em linhas gerais, espinores foram introduzidos na Mecânica Quântica como certas funções de onda descrevendo, via acoplamentos específivos no termo potencial da equação de Schrödinger, um grau de liberdade interno, o spin. A evolução conceitual que marca a incorporação da dinâmica relativística e o conceito de campo à Mecânica Quântica, culminando com a Teoria Quântica de Campos, levou também a refinamentos do conceito de espinor em física. O operador de campo quântico espinorial utilizado na descrição do setor de matéria do Modelo Padrão carrega consigo coeficientes de expansão espinoriais cuja propriedade definidora flerta com um viés matemático bem estabelecido para espinores: entidades que carregam a representação dos grupos de rotação em um espaço de dimensão finita. No caso específico dos coeficentes de expansão, estamos nos referindo às rotações que compõe o grupo de Lorentz.

Na década de sessenta do século precedente, para nos atermos a um determinado recorte histórico, a Teoria Quântica de Campos passou por ampla formalização, colocando suas bases teóricas em um patamar sólido. Entretanto, no que diz respeito ao uso de espinores, uma possível fenda investigativa se abre se os utilizamos como objetos que carregam representações de certos subgrupos do grupo de Lorentz, tais como os que erigem a chamada *Very Special Relativity.* É nesse sentido que contextualizamos essa primeira parte da citação, fazendo-lhe coro: a descrição de campos fermiônicos cujos coeficientes de expansão apresentam modificações sutis, porém importantes, dos conceitos usuais, tem levado a um amplo campo de novas perspectivas. Tais novos campos fermiônicos apresentam (ou são construídos para que apresentem) propriedades que lhes faculta sua aplicação na descrição da matéria escura. No trajeto formal de desenvolvimento da teoria de campos incorporando esses novos campos há, acreditamos, muito a ser compreendido e formalizado.

> *Their algebra is formally understood but their general significance is mysterious.*

Do ponto de vista algébrico, há uma classificação espinorial com forte apelo físico. Sabemos que em física de altas energias, um férmion sozinho (descrito por um espinor) não é passível de detecção. Estados acessíveis experimentalmente são aqueles formados pelos bilineares covariantes. O entendimento concreto de espinores como elementos (fibras) de um fibrado principal contendo transformações de $SL(2,\mathbb{C})$ e com características bem definidas do espaço de representação, possibilitou a execução de um programa algébrico profundo na década de oitenta, levando à categorização sistemática dos espinores de acordo com os valores assumidos pelos seus bilineares.

Sem dúvida, no que é concernente a esse trabalho, podemos endossar a asserção de que a álgebra espinorial é bem entendida. No entanto, ainda há espaço para mistério: essa mesma formalização algébrica revela a existência de uma classe de espinores cuja contrapartida física ainda não foi completamente explorada. De fato, e vamos nos remeter a esse ponto em momento oportuno nessa tese, há até então apenas um sistema físico conhecido cujo setor espinorial recai sobre essa classe fugidia.

In some sense they describe the "square root" of geometry ⋯

Um programa investigativo levado a termo por Eliè Cartan mostra uma faceta bastante intrigante relacionada aos espinores. De fato, espinores podem ser entendidos como objetos pré-geométricos, no sentido de que pontos do espaço-tempo podem ser escritos como composições de espinores. Nesse contexto eles seriam a "raíz quadrada da geometria", objetos mais fundamentais do que pontos que compõe o espaço pseudo-euclidiano. Mais do que curioso, tal resultado permite uma abordagem interessante para a existência dos chamados espinores exóticos.

A existência de espinores exóticos está vinculada à não trivialidade da variedade de base na qual a teoria toma forma. Por exemplo, a dinâmica espinorial em um espaço de Minkowski não simplesmente conexo abre a possibilidade de existência de outras estruturas espinoriais. Tais estruturas possuem dinâmica essencialmente usual, porém com uma correção de origem topológica, mas que pode ser entendida como um acoplamento adicional com um campo vetorial externo. É precisamente aqui que entra novamente a relevância dos espinores escuros. Uma propriedade essencial para um candidato a matéria escura é a impossibilidade de acoplamento com campos usuais do Modelo Padrão. Logo, diferentemente do que acontece com espinores usuais, o estudo de espinores escuros exóticos traz informação genuína sobre a topologia do sistema.

Para que possamos abordar com propriedade o que foi dito nessa curta exposição, separamos nossas contribuições em duas frentes de trabalho, a saber: uma voltada às propriedades algébricas dos espinores escuros, e outra enfocando o entendimento das propriedades físicas dos mesmos. Nesse último caso, fazemos ainda outra ramificação separando nossas contribuições em duas subáreas: teoria geral de partículas e campos e cosmologia. Percorreremos essa trajetória em dez trabalhos que tipificam a atuação e contribuição que construímos (e continuamos a construir) nessa área. Ainda que uma tal divisão seja imperfeita, pois vários dos trabalhos apresentados transitam entre essas diversas classificações, ela será útil na categorização geral.

O texto está organizado da seguinte maneira: no Capítulo 1 expomos nossas contribuições acerca de espinores escuros e exóticos com mais foco nos aspectos algébricos para o estabelecimento dos mesmos. Essa exposição será composta de quatro trabalhos cujo mote central é formal. O Capítulo 2 será destinado a contribuições voltadas a teoria de partículas. Serão três trabalhos, com propostas de estudos específicos de sinais em aceleradores e uma aplicação, com o cálculo da radiação Hawking. O Capítulo 3 fica ao encargo de estudos de alguns efeitos dos espinores escuros em Cosmologia. Veremos também três trabalhos, sendo dois eminentemente de cunho cosmológico e outro onde propomos um modelo sigma para tais férmions, estudando alguma aplicação em cosmologia. Uma vez que parte desse texto pretende ter um aspecto crítico, reservamos o capítulo final para tal fim.

Capítulo 2

Aspectos Formais

Em uma perspectiva bastante abarcativa e ambiciosa de trabalho, o estudo desde a álgebra espinorial até a extração de algum possível observável físico se mostra robusto. Gostaríamos de iniciar esse programa nos remetendo neste Capítulo à fundamentação algébrica do nosso estudo. Aqui veremos aspectos relevantes de espinores exóticos escuros, bem como a apreciação de um sistema físico levando a um tipo de espinor jamais utilizado. Finalizaremos estudando novos possíveis espinores escuros do ponto de vista algébrico.

Cada um desses tópicos será exposto em uma seção deste capítulo, que contará sempre com um texto introdutório discutindo aspectos de relevância do trabalho. Começamos com o que podemos entender como uma introdução às demais seções: um trabalho curto contendo uma descrição de vários temas a que este capítulo diz respeito.

Physics Letters B 718 (2013) 1519–1523

Contents lists available at SciVerse ScienceDirect

Physics Letters B

www.elsevier.com/locate/physletb

Unfolding physics from the algebraic classification of spinor fields

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ARTICLE INFO

Article history:
Received 11 September 2012
Received in revised form 10 December 2012
Accepted 11 December 2012
Available online 19 December 2012
Editor: S. Dodelson

ABSTRACT

After reviewing the Lounesto spinor field classification, according to the bilinear covariants associated to a spinor field, we call attention and unravel some prominent features involving unexpected properties about spinor fields under such classification. In particular, we pithily focus on the new aspects — as well as current concrete possibilities. They mainly arise when we deal with some non-standard spinor fields concerning, in particular, their applications in physics.

1. Introduction

From the classical point of view, the definition of spinors is based upon irreducible representations of the group $Spin_+(p,q)$, where $p+q=n$ is the spacetime dimension. Due to the immediate physical interest, mainly the Minkowski spacetime $\mathbb{R}^{1,3}$ has being regarded since the 1920s. On the another hand, the representation space associated to an irreducible regular representation in a Clifford algebra is a minimal left ideal. Its elements are the so-called algebraic spinors. Another possible definition of a spinor, which is denominated operatorial, can be introduced from another representation — distinct of the regular representation — of a Clifford algebra, using the representation space associated to the even subalgebra. This definition is equivalent to the classical and algebraic ones, in particular in the cases of great interest for physical applications. The classical definition of spinor is the customary approach in several superb textbooks in physics, e.g., [1]. There is no damage in asserting that, in Minkowski spacetime, classical spinors are irreducible representations of the Lorentz group $Spin_+(1,3) \simeq SL(2,\mathbb{C})$. Notwithstanding, this paradigm severely restricts the analysis to the usual Dirac, Weyl, and Majorana spinors.

A new possibility involving the spinor fields classification was introduced by Lounesto [2], as a palpable paradigm shift. It is based upon the bilinear covariants and their underlying multivector structure. In particular, this classification evinces the existence of a new type of spinor field, the so-called flag-dipole spinor fields. Furthermore, it additionally presents another class of spinor fields (the flagpoles) that accommodates Elko spinor fields, which are prime candidates to the dark matter description [3]. They generalize Majorana spinor fields. As it is well known, any spin-half spinor field, that potentially describes the dark matter, respects the symmetries of the Poincaré group in the sense of Weinberg, if it is an element of a standard Wigner class of representations of the Poincaré group. As it will be reported, Elko spinor fields do not belong to the standard Wigner class. Among a significant amount of unexpected and interesting properties, it was recently demonstrated that the topological exotic spacetime structure can be probed uniquely by Elko spinor fields: they are, hence, suitable to investigate the eventual non-trivial topology of the universe [4]. By such exoticness, dynamical constraints converted into a dark spinor mass generation mechanism, with the encrypted VSR symmetries holding as well.

The aim of this work is to report some of the recent advances in this field of research, calling special attention to the interesting features associated to the new spinor fields appearing in the Lounesto's classification. In order to accomplish that, we organize this work as follows: in the next section we review the formal and necessary aspects regarding the Lounesto spinor classification. In Section 3, we explore some of the odd and captivating aspects associated to Elko and flag-dipole spinor fields. In the final section we conclude.

2. Classifying spinor fields

We start this section reviewing some indispensable preliminary concepts. For a deeper approach see, e.g., [5]. Consider the tensor algebra $T(V)=\bigoplus_{i=0}^{\infty} T^i(V)$, where V is a finite n-dimensional real vector space. Henceforth V is regarded as being the tangent space on a point on a manifold. Let $\Lambda^k(V)$ denote the antisymmetric k-tensors space, indeed the k-forms vector space. In this way

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0370-2693/$ – see front matter
http://dx.doi.org/10.1016/j.physletb.2012.12.026

$\Lambda(V) = \bigoplus_{k=0}^{n} \Lambda^k(V)$ is the space of the differential forms over V. For any $\psi \in \Lambda(V)$, the reversion is defined by $\tilde{\psi} = (-1)^{[k/2]}\psi$ (the integer part of m is denoted by $[m]$), which is an anti-automorphism in $\Lambda(V)$. Moreover, $\hat{\psi} = (-1)^k\psi$ denotes the graded involution, also called main automorphism. It is possible to use the metric $g : V^* \times V^* \to \mathbb{R}$ extended to the k-forms space, in order to define the left and right contractions. Hence, for $\psi = \bigwedge_{i=1}^{p} \mathbf{u}^i \equiv \mathbf{u}^1 \wedge \cdots \wedge \mathbf{u}^p$ and $\phi = \bigwedge_{j=1}^{r} \mathbf{v}^r$, with $\mathbf{u}^i, \mathbf{v}^j \in V^*$, the extension of g to $\Lambda(V)$ reads $g(\psi, \phi) = \det(g(\mathbf{u}^i, \mathbf{v}^j))$ for $p = r$, and zero otherwise. Now one defines the left contraction by

$$g(\psi \lrcorner \varphi, \chi) = g(\varphi, \tilde{\psi} \wedge \chi), \quad \text{for } \psi, \varphi, \chi \in \Lambda(V). \tag{1}$$

For $\mathbf{v} \in V$, the Leibniz rule for the contraction is

$$\mathbf{v} \lrcorner (\psi \wedge \varphi) = (\mathbf{v} \lrcorner \psi) \wedge \varphi + \hat{\psi} \wedge (\mathbf{v} \lrcorner \varphi) \tag{2}$$

respectively. The Clifford product between $\mathbf{v} \in V$ and $\chi \in \Lambda(V)$ is $\mathbf{v}\chi = \mathbf{w} \wedge \chi + \mathbf{v} \lrcorner \chi$ and the pair $(\Lambda(V), g)$, endowed with the Clifford product, is denoted by $Cl(V, g)$ ($Cl_{p,q}$ is a notation that shall be reserved to the Clifford algebra when $V \simeq \mathbb{R}^{p,q}$).

In order to properly revisit the bilinear covariants let us fix the gamma matrices notation. All the formalism in representation independent, and hence we use hereon the Weyl (or chiral) representation of γ^μ: $\gamma_0 = \gamma^0 = \begin{pmatrix} \mathbb{O} & \mathbb{I} \\ \mathbb{I} & \mathbb{O} \end{pmatrix}$, $\gamma_k = -\gamma^k = \begin{pmatrix} \mathbb{O} & \sigma_k \\ -\sigma_k & \mathbb{O} \end{pmatrix}$, where $\mathbb{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\mathbb{O} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and the σ_i are the Pauli matrices. Moreover $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$. All the spinor fields in this work are placed in the Minkowski spacetime $(M \simeq \mathbb{R}^{1,3}, \eta, D, \tau, \uparrow)$, where $\eta = \mathrm{diag}(1, -1, -1, -1)$ is a metric which has a compatible (Levi-Civita) connection D associated. Besides, M has spacetime orientation induced by the volume element τ as well as time orientation denoted by $\uparrow$. We denote by $\{x^\mu\}$ global coordinates, in terms of which an inertial frame — a section of the frame bundle $\mathbf{P}_{SO_{1,3}}(M)$ — reads $\mathbf{e}_\mu = \partial/\partial x^\mu$.

At this point we recall that classical spinor fields are sections of the vector bundle $\mathbf{P}_{Spin_{1,3}} \times \mathbb{C}^2$, where the specific representation of $SL(2, \mathbb{C}) \simeq Spin_{1,3}$ in $\mathbb{C}^2$ is implicit. In this framework, the bilinear covariants associated to a spinor field $\psi \in \mathbf{P}_{Spin_{1,3}} \times \mathbb{C}^2$ are sections of $\Lambda(TM)$ into the Clifford bundle of multiform fields, given by

$$\begin{aligned}
&\sigma = \psi^\dagger\gamma_0\psi, \qquad \mathbf{J} = J_\mu\theta^\mu = \psi^\dagger\gamma_0\gamma_\mu\psi\theta^\mu, \\
&\mathbf{S} = S_{\mu\nu}\theta^{\mu\nu} = \frac{1}{2}\psi^\dagger\gamma_0 i\gamma_{\mu\nu}\psi\theta^\mu \wedge \theta^\nu, \\
&\mathbf{K} = K_\mu\theta^\mu = \psi^\dagger\gamma_0 i\gamma_{0123}\gamma_\mu\psi\theta^\mu, \qquad \omega = -\psi^\dagger\gamma_0\gamma_{0123}\psi,
\end{aligned} \tag{3}$$

where $\{\theta^\mu\}$ is the dual basis of $\{\mathbf{e}_\mu\}$. The bilinear covariants obey quadratic equations, the so-called Fierz–Pauli–Kofink identities [2]

$$\begin{aligned}
&\mathbf{J} \llcorner \mathbf{K} = 0, \qquad \mathbf{J}^2 = \omega^2 + \sigma^2, \\
&\mathbf{J} \wedge \mathbf{K} = -(\omega + \sigma\gamma_{0123})\mathbf{S}, \qquad \mathbf{K}^2 = -\mathbf{J}^2,
\end{aligned} \tag{4}$$

which are particularly interesting in what follows. The Fierz aggregate Z is defined by

$$Z = \sigma + \mathbf{J} + i\mathbf{S} - i\gamma_{0123}\mathbf{K} + \gamma_{0123}\omega. \tag{5}$$

Eqs. (3) may be recast in terms of Z, yielding

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$$\begin{aligned}
&Z^2 = 4\sigma Z, \qquad Z\gamma_\mu Z = 4J_\mu Z, \qquad Zi\gamma_{\mu\nu}Z = 4S_{\mu\nu}Z, \\
&Z\gamma_{0123}Z = -4\omega Z, \qquad Zi\gamma_{0123}\gamma_\mu Z = 4K_\mu Z.
\end{aligned} \tag{6}$$

Therefore, it is possible to categorize different spinor fields by different Z's, or similarly by distinct bilinear covariants. The Lounesto spinor field classification provides the following spinor field classes [2]:

1) $\sigma \neq 0, \quad \omega \neq 0$; 4) $\sigma = 0 = \omega, \quad \mathbf{K} \neq 0, \quad \mathbf{S} \neq 0$;
2) $\sigma \neq 0, \quad \omega = 0$; 5) $\sigma = 0 = \omega, \quad \mathbf{K} = 0, \quad \mathbf{S} \neq 0$;
3) $\sigma = 0, \quad \omega \neq 0$; 6) $\sigma = 0 = \omega, \quad \mathbf{K} \neq 0, \quad \mathbf{S} = 0$.

The first three classes are composed by Dirac spinor fields and it is implicit that in this case $\mathbf{J}, \mathbf{K}, \mathbf{S} \neq 0$. In particular, for a Dirac spinor fields describing an electron, $\mathbf{J}$ is a future-oriented timelike current vector providing the current of probability; $\mathbf{S}$ is the distribution of intrinsic angular momentum, and the spacelike vector $\mathbf{K}$ is associated to the direction of the electron spin.

A Majorana spinor field belongs to the class (5), while Weyl spinor fields are in the class (6). Type-(4) spinor fields are the so-called flag-dipole spinor fields. Furthermore, if ψ is a typical Dirac spinor field and ζ is an arbitrary spinor field such that $\zeta^\dagger\gamma_0 \neq 0$, ψ is herewith proportional to $Z\zeta$, where Z is given by Eq. (5).

Before delving deeper into the investigation of some interesting outputs in this approach, let us first emphasize that there are no other possible classes for the spinor fields based on different bilinear covariants. In fact, when $\sigma \neq 0$ and/or $\omega \neq 0$, it implies that $\mathbf{S} \neq 0$ and $\mathbf{K} \neq 0$ — note that $J^0 > 0$ and hence $\mathbf{J}$ does not equal zero. Besides, the constraint $\omega = 0 = \sigma$ implies that $Z = \mathbf{J}(1 + i(\mathbf{s} + h\gamma_{0123}))$, where $(\mathbf{s} + h\gamma_{0123})^2 = -1$, $\mathbf{s}$ is a spacelike vector, and h a real number given by $h = \pm\sqrt{1 + \mathbf{s}^2}$. In this vein $\mathbf{J}(\mathbf{s} + h\gamma_{0123}) = \mathbf{S} + \mathbf{K}\gamma_{0123}$. It is useful to provide further features of type-(4) spinor fields. For flag-dipole spinor fields, Eq. (5) gives $Z = \mathbf{J} + i\mathbf{J}s - ih\gamma_{0123}\mathbf{J}$, where $s = \|\mathbf{s}\|$. It implies forthwith that $(1 + is - ih\gamma_{0123})Z = 0$, and taking into account that $\mathbf{J}^2 = 0$ for type-(4) spinor fields, Z is shown to be Clifford multivector satisfying $Z^2 = 0$. Such spinor fields were widely investigated in [15] in a more topological geometric context, as well as some interesting applications.

The bilinear covariant $\mathbf{S}$ in (3) is given by $\mathbf{S} = \mathbf{J} \wedge \mathbf{s}$. For type-(4) spinor fields the real coefficient satisfies $h \neq 0$. Lounesto shows that either $\mathbf{J}^2 = 0$ or $(s - ih\gamma_{0123})^2 = -1$. The helicity h relates $\mathbf{K}$ and $\mathbf{J}$ by $\mathbf{K} = h\mathbf{J}$. The definition of helicity h in terms of bilinear covariants precedes and implies the definition of helicity in quantum mechanics, as well the equivalent relation for antiparticles [6]. Such approach further prov ides a straightforward form for the Hamiltonian describing the one-layer superconductor graphene, given by $\mathrm{Tr}(\gamma^5\mathbf{K}\gamma^0)$ [6].

3. Peculiar features

Roughly speaking, the framework of Lounesto's classification allows a twofold approach: on the one hand it is possible to study and classify new spinor fields recently discovered in the literature. Moreover, their geometric content can be explored and it sheds new light in the investigation on their physical content. We shall deal with this aspect in the following two subsections. On the another hand, it permits the exploration of genuinely different spinor fields, without any physical counterpart. We delve into this issue in the third subsection.

3.1. Elko spinor fields and its properties

Imagine a mass dimension one spinor field with 1/2 spin, obeying the Klein–Gordon, but *not the Dirac* field equations. Endowed with such predicates, it is indeed possible to call that spinor field as *strange*. In what follows, however, we shall argue that the strangeness of such spinor, the so-called Elko spinor, is far from pejorative.

Elko spinor fields are eigenspinors of the charge conjugation operator with eigenvalues ± 1. The plus [minus] sign stands for self-conjugate [anti-self-conjugate] spinors $\lambda^S(\mathbf{p})$ $[\lambda^A(\mathbf{p})]$. Elko spinor fields arise from the equation of helicity $(\sigma \cdot \hat{\mathbf{p}})\phi^{\pm}(\mathbf{0}) = \pm\phi^{\pm}(\mathbf{0})$ [3]. The four spinor fields are given by

$$\lambda^{S/A}_{\{\mp,\pm\}}(\mathbf{p}) = \sqrt{\frac{E+m}{2m}}\left(1 \mp \frac{\mathbf{p}}{E+m}\right)\lambda^{S/A}_{\{\mp,\pm\}}(\mathbf{0}), \tag{7}$$

where $\lambda^{S/A}_{\{\mp,\pm\}}(\mathbf{0}) = \begin{pmatrix} \pm i\Theta[\phi^{\pm}(\mathbf{0})]^* \\ \phi^{\pm}(\mathbf{0}) \end{pmatrix}$. The operator Θ denotes the Wigner's spin-1/2 time reversal operator. As $\Theta[\phi^{\pm}(\mathbf{0})]^*$ and $\phi^{\pm}(\mathbf{0})$ present opposite helicities, Elko cannot be an eigenspinor field of the helicity operator, and indeed carries both helicities. In order to guarantee an invariant real norm, as well as positive definite norm for two Elko spinor fields, and negative definite norm for the other two, the Elko dual is given by [3]

$$\overset{\neg}{\lambda}{}^{S/A}_{\{\mp,\pm\}}(\mathbf{p}) = \pm i\left[\lambda^{S/A}_{\{\pm,\mp\}}(\mathbf{p})\right]^{\dagger}\gamma^0. \tag{8}$$

It is useful to choose $i\Theta = \sigma_2$, as in [3], in such a way that it is possible to express $\lambda(\mathbf{p}) = \begin{pmatrix} \sigma_2\phi_L^*(\mathbf{p}) \\ \phi_L(\mathbf{p}) \end{pmatrix}$. The dual is defined in such way that the product $(\lambda^{S/A}_{\{\mp,\pm\}})^{\dagger}\zeta\lambda^{S/A}_{\{\pm,\mp\}}$ remains invariant under Lorentz transformations. This requirement implies $\zeta = \pm i\gamma^0$ for the Elko case, since it belongs to the *right* $\oplus$ *left* representation space [7]. Endowed with a new dual, Elko respects different orthonormality relations, which engenders non-standard spin sums. Following this reasoning it is possible to envisage the Elko nonlocality (see [7] for the details). Denoting by $\Lambda(\mathbf{x},t)$ the quantum field constructed out of Elko spinor fields as the expansion coefficients and $\Pi(\mathbf{x},t)$ its conjugate momentum, although the following property

$$\left\{\Lambda(\mathbf{x},t), \Lambda\left(\mathbf{x}',t\right)\right\} = 0 = \left\{\Pi(\mathbf{x},t), \Pi\left(\mathbf{x}',t\right)\right\} \tag{9}$$

holds, an unexpected anti-commutation relation is elicited [3]:

$$\left\{\Lambda(\mathbf{x},t), \Pi\left(\mathbf{x}',t\right)\right\} = i\int \frac{d^3p}{(2\pi)^2}\frac{1}{2m}e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')}2m\left[\mathbb{1} + G(\mathbf{p})\right]. \tag{10}$$

Here $\mathbb{1}$ stands for the identity matrix and $G(\mathbf{p}) = \gamma^5\gamma_\mu n^\mu$ is a factor arising from the spin sums. The vector $n^\mu = (0,\mathbf{n})$ defines some preferential direction [3], where $\mathbf{n} = \frac{1}{\sin\theta}\frac{d\hat{\mathbf{p}}}{d\phi}$. It was recently demonstrated [9], by explicitly calculation, that the integration over the second term of Eq. (10) equals zero. This is a crucial point, since this term decides the locality structure of the quantum field.

The mass dimension one related to such spinor fields severely suppresses the possible couplings to other fields of the standard model. In fact, if we keep in mind power counting arguments, Elko spinor fields may interact — in a perturbative renormalizable way — with itself and with a scalar (Higgs) field. Obviously, the former type of interaction means an unsuppressed quartic self interaction. At this point it is important to remark that this feature (quartic self interaction) is present in the dark matter characteristics observations [10]. Therefore Elko spinor fields seems to perform an adequate fermionic dark matter candidate.

It is worth notice that the appearance of the $G(\mathbf{p})$ function in the spin sums, however, shall not be underestimated. Its presence turns out to be impossible to conciliate Elko quantum field to the full Lorentz group. Nevertheless, Elko fields are, in fact, a spinor representation under the $SIM(2)$ avatar [11] of Very Special Relativity (VSR) [12]. The group $SIM(2)$ is the largest possible subgroup of VSR which is necessary to define a quantum theory when parity symmetry is violated. Hence, understanding Elko as a dark matter prime candidate, it may signalize that in the dark matter sector the Lorentz group may not be the underlying relevant group. Indeed, using the Lounesto framework previously outlined, Elko are classified as type-(5) spinor fields, a generalization of Majorana spinor fields carrying both helicities [13]. As mentioned in the Introduction, Lounesto classification goes beyond the standard classification by irreducible representations of the Lorentz group $Spin_+(1,3)$. From this perspective, it is quite conceivable that the quantum fields, constructed out from expansion coefficients which do not belong to Lorentz representation, do not respect Lorentz symmetries themselves.

3.2. The usefulness of topologically exotic terms

Among an extended inventory of relevant new physical possibilities arising from the use of the non-standard spinor fields, we can branch the role of Elko spinor fields as a detector of exotic spacetime structures [4]. If the base manifold M upon which the theory is built is simply connected, then the first homotopy group $\pi_1(M)$ is well known to be trivial. In this case, supposing that M satisfies the assumptions in the Geroch theorem [14], there exists merely one possible spin structure. Consequently, the spin-Dirac operator in the formalism is the standard one. Notwithstanding, when non-trivial topologies on M are regarded, there is a non-trivial line bundle on M. The set of line bundles and the set of inequivalent spin structures are labeled by elements of the cohomology group $H^1(M,\mathbb{Z}_2)$ — the group of the homomorphisms of $\pi_1(M)$ into $\mathbb{Z}_2$. In this regard, there are several globally different spin structures arising from the different (and inequivalent) patches of the local coverings. The spin-Dirac operator has in this case an additional term, essentially a one-form field, that reflects the non-trivial topology. Spinor fields associated to these inequivalent spin structures are called *exotic* spinor fields.

Let us make those considerations more precise. Throughout this section we denote by $Spin_{1,3}$ and $SO_{1,3}$ the components of such groups connected to the identity, for the sake of conciseness. Given the fundamental map, in fact a two-fold covering relating the orthonormal coframe bundle and the spinor bundle[1] $s : P_{Spin_{1,3}}(M) \to P_{SO_{1,3}}(M)$, a spin structure on M is a principal fiber bundle $\pi_s : P_{Spin_{1,3}}(M) \to M$ satisfying: (i) $\pi(s(p)) = \pi_s(p)$ for every point p of $P_{Spin_{1,3}}(M)$, where π is the projection of $P_{SO_+(1,3)}(M)$ on M, and (ii) $s(p\phi) = s(p)Ad_\phi$. Here given $\phi \in Spin_{1,3}(M)$, we have $Ad_\phi(\kappa) = \phi\kappa\phi^{-1}$, for all $\kappa \in Cl_{1,3}$. A spin structure $P := (P_{Spin_{1,3}}(M), s)$ exists solely when the second Stiefel–Whitney class satisfies specific criteria. To our presentation, however, it is remarkable that if $H^1(M,\mathbb{Z}_2)$ is not trivial, then the spin structure is not uniquely defined. Two spin structures, say P and $\tilde{P}$, are said to be equivalent if there exists a map $\chi : P \to \tilde{P}$ compatible with s and $\tilde{s}$; they are said to be inequivalent otherwise. Given an arbitrary spinor field $\psi \in \sec P_{Spin_{1,3}}(M) \times \mathbb{C}^4$, where "sec" means "section of", to each element of the non-trivial $H^1(M,\mathbb{Z}_2)$ one can associate a Dirac operator ∇. This construction provides an one-to-one correspondence between elements of $H^1(M,\mathbb{Z}_2)$ and inequivalent spin structures (for more details see [8,4,14]).

A crucial difference between the exotic and the standard spinor field is the action of the Dirac operator on exotic spinor fields. In a non-trivial topology scenario, the Dirac operator changes by an additional one-form field, which is a manifestation of the non-trivial

[1] Let $P_{SO_{1,3}}(M)$ denote the orthonormal coframe bundle, that always exist on spin manifolds. Sections of $P_{SO_{1,3}}(M)$ are orthonormal coframes, and sections of $P_{Spin_{1,3}}(M)$ are also orthonormal coframes such that although two coframes differing by a 2π rotation are distinct, two coframes differing by a 4π rotation are identified.

topology. The exotic structure endows the Dirac operator with an additional term given by $a^{-1}(x)\,da(x)$, where $x \in M$ and d denotes the exterior derivative operator. The term $\frac{1}{2i\pi}a^{-1}(x)\,da(x)$ is real, closed, and defines an integer Cěch cohomology class [16]. Using the relation between the Cěch and the de Rham cohomologies, it follows that

$$\oint \frac{1}{2i\pi}a^{-1}(x)\,da(x) \in \mathbb{Z}. \tag{11}$$

When Dirac spinor fields are regarded, the exotic term can be absorbed into a new shifted potential $A \mapsto A + \frac{1}{2i\pi}a^{-1}(x)\,da(x)$: the exotic term may be understood as an external electromagnetic potential that is summed to the physical electromagnetic potential, which plays the role of a disguise for the exotic term. In this way the exotic spacetime structures cannot be probed by Dirac spinor fields, which perceive the exotic term as an effective electromagnetic potential.

From the perspective of Elko spinor fields, however, the situation changes drastically. The reason is that the spinor field discussed in the previous section is an eigenspinor of the charge conjugation operator. Therefore it does not carry local $U(1)$ charge of the standard type. Hence, any type of extra term present in the Dirac operator cannot be absorbed into the electromagnetic potential. As it is extensively discussed in [14], the exotic term may be expressed as $\frac{a(x)}{\sqrt{2\pi}} = \exp(i\theta(x)) \in U(1)$. It yields

$$\begin{aligned}\frac{1}{2\pi}a^{-1}(x)\,da(x) &= \exp\big(-i\theta(x)\big)\big(i\gamma^{\mu}\nabla_{\mu}\theta(x)\big)\exp\big(i\theta(x)\big)\\ &= i\gamma^{\mu}\partial_{\mu}\theta(x).\end{aligned} \tag{12}$$

Now, making the conceivable exigency that the exotic Dirac operator must be considered the square root of the Klein–Gordon operator, we have[2]

$$\begin{aligned}&\big[i\gamma^{\mu}(\nabla_{\mu}+\partial_{\mu}\theta) \pm m\big]\big[i\gamma^{\nu}(\nabla_{\nu}+\partial_{\nu}\theta) \mp m\big]\lambda\\ &\quad = \big(g^{\mu\nu}\nabla_{\mu}\nabla_{\nu} + m^2\big)\lambda = 0.\end{aligned} \tag{13}$$

Therefore, the corresponding Klein–Gordon equation for the exotic Elko spinor field reads

$$\big(\Box + m^2 + g^{\mu\nu}\nabla_{\mu}\nabla_{\nu}\theta + \partial^{\mu}\theta\nabla_{\mu} + \partial^{\mu}\theta\partial_{\mu}\theta\big)\lambda = 0. \tag{14}$$

Finally, in order to have the Klein–Gordon propagator for the exotic Elko, as in the standard one, it follows from Eq. (14) that

$$\big(\Box\theta(x) + \partial^{\mu}\theta(x)\nabla_{\mu} + \partial^{\mu}\theta(x)\partial_{\mu}\theta(x)\big)\lambda = 0. \tag{15}$$

The result encoded in Eq. (15) makes Elko spinor field a very useful tool to explore unusual topologies in many contexts. Indeed Eq. (15) asserts that the Elko spinor structure constrains the exotic term related to the non-trivial spacetime topology. The possibility of extracting information from the subjacent topology without using any additional (sometimes ill defined) shifted potentials is, in fact, quite attractive. Eq. (15) further encompasses the relationship between gravitational sources induced by exotic topologies. Recently the combined action of a spinor field coupled to the gravitational field was obtained in [17]. Furthermore, Eq. (15) complies with the differential-topological restrictions on the spacetime for the evolution of our Universe. The differential-geometric description of matter by differential structures of spacetime might leads to a unifying model of matter, dark matter and dark energy. Indeed, by taking into account exotic differential structures, it may be the source of the observed anomalies without modifying the Einstein equations or introducing unusual types of matter, as a vast resource of possible explanations for recently observed surprising astrophysical data at the cosmological scale, merely provided by differential topology [17].

Furthermore, such exoticness induces a dynamical mass which is embedded in the VSR framework [18]. It is accomplished by identifying the VSR preferential direction with a dynamical dependence on the *kink* solution of a $\lambda\phi^4$ theory, for a scalar field ϕ. The exotic term $\partial_{\mu}\theta$ is chosen to be $v_{\mu}\phi$, where v_{μ} provides a preferential direction, an inherent preferred axis — along which Elko is local. This is solely one among various possible scenarios, using exotic couplings among dark spinor fields and scalar field topological solutions [18].

3.3. The appearance of new spinors

In the specific context of $f(R)$-cosmology, it was recently reported a solution for the Dirac equation with torsion, considering Bianchi type-I cosmological models [19]. The gravitational dynamics of the theory may be described by the metric and its compatible connection, or alternatively by the tetrad field and the spin-connection as well. The equations of motion are

$$\begin{aligned}&f'(R)R_{\rho\sigma} - \frac{1}{2}f(R)g_{\rho\sigma} = \Sigma_{\rho\sigma},\\ &\frac{1}{2}\left(\frac{\partial f'(R)}{\partial x^{\alpha}} + S_{\alpha\gamma}{}^{\gamma}\right)\left(\delta^{\alpha}_{\sigma}\delta^{\beta}_{\rho} - \delta^{\alpha}_{\rho}\delta^{\beta}_{\sigma}\right) + S_{\rho\sigma}{}^{\beta} = f'(R)T_{\rho\sigma}{}^{\beta},\end{aligned}$$

where $R_{\rho\sigma}$ is the Ricci tensor and $T_{\rho\sigma}{}^{\beta}$ stands for the torsion tensor. The quantities $\sigma_{\rho\sigma}$ and $S_{\rho\sigma}{}^{\alpha}$ are the stress–energy and spin tensors of the matter fields. The energy–momentum tensor is given by $\Sigma_{\rho\sigma}$. The idea is to couple $f(R)$-gravity to spinor fields and to a spinless perfect fluid. These spinor fields are shown *not* to be Dirac spinor fields [20]. In addition the second equation of motion assents the existence of torsion even in the absence of spinor fields. Implementing all the necessary constraints, it is possible to show that the spinor solutions reads

$$\psi_1 = \frac{1}{\sqrt{2\tau}}\begin{pmatrix}\sqrt{A-B}\cos\zeta_1 e^{i\theta_1}\\ 0\\ 0\\ \sqrt{A+B}\sin\zeta_2 e^{i\theta_2}\end{pmatrix}, \tag{16}$$

$$\psi_2 = \frac{1}{\sqrt{2\tau}}\begin{pmatrix}0\\ \sqrt{A+B}\cos\zeta_1 e^{i\vartheta_1}\\ \sqrt{A-B}\sin\zeta_2 e^{i\vartheta_2}\\ 0\end{pmatrix}, \tag{17}$$

where A and B are constants, the angular functions have time dependence, and τ is defined as the product of the scale factors appearing in the Bianchi type-I model (not relevant to our purposes). The point to be stressed is that, after a tedious calculation, the bilinear covariants associated to ψ_1 and ψ_2 classify the spinor fields (16) as type-(4): legitimate flag-dipole spinor fields that are obtained when the Dirac equation with torsion is regarded in the $f(R)$-cosmological scenario [21]. It is the first time, up to our knowledge, that a physical solution corresponds to a type-(4) spinor.[3] Eq. (16) evinces a physical manifestation of

[2] Hereon we are not going to specify the different Elko types, which simplify the content of indexes in Eq. (13). Again, for a complete discussion, see [4].

[3] This fact is more remarkable than it may sound. Several spinor solutions are of the form presented in (16). Notwithstanding, after all, the class under Lounesto's classification appears to be other than type-(4). For instance, on p. 65 of [22] it is possible to find similar structured spinor fields. Twenty pages of calculations led the authors to the very exciting conclusion that they belong to the type-(4) set. After some ponderation, however, we were brought back to the Earth: professor Leite Lopes' book was not written using the Weyl representation!

type-(4), or flag-dipole, spinor fields according to Lounesto's classification.

We finalize this section by pointing out a provocative interpretation of the type-(4) spinor fields as manifested via Eq. (16). There is no quantum field constructed out yet with type-(4) spinor fields and it is certainly an interesting branch of research. In view of the analysis of Section 3.1, such a quantum field shall not respect Lorentz symmetry. From this perspective, it would be the darkest possible candidate to dark matter. Being more conservative, without making any reference to its possible quantum field, type-(4) spinor fields, as it appears, are also quite provocative. Usually, generalizations of General Relativity are studied to give account of cosmological problems, without appealing to the existence of dark matter, for instance. Nevertheless, as we have mentioned, type-(4) spinor fields appeared only in a (double) generalization of General Relativity. Moreover, the presence of torsion in an $f(R)$ gravity is crucial to the functional form of these spinor fields as explicit in (16). Hence, type-(4) spinor fields, an essentially dark spinor (we restrain to say dark matter), comes up in a far from usual gravitational theory, which is commonly investigated to preclude the necessity of "dark" objects.

4. Final remarks

A plethora of open questions still haunts (in particular) theoretical physicists. The non-standard spinor fields — both under Lounesto as well as Wigner classification — are an evidently useful alternative to pave the road to solve some questions, mainly in field theory and cosmology/gravitation. It brings some nice and unexpected properties, like the existence of fermions with mass dimension one and a subtle Lorentz symmetry breaking, for instance. Facing such paradigm shift seems to upheaval what we know already about field theory and the elementary particles description, which were restricted to Dirac, Majorana and Weyl spinor fields heretofore, in Minkowski spacetime. As we have shown, flag-dipole type-(4) spinor fields are physical solutions of the Dirac equation with torsion in the context of $f(R)$-cosmology. Furthermore, Elko spinor fields representing type-(5), abreast of Majorana spinor fields, are evinced to be prime candidates to describe dark matter. We moreover have introduced the exotic dark spinor fields, which dynamics constraints both the spacetime metric structure and the non-trivial topology of the universe. In particular, it brings exotic couplings among dark spinor fields and scalar field topological solutions. The topics here introduced are merely the tip of the iceberg, and there are more useful properties on spinor fields (and their application in physics) still to be explored.

Acknowledgements

The authors would like to thanks Prof. José Abdalla Helayël-Neto for the continuous motivation. R. da Rocha is grateful to Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq) grants 476580/2010-2 and 304862/2009-6 for financial support.

References

[1] S. Weinberg, The Quantum Theory of Fields. Vol. 1: Foundations, Cambridge Univ. Press, Cambridge, UK, 1995.
[2] P. Lounesto, Clifford Algebras and Spinors, 2nd ed., Cambridge Univ. Press, Cambridge, 2002.
[3] D.V. Ahluwalia, D. Grumiller, JCAP 0507 (2005) 012, arXiv:hep-th/0412080;
D.V. Ahluwalia, D. Grumiller, Phys. Rev. D 72 (2005) 067701, arXiv:hep-th/0410192.
[4] R. da Rocha, A.E. Bernardini, J.M. Hoff da Silva, JHEP 1104 (2011) 110, arXiv:1103.4759 [hep-th].
[5] R.A. Mosna, W.A. Rodrigues Jr., J. Math. Phys. 45 (2004) 2945, arXiv:math-ph/0212033.
[6] C.G. Boehmer, L. Corpe, J. Phys. A: Math. Theor. 45 (2012) 205206, arXiv:1204.0135 [math-ph].
[7] D.V. Ahluwalia, C.-Y. Lee, D. Schritt, T.F. Watson, Phys. Lett. B 687 (2010) 248, arXiv:0804.1854 [hep-th].
[8] T. Friedrich, Dirac Operators in Riemannian Geometry, Graduate Studies in Mathematics, vol. 25, AMS, Providence, USA, 2000.
[9] E. Capelas de Oliveira, W.A. Rodrigues Jr., arXiv:1210.7207 [math-ph].
[10] D.N. Spergel, P.J. Steinhardt, Phys. Rev. Lett. 84 (2000) 3760;
P.K. Samal, R. Saha, P. Jain, J.P. Ralston, Mon. Not. Roy. Astron. Soc. 396 (2009) 511, arXiv:0811.1639 [astro-ph];
M. Frommert, T.A. Ensslin, Mon. Not. Roy. Astron. Soc. 403 (2010) 1739, arXiv:0908.0453 [astro-ph].
[11] D.V. Ahluwalia, S.P. Horvath, JHEP 1011 (2010) 078, arXiv:1008.0436 [hep-ph].
[12] A.G. Cohen, S.L. Glashow, Phys. Rev. Lett. 97 (2006) 021601, arXiv:hep-ph/0601236.
[13] R. da Rocha, W.A. Rodrigues Jr., Mod. Phys. Lett. A 21 (2006) 65, arXiv:math-ph/0506075.
[14] R.P. Geroch, J. Math. Phys. 9 (1968) 1739;
R.P. Geroch, J. Math. Phys. 11 (1970) 343.
[15] R. da Rocha, J.M. Hoff da Silva, Adv. Appl. Clifford Algebras 20 (2010) 847, arXiv:0811.2717 [math-ph].
[16] S.J. Avis, C.J. Isham, Nucl. Phys. B 156 (1979) 441.
[17] T. Asselmeyer-Maluga, C.H. Brans, Gen. Rel. Grav. 34 (2002) 1767, arXiv:gr-qc/0110043;
C.H. Brans, Class. Quant. Grav. 11 (1994) 1785, arXiv:gr-qc/9404003;
C.H. Brans, J. Math. Phys. 35 (1994) 5494, arXiv:gr-qc/9405010.
[18] A.E. Bernardini, R. da Rocha, Phys. Lett. B 717 (2012) 238, arXiv:1203.1049 [hep-th].
[19] S. Vignolo, L. Fabbri, R. Cianci, J. Math. Phys. 52 (2011) 112502, arXiv:1106.0414 [gr-qc].
[20] L. Fabbri, S. Vignolo, Class. Quant. Grav. 28 (2011) 125002, arXiv:1012.1270 [gr-qc].
[21] R. da Rocha, R.T. Cavalcanti, J.A. Silva-Neto, L. Fabbri, J.M. Hoff da Silva, in preparation.
[22] J. Leite Lopes, Introducción a la Electrodinámica Cuántica, Ed. Trillas, México, 1977.

2.1 Espinores Exóticos

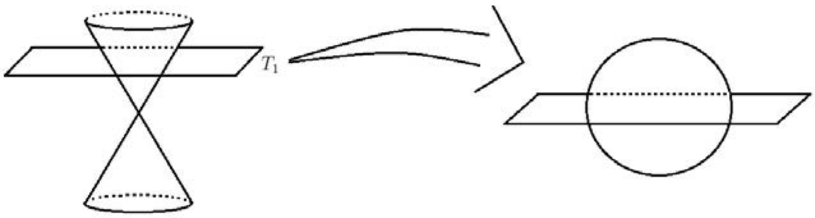
Um dos tópicos abordados no trabalho introdutório diz respeito à relevância do estudo de espinores escuros exóticos. Sobre espinores escuros, e suas propriedades definidoras por assim dizer, falaremos detidamente nos capítulos ulteriores. Assim, uma vez que a formulação para a compreensão das estruturas exóticas faz-se a mesma para todos os espinores vamos nos ater a apresentá-la, chamando a atenção, posteriormente, às peculiariedades dos espinores escuros exóticos.

No trabalho que reproduzimos nesta seção há uma abordagem bastante formal e sólida da adequação de espinores exóticos ao caso de espinores escuros, com um posterior vislumbre de aplicação em cosmologia. Gostaríamos aqui, entretanto, de aproveitar o ensejo para realizar uma introdução menos precisa, porém mais intuitiva, da noção de espinor exótico. Também, devido ao fato dos termos de exoticidade espinorial serem pouco vistos na literatura corrente, estenderemo-nos um pouco mais na sua apresentação.

Comecemos por trabalhar o conceito de espinor do ponto de vista puramente geométrico. Devido ao caráter pseudo-euclideano do espaço-tempo sabemos haver vetores tipo-luz que, sob atuação da métrica, levam ao conceito de cone-de-luz[1]. Considerando uma intersecção de um dado hiperplano $(T_1 = 1, X, Y, Z)$ (com $T_1 = 1$ por simplicidade) com o cone de luz, temos como resultado uma casca esférica de raio unitário, a esfera de Riemann (ver figura 2.1). Em seguida, consideremos um mapa injetivo associando a cada ponto na esfera um dado ponto em um plano complexo que intersepta a esfera em $Z = 0$. Essa é a chamada projeção estereográfica. Nessa projeção, as coordenadas (X, Y, Z) na esfera podem ser descritas por

[1] A abordagem que descreveremos é válida para qualquer tipo de vetor, mas para os tipo-luz ela é mais simples.

Figura 2.1: Origem da esfera de Riemann.



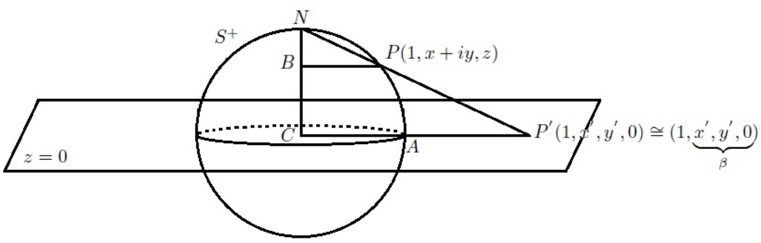
um número complexo $\beta = X' + iY'$. A figura 2.2 mostra como podemos construir o mapeamento a partir dos triângulos $P'CN$ e PBN de modo que

$$\beta = \frac{X - iY}{1 - Z}.$$

Entretanto para que se possa descrever o polo norte ($\beta = \infty$) é conveniente se associar aos pontos da esfera não apenas um único complexo, mas um par (η,ξ) tal que $\beta = \xi/\eta$. Desse modo, o polo norte é obtido pela coordenada

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Figura 2.2: Construção do mapeamento.



O ponto de vista formal que queremos apreciar nesta seção pode agora ser anunciado: espinores são, de fato, as coordenadas projetivas da projeção estereográfica de uma seção do cone de luz (com a ressalva da nota de rodapé da página 13) no plano complexo. É uma questão de simples álgebra agora se ver que

$$X = \frac{\xi\bar{\eta} + \eta\bar{\xi}}{\xi\bar{\xi} + \eta\bar{\eta}}, \qquad Y = \frac{\xi\bar{\eta} - \eta\bar{\xi}}{i(\xi\bar{\xi} + \eta\bar{\eta})}, \qquad Z = \frac{\xi\bar{\xi} - \eta\bar{\eta}}{\xi\bar{\xi} + \eta\bar{\eta}},$$

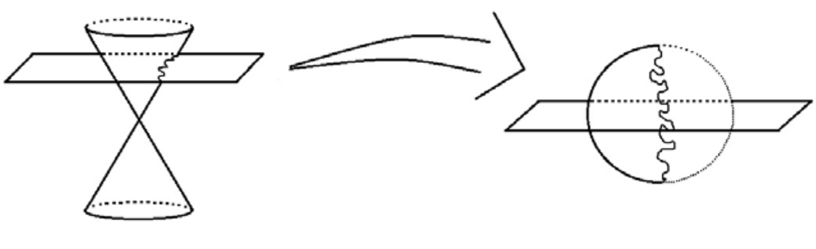
onde a barra indica conjugação. Nota-se agora o caráter especial de "raíz quadrada da geometria" atribuído aos espinores. A concepção padrão de entendimento de espinores como elementos que carregam representações irredutíveis do grupo de Lorentz pode ser obtida da análise acima como se segue: considere um ponto dado por $(1,X,Y,Z)(\xi\bar{\xi} - \eta\bar{\eta})/\sqrt{2}$. Uma transformação de $SL(2,\mathbb{C})$ nas coordenadas do ponto (equivalentemente, em (ξ, η)) deixa invariante o determinante

$$\det\left[\begin{pmatrix}\xi\\ \eta\end{pmatrix}(\bar{\xi}\ \ \bar{\eta})\right],$$

que, traduzido em termos das coordenadas do espaço-tempo, nada mais é do que a métrica de Minkowski.

Voltando ao tópico central de espinores exóticos, notemos que a existência de "buracos" no espaço-tempo (levando a uma topologia não-trivial) inviabiliza a concepção usual de espinores (ver Figura 2.3). A topologia não-trivial é refletida

Figura 2.3: Visualização da topologia não-trivial na estrutura espinorial.



(dentre outros efeitos) por um primeiro grupo de homotopia não-trivial do espaço-tempo $\pi_1(M)$. Por outro lado o grupo de homomorfismos de $\pi_1(M)$ em $\mathbb{Z}_2$ rotula os diferentes (e não equivalentes) cobrimentos locais necessários para se contornar a região com o "buraco". Esse grupo de homomorfismos é o primeiro grupo de cohomologia do espaço-tempo, e sua não trivialidade é herdada do fato de $\pi_1(M)$ ser não-trivial. Assim a topologia não-trivial dá origem a cobrimentos inequivalentes, que por sua vez levam a projeções estereográficas também inequivalentes e, portanto, espinores diferentes surgem. Dá-se assim origem aos espinores exóticos. Por fim, enfatizamos que a única diferença na dinâmica de ambos os espinores se dá na conexão relacionada ao espinor exótico, que deve levar em conta a topologia não-trivial.

Published for SISSA by Springer

Received: March 25, 2011
Accepted: April 13, 2011
Published: April 26, 2011

Exotic dark spinor fields

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Abstract: Exotic dark spinor fields are introduced and investigated in the context of inequivalent spin structures on arbitrary curved spacetimes, which induces an additional term on the associated Dirac operator, related to a Čech cohomology class. For the most kinds of spinor fields, any exotic term in the Dirac operator can be absorbed and encoded as a shift of the electromagnetic vector potential representing an element of the cohomology group $H^1(M,\mathbb{Z}_2)$. The possibility of concealing such an exotic term does not exist in case of dark (ELKO) spinor fields, as they cannot carry electromagnetic charge, so that the full topological analysis must be evaluated. Since exotic dark spinor fields also satisfy Klein-Gordon propagators, the dynamical constraints related to the exotic term in the Dirac equation can be explicitly calculated. It forthwith implies that the non-trivial topology associated to the spacetime can drastically engender — from the dynamics of dark spinor fields — constraints in the spacetime metric structure. Meanwhile, such constraints may be alleviated, at the cost of constraining the exotic spacetime topology. Besides being prime candidates to the dark matter problem, dark spinor fields are shown to be potential candidates to probe non-trivial topologies in spacetime, as well as probe the spacetime metric structure.

Keywords: Cosmology of Theories beyond the SM, Differential and Algebraic Geometry, Topological Field Theories

ArXiv ePrint: 1103.4759

doi:10.1007/JHEP04(2011)110

JHEP04(2011)110

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1 Introduction

ELKO — *Eigenspinoren des Ladungskonjugationsoperators* — spinor fields[1] describe a non-standard Wigner class of fermions, for which charge conjugation and parity are commuting operators, rather than anticommuting ones [1–4]. They support two types of dispersion relations, accomplish dual-helicity eigenspinors of the spin-$1/2$ charge conjugation operator, and carry mass dimension one, besides having non-local properties. At low-energy limits, ELKO behaves as a representation of the Lorentz group through the setup of a preferred frame related to its wave equation [3–6]. Ahluwalia-Khalilova and Grumiller embedded ELKO [1] into the quantum field theory, from which large applications in cosmology and gravity can be outlined. The corresponding ELKO Lagrangian neither predicts interactions with Standard Model (SM) fields nor shows coupling with gauge fields. Otherwise, exotic interactions with the Higgs boson can somehow be depicted in order to endow such spinor fields to be prime candidates to describe dark matter [7]. In particular, observational aspects on such a possibility has been proposed at LHC: dark (ELKO) spinor fields can be observed, at center of mass energy around 7 TeV and total luminosity from $1\,fb^{-1}$ to $10\,fb^{-1}$, indicating that the number of events is large enough to motivate a detailed analysis about ELKO particle at high energy experiments [8].

In addition, the embedding of dark spinor fields into the SM [10, 11] was introduced. ELKO spinor fields dominant interaction via the gravitational field makes them naturally dark, and recently [12–14] dark spinor fields were investigated in a cosmological setting, where interesting solutions and also models where the spinor is coupled conformally to gravity are provided. Some additional applications of ELKO spinor fields to cosmology

[1] ELKO is the German acronym for eigenspinors of the charge conjugation operator.

can be seen, e.g., in [15–27]. In particular, possible applications of ELKO spinor fields to more general $f(R)$ gravitational theories were accomplished in [28], and supersymmetric models concerning ELKO were introduced in [29].

The main aim of our manuscript is to investigate dark spinor fields in spacetimes with non-trivial topologies, in order to clarify how the dynamics of such dark spinor fields can induce constraints on the metric spacetime structure, as well as in the non-trivial topology itself. Physical applications of non-trivial topologies on spacetime, including thermodynamics, superconductivity, and condensed matter have been extensively explored in the last years. For instance, the quantum theory of fields propagating on a manifold M not simply connected was investigated in [30]. The existence of a nontrivial line bundle on a manifold M, whose sections may be regarded as a generalization of the concept of a scalar field, is inherent in multiply connected manifolds, which in addition can imply in the existence of inequivalent spin structures. It is well known that the set of real line bundles on M and the set of inequivalent spin structures are both labeled by elements of the cohomology group $H^1(M, \mathbb{Z}_2)$ — the group of homomorphisms of the fundamental group $\pi_1(M)$ into $\mathbb{Z}_2$. Namely, there are many globally different spin structures which arise from inequivalent patchings of the local double coverings, see, e.g., [31]. The use of generalized spin structures has been discussed in [32–36], in particular the ones considered by [37] for the case where the original fiber bundle has no spin structure at all.

Whereas the manifold M may have no spinor bundle — when it does not satisfy Geroch theorem hypotheses [38, 39] — it may have also many of them, which are split into equivalence classes — called spin structure. The formal aspects about the inequivalent spin structures in spacetime in terms of the different possible spin connections [30, 36, 40, 41] have been explored and reveal prominent physical applications. In an arbitrary spacetime that admits spin structure we delve into the problem of how to select a particular spin structure and its corresponding Dirac equation. In [30] it is discussed in what content the quantum field theory associated with a spinor field must involve some kind of "average" of all the spin connections.[2]

On simply connected spacetimes, the associated fundamental group satisfies $\pi_1(M) = 0$, therefore there is only one spin structure. Although compact simply connected 4-dimensional manifolds admitting a spin structure can be classified in terms of the Euler and Pontryagin numbers [42], multiply connected spacetimes are devoid in general of such a classification. As argued in [30, 40, 41], since Nature seems to use all mutually consistent degrees of freedom in a physical system, the Feynman path integral formalism, for instance, should also include multiply connected manifolds — which is among other prominent motivations to investigate quantum field theory on such spacetimes, which in addition are used in quantum gravity at both cosmological and Planck length scales [42, 43]. Multiply connected manifolds are also elicited in the theory of superconductivity. In fact, in [36] the inequivalent generalized spin structures are investigated in order to explain Cooper pairing phenomena in superconductors. In addition, such manifolds are used in the instanton compactification [44–46] of 4-dimensional Euclidean spacetimes and also in t' Hooft's

[2] As it must be summed over all topological sectors in instanton physics [30].

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treatment of confinement. In [47, 48], the finite-temperature stress-energy-momentum for a conformally coupled massive scalar field in multiply connected spaces was calculated and cogently investigated from a thermodynamic viewpoint. Also, in [49] and [50–54] covariant Casimir calculations were performed for the massless scalar field in several flat multiply connected spaces, expressing the stress expectation values as the coincidence limit of a bilinear operator acting on the Feynman propagator for the manifold, motivating the introduction of a robust mathematical approach [55, 56]. Finally, in [36] exotic spinor fields provides pure geometrical explanation of the charge dependence on the quantized flux and also the Joseph current in superconductivity.

To summarize, one of the main outstanding exotic spinor fields features is that they must be taken into account and employed in a variety of problems, wherein standard spinor fields cannot. For instance, when the vacuum polarization tensor of spinor electrodynamics is calculated [57, 58], it was found that the two types of spinor fields — standard and exotic — generate different vacuum polarization effects, which are physically inequivalent. When the effect of the vacuum polarization upon photon propagation is considered, it is shown that standard spinor fields give rise to non causal photon propagation, whereas exotic spinor fields do not. Even when the vacuum energy for a free spinor field is calculated, it is found that the exotic configuration gives rise to a vacuum state of lower energy than the standard one. These prominent features make exotic spinor fields as a broad audience candidate for concrete physical problems.

We shall address to the question about such inequivalent spin structures and their consequences to the coupled system of Dirac equations satisfied by the four types of dark spinor fields. The physical assumption that — under the exotic spin structure — the exotic dark spinor fields satisfy the Klein-Gordon propagator brings up some constraint on the metric spacetime structure, as well as in the exotic topology, both arbitrary *a priori*. The characterization of dark (ELKO) spinor fields, and its inherent analysis is obtained through the natural introduction — topologically impelled — of an exotic term in the Dirac operator that, contrary to the case of the Dirac spinor field, cannot be absorbed in any external electromagnetic vector field. For Dirac fields, such term can be concealed and encoded as a shift of the electromagnetic potential.[3] Therefore, besides addressing feasible aspect to the dark matter problem, dark spinor fields are also useful to probe non-trivial topological properties in spacetime.

The manuscript is organized as follows. After briefly presenting some algebraic preliminaries in section 2 regarding inequivalent spin structures, in section 3, the ELKO properties are introduced, together with the bilinear covariants that completely characterize a spinor field through the Fierz identities. In section 4 the exotic structure is introduced and the corresponding implications on the behaviour of ELKO are depicted. Dark spinor dynamics not only constrains the possibilities for the exotic topology but also induces constraints in the spacetime geometry through the exotic topology coming from the dynamics of dark spinor fields. We prove that it brings up some subtle consequences on the spacetime geometry. A brief summary of important useful results throughout the manuscript is described in the appendices A and B, and we draw our conclusions in section 5.

[3]Representing an element of the cohomology group $H^1(M, \mathbb{Z}_2)$.

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2 Preliminaries: exotic spin structures

In this section we review some results concerning general inequivalent spin structures in spinor bundles. For more details see the appendix.

One denotes by $(M, g, \nabla, \tau_g, \uparrow)$ the spacetime structure [59, 60]: M denotes a 4-dimensional manifold — which we shall assume as a compact, paracompact, pseudo-Riemannian manifold which is both space and time orientable and which admits spinor fields — g is the metric, ∇ denotes the connection associated to g, τ_g defines a spacetime orientation and $\uparrow$ refers a time orientation. As usual T^*M $[TM]$ denotes the cotangent [tangent] bundle over M, $F(M)$ denotes the principal bundle of frames, and $P_{\mathrm{SO}^e_{1,3}}(M)$ denotes the orthonormal coframe bundle. Such bundles do exist on spin manifolds. Sections of $P_{\mathrm{SO}^e_{1,3}}(M)$ are orthonormal coframes, and sections of $P_{\mathrm{Spin}^e_{1,3}}(M)$ are also orthonormal coframes such that although two coframes differing by a 2π rotation are distinct, two coframes differing by a 4π rotation are identified.

A spin structure on M consists of a principal fiber bundle $\pi_s : P_{\mathrm{Spin}^e_{1,3}}(M) \to M$, with group $\mathrm{Spin}^e_{1,3}$, and the fundamental map — indeed a two-fold covering

$$s : P_{\mathrm{Spin}^e_{1,3}}(M) \to P_{\mathrm{SO}^e_{1,3}}(M),$$

satisfying the following conditions:

1. $\pi(s(p)) = \pi_s(p)$, $\forall p \in P_{\mathrm{Spin}^e_{1,3}}(M)$; π is the projection map of $P_{\mathrm{SO}^e_{1,3}}(M)$ on M.

2. $s(p\phi) = s(p)\mathrm{Ad}_\phi$, $\forall p \in P_{\mathrm{Spin}^e_{1,3}}(M)$ and $\mathrm{Ad} : \mathrm{Spin}^e_{1,3} \to \mathrm{Aut}(\mathcal{C}\ell_{1,3})$, $\mathrm{Ad}_\phi : \Xi \mapsto \phi\Xi\phi^{-1} \in \mathcal{C}\ell_{1,3}$ [59].

Namely, the following diagram

$$\begin{tikzcd}
P_{\mathrm{Spin}^e_{1,3}}(M) \arrow[rr, "s"] \arrow[dr, "\pi_s"'] & & P_{\mathrm{SO}^e_{1,3}}(M) \arrow[dl, "\pi"] \\
& M &
\end{tikzcd}$$

commutes. The conditions for existence of a spin structure in a general manifold are discussed in [61, 62].

It is well known that a spin structure $(P_{\mathrm{Spin}^e_{1,3}}(M), s)$ exists if and only if the second Stiefel-Whitney class associated to M satisfies certain properties. If $H^1(M, \mathbb{Z}_2)$ is not trivial, the spin structure is not uniquely defined,[4] and all the other inequivalent spin structures can be provided from $(P_{\mathrm{Spin}^e_{1,3}}(M), s)$.

[4]Up to trivial bundle isomorphisms.

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Two spin structures $P{:=}(P_{\mathrm{Spin}^e_{1,3}}(M), s)$ and $\mathring{P} := ((\mathring{P}_{\mathrm{Spin}^e_{1,3}}(M), \mathring{s})$ are said to be equivalent if there exists a $\mathrm{Spin}^e_{1,3}$-equivariant map $\zeta : P \to \mathring{P}$ compatible with s and $\mathring{s}$:

$$\begin{tikzcd}
P \arrow[rr, "\zeta"] \arrow[dr, "s"'] & & \mathring{P} \arrow[dl, "\mathring{s}"] \\
& P_{\mathrm{SO}^e_{1,3}}(M) &
\end{tikzcd}$$

Now we briefly review some few definitions necessary to introduce exotic spinor fields. Spinor fields are sections of vector bundles associated with the principal bundle of spinor coframes. A complex spinor bundle for M is a vector bundle $S_c(M) = P_{\mathrm{Spin}^e_{1,3}}(M) \times_{\mu_c} \mathcal{M}_c$, where $\mathcal{M}_c$ is a complex left module for $\mathbb{C}\otimes\mathcal{C}\ell_{1,3} \simeq \mathrm{M}(4,\mathbb{C})$, and where μ_c is a representation of $\mathrm{Spin}^e_{1,3}$ in $\mathrm{End}(\mathcal{M}_c)$ given by left multiplication by elements of $\mathrm{Spin}^e_{1,3}$. When $\mathcal{M}_c = \mathbb{C}^4$ and μ_c the $D^{(1/2,0)} \oplus D^{(0,1/2)}$ representation of $\mathrm{Spin}^e_{1,3} \simeq \mathrm{SL}(2,\mathbb{C})$ in $\mathrm{End}(\mathbb{C}^4)$, we immediately recognize the usual definition of the covariant spinor bundle of M as given, e.g., in [61] and [62].

Classical spinor fields[5] carrying a $D^{(1/2,0)} \oplus D^{(0,1/2)}$, or $D^{(1/2,0)}$, or $D^{(0,1/2)}$ representation of $\mathrm{SL}(2,\mathbb{C})$ are sections of the vector bundle $P_{\mathrm{Spin}^e_{1,3}}(M) \times_\rho \mathbb{C}^4$, where ρ stands for the $D^{(1/2,0)} \oplus D^{(0,1/2)}$ (or $D^{(1/2,0)}$, or $D^{(0,1/2)}$) representation of $\mathrm{SL}(2,\mathbb{C})$ in $\mathbb{C}^4$. Other important spinor fields, like Weyl spinor fields are obtained by imposing some constraints on the sections of $P_{\mathrm{Spin}^e_{1,3}}(M) \times_\rho \mathbb{C}^4$, see, e.g., [70, 71] for details.

Two spin structures $(P_{\mathrm{Spin}^e_{1,3}}(M), s)$ and $(\mathring{P}_{\mathrm{Spin}^e_{1,3}}(M), \mathring{s})$ are respectively described by the maps h_{jk} and $\mathring{h}_{jk}$ from $U_i{\cap}U_j$ to $\mathrm{Spin}^e_{1,3}$, both satisfying eq. (A.1), and also the property $\varsigma \circ h_{jk} = a_{jk} = \varsigma \circ \mathring{h}_{jk}$. The following diagram illustrates such relations, summarizing what should be emphasized heretofore:

$$\begin{tikzcd}
U_i \cap U_j \subset M \arrow[rr, "h_{ij}"] \arrow[dr, "a_{ij}"] \arrow[dd, "\mathring{h}_{ij}"'] & & \mathrm{Spin}^e_{1,3} \arrow[dl, "\varsigma"'] \arrow[dr, hook] \arrow[dd] & \\
& \mathrm{SO}^e_{1,3} \arrow[dd, hook] & & P_{\mathrm{Spin}^e_{1,3}}(M) \arrow[dd, "s"] \\
\mathring{\mathrm{Spin}}^e_{1,3} \arrow[ur, "\varsigma"] \arrow[rr] & & \mathring{P}_{\mathrm{Spin}^e_{1,3}}(M) \arrow[ur, "\zeta"'] \arrow[dr, "\mathring{s}"'] & \\
& P_{\mathrm{SO}^e_{1,3}}(M) \arrow[rr, "id"] & & P_{\mathrm{SO}^e_{1,3}}(M)
\end{tikzcd}$$

[5]Quantum spinor fields are operator valued distributions, as well known. It is not necessary to introduce quantum fields in order to know the algebraic classification of ELKO spinor fields.

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Here another identical copy of $\mathrm{Spin}^e_{1,3}$ is denoted by $\mathring{\mathrm{Spin}}^e_{1,3}$, in order to become clearer the analysis about the inequivalent spin structures.

Now one defines a map c_{jk} by the relation $h_{ij}(x) = \mathring{h}_{ij}(x)c_{ij}$ such that $c_{ij} : U_i \cap U_j \to \ker \varsigma = \mathbb{Z}_2 \hookrightarrow \mathrm{Spin}^e_{1,3}$, satisfying $c_{ij} \circ c_{jk} = c_{ik}$. Such a map c_{ij} defines an 1-dimensional real bundle denoted [72]. Given the irreducible representation $\rho : \mathcal{C}\ell_{1,3} \to \mathrm{M}(4, \mathbb{C})$ in $P_{\mathrm{Spin}^e_{1,3}}(M) \times_\rho \mathbb{C}^4$, as the map $c_{ij}(x)$ is an element of $\mathbb{Z}_2$, it follows that $\rho(c_{ij}(x)) = \pm 1$, since ρ is faithful. When ρ is restricted to $\mathrm{Spin}^e_{1,3}$, it is called Dirac representation.

We assume as in [30, 36, 37, 40, 41] that there is a set of functions $\xi_i : U_i \to \mathbb{C}$ such that $\|\xi_i(x)\| = 1$, namely $\xi_i(x) \in \mathrm{U}(1)$, and

$$\xi_i(x)(\xi_j(x))^{-1} = \rho(c_{ij}(x)) = \pm 1. \tag{2.1}$$

In the case where the second integral cohomology $H^2(M, \mathbb{Z}_2)$ has no 2-torsion, such functions always do exist [30, 36, 40, 41, 72], and $\xi_i^2(x) = \xi_j^2(x)$, $x \in U_i \cap U_j$. Consequently the local functions ξ_i define a unique unimodular function $\xi : M \to \mathbb{C}$ such that for all $x \in U_i$ it follows that $\xi(x) = \xi_i^2(x)$.

Given now an arbitrary spinor field $\psi \in \sec P_{\mathrm{Spin}^e_{1,3}}(M) \times_\rho \mathbb{C}^4$, to each element of $H^1(M, \mathbb{Z}_2)$, associate a covariant derivative ∇. This construction provides indeed a one-to-one correspondence between elements of $H^1(M, \mathbb{Z}_2)$ and inequivalent spin structures.

A local component $\psi_i : U_i \to \mathbb{C}^4$ of a spinor field in $P_{\mathrm{Spin}^e_{1,3}}(M) \times_\rho \mathbb{C}^4$ is the unique function such that $\rho(\ell_i, \psi_i(x)) = \psi(x)$, given local sections $\ell_i : U_i \to (P_{\mathrm{Spin}^e_{1,3}}(M), s)$, we have the transition law

$$\psi_i(x) = \rho(h_{ij}(x))\psi_j(x), \qquad \text{where } x \in U_i \cap U_j.$$

A system of local sections $\mathring{\ell}_i : U_i \to \mathring{P}_{\mathrm{Spin}^e_{1,3}}(M)$ can be constructed from the standard ones ℓ_i in such a way that $s \circ \ell_i = \mathring{s} \circ \mathring{\ell}_i$, as presented in the following diagram:

$$\begin{tikzcd}
 & U_i \arrow[dl, "\mathring{\ell}_i"'] \arrow[d, "m_i"] \arrow[dr, "\ell_i"] & \\
\mathring{P}_{\mathrm{Spin}^e_{1,3}}(M) \arrow[r, "\mathring{s}"] & P_{\mathrm{SO}^e_{1,3}}(M) & P_{\mathrm{Spin}^e_{1,3}}(M) \arrow[l, "s"']
\end{tikzcd}$$

It enables the (exotic) local spinor field components to present the respective transition property

$$\mathring{\psi}_j(x) = \rho(\mathring{h}_{ij}) = \rho(h_{ij}(x))\rho(c_{ij}(x))\mathring{\psi}_i(x), \qquad \text{where } x \in U_i \cap U_j. \tag{2.2}$$

From eq. (2.1) it follows that $\rho(\xi_i) = \rho(c_{ij}(x))\rho(\xi_j)$ and if we compare it with eq. (2.2), it is clear that $\rho(\xi_i)\mathring{\psi}_i$ transforms as the local component ψ_i of $P_{\mathrm{Spin}^e_{1,3}}(M) \times_\rho \mathbb{C}^4$, which subsequently induces a bundle map

$$\begin{aligned} f : \mathring{P}_{\mathrm{Spin}^e_{1,3}}(M) \times_\rho \mathbb{C}^4 &\to P_{\mathrm{Spin}^e_{1,3}}(M) \times_\rho \mathbb{C}^4 \\ \mathring{\psi}_i &\mapsto f(\mathring{\psi}_i) := \rho(\xi_i)\mathring{\psi}_i = \psi_i \end{aligned} \tag{2.3}$$

such that

$$\overset{\circ}{\nabla}_X f(\overset{\circ}{\psi}) = f(\nabla_X \overset{\circ}{\psi}) + \frac{1}{2}(X \lrcorner (\xi^{-1} d\xi)) f(\overset{\circ}{\psi}) \tag{2.4}$$

holds for all sections $\psi \in P_{\mathrm{Spin}^e_{1,3}}(M) \times_\rho \mathbb{C}^4$ and all vector fields X. Details on how to derive eq. (2.4) are comprehensively given in, e.g., [30, 36, 37, 40, 41, 72, 73].

3 Dark (ELKO) spinor fields

This section is devoted to a brief review of the bilinear covariants through the programme introduced in [70, 71, 74]. The spinor fields classification is provided by a brief review of [10, 11, 74–76].

Given a spinor field $\psi \in \sec P_{\mathrm{Spin}^e_{1,3}}(M) \times_\rho \mathbb{C}^4$, the bilinear covariants are the following sections of $\Lambda(TM) = \oplus_{r=0}^4 \Lambda^r(TM) \hookrightarrow C\ell(M,g)$ [59, 77, 78]:

$$\begin{aligned} &\sigma = \psi^\dagger \gamma_0 \psi, \qquad \mathbf{J} = J_\mu \mathbf{e}^\mu = \psi^\dagger \gamma_0 \gamma_\mu \psi \mathbf{e}^\mu, \quad \mathbf{S} = S_{\mu\nu} \mathbf{e}^{\mu\nu} = \frac{1}{2} \psi^\dagger \gamma_0 i \gamma_{\mu\nu} \psi \mathbf{e}^\mu \wedge \mathbf{e}^\nu, \\ &\mathbf{K} = \psi^\dagger \gamma_0 i \gamma_{0123} \gamma_\mu \psi \mathbf{e}^\mu, \quad \omega = -\psi^\dagger \gamma_0 \gamma_{0123} \psi, \end{aligned} \tag{3.1}$$

with $\sigma, \omega \in \sec \Lambda^0(TM)$, $\mathbf{J}, \mathbf{K} \in \sec \Lambda^1(TM)$ and $\mathbf{S} \in \sec \Lambda^2(TM)$. In the formulæ appearing in eq. (3.1) the set $\{\gamma_\mu\}$ refers to the Dirac matrices in chiral representation (see eq. (3.3)). Also, $\{1, \mathbf{e}^\mu, \mathbf{e}^\mu\mathbf{e}^\nu, \mathbf{e}^\mu\mathbf{e}^\nu\mathbf{e}^\rho, \mathbf{e}^0\mathbf{e}^1\mathbf{e}^2\mathbf{e}^3\}$, where $\mu, \nu, \rho = 0, 1, 2, 3$, and $\mu < \nu < \rho$ is a basis for $C\ell(M,g)$, and $\{\mathbf{1}_4, \gamma_\mu, \gamma_\mu\gamma_\nu, \gamma_\mu\gamma_\nu\gamma_\rho, \gamma_0\gamma_1\gamma_2\gamma_3\}$ is a basis for $\mathrm{M}(4,\mathbb{C})$. In addition, these bases satisfy the respective Clifford algebra relations [70] $\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2g_{\mu\nu}\mathbf{1}_4$ and $\mathbf{e}^\mu\mathbf{e}^\nu + \mathbf{e}^\nu\mathbf{e}^\mu = 2g^{\mu\nu}$, where $\mathbf{1}_4$ is the identity matrix. When there is no opportunity for confusion we shall omit the $\mathbf{1}_4$ identity matrix in our formulæ. For the orthonormal covector fields $\mathbf{e}^\mu$ and $\mathbf{e}^\nu$, $\mu \neq \nu$, their Clifford product $\mathbf{e}^\mu\mathbf{e}^\nu$ is equal to the exterior product of those vectors, i.e., $\mathbf{e}^\mu\mathbf{e}^\nu = \mathbf{e}^\mu \wedge \mathbf{e}^\nu = \mathbf{e}^{\mu\nu}$. Also, for $\mu \neq \nu \neq \rho$, $\mathbf{e}^{\mu\nu\rho} = \mathbf{e}^\mu\mathbf{e}^\nu\mathbf{e}^\rho$, etc. More details on our notations, if needed, can be found in [59, 60].

In Minkowski spacetime, the case of the electron is described by Dirac spinor fields (classes 1, 2 and 3 below), $\mathbf{J}$ is a future-oriented timelike current vector which gives the current of probability, and $\mathbf{J}^2 = J_\mu J^\mu > 0$. Furthermore, for the case of Dirac spinor fields, the bivector $\mathbf{S}$ is associated with the distribution of intrinsic angular momentum, and the spacelike vector $\mathbf{K}$ is associated with the direction of the electron spin. For a detailed discussion concerning such entities, their relationships and physical interpretation, and generalizations, see, e.g., [70, 71, 77–79].

The bilinear covariants satisfy the Fierz identities[6] [70, 71, 77–79]

$$\mathbf{J}^2 = \omega^2 + \sigma^2, \quad \mathbf{K}^2 = -\mathbf{J}^2, \quad \mathbf{J} \llcorner \mathbf{K} = 0, \quad \mathbf{J} \wedge \mathbf{K} = -(\omega + \sigma\gamma_{0123})\mathbf{S}.$$

A spinor field such that not both ω and σ are null is said to be regular. When $\omega = 0 = \sigma$, a spinor field is said to be singular, and in this case the Fierz identities are in

[6] Given the spacetime metric g, it is possible to extend g to the exterior bundle $\Lambda(TM)$. Given $\psi = u^1 \wedge \cdots \wedge u^k$ and $\phi = v^1 \wedge \cdots \wedge v^l$, for $u^i, v^j \in \sec TM$, one defines $g(\psi,\phi) = \det(g(u^i, v^j))$ if $k = l$ and $g(\psi,\phi) = 0$ if $k \neq l$. Given $\psi, \phi, \xi \in \Lambda(TM)$, the left contraction is defined implicitly by $g(\psi \lrcorner \phi, \xi) = g(\phi, \tilde{\psi} \wedge \xi)$.

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general replaced by the more general conditions [79]

$$Z^2 = 4\sigma Z, \quad Z\gamma_\mu Z = 4J_\mu Z, \quad Zi\gamma_{\mu\nu}Z = 4S_{\mu\nu}Z, \quad Zi\gamma_{0123}\gamma_\mu Z = 4K_\mu Z, \quad Z\gamma_{0123}Z = -4\omega Z,$$

where $Z = \sigma + \mathbf{J} + i\mathbf{S} + i\mathbf{K}\gamma_{0123} + \omega\gamma_{0123}$.

Lounesto spinor field classification is given by the following spinor field classes [70, 71], where in the first three classes it is implicit that $\mathbf{J}, \mathbf{K}, \mathbf{S} \neq 0$:

1) $\sigma \neq 0, \quad \omega \neq 0.$

2) $\sigma \neq 0, \quad \omega = 0.$

3) $\sigma = 0, \quad \omega \neq 0.$

4) $\sigma = 0 = \omega, \quad \mathbf{K} \neq 0, \quad \mathbf{S} \neq 0.$

5) $\sigma = 0 = \omega, \quad \mathbf{K} = 0, \quad \mathbf{S} \neq 0.$

6) $\sigma = 0 = \omega, \quad \mathbf{K} \neq 0, \quad \mathbf{S} = 0.$

The current density $\mathbf{J}$ is always non-zero. Type 1, 2 and 3 spinor fields are denominated Dirac spinor fields for spin-1/2 particles and type 4, 5, and 6 are respectively called flag-dipole, flagpole and Weyl spinor fields. Majorana spinor fields are a particular case of a type 5 spinor field. It is worthwhile to point out a peculiar feature of types 4, 5 and 6 spinor fields: although $\mathbf{J}$ is always non-zero, $\mathbf{J}^2 = -\mathbf{K}^2 = 0$. It shall be seen below that the bilinear covariants related to an ELKO spinor field, satisfy $\sigma = 0 = \omega$, $\mathbf{K} = 0$, $\mathbf{S} \neq 0$ and $\mathbf{J}^2 = 0$.

Since Lounesto proved that there are no other classes based on distinctions among bilinear covariants, ELKO spinor fields must belong to one of the disjoint six classes. In [74] it is shown that ELKO spinor fields are indeed in class 5 above.

Some properties of dark (ELKO) spinor fields,[7] as introduced in [1, 2, 7] can be now briefly reviewed. An ELKO Ψ corresponding to a plane wave with momentum $p = (p^0, \mathbf{p})$ can be written, without loss of generality, as $\Psi(p) = \lambda(\mathbf{p})e^{\pm ip\cdot x}$ where

$$\lambda(\mathbf{p}) = \begin{pmatrix} i\Theta\phi^*(\mathbf{p}) \\ \phi(\mathbf{p}) \end{pmatrix}, \tag{3.2}$$

and given the rotation generators $\mathfrak{J}$, the Wigner's spin-1/2 time reversal operator Θ satisfies $\Theta\mathfrak{J}\Theta^{-1} = -\mathfrak{J}^*$. Hereon, as in [1], the Weyl representation of γ^μ is used

$$\gamma_0 = \gamma^0 = \begin{pmatrix} \mathbb{O} & \mathbb{I} \\ \mathbb{I} & \mathbb{O} \end{pmatrix}, \quad -\gamma_k = \gamma^k = \begin{pmatrix} \mathbb{O} & -\sigma_k \\ \sigma_k & \mathbb{O} \end{pmatrix}, \quad \gamma^5 = -i\gamma^0\gamma^1\gamma^2\gamma^3 = -i\gamma^{0123} = \begin{pmatrix} \mathbb{I} & \mathbb{O} \\ \mathbb{O} & -\mathbb{I} \end{pmatrix}, \tag{3.3}$$

[7] Hereon throughout the text the term dark spinor field and ELKO are alternatively used having the same meaning, since ELKO is a candidate to describe dark matter, as comprehensively proposed, derived, and investigated in, e.g., [1–4, 7, 8, 10–14, 19–24, 74, 75, 80–82]. We choose the acronym ELKO to denote field theoretical and more formal properties of such a spinor field, whereas the naming dark spinor fields shall be used hereon alternatively to ELKO, in order to present and investigate the potentially cosmological applications as well as its usefulness as an attempt to the dark matter problem.

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where $\mathbb{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\mathbb{O} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. The σ_i are the Pauli matrices.

ELKO spinor fields are eigenspinors of the charge conjugation operator C

$$C\lambda(\mathbf{p}) = \pm\lambda(\mathbf{p}), \qquad \text{for} \quad C = \begin{pmatrix} \mathbb{O} & i\Theta \\ -i\Theta & \mathbb{O} \end{pmatrix} K.$$

The operator K $\mathbb{C}$-conjugates 2-component spinor fields appearing on the right. The plus sign stands for self-conjugate spinor fields, $\lambda^S(\mathbf{p})$, while the minus yields anti self-conjugate spinor fields, $\lambda^A(\mathbf{p})$. Explicitly, the complete form of ELKO can be found by solving the equation of helicity $(\sigma \cdot \widehat{\mathbf{p}})\phi^{\pm} = \pm\phi^{\pm}$ in the rest frame and subsequently make a boost, to recover the result for any $\mathbf{p}$ [1]. Here $\widehat{\mathbf{p}} := \mathbf{p}/\|\mathbf{p}\|$. The boosted four spinor fields are

$$\lambda^{S/A}_{\{\mp,\pm\}}(\mathbf{p}) = \sqrt{\frac{E+m}{2m}}\left(1 \mp \frac{p}{E+m}\right)\lambda^{S/A}_{\{\mp,\pm\}}(\mathbf{0}), \quad \text{where} \quad \lambda^{S/A}_{\{\mp,\pm\}}(\mathbf{0}) = \begin{pmatrix} \pm i\Theta[\phi^{\pm}(\mathbf{0})]^* \\ \phi^{\pm}(\mathbf{0}) \end{pmatrix}. \tag{3.4}$$

One should notice that, since $\Theta[\phi^{\pm}(\mathbf{0})]^*$ and $\phi^{\pm}(\mathbf{0})$ have opposite helicities, ELKO cannot be an eigenspinor field of the helicity operator. The ELKO dual is given by [1]

$$\overset{\neg S/A}{\lambda}_{\{\mp,\pm\}}(\mathbf{p}) = \pm i\left[\lambda^{S/A}_{\{\pm,\mp\}}(\mathbf{p})\right]^{\dagger}\gamma^0. \tag{3.5}$$

Now let one denotes the eigenspinors of the Dirac operator for particles and antiparticles respectively by $u_{\pm}(\mathbf{p})$ and $v_{\pm}(\mathbf{p})$. The subindex $\pm$ regards the eigenvalues of the helicity operator $(\sigma \cdot \widehat{\mathbf{p}})$. The parity operator acts as

$$Pu_{\pm}(\mathbf{p}) = +\,u_{\pm}(\mathbf{p}), \qquad\qquad Pv_{\pm}(\mathbf{p}) = -\,v_{\pm}(\mathbf{p}),$$

which implies that $P^2 = \mathbb{I}$ in this case. The action of C on these spinors is given by

$$C(u_{\pm 1/2}(\mathbf{p})) = \mp v_{\mp}(\mathbf{p}), \qquad\qquad C(v_{\pm 1/2}(\mathbf{p})) = \pm u_{\mp 1/2}(\mathbf{p}). \tag{3.6}$$

which implies that $\{C, P\} = 0$.

On the another hand the parity operator P acts on ELKO by

$$P\lambda^{\mathrm{S}}_{\mp,\pm}(\mathbf{p}) = \pm\, i\,\lambda^{\mathrm{A}}_{\pm,\mp}(\mathbf{p})\,, \qquad P\lambda^{\mathrm{A}}_{\mp,\pm}(\mathbf{p}) = \mp\, i\,\lambda^{\mathrm{S}}_{\pm,\mp}(\mathbf{p}), \tag{3.7}$$

and it follows that $[C, P] = 0$.

Denoting [1] for Dirac spinor fields

$$u_+(\mathbf{p}) = d_1, \quad u_-(\mathbf{p}) = d_2, \quad v_+(\mathbf{p}) = d_3 \quad v_-(\mathbf{p}) = d_4,$$

and for the ELKO

$$\lambda^{\mathrm{S}}_{\{-,+\}}(\mathbf{p}) = e_1, \quad \lambda^{\mathrm{S}}_{\{+,-\}}(\mathbf{p}) = e_2, \quad \lambda^{\mathrm{A}}_{\{-,+\}}(\mathbf{p}) = e_3, \quad \lambda^{\mathrm{A}}_{\{+,-\}}(\mathbf{p}) = e_4,$$

it is possible to write ELKO as [80]

$$e_i = \sum_{j=1}^{4}\Omega_{ij}d_j, \quad i = 1,2,3,4, \quad \text{where} \;\; \Omega_{ij} \underset{26}{=} \begin{cases} +\,(1/2m)\,\overline{d}_j\,e_i\mathbb{I}, & \text{for } j = 1,2\,, \\ -\,(1/2m)\,\overline{d}_j\,e_i\mathbb{I}, & \text{for } j = 3,4\,. \end{cases} \tag{3.8}$$

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In matrix form, Ω reads

$$\Omega = \frac{1}{2}\begin{pmatrix} \mathbb{I} & -i\mathbb{I} & -\mathbb{I} & -i\mathbb{I} \\ i\mathbb{I} & \mathbb{I} & i\mathbb{I} & -\mathbb{I} \\ \mathbb{I} & i\mathbb{I} & -\mathbb{I} & i\mathbb{I} \\ -i\mathbb{I} & \mathbb{I} & -i\mathbb{I} & -\mathbb{I} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \mathcal{B} & -\mathcal{B}^* \\ \mathcal{B}^* & -\mathcal{B} \end{pmatrix} \otimes \mathbb{I}, \tag{3.9}$$

where $\mathcal{B} := (\mathbb{I} + \sigma_2)$, Such results show that ELKO can be expressed somehow as a linear combination of the Dirac particle and antiparticle spinor fields. It reinforces the Lounesto theorems, showing that classes of spinor fields under Lounesto spinor fields classification are not preserved by sum (for details see [70, 71, 83]). In order to obtain the ELKO evolution, a prescription where the momentum is written in terms of the covariant derivative as $p_\mu \mapsto i\nabla_\mu$ is regarded. As one shall see in the following section, such a prescription is convenient when one considers the coordinate representation $\lambda^{\rm S/A}(x) = \lambda^{\rm S/A}(\mathbf{p})\ \exp\left(\varepsilon^{\rm S/A}\ ip_\mu x^\mu\right)$. However, that is not the only way to prescribe the ELKO evolution. The momentum can also be replaced with the derivatives times the γ^5 matrix as performed, for instance, in the investigation of ELKO auto interactions when one considers the ELKO field interacting with its own spin density via contorsional auto interactions [84].

4 ELKO dynamics in the exotic spin structure

In spacetimes with non-trivial topology it is well known that there is an additional degree of freedom for fermionic particles [85]. Albeit in the classical level it might be naively suggested that exotic spinor fields describe different particles, the breakthrough idea proposed is that, in the quantum framework, a new partition function which is the sum over all possibilities must be taken into account. See [30, 85] and references therein for more details.

In this section it is thoroughly shown that dark spinor fields are a natural probe of the non-trivial topology and also provide, from their inherent dynamics, constraints either in the spacetime metric structure or in its topology, or in both.

Essentially, exotic spinor fields are parallel transported like standard spinor fields, but an outstanding property distinguishes both kinds of spinor fields: the covariant derivative acting on these exotic spinor fields changes by an additional one-form field that is manifestation of the non-trivial topology, as it was shown in section 2. The exotic structure endows the Dirac operator with an additional term $\xi^{-1}(x)d\xi(x), x \in M$, where $d : \sec\Lambda^0(TM) \to \sec\Lambda^1(TM)$ denotes the exterior derivative operator. The term $\frac{1}{2\pi i}\xi^{-1}(x)d\xi(x)$ is real and closed, but not exact, and defines an integer cohomology class in the Čech sense [30, 36, 37, 40, 41]. Using the relation between Čech and de Rham cohomologies, the integral of $\frac{1}{2\pi i}\xi^{-1}(x)d\xi(x)$ around any closed curve is an integer. In the context of the exotic Dirac equation, the electromagnetic vector potential A term is affected by the transformation $A \mapsto A + \frac{1}{2\pi i}\xi^{-1}d\xi$, which exactly corresponds to the addition of another electromagnetic potential, when Dirac spinor fields are taken into account. In such case the exotic term may be then absorbed in an external electromagnetic potential, representing an element of $H^1(M, \mathbb{Z}_2)$ [36, 40, 41, 73, 86]. Namely, in this case the interaction is encoded as a shift in the vector potential.

The importance to analyze dark spinor fields in this context is that this possibility is not present if ELKO spinor fields are employed, as they cannot carry electromagnetic charge and the full topological treatment is appropriate in this case [73].

In addition to the ELKO spinor fields $\lambda(x)$ — that was indeed defined as sections in the bundle $P_{\mathrm{Spin}^e_{1,3}}(M) \times_\rho \mathbb{C}^4$, in section 2 — one can get a second type of ELKO $\mathring{\lambda}(x)$, which can be described by sections in the inequivalent spin structure-induced spinor bundle

$$\mathring{P}_{\mathrm{Spin}^e_{1,3}}(M) \times_\rho \mathbb{C}^4, \tag{4.1}$$

with a variation of the covariant derivative, given by [36]

$$\mathring{\nabla}_X \mathring{\lambda}(x) = \nabla_X \mathring{\lambda}(x) - \frac{1}{2}\left[X \lrcorner \left(\xi^{-1}(x) d\xi(x)\right)\right] \mathring{\lambda}(x), \tag{4.2}$$

where X denotes a vector field in M.

In general, the exotic term in eq. (4.2) is assumed — in order to be an integer of a Čech cohomology class — to be indeed $\frac{1}{2\pi i}\left(\xi^{-1}(x)d\xi(x)\right)$ [30, 36, 37, 40, 41]. We henceforth redefine $\xi(x) \mapsto \xi(x)/\sqrt{2\pi}$ in such a way that the exotic Dirac operator can be written as (see eq. (2.4))

$$i\gamma^\mu \mathring{\nabla}_\mu = i\gamma^\mu \nabla_\mu + \xi^{-1}(x) d\xi(x). \tag{4.3}$$

The exotic Dirac equation is given by

$$(i\gamma^\mu \nabla_\mu + (\xi^{-1}(x)\, d\xi(x)) - m\mathbb{I})\psi(x) = 0, \qquad \text{where } \psi \text{ denotes a Dirac spinor field.}$$

The exotic Dirac spinor fields are annihilated by $\left(i\gamma^\mu \nabla_\mu + (\xi^{-1}(x)\, d\xi(x)) \pm m\mathbb{I}\right)$

$$\begin{cases} \text{For particles:} & \left(i\gamma^\mu \nabla_\mu + (\xi^{-1}(x)\, d\xi(x)) - m\mathbb{I}\right) u(x) = 0\,, \\ \text{For antiparticles:} & \left(i\gamma^\mu \nabla_\mu + (\xi^{-1}(x)\, d\xi(x)) + m\mathbb{I}\right) v(x) = 0\,. \end{cases} \tag{4.4}$$

Hereon we denote $\xi^{-1}(x)\, d\xi(x)$ by $a(x)$ in order to shorten all formulæ notations.

Now it is straightforward to show that ELKO can not be eigenspinors of the exotic Dirac operator $i\gamma^\mu \nabla_\mu + a(x)$. Indeed, denoting

$$e := \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{pmatrix}, \qquad d := \begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{pmatrix},$$

and $\Gamma := \mathbb{I} \otimes (i\gamma^\mu \nabla_\mu + a(x))$, eq. (3.8) becomes $e = \Omega d$. Using $[\Gamma, \Omega] = 0$ yields $\Gamma e = \Omega \Gamma d$. Eqs. (4.4) imply $\Gamma d = m\,\gamma^5 \otimes \mathbb{I}\, d$ and then $\Gamma e = \Omega\left(m\,\gamma^5 \otimes \mathbb{I}\right)\Omega^{-1} e$. An explicit evaluation of $\mu := \Omega\left(m\,\gamma^5 \otimes \mathbb{I}\right)\Omega^{-1}$ reveals

$$\mu = m \begin{pmatrix} \sigma_2 & \mathbb{O} \\ \mathbb{O} & -\sigma_2 \end{pmatrix} \otimes \mathbb{I}\,.$$

Thus, making the direct product explicit again, finally one reaches the result

$$\begin{pmatrix} i\gamma^\mu\nabla_\mu+a(x) & \mathbb{O} & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & i\gamma^\mu\nabla_\mu+a(x) & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & \mathbb{O} & i\gamma^\mu\nabla_\mu+a(x) & \mathbb{O} \\ \mathbb{O} & \mathbb{O} & \mathbb{O} & i\gamma^\mu\nabla_\mu+a(x) \end{pmatrix}\begin{pmatrix} \mathring{\lambda}^{\rm S}_{\{-,+\}} \\ \mathring{\lambda}^{\rm S}_{\{+,-\}} \\ \mathring{\lambda}^{\rm A}_{\{-,+\}} \\ \mathring{\lambda}^{\rm A}_{\{+,-\}} \end{pmatrix} - im\mathbb{I}\begin{pmatrix} -\mathring{\lambda}^{\rm S}_{\{+,-\}} \\ \mathring{\lambda}^{\rm S}_{\{-,+\}} \\ \mathring{\lambda}^{\rm A}_{\{+,-\}} \\ -\mathring{\lambda}^{\rm A}_{\{-,+\}} \end{pmatrix} = 0 \quad (4.5)$$

which establishes that $(i\gamma^\mu\nabla_\mu + a(x) \pm m\mathbb{I})$ do not annihilate the ELKO (dark) spinor fields. The antisymmetric symbol defined as $\varepsilon^{\{-,+\}}_{\{+,-\}} := -1$, the above equations reduces to

$$\Big((i\gamma^\mu\nabla_\mu+a(x))\delta^\beta_\alpha+m\mathbb{I}\varepsilon^\beta_\alpha\Big)\,\mathring{\lambda}^{\rm S}_\beta(x) = 0, \qquad \Big((i\gamma^\mu\nabla_\mu+a(x))\delta^\beta_\alpha-m\mathbb{I}\varepsilon^\beta_\alpha\Big)\,\mathring{\lambda}^{\rm A}_\beta(x) = 0, \quad (4.6)$$

which are the inherent counterparts of eqs. (4.4). The term of δ^β_α is $i\gamma^\mu\nabla_\mu + a(x)$, and the existence of ε^β_α in the mass term forbids ELKO spinor fields to be eigenspinors of the $i\gamma^\mu\nabla_\mu + a(x)$ operator. Namely, the mass terms carry opposite signs and consequently ELKO cannot be annihilated by $(i\gamma^\mu\nabla_\mu + a(x) \pm m\mathbb{I})$, because the term ε^β_α in eq. (4.6), which implies that $\epsilon^{\rm S} = -1$ and $\epsilon^{\rm A} = +1$.

Furthermore, as comprehensively discussed in, e.g., [36, 87], we can express $\xi(x) = \exp(i\theta(x)) \in {\rm U}(1)$, $x \in M$. The exotic spin structure term in this way reads

$$\xi^{-1}(x)d\xi(x) = \exp(-i\theta(x))(i\gamma^\mu\nabla_\mu\theta(x))\exp(i\theta(x)) = i\gamma^\mu\partial_\mu\theta(x). \quad (4.7)$$

From eq. (4.7), eqs. (4.6) are written as

$$\Big((i\gamma^\mu\nabla_\mu + i\gamma^\mu\partial_\mu\theta)\delta^\beta_\alpha \pm m\mathbb{I}\varepsilon^\beta_\alpha\Big)\,\mathring{\lambda}^{\rm S/A}_\beta(x) = 0\,. \quad (4.8)$$

The exotic Dirac operator $i\gamma^\mu\nabla_\mu + i\gamma^\mu\partial_\mu\theta - m\mathbb{I}$, annihilates each of the four exotic Dirac spinor fields $u_\pm(x)$ and $v_\pm(x)$, but as the wave operator in (4.8) couples the $\{\pm,\mp\}$ degrees of freedom such exotic Dirac operator does not annihilate ELKO.

Much has been extensively discussed about the subtle differences between Majorana and ELKO spinor fields, see e. g., [74]. Both in the Lounesto spinor field classification are type-(5) spinor fields, satisfying (3).

We now shall discuss whether the exotic Dirac operator can be considered as a square root of the Klein–Gordon operator – in the sense that $(i\gamma^\mu\nabla_\mu + i\gamma^\mu\partial_\mu\theta - m\mathbb{I})(i\gamma^\mu\nabla_\mu + i\gamma^\mu\partial_\mu\theta + m\mathbb{I}) = (g^{\mu\nu}\nabla_\mu\nabla_\nu + m^2)\mathbb{I}$. This feature must remain true for the ELKO and its exotic partner:

$$((i\gamma^\mu\nabla_\mu+i\gamma^\mu\partial_\mu\theta)\delta^\beta_\alpha \pm m\mathbb{I}\varepsilon^\beta_\alpha)((i\gamma^\mu\nabla_\mu+i\gamma^\mu\partial_\mu\theta)\delta^\beta_\alpha \mp m\mathbb{I}\varepsilon^\beta_\alpha) = (g^{\mu\nu}\nabla_\mu\nabla_\nu+m^2)\mathbb{I}\,\delta^\beta_\alpha, \quad (4.9)$$

since the introduction of an exotic spin structure does not modify the Klein–Gordon propagator fulfillment by dark spinor fields.

The corresponding Klein-Gordon equation is given by

$$(\Box + m^2 + g^{\mu\nu}\nabla_\mu\nabla_\nu\theta + \partial^\mu\theta\nabla_\mu + \partial^\mu\theta\partial_\mu\theta)\mathring{\lambda}(x)^{S/A}_{\{\pm,\mp\}} = 0, \quad (4.10)$$

where $\Box$ denotes the square of the spin-Dirac operator, that can be related to the Laplace-Beltrami operator by the Lichnerowicz formula [88–90]. In order that the Klein-Gordon propagator for the exotic ELKO remains the same as the standard Klein-Gordon propagator for the ELKO spinor field, from eq. (4.10) it follows that

$$(\Box\theta + \partial^\mu\theta\nabla_\mu + \partial^\mu\theta\partial_\mu\theta)\mathring{\lambda}^{S/A}_{\{\pm,\mp\}}(x) = 0. \tag{4.11}$$

Explicitly, for consistency with the standard formalism it can be written that

$$\mathring{\lambda}(x) = \begin{pmatrix} \sigma_2\phi^*(x) \\ \phi(x) \end{pmatrix}, \qquad \text{where} \quad \phi(x) = \begin{pmatrix} \alpha(x) \\ \beta(x) \end{pmatrix}, \quad \alpha(x), \beta(x) \in \mathbb{C}, \tag{4.12}$$

implying that

$$\begin{pmatrix} \Box\theta + i\partial^\mu\theta\nabla_\mu - i\partial^\mu\theta\partial_\mu\theta & 0 \\ 0 & \Box\theta + i\partial^\mu\theta\nabla_\mu - i\partial^\mu\theta\partial_\mu\theta \end{pmatrix} \begin{pmatrix} \beta - i\beta^* \\ \alpha + i\alpha^* \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{4.13}$$

Still, the carrier of the representation space can be written as

$$\mathring{\lambda}^{S/A}_{\{\pm,\mp\}}(x) = \begin{pmatrix} (\beta_1 \mp \beta_2)\exp(\mp i\frac{\pi}{4}) \\ (\alpha_1 \pm \alpha_2)\exp(\pm i\frac{\pi}{4}) \end{pmatrix}, \qquad \beta = \beta_1 + i\beta_2, \quad \alpha = \alpha_1 + i\alpha_2. \tag{4.14}$$

Note that the condition in eq. (4.13) is independent of the function $\theta(x)$ in the case where $\mathrm{Im}(\alpha) = -\mathrm{Re}(\alpha)$ and $\mathrm{Im}(\beta) = \mathrm{Re}(\beta)$, by eq. (4.14). As this condition is too stringent, since we want to analyze the function $\theta(x)$ for an arbitrary ELKO and not for such so particular case, we demand the most general condition given by eq. (4.11) when arbitrary exotic dark spinor fields are taken into account, since the general case must be formulated without restricting the theory on any particular case as in eq. (4.14).

Our analysis hereon sheds new light on the character of the function θ — that is *a priori* arbitrary — that defines the exotic topology. Furthermore, it delves into the way how the exotic topology can be constrained to the spacetime metric structure, via the dynamics of exotic ELKO spinor fields.

Since eq. (4.11) holds for every exotic dark spinor field $\mathring{\lambda}^{S/A}_{\{\pm,\mp\}}(x)$, in particular let us analyze the solutions of eq. (4.11) applied to, for instance, $\mathring{\lambda}^{S}_{\{-,+\}}(x)$. We omit hereon the argument "(x)" for simplicity. Using the expression[8]

$$\nabla_\mu\mathring{\lambda}^{S/A}_{\{\mp,\pm\}} = \partial_\mu\mathring{\lambda}^{S/A}_{\{\mp,\pm\}} - \frac{1}{4}\Gamma_{\mu\rho\sigma}\gamma^\rho\gamma^\sigma\mathring{\lambda}^{S/A}_{\{\mp,\pm\}}, \tag{4.15}$$

[8]Assume a metric compatible covariant derivative operator, we emphasize here that the connection is not required to be symmetric, and it is decomposed into the Christoffel symbol and the contortion tensor.

for such case, after some calculation[9] — denoting $x^0 = t$ — it follows that

$$(\Box\theta)\,\mathring{\lambda}^{S/A}_{\{\mp,\pm\}} + (\partial_0\theta)\Big[\partial_0\mathring{\lambda}^{S/A}_{\{\mp,\pm\}} - \frac{1}{4}\Big((\Gamma_{000} - \Gamma_{011} - \Gamma_{022} - \Gamma_{033})\mathring{\lambda}^{S/A}_{\{\mp,\pm\}} + i\Gamma_{001}\mathring{\lambda}^{A/S}_{\{\pm,\mp\}} \\ + \Gamma_{002}\mathring{\lambda}^{S/A}_{\{\pm,\mp\}} \mp \Gamma_{003}\mathring{\lambda}^{S/A}_{\{\mp,\pm\}} \pm i\Gamma_{012}\mathring{\lambda}^{A/S}_{\{\mp,\pm\}} + i\Gamma_{013}\mathring{\lambda}^{A/S}_{\{\pm,\mp\}} \mp \Gamma_{023}\mathring{\lambda}^{S/A}_{\{\pm,\mp\}}\Big)\Big] \\ -g^{00}(\partial_0\theta)^2\mathring{\lambda}^{S/A}_{\{\mp,\pm\}} = 0 \quad (4.16)$$

The equation above couples *again* all the four exotic spinor fields $\mathring{\lambda}^{S/A}_{\{\pm,\mp\}}$, in the case of spacetimes which the associated connection are non zero.

As proposed in, e. g., [12–14, 21, 22], it is possible for cosmological applications, to assume that the dark spinor fields depend only on the time variable t via a matter field $\kappa(t)$ compatible with homogeneity and isotropy [22] and acts as the only dynamical cosmological variable, in such a way that $\mathring{\lambda}^{S/A}_{\{\pm,\mp\}}(x)$ can be explicitly written as

$$\mathring{\lambda}^{A/S}_{\{-,+\}}(x) \;=\; \kappa(t)\,\chi^{A/S}_{\{-,+\}}, \qquad\qquad \mathring{\lambda}^{A/S}_{\{+,-\}}(x) \;=\; \kappa(t)\,\zeta^{A/S}_{\{+,-\}}, \tag{4.17}$$

where $\zeta^{S/A}$ and $\chi^{S/A}$ are linearly independent constant spinor fields given by [22]

$$\chi^S_{\{-,+\}} = \begin{pmatrix} 0\\ i\\ 1\\ 0\end{pmatrix}, \quad \chi^A_{\{-,+\}} = \begin{pmatrix} 0\\ -i\\ 1\\ 0\end{pmatrix}, \quad \zeta^S_{\{+,-\}} = \begin{pmatrix} 1\\ 0\\ 0\\ -i\end{pmatrix}, \quad \zeta^A_{\{+,-\}} = -\begin{pmatrix} 1\\ 0\\ 0\\ i\end{pmatrix}. \tag{4.18}$$

The matter field $\kappa(t)$ was introduced and satisfies a first order ordinary differential equation in time derivative, involving the time component of the total energy-momentum tensor Σ_{tt}, the Planck mass, and the Hubble constant. In the limits proposed in [22] we can write

$$\frac{\dot\kappa}{\kappa} = -\frac{1}{3}\sqrt{\frac{1}{3M^2_{\rm Pl}}\Sigma_{tt}} + \mathcal{O}(\kappa^4). \tag{4.19}$$

where $M^{-2}_{\rm Pl} = 8\pi G$ is the coupling constant. The last term in the right hand side of the equation above is in ref. [22] kept apart, and such an approximation gives robust cosmological results in full compliance with the references therein. The most general case shall be considered still in this section.

Therefore we can write $\kappa(t) = \exp(at)$, where a is the constant given in the equation above. Using now eqs. (4.17) and (4.18), and considering each one of the four exotic dark

[9]Here we consider the torsionless connection. For the torsion case it must be written $\nabla_\mu\lambda = \partial_\mu\lambda - \frac{1}{4}\Gamma_{\mu\rho\sigma}[\gamma^\mu,\gamma^\sigma]\lambda + \frac{1}{4}K_{\mu\rho\sigma}\gamma^\rho\gamma^\sigma\lambda$, where $K_{\mu\rho\sigma}$ are the contorsion tensor coefficients. Such case is used in the analysis of dark spinor fields in Cosmology in [19, 20, 22], and it is important to remark that in the presence of torsion an additional dynamical term appears [91]. Such more general formalism for while is unnecessary here, since our main aim now is to verify that exotic dark spinor fields dynamics indeed can constraint the metric spacetime structure. By now, we just call some attention to the fact that the torsion fields may act in order to cancel the connection effects in the constraint equation.

spinor $\lambda^{S/A}_{\{-,+\}}$ components in eq. (4.16), we have the system

$$\begin{cases} (-i\Gamma_{001}+\Gamma_{002}-i\Gamma_{013}-\Gamma_{023})\partial_0\theta=0 \\ i\Box\theta+\partial_0\theta\left(i-\frac{1}{4}\left(i\Gamma_{000}-i\Gamma_{011}-i\Gamma_{022}-i\Gamma_{033}+\Gamma_{012}-i\Gamma_{003}\right)\right)-i(\partial_0\theta)^2=0 \\ \Box\theta+\partial_0\theta\left(1-\frac{1}{4}\left(\Gamma_{000}-\Gamma_{011}-\Gamma_{022}-\Gamma_{033}+i\Gamma_{012}-\Gamma_{003}\right)\right)-(\partial_0\theta)^2=0 \\ (\Gamma_{001}-i\Gamma_{002}+\Gamma_{013}+i\Gamma_{023})\partial_0\theta=0. \end{cases} \tag{4.20}$$

The first and fourth equations above together imply that

$$(\partial_0\theta)\Gamma_{012}=0, \tag{4.21}$$

what means that if θ is time dependent, it necessarily means that $\Gamma_{012}=0$. Otherwise, in the case where θ does not depends on time, it implies that $\partial_0\theta=0$, and then we obtain the Laplace equation for θ

$$\nabla^2\theta=0. \tag{4.22}$$

It is worthwhile to note, by passing, that eq. (4.16) in spacetimes where the connection symbols above are zero — in the Minkowski space with Cartesian coordinates, for instance — is reduced to

$$\left(\Box\theta+(\partial_0\theta)a-(\partial_0\theta)^2\right)\mathring{\lambda}^{S/A}_{\{\mp,\pm\}}=0, \tag{4.23}$$

and in this way the dark spinor field dynamics imposes constraints only on the topological sector determined by θ, and there is no coupling among the four exotic spinor fields.

On the another hand, the second and the third equations in the system above together imply that

$$\Box\theta+(\partial_0\theta)\left(1-\frac{1}{2}(\Gamma_{000}-\Gamma_{011}-\Gamma_{022}-\Gamma_{033}-\Gamma_{003})\right)-(\partial_0\theta)^2=0, \tag{4.24}$$

what means that if $\theta=\theta(t)$, so necessarily $4-(\Gamma_{000}-\Gamma_{011}-\Gamma_{022}-\Gamma_{033}-\Gamma_{003})=\partial_0\theta$. Otherwise, again eq. (4.22) holds.

Now, using eqs. (4.17) and (4.18) and considering each one of the four exotic dark spinor $\lambda^{S/A}_{\{+,-\}}$ components in eq. (4.16), we have the same results as for the $\lambda^{S/A}_{\{-,+\}}$. It shows that the exotic topology induces constraints in the spacetime geometry, coming from the dynamics of dark spinor fields. Indeed it is the case in such an approach when the function $\theta(x)$ that generates the exotic structure — realized by eq. (4.7) — is most general, time dependent.

As previously observed, eq. (4.15) was solved for $\mathring{\lambda}^{S/A}_{\{\mp,\pm\}}(x)$, in order to illustrate the exotic dark spinor fields dynamics. It evinces the constraints either on the spacetime metric structure — given an arbitrary 1-form field in spacetime, manifestation of the exotic topology encrypted in the term $\theta(x)$ in eq. (4.7) — or on the exotic parameter $\theta(x)$.

To the most general case, it is not necessary indeed to consider any particular case about $\kappa(t)$, and the system (4.20) is written as

$$\begin{cases} (-i\Gamma_{001}+\Gamma_{002}-i\Gamma_{013}-\Gamma_{023})\partial_0\theta=0 \\ \Box\theta-\partial_0\theta\left(\frac{1}{4}\left(\Gamma_{000}-\Gamma_{011}-\Gamma_{022}-\Gamma_{033}+i\Gamma_{012}-\Gamma_{003}\right)\right)-(\partial_0\theta)^2=-(\partial_0\theta)\frac{\dot\kappa(t)}{\kappa(t)} \\ \Box\theta+\partial_0\theta\left(-\frac{1}{4}\left(\Gamma_{000}-\Gamma_{011}-\Gamma_{022}-\Gamma_{033}+i\Gamma_{012}-\Gamma_{003}\right)\right)-(\partial_0\theta)^2=-(\partial_0\theta)\frac{\dot\kappa(t)}{\kappa(t)} \\ (\Gamma_{001}-i\Gamma_{002}+\Gamma_{013}+i\Gamma_{023})\partial_0\theta=0. \end{cases} \tag{4.25}$$

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The analysis that evinces the constraints among topological and geometrical terms is similar, except for the term $-(\partial_0\theta)\frac{\dot\kappa(t)}{\kappa(t)}$ on the right hand side of the second and third equations above. For instance, in the case previously analyzed, all terms in eq. (4.19) are given by

$$\frac{\dot\kappa}{\kappa} = -\frac{\sqrt{1/M_{\rm pl}^2}}{4\sqrt{3}}\left(\frac{8+3\kappa^4/M_{\rm pl}^4}{12-\kappa^4/M_{\rm pl}^4}\right)\sqrt{4-\kappa^4/M_{\rm pl}^4}, \tag{4.26}$$

and cogently the exotic dark spinor fields dynamics constraints the spacetime topology, or the spacetime metric structure, or both, whatever the form of $\kappa(t)$, and also even for the most general dark spinor fields $\mathring{\lambda}^{S/A}_{\{\mp,\pm\}}$, predicted by eq. (4.11).

5 Concluding remarks and outlook

Given an a priori arbitrary manifold M with non-trivial topology, and using the fact that the inequivalent spin structures give rise to the exotic term endowing the Dirac operator — in our analysis, the exotic term in the Dirac operator (evinced when an arbitrary inequivalent spin structure is taken into account) — we have shown that the exotic dark spinor fields dynamics indeed can constraint the metric spacetime structure. Such constraints can be mitigated for some particular choices of the exotic term θ in (4.7) — but in the most general case both the spacetime metric structure and the non-trivial topology are constrained by the exotic dark spinor field dynamics.

Much has been discussed the about equations constraining the dynamics and the spinor structures, and some questions were addressed about the validity of Klein-Gordon propagator globally, but not locally, for dark spinor fields [12–14]. The formalism here introduced is promising to derive and provide open questions on the dark spinor fields models structures and their subsequent application in cosmology — in particular the dark matter problem.

Eq. (4.17) is successful to decouple topological terms evinced by the exotic θ function and the geometrical terms given by the connection symbols, in some particular cases analyzed from eq. (4.20) on. As in such situations the connection symbols are constrained, it also induces constraints among Christoffel symbols and contorsion tensor components, in the case where torsion is taken into account in the covariant derivative. In addition, eq. (4.16) is the most general coupling between topological and geometrical terms when no particular exotic dark spinor field is considered.

Besides analyzing the exotic dark spinor fields elicited from a non-trivial topology endowed manifold, such additional term in the Dirac operator may be useful to solve some open questions addressed in the current literature [1, 2, 5–7, 9, 12–14, 21–23, 25, 26].

In certain sense, that idea that is in the background of the theoretical tools through which one set the quoted constraints can be identified with the problem of constraints in gauge theories known as the Velo-Zwanziger problem [92]. In the context of such theories, to avoid algebraic inconsistencies originated from a kind of exotic interactions, one sets the constraints independently of the equations of motion. Furthermore, in [93] it is used a similar prescription to introduce a more suitable Dirac operator, that can be also related to the one introduced in [94]. In some cases, the Lagrangian device by itself does not provide

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satisfactory wave equations [93], a problem that is given an adequate interpretation, and we expect to have overcome.

Acknowledgments

R. da Rocha is grateful to Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq) grants 472903/2008-0 and 304862/2009-6 for financial support. A. E. B. would like to thank the financial support FAPESP 2008/50671-0 and CNPq grant 300233/2010-8.

A Clifford bundles

One thus introduces the Clifford bundle of differential forms $\mathcal{C}\ell(M,g)$, which is a vector bundle associated with $P_{\mathrm{Spin}^e_{1,3}}(M)$ [90, 95]. Sections of the Clifford bundles are sums of non-homogeneous differential forms, called Clifford fields. Remember that $\mathcal{C}\ell(M,g) = P_{\mathrm{SO}^e_{1,3}}(M) \times_{\mathrm{Ad}'} \mathcal{C}\ell_{1,3}$, where $\mathcal{C}\ell_{1,3} \simeq \mathrm{M}(2,\mathbb{H})$ is the spacetime algebra and $\mathbb{H}$ denotes the quaternions. Details of the bundle structure are as follows:

1. Let $\pi_c : \mathcal{C}\ell(M,g) \to M$ be the canonical projection of $\mathcal{C}\ell(M,g)$ and let $\cup_{i\in I} U_i$ be an open simple covering of M, together with a set of transition functions $a_{ij} : U_i \cap U_j \to \mathrm{SO}^e_{1,3}$ such that $a_{ij} \circ a_{jk} = a_{ik}$ in $U_i \cap U_j \cap U_k$ and $a_{jj} = \mathrm{id}$. There are trivialization mappings $\psi_i : \pi_c^{-1}(U_i) \to U_i \times \mathcal{C}\ell_{1,3}$ of the form $\psi_i(p) = (\pi_c(p), \psi_{i,x}(p)) = (x, \psi_{i,x}(p))$. If $x \in U_i \cap U_j$ and $p \in \pi_c^{-1}(x)$, then
$$\psi_{i,x}(p) = h_{ij}(x)\psi_{j,x}(p),$$
for $h_{ij}(x) \in \mathrm{Aut}(\mathcal{C}\ell_{1,3})$, where $h_{ij} : U_i \cap U_j \to \mathrm{Aut}(\mathcal{C}\ell_{1,3})$ are the transition mappings of $\mathcal{C}\ell(M,g)$. Since every automorphism of $\mathcal{C}\ell_{1,3}$ is inner, then $h_{ij}(x)\psi_{j,x}(p) = a_{ij}(x)\psi_{i,x}(p)a_{ij}(x)^{-1}$ for some $a_{ij}(x) \in \mathcal{C}\ell^{\star}_{1,3}$, the group of invertible elements of $\mathcal{C}\ell_{1,3}$. In particular, a spin structure $(P_{\mathrm{Spin}^e_{1,3}}(M), s)$ on M is precisely comprised by the system of transition functions $h_{ij} : U_i \cap U_j \to \mathrm{Spin}^e_{1,3}$ such that
$$\varsigma \circ h_{ij} = a_{ij}, \qquad h_{ij} \circ h_{jk} = h_{ik}, \qquad h_{ii} = \mathrm{id}, \tag{A.1}$$
where ς is defined in eq. (A.2).

2. Since $\mathcal{C}\ell^{\star}_{1,3}$ acts naturally on $\mathcal{C}\ell_{1,3}$ as an algebra automorphism through its adjoint representation, the group $\mathrm{SO}^e_{1,3}$ has a natural extension in the Clifford algebra $\mathcal{C}\ell_{1,3}$. A set of lifts of the transition functions in $\mathcal{C}\ell(M,g)$ is a set of elements $\{a_{ij}\} \subset \mathcal{C}\ell^{\star}_{1,3}$ such that, if[10]
$$\begin{aligned}&\mathrm{Ad} : \phi \mapsto \mathrm{Ad}_\phi\\ &\mathrm{Ad}_\phi(\Xi) = \phi\Xi\phi^{-1}, \quad \forall \Xi \in \mathcal{C}\ell_{1,3},\end{aligned}$$
then $\mathrm{Ad}_{a_{ij}} = h_{ij}$ in all intersections.

[10] It is well known that $\mathrm{Spin}^e_{1,3} = \{\phi \in \mathcal{C}\ell^0_{1,3} : \phi\tilde{\phi} = 1\} \simeq \mathrm{SL}(2,\mathbb{C})$ is the universal covering group of the restricted Lorentz group $\mathrm{SO}^e_{1,3}$. Notice that $\mathcal{C}\ell^0_{1,3} \simeq \mathcal{C}\ell_{3,0} \simeq \mathrm{M}(2,\mathbb{C})$, the even subalgebra of $\mathcal{C}\ell_{1,3}$ is the Pauli algebra.

3. The application $\mathrm{Ad}|_{\mathrm{Spin}^e_{1,3}}$ defines a group homomorphism

$$\varsigma : \mathrm{Spin}^e_{1,3} \to \mathrm{SO}^e_{1,3}, \qquad \text{which is onto and } \ker \varsigma = \mathbb{Z}_2. \tag{A.2}$$

Then $\mathrm{Ad}_{\pm 1}$ = identity, and $\mathrm{Ad} : \mathrm{Spin}^e_{1,3} \to \mathrm{Aut}(\mathcal{C}\ell_{1,3})$ descends to a representation of $\mathrm{SO}^e_{1,3}$. Let us call Ad' this representation, i.e., $\mathrm{Ad}' : \mathrm{SO}^e_{1,3} \to \mathrm{Aut}(\mathcal{C}\ell_{1,3})$. Then $\mathrm{Ad}'_{\varsigma(\phi)}\Xi = \mathrm{Ad}_\phi \Xi = \phi \Xi \phi^{-1}$.

4. It is clear that the structure group of the Clifford bundle $\mathcal{C}\ell(M, g)$ is reducible from $\mathrm{Aut}(\mathcal{C}\ell_{1,3})$ to $\mathrm{SO}^e_{1,3}$. The transition maps of the principal bundle $P_{\mathrm{SO}^e_{1,3}}(M)$ can thus be — through Ad' — taken as transition maps for the Clifford bundle. It follows that [63–65]

$$\mathcal{C}\ell(M, g) = P_{\mathrm{SO}^e_{1,3}}(M) \times_{\mathrm{Ad}'} \mathcal{C}\ell_{1,3},$$

i.e., the Clifford bundle is a vector bundle associated with the principal bundle $P_{\mathrm{SO}^e_{1,3}}(M)$ of orthonormal coframes.

B Principal bundles and associated vector bundles

In this section it is reviewed the main definitions and concepts of the theory of principal bundles and their associated vector bundles, which is needed to introduce the Clifford and spin-Clifford bundles used in this paper. Propositions are in general presented without proofs, which can be found, e.g., in [90, 96, 97].

A fiber bundle on a manifold M with Lie group G is denoted by (E, M, π, G, F). E is a topological space called the total space of the bundle, $\pi : E \to M$ is a continuous surjective map, called the canonical projection, and F is the typical fiber. The following conditions must be satisfied:

a) $\pi^{-1}(x)$, the fiber over x, is homeomorphic to F.

b) Let $\{U_i,\, i \in \mathfrak{I}\}$, where $\mathfrak{I}$ is an index set, be a covering of M, such that:

   b1) Locally a fiber bundle E is trivial, namely it is diffeomorphic to a product bundle $\pi^{-1}(U_i) \simeq U_i \times F$ for all $i \in \mathfrak{I}$.

   b2) The diffeomorphisms $\Phi_i : \pi^{-1}(U_i) \to U_i \times F$ have the form

   $$\Phi_i(p) = (\pi(p), \phi_{i,x}(p)), \qquad \phi_i|_{\pi^{-1}(x)} \equiv \phi_{i,x} : \pi^{-1}(x) \to F \text{ is onto} \tag{B.1}$$

   The collection $\{(U_i, \Phi_i)\}$, $i \in \mathfrak{I}$, are said to be a family of local trivializations for E.

   b3) The group G acts on the typical fiber. Considering $x \in U_i \cap U_j$, then, $\phi_{j,x} \circ \phi^{-1}_{i,x} : F \to F$ must coincide with the action of an element of G, for all $x \in U_i \cap U_j$ and $i, j \in \mathfrak{I}$.

   b4) One calls transition functions of the bundle the continuous induced mappings

   $$a_{ij} : U_i \cap U_j \to G, \text{ where } a_{ij}(x) = \phi_{i,x} \circ \phi^{-1}_{j,x}. \tag{B.2}$$

For consistence of the theory the transition functions must satisfy the cocycle condition $a_{ij}(x)a_{jk}(x) = a_{ik}(x)$.

The 5-tuple $(P, M, \pi, G, F \equiv G) \equiv (P, M, \pi, G)$ is called a principal fiber bundle (PFB) if all the conditions about fiber bundles are fulfilled and, moreover, if there is a right action of G on elements $p \in P$, such that:

a) the mapping (defining the right action) $P \times G \ni (p, g) \mapsto pg \in P$ is continuous.

b) given $g, g' \in G$ and $\forall p \in P$, $(pg)g' = p(gg')$.

c) $\forall x \in M, \pi^{-1}(x)$ is invariant under the action of G: each element of $p \in \pi^{-1}(x)$ is mapped into $pg \in \pi^{-1}(x)$, i.e., it is mapped into an element of the same fiber.

d) G acts free and transitively on each fiber $\pi^{-1}(x)$, which means that all elements within $\pi^{-1}(x)$ are obtained by the action of all the elements of G on any given element of the fiber $\pi^{-1}(x)$. This condition is, of course, necessary for the identification of the typical fiber with G.

A bundle $(E, M, \pi,\ G = \mathrm{GL}(n, \mathbb{K}), V)$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, and V is an n-dimensional vector space over $\mathbb{K}$ is called a vector bundle.

A vector bundle (E, M, π, G, F) denoted $E = P \times_\rho F$ is said to be associated to a *PFB* bundle (P, M, π, G) by the linear representation $\rho : G \to \mathrm{GL}(V)$ — which is called the carrier space of the representation — if its transition functions are the images under ρ of the corresponding transition functions of the *PFB* (P, M, π, G). This precisely means the following: consider the following local trivializations of P and E respectively

$$\Phi_i : \pi^{-1}(U_i) \to U_i \times G, \qquad \Xi_i : \pi_1^{-1}(U_i) \to U_i \times F, \tag{B.3}$$

$$\Xi_i(q) = (\pi_1(q), \chi_i(q)) = (x, \chi_i(q)), \qquad \chi_i|_{\pi_1^{-1}(x)} \equiv \chi_{i,x} : \pi_1^{-1}(x) \to F, \tag{B.4}$$

where $\pi_1 : P \times_\rho F \to M$ is the projection of the bundle associated to (P, M, π, G). Then, for all $x \in U_i \cap U_j$, $i, j \in \mathfrak{I}$, it follows that

$$\chi_{j,x} \circ \chi_{i,x}^{-1} = \rho(\phi_{j,x} \circ \phi_{i,x}^{-1}). \tag{B.5}$$

In addition, the fibers $\pi^{-1}(x)$ are vector spaces isomorphic to the representation space V.

Let (E, M, π, G, F) be a fiber bundle and $U \subset M$ an open set. A local section of the fiber bundle (E, M, π, G, F) on U is a mapping

$$s : U \to E \quad \text{such that} \quad \pi \circ s = Id_U, \tag{B.6}$$

If $U = M$ s is said to be a global section.

There is a relation between sections and local trivializations for principal bundles. Indeed, each local section s (on $U_i \subset M$) for a principal bundle (P, M, π, G) determines a local trivialization $\Phi_i : \pi^{-1}(U) \to U \times G$, of P by setting $\Phi_i^{-1}(x, g) = s(x)g = pg = R_g p$. Conversely, Φ_i determines s since

$$s(x) = \Phi_i^{-1}(x, e). \tag{B.7}$$

A principal bundle is trivial if and only if it has a global cross section. A vector bundle is trivial if and only if its associated principal bundle is trivial. Any fiber bundle (E, M, π, G, F) such that M is a paracompact manifold and the fiber F is a vector space admits a cross section. Then, any vector bundle associated to a trivial principal bundle has non-zero global sections. Note however that a vector bundle may admit a non-zero global section even if it is not trivial. Indeed, as shown in the main text, any Clifford bundle possesses a global identity section, and some spin-Clifford bundles admits also identity sections once a trivialization is given.

References

[1] D.V. Ahluwalia and D. Grumiller, *Spin half fermions with mass dimension one: theory, phenomenology and dark matter*, *JCAP* **07** (2005) 012 [hep-th/0412080] [SPIRES].

[2] D.V. Ahluwalia and D. Grumiller, *Dark matter: a spin one half fermion field with mass dimension one?*, *Phys. Rev.* **D 72** (2005) 067701 [hep-th/0410192] [SPIRES].

[3] D.V. Ahluwalia, *Theory of neutral particles: McLennan-Case construct for neutrino, its generalization and a fundamentally new wave equation*, *Int. J. Mod. Phys.* **A 11** (1996) 1855 [hep-th/9409134] [SPIRES].

[4] D.V. Ahluwalia, *Extended set of Majorana spinors, a new dispersion relation and a preferred frame*, hep-ph/0305336 [SPIRES].

[5] A.E. Bernardini and R. da Rocha, *Lorentz-violating dilatations in the momentum space and some extensions on non-linear actions of Lorentz algebra-preserving systems*, *Phys. Rev.* **D 75** (2007) 065014 [hep-th/0701094] [SPIRES].

[6] A.E. Bernardini and R. da Rocha, *Obtaining the equation of motion for a fermionic particle in a generalized Lorentz-violating system framework*, *Europhys. Lett.* **81** (2008) 40010 [hep-th/0701092] [SPIRES].

[7] D.V. Ahluwalia, *Dark matter and its darkness*, *Int. J. Mod. Phys.* **D 15** (2006) 2267 [astro-ph/0603545] [SPIRES].

[8] M. Dias, F. de Campos and J.M. Hoff da Silva, *Exploring light Elkos signal at accelerators*, arXiv:1012.4642 [SPIRES].

[9] D.V. Ahluwalia and S.P. Horvath, *Very special relativity as relativity of dark matter: the Elko connection*, *JHEP* **11** (2010) 078 [arXiv:1008.0436] [SPIRES].

[10] J.M. Hoff da Silva and R. da Rocha, *From Dirac action to ELKO action*, *Int. J. Mod. Phys.* **A 24** (2009) 3227 [arXiv:0903.2815] [SPIRES].

[11] R. da Rocha and J.M. Hoff da Silva, *From Dirac spinor fields to ELKO*, *J. Math. Phys.* **48** (2007) 123517 [arXiv:0711.1103] [SPIRES].

[12] C.G. Boehmer, J. Burnett, D.F. Mota and D.J. Shaw, *Dark spinor models in gravitation and cosmology*, *JHEP* **07** (2010) 053 [arXiv:1003.3858] [SPIRES].

[13] C.G. Boehmer, G. Caldera-Cabral, R. Lazkoz and R. Maartens, *Dynamics of dark energy with a coupling to dark matter*, *Phys. Rev.* **D 78** (2008) 023505 [arXiv:0801.1565] [SPIRES].

[14] L. Fabbri, *Conformal gravity with the most general ELKO Fields*, arXiv:1101.2566 [SPIRES].

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JHEP04(2011)110

[15] D. Gredat and S. Shankaranarayanan, *Consistency relation between the scalar and tensor spectra in spinflation*, *JCAP* **01** (2010) 008 [arXiv:0807.3336] [SPIRES].

[16] S. Shankaranarayanan, *What-if inflaton is a spinor condensate?*, *Int. J. Mod. Phys.* **D 18** (2009) 2173 [arXiv:0905.2573] [SPIRES].

[17] S. Shankaranarayanan, *Dark spinor driven inflation*, arXiv:1002.1128 [SPIRES].

[18] H. Wei, *Spinor dark energy and cosmological coincidence problem*, *Phys. Lett.* **B 695** (2011) 307 [arXiv:1002.4230] [SPIRES].

[19] C.G. Boehmer, *The Einstein-Elko system — can dark matter drive inflation?*, *Annalen Phys.* **16** (2007) 325 [gr-qc/0701087] [SPIRES].

[20] C.G. Boehmer, *The Einstein-Cartan-Elko system*, *Annalen Phys.* **16** (2007) 38 [gr-qc/0607088] [SPIRES].

[21] C.G. Boehmer, *Dark spinor inflation — theory primer and dynamics*, *Phys. Rev.* **D 77** (2008) 123535 [arXiv:0804.0616] [SPIRES].

[22] C.G. Boehmer and J. Burnett, *Dark energy with dark spinors*, *Mod. Phys. Lett.* **A 25** (2010) 101 [arXiv:0906.1351] [SPIRES].

[23] C.G. Boehmer and J. Burnett, *Dark spinors with torsion in cosmology*, *Phys. Rev.* **D 78** (2008) 104001 [arXiv:0809.0469] [SPIRES].

[24] D.V. Ahluwalia, C.-Y. Lee, D. Schritt and T.F. Watson, *Dark matter and dark gauge fields*, arXiv:0712.4190 [SPIRES].

[25] D.V. Ahluwalia, C.-Y. Lee, D. Schritt and T.F. Watson, *Elko as self-interacting fermionic dark matter with axis of locality*, *Phys. Lett.* **B 687** (2010) 248 [arXiv:0804.1854] [SPIRES].

[26] D.V. Ahluwalia, C.-Y. Lee and D. Schritt, *Self-interacting Elko dark matter with an axis of locality*, *Phys. Rev.* **D 83** (2011) 065017 [arXiv:0911.2947] [SPIRES].

[27] D.V. Ahluwalia, *Towards a relativity of dark-matter rods and clocks*, *Int. J. Mod. Phys.* **D 18** (2009) 2311 [arXiv:0904.0066] [SPIRES].

[28] L. Fabbri and S. Vignolo, *The most general ELKOs in torsional f(R)-theories*, arXiv:1012.4282 [SPIRES].

[29] K.E. Wunderle and R. Dick, *A supersymmetric Lagrangian for Fermionic fields with mass dimension one*, arXiv:1010.0963 [SPIRES].

[30] S.J. Avis and C.J. Isham, *Lorentz gauge invariant vacuum functionals for quantized spinor fields in nonsimply connected space-times*, *Nucl. Phys.* **B 156** (1979) 441 [SPIRES].

[31] J.W. Milnor, *Spin structures on manifolds*, *L' Enseignement Math.* **9** (1963) 198.

[32] M.F. Atiyah, R. Bott and V.K. Patodi, *On the heat equation and the index theorem*, *Invent. Math.* **19** (1973) 279.

[33] S.W. Hawking and C.N. Pope, *Generalized spin structures in quantum gravity*, *Phys. Lett.* **B 73** (1978) 42 [SPIRES]

[34] N. Seiberg and E. Witten, *Spin structures in string theory*, *Nucl. Phys.* **B 276** (1986) 272 [SPIRES].

[35] A. Back, P.G.O. Freund and M. Forger, *New gravitational instantons and universal spin structures*, *Phys. Lett.* **B 77** (1978) 181 [SPIRES].

JHEP04(2011)110

[36] H.R. Petry, *Exotic spinors in superconductivity*, *J. Math. Phys.* **20** (1979) 231.

[37] S.J. Avis and C.J. Isham, *Generalized spin structures on four-dimensional space-times*, *Commun. Math. Phys.* **72** (1980) 103 [SPIRES].

[38] R.P. Geroch, *Spinor structure of space-times in general relativity. I*, *J. Math. Phys.* **9** (1968) 1739 [SPIRES].

[39] R.P. Geroch, *Spinor structure of space-times in general relativity. II*, *J. Math. Phys.* **11** (1970) 343 [SPIRES].

[40] C.J. Isham, *Twisted quantum fields in a curved space-time*, *Proc. R. Soc. London, Ser.* **A 362** (1978) 383.

[41] C.J. Isham, *Spinor fields in four-dimensional space-time*, *Proc. R. Soc. London, Ser.* **A 364** (1978) 591.

[42] S.W. Hawking, *Space-time foam*, *Nucl. Phys.* **B 144** (1978) 349 [SPIRES].

[43] S.M. Christensen and M.J. Duff, *Flat space as a gravitational instanton*, *Nucl. Phys.* **B 146** (1978) 11 [SPIRES].

[44] M. Lüscher, SO(4) *symmetric solutions of Minkowskian Yang-Mills field equations*, *Phys. Lett.* **B 70** (1977) 321 [SPIRES].

[45] R. Sasaki, *Exact classical solutions of the massless σ-model with gauge fields in Minkowski space*, *Phys. Lett.* **B 80** (1978) 61 [SPIRES].

[46] B.M. Schechter, *Yang-Mills theory on the hypertorus*, *Phys. Rev.* **D 16** (1977) 3015 [SPIRES].

[47] S.D. Unwin, *Thermodynamics in multiply connected spaces*, *J. Phys.* **A 12** (1979) L309.

[48] S.D. Unwin, *Quantised spin-1 field in flat Clifford-Klein space-times*, *J. Phys.* **A 13** (1980) 313.

[49] B.S. DeWitt, *Quantum field theory in curved space-time*, *Phys. Rept.* **19** (1975) 295 [SPIRES].

[50] J.S. Dowker and R. Banach, *Quantum field theory on Clifford-Klein space-times. The effective Lagrangian and vacuum stress energy tensor*, *J. Phys.* **A 11** (1978) 2255 [SPIRES].

[51] R. Banach and J.S. Dowker, *The vacuum stress tensor for automorphic fields on some flat space-times*, *J. Phys.* **A 12** (1979) 2545 [SPIRES].

[52] R. Banach and J.S. Dowker, *Automorphic field theory: some mathematical issues*, *J. Phys.* **A 12** (1979) 2527 [SPIRES].

[53] J.S. Dowker and R. Critchley, *Vacuum stress tensor in an Einstein universe. Finite temperature effects*, *Phys. Rev.* **D 15** (1977) 1484 [SPIRES].

[54] J.S. Dowker and R.Critchley, *Covariant Casimir calculations*, *J. Phys.* **A 9** (1976) 535.

[55] R. Banach, *Effective potentials for twisted fields*, *J. Phys.* **A 14** (1981) 901 [SPIRES].

[56] R. Banach, *The quantum theory of free automorphic fields*, *J. Phys.* **A 13** (1980) 2179 [SPIRES].

[57] L.H. Ford, *Twisted scalar and spinor strings in Minkowski space-time*, *Phys. Rev.* **D 21** (1980) 949 [SPIRES]. 39

[58] L.H. Ford, *Vacuum polarization in a nonsimply connected space-time*, *Phys. Rev.* **D 21** (1980) 933 [SPIRES].

[59] R.A. Mosna and W.A. Rodrigues, Jr, *The bundles of algebraic and Dirac-Hestenes spinor fields*, *J. Math. Phys.* **45** (2004) 2945 [math-ph/0212033] [SPIRES].

[60] W.A. Rodrigues, Jr, *Algebraic and Dirac-Hestenes spinors and spinor fields*, *J. Math. Phys.* **45** (2004) 2908 [math-ph/0212030] [SPIRES].

[61] G.L. Naber, *Topology, geometry and gauge fields. Interactions*, *Appl. Math. Sci.* **141**, Springer-Verlag, New York U.S.A. (2000).

[62] M. Nakahara, *Geometry, topology and physics*, Institute of Physics Publ., Bristol U. K. (1990).

[63] H.B. Lawson, Jr. and M.L. Michelson, *Spin geometry*, Princeton University Press, Princeton U.S.A. (1989).

[64] R.A. Mosna, D. Miralles, J. Vaz, Jr., *Multivector Dirac equations and $\mathbb{Z}_2$-gradings of Clifford algebras*, *Int. J. Theor. Phys.* **41** (2002) 1651.

[65] R.A. Mosna, D. Miralles, J. Vaz, Jr., *$\mathbb{Z}_2$-gradings of Clifford algebras and multivector structures*, *J. Phys.* **A 36** (2003) 4395 [math.PH/0212020]

[66] E. Notte-Cuello, R. da Rocha and W.A. Rodrigues, *The effective Lorentzian and teleparallel spacetimes generated by a free electromagnetic field*, *Rept. Math. Phys.* **62** (2008) 69 [gr-qc/0612098] [SPIRES].

[67] V.V. Fernandez, W.A. Rodrigues, Jr., A.M. Moya, and R. da Rocha, *Clifford and extensor calculus and the Riemann and Ricci extensor fields in of deformed structures*, *Int. J. Geom. Meth. Mod. Phys.* **4** (2007) 1159 [math.0502003/].

[68] E.A. Notte-Cuello, R. da Rocha and W.A. Rodrigues, Jr., *Some thoughts on geometries and on the nature of the gravitational field*, *J. Phys. Math.* **2** (2010) P100506 [arXiv:0907.2424] [SPIRES].

[69] R. da Rocha and W.A. Rodrigues, Jr., *The Dirac-Hestenes equation for spherical symmetric potentials in the spherical and Cartesian gauges*, *Int. J. Mod. Phys.* **A 21** (2006) 4071 [math.PH/0601018]

[70] P. Lounesto, *Clifford algebras, relativity and quantum mechanics*, in *Gravitation: the spacetime structure, Proc. of the* 8^{th} *Latin American Symposium on Relativity and Gravitation*, P. Letelier and W.A. Rodrigues, Jr. eds. , Águas de Lindóia Brazil, 25-30 July 1993, World-Scientific, London U. K. (1993).

[71] P. Lounesto, *Clifford algebras and spinors*, 2^{nd} ed., Cambridge University Press, Cambridge U. K. (2002) pg. 152-173.

[72] T. Friedrich, *Dirac operators in Riemannian geometry*, Graduate Studies in Mathematics **25**, American Mathematical Society, Providence U.S.A. (2000).

[73] A. Chockalingham and C.J. Isham, *Twisted supermultiplets*, *J. Phys.* **A 13** (1980) 2723 [SPIRES].

[74] R. da Rocha and W.A. Rodrigues, Jr., *Where are ELKO spinor fields in Lounesto spinor field classification?*, *Mod. Phys. Lett.* **A 21** (2006) 65 [math-ph/0506075] [SPIRES].

[75] R. da Rocha and J.M. Hoff da Silva, *ELKO spinor fields: Lagrangians for gravity derived*

JHEP04(2011)110

from supergravity, Int. J. Geom. Meth. Mod. Phys. **6** (2009) 461 [arXiv:0901.0883] [SPIRES].

[76] R. da Rocha and J.G. Pereira, *The quadratic spinor Lagrangian, axial torsion current and generalizations*, Int. J. Mod. Phys. **D 16** (2007) 1653 [gr-qc/0703076] [SPIRES].

[77] P.R. Holland, *Relativistic algebraic spinors and quantum motions in phase space*, *Found. Phys.* **16** (1986) 708.

[78] P.R. Holland, *Minimal ideals and Clifford algebras in the phase space representation of spin-1/2 fields*, in the *Proceedings of the Workshop on Clifford Algebras and their Applications in Mathematical Physics, Canterbury 1985*, J.S.R. Chisholm and A.K. Common eds. , Reidel Dordrecht Holland (1986) pg. 273-283.

[79] J.P. Crawford, *On the algebra of Dirac bispinor densities: factorization and inversion theorems*, *J. Math. Phys.* **26** (1985) 1429.

[80] D.V. Ahluwalia and M. Sawicki, *Front form spinors in the Weinberg-Soper formalism and generalized Melosh transformations for any spin*, Phys. Rev. **D 47** (1993) 5161 [nucl-th/9603019] [SPIRES].

[81] L. Fabbri, *Causality for ELKOs*, Mod. Phys. Lett. **A 25** (2010) 2483 [arXiv:0911.5304] [SPIRES].

[82] L. Fabbri, *Causal propagation for ELKO fields*, Mod. Phys. Lett. **A 25** (2010) 151 [arXiv:0911.2622] [SPIRES].

[83] R. da Rocha and J.M. Hoff da Silva, *ELKO, flagpole and flag-dipole spinor fields and the instanton Hopf fibration*, Adv. Appl. Clifford Algebras **20** (2010) 847 [arXiv:0811.2717] [SPIRES].

[84] L. Fabbri, *Zero energy of plane-waves for ELKOs*, arXiv:1008.0334 [SPIRES].

[85] T. Asselmeyer and G. Hess, *Fractional quantum hall effect, composite fermions and exotic spinors*, cond-mat/9508053 [SPIRES].

[86] R. Grimm, *Geometry of supergravity-matter coupling*, Nucl. Phys. **B 18** (1990) 113.

[87] G. Hess, *Exotic Majorana spinors in* $(3+1)$*-dimensional space-times*, J. Math. Phys. **35** (1994) 4848 [SPIRES].

[88] A. Lichnerowicz, *Spineurs harmonique*, *C. R. Acad. Sci. Paris Sér.* **A257** (1963) 7.

[89] E.A. Notte-Cuello, W.A. Rodrigues, Jr. and Q.A.G. de Souza, *The square of the Dirac and spin-Dirac operators on a Riemann-Cartan space(time)*, Rept. Math. Phys. **60** (2007) 135 [math-ph/0703052] [SPIRES].

[90] W.A. Rodrigues, Jr. and E. Capelas de Oliveira, *The many faces of Maxwell, Dirac and Einstein equations. A Clifford bundle approach*, Lecture Notes in Physics **722**, Springer, New York U.S.A. (2007).

[91] L. Fabbri, *The most general theory for ELKOs*, arXiv:1011.1637 [SPIRES].

[92] G. Velo and D. Zwanziger, *Propagation and quantization of Rarita-Schwinger waves in an external electromagnetic potential*, Phys. Rev. **186** (1969) 1337 [SPIRES].

[93] G. Velo and D. Zwanziger, *Noncausality and other defects of interaction lagrangians for particles with spin one and higher*, Phys41Rev. **188** (1969) 2218 [SPIRES].

[94] M. Fierz and W. Pauli, *On relativistic wave equations for particles of arbitrary spin in an electromagnetic field*, *Proc. Roy. Soc. London* **A 173** (1939) 211.

[95] W.A. Rodrigues, Jr., R. da Rocha and J. Vaz, Jr., *Hidden consequence of active local Lorentz invariance*, *Int. J. Geom. Meth. Mod. Phys.* **2** (2005) 305 [math-ph/0501064] [SPIRES].

[96] Y. Choquet-Bruhat, C. DeWitt-Morette and M. Dillard-Bleick, *Analysis, manifolds and physics (revised edition)*, North-Holland, Amsterdam Netherlands (1977).

[97] S. Kobayashi and K. Nomizu, *Foundations of differential geometry* **1**, Interscience Publishers, New York U.S.A. (1963).

2.2 Álgebra Espinorial

Para os dois trabalhos que se seguirão nesse capítulo, gostaríamos de realizar uma contextualização conjunta. Isso se deve ao fato de ambos tratarem, de modo geral, da classificação algébrica de espinores baseada em seus bilineares covariantes.

Aqui, entenderemos espinores como seções do fibrado principal $P_{SL(2,\mathbb{C})} \times_\rho \mathbb{C}^4$, onde ρ diz respeito ao espaço de representação (1/2,0), (0,1/2) ou (1/2,0)$\oplus$(0,1/2). Fato importante, cuja demonstração remonta à década de oitenta, é que um determinado espinor pode ser escrito em termos de seus bilineares. Note-se que essa concepção é oposta à que usualmente se tem no trabalho usual com espinores. De fato, bastante comum é a utilização de espinores para o computo dos bilineares a ele associados. O resultado do qual nos valemos, entretanto, utiliza os bilineares para compor o espinor que lhes deu origem. Essa é a tônica do chamado Teorema da Inversão.

De modo taquigráfico, mas bastante ilustrativo, o teorema da inversão permite que escrevamos um determinado espinor ψ da seguinte forma:

$$\psi \simeq \{``Soma'' \quad dos \quad Bilineares \quad Covariantes\} e^{i\theta}\eta,$$

onde η é um espinor arbitrário, θ uma fase (cuja presença é necessária para que se obtenha os graus de liberdade corretos) e os bilineares estão contraídos com a base apropriada da álgebra de Clifford para permitir a soma de elementos pertencentes a diferentes seções da álgebra exterior.

As diferentes classes de espinores surgem quando consideramos diferentes combinações dos bilineares. Entretanto, nem toda combinação de bilineares leva a um novo espinor. A razão é que os bilineares precisam satisfazer um conjuntos de identidades algébricas, as identidades de Fierz-Pauli-Kofink. Levando-se em contas tais identidades, mostra-se a existência de seis, e somentes seis, classes distintas de espinores. As três primeiras classes são reservadas aos chamados espinores regulares, isto é aqueles que possuem $\sigma = \bar{\psi}\psi$ e/ou $\omega = \bar{\psi}\gamma^5\psi$ não nulos e os outros bilineares podendo ou não ser nulos (exceção feita a $\mathbf{J} = j_\mu\gamma^\mu$ que é sempre não-nulo na formulação usual), desde que as identidades de Fierz-Pauli-Kofink sempre sejam satisfeitas. As demais categorias de espinores perfazem os chamados espinores singulares, para os quais $\sigma = 0 = \omega$ com os outros bilineares respeitando os mesmos preceitos do caso anterior.

Em primeiro lugar, é relevante se salientar que a classificação acima descrita é bastante relevante do ponto de vista físico, uma vez que se vale dos bilineares, ou seja dos observáveis fermiônicos, para ser estabelecida. Há dois pontos, contudo, que merecem ser enfatizados aqui, haja visto o fato que a observação de cada um deles deu origem a um dos trabalhos que se seguirão. O primeiro ponto é um resultado da classificação: o caso particular do espinor tipo-4 nunca havia tido uma contrapartida física. Dito de outro modo, nunca havia se encontrado um sistema físico envolvendo espinores que se encaixavam na descrição de um tipo-4. O próprio autor da classificação chama a atenção para esse ponto. O primeiro dos dois trabalhos a seguir trata exatamente de um sistema físico cujos espinores se enquadram na descrição de um tipo-4.

O sistema que possibilita um tal resultado é, entretanto, bastante não usual. Trata-se de férmions em espaços curvos cujo *background* é dado pela gravitação $f(R)$ com campos de torção. Apesar da complexidade, o sistema admite solução; dada exatamente em termos dos referidos espinores tipo-4. Obviamente é um caso restrito, por tratar-se de um modo do campo espinorial. Ainda assim, dada o caráter do espinor, é digno de nota. Antes de nos voltarmos ao segundo ponto relevante da classificação, gostaríamos apenas de enfatizar o fato de que espinores em espaços curvos também são passíveis de classificação através dos bilineares covariantes frente transformações de Lorentz bastando que, para tanto, procedamos à construção do fibrado tangente à variedade de base em questão.

O ponto que nos remete ao segundo trabalho a seguir reside no fato de que toda a classificação de espinores baseada nos bilineares se utiliza do vínculo $\mathbf{J} \neq 0$. Essa imposição não é algébrica, mas física. De fato, levando-se em conta espinores que obedecem à equação de Dirac, as componentes j_μ do bilinear $\mathbf{J}$ são as componentes da corrente conservada. Logo, $\mathbf{J} = 0$ implicaria a não existência de corrente fermiônica conservada o que, já no caso mais simples, levaria à inexistência da partícula fermiônica a que se quer descrever com o espinor. O raciocínio que embasa o segundo trabalho é, então, o seguinte: se $\mathbf{J} = 0$ o espinor em questão não pode satisfazer a equação de Dirac (pelos motivos anteriormente expostos). Assim, levando-se em conta que toda a classificação é relativista, o espinor com $\mathbf{J} = \mathbf{0}$ pode obedecer apenas à equação de Klein-Gordon. Logo, se as identidades de Firez-Pauli-Kofink permitirem, seria possivel se classificar espinores que satisfazem exclusivamente a equação de Klein-Gordon. Isso de fato foi verificado, e via formalismo operatorial encontramos três diferentes classes de espinores nesse caso, todos eles singulares.

A partir daí extrapolamos a análise, do ponto de vista semântico, chamando tal categorização de classificação de (alguns) espinores de dimensão canônica de massa um. Nesse caso, os espinores que encontramos seriam os coeficientes de expansão do campo quântico correspondente cuja dinâmica seria herdada da análise anterior.

JOURNAL OF MATHEMATICAL PHYSICS **54**, 102505 (2013)

Flag-dipole spinor fields in ESK gravities

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(Received 7 March 2013; accepted 8 October 2013; published online 29 October 2013)

We consider the Riemann-Cartan geometry as a basis for the Einstein-Sciama-Kibble theory coupled to spinor fields: we focus on $f(R)$ and conformal gravities, regarding the flag-dipole spinor fields, type-(4) spinor fields under the Lounesto classification. We study such theories in specific cases given, for instance, by cosmological scenarios: we find that in such background the Dirac equation admits solutions that are not Dirac spinor fields, but in fact the aforementioned flag-dipoles ones. These solutions are important from a theoretical perspective, as they evince that spinor fields are not necessarily determined by their dynamics, but also a discussion on their structural (algebraic) properties must be carried off. Furthermore, the phenomenological point of view is shown to be also relevant, since for isotropic Universes they circumvent the question whether spinor fields do undergo the Cosmological Principle.
 [http://dx.doi.org/10.1063/1.4826499]

I. INTRODUCTION

In a geometry that incorporates a differential structure, the introduction of covariant derivatives is as inevitable as the definition of the metric. Moreover, the connection in the most general case is not symmetric as well as the metric is not a constant, giving rise respectively to torsion and curvature.

On the other hand, torsion might play an important role from a genuine physical point of view, as the spacetime curvature already does and is undeniably measured in numerous experiments. In fact, according to the Wigner classification of particles in terms of their masses and spin, physical fields are known to be characterized by both the energy and the spin density. In the most general case, all geometric quantities can be coupled to corresponding physical fields, through specific field equations. Therefore, in the same spirit in which Einstein gravity couples curvature to energy, in the most general case this coupling is still valid. Besides, it is also accompanied by a correspondent coupling between torsion and spin. Then Einstein gravity is not the most general case but the most general spinless situation, in the sense that it is only the most general dynamical solution in absence of any spinning matter of Einstein-Sciama-Kibble theory.[1] Here, by Einstein-Sciama-Kibble (ESK) theory we assume it in a broad sense as any torsional completion of gravity, no matter the order-derivatives of the field equations that define it.

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0022-2488/2013/54(10)/102505/12/$30.00 **54**, 102505-1

The structure of ESK gravity is then constructed on the scheme for which we have curvature-energy as well as torsion-spin field equations. What is known to be the ESK theory in the strict sense is realized by insisting that those field equations have the least-order derivative possible, but more general ESK-like theories are possible by relaxing this condition. Of course all possible ESK-like theories are infinite, not all of them are physical, and even among the physical ones, not all of them can be sensibly considered. So a choice is to be made, and ours shall be on those that at the moment are the most in fashion: $f(R)$-gravity and the conformally gravity.

Motivated by those considerations, the ESK theory may be considered, both in its $f(R)$ and in its conformal realization, with general spinning matter fields. For the case of spinorial matter, throughout the paper we shall employ spin-$\frac{1}{2}$ spinor fields, solely. As for the case of gravity we allowed ourselves to consider higher-order derivative extensions, for the case of matter fields we shall do the same by allowing ourselves to take into account higher-order derivative spinorial matter field equations, but we shall not take into account higher-spin fields. This restriction can be dictated by the fact that such higher-spin fields may be unphysical, displaying inconsistency, non-causality, and other problems,[2] or simply because it is not possible to consider all possibilities and a choice must be made. Nevertheless, as just noticed, we shall allow ourselves to go beyond the first-order derivative field equations.

In this paper, the spinor fields shall not be called Dirac spinors, as many more possibilities can be met.[3,20] In Ref. 3, Lounesto proceeds with the classification of the possible spin-$\frac{1}{2}$ spinors, categorizing them within six classes: Dirac fields, in various forms, belong to the first three of them; flag-dipoles and flagpoles are the fourth and fifth type of spinor fields, disseminated in the literature as a mathematical apparatus to support Penrose flags,[7] among other interesting applications; Weyl spinors is within the sixth class. As the first three classes, as well as the fifth and sixth ones, are prominently relevant in quantum field theory and its phenomenology, the fourth class should be better understood, and hereon we shall therefore focus mainly on the flag-dipoles.

No matter what spinor field we consider, in ESK theories torsion shall always be coupled to the spin density of the matter field. Therefore, after that all terms involving the covariant derivatives and the curvature are split in their torsionless counterparts plus torsional contributions, the latter can be substituted through the torsion-spin field equations in terms of the spin density of the spinorial matter field. All field equations of the ESK theory thus reduce to the same equations of the torsionless theory complemented by spin-spin self-interacting potentials, and thus nonlinearities appear in the matter field equations. This is general, and the specific gravitational background ($f(R)$ of conformal) and type of spinor (Dirac or flag-dipole, or other still) shall determine the exact structure of these nonlinearities in the matter field equations. For example, in the least-order derivative ESK gravity with Dirac fields, the nonlinearities are given in terms of axial current squared contact interactions, that is with the structure of the Nambu-Jona-Lasinio (NJL) potential.[5] As we shall see, in $f(R)$ gravity they shall turn out to be structurally similar apart from a scaling function as a running coupling, while in conformal gravity they might be entirely different. In all these cases, however, when the spinor field is a flag-dipole, the interaction is shown to change.

In what follows, we aim to study the flag-dipole type-(4) spinor fields dynamics in the case of an ESK theory, whether $f(R)$ or conformal gravity: we plan to show that in such a context, matter fields which are solutions of the Dirac equation are not necessarily Dirac spinor fields by exhibiting a physical solution of the Dirac equation that is instead a flag-dipole spinor field; in this case then, we shall be able to show, through a specific example, that a spinor field is not fully determined by its dynamics since spinor fields obeying the Dirac equation are not necessarily Dirac spinor fields.

This paper is organized as follows: in Sec. II we introduce the Lounesto classification program according to the bilinear covariants and provide some necessary concepts concerning type-(4) spinor fields. In Sec. III we study the spinor fields solutions in the context of torsional $f(R)$ gravity and conformal gravity cases, showing that they are non-standard singular classes under Lounesto spinor field classification. In Sec. IV we conclude. In the Appendix we show how to construct the most general type-(4) flag-dipole spinor field.

II. NON-STANDARD (FLAG-DIPOLE) SPINOR FIELDS

This section is devoted to briefly provide some properties on the flag-dipole spinor fields, where the most relevant general properties regarding such spinor fields, and the notation fixed throughout the text as well, are introduced.

Classical spinor fields carry the (1/2, 0) $\oplus$ (0, 1/2) representation of the Lorentz group $SL(2,\mathbb{C}) \simeq \mathrm{Spin}^e_{1,3}$. They are sections of the vector bundle $\mathbf{P}_{\mathrm{Spin}^e_{1,3}}(M) \times_\rho \mathbb{C}^4$, where ρ denotes the (1/2, 0) $\oplus$ (0, 1/2) representation of $SL(2,\mathbb{C})$ in $\mathbb{C}^4$. Furthermore, classical spinor fields can be sections of the vector bundle $\mathbf{P}_{\mathrm{Spin}^e_{1,3}}(M) \times_{\rho'} \mathbb{C}^2$, where ρ' is the (1/2, 0) or the (0, 1/2) representation of $SL(2,\mathbb{C})$ in $\mathbb{C}^2$. Given a spinor field ψ, the bilinear covariants are defined as

$$\sigma = \psi^\dagger\gamma_0\psi, \quad \mathbf{J} = J_\mu\theta^\mu = \psi^\dagger\gamma_0\gamma_\mu\psi\theta^\mu, \quad \mathbf{S} = S_{\mu\nu}\theta^{\mu\nu} = \frac{1}{2}\psi^\dagger\gamma_0 i\gamma_{\mu\nu}\psi\theta^\mu\wedge\theta^\nu,$$
$$\mathbf{K} = K_\mu\theta^\mu = \psi^\dagger\gamma_0 i\gamma_{0123}\gamma_\mu\psi\theta^\mu, \quad \omega = -\psi^\dagger\gamma_0\gamma_{0123}\psi. \tag{1}$$

Here $\{\gamma_\mu\}$ denotes to the Dirac matrices, and the objects in (1) satisfy the Fierz identities[3,8,9] $\mathbf{J}^2 = \omega^2 + \sigma^2$, $\mathbf{J}\llcorner\mathbf{K} = 0$, $\mathbf{K}^2 = -\mathbf{J}^2$, and $\mathbf{J}\wedge\mathbf{K} = -(\omega + \sigma\gamma_{0123})\mathbf{S}$. A spinor field such that at least one of the ω and the σ are null [not null] is said to be singular [regular]. The Lounesto spinor field classification is provided by the following spinor field classes:[3]

(1) $\sigma \neq 0,\ \omega \neq 0$	(4) $\sigma = 0 = \omega,\quad \mathbf{K} \neq 0,\quad \mathbf{S} \neq 0$
(2) $\sigma \neq 0,\ \omega = 0$	(5) $\sigma = 0 = \omega,\quad \mathbf{K} = 0,\quad \mathbf{S} \neq 0$
(3) $\sigma = 0,\ \omega \neq 0$	(6) $\sigma = 0 = \omega,\quad \mathbf{K} \neq 0,\quad \mathbf{S} = 0$

Types-(1), -(2), and -(3) are named Dirac spinor fields in the Lounesto classification, and in these cases it is implicit that $\mathbf{J}, \mathbf{K}, \mathbf{S} \neq 0$. Types-(4), -(5), and -(6) are, respectively, called flag-dipole, flagpole, and Weyl spinor fields. For Dirac spinor fields, $\mathbf{S}$ is the distribution of intrinsic angular momentum; $\mathbf{J}$ is associated with the current of probability, and $\mathbf{K}$ is associated with the direction of the electron spin.[3,8,9] By introducing the element $Z = \sigma + \mathbf{J} + i\mathbf{S} + i\mathbf{K}\gamma_{0123} + \omega\gamma_{0123}$, Z is denominated a boomerang whenever it satisfies $\gamma_0 Z^\dagger\gamma_0 = Z$. When a spinor field is singular, namely it satisfies $\sigma = 0 = \omega$, the Fierz identities are substituted by the more general identities:[8]

$$Z^2 = \sigma Z, \quad Zi\gamma_{\mu\nu}Z = 4S_{\mu\nu}Z, \quad Z\gamma_\mu Z = 4J_\mu Z, \quad Zi\gamma_{0123}\gamma_\mu Z = 4K_\mu Z, \quad Z\gamma_{0123}Z = -4\omega Z. \tag{2}$$

When one considers a type-(4) flag-dipole spinor field, the distribution of intrinsic angular momentum is provided by $\mathbf{S} = \mathbf{J}\wedge s$, where s is a spacelike vector orthogonal to $\mathbf{J}$. The real number $h \neq 0$ is such that $\mathbf{K} = h\mathbf{J}$, evincing thus the definition of helicity. It satisfies $h^2 = 1 + s^2$, implying the definition of helicity h in quantum mechanics.[10] Type-(5) spinor fields are a particular case where $h = 0$. Indeed, by $\mathbf{K} = h\mathbf{J}$, when $h = 0$ the expressions $\omega = 0 = \sigma$, $\mathbf{K} = 0$, $\mathbf{J} \neq 0$ hold. Type-(5) spinor fields are therefore limiting cases of type-(4) spinor fields. More details on the most general form of type-(4) spinor fields are provided in the Appendix.

III. MATTER FIELDS IN RIEMANN-CARTAN GEOMETRIES

Once some features related to type-(4) spinor fields are introduced, we shall take into account the Wigner classification, to further study the spinor fields properties. According to the Wigner classification, in terms of irreducible representations of the Poincaré group, quantum particles are classified in terms of their mass and spin labels. The corresponding quantities for the quantum fields are given in terms of energy and spin densities. If one wishes to pursue the same spirit that Einstein followed to develop a theory for gravity, expressing the field equations by coupling the curvature to energy, in the most general case where torsion is present, one is compelled to recover the field equations coupling the curvature to energy but accompanied by similar field equations coupling the torsion to spin. When this is accomplished in the most straightforward way, the Einstein equations for the curvature-energy coupling are generalized as to include the Sciama-Kibble equations for the torsion-spin coupling. Namely, the ESK system of field equations, which can be obtained by generalizing the Ricci scalar written in terms of the metric $R(g)$ by the Ricci scalar written in terms of both metric and torsion $R(g, T)$ in the action, and subsequently varying it with respect to the two independent fields.

Notwithstanding, this is merely the most straightforward generalization of gravity with torsion. Other more general theories can be obtained by adding torsion not only implicitly through the curvature, but explicitly as well, as quadratic terms beside the curvature $R(g, T) + T^2$ in the action. Once the field equations are written down, and all torsional contributions are separated and evinced as spinor interactions, the effects of these extensions are reduced to a simple scaling of the torsional terms, or equivalently of the spinor interaction. It is evinced by introducing new coupling constants for such spin potentials. One of the most important problems about torsion in gravity, namely the fact that torsion should have been relevant only at the Planck scales, can thus be overcome since in these theories torsion has its own coupling constant, that does not necessarily coincide with the gravitational constant.[11,12]

On the other hand, however, those theories do not encompass the possibility to have *dynamical* extensions, such as those provided by higher-order derivative field equations. The two most important ones are the case for which the Ricci scalar R is replaced by an arbitrary function $f(R)$ in the action,[13] and the one that is capable of implementing the conformal symmetry in the action itself.[14,15] In the following we shall deal with both of them.

A. Torsional *f*(*R*)-Gravity

The extension of the Einstein-Hilbert action regarding an arbitrary function $f(R)$ is captivating, since it is the most general whenever one restricts the Ricci scalar as the sole source of dynamical information. In the case where both the metric and the torsion as well are taken into account, the variation with respect to an arbitrary metric g and a g-compatible connection Γ (or equivalently a tetrad field e and a spin-connection ω) yields the metric-affine (or tetrad-affine) approach(es).[16–19] The correspondent field equations are

$$T_{ij}{}^{h} = \frac{1}{f'(R)}\left[\frac{1}{2}\left(\frac{\partial f'(R)}{\partial x^{p}} + S_{pq}{}^{q}\right)\epsilon_{r}{}^{ph}\epsilon_{i}{}^{r}{}_{j} + S_{ij}{}^{h}\right], \tag{3a}$$

$$\Sigma_{ij} = f'(R)R_{ij} - \frac{1}{2}f(R)g_{ij}, \tag{3b}$$

where R_{ij}, ϵ_{ijk}, and $T_{ij}{}^{h}$ are the Ricci, the Levi-Civita, and the torsion tensors, respectively. The Σ_{ij} and $S_{ij}{}^{h}$ denote the stress-energy and spin density tensors associated to the matter fields: the conservation laws

$$\nabla_i \Sigma^{ij} + T_i \Sigma^{ij} - \Sigma_{pi} T^{jpi} - \frac{1}{2} S_{sti} R^{stij} = 0, \tag{4a}$$

$$\nabla_h S^{ijh} + T_h S^{ijh} + \Sigma^{ij} - \Sigma^{ji} = 0, \tag{4b}$$

come from the Bianchi identities.[13] In Eq. (4) the symbols ∇_i and R^{ijkl} denote, respectively, the covariant derivative and the curvature tensor, with respect to the dynamical connection Γ. By denoting $\Gamma^i = e^i_\mu \gamma^\mu$, where e^μ_i is a tetrad associated with the metric, and by introducing $\mathrm{S}_{\mu\nu} := \frac{1}{8}[\gamma_\mu, \gamma_\nu]$, the covariant derivatives of the matter field ψ and its Dirac adjoint are denoted by $D_i\psi = \frac{\partial\psi}{\partial x^i} + \omega_i{}^{\mu\nu}\mathrm{S}_{\mu\nu}\psi$ and $D_i\bar{\psi} = \frac{\partial\bar{\psi}}{\partial x^i} - \bar{\psi}\omega_i{}^{\mu\nu}\mathrm{S}_{\mu\nu}$, where $\omega_i{}^{\mu\nu}$ is the spin connection. One can furthermore indite $D_i\psi = \frac{\partial\psi}{\partial x^i} - \Omega_i\psi$ and $D_i\bar{\psi} = \frac{\partial\bar{\psi}}{\partial x^i} + \bar{\psi}\Omega_i$

$$\Omega_i := -\frac{1}{4} g_{jh}\left(\Gamma_{ik}{}^{j} - e^j_\mu \partial_i e^\mu_k\right)\Gamma^h\Gamma^k, \tag{5}$$

where $\Gamma_{ik}{}^{j}$ denote the coefficients of the linear connection Γ, since the relation between linear and spin connection is provided by $\Gamma_{ij}{}^{h} = \omega_i{}^{\mu}{}_{\nu} e^h_\mu e^\nu_j + e^h_\mu \partial_i e^\mu_j$, as can be immediately calculated. In the case of matter fields, the spin density tensor is given by $\mathrm{SS}_{ij}{}^{h} = \frac{i}{2}\overline{\psi}\left\{\Gamma^h, \mathrm{S}_{ij}\right\}\psi \equiv -\frac{1}{4}\eta^{\mu\sigma}\epsilon_{\sigma\nu\lambda\tau}K^{\tau}e^{h}_{\mu}e^{\nu}_{i}e^{\lambda}_{j}$. Remember that K^{τ} is the component of the pseudo-vector bilinear covariant defined at (1). The stress-energy tensor components of the matter fields are hence described as

$$\Sigma^{D}_{ij} := \frac{i}{4}\left(\overline{\psi}\Gamma_i D_j\psi - D_j\overline{\psi}\Gamma_i\psi\right) \qquad \text{and} \qquad \Sigma^{F}_{ij} := (\rho + p)U_iU_j - pg_{ij}. \tag{6}$$

In Eq. (6), ρ, p, and U_i denote, respectively, the matter-energy density, the pressure, and the four-velocity of the fluid. The trace of Eq. (3b), given by

$$f'(R)R - 2f(R) = \Sigma, \tag{7}$$

is supposed to relate the Ricci scalar curvature R and the trace Σ of the stress-energy tensor, as in Ref. 13 and 16–18. Furthermore, it is assumed that $f(R) \neq kR^2$—since the case $f(R) = kR^2$ is solely compatible to the condition $\Sigma = 0$. Now, from Eq. (7) it is possible to express $R = F(\Sigma)$, where F is an arbitrary function. Furthermore, introducing the scalar field $\varphi := f'(F(\Sigma))$ as well as the effective potential $V(\varphi) := \frac{1}{4}\left[\varphi F^{-1}((f')^{-1}(\varphi)) + \varphi^2(f')^{-1}(\varphi)\right]$, the field equations (3b) are written in the Einstein-like form

$$\begin{aligned} \mathring{R}_{ij} - \frac{1}{2}\mathring{R}g_{ij} = \frac{1}{\varphi}\Sigma^{F}_{ij} + \frac{1}{\varphi}\Sigma^{D}_{ij} + \frac{1}{\varphi^2}\Big(&-\frac{3}{2}\varphi_i\varphi_j + \varphi\mathring{\nabla}_j\varphi_i + \frac{3}{4}\varphi_h\varphi_k g^{hk}g_{ij} \\ &-\varphi\mathring{\nabla}^h\varphi_h g_{ij} - V(\varphi)g_{ij}\Big) + \mathring{\nabla}_h\hat{\mathrm{S}}_{ji}{}^{h} + \hat{\mathrm{S}}_{hi}{}^{p}\hat{\mathrm{S}}_{jp}{}^{h} - \frac{1}{2}\hat{\mathrm{S}}_{hq}{}^{p}\hat{\mathrm{S}}^{q}{}_{p}{}^{h}g_{ij}, \end{aligned} \tag{8}$$

where $\mathring{R}_{ij}$, $\mathring{R}$, and $\mathring{\nabla}_i$ denote, respectively, the Ricci tensor, the Ricci scalar curvature, and the covariant derivative of the Levi–Civita connection. Here $\hat{\mathrm{S}}_{ij}{}^{h} := -\frac{1}{2\varphi}\mathrm{S}_{ij}{}^{h}$ and $\varphi_i := \frac{\partial\varphi}{\partial x^i}$. In addition, the generalized Dirac equations for the spinor field are in this context

$$i\Gamma^h D_h\psi + \frac{i}{2}T_h\Gamma^h\psi - m\psi = 0, \tag{9}$$

where $T_h := T_{hj}{}^{j}$ is the axial torsion.[30] The symmetrized part of the Einstein-like equations (8) as well as the Dirac equations (9) are written as[13]

$$\begin{aligned} \mathring{R}_{ij} - \frac{1}{2}\mathring{R}g_{ij} = \frac{1}{\varphi}\Sigma^{F}_{ij} + \frac{1}{\varphi}\mathring{\Sigma}^{D}_{ij} + \frac{1}{\varphi^2}\Big(&-\frac{3}{2}\varphi_i\varphi_j + \varphi\mathring{\nabla}_j\varphi_i + \frac{3}{4}\varphi_h\varphi_k g^{hk}g_{ij} \\ &-\varphi\mathring{\nabla}^h\varphi_h g_{ij} - V(\varphi)g_{ij}\Big) + \frac{3}{64\varphi^2}K^{\tau}K_{\tau}g_{ij} \end{aligned} \tag{10}$$

and

$$i\Gamma^h\mathring{D}_h\psi - \frac{3}{16\varphi}\left[\sigma + i\omega\gamma_5\right]\psi - m\psi = 0, \tag{11}$$

where $\mathring{\Sigma}^{D}_{ij} := \frac{i}{4}\left[\overline{\psi}\Gamma_{(i}\mathring{D}_{j)}\psi - \left(\mathring{D}_{(j}\overline{\psi}\right)\Gamma_{i)}\psi\right]$ and $\mathring{D}_i$ is the covariant derivative of the Levi–Civita connection.

As spinor fields satisfying the Dirac equation in this scenario are incompatible with stationary spherical symmetry,[21] the simplest choice for the background must be at least an axially symmetric Bianchi-I type metric, given by the form $ds^2 = dt^2 - a^2(t)\,dx^2 - b^2(t)\,dy^2 - c^2(t)\,dz^2$, where the $\Gamma^i = e^{i}_{\mu}\gamma^{\mu}$ are given by

$$\Gamma^0 = \gamma^0, \qquad \Gamma^1 = \frac{1}{a(t)}\gamma^1, \qquad \Gamma^2 = \frac{1}{b(t)}\gamma^2, \qquad \Gamma^3 = \frac{1}{c(t)}\gamma^3, \tag{12}$$

and the tetrad field is given by $e^{\mu}_{0} = \delta^{\mu}_{0}$, $e^{\mu}_{1} = a(t)\delta^{\mu}_{1}$, $e^{\mu}_{2} = b(t)\delta^{\mu}_{2}$, and $e^{\mu}_{3} = c(t)\delta^{\mu}_{3}$, for $\mu = 0, 1, 2, 3$. The spin-Dirac operator acts on spinor fields and their conjugates, respectively, as $\mathring{D}_i\psi = \partial_i\psi - \mathring{\Omega}_i\psi$ and $\mathring{D}_i\overline{\psi} = \partial_i\overline{\psi} + \overline{\psi}\mathring{\Omega}_i$, where the spin connection coefficients $\mathring{\Omega}_i$ are given by (introducing the notation $a_1 = a$, $a_2 = b$, and $a_3 = c$)

$$\mathring{\Omega}_0 = 0, \qquad \mathring{\Omega}_i = \frac{1}{2}\dot{a}_i\gamma^i\gamma^0.$$

Therefore, the Einstein-like equation (10) reads

$$\frac{\dot a}{a}\frac{\dot b}{b}+\frac{\dot b}{b}\frac{\dot c}{c}+\frac{\dot a}{a}\frac{\dot c}{c}=\frac{\rho}{\varphi}-\frac{3}{64\varphi^2}K^\sigma K_\sigma+\frac{1}{\varphi^2}\left[-\frac{3}{4}\dot\varphi^2-\varphi\dot\varphi\frac{\dot\tau}{\tau}-V(\varphi)\right], \tag{13a}$$

$$\frac{\ddot a_r}{a_r}+\frac{\ddot a_s}{a_s}+\frac{\dot a_r}{a_r}\frac{\dot a_s}{a_s}=-\frac{p}{\varphi}+\frac{1}{\varphi^2}\left[\varphi\dot\varphi\frac{\dot a_t}{a_t}+\frac{3}{4}\dot\varphi^2-\varphi\left(\ddot\varphi+\frac{\dot\tau}{\tau}\dot\varphi\right)-V(\varphi)\right]+\frac{3}{64\varphi^2}K^\sigma K_\sigma, \tag{13b}$$

where r, s, t denote indexes 1, 2, 3 different from each other. The Dirac field equation (11) assumes the form

$$\dot\psi+\frac{\dot\tau}{2\tau}\psi+im\gamma^0\psi-\frac{3i}{16\phi}(\sigma\gamma^0+i\omega\gamma^0\gamma^5)\psi=0, \tag{14}$$

where $\tau := abc$.[22,23] Together with the conditions

$$\mathring{\Sigma}^D_{rs}=0 \quad\Rightarrow\quad a_r\dot a_s-a_s\dot a_r=0 \quad\cup\quad K^{\mathsf{T}}=0, \tag{15}$$

the equations $\mathring{\Sigma}^D_{0i}=0$ are automatically satisfied. Finally, the conservation laws together with an equation of state of the kind $p=\lambda\rho$ (here λ is a number between 0 and 1) yield $\dot\rho+\frac{\dot\tau}{\tau}(1+\lambda)\rho=0$, which completes the whole set of field equations, having the general solution given by

$$\rho=\rho_0\tau^{-(1+\lambda)}, \qquad \rho_0=\text{constant}. \tag{16}$$

The matter field in such axially symmetric background is such that conditions (15) are constraints imposed on the metric or on the matter field. They exist if and only if one of the following conditions holds:

a) by imposing constraints of purely geometrical origin, as $a\dot b-b\dot a=0$, $a\dot c-c\dot a=0$, $c\dot b-b\dot c=0$. In this scenario there are fermionic matter fields in an isotropic Universe, which might *a priori* cause some pathology, as Dirac fields are well known not to undergo the Cosmological Principle.[24] But the result by Tsamparlis,[24] although valid for Dirac spinor fields, does not hold for the other spinor field classes, according to Lounesto classification;
b) another condition is to impose constraints of purely material origin by requiring that the spatial components of the spin direction satisfy $K_i=0$. This represents an anisotropic Universe devoid of terms coupling matter to the axial torsion. In this case there is no fermionic torsional interactions. Indeed, the particle spin interacts with the axial component of the torsion tensor, and when the spatial components of the spin direction equal zero it implies that such particles described by the field ψ do not interact to the torsion. Besides, if Dirac fields are absent then it is not clear what may then justify anisotropies;[6]
c) the last situation would be originated by the geometry and the matter as well, by insisting that, for instance, $a\dot b-b\dot a=0$ and $K_1=0=K_2$. It provides partial isotropy for only two axes, with the corresponding components of the spin vector vanishing. It describes a Universe an ellipsoid of rotation about the axis along which the spin vector does not vanish. By insisting on the proportionality between two pairs of axes we inevitably get the total isotropy of the 3-dimensional space. Therefore, the situation in which we have $a=b$, with $K_1=0=K_2$, is the only one be entirely satisfactory. Henceforth, this situation shall be considered, where the sole spatial component of the spin direction is $K_3\neq 0$.

Here, the Dirac and Einstein-like equations (13) and (14) can be worked out as in Refs. 22 and 23: for instance, through suitable combinations of (13) we obtain the equations

$$\frac{d}{dt}(J_0\tau)=0=\frac{d}{dt}(\sigma\tau)+\frac{3\omega K_0\tau}{8\varphi}, \tag{17a}$$

$$-\frac{d}{dt}(\omega\tau)+\left[2m+\frac{3\sigma}{8\varphi}\right]K_0\tau=0=\frac{d}{dt}(K_0\tau)+2m\omega\tau. \tag{17b}$$

while from Eqs. (17) it is straightforward to deduce that

$$(K_3)^2 = \sigma^2 + \omega^2 + (K_0)^2 = \frac{C^2}{\tau^2}, \qquad\qquad (J_0)^2 = \frac{D^2}{\tau^2}, \tag{18}$$

with C and D constants. It is worthwhile to emphasize that in this special case the theory has an additional discrete symmetry provided by the transformation $\psi \mapsto \gamma^5\gamma^0\gamma^1\psi$, making all field equations are invariant. In the Dirac equation the four complex components is in this case reduced to two complex components. Such assertion is equivalent to take flagpole spinor fields, that have four real parameters. Hence (17) are the field equations to be solved. The compatibility to all constraints allows only three classes of spinor fields, each of which has a general member written in one of the following forms:

$$\psi = \frac{1}{\sqrt{2\tau}}\begin{pmatrix} \sqrt{K-C}\cos\zeta_1 e^{i\theta_1} \\ \sqrt{K+C}\cos\zeta_2 e^{i\vartheta_1} \\ \sqrt{K-C}\sin\zeta_1 e^{i\vartheta_2} \\ \sqrt{K+C}\sin\zeta_2 e^{i\vartheta_2} \end{pmatrix},$$

with constraints $\tan\zeta_1\tan\zeta_2 = (-1)^{n+1}$ and $\theta_1 + \theta_2 - \vartheta_1 - \vartheta_2 = \pi n$ for any n integer, and also

$$\psi = \frac{1}{\sqrt{2\tau}}\begin{pmatrix} \sqrt{K-C}\cos\zeta_1 e^{i\theta_1} \\ 0 \\ 0 \\ \sqrt{K+C}\sin\zeta_2 e^{i\theta_2} \end{pmatrix} \quad \text{and} \quad \psi = \frac{1}{\sqrt{2\tau}}\begin{pmatrix} 0 \\ \sqrt{K+C}\cos\zeta_1 e^{i\vartheta_1} \\ \sqrt{K-C}\sin\zeta_2 e^{i\vartheta_2} \\ 0 \end{pmatrix}. \tag{19}$$

where $\zeta_1, \zeta_2, \theta_1, \theta_2, \vartheta_1, \vartheta_2$ are time dependent. The most interesting case is one the provided by (19). For instance, the second spinor field at (19) is

$$\psi = \frac{1}{\sqrt{2\tau}}\begin{pmatrix} 0 \\ \sqrt{K+C}e^{i\beta(t)} \\ \sqrt{K-C}e^{-i\beta(t)} \\ 0 \end{pmatrix}, \tag{20}$$

for $\beta(t) = -mt - \frac{3C}{16}\int\frac{dt}{\tau}$. There are further constraints $\sigma = \frac{C}{\tau}$, $\psi^\dagger\psi = \frac{K}{\tau}$ and $\omega = 0 = K_0$. Such a spinor field *is a type-(4) flag-dipole spinor field, according to the Lounesto spinor fields classification.*[25] This is a remarkable fact: once it is assumed a spinor field ψ in a *f*(*R*) Riemann-Cartan cosmology, some type-(4) spinor fields are obtained as the spinor fields (19). Indeed, there is no assumption in Eq. (9) that makes ψ a legitimate Dirac spinor field, as it merely regards *a priori* a spinor field ψ that satisfies the Dirac equation. As far as we know, this is up to now the unique physical system whose acceptable solution is given in terms of such spinor fields.

On the other hand, when one imposes $K_3 = 0$ as a constraint of purely material origin, Eq. (18) implies that $K_0 = 0$. Therefore, $K^\mu = 0$ and we obtain a type-(5) spinor field under Lounesto spinor field classification, which encompasses Majorana and Elko dark spinor fields. It must be stressed that the condition $K_3 = 0$ does not necessarily imply that in this case there is no fermion fields satisfying the Dirac equation (9). In fact, Elko fields do not satisfy the Dirac equation at all.[31]

In summary, by the solutions above, the so-called Dirac field ψ in (19) and (20) is not a Dirac spinor field according to Lounesto classification, but a type-(4) flag-dipole spinor field. Besides, since $K_i = 0$ and in particular $K_3 = 0$, by (18) it implies that we are concerning now a type-(5) spinor field, which is a flagpole. But in this case, it is well-known that type-(5) encompasses Elko, Majorana, and the complementary spinor fields, presented at (A7). Elko, however, is well known not to satisfy Dirac equations, so as we departure from (18), Elko is excluded to be a solution of such system. The point to be stressed here is that according to the Lounesto spinor field classification, ψ

can be allocated in any of the six disjoint classes and there is no *ab initio* relation between the type of the spinor field and the associated dynamics. As mentioned, for instance, the types-(1), (2), and (3) are Dirac spinor fields in the Lounesto classification, having some subset satisfying the Dirac equation. By the same token, type-(6) spinor fields encompass Weyl spinor fields, that indeed satisfy Dirac equations. Nevertheless, it was an open problem whether type-(4) spinor fields satisfy or not the Dirac equations, but the Dirac equations is shown to be dynamically forbidden for the solutions found.[6]

B. Torsional conformal-gravity

It is worth to point some recent progress in the study of spinor fields in generalized gravity, as well as some open issues which are under current investigation. While it is somewhat apart of the main theme of the paper, it is certainly enriching from the bookkeeping purposes. In this vein, another interesting higher-order theory of gravity is the one with two curvatures, because this is the only case in which conformal invariance can be obtained.[14] As it turns out, there are two ways to implement conformal transformations for torsion: the first is to require the most general (reasonable) conformal transformation for torsion (where by reasonable we mean reasonable according to what is discussed, for instance, in Ref. 27). The another is to insist on the fact that no conformal transformation is to be given to torsion (because conformal transformations are of metric origin while torsion is independent on the metric). In the former case, because conformal transformations link the metric to torsion, one must modify the Riemann curvature with quadratic-trace torsion terms in order to get a curvature whose irreducible part is conformally invariant.[14] In the latter case, torsion and curvature are separated and essentially independent. Consequently, in the former case[14] the field equations are closely intertwined together, while in the latter case the field equations are independent thus maintaining the curvature-energy and torsion-spin coupling in the spirit of the ESK field equations.

1. Torsion with general conformal transformations

In the first case the coupling to the Dirac field has been studied in Ref. 14. However, as in this case the field equations that couple torsion to spin are not invertible in general, torsion cannot be substituted by the spin density into the Dirac field equations, which therefore remain of the general form

$$i\gamma^\mu \mathring{D}_\mu \psi + \tfrac{3}{4} W_\sigma \gamma^5 \gamma^\sigma \psi = 0, \tag{21}$$

where W_σ is the axial vector dual of the completely antisymmetric part of the torsion tensor. Hence the arguments used in Ref. 21 cannot be recovered, and therefore stationary spherically symmetric symmetries are possible. However in such a case, the complete antisymmetry of the Dirac field does not turn into the complete antisymmetry of torsion. Instead, rather in constraints for the gravitational fields that cannot be satisfied in general situations. In this case of general torsional conformal transformations the Dirac field appears to be ill-defined.

An alternative situation is therefore to study Elko fields, which has been accomplished in Ref. 15. However, their dynamics in terms of cosmological solutions has not been studied yet.

2. Torsion with no conformal transformations

The coupling to the Dirac field was studied,[14] showing that the complete antisymmetry of the spin density results into the complete antisymmetry of the torsion tensor, which dual is an axial vector given by

$$W_\rho = \left(\tfrac{4a}{\hbar} K^\mu K_\mu\right)^{-1/3} J_\rho, \tag{22}$$

so that torsion can be replaced with the spin density of the spinor field, and the Dirac field equation becomes

$$i\gamma^\mu \mathring{D}_\mu \psi - \left(\tfrac{256a}{27} K^\rho K_\rho\right)^{-\frac{1}{3}} \overline{\psi}\gamma_\nu\psi\gamma^\nu\psi = 0, \tag{23}$$

with a nonlinear self-interaction that is renormalizable nonetheless. After a straightforward Fierz rearrangement they can be written as

$$i\gamma^{\mu}\mathring{D}_{\mu}\psi - \left(\tfrac{27}{256a}\right)^{1/3}\left(\sigma^{2} + i\omega^{2}\right)^{-1/3}\left(\sigma\mathbb{I} - \omega\gamma_{5}\right)\psi = 0, \tag{24}$$

clearly showing that the type-(4) spinor fields would verify a Dirac field equation of the form $i\gamma^{\mu}\mathring{D}_{\mu}\psi = 0$, as if torsion were never present, precisely like the ESK theory. In this case, it again happens that the reasoning performed in Ref. 21 does not apply, and stationary spherically symmetric solutions are possible, the gravitational field equations would reduce to the torsionless spherically symmetric Weyl field equations in a Schwarzschild spacetime. For this type of conformal gravity, the case of Elko fields has not been studied yet.

IV. CONCLUSIONS

In this paper, we have explored both the regular and singular spinor fields, establishing the general gravitational background with torsion in which the spinor fields are supposed to live in. We proved that some singular flag-dipoles spinor fields are physical solutions for the Dirac equation in ESK theories: in particular this has been obtained in $f(R)$-gravity but it could not be recovered in conformal gravity as well.

In the case of cosmology, when considering Dirac-like fields in $f(R)$-gravity, the presence of torsion imposes the use of an anisotropic background in which the geometric side is diagonal, while the energy tensor is not, due to intrinsic features of the spinor field. In this circumstance, the non-diagonal part of the gravitational field equations results into the constraints (15) characterizing the structure of the spacetime, or the helicity of the spinor field, or both. In our understanding, the only physically meaningful situation is the one in which two axes are equal and one spatial component of the axial vector torsion does not vanish. It provides a Universe that is spatially an ellipsoid of rotation revolving about the only axis along which the spin density is not equal to zero.

In the case of conformal gravity, except for the case of torsional conformal transformations, for which the Dirac field seems not well-defined, the case of torsion without conformal transformations appears to be well-posed. In this case, the gravitational background is much like the torsionless one, and although we have not proved it mathematically, there are reasons to believe that singular type-(4) spinor fields may still emerge.

In summary, the presence of torsion induces nonlinear interactions, whose details depend on whatever conformal of $f(R)$ gravitational background is used, but in general such torsionally-induced self-interactions for the spinors affect the dynamics of the spinor itself: specifically, it is possible to find perfectly physical solutions of the Dirac equation which are nevertheless not Dirac fields, but flag-dipoles, and thus singular. We have also found that, in addition, the new solutions encompass Elko and Majorana spinor fields, when the associated spin direction vanishes, providing an anisotropic Universe without fermionic torsional interactions.

However, we believe that the main message that is to be taken is that a spinor field satisfying the Dirac field equation is not necessarily nonsingular: with a metaphoric analogy, we may say that the Dirac equations does not necessarily take care of itself by forbidding singular solutions.

To remove them, an even deeper analysis must be carried over at an algebraic level.

ACKNOWLEDGMENTS

R. da Rocha and is grateful to CNPq Grant No. 451042/2012-3 for financial support. R. T. Cavalcanti is grateful to CAPES and to UFABC for financial support. J. A. Silva-Neto is grateful to UFABC for financial support. J. M. Hoff da Silva in grateful to CNPq for financial support.

APPENDIX: CLIFFORD ALGEBRAS AND GENERAL TYPE-(4) AND TYPE-(5) SPINOR FIELDS

Let V be a finite n-dimensional real vector space and $\Lambda(V) = \bigoplus_{k=0}^{n} \Lambda^k(V)$ the space of multivectors over V, where $\Lambda^k(V)$ denotes the k-forms vector space. By defining the reversion, given $\tau, \psi, \xi \in \Lambda(V)$, the left contraction is defined implicitly by $\eta(\tau\lrcorner\psi, \xi) = \eta(\psi, \tilde{\tau} \wedge \xi)$. The Clifford product between $\mathbf{v} \in V$ and ψ is provided by $\mathbf{v}\psi = \mathbf{v} \wedge \psi + \mathbf{v}\lrcorner\psi$. Given a metric η, the pair $(\Lambda(V), \eta)$ endowed of the Clifford product is the Clifford algebra $\mathcal{C}\ell_{1,3}$ of $\mathbb{R}^{1,3}$. All spinor fields are placed in a manifold which locally is a Minkowski spacetime $(M, \eta, \mathring{D}, \tau_\eta, \uparrow)$ in what follows, where M is a manifold, $\mathring{D}$ denotes the Levi-Civita connection associated with η, M is oriented by the 4-volume element τ_η and time-oriented by $\uparrow$. Furthermore, $\{\mathbf{e}_\mu\}$ is a section of the frame bundle $\mathbf{P}_{\mathrm{SO}^e_{1,3}}(M)$. $\{\mathbf{e}^\mu\}$ is the dual frame: $\mathbf{e}^\mu(\mathbf{e}_\nu) = \delta^\mu_\nu$, with $\{\theta^\mu\}$ and $\{\theta_\mu\}$, respectively, the dual bases of $\{\mathbf{e}_\mu\}$ and $\{\mathbf{e}^\mu\}$. Hereupon we denote $\mathbf{e}_{\mu\nu} = \mathbf{e}_\mu\mathbf{e}_\nu$ and $\mathbf{e}_{\mu\nu\rho} = \mathbf{e}_\mu\mathbf{e}_\nu\mathbf{e}_\rho$.

In order to better understand the structure of type-(4) and their limiting case type-(5) spinor fields, the question is: what is the general form of these spinor fields? In order to answer it, let us take a general spinor given by $\psi = (f, g, \eta, \xi)^\intercal$, with $f, g, \eta, \xi \in \mathbb{C}$, and the definition of these spinor types given by Lounesto classification.[3]

1. Spinor fields of type-(4)

As we aim to characterize the most general type-(4) flag-dipole spinor field, the conditions $\sigma = 0 = \omega$ results $\eta f^* + \xi g^* = 0$. We have to analyze the possibilities evinced from this equation. If $f = 0 = g$ or $\eta = 0 = \xi$, it implies a type-(6) spinor field, with $\mathbf{S} = 0$, and therefore this possibility must be dismissed here. It remains the conditions: either $\eta = 0 = \xi, f = 0 = g$, or none of the components can be zero. In this last case, one can isolate a part of them, for example $f = \frac{g\eta\xi^*}{\|\eta\|^2}$. Further, the condition $\mathbf{K} \neq 0$ induces the following possibilities:

a) If $\eta = 0 = \xi$, hence $K_1 = K_2 = 0$, and $K_0 \neq 0 \neq K_3 \Rightarrow \|f\|^2 \neq \|\xi\|^2$;
b) If $f = 0 = \xi$, it implies that $K_1 = K_2 = 0$, and $K_0 \neq 0 \neq K_3 \Rightarrow \|g\|^2 \neq \|\eta\|^2$;
c) If all the components are not zero, $K_1 \neq 0 \neq K_2 \Rightarrow \|g\|^2 \neq \|\eta\|^2$.

In the third case, if $\|g\|^2 = \|\eta\|^2$, therefore $\mathbf{K} = 0$. Furthermore, still in the third case, $\|g\|^2 \neq \|\eta\|^2 \Leftrightarrow \|f\|^2 \neq \|\xi\|^2$. Thus, the possible type-(4) spinor fields are

$$
\begin{aligned}
\psi_{(4)} &= (f, 0, 0, \xi)^\intercal\ , && \|f\|^2 \neq \|\xi\|^2\ , \quad \text{or}\\
\psi_{(4)} &= (0, g, \eta, 0)^\intercal\ , && \|g\|^2 \neq \|\eta\|^2\ , \quad \text{or}\\
\psi_{(4)} &= \left(\frac{g\eta\xi^*}{\|\eta\|^2}\ , g, \eta, \xi\right)^\intercal, && \|g\|^2 \neq \|\eta\|^2\ .
\end{aligned} \tag{A1}
$$

If some inequality associated to one of the spinors above does not hold, it turns forthwith to be a type-(5), which shall be analyzed in what follows.

2. Spinor fields of type-(5)

We start by noticing how the conditions on the bilinear covariants associated to a type-(5) spinor field imply the following conditions on the spinor field components:

$$\sigma = \overline{\psi}\psi = 0 = -\overline{\psi}\gamma_{0123}\psi = \omega \Rightarrow \eta f^* + \xi g^* = 0, \tag{A2}$$

$$K_1 = \overline{\psi}i\gamma_{0123}\gamma_1\psi = 0 = \overline{\psi}i\gamma_{0123}\gamma_2\psi = K_2 \Rightarrow g f^* + \xi\eta^* = 0, \tag{A3}$$

$$K_0 = \overline{\psi}i\gamma_{0123}\gamma_0\psi = 0 = \overline{\psi}i\gamma_{0123}\gamma_3\psi = K_3 \Rightarrow \|f\|^2 = \|\xi\|^2 \text{ and } \|g\|^2 = \|\eta\|^2. \tag{A4}$$

Equation (A4) can be obtained from (A2) and (A3), which are therefore essential to characterize type-(5) spinor fields. In this vein, an equation candidate to describe the dynamics of these general spinor fields must keep (A2) and (A3) invariant. Elko spinor fields obey these equations.

By performing a straightforward calculation with the aid of Eqs. (A2) and (A3) it is possible to obtain

$$f = -\xi^*(\eta + g)(\eta^* + g^*)^{-1} = -\xi^* \left(\frac{\eta + g}{\|\eta + g\|} \right)^2, \tag{A5}$$

and by taking $\tan\varphi_1 = -i\,\frac{\eta+g-(\eta+g)^*}{\eta+g+(\eta+g)^*}$, we can write $f = -\xi^* e^{2i\varphi_1}$ and $g = -\eta^* e^{2i\varphi_2}$, where φ_1 and φ_2 are related by[32]

$$\tan\varphi_2 = -i\,\frac{\xi(1 + e^{-2i\varphi_1}) - [\xi(1 + e^{-2i\varphi_1})]^*}{\xi(1 - e^{-2i\varphi_1}) + [\xi(1 - e^{-2i\varphi_1})]^*} = -\cot\varphi_1.$$

However, $\tan\varphi_2 = -\cot\varphi_1 \Rightarrow \varphi_2 = \varphi_1 + (2k+1)\frac{\pi}{2}$, and then $e^{2i\varphi_2} = e^{2i\varphi_1} e^{i(2k+1)\pi} = -e^{2i\varphi_1}$, for every $k = 0, 1, 2, \ldots$. Hence a general type-(5) spinor can be represented by

$$\psi_{(5)} = \left(-\xi^* e^{2i\varphi_1}, \eta^* e^{2i\varphi_1}, \eta, \xi\right)^{\intercal}. \tag{A6}$$

Writing $\psi_{(5)} = (\chi_2, \chi_1)^{\intercal}$, it is straightforward to realize that $\chi_2 = -i\sigma_2 \chi_1^* e^{2i\varphi_1} = \sigma_2 \chi_1^* e^{i(2\varphi_1 - \frac{\pi}{2})}$. By taking $\varphi \equiv 2\varphi_1 - \frac{\pi}{2}$ a more compact form of (A6) is

$$\psi_{(5)} = \left(e^{i\varphi}\sigma_2\chi_1^*, \chi_1\right)^{\intercal}. \tag{A7}$$

By acting now the charge conjugation operator,[4,26] with $i\Theta = \sigma_2$, it yields

$$C\psi_{(5)} = \mu\psi_{(5)}, \qquad \text{for} \quad C = \begin{pmatrix} \mathbb{O} & i\Theta \\ -i\Theta & \mathbb{O} \end{pmatrix} K \qquad \text{and} \qquad \mu = -e^{i\varphi}.$$

Here K conjugates the spinor components. Hence the eigenvalues take place on the sphere S^1. When these eigenvalues are real and χ_1, χ_2 are dual helicity eigenstates, Elko spinor fields are obtained. The type-(5) flagpole spinor fields were shown to have a prominent role on the derivation of all Lagrangians for the gravity from the one for supergravity.[28,29]

[1] F. W. Hehl, P. Von Der Heyde, G. D. Kerlick, and J. M. Nester, Rev. Mod. Phys. **48**, 393 (1976).
[2] G. Velo and D. Zwanziger, Phys. Rev. **186**, 1337 (1969).
[3] P. Lounesto, *Clifford Algebras and Spinors*, 2nd ed. (Cambridge University Press, Cambridge, 2002).
[4] D. V. Ahluwalia-Khalilova and D. Grumiller, Phys. Rev. D **72**, 067701 (2005); e-print arXiv:hep-th/0410192.
[5] Y. Nambu and G. Jona–Lasinio, Phys. Rev. **122**, 345 (1961).
[6] S. Vignolo, L. Fabbri, and R. Cianci, J. Math. Phys. **52**, 112502 (2011); e-print arXiv:1106.0414 [gr-qc].
[7] R. Penrose and W. Rindler, *Two-Spinor Calculus and Relativistic Fields* (Cambridge University Press, Cambridge, 1986); R. da Rocha and J. Vaz, Int. J. Geom. Methods Mod. Phys. **04**, 547 (2007); e-print arXiv:math-ph/0412074.
[8] J. P. Crawford, J. Math. Phys. **26**, 1439 (1985).
[9] P. R. Holland, Found. Phys. **16**, 701 (1986).
[10] C. G. Boehmer and L. Corpe, J. Phys. A **45**, 205206 (2012); e-print arXiv:1204.0135 [math-ph].
[11] L. Fabbri, Gen. Relativ. Gravit. **45**, 1285 (2013); e-print arXiv:1108.3046 [gr-qc].
[12] L. Fabbri and S. Vignolo, Int. J. Theor. Phys. **51**, 3186 (2012); e-print arXiv:1201.5498 [gr-qc].
[13] L. Fabbri and S. Vignolo, Class. Quantum Grav. **28**, 125002 (2011); e-print arXiv:1012.1270 [gr-qc].
[14] L. Fabbri, Phys. Lett. B **707**, 415 (2012); e-print arXiv:1101.1761 [gr-qc].
[15] L. Fabbri, Phys. Rev. D **85**, 047502 (2012); e-print arXiv:1101.2566 [gr-qc].
[16] S. Capozziello, R. Cianci, C. Stornaiolo, and S. Vignolo, Class. Quantum Grav. **24**, 6417 (2007); e-print arXiv:0708.3038 [gr-qc].
[17] S. Capozziello, R. Cianci, C. Stornaiolo, and S. Vignolo, Int. J. Geom. Methods Mod. Phys. **05**, 765 (2008); e-print arXiv:0801.0445 [gr-qc].
[18] S. Capozziello and S. Vignolo, Ann. Phys. **19**, 238 (2010); e-print arXiv:0910.5230 [gr-qc].
[19] G. F. Rubilar, Class. Quantum Grav. **15**, 239 (1998).
[20] R. da Rocha, A. E. Bernardini, and J. M. Hoff da Silva, J. High Energy Phys. 04 (2011) 110; e-print arXiv:1103.4759.
[21] L. Fabbri, Int. J. Theor. Phys. **52**, 634 (2013); e-print arXiv:1106.4695 [gr-qc].
[22] B. Saha and G. N. Shikin, J. Math. Phys. **38**, 5305 (1997); e-print arXiv:gr-qc/9609055.
[23] B. Saha and T. Boyadjiev, Phys. Rev. D **69**, 124010 (2004); e-print arXiv:gr-qc/0311045.
[24] M. Tsamparlis, Phys. Lett. A **75**, 27 (1979).
[25] J. M. Hoff da Silva and R. da Rocha, Phys. Lett. B **718**, 1519 (2013); e-print arXiv:1212.2406 [hep-th].
[26] D. V. Ahluwalia-Khalilova and D. Grumiller, J. Cosmol. Astropart. Phys. 07 (2005) 012; e-print arXiv:hep-th/0412080.

[27] I. L. Shapiro, Phys. Rept. **357**, 113 (2002); e-print arXiv:hep-th/0103093.
[28] R. da Rocha and J. M. Hoff da Silva, Int. J. Geom. Meth. Mod. Phys. **06**, 461 (2009); e-print arXiv:0901.0883 [math-ph].
[29] R. da Rocha and J. G. Pereira, Int. J. Mod. Phys. D **16**, 1653 (2007); e-print arXiv:gr-qc/0703076 [gr-qc].
[30] It is interesting to note that at this point it is not formally explicit by (9) whether we are dealing to Dirac equation with torsion in a simply connected space or with a Dirac equation without torsion in a multiply connected space-time.[20] As both descriptions are mathematically equivalent, we can transpose one formalism into another, in order to circumvent such question.
[31] In fact, Elko spinor fields[4,26] are eigenspinors of the charge conjugation operator and do not satisfy Dirac equation.[4,26] Some important applications are provided, for instance, at Ref. 27. There is still the complementary set of Elko and Majorana fields, with respect to the type-(5) spinor, whose dynamics is still unknown. Its general form is provided in the Appendix.
[32] When $\varphi_1 \neq n\pi$, that is, $\eta + \text{g}$ is not real.

Eur. Phys. J. C (2015) 75:266
DOI 10.1140/epjc/s10052-015-3498-2

THE EUROPEAN PHYSICAL JOURNAL C

Regular Article - Theoretical Physics

Questing mass dimension 1 spinor fields

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Received: 29 April 2015 / Accepted: 30 May 2015 / Published online: 18 June 2015

Abstract This work deals with new classes of spinors of mass dimension 1 in Minkowski spacetime. In order to accomplish it, Lounesto's classification scheme and the inversion theorem are going to be used. The algebraic framework shall be revisited by explicating the central point performed by the Fierz aggregate. Then the spinor classification is generalized in order to encompass the new mass dimension 1 spinors. The spinor operator is shown to play a prominent role to engender the new mass dimension 1 spinors, accordingly.

1 Introduction

There is a spinor classification due to Lounesto [1], which is particularly interesting for physicists due to its twofold ubiquitous aspect: on the one hand it is based upon bilinear covariants, and thus upon physical observables. On the other hand, by a peculiar multivector structure—the Fierz aggregate—that leads to the so-called boomerang [1], a quite elegant geometrical interpretation may be added to the classification. Moreover, with the aid of the boomerang it is possible likewise to prove that there are precisely six different classes of spinors in Lounesto's classification [1]. The most general forms of the respective spinors in each class were introduced in [2]. Lounesto's spinor classification was further employed to derive all the Lagrangians for gravity from the quadratic spinor Lagrangian [3]. Higher dimensional spaces have a similar spinor classification [4], however, the so-called geometric Fierz identities [5] obstruct the proliferation of new spinors classes in higher dimensions [4].

Within the Lounesto classification, a specific bilinear covariant plays a crucial role, since it cannot be zero. This bilinear represents the current density, at least for the case of a regular spinor describing the electron. Its components read $\mathbf{J} = J_\mu e^\mu = \psi^\dagger\gamma_0\gamma_\mu\psi e^\mu$, where ψ denotes a spinor and e^μ is a dual basis in $\mathscr{C}\ell_{1,3}$. Additionally, it is valuable to remark that $\mathbf{J} = J_\mu e^\mu$ is essential for the definition of the boomerang structure. Regarding the electron theory, it is straightforward to realize the physical argument to explain why $\mathbf{J}$ must not vanish. Indeed, $\mathbf{J}$ is the conserved current in this case and therefore if $\mathbf{J} = 0$ there is no associated particle [6]. In particular the time component $J_0 = \psi^\dagger\psi$ provides the probability density of the electron, and when integrated over the spacetime it should obviously be non-null.

One of the main points that shall be pursued in this work is that $\mathbf{J}$ can be understood as a conserved current solely when the considered spinor obeys the usual dynamics rules by the Dirac equation, namely, it is an eigenspinor of the Dirac operator or, equivalently, it is described by the Dirac Lagrangian. The canonical mass dimension in this case is the same mass dimension 3/2 associated to usual spin-1/2 fermions in the standard model. Since we are looking for possible manifestations of mass dimension 1 fermions in Minkowski spacetime, it is indeed possible to set $\mathbf{J} = 0$, accordingly. In fact, by accomplishing it, even the previously mentioned algebraic argument precluding new spinor classes may be circumvented. Nevertheless, in this novel context, we should emphasize that the underlying dynamics shall not be dictated by the well-known Dirac equation. As the construction is relativistic, the spinors arising from the analysis with $\mathbf{J} = 0$ shall respect a priori merely the Klein–Gordon equation. Actually, in a very conventional scheme, they must do so. Hence, the epigraph is now explained: the resulting spinors must have mass dimension 1. Clearly by "mass dimension" we mean the canonical mass dimension of the associated quantum field, which inherits this property from the dynamics respected by its expansion coefficients.

Mass dimension 1 spinors have attracted attention mainly due to the fact that they can be coupled only to gravity, and to scalar fields as well, in a perturbatively renormalizable way.

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It thus makes it suitable for exploration under the ensign of dark matter. Mass dimension 1 spinors in Minkowski spacetime known in the literature are the so-called Elko spinors, which have been studied in a comprehensive context. They comprise prominent applications in 4D gravity and cosmology [3,7–11], and in brane-world models as well [12,13], besides their exotic counterparts [14,15]. Moreover, in spite of the robust and rich framework already developed [16–20], Elko has been predicted to be measured in Higgs processes at LHC [21,22] and explored in tunneling methods concerning black holes [23]. Massive spin-1/2 fields of mass dimension were obtained by constructing quantum fields from higher-spin Elkos, however, these fields are still linked to the Elko construct. We stress, however, that the spinors to be found here are intrinsically different from the Elkos by the simple fact that $\mathbf{J} \neq 0$ in the Elko case.

The classification of mass dimension 1 spinors is performed by a possible and consistent modification in the Lounesto classification. However, in order to have an explicit form for them it is necessary the use of the so-called inversion theorem [24,25].

This paper is organized as follows: in the next section the main steps of the framework which supports our analysis shall be revisited, namely the standard Lounesto classification and the inversion theorem. In Sect. 3 we show the existence of three new classes of mass dimension 1 spinors, obtaining the algebraic form in each case accordingly. In the last section we make our concluding remarks and present a brief outlook.

2 The framework

In order to properly address the problem to be approached and solved, it is pivotal to review some key aspects of the standard formalism, highlighting the structures to be studied and generalized. To start, Lounesto's spinor classification shall be revisited, and subsequently the inversion theorem algorithm shall be thereafter employed, accordingly.

2.1 The Lounesto's spinors classification and generalizations

Consider the Minkowski spacetime $(M, \eta_{\mu\nu})$ and its tangent bundle TM. Denoting sections of the exterior bundle by $\sec \Lambda(TM)$, given a k-vector $a \in \sec \Lambda^k(TM)$, the reversion is defined by $\tilde{a} = (-1)^{|k/2|}a$, while the grade involution reads $\hat{a} = (-1)^k a$, where $|k|$ stands for the integral part of k. By extending the Minkowski metric from $\sec \Lambda^1(TM) = \sec T^*M$ to $\sec \Lambda(TM)$, and considering $a_1, a_2 \in \sec \Lambda(V)$, the left contraction is given by $g(a \lrcorner a_1, a_2) = g(a_1, \tilde{a} \wedge a_2)$. The well-known Clifford product for (the dual of) a vector field $v \in \sec \Lambda^1(TM)$ and a multivector is prescribed by $va = v \wedge a + v \lrcorner a$, defining thus the spacetime Clifford algebra $\mathscr{C}\ell_{1,3}$. The set $\{e_\mu\}$ represents sections of the frame bundle $\mathbf{P}_{\mathrm{SO}^e_{1,3}}(M)$ and $\{\gamma^\mu\}$ can be further thought of as being the dual basis $\{e_\mu\}$, namely, $\gamma^\mu(e_\mu) = \delta^\mu_\nu$. Classical spinors are objects of the space that carries the usual $\tau = (1/2, 0) \oplus (0, 1/2)$ representation of the Lorentz group, which can be thought of as being sections of the vector bundle $\mathbf{P}_{\mathrm{Spin}^e_{1,3}}(M) \times_\tau \mathbb{C}^4$.

Given a spinor field $\psi \in \sec \mathbf{P}_{\mathrm{Spin}^e_{1,3}}(M) \times_\tau \mathbb{C}^4$, the bilinear covariants are sections of the bundle $\Lambda(TM)$ [1,24]. Indeed, the well-known Lounesto spinors classification is based upon bilinear covariants and the underlying multivector structure. The physical nature of the classification focuses on the bilinear covariants, which are physical observables, characterizing types of fermionic particles. The observable quantities are given by the following multivector structure:

$$\begin{aligned} &\sigma = \psi^\dagger \gamma_0 \psi, \quad \omega = -\psi^\dagger \gamma_0 \gamma_{0123} \psi, \\ &J_\mu = \psi^\dagger \gamma_0 \gamma_\mu \psi, \quad K_\mu = \psi^\dagger \gamma_0 i \gamma_{0123} \gamma_\mu \psi, \\ &S_{\mu\nu} = \frac{1}{2} \psi^\dagger \gamma_0 i \gamma_{\mu\nu} \psi, \end{aligned} \tag{1}$$

where $\gamma_{0123} := i\gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3$. The set $\{\mathbf{1}, \gamma_I\}$ (where $I \in \{\mu, \mu\nu, \mu\nu\rho, 5\}$ is a composed index) is a basis for $\mathscr{M}(4, \mathbb{C})$ satisfying $\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\eta_{\mu\nu}\mathbf{1}$.

The above bilinear covariants in the Dirac theory are interpreted, respectively, as the mass of the particle (σ), the pseudo-scalar (ω) relevant for parity-coupling, the current of probability ($\mathbf{J}$), the direction of the electron spin ($\mathbf{K}$), and the probability density of the intrinsic electromagnetic moment ($\mathbf{S}$) associated to the electron. The most important bilinear covariant for our goal here is $\mathbf{J}$, although with a different meaning. In fact, in the next section we shall set $\mathbf{J} = 0$, enabling the extension of the standard Lounesto classification to this case.

A prominent requirement for Lounesto's spinors classification is that the bilinear covariants satisfy quadratic algebraic relations, namely, the so-called Fierz–Pauli–Kofink (FPK) identities, which read

$$\begin{aligned} &J_\mu J^\mu = \sigma^2 + \omega^2, \quad J_\mu J^\mu = -K_\mu K^\mu, \\ &J_\mu K^\mu = 0, \quad \mathbf{J} \wedge \mathbf{K} = -(\omega + \sigma\gamma_{0123})\mathbf{S}. \end{aligned} \tag{2}$$

It is worth to remark that the above identities are fundamental, not merely for the aims regarding the classification, but, moreover, for asserting the inversion theorem, as we are going to see in the next subsection.

Within the Lounesto classification scheme, a non-vanishing $\mathbf{J}$ is crucial, since it enables one to define the so-called boomerang [1], which has an ample geometrical meaning in asserting that there are precisely six different classes of spinors. This is a prominent consequence of the definition of a boomerang [1]. As far as the boomerang is concerned, it is not possible to exhibit more than six types of spinors, according to the bilinear covariants. Indeed, Lounesto's spinor classification splits regular and singular spinors. The regular spinors are those which have at least one of the bilinear covariants σ and ω non-null. On the other hand, singular spinors correspond to $\sigma = 0 = \omega$, and in this case the Fierz identities are in general replaced by the more general conditions [24]:

$$Z^2 = 4\sigma Z, \quad Z\gamma_\mu Z = 4J_\mu Z, \quad Z\gamma_{0123}Z = -4\omega Z$$
$$Zi\gamma_{\mu\nu}Z = 4S_{\mu\nu}Z, \quad Zi\gamma_{0123}\gamma_\mu Z = 4K_\mu Z. \tag{3}$$

When an arbitrary spinor ξ satisfies $\widetilde{\xi}^*\psi \neq 0$ and belongs to $\mathbb{C}\otimes\mathscr{C}\ell_{1,3}$—or equivalently when $\xi^\dagger\gamma_0\psi \neq 0 \in \mathscr{M}(4,\mathbb{C})$—it is possible to recover the original spinor ψ from its aggregate $\mathbf{Z}$ given by

$$\mathbf{Z} = \sigma + \mathbf{J} + i\mathbf{S} + i\mathbf{K}\gamma_{0123} + \omega\gamma_{0123} \tag{4}$$

and the spinor ξ by the so-called Takahashi algorithm [25] likewise. In fact, the spinor ψ and the multivector $\mathbf{Z}\xi$ differ solely by a multiplicative constant, and can be thus written as

$$\psi = \frac{1}{2\sqrt{\xi^\dagger\gamma_0\mathbf{Z}\xi}}\, e^{-i\theta}\mathbf{Z}\xi, \tag{5}$$

where $e^{-i\theta} = 2(\xi^\dagger\gamma_0\mathbf{Z}\xi)^{-1/2}\xi^\dagger\gamma_0\psi \in U(1)$. For more details see, e.g., [24]. Equivalently to Eq. (5), we shall use hereupon the notation $\psi \backsim \mathbf{Z}\xi$ to say that both sides of this equivalence are in the same equivalence class with respect to the quotient by $\mathbb{C}$. Moreover, when σ, ω, $\mathbf{J}$, $\mathbf{S}$, $\mathbf{K}$ satisfy the Fierz identities, then the complex multivector operator $\mathbf{Z}$ is named a Fierz aggregate. When $\gamma_0\mathbf{Z}^\dagger\gamma_0 = \mathbf{Z}$, thus $\mathbf{Z}$ is said to be a boomerang [1].

The Takahashi algorithm reveals the importance of the aggregate. Moreover, the inversion theorem (to be regarded in the next subsection) is inspired on this spinor representation (5). More significantly here, the aggregate plays a central role within the Lounesto classification since, in order to complete the classification itself, $\mathbf{Z}$ has to be promoted to a boomerang, satisfying

$$\mathbf{Z}^2 = 4\sigma\mathbf{Z}. \tag{6}$$

Obviously, for the regular spinors case the above condition is satisfied and $\mathbf{Z}$ is automatically a boomerang. However, for singular spinors it is not so straightforward. Indeed, for singular spinors we must envisage the geometric structure underlying the multivector. From the geometric point of view the following relations between the bilinear covariants must be fulfilled in order to ensure that the aggregate be a boomerang: $\mathbf{J}$ must be parallel to $\mathbf{K}$ and both are in the plane formed by the bivector $\mathbf{S}$. Hence, using Eq. (4) and taking into account that we are dealing with singular spinors, it is straightforward to see that the aggregate can be recast in the form 61

$$\mathbf{Z} = \mathbf{J}(1 + i\mathbf{s} + ih\gamma_{0123}), \tag{7}$$

where $\mathbf{s}$ is a space-like vector orthogonal to $\mathbf{J}$, and h is a real number. The multivector as expressed in Eq. (7) is a boomerang [19]. By inspecting the condition (6) we see that for singular spinors $\mathbf{Z}^2 = 0$. However, in order for the FPK identities to hold it is also necessary that both conditions[1] $\mathbf{J}^2 = 0$ and $(\mathbf{s} + h\gamma_{0123})^2 = -1$ must be satisfied. These considerations are important in order to constrain the possible spinor classes.

Now, let us make explicit that from (5) one can see that different bilinear covariants combinations may lead to different spinors, taking into account the constraints coming from the FPK identities. Altogether, the algebraic constraints reduce the possibilities to six different spinor classes, namely:

1. $\sigma \neq 0, \quad \omega \neq 0$;
2. $\sigma \neq 0, \quad \omega = 0$;
3. $\sigma = 0, \quad \omega \neq 0$;
4. $\sigma = 0 = \omega, \quad \mathbf{K} \neq 0, \quad \mathbf{S} \neq 0$;
5. $\sigma = 0 = \omega, \quad \mathbf{K} = 0, \quad \mathbf{S} \neq 0$;
6. $\sigma = 0 = \omega, \quad \mathbf{K} \neq 0, \quad \mathbf{S} = 0$.

The spinors types-(1), (2), and (3), are called Dirac spinor fields (regular spinors). The spinor field (4) is called flag-dipole [26], while the spinor field (5) is named flag-pole [27]. Majorana [28] and Elko [16,19] spinors are elements of the flag-pole class. Finally, the type (6) dipole spinors are exemplified by Weyl spinors. Note that there are only six different spinor fields. To see that, notice that for the regular case, since $\mathbf{J} \neq 0$, it follows that $\mathbf{S} \neq 0$ and $\mathbf{K} \neq 0$ as impositions from the identities (2). On the other hand, for the singular case, the geometry asserts that $\mathbf{J}(\mathbf{s} + h\gamma_{0123}) = \mathbf{S} + \mathbf{K}\gamma_{0213}$. Hence, as far as $\mathbf{J} \neq 0$, we have already considered all the possibilities.

As is clear from the above reasoning, $\mathbf{J} \neq 0$ is much more a matter of taste. There is instead algebraic necessity of demonstrating the existence of six different classes. In fact, however, a non-vanishing $\mathbf{J}$ is indispensable only for the regular spinor case. As mentioned, the above classification makes use of this constraint in all the cases, since the very idea of the classification was to categorize spinors which could be related to Dirac particles in some respect. As far as we leave this (physical) concept, more spinors can be found.

By taking $\mathbf{J} = 0$, we cannot describe Dirac particles anymore. Therefore, the spinors arising from this consideration must be merely ruled by the Klein–Gordon dynamics and, therefore, they must have mass dimension 1. We finalize by stressing that the resulting spinors (see Sect. 3) have to be singular, as in the contrary case they would violate the FPK identities and, besides, the geometrical aspects underlying the algebraic structure need to be reconsidered.

[1] We remark that $\mathbf{J}$ must be different from zero in the Lounesto classification.

2.2 The inversion theorem

It is well known, in the quantum mechanical context, that all the physical observables are represented by quadratic quantities of the wave function, for example the probability density. In the specific case of the Dirac particle, represented by a four-component spinor wave function ψ, we can write sixteen real quadratic forms, called bilinear covariants $\rho_i = \widetilde{\psi}\Gamma_i\psi$. The bilinear covariants are represented in the set of Eq. (1). The bilinear covariants are not individual quantities [25], since their structure depends on the spinor itself. Crawford makes use of the FPK identities to define the inversion theorem, which asserts that the general form of an arbitrary spinor may be expressed in terms of the bilinear covariants as

$$\begin{aligned}\psi &= \mathrm{e}^{-i\varphi}\left(\Sigma - i\Pi\gamma_5 + J_\mu\gamma^\mu - K_\mu\gamma_5\gamma^\mu + \frac{1}{2}S_{\mu\nu}\sigma^{\mu\nu}\right)\xi,\\ &= \mathrm{e}^{-i\varphi}R^i\Gamma_i\xi, \end{aligned} \tag{8}$$

where the set $\{\varphi, R^i\}$, contains real functions, and ξ is an arbitrary constant spinor. It is clear that even if we choose a specific spinor ξ, we have the freedom to choose a set $\{\varphi, R^i\}$, since the function ψ contains only eight independent functions. Another important assertion, taken into account by Crawford, is that the set of functions R^i must always satisfy the corresponding equations from the FPK identities. A proof for this statement can be found in Ref. [24].

It is important to stress that the alluded inversion is not unique, since we can choose an arbitrary phase φ, and the constant spinor ξ. Thus, concerning the inversion program, it is fairly important to bear in mind that it is useful within the formal algebraic context. In the next section, we shall apply the inversion theorem in order to recover mass dimension 1 spinors coming from a suitable modification of Lounesto's scheme.

3 Algebraic construction of new spinors

After briefly revisiting the equivalence among the classical, algebraic, and operator spinor formulations in what follows, we shall be able to analyze the possible constructions for the new mass dimension 1 spinors. Let us hence start by expressing an arbitrary multivector in $\mathscr{C}\ell_{1,3}$ as (henceforth $e_\mu e_\nu e_\lambda = e_{\mu\nu\lambda}$)

$$\Gamma = \alpha + \alpha^\mu e_\mu + \alpha^{\mu\nu}e_{\mu\nu} + \alpha^{\mu\nu\sigma}e_{\mu\nu\sigma} + \alpha^{0123}e_{0123}. \tag{9}$$

Given the isomorphism $\mathscr{C}\ell_{1,3} \simeq \mathscr{M}(2,\mathbb{H})$, where $\mathbb{H}$ denotes the quaternionic ring, and a primitive idempotent $f = \frac{1}{2}(1 + e_0)$ is taken to define a minimal left ideal $\mathscr{C}\ell_{1,3}f$. This is relevant, in particular, to attain a spinor representation of $\mathscr{C}\ell_{1,3}$. The most general multivector in $\mathscr{C}\ell_{1,3}f$ reads

$$\begin{aligned}\zeta &= (\beta^1 + \beta^2 e_{23} + \beta^3 e_{31} + \beta^4 e_{12})f\\ &\quad + (\beta^5 + \beta^6 e_{23} + \beta^7 e_{31} + \beta^8 e_{12})e_{0123}f.\end{aligned} \tag{10}$$

Since the identification $\zeta = \Gamma f \in \mathscr{C}\ell_{1,3}f$ holds, it implies the following equivalence between their respective components:

$$\begin{aligned}&\beta^1 = \alpha + \alpha^0, \quad \beta^2 = \alpha^{23} + \alpha^{023}, \quad \beta^3 = -\alpha^{13} - \alpha^{013},\\ &\beta^4 = \alpha^{12} + \alpha^{012}, \quad \beta^5 = -\alpha^{123} + \alpha^{0123}, \quad \beta^6 = \alpha^1 - \alpha^{01},\\ &\beta^7 = \alpha^2 - \alpha^{02}, \quad \beta^8 = \alpha^3 - \alpha^{03}.\end{aligned} \tag{11}$$

By denoting $\mathfrak{i} = e_2e_3$, $\mathfrak{j} = e_3e_1$, and $\mathfrak{k} = e_1e_2$, it is clear that the set $\{1, \mathfrak{i}, \mathfrak{j}, \mathfrak{k}\}$ is a basis for the quaternion algebra $\mathbb{H}$. The two quaternions appear as coefficients in (10), namely,

$$\begin{aligned}q_1 &= \beta^1 + \beta^2 e_{23} + \beta^3 e_{31} + \beta^4 e_{12},\\ q_2 &= \beta^5 + \beta^6 e_{23} + \beta^7 e_{31} + \beta^8 e_{12} \in \mathbb{H},\end{aligned} \tag{12}$$

where $\mathbb{H} = f\mathscr{C}\ell_{1,3}f = \mathrm{span}_{\mathbb{R}}\{1, e_{23}, e_{31}, e_{12}\}$ commutes with f and e_{0123}. This yields the equality $q_1 f + q_2 e_{0123} f = fq_1 + e_{0123}fq_2$, evincing that the left ideal $\mathscr{C}\ell_{1,3}f$ is in fact a right module over $\mathbb{K}$ with a basis $\{f, e_{0123}f\}$. Moreover, the orthonormal basis $\{e_\mu\}$ has an immediate standard representation,

$$e_0 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \quad e_1 = \begin{pmatrix}0 & \mathfrak{i}\\ \mathfrak{i} & 0\end{pmatrix}, \quad e_2 = \begin{pmatrix}0 & \mathfrak{j}\\ \mathfrak{j} & 0\end{pmatrix},$$
$$e_3 = \begin{pmatrix}0 & \mathfrak{k}\\ \mathfrak{k} & 0\end{pmatrix},$$

which consequently induces representations for the idempotent f and the multivector $e_{0123}f$:

$$[f] = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} \quad \text{and} \quad [e_{0123}f] = \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}.$$

Therefore, a general element $\Gamma \in \mathscr{C}\ell_{1,3}$ can be expressed as

$$\begin{pmatrix}q_1 & q_2\\ q_3 & q_4\end{pmatrix} \in \mathscr{M}(2,\mathbb{H}) \tag{13}$$

where $q_1 = \alpha + \alpha^0 + (\alpha^{23} + \alpha^{023})\mathfrak{i} - (\alpha^{13} + \alpha^{013})\mathfrak{j} + (\alpha^{12} + \alpha^{012})\mathfrak{k}$, $q_2 = (\alpha^{0123} - \alpha^{123}) + (\alpha^1 - \alpha^{01})\mathfrak{i} + (\alpha^2 - \alpha^{02})\mathfrak{j} + (\alpha^3 - \alpha^{03})\mathfrak{k}$, $q_3 = -(\alpha^{123} + \alpha^{0123}) + (\alpha^1 + \alpha^{01})\mathfrak{i} + (\alpha^2 + \alpha^{02})\mathfrak{j} + (\alpha^3 + \alpha^{03})\mathfrak{k}$ and $q_4 = (\alpha - \alpha^0) + (\alpha^{23} - \alpha^{023})\mathfrak{i} + (\alpha^{013} - \alpha^{13})\mathfrak{j} + (\alpha^{12} - \alpha^{012})\mathfrak{k}$.

A multivector Ψ in the even subalgebra $\mathscr{C}\ell^+_{1,3}$ is named spinor operator; it reads

$$\Psi = \alpha + \alpha^{\mu\nu}e_{\mu\nu} + \alpha^{0123}e_{0123}. \tag{14}$$

From the point of view of Eq. (13) it yields

$$[\Psi] = \begin{pmatrix}q_1 & -q_2\\ q_2 & q_1\end{pmatrix}$$
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$$= \begin{pmatrix}\alpha + \alpha^{23}\mathfrak{i} - \alpha^{13}\mathfrak{j} + \alpha^{12}\mathfrak{k} & -\alpha^{0123} + \alpha^{01}\mathfrak{i} + \alpha^{02}\mathfrak{j} + \alpha^{03}\mathfrak{k}\\ \alpha^{0123} - \alpha^{01}\mathfrak{i} - \alpha^{02}\mathfrak{j} - \alpha^{03}\mathfrak{k} & \alpha + \alpha^{23}\mathfrak{i} - \alpha^{13}\mathfrak{j} + \alpha^{12}\mathfrak{k}\end{pmatrix}.$$

The isomorphisms $\mathscr{C}\ell_{1,3}\frac{1}{2}(1+e_0) \simeq \mathscr{C}\ell_{1,3}^{+} \simeq \mathbb{H}^2 \simeq \mathbb{C}^4$ among vector spaces, respectively, evince the correspondence among the algebraic, the operatorial, and the classical definitions of a spinor in Minkowski spacetime. Indeed, the spinor space $\mathbb{H}^2$ carries the $(1/2, 0) \oplus (0, 1/2)$ (or $(1/2, 0)$ or $(0, 1/2)$) representations of the Lorentz group, and it is isomorphic both to the minimal left ideal $\mathscr{C}\ell_{1,3}\frac{1}{2}(1+e_0)$, which is equivalent to the algebraic spinor, and to the even subalgebra $\mathscr{C}\ell_{1,3}^{+}$, which corresponds to the space of spinor operators [29,30]. Thus the Dirac spinor is expressed equivalently as

$$\begin{pmatrix} q_1 & -q_2 \\ q_2 & q_1 \end{pmatrix}[f] = \begin{pmatrix} q_1 & 0 \\ q_2 & 0 \end{pmatrix} \cong \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$$
$$= \begin{pmatrix} \alpha + \alpha^{23}\mathfrak{i} - \alpha^{13}\mathfrak{j} + \alpha^{12}\mathfrak{k} \\ \alpha^{0123} - \alpha^{01}\mathfrak{i} - \alpha^{02}\mathfrak{j} - \alpha^{03}\mathfrak{k} \end{pmatrix} \in \mathscr{C}\ell_{1,3} f \simeq \mathbb{H}^2. \quad (15)$$

Now by employing the usual representation

$$1 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathfrak{i} \mapsto \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \mathfrak{j} \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$
$$\mathfrak{k} \mapsto \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$

in 2×2 complex matrices, the spinor operator Ψ in (14) can be viewed furthermore as a 4×4 matrix, as follows:

$$\begin{pmatrix} \alpha+\alpha^{23}i & -\alpha^{13}+\alpha^{12}i & -\alpha^{0123}+\alpha^{01}i & \alpha^{02}+\alpha^{03}i \\ \alpha^{13}+\alpha^{12}i & c-\alpha^{23}i & -\alpha^{02}+\alpha^{03}i & -\alpha^{0123}-\alpha^{01}i \\ \alpha^{0123}-\alpha^{01}i & -\alpha^{02}-\alpha^{03}i & \alpha+\alpha^{23}i & -\alpha^{13}+\alpha^{12}i \\ \alpha^{02}-\alpha^{03}i & \alpha^{0123}+\alpha^{01}i & \alpha^{13}+\alpha^{12}i & \alpha-\alpha^{23}i \end{pmatrix}$$
$$\equiv \begin{pmatrix} \psi_1 & -\psi_2^* & -\psi_3 & \psi_4^* \\ \psi_2 & \psi_1^* & -\psi_4 & -\psi_3^* \\ \psi_3 & -\psi_4^* & \psi_1 & -\psi_2^* \\ \psi_4 & \psi_3^* & \psi_2 & \psi_1^* \end{pmatrix}. \quad (16)$$

The spinor ψ lives in the left (minimal) ideal $(\mathbb{C}\otimes\mathscr{C}\ell_{1,3})f$, where $f = \frac{1}{4}(1+e_0)(1+ie_{12})$ is an idempotent that equals diag$(1, 0, 0, 0)$ in the Dirac representation, making $e_\mu \mapsto \gamma_\mu \in \mathscr{M}(4, \mathbb{C})$. Hence it follows that

$$\psi \simeq \begin{pmatrix} \psi_1 & 0 & 0 & 0 \\ \psi_2 & 0 & 0 & 0 \\ \psi_3 & 0 & 0 & 0 \\ \psi_4 & 0 & 0 & 0 \end{pmatrix} \in (\mathbb{C}\otimes\mathscr{C}\ell_{1,3})f, \quad \text{or} \quad \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} \in \mathbb{C}^4,$$

illustrating the usual prescription between the multivector ψ and the classical Dirac spinor field.

In this context, the posed conundrum is thus reduced to the calculation of the spinor operator (14), finding ψ [1,31]. Prior to accomplishing it, however, it is necessary to define the bilinear covariants in terms of the spinor operator Ψ [29]:

$$\sigma = \langle \Psi\tilde{\Psi}\rangle_0, \quad \omega = -\langle \Psi e_5\tilde{\Psi}\rangle_0, \quad J = \Psi e_0\tilde{\Psi},$$ 63
$$S = \Psi e_1 e_2\tilde{\Psi}, \quad K = \Psi e_3\tilde{\Psi}, \quad (17)$$

where $e_5 = e_0e_1e_2e_3$ and $\langle\ \cdot\ \rangle_0$ denotes the scalar part of the multivector taken into account.

It is important to highlight that the bilinear covariants in (1) provide 16 independent quantities. On the other hand, it is also possible to express the spinor as a function of such bilinear covariants with an arbitrary phase (see Sect. 2.2), according to the Takahashi theorem [25]. Thus, keeping in mind that the spinor exhibits only 8 degrees of freedom and the bilinear covariants have 16 degrees of freedom, it is necessary to use the Fierz identities. Such identities reduce the degrees of freedom to 7, being the extra degree of freedom associated to a phase factor.[2] Taking into account Eq. (15), it is usual, in order to reduce the degrees of freedom of Ψ, to define the following relation:

$$\alpha\exp(e_{12}\theta) \cong \frac{1}{4}\big(\Psi + e_0\Psi e_0 + e_{21}\Psi e_{12} + e_{210}\Psi e_{012}\big), \quad (18)$$

where α is a constant and θ is an arbitrary phase. To find the constant α, we use the complex conjugate of Eq. (18), which for the algebra here considered is equivalent to the reversion. It yields the following expression:

$$\alpha\exp(e_{21}\theta) \cong \frac{1}{4}\big(\tilde{\Psi} + e_0\tilde{\Psi}e_0 + e_{12}\tilde{\Psi}e_{21} + e_{012}\tilde{\Psi}e_{210}\big), \quad (19)$$

and by multiplying Eqs. (18) and (19) we obtain

$$\alpha^2 = \frac{1}{16}\big(\sigma + e_5\omega + \mathbf{J}e_0 + \mathbf{S}e_{21} - e_{0123}\mathbf{K}e_{210} + \mathbf{J}e_0 + \sigma$$
$$+ e_5\omega - e_0e_{0123}\mathbf{K}e_{21} + e_0\mathbf{S}e_{210} - e_{21}(\sigma + e_5\omega)e_{21}$$
$$+ e_{21}\mathbf{S} - e_{21}e_{0123}\mathbf{K}e_0 - e_{21}\mathbf{J}e_{210} - e_{210}e_{0123}\mathbf{K}$$
$$+ e_{210}\mathbf{S}e_0 - e_{210}\mathbf{J}e_{21} - e_{210}(\sigma + e_5\omega)e_{210}\big).$$

Making use of $e_\mu e_\nu + e_\nu e_\mu = 2\eta_{\mu\nu}$, it yields

$$\alpha = \frac{1}{2}\big(\sigma + e_5\omega + \mathbf{J}e_0 - \mathbf{K}e_3 - \mathbf{S}e_{12}\big)^{1/2}. \quad (20)$$

The final step to determine Ψ in terms of α and its bilinear covariants is to multiply Eq. (19), from which we get

$$\Psi\alpha\exp(e_{21}\theta) \cong \frac{1}{4}\big(\Psi\tilde{\Psi} + \Psi e_0\tilde{\Psi}e_0 + \Psi e_{12}\tilde{\Psi}e_{21}$$
$$+ \Psi e_{012}\tilde{\Psi}e_{210}\big). \quad (21)$$

By using the relations (17), the expression for Ψ is given by

$$\Psi = \frac{1}{4\alpha}\big(\sigma + e_5\omega + \mathbf{J}e_0 - \mathbf{K}e_3 - \mathbf{S}e_{12}\big)\exp(e_{12}\theta).$$

Through Eq. (14), it is possible to define the algebraic spinor ψ by

[2] For completeness, by considering Pauli spinors we have 4 degrees of freedom, while the Fierz identities take account of 3 of them. Again, the extra degree of freedom is associated to a phase [31].

$$\psi = \frac{1}{4\alpha}\big(\sigma + e_5\omega + \mathbf{J}e_0 - \mathbf{K}e_3 - \mathbf{S}e_{12}\big)\exp(e_{12}\theta)\begin{pmatrix}1\\0\\0\\0\end{pmatrix}. \tag{22}$$

By means of Eq. (22) it is possible to recover the algebraic spinor from its bilinear covariants via the inversion theorem setup. Having completed the above program for the general case, the application to new mass dimension 1 spinors follows straightforwardly.

As remarked in Sect. 2, the Lounesto classification is based upon the FPK identities. As far as these relations are satisfied, novel possibilities involving spinors can be considered. We propose a classification of new spinors, arising from considering that the bilinear covariant $\mathbf{J}$ is always null and the aggregate associated ($\mathbf{Z}$) is no longer a boomerang as well. On the other hand, the bilinear covariants still satisfy the identities (2). As emphasized by the previous analysis, this last requirement is important, since we shall express the new algebraic spinors functional form.

The consideration that the bilinear covariants must satisfy the FPK identities with $\mathbf{J} = 0$ reveals the existence of three new spinors. We shall finalize this section by evincing their bilinears and their algebraic structure.

Case 1: $\sigma = 0 = \omega$, $\mathbf{J} = 0$, $\mathbf{K} \neq 0$ and $\mathbf{S} \neq 0$. It can be verified that all the FPK identities (2) are satisfied. Moreover, the aggregate (not a boomerang) associated with this spinor reads

$$\mathbf{Z} = i(\mathbf{S} + \mathbf{K}e_{0123}). \tag{23}$$

Finally, considering this particular arrangement of the bilinear covariants, the spinor operator is given by

$$\Psi \cong \frac{1}{2\sqrt{-K_3 - S_{21}}}(-Ke_3 - Se_{21})\exp(e_{12}\theta),$$

and the algebraic spinor turns out to be

$$\psi = \frac{1}{2\sqrt{-K_3 - S_{21}}}(-Ke_3 - Se_{21})\exp(e_{12}\theta)\begin{pmatrix}1\\0\\0\\0\end{pmatrix}.$$

The next cases follow in straightforward analogy.

Case 2: $\sigma = 0 = \omega$, $\mathbf{J} = 0$, $\mathbf{K} = 0$ and $\mathbf{S} \neq 0$. Here, the FPK identities are also satisfied and the aggregate associated is simply given by

$$\mathbf{Z} = i\mathbf{S}. \tag{24}$$

The spinor operator reads

$$\Psi \cong \frac{1}{2\sqrt{-S_{21}}}(-Se_{21})\exp(e_{12}\theta),$$

and the algebraic spinor can be written as

$$\psi = \frac{1}{2\sqrt{-S_{21}}}(-Se_{21})\exp(e_{12}\theta)\begin{pmatrix}1\\0\\0\\0\end{pmatrix}.$$

Case 3: $\sigma = 0 = \omega$, $\mathbf{J} = 0$, $\mathbf{K} \neq 0$, and $\mathbf{S} = 0$, again the FPK identities hold, and the associated spinor operator has the following form:

$$\Psi \cong \frac{1}{2\sqrt{-K_3}}(-Ke_3)\exp(e_{12}\theta),$$

leading to the following algebraic spinor:

$$\psi = \frac{1}{\sqrt{-K_3}}(-Ke_3)\exp(e_{12}\theta)\begin{pmatrix}1\\0\\0\\0\end{pmatrix}.$$

The cases we have shown demonstrate the existence of three new classes of spinors not cataloged previously, which in particular, present mass dimension 1 in Minkowski spacetime. These spinors have the specific bilinear covariant $\mathbf{J}$ equal to zero. Since for spinors respecting the Dirac dynamics $\mathbf{J}$ is the conserved current, here we must be dealing with spinors obeying only the Klein–Gordon equation. Notice that it is a natural consequence, since a given spinor in this context is nothing but a section of the bundle comprised by SL(2, $\mathbb{C}$) and $\mathbb{C}^4$. Thus, it must respect relativistic dynamics. From the mathematical point of view, instead, $\mathbf{J} \neq 0$ is also a necessary condition to promote the Fierz aggregate to a more meaningful quantity (in the geometrical context), the boomerang which, in turn, is essential in reducing the number of different spinor classes to six in the Lounesto classification. In the consideration of $\mathbf{J} = 0$ the classification itself is rebuilt and new spinors arise.

4 Concluding remarks and outlook

We have shown the existence of three new spinors of mass dimension 1, via the inversion theorem and a consistent modification of the Lounesto spinor field classification. This has been achieved considering the specific bilinear covariant $\mathbf{J}$ to be equal to zero. Physically, it means that the new spinors cannot respect the Dirac dynamics, only the Klein–Gordon one, enabling thus the canonical mass dimension to be equal to 1.

A word of caution may be added to these final remarks. As remarked along in the text, the adopted procedure is consistent; and bearing in mind the precedent opened by previous mass dimension 1 spinors (the Elkos), the spinors found may have several physically relevant aspects to be explored [21].

This is, in fact, our belief concerning the generalization presented here. However, one must take into account that the classification and the algebraic functional form do not say much about the emergence of these spinors in nature. As it is, the quantities described in the cases 1, 2, and 3 of the previous section are mathematically well-defined structures whose associated physical field would have interesting properties. The possibility of a physical manifestation of such spinors is currently under investigation.

Acknowledgments CHCV thanks to CAPES (PEC-PG) for the financial support, JMHS thanks to CNPq (308623/2012-6; 445385/2014-6) for partial support. RdR is grateful to CNPq Grants 473326/2013-2 and 303027/2012-6. RdR is grateful to Prof. Loriano Bonora, for the fruitful discussions.

References

1. P. Lounesto, *Clifford Algebras and Spinors*, 2nd edn. (Cambridge University Press, Cambridge, 2001)
2. R.T. Cavalcanti, Int. J. Mod. Phys. D **23**, 1444002 (2014)
3. R. da Rocha, J.G. Pereira, Int. J. Mod. Phys. D **16**, 1653 (2007)
4. L. Bonora, K.P.S. de Brito, R. da Rocha, JHEP **1502**, 069 (2015)
5. C.I. Lazaroiu, E.M. Babalic, I.A. Coman, JHEP **1309**, 156 (2013)
6. J.D. Bjorken, S.D. Drell, *Relativistic Quantum Mechanics* (McGraw Hill, New York, 1964)
7. H. Wei, Phys. Lett. B **695**, 307 (2011)
8. C.Y. Lee, arXiv:1404.5307 [hep-th]
9. J.M. Hoff da Silva, S.H. Pereira, JCAP **1403**, 009 (2014)
10. A. Pinho S. S., S.H. Pereira, J.F. Jesus, Eur. Phys. J. C **75**(1), 36 (2015)
11. B. Agarwal, A.C. Nayak, R.K. Verma, P. Jain, arXiv:1407.0797 [hep-ph]
12. Y.X. Liu, X.N. Zhou, K. Yang, F.W. Chen, Phys. Rev. D **86**, 064012 (2012)
13. I.C. Jardim, G. Alencar, R.R. Landim, R.N.C. Filho, Phys. Rev. D **91**, 085008 (2015)
14. R. da Rocha, A.E. Bernardini, J.M. Hoff da Silva, JHEP **1104**, 110 (2011)
15. A.E. Bernardini, R. da Rocha, Phys. Lett. B **717**, 238 (2012)
16. D.V. Ahluwalia, D. Grumiller, JCAP **0507**, 012 (2005)
17. D.V. Ahluwalia, S.P. Horvath, JHEP **1011**, 078 (2010)
18. D.V. Ahluwalia, C.Y. Lee, D. Schritt, T.F. Watson, Phys. Lett. B **687**, 248 (2010)
19. R. da Rocha, J.M. Hoff da Silva, Adv. Appl. Clifford Algebr. **20**, 847 (2010)
20. K.E. Wunderle, R. Dick, Can. J. Phys. **90**, 1185 (2012)
21. A. Alves, F. de Campos, M. Dias, J.M. Hoff da Silva, Int. J. Mod. Phys. A **30**, 0006 (2015)
22. J.M. Hoff da Silva, R. da Rocha, Phys. Lett. B **718**, 1519 (2013)
23. R. da Rocha, J.M. Hoff da Silva, Europhys. Lett. **107**, 50001 (2014)
24. J.P. Crawford, J. Math. Phys. **26**, 1439 (1985)
25. Y. Takahashi, Phys. Rev. D **26**, 2169 (1982)
26. R. da Rocha, L. Fabbri, J.M. Hoff da Silva, R.T. Cavalcanti, J.A. Silva-Neto, J. Math. Phys. **54**, 102505 (2013)
27. I.M. Benn, R.W. Tucker, *An Introduction to Spinors and Geometry with Applications in Physics* (Adam Hilger, Bristol, 1987)
28. E. Majorana, Nuovo Cimento **9**, 43 (1932)
29. W.A. Rodrigues Jr, Q.A.G. de Souza, J. Vaz, P. Lounesto, Int. J. Theor. Phys. **35**, 1849 (1996)
30. R. da Rocha, J. Vaz Jr, Int. J. Geom. Methods Mod. Phys. **4**, 547 (2008)
31. J. Vaz Jr., Construction of monopoles and instantons by using spinors and the inversion theorem, in *Clifford Algebras and Their Application in Mathematical Physics*. Fundamental theories of physics, vol. 94, ed. by V. Dietrich, K. Habetha, G. Jank (Kluwer Academic Publishers, Amsterdam, 1998), pp. 401–421

Capítulo 3

Física dos Espinores Escuros

Nossa intenção neste capítulo é nos focarmos em algumas das consequências que a estrutura formal de trabalho com os espinores escuros trazem para a física. Vamos nos ater, em particular, aos chamados espinores Elko (sigla em alemão para *Eigenspinoren des LadungsKonjugationsOperators*, autoespinores do operador de conjugação de carga). A sigla remete ao nome que abarca uma de suas principais características. Vamos a elas.

Um espinor pertencente ao espaço de representação completo $(0,1/2)\oplus(1/2,0)$ é composto, via de regra, por dois espinores de Weyl (bi-espinores), um mão direita (frente boosts de Lorentz) outro mão esquerda. Cada um desses bi-espinores pertence a um setor do espaço de representação completo. Usualmente, a relação entre ambos os setores do espaço de representação é dada através do operador de paridade. De fato, se tal procedimento é adotado então a dinâmica do espinor fixa-se como dada pela equação de Dirac. Em outras palavras, a adoção de autoespinores do operador paridade como bi-espinores compondo o quadri-espinor completo implica a dinâmica de Dirac. Entretanto a utilização do operador de paridade não é a única maneira de se relacionar ambas as partes do espaço de representação. Uma identidade algébrica entre as matrizes de Pauli (conhecida na literatura como "mágica das matrizes de Pauli") permite-nos essa correlação, tal como se dá com a utilização do operador de paridade, porém sem a necessidade de introdução de nenhuma simetria discreta.

Um resultado direto da observação feita ao final do parágrafo precedente é o fato de que os espinores resultantes não obedecem a equação de Dirac, mas apenas a equação de Klein-Gordon. Ainda no que diz respeito à estrutura do espinor, uma base explícita pode ser estabelecida via atuação do operador de helicidade no referencial de repouso. Esse detalhe é particularmente relevante, uma vez que vale apenas para espinores de massa não nula, necessariamente. Prosseguindo na formulação básica, uma característica importante é o fato dos espinores encontrados terem fases fixadas pela exigência de que sejam autoespinores do operador conjugação de carga. Isso implica, em última análise, a neutralidade desejada. Sendo neutro, o espinor não carregará cargas de gauge em geral, e em particular no que diz respeito ao grupo $U(1)$. Logo, o termo "escuro" torna-se preciso para a descrição desse espinor.

Outra característica relevante advém da construção do dual (adjunto) associado ao espinor Elko. Com efeito, um procedimento bastante criterioso para a construção do dual, impondo apenas condições físicas, mostra a existência de um novo dual para o Elko cuja principal consequência está relacionada ao campo quântico. Notemos, antes, que assim como acontece com espinores de Dirac, é possível se mostrar (via identidade algébrica de primeira ordem na derivada) que os Elkos não se transformam de maneira unitária. Portanto não podemos associá-los a um estado quântico, exigindo assim o procedimento de segunda quantização.

Nesse ponto, é importante explicitarmos que o procedimento usual de se extrair a teoria de campos da lagrangiana respeitada pelo campo em questão não deve ser adotado *a priori*. A razão para tal é que todo o processo de construção do Elko é feito em uma perspectiva *bottom-up* dotando a teoria, o quanto possível, de um caráter de inevitabilidade. Assim, uma vez estabelecido o campo quântico cujos coeficientes espinoriais de expansão são os espinores descritos até aqui, podemos calcular o propagador de Feymann-Dyson e os correlatores a tempo fixo. O cálculo do propagador revela, de fato, possuir o campo dimensão canônica de massa um, evidenciando, uma vez mais, o fato do Elko obedecer exclusivamente à equação de Klein-Gordon. Esse fato será importante para o desenvolvimento de parte desse capítulo. Antes, porém, remetemo-nos a um aspecto sutil, porém importante, da formulação. As somas de spin do Elko possuem um fator não usual que depende da parametrização do momento, levando a uma quebra da simetria de Lorentz. Fato é que tal fator é invariante por transformações do grupo $SIM(2)$, um subgrupo do grupo de Lorentz obtido quando as simetrias discretas são retiradas do grupo e os geradores do setor ortócrono-próprio são rearranjados. É mister relembrarmos que na formulação primeva do espinor o operador de paridade não foi levado em conta, logo não há inconsistência interna na formulação da teoria. A modificação nas somas de spin, entretanto, leva a uma consequência direta na localidade do campo quântico: o cálculo dos correlatores mostra a existência de um eixo de localidade, cuja existência novamente encontra respaldo no grupo $SIM(2)$.

Nas próximas seções exporemos dois trabalhos que visam extrair algum possível sinal do Elko no LHC e, em seguida, um trabalho (em um contexto inteiramente diverso) que trata de forma aproximada a radiação Hawking de Elkos.

3.1 Sinais no LHC

Como mencionado ao longo das páginas precedentes, a formulação de um campo espinorial escuro, e particularmente do Elko, é tal que os habilita a candidatos à matéria escura. Assim, a própria formulação precisa garantir (por construção) limitações severas nas possibilidades de interação do campo (pois do contrário o campo não seria escuro). A questão que se coloca aqui, então, é: como detectar um campo cuja principal característica é ter pouca possibilidade de detecção? Vejamos o panorama geral.

Primeiramente, o fato dos coeficientes de expansão do campo espinorial serem autoespinores do operador de conjugação de carga impede que a imposição de simetrias de gauge sejam efetuadas. Deste modo, o campo é neutro com relação a cargas de gauge e não interage, portanto, com o campo eletromagnético. Não há então a possibilidade de termos de interação do tipo $espinor \;\; dual \; A^{\mu}A_{\mu} \; espinor$.

Ademais, a dimensão canônica de massa do campo reduz drasticamente outros possíveis acoplamentos. De fato, a fim de que tenhamos acoplamentos renormalizáveis perturbativamente, não devemos ter dimensão de massa negativa nas constantes de acoplamento[1]. Assim sendo, resta-nos apenas o acoplamento com o campo gravitacional (do qual nos ocuparemos no próximo capítulo) e com o campo de Higgs. Nos próximos dois trabalhos nos ocuparemos com a interação do Elko com o campo de Higgs, visando sua detecção no LHC.

No primeiro dos dois próximos trabalhos duas características foram investigadas: a não localidade do Elko, decorrente da sutil violação de Lorentz do campo, estudada via correções radiativas de sua interação com o campo de Higgs, e o

[1]Notemos que acoplamentos de ordem superior serão severamente suprimidos por ordens de magnitude da escala de Planck.

estudo em nível de árvore de uma fusão de bósons de Higgs produzindo Elkos, e múons e *missing energy* no estado final. A criação de Elkos diminui a quantidade de energia perdida ao final do processo, levando a um indicativo de sua presença na reação.

No segundo trabalho, extendemos a análise anterior e investigamos ainda outro acoplamento do Elko com o campo de Higgs: o acoplamento Elko-Elko-Higgs com energia de centro de massa de $14TeV$. Mostramos que esse último acoplamento apresenta um sinal bastante promissor quando comparado aos anteriores.

Physics Letters B 706 (2012) 352–359

Contents lists available at SciVerse ScienceDirect

Physics Letters B

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Exploring Elko typical signature

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ARTICLE INFO

Article history:
Received 9 September 2011
Received in revised form 10 November 2011
Accepted 15 November 2011
Available online 20 November 2011
Editor: S. Dodelson

ABSTRACT

We study the prospects of observing the presence of a relatively light Elko particle as a possible dark matter candidate, by pointing out a typical signature for the process encompassing the Elko non-locality, exploring some consequences of the unusual Elko propagator behavior when analyzed outside the Elko axis of propagation. We also consider the production of a light Elko associated to missing energy and isolated leptons at the LHC, with center of mass energy of 7 and 14 TeV and total luminosity from 1 fb^{-1} to 10 fb^{-1}. Basically, the Elko non-locality engenders a peculiar signal in the missing energy turning it sensible to the angle of detection.

1. Introduction

Elko spinor fields are unexpected spin one-half matter fields endowed with mass dimension 1 [1,2]. Since its recent theoretical discovery, it has attracted much attention, in part by the wide range of possibility opened by such peculiar matter fields in cosmology and physics [3] and in part from the mathematical point of view [4]. The word Elko is the acronym for *Eigenspinoren des Ladungskonjugationsoperators* or Dual-helicity eigenspinors of the charge conjugation operator (see Eq. (2)).

The two aforementioned characteristics of Elko (namely, spin one-half and mass dimension 1) makes quite reduced the possible coupling to the Standard Model fields. In fact, keeping in mind that interaction terms with mass dimension greater than four should be severely suppressed by some fundamental mass scale and focusing in simple power counting renormalizable arguments, it turns out that Elko spinor fields may have quartic self-interaction and an Elko–Higgs (doublet) interaction.[1] In this vein, such spinor field may act as a dark matter candidate.

Another interesting feature about Elko is its non-locality. Elko spinor fields do not belong to a standard Wigner's class [5]. It was demonstrated, however, that Elko breaks Lorentz symmetry (in a subtle way) by containing a preferred direction [6]. It is worth to note that the existence of a preferred direction – the so-called 'axis of evil' – (as well as a self-interaction) is believed to be a property of dark matter [7]. We also remark, for completeness, that the quantum field associated to the Elko spinor is now better understood in the scope of Very Special Relativity (VSR) framework [8]. In fact, it is possible to describe, or construct, Elko spinor fields as the spinor representation of *SIM*(2) subgroup of VSR [9]. In this vein, since *SIM*(2) is the largest subgroup of VSR encompassing all the necessary physical symmetries except some (violated) discrete symmetry, the tension between Elko and Lorentz symmetries disappears.

On the other hand, it is well known that accelerators will test, in an incontestable way, theories in the scope of physics beyond the Standard Model as well as shed some light to the mass generation problem [10–13]. Candidates of dark matter predicted in particle physics theories, like supersymmetry, are on the focus of such studies and the answers will provide additional information for a deeper level of our understanding on astrophysics and cosmology. In such a way, the CERN Large Hadron Collider (LHC) results are fundamental for any study connecting high energy physics and astrophysics/cosmology. The LHC will provide center-of-mass energy enough to probe directly the weak scale and the origin of mass. Therefore, since we still have the open question of the dark matter nature, it is possible the study of the origin of mass as well as the candidate to the dark matter in the search of Elko. In considering some specific process for Elko production, radiative corrections must be taken into account. In this case, as we will see, the Elko non-locality is manifest leading to an exclusive output in the final signature. At phenomenological grounds, such a behavior suggests a different analysis for the search of Elko at accelerators. So, we consider in some detail a tree level process (where the non-locality is absent) concerning to the Elko production at the LHC, whose signature is $\mu^+ + \mu^- + 2\varsigma$. Such process includes the quartic self-interaction and a coupling with the Higgs scalar field.

This Letter is organized as follows: In the next section we introduce some formal aspects of the Elko spinor fields calling attention to the main characteristics that will be relevant in the subsequent

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[1] We shall emphasize that Elko does not carry standard $U(1)$ gauge invariance [1].

0370-2693/$ – see front matter
doi:10.1016/j.physletb.2011.11.030

analysis. In Section 3 we explore the Elko non-locality, when considering radiative corrections. In Section 4 we analyze the tree level case of a viable cross-section for Elko production at the LHC. Then, we move forward investigating some peculiar aspects of our signal. In the last section we conclude.

2. Elko spinor fields

In this section we briefly introduce the main aspects concerning the construction of Elko spinor fields. Its formal structure may be outlined as follows. Let C be the charge conjugation operator given, in Weyl realization, by

$$C = \begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix} K, \tag{1}$$

being K the operator that complex conjugate a spinor which appears on its right and σ_2 the usual Pauli matrix. The Elko spinor, $\lambda(\mathbf{p})$, is defined by

$$C\lambda(\mathbf{p}) = \pm\lambda(\mathbf{p}), \tag{2}$$

where plus sign yields self-conjugate spinors ($\lambda^S(\mathbf{p})$) and minus anti self-conjugate spinors ($\lambda^A(\mathbf{p})$)

$$\lambda(\mathbf{p}) = \begin{pmatrix} \pm\sigma_2\phi_L^*(\mathbf{p}) \\ \phi_L(\mathbf{p}) \end{pmatrix}. \tag{3}$$

In the above equation $\phi_L(\mathbf{p})$ transforms as a left-handed (Weyl) spinor, hence $\sigma_2\phi_L^*(\mathbf{p})$ transforms as a right-handed spinor. In this vein, Elko spinor belongs to the $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ representation space. Now, let us set the explicit form of Elko, in the rest frame[2] ($\mathbf{p} = \mathbf{0}$). In order achieve the formal profile of Elko, one may look at the helicity equation $(\sigma \cdot \hat{\mathbf{p}})\phi^{\pm}(\mathbf{0}) = \pm\phi^{\pm}(\mathbf{0})$. Taking $\hat{\mathbf{p}} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ we arrive at four spinors, following the standard notation, given by

$$\begin{aligned}
\lambda^S_{\{+,-\}}(\mathbf{0}) &= \begin{pmatrix} +\sigma_2[\phi_L^-(\mathbf{0})]^* \\ \phi_L^-(\mathbf{0}) \end{pmatrix}, \\
\lambda^S_{\{-,+\}}(\mathbf{0}) &= \begin{pmatrix} +\sigma_2[\phi_L^+(\mathbf{0})]^* \\ \phi_L^+(\mathbf{0}) \end{pmatrix}, \\
\lambda^A_{\{+,-\}}(\mathbf{0}) &= \begin{pmatrix} -\sigma_2[\phi_L^-(\mathbf{0})]^* \\ \phi_L^-(\mathbf{0}) \end{pmatrix}, \\
\lambda^A_{\{-,+\}}(\mathbf{0}) &= \begin{pmatrix} -\sigma_2[\phi_L^+(\mathbf{0})]^* \\ \phi_L^+(\mathbf{0}) \end{pmatrix},
\end{aligned} \tag{4}$$

with phases adopted such that

$$\phi_L^+(\mathbf{0}) = \sqrt{m_\varsigma}\begin{pmatrix} \cos(\theta/2)e^{-i\phi/2} \\ \sin(\theta/2)e^{i\phi/2} \end{pmatrix} \tag{5}$$

and

$$\phi_L^-(\mathbf{0}) = \sqrt{m_\varsigma}\begin{pmatrix} -\sin(\theta/2)e^{-i\phi/2} \\ \cos(\theta/2)e^{i\phi/2} \end{pmatrix}. \tag{6}$$

We remark that $-i\sigma_2[\phi_L^{\pm}(\mathbf{0})]^*$ and $\phi_L^{\pm}(\mathbf{0})$ present opposite helicities and, hence, Elko carries *both* helicities. Another important formal aspect of Elko spinor fields is its dual spinor. In order to guarantee an invariant real norm, being positive definite for two Elko spinor fields and negative definite norm for the other two, the dual for Elko is defined by

$$\overline{\lambda}^{S/A}_{\{\mp,\pm\}}(\mathbf{p}) = \pm i\big[\lambda^{S/A}_{\{\pm,\mp\}}\big((\mathbf{0})\big)\big]^\dagger\gamma^0. \tag{7}$$

With such a definition for the Elko dual, one arrives at the following spin sums [1]

$$\begin{aligned}
\sum_\kappa \lambda^S_\kappa\overline{\lambda}^S_\kappa &= +m_\varsigma\big[\mathbb{I} + \mathcal{G}(\phi)\big], \\
\sum_\kappa \lambda^A_\kappa\overline{\lambda}^A_\kappa &= -m_\varsigma\big[\mathbb{I} - \mathcal{G}(\phi)\big],
\end{aligned} \tag{8}$$

where $\mathcal{G}(\phi)$ is given by [6]

$$\mathcal{G}(\phi) = \gamma^5(\gamma_1\sin\phi - \gamma_2\cos\phi), \tag{9}$$

and the gamma matrices are

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & -\sigma^i \\ \sigma^i & 0 \end{pmatrix}, \tag{10}$$

being $\gamma^5 = -i\gamma^0\gamma^1\gamma^2\gamma^3$. Spin sums entering in a profound level into the local structure, as well as the statistic, of the theory. It is important to note that the right-hand side of Eqs. (8) is not proportional (or unitary connected) to the momentum operators.[3] Therefore the relations (8) are responsible for the peculiar characteristics of Elko locality structure, as well as its breaking of Lorentz invariance. Such peculiarity, obviously, brings important modifications in the S-matrix calculations (see next section).

After studying the formal structure of Elko spinor fields, we shall examine the quantum field associated to such spinor. It is possible to define an Elko-based quantum field, respecting its formal properties, by

$$\begin{aligned}
\eta(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2mE(\mathbf{p})}} \\
\times \sum_\alpha\big[c_\alpha(\mathbf{p})\lambda^S_\alpha(\mathbf{p})e^{-ip_\mu x^\mu} + c^\dagger_\alpha(\mathbf{p})\lambda^A_\alpha e^{+ip_\mu x^\mu}\big],
\end{aligned} \tag{11}$$

being $c^\dagger_\alpha(\mathbf{p})$ and $c_\alpha(\mathbf{p})$ the creation and annihilation operators, respectively, satisfying the fermionic anticommutation relations

$$\big\{c_\alpha(\mathbf{p}), c^\dagger_{\alpha'}(\mathbf{p}')\big\} = (2\pi)^3\delta^3(\mathbf{p} - \mathbf{p}')\delta_{\alpha\alpha'}, \tag{12}$$

$$\big\{c^\dagger_\alpha(\mathbf{p}), c^\dagger_{\alpha'}(\mathbf{p}')\big\} = \big\{c_\alpha(\mathbf{p}), c_{\alpha'}(\mathbf{p}')\big\} = 0. \tag{13}$$

The Elko dual $\overline{\eta}$ is obtained by replacing λ by its dual, c by $c^\dagger$ and $ip_\mu x^\mu$ by $-ip_\mu x^\mu$ (and vice versa). There is a crucial identity obeyed by Elko, given by the application of the $\gamma_\mu p^\mu$ operator to $\lambda^{S/A}(\mathbf{p})$:

$$\big(\gamma_\mu p^\mu\delta^\beta_\alpha \pm im\mathbb{I}\varepsilon^\beta_\alpha\big)\lambda^{S/A}_\beta(\mathbf{p}) = 0, \tag{14}$$

where $\varepsilon^{\{-,+\}}_{\{+,-\}} := -1$ and δ^β_α is the usual Kronecker symbol. In view of (the simply algebraic) Eq. (14) it turns out that Elko satisfies the Klein–Gordon (not Dirac) equation and, therefore, it must be associated to a Klein–Gordon-like Lagrangian:

$$\mathcal{L}^{free} = \partial^\mu\overline{\eta}(x)\partial_\mu\eta(x) - m^2_\varsigma\overline{\eta}(x)\eta(x). \tag{15}$$

As already mentioned in the Introduction, we shall study the coupling between Elko and Higgs fields, since it is the unique

[2] Of course, the explicit form for any momentum is obtained by performing a boost in $\lambda(\mathbf{p})$.

[3] In acute contrast with the usual Dirac case.

renormalizable (perturbatively) Elko coupling. Therefore, in the next section we shall explore the features of the (15) Lagrangian, plus the interaction given by

$$\mathcal{L}^{int} = \lambda_{\varsigma}\phi^2(x)\overline{\eta}(x)\eta(x). \tag{16}$$

In this work, and consequently to obtain the Feynman rules relevant to it (see Ref. [15]), our object of study is (15) and (16) added with the usual kinetic and interaction terms for the Higgs boson, the Z vector field and summing over all the quarks in the theory, as they appear in the Standard Model after symmetry breaking.

As the last remark we emphasize that, in general, Eqs. (8) and (9) suggest that there is a preferred axis for Elko. In fact, it is possible to show that Elko enjoys locality in the direction perpendicular to its plane [6], or, equivalently, along the preferred axis $\hat{z}_e$. Let us give an example coming from the canonical structure of Elko fields in order to clarify this point. The canonical conjugate momenta to the Elko fields are given by

$$\Pi(x) = \frac{\partial \mathcal{L}_{KG}}{\partial \dot{\eta}} = \frac{\partial \overline{\eta}}{\partial t}, \tag{17}$$

where $\mathcal{L}_{KG}$ stands for a Klein–Gordon-like Lagrangian. The equal time anticommutator for $\eta(x)$ and its conjugate momentum is

$$\begin{aligned}\{\eta(\mathbf{x},t), \Pi(\mathbf{x}',t)\} = i\int \frac{d^3p}{(2\pi)^3}\frac{1}{2m}e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')} \\ \times \sum_{\alpha}\big[\lambda^S_\alpha(\mathbf{p})\overline{\lambda}^S_\alpha(\mathbf{p}) - \lambda^A_\alpha(-\mathbf{p})\overline{\lambda}^A_\alpha(-\mathbf{p})\big],\end{aligned} \tag{18}$$

which, in the light of the spin sums, may be recast in the following form

$$\{\eta(\mathbf{x},t), \Pi(\mathbf{x}',t)\} = i\delta^3(\mathbf{x}-\mathbf{x}')\mathbb{I} + i\int \frac{d^3p}{(2\pi)^3 e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')}}\mathcal{G}. \tag{19}$$

The existence of a preferred axis is now evident, since the second integral in the right-hand side of Eq. (19) vanishes along the $\hat{z}_e$. So, this preferred axis may be understood as an axis of locality.

3. Exploring Elko non-locality

According to its typical Lagrangian Elko spinor fields couple only to the Higgs boson and, hence, any production mechanism of such particle must occur via Higgs production or decay process. A very specific feature of Elko production is its non-locality, encoded in the propagator behavior which has a different form (the $\mathcal{G}(\phi)$ term appears explicitly) when computed outside its axis of propagation. In order to explore a little further this effect, let us consider for instance the first graph of a cascade production of Elko particles (Fig. 1).

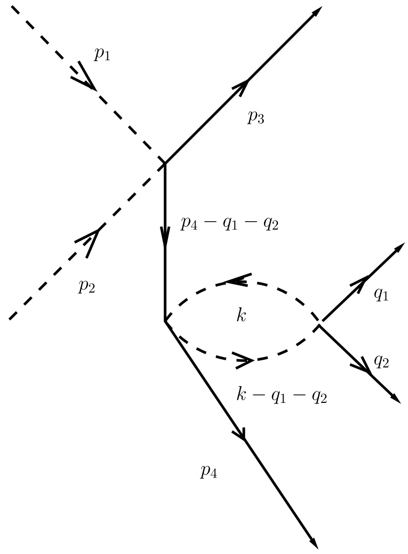

Fig. 1. Example of higher order graphic relevant to Elko production and its non-locality. Dotted lines stand for Higgs boson and continuous lines for Elko.

If one chooses to compute (or measure) such a higher order process in the same plane where the intermediary Elko is propagating, the amplitude reads

$$\begin{aligned}i\mathcal{M} &= \lambda^3_\varsigma \frac{\lambda^A_\alpha(p_3)\lambda^A_\rho(q_1)\overline{\lambda}^S_\beta(p_4)\overline{\lambda}^S_\sigma(q_2)}{(p_4+q_1+q_2)^2 - m^2_\varsigma} \\ &\quad\times \int \frac{d^4k}{(2\pi)^4}\frac{1}{[k^2-m^2_H][(k-q_1-q_2)^2-m^2_H]}.\end{aligned}$$

Otherwise, there is also in the amplitude the presence of the $\mathcal{G}(\phi)$ term

$$\begin{aligned}i\mathcal{M} &= \lambda^3_\varsigma \frac{\lambda^A_\alpha(p_3)\lambda^A_\rho(q_1)[1+\mathcal{G}(\phi)]\overline{\lambda}^S_\beta(p_4)\overline{\lambda}^S_\sigma(q_2)}{(p_4+q_1+q_2)^2 - m^2_\varsigma} \\ &\quad\times \int \frac{d^4k}{(2\pi)^4}\frac{1}{[k^2-m^2_H][(k-q_1-q_2)^2-m^2_H]}.\end{aligned}$$

The divergence appearing in the above amplitude was treated via Pauli–Villars regularization, subtracted this amplitude from its value at $q_1 = q_2 = 0$. The result is given by

$$\begin{aligned}i\mathcal{M}_{RG} &= \lambda^3_\varsigma \frac{\lambda^A_\alpha(p_3)\lambda^A_\rho(q_1)[\mathbb{I}+\mathcal{G}(\phi)]\overline{\lambda}^S_\beta(p_4)\overline{\lambda}^S_\sigma(q_2)}{(p_4+q_1+q_2)^2 - m^2_\varsigma} \\ &\quad\times \int_0^1 \ln\left(\frac{(q_1+q_2)^2x(x-1)+m^2_H}{m^2_H}\right).\end{aligned} \tag{20}$$

Computing the traces (where E_1 and E_2 are, respectively q_1 and q_2 particle energies) the average spin squared sum is

$$\begin{aligned}\frac{1}{16}\sum_{spins}|\mathcal{M}_{RG}|^2 \\ &= \frac{E_2E_4(E_3+p_3)(E_1+q_1)trace[(\mathbb{I}-\mathcal{G}(\phi))(\mathbb{I}+\mathcal{G}(\phi))(\mathbb{I}+\mathcal{G}(\phi))]trace[\mathbb{I}-\mathcal{G}(\phi)]}{[(p_4+q_1+q_2)^2-m^2_\varsigma]^2} \\ &\quad\times\left[\int_0^1 \ln\left(\frac{(q_1+q_2)^2x(x-1)+m^2_H}{m^2_H}\right)\right]^2\lambda^6_\varsigma \\ &= \lambda^6_\varsigma\frac{8E_2E_4(E_3+p_3)(E_1+q_1)}{[(p_4+q_1+q_2)^2-m^2_\varsigma]^2} \\ &\quad\times\left[\int_0^1 \ln\left(\frac{(q_1+q_2)^2x(x-1)+m^2_H}{m^2_H}\right)\right]^2.\end{aligned} \tag{21}$$

Note that if one lies on the $\vec{p}_4+\vec{q}_1+\vec{q}_2$ direction the obtained result is divided by two. Since the decay rate is proportional to the average spin squared amplitude integrated over the four-body phase space, the Elko particle decays in a preferred axis. Besides, the decay process in such a channel is one half lower than in any other direction.

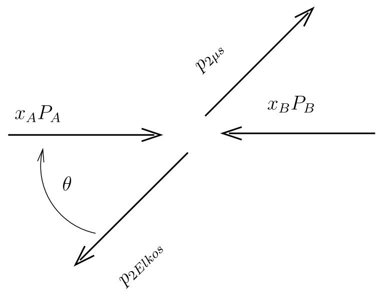

Fig. 2. Kinematics of Elko production.

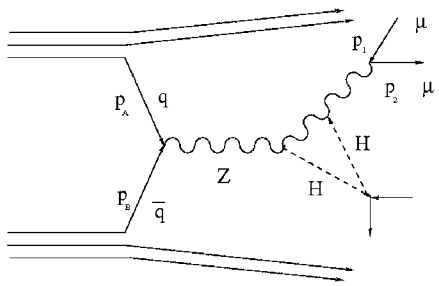

Fig. 3. $q+\bar{q}\rightarrow\mu^{+}+\mu^{-}+2\varsigma$ scattering. The loop is composed by two Higgs and a Z boson.

The above considerations lead to an important result: if the cut applied on ϕ includes the intermediary Elko propagation axis, the measured decay is lower than any other cut in which this specific direction is not included. Therefore it breaks ϕ isotropy which is, obviously, fully observed in all Standard Model particles. Such a process makes then manifest the Elko non-locality, giving also a clue for its signature. We should also note another feature in this production, as reflect of momentum conservation, represented in Fig. 2. An increase in the Elko production, in a preferred direction, should implicate a decrease of the remain particles final momentum in the same direction (as a missing energy in the detector), reflecting in a complementary angular distribution, when compared with its possible *background*.

4. Tree level case

For tree level calculations, the non-locality effect is not manifest, and the study of possible signals of Elko decay at accelerators is addressed to the standard searching. For this purposes, we have considered the case where Elko can be produced at the LHC through the Higgs boson fusion, via quartic coupling as depicted in Fig. 3. In both cases (Higgs production or decay process), however, the production is suppressed according to the value of the coupling constant, leaving the number of events and the signature of the decay expressed as a function of two fundamental parameters of the model: the Elko mass and the Elko–Higgs boson coupling constant, which will be taken as less than or equal to one, in order to ensure renormalizability. At the LHC, signatures with leptons as a final state are preferred, specially muons, whose background can be calculated directly from the Standard Model. Besides, the identification of muons are well given as, for example, at CMS technical proposal. In this vein, we will be focused in a two muons + Elko signal, according to the process illustrated in the graph (Fig. 3). In this case the process is $q+\bar{q}\rightarrow\mu^{+}+\mu^{-}+2\varsigma$, where 2ς stands for the two Elko particles with mass m_ς produced in the threshold were they will be on rest in the CoM frame. We do not considered here the direct production of two Higgs from Elko fusion, since the Higgs boson is, indeed, the key block to be detected

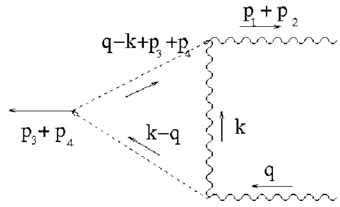

Fig. 4. Performed loop calculation.

at the LHC. We have fixed the Higgs mass boson in the experimental limit [14] and also considered jets with high energy and momentum. In such case, they will emerge almost collinear with the beam. The interaction rate is proportional to the cross section calculated as follows:

We shall label $p_A = x_A P_A$ and $p_B = x_B P_B$, respectively, as the momentum for the quark and anti-quark, related to the initial protons $P_{A,B}$ and the muons with momentum p_1 and p_2. The amplitude is given by:

$$
\begin{aligned}
i\mathcal{M} = {} & q^{r}(p_A)\left[\frac{ig_Z}{2}\gamma^{\mu}\left(c_V^f - c_A^f\gamma^5\right)\right] \\
& \times \bar{q}^{r'}(p_B)\left[\frac{-i}{q^2-m_Z^2}\left(g_{\mu\nu}-\frac{q_\mu q_\nu}{m_Z^2}\right)\right] \\
& \times \frac{igm_Z g^{\nu\rho}}{2\cos(\theta_W)}\left[\frac{-i}{k^2-m_Z^2}\left(g_{\rho\sigma}-\frac{k_\rho k_\sigma}{m_Z^2}\right)\frac{i}{(q-k)^2-m_H^2}\right] \\
& \times \frac{igm_Z g^{\sigma\gamma}}{2\cos(\theta_W)}\left[\frac{i}{(q-k)-m_H^2}\right]\lambda_\varsigma \bar{\lambda}^S{}_{\Lambda}\lambda^A_{\Omega} \\
& \times \left[\frac{-i}{q^2-m_Z^2}\left(g_{\gamma\delta}-\frac{q_\gamma q_\delta}{m_Z^2}\right)\right] \\
& \times \frac{-ig_Z}{2}\gamma^{\delta}\left(-\frac{1}{2}+2\sin^2(\theta_W)+\frac{1}{2}\gamma^5\right)\bar{u}^s(p_1)v^{s'}(p_2),
\end{aligned}
\tag{22}
$$

following the conventions of Ref. [15], where the factors for quarks read

$$
\begin{aligned}
u \quad &\Rightarrow \quad c_A^f = 1/2, \qquad c_V^f = 1/2 - 4/3\sin^2(\theta_W), \\
d \quad &\Rightarrow \quad c_A^f = -1/2, \qquad c_V^f = -1/2 + 2/3\sin^2(\theta_W).
\end{aligned}
$$

On partonic CoM reference frame and $p_A = p_B = p_1 = p_2 \approx 0$ we can set

$$
\begin{aligned}
& p_A = \frac{\sqrt{\hat{s}}}{2}(1,0,0,1), \qquad p_B = \frac{\sqrt{\hat{s}}}{2}(1,0,0,-1), \\
& p_1 = \left(\frac{\sqrt{\hat{s}}}{2}-m_\varsigma\right)\left(1,\sin(\theta),0,\cos(\theta)\right), \\
& p_2 = \left(\frac{\sqrt{\hat{s}}}{2}-m_\varsigma\right)\left(1,-\sin(\theta),0,-\cos(\theta)\right), \\
& p_3 = m_\varsigma(1,0,0,0), \qquad p_4 = m_\varsigma(1,0,0,0),
\end{aligned}
$$

where $q = \sqrt{\hat{s}}$.

Looking at Fig. 4 we can identify

$$
\begin{aligned}
& P_1 = (k-q)^2 - m_H^2, \\
& P_2 = (q+2m_\varsigma-k)^2 - m_H^2, \\
& P_3 = k^2 - m_Z^2 = l_0^2 - l_\perp^2 - m_Z^2,
\end{aligned}
\tag{23}
$$

as the denominators for the function to be integrated. In order to use the functions well established (`OneLoop2Pt`) on `xloops` package [16] we need to reduce the number of functions on denominator (23), using Feynman trick,

$$
\begin{aligned}
&\frac{1}{P_1P_2P_3}=\int_0^1\frac{1}{P_3}\frac{dx}{[P_1x+P_2(1-x)]^2}=\int_0^1\frac{1}{P_3}\frac{dx}{(k+q')^2-m^2},\\
&q'=-x\sqrt{\hat{s}}+(x-1)(\sqrt{\hat{s}}+2m_\varsigma)\\
&\quad\Rightarrow\quad e_0^\mu=\frac{q'^\mu}{\|q'\|}=-(1,0,0,0),\\
&m^2=\big[x\sqrt{\hat{s}}+(1-x)(\sqrt{\hat{s}}+2m_\varsigma)\big]+m_H^2-x\sqrt{\hat{s}}\\
&\qquad-(1-x)(\sqrt{\hat{s}}+2m_\varsigma)^2,
\end{aligned}
\tag{24}
$$

where x integration was performed with `Maple` using of the approximation where $m_\varsigma/\sqrt{\hat{s}}\approx 0$. Obviously, such an approximation in the Elko mass is largely justified in order to guarantee Elko spinor fields as a dark matter candidate. This choice restrict the experimental analysis to events with low energy QCD jets in its final state, since almost all momentum is transferred to the initial partons, providing a signature for the Elko production. One can expect missing energy on detectors, due to the fact that Elko particles will be unobserved by detectors and the only impact in its production reduces the final $\mu^+ + \mu^-$ quadrimomentum. With this expression at hands, it is necessary to multiply by its conjugate and perform the respective polarization sums (8), taking into account, obviously, the terms $\mathcal{G}(\phi)$ responsible for the nonlocality outside $\hat{z}$ axis. Is straightforward to perform those traces for Elko polarization sums using the Elko dual definitions and the spin sums [1]

$$
\begin{aligned}
&\sum_\kappa \bar{\lambda}^S_\kappa(\bar{\lambda}^S_\kappa)^\dagger\\
&\quad=\sum_\kappa\big(i\epsilon^\rho_\kappa\lambda^{S\dagger}_\rho\gamma^0\big)\big(i\epsilon^\sigma_\kappa\lambda^{S\dagger}_\sigma\gamma^0\big)^\dagger=\sum_\kappa\epsilon^\rho_\kappa\epsilon^\sigma_\kappa\lambda^{S\dagger}_\rho\lambda^S_\sigma
\end{aligned}
\tag{25}
$$

$$
\quad=\lambda^S_{\{-,+\}}{}^\dagger\lambda^S_{\{-,+\}}+\lambda^S_{\{+,-\}}{}^\dagger\lambda^S_{\{+,-\}}=4E\mathbb{I},
\tag{26}
$$

where $\epsilon^{\{-,+\}}_{\{+,-\}}=-\epsilon^{\{+,-\}}_{\{-,+\}}=-1$.

After squaring, taking traces and averaging over the spin of the initial and final particles (we approximate the masses for quarks and muons to zero), we should obtain $\sum_{r,r'}\sum_{s,s',\Omega,\Lambda}|\mathcal{M}|^2$. One could use it to calculate

$$
d\hat{\sigma}=\frac{1}{2E_A2E_B}\frac{1}{2}\left(\frac{1}{64}\sum_{spin}|\mathcal{M}|^2\right)dP\,S,
$$

where $dP\,S$ is the phase space for two muons with momentum p_1 and p_2 and two Elkos with mass m_ς on rest, i.e.,

$$
\begin{aligned}
dP\,S&=(2\pi)^4\delta^4(p_A+p_B-p_1-p_2-p_3-p_4)\\
&\qquad\times\frac{d^3p_1}{(2\pi)^3(2E_1)}\frac{d^3p_2}{(2\pi)^3(2E_2)}\\
&=\frac{1}{4(2\pi)^2}\delta(\sqrt{\hat{s}}-E_1-E_2-2m_\varsigma)\frac{p_1^2\,dp_1\,d\Omega}{E_1E_2}\\
&=\frac{1}{32\pi^2}\frac{\sqrt{\hat{s}}}{\sqrt{\hat{s}}-2m_\varsigma}d\Omega,
\end{aligned}
\tag{27}
$$

where $|p_1|dp_1=E_1\,dE_1$, being $E_1=(\sqrt{\hat{s}}/2-m_\varsigma)$. We emphasize that we are working within $m_\varsigma\approx 0$ approximation. We also stress that $d\hat{\sigma}$ has no dependence on angular coordinates, so the integration on $d\Omega$ gives a multiplicative factor 4π for the total cross section. Our final result, however, is to much huge to be presented here.

On the hadronic frame, $P_A=\frac{\sqrt{s}}{2}(1,0,0,1)$ and $P_A=\frac{\sqrt{s}}{2}(1,0,0,-1)$. Thus

$$
s=(P_A+P_B)^2=\frac{\hat{s}}{x_Ax_B},
$$

and we will integrate using `Cuba` routines [17]

$$
\begin{aligned}
\sigma_{(p+p\to\mu^++\mu^-+2\varsigma)}=\sum_q\int_0^1\int_0^1 dx_A\,dx_B\,\big[f_q(x_A)f_{\bar{q}}(x_B)\\
+f_{\bar{q}}(x_A)f_q(x_B)\big]\hat{\sigma}(\hat{s})\delta(\hat{s}-x_Ax_Bs).
\end{aligned}
$$

With the hadronic total cross section at hands, it is straightforward to obtain the event rate R by multiplying σ by the integrated luminosity $\mathcal{L}$, estimated in 1 fb^{-1} and 10 fb^{-1}.

The results of the studied process are presented in Fig. 5. We show the total expected event rate for 2 Elkos $+\,\mu^+\mu-$ via the Higgs boson fusion, at the LHC, for two different values of the center-of-mass energy, as well as two different values for the total luminosity. The total number of events is presented as a function of the Elko mass. The main case we consider, with total luminosity of 10 fb^{-1}, at 7 TeV, for a coupling constant of an order of 1 shows a quite optimistic number of events, around a thousand. For a smaller coupling constant, $O(10^{-2})$, the number of events is also large. In this sense, we can consider the LHC, for instance, as a good scenario to study both, the Higgs boson and the Elko production in order to shed some light to the dark matter problem. For a 14 TeV center-of-mass energy case, in both 1 fb^{-1} and 10 fb^{-1} cases, the total number of events produced at the LHC is even bigger, for the different values of the coupling constant. By now, since the number of events is encouraging, we shall keep our attention in the exploration of a typical signature encoding the Elko nonlocality.

5. Detection possibility at LHC

Even though the decay in the preferred axis is estimated as one half lower than in any other direction, a poor detector angular resolution on this decay will smear out this effect, either due to the detector tracking, or due to the poor event reconstruction. Therefore it is mandatory to make an estimation of the minimum angular resolution requirement to detect this effect. At the LHC, the minimum angular resolution at, e.g., the CMS detector $\Delta\phi_{res}=10$ mrad [18]. The relative significance on this interval for an integrated luminosity L, taking into account our background will be given by

$$
S_{rel}=\frac{S}{\sqrt{B}},
\tag{28}
$$

since the background is isotropically distributed in the azimuthal angle and the efficiency on the muon measurement is about 98%. In Eq. (28), S stands for the number of events produced in the Elko decay and B denotes the number of events related to the background.

The signal is characterized by a dimuon in the final state reconstructed in a Z boson and some missing energy in the final state. Thus the irreducible SM background consists of the ZZ decaying in two muons and two neutrinos, as already studied in Ref. [19]. The background processes for the signal, considering next-to-leading order cross section are presented in Table 1 (see [18]). The irreducible SM background for the signal is the ZZ process, where one of the Z bosons decays into neutrinos. Since we

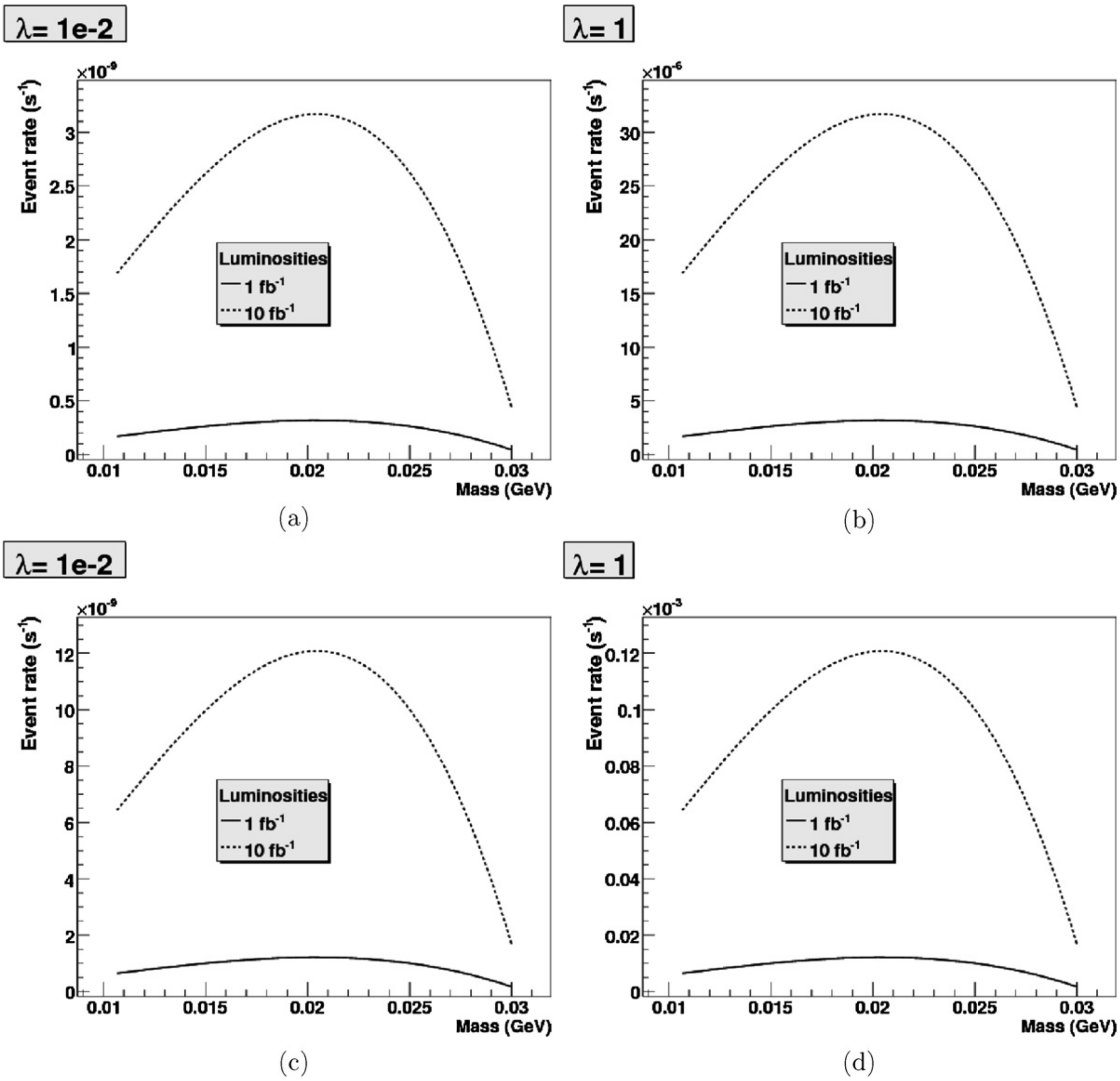

Fig. 5. Event rate ($1/s$) versus mass (GeV) for two luminosity values and the center-of-mass energy at the LHC for 7 TeV (a)–(b) and 14 TeV (c)–(d). The range for mass was chosen to guarantee the fact that the Elko can be a possible candidate for dark matter [1]. We have considered two values for λ_ς, namely, 1 and 10^{-2}.

Table 1
Background estimative for the high-order process under study.

Channel	Cross section (pb)
$qq \to WW \to \mu^+\mu^-$	11.7
$t\bar{t}$	840
$gg \to WW \to \mu^+\mu^-$	0.54
γ^*, Z	145 000
$b\bar{b} \to \mu^+\mu^-$	710
$ZW \to \mu^+\mu^- l^\pm$	1.63
$tWb \to \mu^+\mu^-$	3.4
$ZZ \to \mu^+\mu^-$	1.52

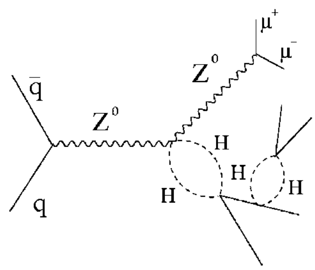

Fig. 6. Feynman diagram for the production of 4 missing Elko bosons (solid lines) and two muons, associated with some Higgs intermediary process.

are interested in making an estimate of the signal, taking into account the background without defining cuts for a detailed analysis, we only consider the ZZ process, as it is much larger than the signal. In such a case, the number of events for the background considering a luminosity of 2×10^{33} cm^{-2} s^{-1}, is around 99 000 events.

In order to explore the claimed angular dependence for the signal, we study the process $q\bar{q} \to Z^*\varsigma\varsigma^* \to 2\mu 4\varsigma$, which is shown in Fig. 6. The two final muons inherit the sensibility on azimuthal angle by momentum conservation on the final states. Actually this process is nothing but that one described in Fig. 3 followed by the decay of Fig. 1, mediated by two loops involving Higgs particles.

An analytic expression for this process can be obtained using the equation for $\frac{1}{16}\sum_{spins}|\mathcal{M}_{RG}|^2$ (see Section 3) and supposing the limit $\frac{q_1+q_2}{m_E} \approx 0, q_1 + q_2 > p_4$. One can expand the integrand, and proceed with the integration for the first term to obtain

$$\frac{1}{16}\sum_{spins}|\mathcal{M}_{RG}|^2 \approx \lambda_\varsigma^6 \frac{8E_2E_4(E_3+p_3)(E_1+q_1)}{(p_4+q_1+q_2)^4}\frac{(q_1+q_2)^4}{36m_H^4}$$
$$\approx \lambda_\varsigma^2 \frac{2E_2E_4(E_3+p_3)(E_1+q_1)}{9m_H^4}.$$

In the limit $p_3, p_4 \to 0$, and with all final state energies near to the Elko mass, we obtain a lower bound to this values given by

$$\frac{1}{16}\sum_{spins}|\mathcal{M}_{RG}|^2 \geqslant \lambda_\varsigma^6 \frac{2m_E^4}{9m_H^4}, \tag{29}$$

which shall be multiplied by the cross section obtained numerically before.

For the Elko production, on the simple decay $2\mu + 2\varsigma$ and fixing the coupling constant at its maximum value ($\lambda_\varsigma = 1$) as well as $m_\varsigma = 0.09$ GeV, we have $\sigma_{signal} = 5.06$ fb. At the LHC, for a 1 fb^{-1} integrated luminosity, one should obtain a ratio $\frac{S}{\sqrt{B}} = \frac{\sigma_{signal}\sqrt{L}}{\sqrt{\sigma_{bckg}}}$ around 5, where S stands for the number of events for the signal and B the number of events for the background (or actually one half of this value, taking account the angular asymmetry).

However as the detection of this process signature depends on the coupling constant, since under a certain value it would be required a better angular resolution in the detector to distinguish the signal from the background. For an indirect search of Elko particles via azimuthal angular asymmetry with $2\mu + 4\varsigma$ process, using the angular resolution for the CMS detector ($\Delta\phi \geqslant \Delta\phi_{res}$), $m_\varsigma = 0.09$ GeV and $\sqrt{s} = 7$ TeV, the number of events decreases substantially to $S = 4.4 \times 10^{-15}$ taking (29) into account. Hence, one can see that for these parameters Eq. (28) gives a result which is clearly insufficient to claim a discovery at the LHC. The process on study has actually a dependency on λ_ς^6, so the estimated minimum resolution for the $\lambda_\varsigma = 1 \times 10^{-2}$ case, maintaining $S_{rel} \approx 5$, is $\Delta\phi_{res} \approx 9.1 \times 10^{-11}$ rad. Lower values of λ_ς should require a better resolution on the detector. Of course, for this rough estimate, none type of cuts was performed and a detailed study using a Monte Carlo simulation for the final state Elko momenta would be in order.

The main motivation for this analysis is the possibility of Elko detection in a range of parameters making possible to address Elko as possible dark matter candidate. We now shall look at the following question: what should be the expected missing energy in the dimuon + jet system, if Elko production is occurring taking into account the Elko non-locality? Considering the proton–proton energy as $(\sqrt{s}, 0, 0, 0)$ in Fig. 2 we have the momentum configuration

$$p_{2\mu} = \frac{\sqrt{s}}{2}\left(1 + \frac{m_{2\mu}}{s} - \frac{m_{2\varsigma}}{s}, \beta\sin(\theta), 0, \beta\cos(\theta)\right),$$
$$p_{2\varsigma} = \frac{\sqrt{s}}{2}\left(1 + \frac{m_{2\varsigma}}{s} - \frac{m_{2\mu}}{s}, -\beta\sin(\theta), 0, -\beta\cos(\theta)\right),$$

where $m_{2\varsigma}(m_{2\mu})$ is the invariant mass, for instance $m_{2\varsigma} = 2m_\varsigma^2 - 2\vec{p}_3\cdot\vec{p}_4 + 2E_3E_4$, as the sum of two momentum vectors, and $\beta = \sqrt{1 - 2\frac{m_{2\mu}+m_{2\varsigma}}{s} + \frac{(m_{2\mu}-m_{2\varsigma})^2}{s^2}}$. Therefore the missing energy is

$$E^{miss} = \sqrt{s}\left(1 + \frac{m_{2\mu} - m_{2\varsigma}}{s}\right) - \sqrt{s} = \frac{m_{2\mu} - m_{2\varsigma}}{\sqrt{s}}.$$

An important requirement is imposed by the minimum energy resolution for the search of missing energy on this channel. Considering the same parametrization as used for the CMS detector [18], we suppose that the threshold for the missing energy for the signal is given by

$$E^{miss} = \frac{m_{2\mu} - m_{2\varsigma}}{\sqrt{E}}.$$

In the limit that the two Elkos does not have a significant momenta, it is possible to approximate $m_{2\mu} \approx m_Z = 91.187$ GeV and, then, one should to select only events with $E_{miss} > 25$ GeV. This means that a detailed analysis should take into account both, angular and energy, resolutions.

6. Final remarks

By analyzing the consequences of the unusual Elko propagator behavior, it was possible to derive a typical signature to the Elko production, namely: due to the Elko non-locality, the measured decay depends on the angular cut applied, breaking therefore the angular isotropy (fully observed in all standard model processes).

We shall stress two important points: Fig. 1 may be understood as the first term of a sum involving internal Elko productions of the same type (a "cascade" of a "fork"), what means that its contribution can be improved by the sum of those graphs, faced as a finite geometric series on λ^2; second, it should be stressed for completeness, that another factor resulting as an unexpected asymmetry on ϕ (for graphs involving four Elkos coupling) arises from the inclusion of the $\vec{\eta}\vec{\eta}$ and $\eta\eta$ type propagators, which are proportional to $N(p')$ and $M(p)$ matrices, the "twisted spin sums":

$$M(p) = \begin{bmatrix} e^{-i\phi}p\cos(\theta) & p\sin(\theta) & 0 & -iE \\ p\sin(\theta) & -e^{i\phi}p\cos(\theta) & iE & 0 \\ 0 & -iE & -e^{-i\phi}p\cos(\theta) & -p\sin(\theta) \\ -iE & 0 & -p\sin(\theta) & e^{i\phi}p\cos(\theta) \end{bmatrix},$$

$$N(p') = \begin{bmatrix} \sqrt{p'^2+m_\varsigma^2} & 0 & ip'\sin(\theta') & -ie^{-i\phi'}p'\cos(\theta') \\ 0 & \sqrt{p'^2+m_\varsigma^2} & -ie^{i\phi'}p'\cos(\theta') & -ip'\sin(\theta') \\ ip'\sin(\theta') & -ie^{-i\phi'}p'\cos(\theta') & -\sqrt{p'^2+m_\varsigma^2} & 0 \\ -ie^{i\phi'}p'\cos(\theta') & -ip'\sin(\theta') & 0 & -\sqrt{p'^2+m_\varsigma^2} \end{bmatrix}. \tag{30}$$

Acknowledgements

We are grateful to Professor D.V. Ahluwalia and Professor A. Alves for valuable suggestions.

References

[1] D.V. Ahluwalia, D. Grumiller, JCAP 0507 (2005) 012, arXiv:hep-th/0412080.
[2] D.V. Ahluwalia, D. Grumiller, Phys. Rev. D 72 (2005) 067701, arXiv:hep-th/0410192.
[3] C.G. Boehmer, Ann. Phys. 16 (2007) 325, arXiv:gr-qc/0701087;
C.G. Boehmer, Ann. Phys. 16 (2007) 38, arXiv:gr-qc/0607088;
C.G. Boehmer, J. Burnett, Mod. Phys. Lett. A 25 (2010) 101, arXiv:0906.1351 [gr-qc];
S. Shankaranarayanan, Int. J. Mod. Phys. D 18 (2009) 2173, arXiv:0905.2573 [astro-ph];
C.G. Boehmer, J. Burnett, Phys. Rev. D 78 (2008) 104001, arXiv:0809.0469 [gr-qc];
D. Gredat, S. Shankaranarayanan, JCAP 1001 (2010) 008, arXiv:0807.3336 [astro-ph];
C.G. Boehmer, Phys. Rev. D 77 (2008) 123535, arXiv:0804.0616 [astro-ph];
78 C.G. Boehmer, D.F. Mota, Phys. Lett. B 663 (2008) 168, arXiv:0710.2003 [astro-ph];
L. Fabbri, Mod. Phys. Lett. A 25 (2010) 2483, arXiv:0911.5304 [gr-qc];
L. Fabbri, arXiv:1008.0334 [gr-qc].
[4] L. Fabbri, Mod. Phys. Lett. A 25 (2010) 151, arXiv:0911.2622 [gr-qc];
R. da Rocha, W.A. Rodrigues Jr., Mod. Phys. Lett. A 21 (2006) 65, arXiv:math-ph/0506075;

R. da Rocha, J.M. Hoff da Silva, J. Math. Phys. 48 (2007) 123517, arXiv:0711.1103 [math-ph];
R. da Rocha, J.M. Hoff da Silva, Adv. Appl. Clifford Alg. arXiv:0811.2717 [math-ph];
J.M. Hoff da Silva, R. da Rocha, Int. J. Mod. Phys. A 24 (2009) 3227, arXiv:0903.2815 [math-ph].
[5] E.P. Wigner, Ann. of Math. 40 (1939) 149.
[6] D.V. Ahluwalia, C.-Y. Lee, D. Schritt, T.F. Watson, Phys. Lett. B 687 (2010) 248, arXiv:0804.1854 [hep-th];
D.V. Ahluwalia, C.-Y. Lee, D. Schritt, arXiv:0911.2947 [hep-ph].
[7] K. Land, J. Magueijo, Mon. Not. Roy. Astron. Soc. 378 (2007) 153, arXiv:astro-ph/0611518;
P.K. Samal, R. Saha, P. Jain, J.P. Ralston, arXiv:0811.1639 [astro-ph], 2008;
M. Fommert, T.A. Ensslin, 2009; arXiv:0908.0453 [astro-ph].
[8] A.G. Cohen, S.L. Glashow, Phys. Rev. Lett. 97 (2006) 021601, hep-ph/0601236.
[9] D.V. Ahluwalia, S.P. Horvath, JHEP 1011 (2010) 078, arXiv:1008.0436 [hep-ph].
[10] P. Nath, et al., Nucl. Phys. B Proc. Suppl. 200–202 (2010) 185, arXiv:1001.2693.
[11] ATLAS inner detector: Technical design report, vol. 1, CERN-LHCC-97-16.
[12] S.P. Martin, arXiv:hep-ph/9709356.
[13] J.A. Aguilar-Saavedra, et al., Eur. Phys. J. C 46 (2006) 43, arXiv:hep-ph/0511344.
[14] The CDF Collaboration, CDF Note 10223, Preliminary results for the ICHEP 2010 Conference.
[15] M.J.G. Veltman, Cambridge Lecture Notes in Physics, vol. 4, Cambridge Univ. Press, Cambridge, UK, 1994, 284 pp.
[16] C. Bauer, H.S. Do, Comput. Phys. Comm. 144 (2002) 154, arXiv:hep-ph/0102231.
[17] T. Hahn, Comput. Phys. Comm. 168 (2005) 78, arXiv:hep-ph/0404043.
[18] CMS Physics Technical Design Report, vol. II: Physics Performance, J. Phys. G: Nucl. Part. Phys. 34 (2007) 995.
[19] K. Cheung, J. Song, Phys. Rev. D 81 (2010) 097703, arXiv:1004.2783 [hep-ph].

International Journal of Modern Physics A
Vol. 30, No. 1 (2015) 1550006 (13 pages)

DOI: 10.1142/S0217751X15500062

Searching for Elko dark matter spinors at the CERN LHC

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Received 11 September 2014
Revised 10 October 2014
Accepted 24 November 2014
Published 9 January 2015

The aim of this paper is to explore the possibility of discovering a fermionic field with mass dimension one, the Elko field, in the Large Hadron Collider. Due to its mass dimension, an Elko can only interact either with Standard Model spinors and gauge fields at one-loop order or at tree level through a quartic interaction with the Higgs field. In this Higgs portal scenario, the Elko is a viable candidate to a dark matter constituent which has been shown to be compatible with relic abundance measurements from WMAP and direct dark matter searches. We propose a search strategy for this dark matter candidate in the channel $pp \to \ell^+\ell^- + \not{E}_T$ at the $\sqrt{s} = 14$ TeV LHC. We show the LHC potential to discover the Elko considering a triple Higgs–Elkos coupling as small as ~ 0.5 after 1 ab^{-1} of integrated luminosity. Some phenomenological consequences of this new particle and its collider signatures are also discussed.

PACS numbers: 13.85.Rm, 12.38.Bx, 95.35.+d

1. Introduction

The so-called Elko spinor fields are a set of four spinors, whose main characteristic is to be eingenspinors of the charge conjugation operator. This construction renders

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these spinors an important property: they are blind with respect to electromagnetic interactions. Moreover, the quantum field associated to an Elko spinor presents the peculiarity of having mass dimension one, severely reducing the perturbatively renormalizable possible couplings of the quantum Elko field,[1] as a matter of fact, only a quartic interaction with two Elkos and two Higgs bosons is allowed in the Standard Model (SM). These two characteristics underlying the formal Elko structure are responsible for making the spinor to deserve the epithet of *a first principle dark matter candidate*.[1] A complementary analysis involving relic density and gravitational collapse of a primordial Elko suggest, further, an Elko mass of order of a few MeV[2] in order the Elko field to be a viable dark matter (DM) candidate.

With the CERN Large Hadron Collider (LHC) in operation and its success in the recent discovery of the Higgs boson, it is natural to ask whether the features of the Elko field can be probed in this collider. In this paper, we investigate the possibility for Elko production at the 14 TeV LHC.

Assuming only a quartic interaction with Higgs bosons makes a model for Elkos a *Higgs portal* type model[3] where DM communicates to the rest of the spectrum only through Higgs interactions. In these scenarios, the search for DM at the LHC has been made mainly looking for mono-X signatures, where X is any SM particle that can serve as a tagging signal, once the DM, as the Elko, would leave no trace in the detectors. These signatures have large SM backgrounds that can be typically cleaned up by imposing a hard cut on the missing transverse momentum of the events.

On the other hand, after electroweak symmetry breaking (EWSB), the Higgs field acquires a nonzero vacuum expectation value (VEV) inducing a renormalizable triple vertex with two Elkos and a single Higgs boson. In this case, the Elko would get its mass from the EWSB mechanism and the production of Elkos pairs is considerably enhanced compared to the quartic coupling scenario.

We present in this paper the prospects for search strategies, at the 14 TeV LHC, in the mono-Z, mono-jet, mono-Higgs, and Weak Boson Fusion (WBF) channels. We found that mono-Z process $pp \to Z^* \to Z + \text{Elkos} \to \ell^+\ell^- + \not{E}_T$, where $\ell = e, \mu$ all charged leptons but the taus, is the most promising in the search for Elkos even after taking into account all the dominant and sub-dominant SM backgrounds.

The paper is organized as follows. In Sec. 2, we shall briefly resume the Elko construction and some of its properties; in Sec. 3, we make the phenomenological analysis of channels mentioned above. In Sec. 4, we present our conclusions.

2. Elko Spinor Field

The charge conjugation operator, in the Weyl representation, is given by

$$C = \begin{pmatrix} \mathbb{O} & i\Theta \\ -i\Theta & \mathbb{O} \end{pmatrix} K\,, \tag{1}$$

where K is responsible for complex conjugate the spinor it is acting on and $\Theta = -i\sigma_2$. The very equation defining the Elko spinors is given by

$$C\lambda(\mathbf{p}) = \pm\lambda(\mathbf{p})\,. \tag{2}$$

It is possible to show that there are four eigenspinors, two corresponding to the $+$ plus — the self-conjugated spinors λ^S — and two related to the $-$ sign, the antiself-conjugated spinors. The functional form for these spinors may be found explicitly, for instance in Refs. 1 and 2. Here, we shall focus on the main properties concerning our analysis. We shall pinpoint that the explicit construction of the Elko spinors defines it as a member of all the $r \oplus l$ Weyl representation space, i.e. the spinor carries both helicities.

It is possible to find the correct dual for the Elko spinor by means of a quite precise criteria, as follows. Let us demand that the product $[\lambda_\alpha]^\dagger \eta \lambda_\beta$ be invariant under arbitrary Lorentz transformations, where the index labels one of the four Elko. This requirement amounts out as the constraints $[J_i, \eta] = 0 = \{K_i, \eta\}$, being J and K the Lorentz transformation generators of rotations and boosts, respectively. It can be readily verified that the unique consistent solution is given by[4] $\eta = \pm i\gamma^0$ and the dual representation so that the norm is well-defined (leading to a positive definite Hamiltonian) is given by

$$\stackrel{\neg}{\lambda}{}^{S/A}_{\{\mp,\pm\}}(\mathbf{p}) := \pm i\Big[\lambda^{S/A}_{\{\pm,\mp\}}(\mathbf{p})\Big]^\dagger \gamma^0\,. \tag{3}$$

With the aid of the above equations, it is possible to set down the orthonormality relations:

$$\stackrel{\neg}{\lambda}{}^S_\alpha(\mathbf{p})\lambda^I_{\alpha'}(\mathbf{p}) = +2m\delta_{\alpha\alpha'}\delta_{SI}\,, \tag{4a}$$

$$\stackrel{\neg}{\lambda}{}^A_\alpha(\mathbf{p})\lambda^I_{\alpha'}(\mathbf{p}) = -2m\delta_{\alpha\alpha'}\delta_{AI}\,, \tag{4b}$$

where $I \in \{S, A\}$ and the completeness relation

$$\frac{1}{2m}\sum_\alpha \Big[\lambda^S_\alpha(\mathbf{p})\stackrel{\neg}{\lambda}{}^S_\alpha(\mathbf{p}) - \lambda^A_\alpha(\mathbf{p})\stackrel{\neg}{\lambda}{}^A_\alpha(\mathbf{p})\Big] = \mathbb{I}\,, \tag{5}$$

with $\alpha = \{+,-\},\{-,+\}$.

It is important to emphasize the emergence of unusual *spin sums*, given by

$$\sum_\alpha \lambda^S_\alpha(\mathbf{p})\stackrel{\neg}{\lambda}{}^S_\alpha(\mathbf{p}) = +m[\mathbb{I} + \mathcal{G}(\mathbf{p})]\,, \tag{6a}$$

$$\sum_\alpha \lambda^A_\alpha(\mathbf{p})\stackrel{\neg}{\lambda}{}^A_\alpha(\mathbf{p}) = -m[\mathbb{I} - \mathcal{G}(\mathbf{p})]\,, \tag{6b}$$

where one can write down the explicit form for $\mathcal{G}(\mathbf{p})$ as

$$\mathcal{G}(\mathbf{p}) = \begin{pmatrix} 0 & 0 & 0 & -ie^{-i\phi} \\ 0 & 0 & ie^{i\phi} & 0 \\ 0 & -ie^{-i\phi} & 0 & 0 \\ ie^{i\phi} & 0 & 0 & 0 \end{pmatrix}.$$

Notice that the spin sums are modified by a rather nontrivial term whose argument is given by the momentum. These spin sums do violate (in a rather subtle way) the full Lorentz invariance, making Elko fields invariant under a subgroup of the Lorentz group (see Ref. 5 for an up to date account on the formalism). We point out that in the case of the subgroup in question, namely HOM(2),[6] rotations and boosts are still present as generators.

It is easy to show that the Elko spinor satisfy a Dirac-like equation which is only an algebraic identity (nothing to do with the field dynamics) given by

$$\mathcal{D}\lambda_\beta^{S/A}(\mathbf{p}) = \left(\gamma_\mu p^\mu \delta_\alpha^\beta \pm im\mathbb{I}\varepsilon_\alpha^\beta\right)\lambda_\beta^{S/A}(\mathbf{p}) = 0\,,$$

where δ_β^α is the usual Kronecker symbol and the antisymmetric symbol ε is defined as $\varepsilon_{\{+,-\}}^{\{-,+\}} := -1$. The importance of such a relation shall not be underestimated. In fact, the very existence of the operator $\mathcal{D}$ acting on the spinor space gives information about the physical content encoded on the Elko spinor: the covariance condition arising from the $\mathcal{D}$ operator is the same of the Dirac one and, therefore, the corresponding transformation on $\lambda^{S/A}$ is not unitary. As a result, $\lambda^{S/A}$ cannot be associated to a quantum state in any sense and (second) quantization is necessary.

The full consistent quantum field associated with the Elko can be written as[1,2]

$$\eta(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2mE(\mathbf{p})}}\sum_\beta \left[c_\beta(\mathbf{p})\lambda_\beta^S(\mathbf{p})\mathrm{e}^{-ip_\mu x^\mu} + c_\beta^\dagger(\mathbf{p})\lambda_\beta^A(\mathbf{p})\mathrm{e}^{+ip_\mu x^\mu}\right].$$

Analogously, its dual $\left(\vec{\eta}(x)\right)$ is obtained by replacing λ for $\vec{\lambda}$, c for $c^\dagger$ and $ip_\mu x^\mu \leftrightarrow -ip_\mu x^\mu$. The anticommutators for the creation and destruction operators, $c_\beta^\dagger(\mathbf{p})$ and $c_\beta(\mathbf{p})$, are:

$$\left\{c_\beta(\mathbf{p}), c_{\beta'}^\dagger(\mathbf{p}')\right\} = (2\pi)^3\delta^3(\mathbf{p}-\mathbf{p}')\delta_{\beta\beta'}\,, \tag{7}$$

$$\left\{c_\beta^\dagger(\mathbf{p}), c_{\beta'}^\dagger(\mathbf{p}')\right\} = \left\{c_\beta(\mathbf{p}), c_{\beta'}(\mathbf{p}')\right\} = 0\,. \tag{8}$$

In order to unveil the dynamics associated to the quantum field, a bottom-up approach is necessary. The best procedure is to calculate the Feynman–Dyson propagator inferring, then, the corresponding Lagrangian. After a slightly modified textbook calculation, one arrives at

$$\mathcal{S}(x-x') = \int \frac{d^4p}{(2\pi)^4}\,\mathrm{e}^{ip_\mu(x^\mu - x'^\mu)}\,\frac{\mathbb{I}}{p_\mu p^\mu - m^2 + i\epsilon}\,, \tag{9}$$

in the absence of a preferred direction, which is nothing but the Klein–Gordon propagator. Hence, the Elko spinor field has mass dimension one and satisfy the Klein–Gordon equation,

$$(p_\mu p^\mu - m^2)\lambda^{S/A}(\mathbf{p}) = 0\,.$$

As a consequence, the perturbatively renormalizable terms in the Lagrangian density are only the mass term and an interaction of a scalar field

$$\mathcal{L} = \partial^\mu\vec{\eta}(x)\partial_\mu\eta(x) - m_\varepsilon^2\,\vec{\eta}(x)\eta(x) + \lambda_E\,\vec{\eta}(x)\eta(x)\phi(x)^2\,,$$

where λ_E is the coupling constant. Thus, the only possible interaction with vector bosons is via production of two Higgs mediated by a loop of these particles, with a subsequent generation of a $\eta\bar{\eta}$ pair, in the form of an effective vertex.

Nevertheless, we can also shift the Higgs by a nonzero VEV, obtaining a triple Higgs(H)–Elko(E)–Elko(E), HEE, vertex

$$\mathcal{L}_{\text{int}} = \alpha_E \, \bar{\eta}(x)\eta(x)\phi(x) \,, \tag{10}$$

with $[\alpha_E] = [\text{mass}]$, also renormalizable. This coupling constant naturally arises in a theory where Elkos also get their masses from the electroweak symmetry breaking mechanism, and its relation with λ_E and the Higgs VEV is

$$\alpha_E = \frac{v}{\sqrt{2}}\lambda_E \,.$$

As we are going to show, Elkos that couple to Higgs bosons according to Eq. (10) have a relevant production cross-section at the 14 TeV LHC, unlike those that couples to Higgs bosons through quartic interactions only.

2.1. *Computation of the one-loop amplitudes*

Let us illustrate the computation of the one-loop amplitude contributing to the production mode $pp \to Z^* \to Z + E\bar{E}$. This computation will be extended for all analysis containing a λ_E coupling. Afterwards, we propose a search strategy for Elkos at the 14 TeV LHC.

The Feynman diagrams contributing to the process $pp \to Z + E\bar{E}$ is shown in Fig. 1.

The effective coupling between two Z bosons and two Elko fields, which gives rise to the amplitude M_1 in Fig. 1, in the on shell renormalization scheme is given by ($\tilde{V}_{ZZ\to\lambda\bar{\lambda}} = V^{\text{eff}}_{ZZ\to\lambda\bar{\lambda}}(p_5, p_6) - V^{\text{eff}}_{ZZ\to\lambda\bar{\lambda}}(0)$), due a Higgs loop, using a cutoff scale,

$$\tilde{V}^{\text{eff}} = \begin{cases} g_1 \dfrac{\lambda_E}{16\pi^2}\left(2 + \sqrt{1-\gamma}\ln\left|\dfrac{1-\sqrt{1-\gamma}}{1+\sqrt{1-\gamma}}\right|\right) + r \,, & \gamma \le 1 \,, \\ g_1 \dfrac{\lambda_E}{16\pi^2}\left(2 - 2\sqrt{\gamma-1}\tan^{-1}(\sqrt{\gamma-1})\right) + r \,, & \gamma > 1 \,, \end{cases} \tag{11}$$

where g_1 is the coupling between two Higgs and two Z bosons, $\gamma = \frac{4m_H^2}{|p_5+p_6|^2}$ and $p_{5,6}$ are the final state Elkos four-momentum. The r function is

$$r = g_2 \frac{\lambda_E}{16\pi^2}\int_0^1 dx \frac{\tan^{-1}\left(\frac{(q+|p^5+p_6|)x+|p_5+p_6|}{\Delta}\right)}{|p_5+p_6|\Delta} - \frac{\tan^{-1}\left(\frac{(q+|p_5+p_6|)x}{\Delta}\right)}{|p_5+p_6|\Delta} \,,$$

where

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$$\Delta = \sqrt{-m_H^2(x-1) + m_Z^2 x - \left(q + \frac{|p^5+p_6|}{2}\right)^2 x} \,,$$

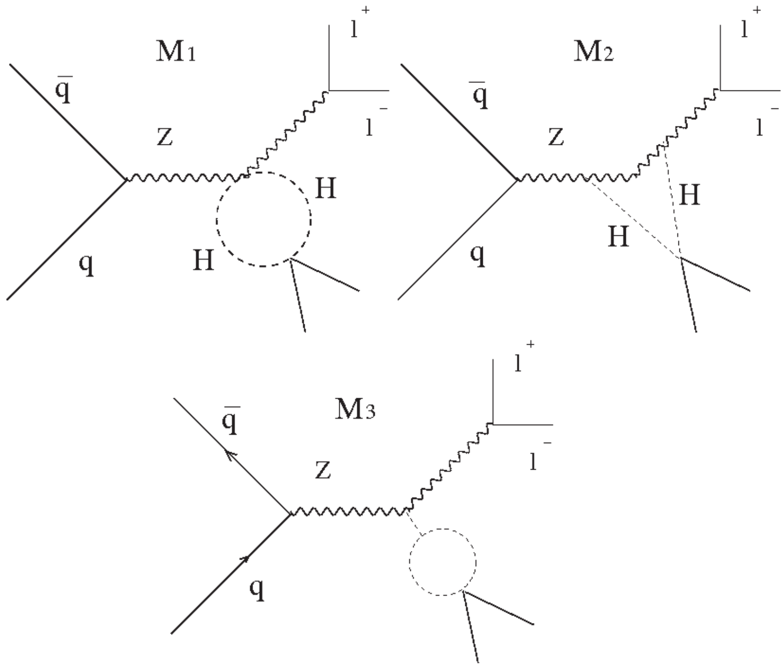

Fig. 1. Feynman diagram for the production of a pair of Elko spinors associated with a SM gauge boson Z_0.

depending on the Z mass, m_Z, the coupling between one Higgs and two Z bosons, g_2, and the quartic HEE coupling λ_E. The external momentum of a vector boson entering in the effective vertex is denoted by q.

The second amplitude contributing to the effective vertex, M_2, is a tadpole diagram involving one Higgs and two Elkos, and is given by

$$\tilde{V}^{\text{eff}}_{H\to\lambda\bar{\lambda}} = \frac{\lambda_E}{16\pi^2}\left(2 - \frac{2m_H \tan^{-1}\left(\frac{|p_5+p_6|}{m_H}\right)}{|p_5+p_6|} + \ln\left|\frac{m_H^2}{m_H^2 + |p_5+p_6|^2}\right|\right). \tag{12}$$

We also will need the spin sum of dimension-one Elko spinors to compute the square amplitude to the partonic process $q\bar{q} \to Z$ and two Elkos depicted in Fig. 1. To perform it, we use the fact that[1]

$$\lambda^{S/A}_{\{\mp,\pm\}}(\vec{p}) = \sqrt{\frac{E+m_\varepsilon}{2m_\varepsilon}}\left(1 \mp \frac{|\vec{p}|}{E+m_\varepsilon}\right)\lambda^{S/A}_{\{\mp,\pm\}}(\vec{0}) = \lambda^{S/A}_{\{\mp,\pm\}}(-\vec{p})\,, \tag{13}$$

where $E = \frac{E_{\text{cm}}}{2}$. Let us fix the spinor indices $a, b = 1, 2, 3, 4$ to write the amplitude of the simplest scattering process — the two Higgs annihilation generating two Elkos — as

$$\mathcal{M} = \frac{\lambda_E}{m_\varepsilon}\lambda^{I,a}_{\alpha}\,\bar{\lambda}^{J,b}_{\alpha'}\delta^{ab}\,,$$

so that

$$|\mathcal{M}|^2 = \frac{\lambda_E^2}{m_\varepsilon^2}\lambda^{I,a}_{\alpha}\lambda^{I,c\dagger}_{\alpha}\,\bar{\lambda}^{J,b}_{\alpha'}\,\bar{\lambda}^{J,d\dagger}_{\alpha'}\delta^{ab}\delta^{cd}\,.$$

The spin and Elko type averages are

$$\frac{1}{16}\sum_{\alpha,\alpha'}\sum_{I,J}|\mathcal{M}|^2 = \frac{\lambda_E}{16m_\varepsilon^2}\sum_{\alpha,\alpha'}\left(\lambda_\alpha^A\lambda_\alpha^{A\dagger} + \lambda_\alpha^S\lambda_\alpha^{S\dagger}\right)_{ac}$$
$$\times\sum_I\left(\bar{\lambda}_{\alpha'}^I\bar{\lambda}_{\alpha'}^{I\dagger}\right)_{bd}\delta^{ab}\delta^{cd}\,. \tag{14}$$

The sum in the first parenthesis of Eq. (14) is given by $2(E\mathbb{I}-P\mathcal{G})_{ac}$ (see Eqs. (B.18) and (B.19) of Ref. 1), while in the second line we have

$$\sum_{\alpha'}\bar{\lambda}_{\alpha'}^I\bar{\lambda}_{\alpha'}^{I\dagger} = \sum_\alpha \lambda_{\alpha'}^{I\,\dagger}\lambda_{\alpha'}^I\,.$$

Thus,

$$\frac{1}{16}\sum_{\alpha,\alpha'}\sum_{I,J}|\mathcal{M}|^2 = \frac{\lambda_E^2}{8m_\varepsilon^2}(E\delta_{bd} - p\mathcal{G}_{bd})\sum_I\sum_\alpha \lambda_\alpha^{I,b\dagger}\lambda_\alpha^{I,d}\,. \tag{15}$$

Now making use of Eqs. (B.24) and (B.25) of Ref. 1

$$\sum_\alpha \lambda_\alpha^{I\,\dagger}\lambda_\alpha^I = 2(E-p) + 2(E+p) = 4E\,,$$

we can write

$$\frac{1}{16}\sum_{\alpha,\alpha'}\sum_{I,J}|\mathcal{M}|^2 = \frac{\lambda_E^2}{16m_\varepsilon^2}\left[16E^2 - 2p\sum_I\sum_\alpha\left(\lambda_\alpha^{I,b\dagger}\mathcal{G}_{bd}\lambda_\alpha^{I,d}\right)\right]$$
$$= \frac{\lambda_E^2}{16m_\varepsilon^2}\left[16E^2 - 2p\,\mathrm{tr}(2E\mathcal{G} + 2p\mathbb{I})\right]$$
$$= \frac{\lambda_E^2}{m_\varepsilon^2}(E^2 - p^2) = \lambda_E^2\,, \tag{16}$$

where the ciclicity property of the trace on the first line was used.

Combining Eqs. (11), (12) and (16) one can construct an effective vertex relating two vector bosons and two Elko fields to obtain relevant signal amplitudes.

3. Searching for Elkos at the LHC

3.1. *Quartic coupling scenario*

Consider only the λ_E type coupling. As a consequence of its sole tree-level coupling being a quartic coupling to Higgs bosons pairs, Elkos can only be pair produced at colliders. The associated $gg \to H^* \to H + E\bar{E}$ production is the most straightforward tree-level mechanism to produce Elkos but with a too small cross-section. Taking into account the cleaner decay modes into photons and massive gauge bosons, the number of events would be very small even for high luminosities.

However at one-loop level several possibilities are opened, including the Higgs boson decay to Elkos. For example, processes like $pp \to H + X$, where the Higgs boson decays to Elkos might lead to mono-X signatures, X = jet, photon, W, Z or a Higgs. The quartic $VVHH$, $V = W$, Z, couplings might give rise to WBF $jj + E\bar{E}$ and associated $V = Z$, W boson-Elkos $V + E\bar{E}$ events by their turn. We are going to discuss the search prospects for these channels now.

Associated Z-Elkos channel

Let us analyze the associated production of a Z boson and two Elkos in the channel $\ell^\pm\ell^\mp + \not{E}_T$, $\ell = e, \mu$. The W-Elkos is a more difficult channel because it is much harder to identify the signal events based solely on missing energy distributions.

We simulate the signal events with `Madgraph5`[8] modifying the SM $ZZHH$ and ZZH vertices to take into account the effective vertices of the Elko case. We stress the fact that the vertex between two Higgs particles and $\lambda_\alpha^{I,a}$ and $\bar{\lambda}_{\alpha'}^{J,b}$, namely

$$\lambda_E \delta^{ab} \,,$$

was chosen in order to maintain this coupling renormalizable. Also, we set the Elko mass to $m_\varepsilon = 0.01$ GeV, in consonance with the Elko mass range estimated in DM direct and indirect searches.[1]

The total cross-section for Elko production in association to a Z boson is around 10^{-5} fb at the 8 TeV LHC, which is far beyond the LHC reach for the current integrated luminosity. Thus, we proceed to the prospects for Elko discovery at the 14 TeV LHC. The factorization scale is set to be $\sqrt{\hat{s}}$, $\hat{s}$ being the parton level center-of-mass energy. We do not expect any large deviations of our partonic level estimates by including detector effects and showering, given the optimal coverage and detection efficiency of the LHC detectors for events with hard electrons and muons, and large missing transverse energy. Yet, a full simulation should be performed to properly evaluate those effects.

In the process of Fig. 1 the final state of interest consists of two opposite-sign electrons or muons and missing energy associated to the Elkos production. The main background contributions come from:[7]

(1) $ZZ, Z\gamma \to \ell^+\ell^- + \nu_\ell\bar{\nu}_\ell$;
(2) $W^+W^- \to \ell^+\ell^- + \nu_\ell\bar{\nu}_l$;
(3) $W^\pm Z \to \ell^\pm\ell^\mp\ell^\pm + \nu_\ell$ with one missing charged lepton;
(4) $t\bar{t} \to W^+W^-b\bar{b} \to \ell^+\ell^-b\bar{b} + \nu_\ell\bar{\nu}_\ell$;
(5) $W^\pm j \to \ell^+\ell^- + \nu_\ell$, with the jet misidentified as a charged lepton in the detector.

We also have generated all these samples using `Madgraph5`.

The reducible backgrounds $W^+Z \to l^+l^-l^+\nu_l$ and $W^-Z \to l^-l^+l^-\bar{\nu}_l$ are suppressed imposing only two opposite sign leptons on the final state since a third charged lepton is rarely outside the fiducial region of the detectors. The acceptance cuts for charged leptons are given by

$$p_{T_\ell} > 10 \text{ GeV}\,, \quad |\eta_\ell| < 2.5\,. \tag{17}$$

Table 1. Cross-sections for signal and backgrounds for three of the most promising channels for Elko discovery at the 14 TeV LHC. Cuts applied to reduce backgrounds are described in the text.

Process	σ (pb)
$pp \to \ell^+\ell^- + \not{E}_T$	
Signal	4.69×10^{-4}
Background	1.969
$pp \to jj + \not{E}_T$ (WBF)	
Signal	0.001119
Background	0.75
$pp \to j + \not{E}_T$	
Signal	0.02531×10^{-5}
Background	2820

Fake leptons are another source of potential background events which arise from a jet enriched environment. In order to estimate the probability of a jet to be misidentified as a lepton we simulated a sample using `Pythia`[10] for jet showering and clustering, and `PGS` for detector simulation, for different signal leptons on the final state. We have found a probability of 10^{-4} for a jet to be misidentified as an isolated lepton, which is consistent to the presented experimental studies.[11,12] Using this result, the background $W^{\pm} + j$ can be eliminated.

In order to increase the signal to background ratio, we demand the events to satisfy the following cut

$$\sum_{\text{visible}} |\vec{p}_T| < 120 \text{ GeV}\,, \tag{18}$$

where $\sum_{\text{visible}} |\vec{p}_T|$ is the scalar sum of the transverse momentum vector of all visible objects, in the case of our signal, the charged leptons. Unfortunately, even after this cut a signal to background ratio relevant for discovery at the LHC seems difficult. The cross-sections are described in Table 1.

Elkos in weak boson fusion

Concerning the Elko plus two jets signal, the WBF channel, we used the `PGS` and `Pythia` on the samples to make our analysis more reliable. We have applied the following cuts on both signal and background:

$$|\eta_j| < 2.5, p_{T_j} > 30 \text{ GeV} \quad \text{and} \quad M_{jj} > 800 \text{ GeV}\,,$$

where M_{jj} in the jets invariant mass.

Table 1 summarizes our results. The background quoted in the table is the irreducible one $pp \to jj\nu_\ell\bar{\nu}_\ell$. Considering the large background, again we are not able to obtain a relevant signal to background significance ratio for this channel at the LHC.

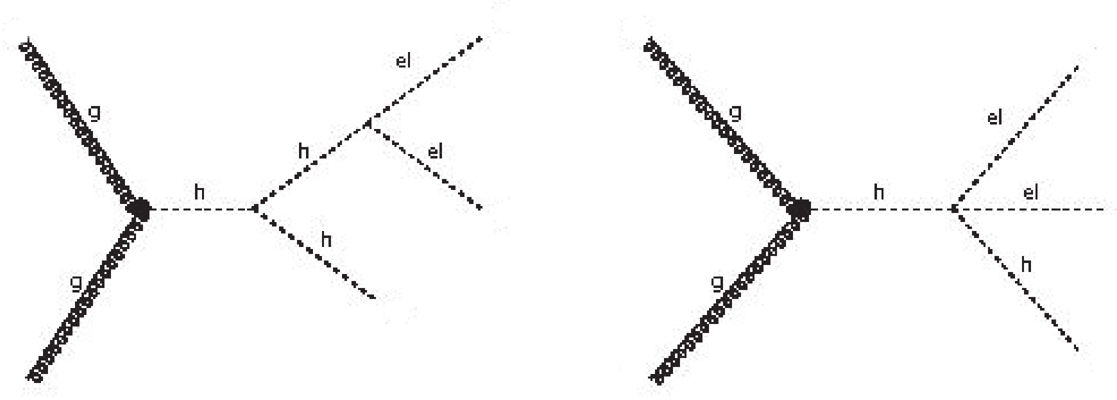

Fig. 2. Process using the effective vertex for Higgs production.

Elkos in gluon fusion

Using a modified version of `heft` model on `MadGraph` we also obtained the cross-section for the process in Fig. 2, where an off-shell Higgs boson is produced in gluon fusion and decays to Elkos and an additional Higgs boson, $\sigma = 7.183 \times 10^{-5}$ pb, for $m_\varepsilon = 10$ MeV and coupling constant between two Higgs and two Elkos set to one. This is a too low cross-section to proceed with a detailed analysis.

Monojet channel

Now, we look for the possibility to observe Elkos in one jet plus missing energy channel. This monojet channel has been intensely studied as a promising way for DM detection at hadron colliders.[14] On Table 1 we quote the cross-sections for signal and background in this case. As we can see, the prospects to get a reasonable signal to background ratio for discovering Elko through this signature are hopeless.

After analyzing the most relevant channels to Elko discovery through quartic interactions with discouraging results, let us now investigate the scenario where the Elko has a triple coupling to the Higgs boson.

3.2. *Triple coupling scenario*

The situation is very different for the coupling described by Eq. (10). Again we simulate the $pp \to l^+l^- + \not{E}_T$. The signal cross-section due the HEE coupling is 0.103 pb, after acceptance cuts of Eq. (17), this time for 10 MeV Elkos and $\alpha_E = 1$. This increase in the production cross-section is consequence of the tree-level coupling involved in the Elkos production. The Higgs branching ratio to Elkos is around 6% which is still comfortably allowed by the LHC data.[16]

After demanding the acceptance cuts of Eq. (17) and the cut of Eq. (18), we have implemented the following additional cuts (20),

$$80 \text{ GeV} < M_{\ell\ell} < 120 \text{ GeV}\,, \tag{19}$$

$$\not{E}_T > 50 \text{ GeV}\,. \tag{20}$$

Table 2. Cross-sections for signal and background in $pp \to \ell^+\ell^- + \not{E}_T$, $\ell = e, \mu$, after all cuts in the triple coupling scenario.

Process	σ (fb) after all cuts
Signal	6.8
Background	133.6

The cut on the leptons invariant mass, $M_{\ell\ell}$, eliminates the events where the lepton pair is not produced by an on-shell Z boson including the subdominant irreducible backgrounds $Z\gamma$ and W^+W^-. By the way, this is the reason why we chose only the associated production of a Z boson and Elkos whereas it would be possible to include the W plus Elkos production as well. As a leptonic W cannot be reconstructed and the Elko events do not present a large amount of missing energy when compared to the resonant W production, the SM background $pp \to W^\pm \to \ell^\pm + \not{E}_T$ would be overwhelming. The missing energy cut, by its turn, is essential for trigger purposes and helps to increase the signal to background ratio.

After imposing this cut we obtained the results in Table 2. The signal acceptance is 0.63 after applying all cuts.

Using $\sqrt{2((S+B)\ln\left(1+\frac{S}{B}\right)-S)}$, S and B the number of signal and backgrounds events, respectively, as the test statistic and considering that the number of events is quadratic in the coupling constant, we obtain the statistical significance in terms of the coupling constant and the integrated luminosity, L_{int}.

We show in Fig. 3 the required luminosity to a 3, 5 and 10σ signal as a function of the α_E coupling for a 10 MeV Elko at the 14 TeV LHC.

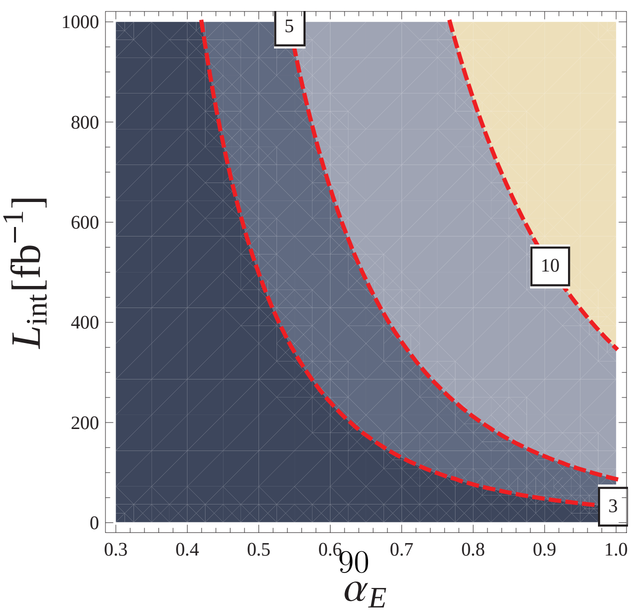

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Fig. 3. Significance as a function of α_E and the integrated luminosity in fb^{-1}.

With 1 ab^{-1}, HEE couplings as small as ~ 0.4 can lead to an evidence signal of 3σ at the LHC, and a 5σ discovery is possible for an enhanced coupling of 0.5. If $\alpha_E \gtrsim 0.6$, an integrated luminosity of ~ 500 fb^{-1} suffices for discovery of the Elko spinor at the 14 TeV LHC. Further detailed analysis are required, including a complete simulation of the detectors in order to confirm the precise limits on the coupling constant.

4. Conclusions

The Elko field can be considered a natural candidate for the main constituent of DM[1] — being a spinor of mass dimension one interacting only weakly to the SM particles via Higgs couplings without imposing any extra symmetries. Its interactions with the Higgs boson open the possibility of discovery at colliders, as the LHC.

We have investigated, in this paper, two scenarios for Elko interactions with Higgs bosons: the quartic coupling scenario, for which a 10 MeV Elko is shown to give rise to the right relic abundance as measured by WMAP, and the triple coupling scenario, where the Elko mass is generated trough the electroweak symmetry breaking mechanism.

The quartic scenario is very challenging even at the 14 TeV LHC for all the most promising channels. On the other hand, if triple couplings are present, the Elko can be easily discovered in the $pp \to \ell^+\ell^- + \not{E}_T$ channel. For example, with a HEE coupling of 0.5, a 10 MeV Elko discovery is possible after 1 ab^{-1}. However, couplings of order 0.6 or larger can be probed with up to 500 fb^{-1}.

In the triple coupling scenario, the Elko search can benefit from DM searches in mono-Z, W, monojet and monophoton channels at the 14 TeV LHC. Recent analyzes based upon the LHC7 and LHC8 is not likely to bound the Elko coupling however, since the signal cross-sections are too small.

Nevertheless, the 14 TeV LHC may open other possibilities as the WBF channel and an improvement on the bound on the Higgs invisible decay branching ratio.

Acknowledgments

This research was supported by resources supplied by the Center for Scientific Computing (NCC/GridUNESP) of the São Paulo State University (UNESP). J. M. H. da Silva thanks to CNPq for partial financial support (482043/2011-3; 308623/2012-6). The work of A. Alves was supported by Fundação de Amparo à Pesquisa do Estado de São Paulo (FAPESP), under the grant 2013/22079-8.

References

1. D. V. Ahluwalia and D. Grumiller, *J. Cosmol. Astropart. Phys.* **0507**, 012 (2005).
2. D. V. Ahluwalia and D. Grumiller, *Phys. Rev. D* **72**, 067701 (2005).
3. A. Djouadi *et al.*, *Phys. Lett. B* **709**, 65 (2012).

4. D. V. Ahluwalia, C.-Y. Lee and D. Schritt, *Phys. Lett. B* **687**, 248 (2010).
5. D. V. Ahluwalia, arXiv:1305.7509 [hep-th].
6. A. G. Cohen and S. L. Glashow, *Phys. Rev. Lett.* **97**, 021601 (2006).
7. T. M. Trefzger and K. Jakobs, at LHC, ATL-PHYS-2000-015.
8. M. Herquet and F. Maltoni, *Nucl. Phys. B* (*Proc. Suppl.*) **179-180**, 211 (2008).
9. E. Conte, B. Fuks and G. Serret, *Comput. Phys. Commun.* **184**, 222 (2013).
10. T. Sjostrand, S. Mrenna and P. Z. Skands, *Comput. Phys. Commun.* **178**, 852 (2008), arXiv:0710.3820 [hep-ph].
11. ATLAS Collab., ATLAS: Detector and physics performance, Technical Design Report, Vol. 1, CERN-LHCC-99-14.
12. ATLAS Collab., ATLAS: Detector and physics performance, Technical Design Report, Vol. 2, CERN-LHCC-99-15.
13. H. Murayama, I. Watanabe and K. Hagiwara, KEK-91-11.
14. CMS Collab., CMS PAS EXO-12-048.
15. A. Djouadi, J. Kalinowski and M. Spira, *Comput. Phys. Commun.* **108**, 56 (1998).
16. G. Belanger, B. Dumont, U. Ellwanger, J. F. Gunion and S. Kraml, *Phys. Rev. D* **88**, 075008 (2013).

3.2 Radiação Hawking com espinores Elko

O próximo trabalho pode ser considerado, para efeitos desta tese, como um interlúdio entre os aspectos estudados um teoria de campos e os investigados em espaços curvos. O trabalho sai da linha investigativa central que queremos enfatizar nesta tese, isto é, o da investigação de aspectos formais (algébricos e físicos) e posterior busca por sinais fenomenológicos. Entretanto constitui parte recorrente do trabalho com espinores escuros, a saber: revisitar resultados físicos estabelecidos com outros campos, atentando para os aspectos novos que daí podem decorrer.

Estudamos a emissão e absorção de Elkos devido à radiação Hawking de um buraco negro. No trabalho em questão o buraco negro é simulado pela extremidade de uma corda negra. A técnica empregada é uma aproximação tipo WKB ao método de tunelamento previamente estabelecido para o computo da radiação Hawking.

Mostra-se que, enquanto a temperatura Hawking é mantida (corroborando uma vez mais seu caráter universal), a probabilidade de tunelamento de Elkos é a mesma para partículas entrando e saindo do horizonte de eventos, não sendo relevante qual tipo de Elko (qual estado de helicidade, ou qual autovalor, $+1$ ou -1, com relação ao operador de conjugação de carga). Cabe ressaltar que o mesmo método quando empregado para férmions de Dirac releva uma distinção de tais probabilidades dependendo do tipo de campo espinorial que é levado em conta.

A LETTERS JOURNAL EXPLORING THE FRONTIERS OF PHYSICS

September 2014

EPL, **107** (2014) 50001
doi: 10.1209/0295-5075/107/50001

www.epljournal.org

Hawking radiation from Elko particles tunnelling across black-strings horizon

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received 9 June 2014; accepted in final form 9 August 2014
published online 22 August 2014

PACS 04.70.Dy – Quantum aspects of black holes, evaporation, thermodynamics
PACS 04.50.Gh – Higher-dimensional black holes, black strings, and related objects

Abstract – We apply the tunnelling method for the emission and absorption of Elko particles in the event horizon of a black-string solution. We show that Elko particles are emitted at the expected Hawking temperature from black strings, but with a quite different signature with respect to the Dirac particles. We employ the Hamilton-Jacobi technique to black-hole tunnelling, by applying the WKB approximation to the coupled system of Dirac-like equations governing the Elko particle dynamics. As a typical signature, different Elko particles are shown to produce the same standard Hawking temperature for black strings. However, we prove that they present the same probability irrespectively of outgoing or ingoing the black-hole horizon. This provides a typical signature for mass-dimension-one fermions, that is different from the mass-dimension-three halves fermions inherent to Dirac particles, as different Dirac spinor fields have distinct inward and outward probability of tunnelling.

Black-hole tunnelling procedures have been placed as prominent methods for calculating black-holes temperature [1–14]. The tunnelling method imparts a dynamical model describing the black-hole radiation, and has been applied to a plenty of black holes, both for the tunnelling of scalar particles [1,2] and Dirac particles as well [12–18]. The first black-hole tunnelling method [3] came after from the seminal paper by Kraus and Wilczek [1,2]. An alternative technique to black-hole tunnelling is the Hamilton-Jacobi one [4], regarding an emitted particle, by picking a suitable ansatz for the action. This method was further extended, by applying the WKB approximation to the Dirac equation [14–18]. The black-hole tunnelling method has some advantages with respect to other techniques for calculating the temperature, and can be successfully further applied to black holes of type Kerr and Kerr-Newman [8,9,15], the 3-dimensional BTZ [12], the Vaidya [19] black hole, and other dynamical black holes [11] as well. The tunnelling method is pivotal, as it provides an intuitive scenario for the black-hole radiation, where a particle follows a trajectory from the inside of the black hole to the outside, which is classically a banned process. By energy conservation, the radius of the black hole shrinks as a function of the energy of the outgoing particle, hence the particle produces its own tunnelling barrier [15,16]. This also provides a dynamical model of black-hole radiation, as the mass of the black hole decreases.

A quantum WKB approach was used to compute the corrections to the Hawking temperature and Bekenstein-Hawking entropy for the Schwarzschild black hole, modifying the Schwarzschild metric which takes into account effects of quantum corrections [20–23]. Furthermore, the black-hole area was shown to have a lower bound [24] in tunnelling formalism. The chirality condition was likewise introduced to unify the anomaly and the tunnelling formalisms for deriving the Hawking effect [25], and the Hawking radiation from the black hole, both in Hořava-Lifshitz and Einstein-Gauss-Bonnet gravities, was discussed in [26,27]. Important achievements have been also accomplished in, *e.g.*, [28] in a non-commutative framework.

The tunnelling method has been employed to provide the Hawking radiation due to photon and gravitino tunnelling [29]. Moreover, this method was extended to model the emission of spin-(1/2) fermions, and the Hawking radiation was deeply analyzed in [30] as tunnelling of a Dirac particle throughout an event horizon, where quantum corrections in the single-particle action are proportional to the usual semiclassical contribution and the modifications to the Hawking temperature and Bekenstein-Hawking entropy were derived for the Schwarzschild black hole. For spin-(1/2) particles, the question of how the spin affects the black hole plays a prominent role [12–18]. Due to the fact that statistically there are particles with the spin in any direction, the effect of the spin of each type of fermion cancels out. Hence, to the lowest WKB order of approximation, the rotation of the black hole does not change. Since a black hole has a well-defined temperature, it should radiate all types of particles with all types of possible spins: as gravity couples democratically to all species of particles, thus the probability that a particle is emitted, compatibly with energy conservation, does not depend on the particle. Of course, this is well known to be only approximately correct owing to the grey-body factors. The authors in [7] argued that the probability of emission of a particle approaches zero when its energy becomes of the order of the mass of the emitting black hole. According to what one would expect from energy conservation, the tunnelling barrier is set by the shrinking of the black-hole horizon with a change in the radius, established by the energy of the emitted particle itself [7], as black holes decrease in mass as energy is emitted. Consequently, the radius of the event horizon decreases [7,11,15], and the usual approximations used in the literature [1–18] remain to be adopted here.

Elko (dark) spinor fields (dual-helicity eigenspinors of the charge conjugation operator [31]) are spin-(1/2) fermions of mass dimension one, with novel features that make them capable to incorporate both the Very Special Relativity (VSR) paradigm [32,33] and the dark matter description as well [31,33,34]. Moreover, an Elko spinor mass generation mechanism has been introduced in [35], by a natural coupling to the kink solution of a $\lambda\phi^4$ field theory. It provides exotic couplings between scalar field topological solutions and Elko spinor fields [35]. Some attempts to detect Elko at the LHC have been proposed [36,37], as well as promising applications in cosmology have been widely investigated [38–43]. Not merely in quantum field theory, and supersymmetry [44], but additionally the Einstein-Hilbert, the Einstein-Palatini, and the Holst actions were shown to be derived from the quadratic spinor Lagrangian, when Elko spinor fields are considered [45,46].

The tunnelling method is used in this paper to model Elko particles emission and absorption from black strings. We show that Elko particles are emitted at the expected Hawking temperature from black holes and black strings, providing further evidence for the universality of black-hole radiation [11,15,16], however with a specific signature that is different from Dirac particles. In fact, we shall prove that Elko particles behave contrastively from Dirac particles, that present different inward and outward probability of tunnelling —depending on a relationship between the spinor components [17,18]. In fact, we shall show that the four distinct Elko particles, being eigenspinors of the charge conjugation operator with dual helicity, manifest the property of presenting the same equations for tunnelling, and consequently the same inward and outward probability of tunnelling. Moreover, the standard Hawking temperature for black strings is obtained in this context. The results presented in this paper for Elko particles differ from Dirac particles, as naturally Elko particles are fields presenting mass dimension one [31,34,39].

String theory has solutions describing extra-dimensional extended objects surrounded by event horizons, namely black strings. These solutions can have unusual causal structure, and provide some insight into the properties of singularities in string theory. Black strings have been studied in the context of supergravity theories, topological defects and low-energy string theories [47–49] and from the pure gravitational context in [50,51], as well as in some realistic generalizations [52–54].

The solution of Einstein equations with a negative cosmological constant in the form of cylindrically symmetric spacetime is provided by [55–57]

$$\mathrm{d}s^2 = -f(r)\mathrm{d}t^2 + \frac{1}{f(r)}\mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + \alpha^2 r^2 \mathrm{d}z^2, \qquad (1)$$

where $\Lambda = -3\alpha^2$ denotes the cosmological constant, M is associated to the ADM mass density of the black string, and $f(r) = \left(\alpha^2 r^2 - \frac{M}{r}\right)$. The event horizon of the black hole is clearly provided by

$$r_0 = \left(\frac{M}{\alpha^2}\right)^{1/3}.$$

Also, this solution was discussed in [58] in the context of Einstein-Maxwell gravity.

In order to analyze the tunnelling of fermions throughout the black-string horizon, we depart from the usual mass-dimension-3/2 fermions, and shall investigate the role that Elko particles play in this context. To accomplish it, the basic features of Elko particles are briefly revisited [31,34,59]. Elko spinor fields $\lambda(p^\mu)$ are eigenspinors of the charge conjugation operator C, namely, $C\lambda(p^\mu) = \pm\lambda(p^\mu)$ (here the momentum space is used just to fix the notation). The Weyl representation of γ^μ is used hereupon. The plus (minus) sign regards self-conjugate, (anti–self-conjugate) spinor fields, denoted by $\lambda^S(p^\mu)$ ($\lambda^A(p^\mu)$). Explicitly, once the rest spinors $\lambda(k^\mu)$ are obtained, for an arbitrary p^μ it yields

$$\lambda(p^\mu) = e^{i\boldsymbol{\kappa}\cdot\boldsymbol{\varphi}}\lambda(k^\mu), \qquad (2)$$

where $k^\mu = \left(m, \lim_{p\to 0}\frac{\boldsymbol{p}}{p}\right)$, for $p = |\boldsymbol{p}|$. The boost operator in (2) is provided by [59]

$$e^{i\boldsymbol{\kappa}\cdot\boldsymbol{\varphi}} = \sqrt{\frac{E+m}{2m}}\,\mathrm{diag}\left(\mathbb{I} + \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{E+m},\ \mathbb{I} - \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{E+m}\right).$$

The $\phi_L(k^\mu)$ are defined to be eigenspinors of the helicity operator $\boldsymbol{\sigma}\cdot\hat{\boldsymbol{p}}$:

$$\boldsymbol{\sigma}\cdot\hat{\boldsymbol{p}}\,\phi_L^\pm(k^\mu) = \pm\phi_L^\pm(k^\mu),$$

where $\hat{\boldsymbol{p}} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$, and the phases are employed [31,34,59] such that

$$\phi_L^+(k^\mu) = \sqrt{m}\begin{pmatrix}\cos(\theta/2)e^{-i\phi/2}\\ \sin(\theta/2)e^{+i\phi/2}\end{pmatrix}, \tag{3}$$

$$\phi_L^-(k^\mu) = \sqrt{m}\begin{pmatrix}-\sin(\theta/2)e^{-i\phi/2}\\ \cos(\theta/2)e^{+i\phi/2}\end{pmatrix}. \tag{4}$$

Elko spinor fields $\lambda(k^\mu)$ are defined by

$$\lambda_\pm^S(k^\mu) = \begin{pmatrix} i\Theta\left[\phi_L^\pm(k^\mu)\right]^*\\ \phi_L^\pm(k^\mu)\end{pmatrix}, \tag{5}$$

$$\lambda_\pm^A(k^\mu) = \pm\begin{pmatrix} -i\Theta\left[\phi_L^\mp(k^\mu)\right]^*\\ \phi_L^\mp(k^\mu)\end{pmatrix}, \tag{6}$$

where the Θ denotes the Wigner time-reversal operator for spin one-half. The expression

$$\boldsymbol{\sigma}\cdot\hat{\boldsymbol{p}}\left(\Theta[\phi_L^\pm(k^\mu)]^*\right) = \mp\left(\Theta[\phi_L^\pm(k^\mu)]^*\right)$$

evinces the helicity of $\Theta[\phi_L(k^\mu)]^*$ to be opposite to that of $\phi_L(k^\mu)$, and therefore

$$\lambda_\pm^S(p^\mu) = \sqrt{\frac{E+m}{2m}}\left(1 \mp \frac{p}{E+m}\right)\lambda_\pm^S(k^\mu), \tag{7}$$

$$\lambda_\pm^A(p^\mu) = \sqrt{\frac{E+m}{2m}}\left(1 \pm \frac{p}{E+m}\right)\lambda_\pm^A(k^\mu) \tag{8}$$

are the expansion coefficients of a mass-dimension-one quantum field. In fact, the Dirac operator $(\gamma_\mu p^\mu \pm m\mathbb{I}_4)$ does not annihilate the $\lambda(p^\mu)$ and the following results hold [31,34,59]:

$$\gamma_\mu p^\mu \lambda_+^S(p^\mu) = im\lambda_-^S(p^\mu), \tag{9}$$

$$\gamma_\mu p^\mu \lambda_-^S(p^\mu) = -im\lambda_+^S(p^\mu), \tag{10}$$

$$\gamma_\mu p^\mu \lambda_-^A(p^\mu) = im\lambda_+^A(p^\mu), \tag{11}$$

$$\gamma_\mu p^\mu \lambda_+^A(p^\mu) = -im\lambda_-^A(p^\mu). \tag{12}$$

Nevertheless, it still implies annihilation of Elko by the Klein-Gordon operator.

Hawking radiation from black holes comprises different types of charged and uncharged particles. We investigate tunnelling of Elko particles from the event horizon of a black-string solution via tunnelling formalism. By taking

$$\nabla_\mu = \partial_\mu + \Omega_\mu, \qquad \Omega_\mu = \frac{1}{2} i\Gamma_\mu^{\alpha\beta}\Sigma_{\alpha\beta},$$

where $\Sigma_{\alpha\beta} = \frac{1}{4}i\left[\gamma^\alpha, \gamma^\beta\right]$ is the spin density tensor and the γ^μ are the usual gamma matrices satisfying the Dirac-Clifford relation for Minkowski spacetime, the matrices

$$\gamma^t = \frac{1}{\sqrt{f}}\gamma^0,\ \gamma^r = \sqrt{f}\,\gamma^3,\ \gamma^\theta = \frac{1}{r}\gamma^1,\ \gamma^z = \frac{1}{\alpha r}\gamma^2, \tag{13}$$

are chosen as usually [17], where $f = f(r)$. In order to find the solution of eqs. (9)–(12) in the background of the black string, we employ the standard form for the Elko particle, through the similar notation $\phi_L^+(k^\mu) = \binom{\alpha}{\beta}$, with α and β defined as in eq. (3):

$$\lambda_+^{\mathrm{S}}(t,r,\theta,z) = \begin{pmatrix}-i\beta^*\\ i\alpha^*\\ \alpha\\ \beta\end{pmatrix}\exp\left(\frac{i}{\hbar}\tilde{I}\right), \tag{14}$$

$$\lambda_-^{\mathrm{S}}(t,r,\theta,z) = \begin{pmatrix}-i\alpha\\ -i\beta\\ -\beta^*\\ \alpha^*\end{pmatrix}\exp\left(\frac{i}{\hbar}\tilde{I}\right), \tag{15}$$

$$\lambda_+^{\mathrm{A}}(t,r,\theta,z) = \begin{pmatrix}i\alpha\\ i\beta\\ -\beta^*\\ \alpha^*\end{pmatrix}\exp\left(\frac{i}{\hbar}\tilde{I}\right), \tag{16}$$

$$\lambda_-^{\mathrm{A}}(t,r,\theta,z) = \begin{pmatrix}-i\beta^*\\ i\alpha^*\\ -\alpha\\ -\beta\end{pmatrix}\exp\left(\frac{i}{\hbar}\tilde{I}\right). \tag{17}$$

Here $\tilde{I} = \tilde{I}(t,r,\theta,z)$ represents the classical action. We use the above forms for the Elko particles in each one of eqs. (9)–(12), and solve this coupled system of equations. Thus, by applying the WKB approximation, where $\frac{i}{\hbar}\tilde{I} = \frac{i}{\hbar}I + I_0 + \mathcal{O}(\hbar)$, and considering terms solely up to the leading order in $\hbar$, by denoting $I_r = \partial I/\partial r$, $I_t = \partial I/\partial t$, $I_\theta = \partial I/\partial\theta$, and $I_z = \partial I/\partial z$, this procedure yields

$$\frac{i\alpha^* I_t}{\sqrt{f}} + \beta\sqrt{f}\,I_r = m\beta^* + \left(\frac{i}{\alpha z}I_z - \frac{1}{r}I_\theta\right)\alpha^*, \tag{18}$$

$$\frac{i\beta I_t}{\sqrt{f}} - \alpha^*\sqrt{f}\,I_r = m\alpha^* - \left(\frac{i}{\alpha z}I_z + \frac{1}{r}I_\theta\right)\beta^*. \tag{19}$$

By taking into account the Killing vectors of the background spacetime we can employ the usual ansatz in refs. [15–18]

$$I(t,r,\theta,z) = -Et + W(r) + l\theta + Jz, \tag{20}$$

where E is the energy of the emitted particles and W is the part of the action $\tilde{I}$ that contributes to the tunnelling probability. Using this ansatz in eqs. (18), (19), and by taking into account that the contribution of J and l to the imaginary part of the action is canceled, as shown in [15–18], the terms in (18), (19) encompassing J and l are dismissed. The same solution for J is obtained for both the outgoing and incoming cases.

As it is comprehensively exposed in [15–18], near the black-string horizon massive particles behave like massless particles (we further analyze it after eq. (31)). Phenomenologically, considering the well-established Elko production by Higgs interactions [36], we proceed as refs. [15–18] and consider the parameter $m \approxeq 0$, without loss of generality, as near the horizon massive particles behave as massless ones. Thus, the function $W(r)$ can be computed merely from eqs. (19) and (20) as

$$\begin{aligned} -i\alpha^* E + \beta f(r) W'(r) &= 0, \\ -i\beta E + \alpha^* f(r) W'(r) &= 0. \end{aligned} \tag{21}$$

In this case for

$$\alpha = i\beta^* \tag{22}$$

we have

$$W'_+(r) = E/f(r), \tag{23}$$

whilst for the choice

$$\alpha = -i\beta^* \tag{24}$$

we get the opposite sign

$$W'_-(r) = -E/f(r). \tag{25}$$

W_+ (W_-) corresponds to outward (inward) solutions (see refs. [15–18]). Equations (23) and (25) imply that

$$W_\pm(r) = \pm \int (E/f(r))\mathrm{d}r, \tag{26}$$

which has a simple pole at $r = r_0$. By integrating around the pole, it yields

$$W_\pm(r) = \frac{\pm\pi i E}{2\alpha^2 r_0 + \frac{M}{r_0^2}}. \tag{27}$$

The probabilities of crossing the horizon in each direction can be given by [4]

$$P_\pm \propto \exp\left(-\frac{2}{\hbar}\mathrm{Im}\, W_\pm(r)\right), \tag{28}$$

where P_+ (P_-) denotes the probability of emission (absorption) by the horizon. While computing the imaginary part of the action, we note that it is the same for both the incoming and outgoing solutions. Now, using eqs. (28), the probability of particles tunnelling from inside to outside the horizon is given by

$$\Gamma \propto \frac{P_+}{P_-} = \exp\left(-\frac{4}{\hbar}\mathrm{Im} W_+(r)\right), \tag{29}$$

where in the last equality we employed eq. (27), implying that

$$\Gamma = \exp\left(\frac{-4\pi E}{2\alpha^2 r_0 + \frac{M}{r_0^2}}\right). \tag{30}$$

The tunnelling probability is given by $\Gamma = \exp(-\beta E)$, where $\beta = T_H^{-1}$, yielding the Hawking temperature formula

$$T_H = \frac{1}{4\pi}\left(2\alpha^2 r_0 + \frac{M}{r_0^2}\right), \tag{31}$$

which is the correct Hawking temperature for black strings [60]. For the massive case, near the horizon the massive particles behave like massless ones. Since the extra contributions vanish at the horizon, the result of integrating around the pole for $W_\pm$ in the massive case is the same as in the massless case and the Hawking temperature is recovered for the fermionic vacuum. Furthermore, as in the Dirac tunnelling, for both the massive and massless cases the Hawking temperature is obtained, implying that the Elko particles $\lambda^S_+, \lambda^A_-, \lambda^S_-, \lambda^A_+$ defined in eqs. (5)–(8) —with explicit components in (14)–(17)— are emitted at the same rate. This endows Elko particles with a different signature with respect to the Dirac particles (see, *e.g.*, refs. [15–18]), which we shall emphasize below.

In the tunnelling formalism the probability of particles crossing the black-hole horizon on either sides is calculated using complex path integrals. The particles cover geodesics which are now allowed classically. Nevertheless, the probability for absorption of particles should actually be equal to one, as this is a path which is permitted classically [3], providing an efficient way for computing the Hawking temperature as well. Solving Elko coupled equations (9)–(12) in the background of black strings and by applying the WKB approximation, we have provided the tunnelling probability of Elko particles and the Hawking temperature associated to it.

Moreover, the tunnelling of Elko particles has a different feature when compared to Dirac particles. The method developed in [15,16] for Dirac particles was further used in [17] in the context of black strings for the very special case where the spinor field is given by

$$\Psi_\uparrow(t, r, \theta, z) = \begin{pmatrix} A(t, r, \theta, z) \\ 0 \\ B(t, r, \theta, z) \\ 0 \end{pmatrix} \exp\left(\frac{i}{\hbar}\tilde{I}\right), \tag{32}$$

where the author shows that there is a constraint between A and B, similarly to (22) and (24). The inward and outward probability of tunnelling depends on the relation between A and B. For each constraint, Dirac particles present just one behavior: either ingoing or outgoing particles. Notwithstanding, Elko particles are eigenspinors of the charge conjugation operator, and all the eigenspinor fields ($\lambda^S_+, \lambda^A_-, \lambda^S_-, \lambda^A_+$) present the same probability either outgoing or ingoing for tunnelling. Notice that Elko spinor field λ^S_+ in (14) differs from λ^A_- in (17) just by the sign in the left-hand component, whereas the Elko spinor field λ^S_- in (15) is different from λ^A_+ in (16) by the sign in the right-hand component, although they are quite different quantum fields [59]. Moreover, all the four Elko particles present the same inward and outward probability of tunnelling and the standard Hawking temperature for black strings is obtained.

* * *

RdR acknowledges CNPq grants 303027/2012-6 and 473326/2013-2, and is also *Bolsista da CAPES Proc. No.* 10942/13-0. JMHdS thanks CNPq (482043/2011-3; 308623/2012-6).

REFERENCES

[1] Kraus P. and Wilczek F., *Nucl. Phys. B*, **433** (1995) 403.
[2] Kraus P. and Wilczek F., *Nucl. Phys. B*, **437** (1995) 231.
[3] Parikh M. K. and Wilczek F., *Phys. Rev. Lett.*, **85** (2000) 5042.
[4] Srinivasan K. and Padmanabhan T., *Phys. Rev. D*, **60** (1999) 24007.
[5] Medved A. J. M., *Phys. Rev. D*, **66** (2002) 124009.
[6] Agheben M., Nadalini M., Vanzo L. and Zerbini S., *JHEP*, **05** (2005) 014.
[7] Arzano M., Medved A. and Vagenas E., *JHEP*, **09** (2005) 037.
[8] Jiang Q.-Q., Wu S.-Q. and Cai X., *Phys. Rev. D*, **73** (2006) 064003.
[9] Zhang J. and Zhao Z., *Phys. Lett. B*, **638** (2006) 110.
[10] Mitra P., *Phys. Lett. B*, **648** (2007) 240.
[11] Kerner R. and Mann R. B., *Phys. Rev. D*, **75** (2007) 084022.
[12] Li R. and Ren J.-R., *Phys. Lett. B*, **661** (2008) 370.
[13] di Criscienzo R. and Vanzo L., *EPL*, **82** (2008) 60001.
[14] Li R. and Ren J.-R., *Class. Quantum Grav.*, **25** (2008) 125016.
[15] Kerner R. and Mann R. B., *Phys. Lett. B*, **665** (2008) 277.
[16] Kerner R. and Mann R. B., *Class. Quantum Grav.*, **25** (2008) 095014.
[17] Ahmed J. and Saifullah K., *JCAP*, **08** (2011) 011.
[18] Gillani U. A. and Saifullah K., *Phys. Lett. B*, **699** (2011) 15.
[19] Ren J., Zhang J. and Zhao Z., *Chin. Phys. Lett.*, **23** (2006) 2019.
[20] Banerjee R. and Majhi B. R., *Phys. Lett. B*, **674** (2009) 218.
[21] Banerjee R. and Majhi B. R., *JHEP*, **06** (2008) 095.
[22] Banerjee R. and Majhi B. R., *Phys. Lett. B*, **675** (2009) 243.
[23] Banerjee R. and Majhi B. R., *Phys. Lett. B*, **662** (2008) 62.
[24] Banerjee R., Majhi B. R. and Vagenas E. C., *EPL*, **92** (2010) 20001.
[25] Banerjee R. and Majhi B. R., *Phys. Rev. D*, **79** (2009) 064024.
[26] Majhi B. R., *Phys. Lett. B*, **686** (2010) 49.
[27] Banerjee R., Majhi B. R. and Vagenas E. C., *Phys. Lett. B*, **686** (2010) 279.
[28] Banerjee R., Majhi B. R. and Samanta S., *Phys. Rev. D*, **77** (2008) 124035.
[29] Majhi B. R. and Samanta S., *Ann. Phys. (N.Y.)*, **325** (2010) 2410.
[30] Majhi B. R., *Phys. Rev. D*, **79** (2009) 044005.
[31] Ahluwalia D. V. and Grumiller D., *JCAP*, **07** (2005) 012.
[32] Cohen A. G. and Glashow S. L., *Phys. Rev. Lett.*, **97** (2006) 021601.
[33] Ahluwalia D. V. and Horvath S. P., *JHEP*, **11** (2010) 078.
[34] Ahluwalia D. V. and Grumiller D., *Phys. Rev. D*, **72** (2005) 067701.
[35] Bernardini A. E. and da Rocha R., *Phys. Lett. B*, **717** (2012) 238.
[36] Dias M., de Campos F. and Hoff da Silva J. M., *Phys. Lett. B*, **706** (2012) 352.
[37] Alves A., de Campos F., Dias M. and Hoff da Silva J. M., *Searching for Elko dark matter spinors at the CERN LHC*, arXiv:1401.1127 [hep-ph].
[38] Fabbri L., *Phys. Lett. B*, **704** (2011) 255.
[39] da Rocha R., Bernardini A. E. and Hoff da Silva J. M., *JHEP*, **04** (2011) 110.
[40] Hoff da Silva J. M. and Pereira S. H., *JCAP*, **03** (2014) 009.
[41] Basak A., Bhatt J. R., Shankaranarayanan S. and Prasantha K. V., *JCAP*, **04** (2013) 025.
[42] Kouwn S., Lee J., Lee T. H. and Oh P., *Mod. Phys. Lett. A*, **28** (2013) 1350121.
[43] Boehmer C. G., Burnett J., Mota D. F. and Shaw D. J., *JHEP*, **07** (2010) 053.
[44] Wunderle K. E. and Dick R., *Can. J. Phys.*, **87** (2009) 909.
[45] da Rocha R. and Hoff da Silva J. M., *Int. J. Geom. Methods Mod. Phys.*, **6** (2009) 461.
[46] da Rocha R. and Pereira J. G., *Int. J. Mod. Phys. D*, **16** (2007) 1653.
[47] Horne J. H. and Horowitz G. T., *Nucl. Phys. B*, **368** (1992) 444.
[48] Anderson W. G. and Kaloper N., *Phys. Rev. D*, **52** (1995) 4440.
[49] Kaloper N., *Phys. Rev. D*, **48** (1993) 4658.
[50] Chamblin A., Hawking S. W. and Reall H. S., *Phys. Rev. D*, **61** (2000) 065007.
[51] Seahra S. S., Clarkson C. and Maartens R., *Phys. Rev. Lett.*, **94** (2005) 121302.
[52] Bazeia D., Hoff da Silva J. M. and da Rocha R., *Phys. Lett. B*, **721** (2013) 306.
[53] da Rocha R., Piloyan A., Kuerten A. M. and Coimbra-Araujo C. H., *Class. Quantum Grav.*, **30** (2013) 045014.
[54] da Rocha R. and Hoff da Silva J. M., *Eur. Phys. J. C*, **72** (2012) 2258.
[55] Lemos J. P. S., *Class. Quantum Grav.*, **12** (1995) 1081.
[56] Lemos J. P. S. and Zanchin V. T., *Phys. Rev. D*, **54** (1996) 3840.
[57] Santos N. O., *Class. Quantum Grav.*, **10** (1993) 2401.
[58] Cai R. G. and Zhang Y., *Phys. Rev. D*, **54** (1996) 4891.
[59] Ahluwalia D. V., *On a local mass dimension one Fermi field of spin one-half and the theoretical crevice that allows it*, arXiv:1305.7509 [hep-th].
[60] Fatima A. and Saifullah K., *Astrophys. Space Sci.*, **341** (2012) 437.

Capítulo 4

Cosmologia com Espinores Elko

Uma vez que queremos viabilizar o campo Elko como um candidato a matéria escura, o estudo de sistemas cosmológicos com tal espinor perfaz terreno apropriado. O acoplamento de espinores Elko com o campo gravitacional é rico em aplicações não só pelo fato de suas características não usuais serem passíveis de exploração, mas também pela caracterísitica primeva do campo. Enquanto campo espinorial, seu acoplamento com campos de torção, por exemplo, pode ser investigado, abrindo boa perspectiva de trabalho.

Neste capítulo abordaremos primeiramente uma releitura de parte da literatura corrente sobre o Elko em sistemas cosmológicos. Tais trabalhos visavam menos uma aplicação direta do espinor na resolução de algum problema estritamente ligado à matéria escura. Com efeito, a ideia central foi entender aspectos básicos associados à dinâmica espinorial em espaços curvos, em um *background* do tipo Friedmann-Lamaitre-Robertson-Walker (FLRW) em particular. A revisitação à literatura passa por dois aspectos aos quais nos remeteremos nas seções à seguir, mas cuja menção é relevante que façamos aqui: a investigação de uma fatoração recorrente na literatura, separando a parte temporal do espinor e trabalhando com um campo depedente do tempo de forma efetiva, e o estudo do Elko como campo responsável também por efeitos de energia escura (além, é claro, de matéria escura) no universo.

Na última seção apresentamos um trabalho que, embora tenha consequências de cunho cosmológico (razão pela qual é aqui catalogado), tem forte apelo de construções típicas em teoria de campos, já que visa a formulação de um modelo sigma não linear para campos fermiônicos de dimensão de massa um, com particularização feita postariormente para os Elkos.

4.1 Aspectos Cosmológicos

Como se sabe, sendo os campos fermiônicos representações irredutíveis do grupo de Poincarè, é natural se tratar espinores no espaço plano. As representações fermiônicas são, por assim dizer, naturalmente acomodadas em tal caso. Para o estudo de sistemas gravitacionais envolvendo espinores, portanto, é necessária a adequação apropriada, seja por meio de tetradas ou (equivalentemente) por redefinições das matrizes gamma de Dirac, de modo a satisfazerem a relação constitutiva da álgebra de Clifford no espaço curvo em questão. Relembramos tal ponto nesse início de seção apenas para enfatizar uma característica que se sobressai no tratamento de férmions em espaços curvos, a dificuldade.

As abordagens iniciais do trato de espinores Elko em cosmologia logo corroboraram essa característica (a da dificuldade). Ademais, por suas características próprias, existem contribuições diferentes (quando comparadas às obtidas através dos espinores de Dirac) advindas do Elko para densidade e pressão escuras. As complicações adicionais levaram a certas estratégias de tratamento das equações, algumas já utilizadas no caso de campos espinoriais usuais, mas que no caso em questão careciam de ulterior análise. Nossa inserção nessa área seguiu essa linha, ou seja, estudar com certa cautela algumas abordagens anteriormente realizadas.

Nos próximos dois trabalhos subsequentes, descreveremos nossa contribuição no estudo dos Elkos em cosmologia na linha de abordagem citada anteriormente. O primeiro de tais trabalhos visa, antes de tudo, o entendimento de uma fatoração amplamente utilizada, a saber: considerar-se o espinor como tendo um fator multiplicativo (e dependente do tempo) comum às suas quatro componentes, de modo a ser fatorado. Com a adicional consideração da parte espinorial restante ser constante, as equações de Friedmann ganham certa simplificação que permite, em alguns casos, um melhor tratamento. O primeiro ponto abordado foi exatamente o estudo da possibilidade de soluções do campos espinorial Elko em um *background* FLRW com diferentes fatores de escala. O resultado obtido, embora não constitua prova formal, foi positivo, no sentido que apresenta a mesma forma funcional de fatoração. Dito de outro modo, encontramos soluções das equações acopladas (sem prévia fatoração) que ficam exatamente na forma fatorada. Ainda nesse trabalho, especulamos sobre a possibilidade do Elko atuar como fonte de energia escura.

No segundo trabalho, estudamos certo arranjo mais abrangente das equações cosmológicas sem particularizar o potencial do campo Elko. A análise feita se vale da teoria de sistemas dinâmicos autônomos, e também revisita (desta vez corrigindo) certos resultados da literatura. O ponto importante a ser destacado é que, via sistemas dinâmicos, é possível se encontrar um atrator das equações acopladas gerais em que um termo de decaimento de energia escura para matéria é introduzido. Assim sendo, o estudo qualitativo das equações dinâmicas mostra ser possível que o Elko atue como energia escura decaindo, na fase de presente aceleração do universo, em matéria. Um tal comportamento é claro ser amplamente desejado para um espinor com o propósito adotado ao Elko. Construído com base na teoria de campos, o Elko visa ser um candidato de primeiros princípios à matéria escura. A possibilidade de que sua atuação cosmológica seja também vinculada aos efeitos de energia escura (com posterior decaimento) é certamente um bônus a toda formulação realizada.

Journal of Cosmology and Astroparticle Physics
An IOP and SISSA journal

Exact solutions to Elko spinors in spatially flat Friedmann-Robertson-Walker spacetimes

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Received October 16, 2013
Revised January 31, 2014
Accepted February 6, 2014
Published March 6, 2014

Abstract. In this paper we present exact solutions to the so-called Elko spinors for three models of expanding universe, namely the de Sitter, linear and the radiation type evolution. The study was restricted to flat, homogeneous and isotropic Friedmann-Robertson-Walker backgrounds. Starting with an Elko spinor we present the solutions for these cases and compare to the case of Dirac spinors. Besides, an attempt to use Elko spinors as a dark energy candidate in the cosmological context is investigated.

Keywords: quantum field theory on curved space, cosmology of theories beyond the SM, particle physics - cosmology connection

ArXiv ePrint: 1401.3252

JCAP03(2014)009

doi:10.1088/1475-7516/2014/03/009

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1 Introduction

Exact solutions to the Dirac equation in curved spacetime is of considerable interest in cosmology and astrophysics, where gravity is believed to play a dominant role in determining the behavior of spin-1/2 particles. A general discussion on the interaction of massless neutrinos and spherically symmetric gravitational fields was performed by Brill and Wheeler [1] in 1957. In the 1970s the phenomenon of particle production in curved spacetime was investigated by Parker [2–6] and in 1974 Hawking discovered the effect of black hole evaporation [7–9], an appropriate example regarding the importance of strong gravitational fields in quantum mechanical processes. Also, the study of the hydrogen atom energy spectrum in curved spacetime was presented by Audretsch and Schäfer [10] and was also studied by Parker in 1980 [11].

Finding exact and analytic solutions of the Dirac equation in curved backgrounds is always a hard task. Some exact solutions have been reported in the middles of 1980 [12–14]. In 1987 Barut and Duru [15] provided an exact solution for the Dirac equation for a spatially flat Friedmann-Robertson-Walker spacetime in three meaningful models of expanding universes, based on the spin connection point of view. Exact solutions of the Dirac equation in open and closed Friedmann-Robertson-Walker spaces were presented in subsequent years for both massive and massless case [16–21]. Solutions for Kasner spacetime was obtained by Srivastava [22] and for an anisotropic Bianchi type VI background was presented by Portugal [23].

In this work we aim to investigate exact solutions for Elko spinors whose dynamics is taken in curved spacetime. More precisely, we study the solutions for the aforementioned spinor field in spatially flat Friedmann-Robertson-Walker spacetimes. Elko spinor fields were introduced in [24, 25] as a possible generalization of Majorana spinor fields. The main property defining Elko spinors is that they are eigenspinors of the charge conjugator operator, making them neutral under electromagnetic interactions by construction. Since the introduction of the Elko spinors, modifications and improvements have been accomplished. The final form for the spinor and its corresponding quantum theory may be found in [26]. There are several works considering Elko spinor fields in the context of curved spacetimes and cosmology. The study of Elko spinors with a possible coupling with torsion fields is presented in [27, 28], as well as its impact on Cosmic Microwave Background anisotropies [29] and its relation to the cosmological principle [30]. Following this reasoning, important consequences of dark spinor models to inflation are studied in [31, 32] and interesting solutions where the dark spinor field leads to slow roll and fast roll de Sitter solutions are presented in [33]. Scalar and tensor cosmological perturbations are discussed in [34, 35] and dark spinor models as a candidate of dark energy are investigated in [36, 37], as well as the cosmological coincidence problem.

As remarked, the endeavour on finding exact solutions of spinor fields in curved spacetimes is important in several contexts. The possibility, raised in the Elko formalism, of understanding this spinor field as a candidate to dark matter (for instance, along with the fact that Elko has a peculiar dynamics) certainly highlights the relevance of studying exact solutions for the Elko dynamics in physically important spacetimes. Furthermore, it is also a robust starting point to investigate Elko particle production in curved backgrounds [38].

This paper is organized as follows: in the next section we give a tutorial and short review about the main aspects of Elko spinor fields. In section III we study exact solutions of Elko dynamics in three different cosmological expanding spacetimes, namely: the de Sitter one, a linear expansion and the radiation dominated universe. Section IV is reserved to the investigation of the obtained solutions in cosmology. In the final section we conclude, comparing the obtained solutions with the usual case of Dirac spinors.

2 Elko spinor fields

In this section we shall review some important aspects concerning Elko spinor fields and its dynamics [24, 26]. As mentioned in the Introduction, the very relation defining Elko spinor fields is given by

$$C\lambda = \pm\lambda, \tag{2.1}$$

where C stands for the charge conjugator operator. Hence, λ is an eigenspinor of C. By solving eq. (2.1), it is possible to recast the spinors as self-conjugate ($+$ sign in (2.1)) $\lambda^S_{\{+,-\}}$, $\lambda^S_{\{-,+\}}$, and anti-self-conjugate ($-$ sign in (2.1)) $\lambda^A_{\{+,-\}}$, $\lambda^A_{\{-,+\}}$. They are given explicitly by

$$\begin{aligned}
\lambda^S_{\{+,-\}}(\vec{0}) &= \begin{pmatrix} +\sigma_2[\phi^-_L(\vec{0})]^* \\ \phi^-_L(\vec{0}) \end{pmatrix}, \\
\lambda^S_{\{-,+\}}(\vec{0}) &= \begin{pmatrix} +\sigma_2[\phi^+_L(\vec{0})]^* \\ \phi^+_L(\vec{0}) \end{pmatrix}, \\
\lambda^A_{\{+,-\}}(\vec{0}) &= \begin{pmatrix} -\sigma_2[\phi^-_L(\vec{0})]^* \\ \phi^-_L(\vec{0}) \end{pmatrix}, \\
\lambda^A_{\{-,+\}}(\vec{0}) &= -\begin{pmatrix} -\sigma_2[\phi^+_L(\vec{0})]^* \\ \phi^+_L(\vec{0}) \end{pmatrix},
\end{aligned} \tag{2.2}$$

with phases adopted such that

$$\phi^+_L(\vec{0}) = \sqrt{m}\begin{pmatrix} \cos(\theta/2)e^{-i\phi/2} \\ \sin(\theta/2)e^{i\phi/2} \end{pmatrix} \tag{2.3}$$

and

$$\phi^-_L(\vec{0}) = \sqrt{m}\begin{pmatrix} -\sin(\theta/2)e^{-i\phi/2} \\ \cos(\theta/2)e^{i\phi/2} \end{pmatrix}. \tag{2.4}$$

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The equations above are valid in the rest frame ($\vec{k}=\vec{0}$), therefore the expressions for arbitrary momenta are obtained by a simple boost. The parameter m denotes the spinor field mass, σ_2 is the usual Pauli matrix, and the momentum parametrization is given by $\hat{k}=(\sin\theta\cos\phi,\sin\theta\sin\phi,\cos\theta)$. It is remarkable that $-i\sigma_2[\phi_L^{\pm}(\vec{0})]^*$ and $\phi_L^{\pm}(\vec{0})$ have opposite helicities. It means that Elko spinor fields belong to the $\left(\frac{1}{2},0\right)\oplus\left(0,\frac{1}{2}\right)$ representation space.

The dual spinor associated to $\lambda^{S/A}$ can be obtained in a very judicious way, by demanding that the product $\overset{\neg}{\lambda}\,\lambda$, being $\overset{\neg}{\lambda}$ the dual, remains invariant under Lorentz transformations [39]. The result reads

$$\overset{\neg}{\lambda}{}^{S/A}_{\{\mp,\pm\}}(\vec{k}) = \pm i\left[\lambda^{S/A}_{\{\pm,\mp\}}(\vec{0})\right]^{\dagger}\gamma^0. \tag{2.5}$$

With the aid of eq. (2.5) it is possible to write down the spin sums

$$\sum_{\kappa}\lambda^S_\kappa\,\overset{\neg}{\lambda}{}^S_\kappa = +m[\mathbb{I}+\mathcal{G}(\phi)],$$
$$\sum_{\kappa}\lambda^A_\kappa\,\overset{\neg}{\lambda}{}^A_\kappa = -m[\mathbb{I}-\mathcal{G}(\phi)], \tag{2.6}$$

where $\mathcal{G}(\phi)$ is given by [39]

$$\mathcal{G}(\phi)=\gamma^5(\gamma_1\sin\phi-\gamma_2\cos\phi). \tag{2.7}$$

In order to unveil the Elko quantum dynamics we need an approach different from the usual textbook ones, inasmuch as we do not know a priori what Lagrangian must be associated to the Elko spinor. The first hint towards its dynamics comes from the following algebraic relation

$$(\gamma_\mu k^\mu\delta_\alpha^\beta \pm im\mathbb{I}\varepsilon_\alpha^\beta)\lambda_\beta^{S/A}(\vec{k})=0, \tag{2.8}$$

which can be obtained by applying $\gamma_\mu k^\mu$ to $\lambda_\beta^{S/A}(\vec{k})$. From eq. (2.8) it is straightforward to see that the application of $\gamma^\nu k_\nu$ from the left leads to

$$(\gamma^\nu\gamma^\mu k_\mu k_\nu - m^2)\lambda^{S/A}_{\{\mp,\pm\}}=0, \tag{2.9}$$

which, by means of $\{\gamma^\mu,\gamma^\nu\}=2\eta^{\mu\nu}$, leads to the Klein-Gordon equation in the momentum. In the following we shall derive the Klein-Gordon equation of the Elko spinor field by a more precise argument.

(2.8) is a Dirac-like equation. It is obviously different from the Dirac equation, but they share the covariant structure. Hence, denoting a spinorial transformation as $\lambda' = S\lambda$ (assuming that λ belongs to a linear representation of, at least, a subgroup of the Lorentz group), and studying the transformation of the expression (2.8) we arrive at the same covariance condition of the standard Dirac equation $S\gamma^\nu S^{-1}\Lambda^\mu{}_\nu=\gamma^\mu$. Therefore, as in the Dirac case, the field λ is not unitarily transformed and cannot be associated to a quantum state. Thus, quantization is necessary.

By keeping some recurrence with the usual spinorial case, we may associate a quantum field by

$$\eta(x)=\int\frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2mE(\vec{k})}}\sum_\alpha[c_\alpha(\vec{k})\lambda^S_\alpha(\vec{k})e^{-ik_\mu x^\mu}+c^\dagger_\alpha(\vec{k})\lambda^A_\alpha(\vec{k})e^{+ik_\mu x^\mu}], \tag{2.10}$$

where $c_\alpha^\dagger(\vec{k})$ and $c_\alpha(\vec{k})$ are the creation and annihilation operators, respectively, satisfying the usual fermionic anti-commutation relations. The quantum dual may be obtained in a rather similar fashion. With the quantum fields at hands we may evaluate the Feynman-Dyson propagator, given by

$$S_{FD}(x'-x) = i\langle|\mathcal{T}\big(\eta(x')\stackrel{\neg}{\eta}(x)\big)|\rangle, \tag{2.11}$$

where $\mathcal{T}$ is the time ordering operator. The calculation is a little tricky due to the subtle aspects of the field. The final result reads [24–26]

$$S_{FD}(x'-x) = -\int \frac{d^4k}{(2\pi)^4} e^{-ik^\mu(x'_\mu - x_\mu)}\left[\frac{1}{k_\mu k^\mu - m^2}\right], \tag{2.12}$$

hence the Elko spinor field must respect (only) the Klein-Gordon Lagrangian, i.e., it has mass dimension one. If we keep ourselves on power counting arguments, then the only perturbatively renormalizable possible terms are the mass one, the self (quartic) interaction $(\stackrel{\neg}{\lambda}\lambda)^2$ and the coupling with a scalar field.

In the following we shall investigate the exact solutions for the Elko spinor field in the context of physically relevant expanding spacetimes.

3 The Elko spinor equation in expanding spacetimes

By the reasons exposed in the previous section, the Elko spinor action in the curved spacetime is given by:

$$S = \frac{1}{2}\int \sqrt{-g}\left(\frac{1}{2}g^{\mu\nu}(\nabla_\mu \stackrel{\neg}{\lambda}\nabla_\nu\lambda + \nabla_\nu \stackrel{\neg}{\lambda}\nabla_\mu\lambda) - V(\stackrel{\neg}{\lambda}\lambda)\right)d^4x\,, \tag{3.1}$$

where $V(\stackrel{\neg}{\lambda}\lambda)$ is the potential and $g \equiv \det g_{\mu\nu}$. The covariant derivatives acting on the Elko spinors are $\nabla_\mu \stackrel{\neg}{\lambda} = \partial_\mu \stackrel{\neg}{\lambda} + \stackrel{\neg}{\lambda}\Gamma_\mu$ and $\nabla_\mu\lambda = \partial_\mu\lambda - \Gamma_\mu\lambda$, where Γ_μ are the spin connections.

The metric in a spatially flat, homogeneous and isotropic Friedmann-Robertson-Walker expanding universe is given by

$$ds^2 = dt^2 - a^2(t)(dx^2 + dy^2 + dz^2)\,, \tag{3.2}$$

thus

$$g_{\mu\nu} = diag(1, -a^2, -a^2, -a^2)\,, \quad g^{\mu\nu} = diag(1, -1/a^2, -1/a^2, -1/a^2)\,, \tag{3.3}$$

with $g^{\mu\alpha}g_{\alpha\nu} = \delta^\mu_\nu$ and $\sqrt{-g} = a^3$. In order to satisfy the defining equations $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$ with respect to the metric (3.3), the Dirac matrices $\gamma^\mu(x)$ are

$$\gamma^0(t) = \gamma_0\,, \qquad \gamma^i(t) = -\frac{1}{a(t)}\gamma_i\,, \quad i = 1, 2, 3\,, \tag{3.4}$$

where γ_μ (lower index) denotes the standard Dirac matrices in the Minkowiski space. The spin connections Γ_μ can be determined as $\Gamma_0 = 0$ and $\Gamma_i = \frac{\dot{a}}{2}\gamma_0\gamma_i$, where a dot denotes a time derivative.

Taking the potential of the form $V = \frac{1}{2}m^2 \stackrel{\neg}{\lambda}\lambda$, the Elko Lagrangian density can be written as

$$\mathcal{L} = \frac{1}{2}\sqrt{-g}\left[g^{\mu\nu}(\nabla_\mu \stackrel{\neg}{\lambda}\nabla_\nu\lambda) - m^2 \stackrel{\neg}{\lambda}\lambda\right]. \tag{3.5}$$

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The equations of motion follow from a principle of least action. For the spinor λ, for instance, we have

$$\partial_\alpha\left[\sqrt{-g}g^{\alpha\nu}(\partial_\nu\lambda-\Gamma_\nu\lambda)\right]+\sqrt{-g}\left[g^{\mu\nu}(\Gamma_\mu\Gamma_\nu\lambda-\Gamma_\mu\partial_\nu\lambda)+m^2\lambda\right]=0\,, \tag{3.6}$$

and the corresponding equation of motion taking into account the metric (3.3) is

$$\ddot{\lambda}+3\left(\frac{\dot{a}}{a}\right)\dot{\lambda}-\frac{1}{a^2}\partial_i^2\lambda-\frac{3}{4}\left(\frac{\dot{a}}{a}\right)^2\lambda+m^2\lambda+\frac{\dot{a}}{a^2}\gamma_0\gamma_i(\partial_i\lambda)=0\,, \tag{3.7}$$

where we have used $\Gamma_i\Gamma_i=\frac{\dot{a}^2}{4}\mathrm{I}$. The corresponding equation for $\stackrel{\neg}{\lambda}$ is

$$\ddot{\stackrel{\neg}{\lambda}}+3\left(\frac{\dot{a}}{a}\right)\dot{\stackrel{\neg}{\lambda}}-\frac{1}{a^2}\partial_i^2\stackrel{\neg}{\lambda}-\frac{3}{4}\left(\frac{\dot{a}}{a}\right)^2\stackrel{\neg}{\lambda}+m^2\stackrel{\neg}{\lambda}-\frac{\dot{a}}{a^2}(\partial_i\stackrel{\neg}{\lambda})\gamma_0\gamma_i=0\,. \tag{3.8}$$

These equations are the generalization of the corresponding equations of motion obtained in [31, 33] for the scalar part of the Elko field, including the non-homogeneous terms of the type $\partial_i\lambda$.

Since a is a function of t only, we can set

$$\lambda(\vec{x},t)=N\frac{\mathrm{e}^{i\vec{k}\cdot\vec{x}}}{a(t)^{3/2}}\begin{pmatrix}\Phi_I(\vec{k},t)\\ \Phi_{II}(\vec{k},t)\end{pmatrix}, \tag{3.9}$$

where N is a normalization constant. It is a fairly simple exercise to constraint the $\Phi_{I,II}$ components of (3.9) by means of the eigenspinor equation (2.1) in the rest frame, in order to ensure the spinor in question as an Elko spinor field indeed. To fix ideas, let us call $\Phi_I^T=(\phi_1(t)\alpha,\phi_2(t)\beta)$, $\Phi_{II}^T=(\phi_3(t)\gamma,\phi_4(t)\delta)$, being $\alpha,\beta,\gamma,\delta$ constants, and work with the positive sign of (2.1). The result is

$$\Phi_I(\vec{k},t)=\begin{pmatrix}\phi_1(t)\alpha(\vec{k})\\ i\phi_3^*(t)\gamma^*(\vec{k})\end{pmatrix},\qquad \Phi_{II}(\vec{k},t)=\begin{pmatrix}\phi_3(t)\gamma(\vec{k})\\ -i\phi_1^*(t)\alpha^*(\vec{k})\end{pmatrix}. \tag{3.10}$$

The functions Φ_I and Φ_{II} will satisfy the following equation,

$$\ddot{\Phi}_{I,II}+\left[\frac{\vec{k}^2}{a^2}+m^2-\frac{3}{2}\left(\frac{\dot{a}}{a}\right)^2-\frac{3}{2}\frac{\ddot{a}}{a}\right]\Phi_{I,II}\pm i\frac{\dot{a}}{a^2}\vec{k}\cdot\vec{\sigma}\,\Phi_{I,II}=0\,, \tag{3.11}$$

where the plus and minus signal stands for Φ_I and Φ_{II}, respectively. It is interesting to note from this equation that the corresponding equations for Φ_I and Φ_{II} are decoupled, but due to the last term the equations for $\phi_1(t)$ and $\phi_3^*(t)$ are coupled, and the same happens for $\phi_3(t)$ and $\phi_1^*(t)$.

To solve this system of coupled equations we make the decomposition $\phi_1(t)=\phi_{1R}(t)+i\phi_{1I}(t)$ and $\phi_3(t)=\phi_{3R}(t)+i\phi_{3I}(t)$, where ϕ_{1R}, ϕ_{3R} stands for the real part of ϕ_1 and ϕ_3, respectively, and ϕ_{1I}, ϕ_{3I} for the imaginary part. Substituting in (3.11) we have the four coupled differential equations:

$$\ddot{\phi}_1(t)+\left[\frac{\vec{k}^2}{a^2}+m^2-\frac{3}{2}\left(\frac{\dot{a}}{a}\right)^2-\frac{3}{2}\frac{\ddot{a}}{a}\right]\phi_1(t)+i\frac{\dot{a}}{a^2}\left(k_3\phi_1(t)+\frac{i\gamma^*}{\alpha}k_-\phi_3^*(t)\right)=0\,, \tag{3.12}$$

$$\ddot{\phi}_3^*(t) + \left[\frac{\vec{k}^2}{a^2} + m^2 - \frac{3}{2}\left(\frac{\dot{a}}{a}\right)^2 - \frac{3}{2}\frac{\ddot{a}}{a}\right]\phi_3^*(t) + i\frac{\dot{a}}{a^2}\left(\frac{\alpha}{i\gamma^*}k_+\phi_1(t) - k_3\phi_3^*(t)\right) = 0\,, \quad (3.13)$$

$$\ddot{\phi}_3(t) + \left[\frac{\vec{k}^2}{a^2} + m^2 - \frac{3}{2}\left(\frac{\dot{a}}{a}\right)^2 - \frac{3}{2}\frac{\ddot{a}}{a}\right]\phi_3(t) - i\frac{\dot{a}}{a^2}\left(k_3\phi_3(t) - \frac{i\alpha^*}{\gamma}k_-\phi_1^*(t)\right) = 0\,, \quad (3.14)$$

$$\ddot{\phi}_1^*(t) + \left[\frac{\vec{k}^2}{a^2} + m^2 - \frac{3}{2}\left(\frac{\dot{a}}{a}\right)^2 - \frac{3}{2}\frac{\ddot{a}}{a}\right]\phi_1^*(t) - i\frac{\dot{a}}{a^2}\left(\frac{\gamma}{-i\alpha^*}k_+\phi_3(t) - k_3\phi_1^*(t)\right) = 0\,, \quad (3.15)$$

with $k_\pm = k_1 \pm ik_2$.

For the anti-self-conjugate spinor $\overset{\neg}{\lambda}$ we use the definition (2.5). The $\pm i$ factor is irrelevant to equation (3.8), thus we set:

$$\overset{\neg}{\lambda}(\vec{x}, t) = N\frac{\mathrm{e}^{-i\vec{k}\cdot\vec{x}}}{a(t)^{3/2}}\left\{\overset{\neg}{\Phi}_I(\vec{k}, t)\,,\quad \overset{\neg}{\Phi}_{II}(\vec{k}, t)\right\} \quad (3.16)$$

where

$$\overset{\neg}{\Phi}_I(\vec{k}, t) = \left\{\phi_3^*(t)\gamma^*(\vec{k})\,,\quad i\phi_1(t)\alpha(\vec{k})\right\}, \quad (3.17)$$

$$\overset{\neg}{\Phi}_{II}(\vec{k}, t) = \left\{\phi_1^*(t)\alpha^*(\vec{k})\,,\quad -i\phi_3(t)\gamma(\vec{k})\right\}. \quad (3.18)$$

The full set of coupled equations for this case is

$$\ddot{\phi}_3^*(t) + \left[\frac{\vec{k}^2}{a^2} + m^2 - \frac{3}{2}\left(\frac{\dot{a}}{a}\right)^2 - \frac{3}{2}\frac{\ddot{a}}{a}\right]\phi_3^*(t) - i\frac{\dot{a}}{a^2}\left(\frac{\alpha}{i\gamma^*}k_+\phi_1(t) - k_3\phi_3^*(t)\right) = 0\,, \quad (3.19)$$

$$\ddot{\phi}_1(t) + \left[\frac{\vec{k}^2}{a^2} + m^2 - \frac{3}{2}\left(\frac{\dot{a}}{a}\right)^2 - \frac{3}{2}\frac{\ddot{a}}{a}\right]\phi_1(t) - i\frac{\dot{a}}{a^2}\left(k_3\phi_1(t) + \frac{i\gamma^*}{\alpha}k_-\phi_3^*(t)\right) = 0\,, \quad (3.20)$$

$$\ddot{\phi}_1^*(t) + \left[\frac{\vec{k}^2}{a^2} + m^2 - \frac{3}{2}\left(\frac{\dot{a}}{a}\right)^2 - \frac{3}{2}\frac{\ddot{a}}{a}\right]\phi_1^*(t) + i\frac{\dot{a}}{a^2}\left(\frac{\gamma}{-i\alpha^*}k_+\phi_3(t) - k_3\phi_1^*(t)\right) = 0\,, \quad (3.21)$$

$$\ddot{\phi}_3(t) + \left[\frac{\vec{k}^2}{a^2} + m^2 - \frac{3}{2}\left(\frac{\dot{a}}{a}\right)^2 - \frac{3}{2}\frac{\ddot{a}}{a}\right]\phi_3(t) + i\frac{\dot{a}}{a^2}\left(k_3\phi_3(t) - \frac{i\alpha^*}{\gamma}k_-\phi_1^*(t)\right) = 0\,. \quad (3.22)$$

3.1 Case $a(t) = a_0\, e^{Ht}$

For the case $a(t) = a_0\,\mathrm{e}^{Ht}$, which represents an inflationary universe or a de Sitter evolution, the coupled equations (3.12)–(3.15) have the following solutions in terms of the Whittaker $M_{\mu,\nu}(z)$ and $W_{\mu,\nu}(z)$ functions [40]:

$$\begin{aligned}\phi_1(t) = \frac{2\mathrm{e}^{\frac{1}{2}Ht}}{\alpha\gamma k_+ + \alpha^*\gamma^* k_-} \Bigg[&\left(c_5\alpha^*\gamma^*k_- + ic_1|\gamma|^2(k_3+k)\right)M_{+1/2,\nu}(z) \\ &+ \left(c_7\alpha^*\gamma^*k_- + ic_3|\gamma|^2(k_3+k)\right)W_{+1/2,\nu}(z) \\ &+ \left(c_6\alpha^*\gamma^*k_- + ic_2|\gamma|^2(k_3-k)\right)M_{-1/2,\nu}(z) \\ &+ \left(c_8\alpha^*\gamma^*k_- + ic_4|\gamma|^2(k_3-k)\right)W_{-1/2,\nu}(z) \Bigg], \end{aligned} \quad (3.23)$$

$$
\begin{aligned}
\phi_3(t) = \frac{2\mathrm{e}^{\frac{1}{2}Ht}}{\alpha\gamma k_+ + \alpha^*\gamma^* k_-} \Bigg[& \Big(c_1\alpha^*\gamma^* k_- - ic_5|\alpha|^2(k_3-k)\Big) M_{+1/2,\nu}(z) \\
& + \Big(c_3\alpha^*\gamma^* k_- - ic_7|\alpha|^2(k_3-k)\Big) W_{+1/2,\nu}(z) \\
& + \Big(c_2\alpha^*\gamma^* k_- - ic_6|\alpha|^2(k_3+k)\Big) M_{-1/2,\nu}(z) \\
& + \Big(c_4\alpha^*\gamma^* k_- - ic_8|\alpha|^2(k_3+k)\Big) W_{-1/2,\nu}(z) \Bigg], \qquad (3.24)
\end{aligned}
$$

where c_i ($i = 1, 2, \ldots, 8$) are integration constants, $k \equiv |\vec{k}|$, $\nu = \sqrt{3 - m^2/H^2}$ and $z = 2ik/(Ha_0\mathrm{e}^{Ht})$.

Finally, we can write the four independent solutions as:

$$
\lambda_1(\vec{x},t) = \frac{2\mathrm{e}^{i\vec{k}\cdot\vec{x}}}{a_0^{3/2}\mathrm{e}^{Ht}} \frac{M_{+1/2,\nu}(z)}{[\alpha\gamma k_+ + \alpha^*\gamma^* k_-]} \begin{pmatrix} c_5|\alpha|^2\gamma^* k_- + ic_1|\gamma|^2\alpha(k_3+k) \\ -c_5|\alpha|^2\gamma^*(k_3-k) + ic_1|\gamma|^2\alpha k_+ \\ c_1|\gamma|^2\alpha^* k_- - ic_5|\alpha|^2\gamma(k_3-k) \\ -c_1|\gamma|^2\alpha^*(k_3+k) - ic_5|\alpha|^2\gamma k_+ \end{pmatrix}, \qquad (3.25)
$$

$$
\lambda_2(\vec{x},t) = \frac{2\mathrm{e}^{i\vec{k}\cdot\vec{x}}}{a_0^{3/2}\mathrm{e}^{Ht}} \frac{W_{+1/2,\nu}(z)}{[\alpha\gamma k_+ + \alpha^*\gamma^* k_-]} \begin{pmatrix} c_7|\alpha|^2\gamma^* k_- + ic_3|\gamma|^2\alpha(k_3+k) \\ -c_7|\alpha|^2\gamma^*(k_3-k) + ic_3|\gamma|^2\alpha k_+ \\ c_3|\gamma|^2\alpha^* k_- - ic_7|\alpha|^2\gamma(k_3-k) \\ -c_3|\gamma|^2\alpha^*(k_3+k) - ic_7|\alpha|^2\gamma k_+ \end{pmatrix}, \qquad (3.26)
$$

$$
\lambda_3(\vec{x},t) = \frac{2\mathrm{e}^{i\vec{k}\cdot\vec{x}}}{a_0^{3/2}\mathrm{e}^{Ht}} \frac{M_{-1/2,\nu}(z)}{[\alpha\gamma k_+ + \alpha^*\gamma^* k_-]} \begin{pmatrix} c_6|\alpha|^2\gamma^* k_- + ic_2|\gamma|^2\alpha(k_3-k) \\ -c_6|\alpha|^2\gamma^*(k_3+k) + ic_2|\gamma|^2\alpha k_+ \\ c_2|\gamma|^2\alpha^* k_- - ic_6|\alpha|^2\gamma(k_3+k) \\ -c_2|\gamma|^2\alpha^*(k_3-k) - ic_6|\alpha|^2\gamma k_+ \end{pmatrix}, \qquad (3.27)
$$

$$
\lambda_4(\vec{x},t) = \frac{2\mathrm{e}^{i\vec{k}\cdot\vec{x}}}{a_0^{3/2}\mathrm{e}^{Ht}} \frac{W_{-1/2,\nu}(z)}{[\alpha\gamma k_+ + \alpha^*\gamma^* k_-]} \begin{pmatrix} c_8|\alpha|^2\gamma^* k_- + ic_4|\gamma|^2\alpha(k_3-k) \\ -c_8|\alpha|^2\gamma^*(k_3+k) + ic_4|\gamma|^2\alpha k_+ \\ c_4|\gamma|^2\alpha^* k_- - ic_8|\alpha|^2\gamma(k_3+k) \\ -c_4|\gamma|^2\alpha^*(k_3-k) - ic_8|\alpha|^2\gamma k_+ \end{pmatrix}. \qquad (3.28)
$$

For the anti-self-conjugate spinor $\overset{\neg}{\lambda}$ we have:

$$
\begin{aligned}
\overset{\neg}{\lambda}_1(\vec{x},t) = \frac{2\mathrm{e}^{-i\vec{k}\cdot\vec{x}}}{a_0^{3/2}\mathrm{e}^{Ht}} \frac{M_{+1/2,\nu}(z)}{[\alpha\gamma k_+ + \alpha^*\gamma^* k_-]} \Big\{ & c_5|\alpha|^2\gamma^* k_- + ic_1|\gamma|^2\alpha(k_3-k), \\
- & c_5|\alpha|^2\gamma^*(k_3+k) + ic_1|\gamma|^2\alpha k_+, \\
& c_1|\gamma|^2\alpha^* k_- - ic_5|\alpha|^2\gamma(k_3+k), \\
- & c_1|\gamma|^2\alpha^*(k_3-k) - ic_5|\alpha|^2\gamma k_+ \Big\}, \qquad (3.29)
\end{aligned}
$$

$$
\begin{aligned}
\overset{\neg}{\lambda}_2(\vec{x},t) = \frac{2\mathrm{e}^{-i\vec{k}\cdot\vec{x}}}{a_0^{3/2}\mathrm{e}^{Ht}} \frac{W_{+1/2,\nu}(z)}{[\alpha\gamma k_+ + \alpha^*\gamma^* k_-]} \Big\{ & c_7|\alpha|^2\gamma^* k_- + ic_3|\gamma|^2\alpha(k_3-k), \\
- & c_7|\alpha|^2\gamma^*(k_3+k) + ic_3|\gamma|^2\alpha k_+, \\
& c_3|\gamma|^2\alpha^* k_- - ic_7|\alpha|^2\gamma(k_3+k), \\
- & c_3|\gamma|^2\alpha^*(k_3-k) - ic_7|\alpha|^2\gamma k_+ \Big\}, \qquad (3.30)
\end{aligned}
$$

$$
\overset{\neg}{\lambda}_3\,(\vec{x},t)=\frac{2\mathrm{e}^{-i\vec{k}\cdot\vec{x}}}{a_0^{3/2}\mathrm{e}^{Ht}}\frac{M_{-1/2,\nu}(z)}{[\alpha\gamma k_+ + \alpha^*\gamma^* k_-]}\;\Big\{\; c_6|\alpha|^2\gamma^* k_- + ic_2|\gamma|^2\alpha(k_3+k)\,, \\
- c_6|\alpha|^2\gamma^*(k_3-k) + ic_2|\gamma|^2\alpha k_+\,, \\
c_2|\gamma|^2\alpha^* k_- - ic_6|\alpha|^2\gamma(k_3-k)\,, \\
- c_2|\gamma|^2\alpha^*(k_3+k) - ic_6|\alpha|^2\gamma k_+ \Big\}\,, \tag{3.31}
$$

$$
\overset{\neg}{\lambda}_4\,(\vec{x},t)=\frac{2\mathrm{e}^{-i\vec{k}\cdot\vec{x}}}{a_0^{3/2}\mathrm{e}^{Ht}}\frac{W_{-1/2,\nu}(z)}{[\alpha\gamma k_+ + \alpha^*\gamma^* k_-]}\;\Big\{\; c_8|\alpha|^2\gamma^* k_- + ic_4|\gamma|^2\alpha(k_3+k)\,, \\
- c_8|\alpha|^2\gamma^*(k_3-k) + ic_4|\gamma|^2\alpha k_+\,, \\
c_4|\gamma|^2\alpha^* k_- - ic_8|\alpha|^2\gamma(k_3-k)\,, \\
- c_4|\gamma|^2\alpha^*(k_3+k) - ic_8|\alpha|^2\gamma k_+ \Big\}\,. \tag{3.32}
$$

3.2 Case $a(t) = a_0 t$

For the case $a(t) = a_0 t$, which represents the limit between the decelerated to the accelerated universe, equations (3.12)–(3.15) have the following linearly independent solutions in terms of the Bessel $J_\nu(z)$ and $Y_\nu(z)$ functions [40]:

$$
\phi_1(t) = \frac{2\sqrt{t}}{\alpha\gamma k_+ + \alpha^*\gamma^* k_-}\;\Big[\;\Big(c_5\alpha^*\gamma^* k_- + ic_1|\gamma|^2(k_3+k)\Big)J_{\nu_-}(z) \\
+ \Big(c_7\alpha^*\gamma^* k_- + ic_3|\gamma|^2(k_3+k)\Big)Y_{\nu_-}(z) \\
+ \Big(c_6\alpha^*\gamma^* k_- + ic_2|\gamma|^2(k_3-k)\Big)J_{\nu_+}(z) \\
+ \Big(c_8\alpha^*\gamma^* k_- + ic_4|\gamma|^2(k_3-k)\Big)Y_{\nu_+}(z)\;\Big]\,, \tag{3.33}
$$

$$
\phi_3(t) = \frac{2\sqrt{t}}{\alpha\gamma k_+ + \alpha^*\gamma^* k_-}\;\Big[\;\Big(c_1\alpha^*\gamma^* k_- - ic_5|\alpha|^2(k_3-k)\Big)J_{\nu_-}(z) \\
+ \Big(c_3\alpha^*\gamma^* k_- - ic_7|\alpha|^2(k_3-k)\Big)Y_{\nu_-}(z) \\
+ \Big(c_2\alpha^*\gamma^* k_- - ic_6|\gamma|^2(k_3+k)\Big)J_{\nu_+}(z) \\
+ \Big(c_4\alpha^*\gamma^* k_- - ic_8|\gamma|^2(k_3+k)\Big)Y_{\nu_+}(z)\;\Big]\,, \tag{3.34}
$$

where $\nu_\pm = (1/2)\sqrt{7 - 4k^2/a_0^2 \pm 4ik/a_0}$ and $z = mt$.

The four independent solutions are:

$$
\lambda_1(\vec{x},t) = \frac{2\mathrm{e}^{i\vec{k}\cdot\vec{x}}}{a_0^{3/2}t}\frac{J_{\nu_-}(z)}{[\alpha\gamma k_+ + \alpha^*\gamma^* k_-]}\begin{pmatrix} c_5|\alpha|^2\gamma^* k_- + ic_1|\gamma|^2\alpha(k_3+k) \\ -c_5|\alpha|^2\gamma^*(k_3-k) + ic_1|\gamma|^2\alpha k_+ \\ c_1|\gamma|^2\alpha^* k_- - ic_5|\alpha|^2\gamma(k_3-k) \\ -c_1|\gamma|^2\alpha^*(k_3+k) - ic_5|\alpha|^2\gamma k_+ \end{pmatrix}\,, \tag{3.35}
$$

$$
\lambda_2(\vec{x},t) = \frac{2\mathrm{e}^{i\vec{k}\cdot\vec{x}}}{a_0^{3/2}t}\frac{Y_{\nu_-}(z)}{[\alpha\gamma k_+ + \alpha^*\gamma^* k_-]}\begin{pmatrix} c_7|\alpha|^2\gamma^* k_- + ic_3|\gamma|^2\alpha(k_3+k) \\ -c_7|\alpha|^2\gamma^*(k_3-k) + ic_3|\gamma|^2\alpha k_+ \\ c_3|\gamma|^2\alpha^* k_- - ic_7|\alpha|^2\gamma(k_3-k) \\ -c_3|\gamma|^2\alpha^*(k_3+k) - ic_7|\alpha|^2\gamma k_+ \end{pmatrix}\,, \tag{3.36}
$$

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$$\lambda_3(\vec{x},t) = \frac{2\mathrm{e}^{i\vec{k}\cdot\vec{x}}}{a_0^{3/2}t}\frac{J_{\nu_+}(z)}{[\alpha\gamma k_+ + \alpha^*\gamma^* k_-]}\begin{pmatrix} c_6|\alpha|^2\gamma^* k_- + ic_2|\gamma|^2\alpha(k_3-k) \\ -c_6|\alpha|^2\gamma^*(k_3+k) + ic_2|\gamma|^2\alpha k_+ \\ c_2|\gamma|^2\alpha^* k_- - ic_6|\alpha|^2\gamma(k_3+k) \\ -c_2|\gamma|^2\alpha^*(k_3-k) - ic_6|\alpha|^2\gamma k_+ \end{pmatrix}, \tag{3.37}$$

$$\lambda_4(\vec{x},t) = \frac{2\mathrm{e}^{i\vec{k}\cdot\vec{x}}}{a_0^{3/2}t}\frac{Y_{\nu_+}(z)}{[\alpha\gamma k_+ + \alpha^*\gamma^* k_-]}\begin{pmatrix} c_8|\alpha|^2\gamma^* k_- + ic_4|\gamma|^2\alpha(k_3-k) \\ -c_8|\alpha|^2\gamma^*(k_3+k) + ic_4|\gamma|^2\alpha k_+ \\ c_4|\gamma|^2\alpha^* k_- - ic_8|\alpha|^2\gamma(k_3+k) \\ -c_4|\gamma|^2\alpha^*(k_3-k) - ic_8|\alpha|^2\gamma k_+ \end{pmatrix}. \tag{3.38}$$

For the anti-self-conjugate spinor $\overset{\neg}{\lambda}$ we have:

$$\begin{aligned}\overset{\neg}{\lambda}_1(\vec{x},t) = \frac{2\mathrm{e}^{-i\vec{k}\cdot\vec{x}}}{a_0^{3/2}t}\frac{J_{\nu_-}(z)}{[\alpha\gamma k_+ + \alpha^*\gamma^* k_-]}\Big\{ &\;c_5|\alpha|^2\gamma^* k_- + ic_1|\gamma|^2\alpha(k_3-k), \\ &- c_5|\alpha|^2\gamma^*(k_3+k) + ic_1|\gamma|^2\alpha k_+, \\ &\;c_1|\gamma|^2\alpha^* k_- - ic_5|\alpha|^2\gamma(k_3+k), \\ &- c_1|\gamma|^2\alpha^*(k_3-k) - ic_5|\alpha|^2\gamma k_+\Big\},\end{aligned} \tag{3.39}$$

$$\begin{aligned}\overset{\neg}{\lambda}_2(\vec{x},t) = \frac{2\mathrm{e}^{-i\vec{k}\cdot\vec{x}}}{a_0^{3/2}t}\frac{Y_{\nu_-}(z)}{[\alpha\gamma k_+ + \alpha^*\gamma^* k_-]}\Big\{ &\;c_7|\alpha|^2\gamma^* k_- + ic_3|\gamma|^2\alpha(k_3-k), \\ &- c_7|\alpha|^2\gamma^*(k_3+k) + ic_3|\gamma|^2\alpha k_+, \\ &\;c_3|\gamma|^2\alpha^* k_- - ic_7|\alpha|^2\gamma(k_3+k), \\ &- c_3|\gamma|^2\alpha^*(k_3-k) - ic_7|\alpha|^2\gamma k_+\Big\},\end{aligned} \tag{3.40}$$

$$\begin{aligned}\overset{\neg}{\lambda}_3(\vec{x},t) = \frac{2\mathrm{e}^{-i\vec{k}\cdot\vec{x}}}{a_0^{3/2}t}\frac{J_{\nu_+}(z)}{[\alpha\gamma k_+ + \alpha^*\gamma^* k_-]}\Big\{ &\;c_6|\alpha|^2\gamma^* k_- + ic_2|\gamma|^2\alpha(k_3+k), \\ &- c_6|\alpha|^2\gamma^*(k_3-k) + ic_2|\gamma|^2\alpha k_+, \\ &\;c_2|\gamma|^2\alpha^* k_- - ic_6|\alpha|^2\gamma(k_3-k), \\ &- c_2|\gamma|^2\alpha^*(k_3+k) - ic_6|\alpha|^2\gamma k_+\Big\},\end{aligned} \tag{3.41}$$

$$\begin{aligned}\overset{\neg}{\lambda}_4(\vec{x},t) = \frac{2\mathrm{e}^{-i\vec{k}\cdot\vec{x}}}{a_0^{3/2}t}\frac{Y_{\nu_+}(z)}{[\alpha\gamma k_+ + \alpha^*\gamma^* k_-]}\Big\{ &\;c_8|\alpha|^2\gamma^* k_- + ic_4|\gamma|^2\alpha(k_3+k), \\ &- c_8|\alpha|^2\gamma^*(k_3-k) + ic_4|\gamma|^2\alpha k_+, \\ &\;c_4|\gamma|^2\alpha^* k_- - ic_8|\alpha|^2\gamma(k_3-k), \\ &- c_4|\gamma|^2\alpha^*(k_3+k) - ic_8|\alpha|^2\gamma k_+\Big\}.\end{aligned} \tag{3.42}$$

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3.3 Case $a(t) = a_0\sqrt{t}$

For the case $a(t) = a_0\sqrt{t}$, which represents a radiation dominated universe, the equations (3.12)–(3.15) has the following linearly independent solutions[1] in terms of the Heun B functions, denoted here by $B_{a,b,\mu,\nu}(z)$, that are solutions of the Heun biconfluent equation [41, 42]:

$$\phi_1(t) = \frac{2te^{imt}}{\alpha\gamma k_+ + \alpha^*\gamma^* k_-} \left[\left(c_3\alpha^*\gamma^* k_- + ic_1|\gamma|^2(k_3+k)\right) B_{2,0,\mu,\nu_-}(z) + \left(c_4\alpha^*\gamma^* k_- + ic_2|\gamma|^2(k_3-k)\right) B_{2,0,\mu,\nu_+}(z)\right], \tag{3.43}$$

$$\phi_3(t) = \frac{2te^{imt}}{\alpha\gamma k_+ + \alpha^*\gamma^* k_-} \left[\left(c_1\alpha^*\gamma^* k_- - ic_3|\alpha|^2(k_3-k)\right) B_{2,0,\mu,\nu_-}(z) + \left(c_2\alpha^*\gamma^* k_- - ic_4|\alpha|^2(k_3+k)\right) B_{2,0,\mu,\nu_+}(z)\right], \tag{3.44}$$

where $\mu = 2ik^2/a_0^2 m$, $\nu_\pm = \pm(2-2i)k/a_0\sqrt{m}$ and $z = (-1+i)\sqrt{mt}$.

The two independent solutions are:

$$\lambda_1(\vec{x},t) = \frac{2e^{i\vec{k}\cdot\vec{x}}t^{1/4}}{a_0^{3/2}} \frac{B_{2,0,\mu,\nu_-}(z)}{[\alpha\gamma k_+ + \alpha^*\gamma^* k_-]} \begin{pmatrix} e^{imt}[c_3|\alpha|^2\gamma^* k_- + ic_1|\gamma|^2\alpha(k_3+k)] \\ e^{-imt}[-c_3|\alpha|^2\gamma^*(k_3-k) + ic_1|\gamma|^2\alpha k_+] \\ e^{imt}[c_1|\gamma|^2\alpha^* k_- - ic_3|\alpha|^2\gamma(k_3-k)] \\ e^{-imt}[-c_1|\gamma|^2\alpha^*(k_3+k) - ic_3|\alpha|^2\gamma k_+] \end{pmatrix}, \tag{3.45}$$

$$\lambda_2(\vec{x},t) = \frac{2e^{i\vec{k}\cdot\vec{x}}t^{1/4}}{a_0^{3/2}} \frac{B_{2,0,\mu,\nu_+}(z)}{[\alpha\gamma k_+ + \alpha^*\gamma^* k_-]} \begin{pmatrix} e^{imt}[c_4|\alpha|^2\gamma^* k_- + ic_2|\gamma|^2\alpha(k_3-k)] \\ e^{-imt}[-c_4|\alpha|^2\gamma^*(k_3+k) + ic_2|\gamma|^2\alpha k_+] \\ e^{imt}[c_2|\gamma|^2\alpha^* k_- - ic_4|\alpha|^2\gamma(k_3+k)] \\ e^{-imt}[-c_2|\gamma|^2\alpha^*(k_3-k) - ic_4|\alpha|^2\gamma k_+] \end{pmatrix}. \tag{3.46}$$

For the anti-self-conjugate spinor $\stackrel{\neg}{\lambda}$ we have:

$$\stackrel{\neg}{\lambda}_1(\vec{x},t) = \frac{2e^{-i\vec{k}\cdot\vec{x}}t^{1/4}}{a_0^{3/2}} \frac{B_{2,0,\mu,\nu_-}(z)}{[\alpha\gamma k_+ + \alpha^*\gamma^* k_-]} \left\{ e^{imt}[c_3|\alpha|^2\gamma^* k_- + ic_1|\gamma|^2\alpha(k_3-k)], \; e^{-imt}[-c_3|\alpha|^2\gamma^*(k_3+k) + ic_1|\gamma|^2\alpha k_+], \; e^{imt}[c_1|\gamma|^2\alpha^* k_- - ic_3|\alpha|^2\gamma(k_3+k),], \; e^{-imt}[-c_1|\gamma|^2\alpha^*(k_3-k) - ic_3|\alpha|^2\gamma k_+] \right\},$$

$$\stackrel{\neg}{\lambda}_2(\vec{x},t) = \frac{2e^{-i\vec{k}\cdot\vec{x}}t^{1/4}}{a_0^{3/2}} \frac{B_{2,0,\mu,\nu_+}(z)}{[\alpha\gamma k_+ + \alpha^*\gamma^* k_-]} \left\{ e^{imt}[c_4|\alpha|^2\gamma^* k_- + ic_2|\gamma|^2\alpha(k_3+k)], \; e^{-imt}[-c_4|\alpha|^2\gamma^*(k_3-k) + ic_2|\gamma|^2\alpha k_+], \; e^{imt}[c_2|\gamma|^2\alpha^* k_- - ic_4|\alpha|^2\gamma(k_3-k)], \; e^{-imt}[-c_2|\gamma|^2\alpha^*(k_3+k) - ic_4|\alpha|^2\gamma k_+] \right\}.$$

[1]There are other independent solutions written in terms of integral equations, but we are omitting these solutions here (see ref. [41, 42] for more details).

4 Elko spinor in cosmology

Some peculiar features of the Elko field have been used in order to extract physical information about cosmological scenarios. For instance, a quite interesting mass upper bound may be found in trying to use Elko fields as dark matter driving inflation [27]. In this section we shall consider a simple model of dark energy, instead, driven by the Elko spinor.

As has been done in recent works [31–33, 36, 37], usually the spinor field is factored out to a real homogeneous scalar field, $\lambda \equiv \varphi(t)\xi$, with the temporal dependence only in $\varphi(t)$, the same for all components and ξ representing a constant and normalized Elko spinor. Due to the homogeneity of the field ($\partial_i \lambda = 0$), the equation (3.7) for $\varphi(t)$ is substantially simplified,

$$\ddot{\varphi} + 3H\dot{\varphi} - \frac{3}{4}H^2\varphi + m^2\varphi = 0\,, \tag{4.1}$$

where $H = \dot{a}/a$. We shall comment on this simplification at the end of this section. The pressure and energy density are given by [33]

$$p_\varphi = \frac{1}{2}\dot{\varphi}^2 - \frac{1}{2}m^2\varphi^2 - \frac{3}{8}H^2\varphi^2 - \frac{1}{4}\dot{H}\varphi^2 - \frac{1}{2}H\varphi\dot{\varphi}\,, \tag{4.2}$$

$$\rho_\varphi = \frac{1}{2}\dot{\varphi}^2 + \frac{1}{2}m^2\varphi^2 + \frac{3}{8}H^2\varphi^2\,. \tag{4.3}$$

The Friedmann equations for $H(t)$ can be written as

$$H^2 = \frac{8\pi G}{3}\rho \qquad \dot{H} = -4\pi G(\rho + p)\,, \tag{4.4}$$

from which follows the conservation equation

$$\dot{\rho} + 3H(\rho + p) = 0\,, \tag{4.5}$$

where ρ and p stands for the total energy density and total pressure of all the matter fields present in the model.

In this simplified model we will consider that all the material content of the universe is the Elko spinor satisfying a dark energy equation of state $p = -\rho$, thus the Friedmann equations reduces to

$$H = \frac{\dot{a}}{a} = \pm\left(\frac{8\pi G}{3}\rho_\varphi\right)^{1/2}, \qquad \dot{H} = 0\,, \tag{4.6}$$

furthermore we have $\dot{\rho}_\varphi = 0$, so that ρ_φ is a constant, implying a de Sitter evolution $a(t) = a_0 \mathrm{e}^{Ht}$, thus we can use the solutions obtained in section 3.1.

Before we proceed, let us examine the restriction imposed by the dark energy equation of state $p_\varphi = -\rho_\varphi$. By using (4.2) and (4.3) we find

$$\dot{\varphi}^2 - \frac{1}{2}H\varphi\dot{\varphi} = 0\,, \tag{4.7}$$

whose solutions are of two types, namely static or dynamic,

$$\varphi(t) = \bar{\varphi} \quad \text{(static)}\,, \qquad \varphi(t) = \varphi_0 \mathrm{e}^{Ht/2} \quad \text{(dynamic)}\,. \tag{4.8}$$

As it can be read from eqs. (3.25)–(3.28), the obtained solutions are already in the time factored form.[2] Therefore, to be able to use the above equations we shall only to apply some limit in order to get the homogeneous solution. This limit can be achieved by taking into account that the inhomogeneity comes from spatial derivatives, giving rise to momentum dependent terms. Hence, the homogeneous limit is obtained by taken $k_{1,2,3}$ to zero carefully. For the argument, it is always possible to restrict the momentum to one direction and then take this momentum to vanish. Once this limit is performed, two things happen: 1) the spinorial part of the solution becomes homogeneous and the normalization can be imputed to the integration constants; 2) the Whittaker $k \to 0$ (or, correspondingly $z \to 0$) limits are in order. As the Whittaker limit $M_{\sigma,\nu}(z \to 0)$ depends on the σ and ν index, we must explore all the possibilities.

Let us start investigating the solution (3.25) in the homogeneous limit. It is easy to see that in this case we have

$$\lambda_1(t) = \frac{2}{a_0^{3/2}} M_{+1/2,\nu}(z \to 0) e^{-Ht} \xi, \tag{4.9}$$

where ξ stands for the constant homogeneous spinorial part of λ_1 (which is completely irrelevant to this application). The unique $z \to 0$ limit allowed for this specific Whittaker function occurs when $2\nu \neq -1, -2, \ldots$. Supposing this is the case we have $M_{+1/2,\nu} \to z^{\nu+1/2}$. Hence, writing the solution as $\lambda_1(t) = \varphi(t)\xi$, bearing in mind that $z = 2ik/a_0 H e^{Ht}$ and absorbing the constant part in φ_0 one gets

$$\varphi = \varphi_0 \mathrm{e}^{-(\nu+3/2)t}. \tag{4.10}$$

Comparing the solution (4.10) with the static solution (4.8) we see that it entails $2\nu = -3$ contradicting the hypothesis for this limit validity. Thus this solution cannot describe the static situation. To fulfil the dynamic solution of (4.8) we must have $\nu = -2$, but as $\nu > 0$ by definition ($\nu = \sqrt{3 - m^2/H^2}$), we shall disregard this solution.

The solution given by (3.27), λ_3, is more interesting. In fact, the time dependent part in this case can be recast as

$$\varphi(t) = \varphi_0 e^{(\nu-3/2)Ht}. \tag{4.11}$$

By comparing eq. (4.11) with (4.8) we see that the dynamical solution requires $\nu = 2$, which is mathematically acceptable, but it leads to $m^2 < 0$, a physically unacceptable condition. It is interesting, however, that massive ghosts solutions to an Einstein-Cartan-Dirac system present the very same behavior [45]. The static case, on the other hand, can be reached if $\nu = 3/2$, leading to an Elko mass given by $m = \frac{\sqrt{3}}{2}H$, providing a physically acceptable solution. In fact, a static solution for the field φ can be directly obtained from (4.1) if the above condition on the mass is set. By analyzing the energy density (4.3) with a static field φ we see that in this case it reproduces exactly a cosmological constant term in the Friedmann equation (4.4).

It can be straightforwardly verified that the remain cases (λ_2 and λ_4) do not contain any novelty, leading to nonphysical solutions or reproducing the static case for $m = \frac{\sqrt{3}}{2}H$.

We would like to conclude this section by tracing some comments on the homogeneity simplification in the spinor solutions. In fact, when this is the case, we have arrived at

[2] Indeed, for all the three cases analysed here, the solutions given by eqs. (3.25)–(3.28), (3.35)–(3.38) and (3.45)–(3.46) are already in the time factored form, which justifies the use of the decomposition $\lambda = \varphi(t)\xi$ in recent works [31–33, 36, 37].

a relationship between the spinor mass and H so that the solution can be applied to this cosmological scenario (also simplified). The net effect of considering the non-homogeneous case is, probably, the obtention of a more general vinculum, this time regarding the mass, the Hubble parameter and the momentum. This dispersion relation-like constraint may, eventually, lead to new possible cosmological applications, but there is no guarantee that it is in fact physically appealing.

5 Concluding remarks

In the present work we have studied the evolution of Elko spinors in a flat Friedmann-Robertson-Walker background finding exact solutions for three different models of expansion, namely a de Sitter, linear and radiation. A very interesting aspect of the solutions we have found is that, contrary to the solutions of the Dirac equation in a spatially flat Friedmann-Robertson-Walker spacetimes [15] where the first two components are coupled to the last two, the equations for the first two components of the Elko spinors are independent of the third and fourth, as can be seen in eq. (3.11).

Still comparing with the Dirac case, we see that the solution for the temporal part of the spinor is totally different in the three cases examined here. For the de Sitter evolution, the Dirac case gives solutions in terms of the Bessel functions, while here we obtain solutions in terms of the Whittaker functions. For both linear and radiation expansion the solutions in the Dirac case are given by means of Whittaker functions, while here we obtain the Bessel functions for the linear evolution and the complicated Heun biconfluent functions as solutions for the radiation. These behaviors illustrate some of the differences between the Dirac and the Elko spinors.

We have investigated a cosmological setting where an homogeneous Elko spinor acts as dark energy in a de Sitter background. It is shown that there are two solutions for this case. A dynamic one, where a constraint in the mass parameter indicates a non-physical scenario and a static solution, in which the spinor field works as an effective cosmological constant. It is important to emphasize that the results of such an application are mainly due to the maintenance of the state equation $p = -\rho$ leading to a de Sitter expansion. Other types of equation of state, giving different expansion rates are not excluded in principle, but dealing with the resulting coupled differential equations system is certainly a difficult endeavour.

We shall finish remembering that here our analysis was restricted to the flat Friedmann-Robertson-Walker geometry. Generalizations involving the parabolic and hyperbolic curved backgrounds may be achieved. Some applications involving particle creation in more general spacetimes are under investigation.

Acknowledgments

SHP is grateful to CNPq for the financial support, process number 477872/2010-7. JMHS thanks CNPq (482043/2011-3, 308623/2012-6) for partial support. The authors are grateful to Professor Roldão da Rocha for careful reading of the manuscript.

References

[1] D.R. Brill and J.A. Wheeler, *Interaction of neutrinos and gravitational fields*, *Rev. Mod. Phys.* **29** (1957) 465 [INSPIRE].

[2] L. Parker, *Particle creation in expanding universes*, Phys. Rev. Lett. **21** (1968) 562 [INSPIRE].

[3] L. Parker, *Quantized fields and particle creation in expanding universes. 1.*, Phys. Rev. **183** (1969) 1057 [INSPIRE].

[4] L. Parker, *Quantized fields and particle creation in expanding universes. 2.*, Phys. Rev. **D 3** (1971) 346 [*Erratum ibid.* **D 3** (1971) 2546-2546] [INSPIRE].

[5] L. Parker, *Particle creation in isotropic cosmologies*, Phys. Rev. Lett. **28** (1972) 705 [*Erratum ibid.* **28** (1972) 1497] [INSPIRE].

[6] L. Parker, *Conformal energy-momentum tensor in Riemannian space-time*, Phys. Rev. **D 7** (1973) 976 [INSPIRE].

[7] S.W. Hawking, *Black hole explosions*, Nature **248** (1974) 30 [INSPIRE].

[8] S.W. Hawking, *Particle Creation by Black Holes*, Commun. Math. Phys. **43** (1975) 199 [*Erratum ibid.* **46** (1976) 206-206] [INSPIRE].

[9] G.W. Gibbons and S.W. Hawking, *Cosmological Event Horizons, Thermodynamics and Particle Creation*, Phys. Rev. **D 15** (1977) 2738 [INSPIRE].

[10] J. Audretsch and G. Schaefer, *Quantum Mechanics of Electromagnetically Bounded Spin* 1/2 *Particles in an Expanding Universe. 1. Influence of the Expansion*, Gen. Rel. Grav. **9** (1978) 243 [INSPIRE].

[11] L. Parker, *One-electron atom as a probe of space-time curvature*, Phys. Rev. **D 22** (1980) 1922 [INSPIRE].

[12] L.P. Chimento and M.S. Mollerach, *Quantum Vacuum Definition for Spin* 1/2 *Fields in Robertson-walker Metrics*, Phys. Rev. **D 34** (1986) 3689 [INSPIRE].

[13] L.P. Chimento and M.S. Mollerach, *Dirac Equation in Bianchi I Metrics*, Phys. Lett. **A 121** (1987) 7 [INSPIRE].

[14] M.A. Castagnino, C.D. El Hasi, F.D. Mazzitelli and J.P. Paz, *On the Dirac Equation in Anisotropic Backgrounds*, Phys. Lett. **A 128** (1988) 25 [INSPIRE].

[15] A.O. Barut and I.H. Duru, *Exact Solutions of the Dirac Equation in Spatially Flat Robertson-walker Space-times*, Phys. Rev. **D 36** (1987) 3705 [INSPIRE].

[16] V.M. Villalba and U. Percoco, *Separation of Variables and Exact Solution to Dirac and Weyl Equations in Robertson-Walker Space-times*, J. Math. Phys. **31** (1990) 715 [INSPIRE].

[17] V.M. Villalba, *Exact solution of the Dirac equation in a reducible Einstein space*, J. Math. Phys. **31** (1990) 1483 [INSPIRE].

[18] D. Parashar, *Dirac equation in Robertson-Walker metric*, Int. J. Theor. Phys. **30** (1991) 1691 [INSPIRE].

[19] D. Parashar, *Dirac equation in curved space*, Nuov. Cim. **A 103** (1990) 1491.

[20] M. Kovalyov and M. Legare, *The Dirac Equation in Robertson-walker Spaces: A Class of Solutions*, J. Math. Phys. **31** (1990) 191 [INSPIRE].

[21] V.M. Villalba and E.I. Catala, *Separation of variables and exact solution of the Klein-Gordon and Dirac equations in an open universe*, J. Math. Phys. **43** (2002) 4909 [gr-qc/0208017] [INSPIRE].

[22] S.K. Srivastava, *Solution of the Dirac Equation in Kasner's Space-time*, J. Math. Phys. **30** (1989) 2838 [INSPIRE].

[23] J. Fjelstad, *On duality and extended chiral symmetry in the* $\mathrm{SL}(2,\mathbb{R})$ *WZW model*, J. Phys. **A 44** (2011) 235404 [arXiv:1102.4196] [INSPIRE].

[24] D.V. Ahluwalia and D. Grumiller, *Dark matter: A Spin one half fermion field with mass dimension one?*, *Phys. Rev.* **D 72** (2005) 067701 [hep-th/0410192] [INSPIRE].

[25] D.V. Ahluwalia and D. Grumiller, *Spin half fermions with mass dimension one: Theory, phenomenology and dark matter*, *JCAP* **07** (2005) 012 [hep-th/0412080] [INSPIRE].

[26] D.V. Ahluwalia, *On a local mass dimension one Fermi field of spin one-half and the theoretical crevice that allows it*, arXiv:1305.7509 [INSPIRE].

[27] C.G. Boehmer, *The Einstein-Cartan-Elko system*, *Ann. Phys.* **16** (2007) 38.

[28] C.G. Boehmer and J. Burnett, *Dark spinors with torsion in cosmology*, *Phys. Rev.* **D 78** (2008) 104001 [arXiv:0809.0469] [INSPIRE].

[29] C.G. Boehmer and D.F. Mota, *CMB Anisotropies and Inflation from Non-Standard Spinors*, *Phys. Lett.* **B 663** (2008) 168 [arXiv:0710.2003] [INSPIRE].

[30] L. Fabbri, *The Most General Cosmological Dynamics for ELKO Matter Fields*, *Phys. Lett.* **B 704** (2011) 255 [arXiv:1011.1637] [INSPIRE].

[31] C.G. Boehmer, *The Einstein-Elko system - Can dark matter drive inflation?*, *Ann. Phys.* **16** (2007) 325.

[32] C.G. Boehmer, *Dark spinor inflation: Theory primer and dynamics*, *Phys. Rev.* **D 77** (2008) 123535 [arXiv:0804.0616] [INSPIRE].

[33] C.G. Boehmer, J. Burnett, D.F. Mota and D.J. Shaw, *Dark spinor models in gravitation and cosmology*, *JHEP* **07** (2010) 053 [arXiv:1003.3858] [INSPIRE].

[34] D. Gredat and S. Shankaranarayanan, *Modified scalar and tensor spectra in spinor driven inflation*, *JCAP* **01** (2010) 008 [arXiv:0807.3336] [INSPIRE].

[35] A. Basak and J.R. Bhatt, *Lorentz invariant dark-spinor and inflation*, *JCAP* **06** (2011) 011 [arXiv:1104.4574] [INSPIRE].

[36] C.G. Boehmer and J. Burnett, *Dark energy with dark spinors*, *Mod. Phys. Lett.* **A 25** (2010) 101 [arXiv:0906.1351] [INSPIRE].

[37] H. Wei, *Spinor Dark Energy and Cosmological Coincidence Problem*, *Phys. Lett.* **B 695** (2011) 307 [arXiv:1002.4230] [INSPIRE].

[38] S.H. Pereira and J.M. Hoff da Silva, in preparation.

[39] D.V. Ahluwalia, C.-Y. Lee and D. Schritt, *Self-interacting Elko dark matter with an axis of locality*, *Phys. Rev.* **D 83** (2011) 065017 [arXiv:0911.2947] [INSPIRE].

[40] M. Abramowitz and I.A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical tables*, Dover, New York U.S.A. (1965).

[41] A. Erdélyi, F. Oberhettinger, W. Magnus and F. Tricomi, *Higher Transcendental functions*, Vol. 3, McGraw Hill, New York U.S.A. (1953).

[42] A. Ronveaux, *Heun's differential equations*, Oxford Science Publications, Oxford U.K. (1995).

[43] P. Lounesto, *Clifford Algebras and Spinors*, Cambridge University Press, Cambridge U.K. (2002).

[44] R. da Rocha and J. Rodrigues, Waldyr Alves, *Where are ELKO spinor fields in Lounesto spinor field classification?*, *Mod. Phys. Lett.* **A 21** (2006) 65 [math-ph/0506075] [INSPIRE].

[45] A. Dimakis and F. Mueller-Hoissen, *Solutions of the Einstein-cartan-Dirac Equations With Vanishing Energy Momentum Tensor*, *J. Math. Phys.* **26** (1985) 1040 [INSPIRE].

Journal of Cosmology and Astroparticle Physics
An IOP and SISSA journal

Some remarks on the attractor behaviour in ELKO cosmology

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Received February 28, 2014
Revised June 19, 2014
Accepted July 21, 2014
Published August 8, 2014

Abstract. Recent results on the dynamical stability of a system involving the interaction of the ELKO spinor field with standard matter in the universe have been reanalysed, and the conclusion is that such system does not exhibit isolated stable points that could alleviate the cosmic coincidence problem. When a constant parameter δ related to the potential of the ELKO field is introduced in the system however, stable fixed points are found for some specific types of interaction between the ELKO field and matter. Although the parameter δ is related to an unknown potential, in order to satisfy the stability conditions and also that the fixed points are real, the range of the constant parameter δ can be constrained for the present time and the coincidence problem can be alleviated for some specific interactions. Such restriction on the ELKO potential opens possibility to apply the ELKO field as a candidate to dark energy in the universe, and so explain the present phase of acceleration of the universe through the decay of the ELKO field into matter.

Keywords: dark matter theory, cosmology of theories beyond the SM, dark energy theory

ArXiv ePrint: 1402.6723

doi:10.1088/1475-7516/2014/08/020

JCAP08(2014)020

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1 Introduction

The relatively recent discovery of the accelerated expansion of the universe has been one of the most active research in cosmology [1–4]. The search for a candidate that can explain the observational data is a challenge that has drawn the attention of many researchers. In general such mysterious component is named Dark Energy (DE) (see [5–9] for a review). The simplest candidate of DE is the cosmological constant Λ, which might explain most of the current astronomical observations. Another open question in cosmology concerns the Dark Matter (DM) problem (see [10, 11] for a review), which is responsible for the great structures in the universe. The so called ΛCDM model, where CDM stands for Cold Dark Matter, is the best model for the present cosmology. Recent results from the Planck satellite [12] fit quite well with this model. However, from the theoretical point of view such model is plagued with some fundamental problems, thereby stimulating the search for alternative dark energy models [13–23]. Among such alternative models, scalar dynamical fields has been proposed recently as possible candidates [24–36].

Another interesting models deal with the possibility of the coupling between DE and DM. The interaction between these completely different fluids has some important consequences, as addressing the coincidence problem, for instance. The coincidence problem could be alleviated on these models by assuming that the DE decays into DM, thus diminishing the difference between the densities of the two components through the evolution of the Universe. In a series of recent papers the possibility of a coupling between DM and DE has been considered [37–56].

Even more recently, a special kind of non standard spinor field has also been studied both as a DM candidate (from the point of view of quantum field theory) as well as DE (in cosmological applications). This spinor field is the so-called ELKO [57–59], which has some interesting and unusual properties. To begin with, this spinor field is formed by a complete set of eigenspinors of the charge conjugator operator, rendering it neutral under U(1) interactions. Moreover, the field obeys only the Klein-Gordon equation. In other words it has mass dimension one. The conjugation of these characteristics made the field quite attractive from many perspectives within the cosmological setup [60–73].

The possible interaction between the matter in the universe with the ELKO field have been studied from the point of view of dynamical systems [71–73], and stable points of the system have been analysed from different aspects, depending on the choice of the dynamical variables. In [71] and [72] the stability analysis for some specific potentials and interactions leads to attractor points just for critical points where, or the universe is totally ELKO dominated or is totally DM dominated, thus these stable points are not scaling solutions, which means they do not allow the coexistence of DM and ELKO field, which could alleviate the cosmic coincidence problem. In [73] a new choice of variables independent of the potential leads to a new set of stable points, but yet not scaling solutions. It is important to emphasise, however, that the dynamical system analysis performed in [73] starts from dynamical equations containing a subtle (but crucial) mistake. The authors of [73] analyse two different cases, and both are plagued with some misleading,[1] which motivated us to the present work. In fact, starting from the proper equations we were able to show that there is not a stable fixed point for the underling dynamical system in the Case II of [73] while the Case I is very strict or even ill-defined. In order to circumvent this situation, we make use of an additional supposition, introducing a constant parameter related to the potential and the constraint it imposes, extracting physically relevant information about the system.

This paper is organised as follows: section 2 is somewhat a short review about the use of the ELKO field in cosmology, making contact with ref. [73]. In order to make explicit our claim about the crucial difference concerning the dynamical equations and their implications, we present in the appendix the right (slightly modified in comparing with [73]) dynamical equations. Two different stability analysis are performed in the sections 3 and 4, where the last one can alleviate the coincidence problem. In the final section we conclude.

2 The ELKO field in cosmology: dynamical equations

The ELKO spinor action in the curved spacetime is given by

$$S = \frac{1}{2}\int \sqrt{-g}\left(\frac{1}{2}g^{\mu\nu}\left(\nabla_\mu \overset{\neg}{\lambda}_E \nabla_\nu\lambda_E + \nabla_\nu \overset{\neg}{\lambda}_E \nabla_\mu\lambda_E\right) - V\left(\overset{\neg}{\lambda}_E \lambda_E\right)\right)d^4x\,, \tag{2.1}$$

where $V(\overset{\neg}{\lambda}_E \lambda_E)$ is the potential and $g \equiv \det g_{\mu\nu}$. The covariant derivatives acting on the ELKO spinors are $\nabla_\mu \overset{\neg}{\lambda}_E = \partial_\mu \overset{\neg}{\lambda}_E + \overset{\neg}{\lambda}_E \Gamma_\mu$ and $\nabla_\mu\lambda_E = \partial_\mu\lambda_E - \Gamma_\mu\lambda_E$, where Γ_μ are

[1]See the last two paragraphs of the next section.

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the spin connections. The metric in a spatially flat, homogeneous and isotropic Friedmann-Robertson-Walker in a expanding universe is given by

$$ds^2 = dt^2 - a^2(t)(dx^2 + dy^2 + dz^2)\,. \tag{2.2}$$

The ELKO Lagrangian density can be writing as

$$\mathcal{L} = \sqrt{-g}\left[\frac{1}{2}g^{\mu\nu}(\nabla_\mu \overset{\neg}{\lambda}_E\, \nabla_\nu\lambda_E) - V(\overset{\neg}{\lambda}_E\, \lambda_E)\right], \tag{2.3}$$

and the equations of motion follows from a principle of least action for $\mathcal{L}$.

As has been done in recent works [64–66, 69, 71], we restrict the ELKO spinor field to the form $\lambda_E \equiv \phi(t)\xi$ and $\overset{\neg}{\lambda}_E \equiv \phi(t)\, \overset{\neg}{\xi}$, where ξ and $\overset{\neg}{\xi}$ are constant spinors. In [70] it has been presented exact solutions to ELKO spinor in spatially flat Friedmann-Robertson-Walker expanding space times, and it has been shown that such factorisation of the time component of the ELKO field is possible for some types of scale factors.

Due to the homogeneity of the field ($\partial_i\phi = 0$), the equation of motion that follows from (2.3) is substantially simplified to,

$$\ddot{\phi} + 3H\dot{\phi} - \frac{3}{4}H^2\phi + V_{,\phi} = 0\,, \tag{2.4}$$

where $H = \dot{a}/a$ and $V_{,\phi} \equiv dV/d\phi$. The pressure and energy density of spinor dark energy are, according to [66], respectively given by

$$p_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi) - \frac{3}{8}H^2\phi^2 - \frac{1}{4}\dot{H}\phi^2 - \frac{1}{2}H\phi\dot{\phi}\,, \tag{2.5}$$

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi) + \frac{3}{8}H^2\phi^2. \tag{2.6}$$

It is supposed that the universe is filled with only two components, namely a matter energy density ρ_m representing the DM a and a ELKO energy density ρ_ϕ, which could represent the DE for the late time acceleration or the inflaton field for the inflationary epoch. The Friedmann equations in a flat background, the ELKO pressure and energy density can be recast in the form[2]

$$H^2 = \frac{\kappa^2}{3}(\rho_m + \rho_\phi)\,, \tag{2.7}$$

$$\dot{H} = -\frac{\kappa^2}{2}(\rho_m + p_m + \rho_\phi + p_\phi)\,, \tag{2.8}$$

$$p_\phi = X - \tilde{V}, \qquad \rho_\phi = X + \tilde{V}, \tag{2.9}$$

where $\kappa^2 \equiv 8\pi G$ and

$$X = \frac{1}{2}\dot{\phi}^2 - \frac{1}{8}\dot{H}\phi^2 - \frac{1}{4}H\phi\dot{\phi}\,, \tag{2.10}$$

$$\tilde{V} = V(\phi) + \frac{1}{8}\dot{H}\phi^2 + \frac{1}{4}H\phi\dot{\phi} + \frac{3}{8}H^2\phi^2. \tag{2.11}$$

[2] Such a decomposition on the pressure and energy density was introduced by Basak et al. [73].

Despite both new variables do not contain pure kinetic and potential elements we shall call, for simplicity, X and $\tilde{V}$ as the kinetic and potential energy of the field ϕ, respectively. The continuity equations for matter and scalar field are, respectively

$$\dot{\rho}_m + 3H(\rho_m + p_m) = Q\,, \tag{2.12}$$

$$\dot{\rho}_\phi + 3H(\rho_\phi + p_\phi) = -Q\,, \tag{2.13}$$

where Q stands for a possible interaction term between the DM and the ELKO field. If $Q = 0$ there is no interaction and the two components evolve separately. If $Q > 0$ there is the decay of ELKO field into DM, an interesting scenery at the inflation, and if $Q < 0$ we have DM decaying into ELKO field (or DE), an interesting approach to late time acceleration. The matter part is described by a perfect fluid with equation of state $p_m = (\gamma - 1)\rho_m$.

Following [73], it is defined the new variables

$$x = \frac{\kappa\sqrt{X}}{\sqrt{3}H}\,, \qquad y = \frac{\kappa\sqrt{\tilde{V}}}{\sqrt{3}H}\,, \qquad v = \frac{\kappa\sqrt{\rho_m}}{\sqrt{3}H}\,, \tag{2.14}$$

the Friedmann equation (2.7) can be written as a constraint equation

$$x^2 + y^2 + v^2 = 1\,, \tag{2.15}$$

or in terms of the densities parameters, $\Omega_\phi + \Omega_m = 1$, where

$$\Omega_\phi = \frac{\kappa^2\rho_\phi}{3H^2} = x^2 + y^2, \qquad \Omega_m = \frac{\kappa^2\rho_m}{3H^2} = v^2. \tag{2.16}$$

In order to satisfy observational data for a FRW flat universe, it will be imposed the additional condition $0 \leq v^2 \leq 1$ and $0 \leq x^2 + y^2 \leq 1$.

The equations (2.8), (2.12) and (2.13) can be written as a dynamical system of the form (see the appendix for a brief deduction):

$$x' = (\epsilon - 3)x - \frac{\lambda}{2H}\frac{y^2}{x} - \frac{Q_1}{x}\,, \tag{2.17}$$

$$v' = \left(\epsilon - \frac{3}{2}\gamma\right)v + \frac{Q_1}{v}\,, \tag{2.18}$$

$$y' = \left(\epsilon + \frac{\lambda}{2H}\right)y\,, \tag{2.19}$$

where

$$\epsilon \equiv -\frac{\dot{H}}{H^2} = 3x^2 + \frac{3}{2}\gamma v^2, \tag{2.20}$$

and $'$ stands for the derivative with respect to $N \equiv \ln a$, such that $f' = \dot{f}/H$ for any function f. We reinforce the appearance of a 1/2 factor in the second term of the right-hand side of eq. (2.17). The following parameters are defined: $\lambda = \frac{\dot{\tilde{V}}}{\tilde{V}}$ and $Q_1 = \frac{\kappa^2 Q}{6H^3}$. ϵ is related to the decelerated parameter q according to

$$q \equiv -\frac{\ddot{a}}{aH^2} = \epsilon - 1\,, \tag{2.21}$$

so that the expansion is accelerated for $q < 0$ (or $\epsilon < 1$) and decelerated for $q > 0$ (or $\epsilon > 1$). Specifically, it is important to note that recent observational results from the Planck satellite measurements of the CMB temperature and lensing-potential power spectra [12] gives $q_0 \simeq -0.527$ for the present deceleration parameter, with $\Omega_m \simeq 0.315$ and $\Omega_\Lambda \simeq 0.685$ (31.5% of dust matter in the universe and 68.5% of dark energy, or cosmological constant, responsible for the present accelerated expansion). Another interesting scenery concerns the inflation, which must have $q \to -1$ (or $\epsilon \to 0$), so that the expansion could be nearly exponential or a de Sitter evolution.

The above three dynamical equations (2.17), (2.18) and (2.19) are exactly the same as obtained by Basak et al. [73], except by the factor 2 in the denominator of the second term in the right-hand side term of eq. (2.17). Such missing factor, as we will show later in next section, took the authors of [73] to a misplaced result about stability in this system.

It is important to notice that, indeed, the above system is not yet in a true dynamical system form, since that it contains the term $\lambda/2H$, which is clearly dependent on the dynamical variables by the term λ, which is defined as $\dot{\tilde{V}}/\tilde{V}$ and $\tilde{V}$ is explicitly y dependent. There are two ways to solve this problem. First we can suppose that such term is a function of the other dynamical variables, namely $\lambda/2H = f(x, v, y)$, so we have a well-defined dynamical system. Another possibility is to setting $\lambda/2H$ as a constant, so that the dynamical system is also well-defined. According to the definition of the λ parameter, such constant is related to the potential $V(\phi)$ of the ELKO field.

In order to study the stability of the above system, a trivial way to satisfy $y' = 0$ is take $y = 0$ (which corresponds to Case I of [73]). However this condition is very restrictive, since it implies $\tilde{V} = 0$, which represents a very particular choice for the potential. The case $y \neq 0$ is much more general, which justifies our new stability analysis.

3 Stability analysis with $\frac{\lambda}{2H} = -\epsilon$

In order to turn the above system of equation in a true dynamical system and study its stability for different types of interaction term Q, we impose the condition[3] $\epsilon = -\frac{\lambda}{2H}$. It is easy to see that $\frac{\lambda}{2H}$ is defined as a dynamical quantity by means of the parameter ϵ. This automatically satisfies $y' = 0$, and the system (2.17) and (2.18) turns to:

$$x' = -3xv^2 + \frac{3}{2}\gamma\frac{v^2}{x}(1-v^2) - \frac{Q_1}{x}\,, \tag{3.1}$$

$$v' = 3vx^2 - \frac{3}{2}\gamma v(1-v^2) + \frac{Q_1}{v}\,. \tag{3.2}$$

The associated linearised matrix, ensured by the topological equivalence settled by the Hartmann-Grobman theorem [75], is given by

$$\begin{pmatrix} \delta x' \\ \delta v' \end{pmatrix} = M \begin{pmatrix} \delta x \\ \delta v \end{pmatrix}, \tag{3.3}$$

where

$$M = \begin{pmatrix} -3v^2 - \frac{3}{2}\gamma\frac{v^2}{x^2}(1-v^2) + \frac{Q_1}{x^2} - \frac{1}{x}\frac{\partial Q_1}{\partial x} & -6vx + \frac{3\gamma v}{x}(1-2v^2) - \frac{1}{x}\frac{\partial Q_1}{\partial v} \\ 6xv + \frac{1}{v}\frac{\partial Q_1}{\partial x} & 3x^2 - \frac{3}{2}\gamma(1-3v^2) - \frac{Q_1}{v^2} + \frac{1}{v}\frac{\partial Q_1}{\partial v} \end{pmatrix}. \tag{3.4}$$

δx and δy are the infinitesimal displacements about the fixed points.

[3]This condition corresponds to the Case II analysed by Basak et al. [73]. But, again, in [73] there is an important missing factor.

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The stability of the system at a fixed point can be obtained from the standard analysis of the determinant (Δ) and the trace (τ) of the matrix M. According to the usual dynamical system theory, if $\Delta < 0$ the eigenvalues are real and have opposite signs, hence the corresponding fixed point is a saddle point. On the other hand, if $\Delta > 0$ and $\tau < 0$ the fixed point is stable, whilst if $\Delta > 0$ and $\tau > 0$ the fixed point is unstable [75]. The fixed points $(\bar{x}, \bar{v})$ for which the above system satisfies $x' = 0$ and $v' = 0$ depends on the choice of the interaction term Q, and several possibilities will be treated in the sequel. Here we consider only the case where the matter part is pressureless, thus we take $\gamma = 1$ from now on.

3.1 $Q_1 = 0$

Here we have $Q = 0$, and consequently, there is no interaction between the standard matter and the ELKO field. Such interaction was treated by Wei [71] with choice of the variables other than (2.14), and no stable point was found. The fixed points of the system (3.1)–(3.2) are given by $[\bar{x} = x,\, \bar{v} = 0]$ and $\left[\bar{x} = \pm\sqrt{\frac{1}{2}(1-\bar{v}^2)},\, \bar{v} = v\right]$. For the first fixed point we have $\Delta = 0$, so we do not have any information about the stability of the system. Furthermore, $\bar{v} = 0$ infers that $\Omega_m = 0$ and $\Omega_\phi = 1$ (from (2.15) and (2.16)) a fully Dark Spinor dominated universe. The second fixed point also has $\Delta = 0$.

3.2 $Q_1 = \beta$

In this case β is constant. If we redefine $\beta = \frac{3}{2}\beta'$, we have $Q = 6\beta\kappa^2 H^3 = 3\beta' H(\rho_\phi + \rho_m)$, an interaction term also treated by Wei [71]. The fixed points are $\left[\bar{x} = \pm\sqrt{\frac{1}{2}(1-\bar{v}^2) - \frac{\beta}{3\bar{v}^2}},\, \bar{v} = v\right]$ and it is easy to show that $\Delta = 0$.

3.3 $Q_1 = \beta v^2$

In this case we have $Q = 2\beta H\rho_m$. The fixed points are $[\bar{x} = x,\, \bar{v} = 0]$ and $\left[\bar{x} = x,\, \bar{v} = \pm\sqrt{(1-2\bar{x}^2) - \frac{2}{3}\beta}\right]$ and nothing can be said about the stability of the fixed points, since $\Delta = 0$ in both cases.

3.4 $Q_1 = \beta x^2$

At this time we have an interaction of the form $Q = \beta H(\rho_\phi + p_\phi)$. The fixed points are $\left[\bar{x} = \pm\sqrt{\frac{3}{2}\frac{(1-\bar{v}^2)}{(3\bar{v}^2+\beta)}}\bar{v},\, \bar{v} = v\right]$, and again $\Delta = 0$.

3.5 $Q_1 = \beta v x^2$

For this kind of interaction $Q = \frac{1}{\sqrt{3}}\beta\kappa\sqrt{\rho_m}(\rho_\phi + p_\phi)$. The fixed points are $\left[\bar{x} = \pm\sqrt{\frac{3}{2}\bar{v}\frac{(1-\bar{v}^2)}{(3\bar{v}+\beta)}},\, \bar{v} = v\right]$, and it can be easily obtained that $\Delta = 0$.

3.6 $Q_1 = \beta x v^2$

In this case we have $Q = \sqrt{\frac{2}{3}}\beta\kappa\rho_m\sqrt{\rho_\phi + p_\phi}$. The fixed points are $[\bar{x} = x,\, \bar{v} = 0]$ and $\left[\bar{x} = x,\, \bar{v} = \pm\sqrt{1 - 2\bar{x}^2 - \frac{2}{3}\beta\bar{x}}\right]$ and $\Delta = 0$ in both cases.

3.7 $Q_1 = \beta x^2 v^2$

Here we have $Q = \frac{1}{3}\beta\kappa^2\rho_m(\rho_\phi + p_\phi)$. The fixed points are $[\bar{x} = x,\ \bar{v} = 0]$ and $\left[\bar{x} = x,\right.$ $\left.\bar{v} = \pm\sqrt{1 - 2\bar{x}^2 - \frac{2}{3}\beta\bar{x}^2}\right]$ and, as the other cases, we have $\Delta = 0$.

Having analysed all the previous cases where the determinant is always zero, we have used an algebraic manipulation software to test different functions. The functions analysed were of the types: $x^n f(v)$ and $v^n f(x)$, with $n = 1, 2, 3, 4$, and for all of them the determinant is always zero. This leads us to the conclusion that the new choice of variables keeps the same results studied by Wei [71], where no point of stability has been found for different types of interaction.

4 Stability analysis with $\frac{\lambda}{2H} = -\delta$

The above study show us that the dynamical system characterised by the equations (2.17)–(2.18) with the Friedmann constraint (2.15) does not presents an isolated fixed point, since a null determinant means that at least one eigenvalue is zero, and there is either a whole line of fixed points on x or v axis. In order to circumvent this situation we shall investigate the subsequent dynamical system for the case in which another (physical) constraint can be used to select attractor points with physical meaning. We suppose that the potential $V(\phi)$ is such that λ satisfies $\frac{\lambda}{2H} = -\delta$, where δ is a constant.

The δ parameter just reflects our ignorance about the potential, since it is related to potential but the specific form of the potential is not required in this analysis. Physically, the conditions of stability satisfied by the parameter δ will show the ranges of possibilities for the potential in order to have a stable system. In other words, what are the restrictions on the potential. Besides the δ parameter, all the interactions Q under analysis are characterised by a coupling constant β. According to (2.12) and (2.13), positive values of β correspond to positive values of Q, which means an increase to DM energy density and a decrease of the ELKO energy density, in other words, decay of ELKO into DM particles. On the other side, negative β values leads to decay of DM into ELKO field.

The cosmic coincidence problem can be alleviated if DM and DE (here represented by the ELKO field) could coexist for the present time of the evolution of the universe. This implies $\rho_m \neq 0$ (which is related to $\bar{v}^2$) simultaneously with $\rho_\phi \neq 0$ (which is related to $\bar{x}^2 + \bar{y}^2$). For this reason, in which follows, we will be interested in fixed points satisfying such conditions.

The corresponding dynamical system obtained from (2.17)–(2.19) is:

$$x' = 3x\left(x^2 - 1 + \frac{\gamma}{2}v^2\right) + \frac{\delta}{x}(1 - x^2 - v^2) - \frac{Q_1}{x}\,, \tag{4.1}$$

$$v' = 3vx^2 - \frac{3}{2}\gamma v(1 - v^2) + \frac{Q_1}{v}\,, \tag{4.2}$$

and the fixed points are chosen such that $\bar{\epsilon} = \delta$, thus the equation for y at the fixed point is $y' = 0$ even for $\bar{y} \neq 0$.

4.1 $Q_1 = 0$

For this case it is possible to find two types of fixed points, however there is only one relevant for present purpose. The first fixed point is $\bar{x} = \pm 1$, which represents $\bar{y} = 0$ and $\bar{v} = 0$. This

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case could represent only the inflationary period and is not a scaling solution. According to our previous discussion we are interested only in the case $\bar{y} \neq 0$. The other fixed point is given by $\left[\bar{x} = \frac{\sqrt{3}}{3}\sqrt{\delta},\, \bar{y} = \sqrt{1 - \frac{1}{3}\delta},\, \bar{v} = 0\right]$, and the conditions to guarantee stability ($\Delta > 0$ and $\tau < 0$) is simply $\delta < \frac{3}{2}$. As we have $\bar{v} = 0$ such fixed point is not a scaling solution too.

4.2 $Q_1 = \beta$

By taking a constant interaction between ELKO and standard matter we find two fixed points that solve the dynamical system. But, as in the last case, there is a restriction in one of them since $\bar{y} = 0$, which lead us to consider only $\left[\bar{x} = \sqrt{\frac{2\delta^2 - 3\delta + 3\beta}{6\delta - 9}},\, \bar{y} = \sqrt{\frac{-2\delta^2 + 9\delta + 3\beta - 9}{6\delta - 9}},\right.$ $\left.\bar{v} = \sqrt{\frac{2\beta}{3 - 2\delta}}\right]$. The conditions to ensure stability are $\beta \geq -\frac{3}{8}$ and $\delta < \frac{9}{4} - \frac{1}{4}\sqrt{9 + 24\beta}$. However, in order to have real fixed points, namely $\bar{v}^2 \geq 0$, $\bar{x}^2 \geq 0$ and $\bar{y}^2 \geq 0$, the condition turns $\frac{3}{4} - \frac{1}{4}\sqrt{9 - 24\beta} < \delta < \frac{3}{4} + \frac{1}{4}\sqrt{9 - 24\beta}$ for $0 < \beta < \frac{3}{8}$. For $\beta \to 0$, we have $0 < \delta < \frac{3}{2}$, and if $\beta \to \frac{3}{8}$ we have $\delta \to \frac{3}{4}$. This shows that such type of interaction can alleviate the coincidence problem if the above conditions are satisfied.

4.3 $Q_1 = \beta x^2$

This case is similar to the last one and we have again two fixed points, being one of them also meaningless because $\bar{y} = 0$. The remaining fixed point is $\left[\bar{x} = \sqrt{\frac{3\delta - 2\delta^2}{9 - 6\delta + 3\beta}},\right.$ $\left.\bar{y} = \sqrt{\frac{9 + 3\beta - 6\delta - 2\beta\delta + 2\delta^2}{9 - 6\delta + 3\beta}},\, \bar{v} = \sqrt{\frac{2\beta\delta}{9 - 6\delta + 3\beta}}\right]$, where it is necessary $\beta \leq -\frac{3}{2}$ and $\delta < 3 + \beta$ or $\beta > -\frac{3}{2}$ and $\delta < \frac{3}{2}$ for such fixed point satisfy the stability condition. However, in order to have real fixed points, these conditions reduce simply to $\beta > 0$ and $0 < \delta < \frac{3}{2}$. For the present time, where $\bar{v}^2 = \Omega_m = 0.315$, we have $\delta = \frac{3}{2}\frac{\Omega_m(3+\beta)}{\beta + 3\Omega_m}$. Thus, in order to satisfy all the conditions we must have $\frac{3}{2}\Omega_m < \delta < \frac{3}{2}$ if $0 < \beta < \infty$, hence scalling solutions for the present time can be obtained only if ELKO field decays into matter ($\beta > 0$) and the δ parameter is limited to the above range. Under these conditions the interaction βx^2 could alleviate the coincidence problem.

4.4 $Q_1 = \beta v^2$

The present case has a fixed available point as being $\left[\bar{x} = \frac{\sqrt{3}}{3}\sqrt{\delta},\, \bar{y} = \sqrt{1 - \frac{1}{3}\delta},\, \bar{v} = 0\right]$. It is easy to see from eq. (4.2) that $\bar{v} = 0$ turns the parameter Q_1 identically equal to zero and trivially satisfies such equation. The conditions for stability for the present fixed point are: (i) $\beta \geq -\frac{3}{2}$ if $\delta < \frac{3}{2} - \beta$; and (ii) $\beta < -\frac{3}{2}$ if $\delta < 3$. Although it has stable points it is not a scaling solution.

The another solution with $\bar{v} \neq 0$ leads to $\bar{y} = 0$.

4.5 $Q_1 = \beta v^2 x^2$

In this interaction we have three types of fixed points. One of them with $\bar{y} = 0$ and two with $\bar{y} \neq 0$. For these last two cases we have $\left[\bar{x} = \frac{\sqrt{3}}{3}\sqrt{\delta},\, \bar{y} = \sqrt{1 - \frac{1}{3}\delta},\, \bar{v} = 0\right]$, under the conditions $\beta < \frac{9 - 6\delta}{2\delta}$ if $0 < \delta < 3$ and $\beta > \frac{9 - 6\delta}{2\delta}$ if $\delta < 0$ for stability. Although it is a stable point it is not a scaling solution. The last point is much more interesting, since that $\bar{v} \neq 0$. It is given by $\left[\bar{x} = \sqrt{\frac{(3 - 2\delta)}{2\beta}},\, \bar{y} = \sqrt{\frac{(3 - 2\delta)(3 + 2\beta)}{6\beta}},\, \bar{v} = \sqrt{\frac{6\delta + 2\delta\beta - 9}{3\beta}}\right]$. The stability conditions leads to the following conditions: (i) $\delta < \frac{9}{2(3+\beta)}$ if $\beta < -3$; (ii) $\frac{9(\beta+6-\sqrt{-\beta^2-3\beta})}{2\beta^2+15\beta+36} < \delta < \frac{3}{2}$ or $\frac{9}{2(3+\beta)} < \delta < \frac{9(\beta+6+\sqrt{-\beta^2-3\beta})}{2\beta^2+15\beta+36}$ if $-\frac{3}{2} < \beta < 0$; and (iii) $\frac{9}{2(3+\beta)} < \delta < \frac{3}{2}$ if $\beta > 0$.

The above three conditions ensures the stability of the system. However, in order to also satisfy the condition of reality of the fixed points, namely $\bar{v}^2 \geq 0$, $\bar{x}^2 \geq 0$ and $\bar{y}^2 \geq 0$, the only possible condition is the last one, $\frac{9}{2(3+\beta)} < \delta < \frac{3}{2}$ for $\beta > 0$. Thus it is only possible to alleviate the cosmic coincidence problem if β is positive, which means the ELKO field decaying into DM. For a small β coupling ($\beta \to 0$), we must have $\delta \to \frac{3}{2}$, while for $\beta \to \infty$ we must have $0 < \delta < \frac{3}{2}$ in order to maintain the stability of the system.

For the present time for instance, where $\bar{v}^2 = \Omega_m = 0.315$, we must have $\delta = \frac{3}{2}\frac{\Omega_m\beta+3}{\beta+3}$ from the fixed point $\bar{v}$. For $\beta \to 0$ we have $\delta \to \frac{3}{2}$ while for $\beta \to \infty$ we have $\delta \to \frac{3}{2}\Omega_m$. Curiously, such condition is the same as the one obtained in case of the interaction βx^2 above.

We have also analysed the interactions given by $Q_1 = \beta v x^2$ and $Q_1 = \beta v^2 x$. For the first one it was found two stable fixed points satisfying $\bar{y} \neq 0$. For the second case there is one stable fixed point. The conditions for stability are very cumbersome, so they are omitted here.

5 Concluding remarks

In this work it has been analysed the dynamical system concerning the study of an interacting Dark Matter model with ELKO fields. Due to an incorrect factor in the evolution equations present in the ref. [73], one of the two cases there analysed leads to an inconsistent result. In their analysis it is possible to find out stable points for some interaction terms between DM and the ELKO field. However, with the correct factor in the evolution equations, we have shown that for several interaction terms there are no attractor points.

Contrary to recent works where the potential is taken as general or assume specific forms but the systems does not present stable points or not represent scaling solutions between the ELKO field and matter, here it is assumed that the potential satisfies a differential equation characterized by a constant parameter δ, and stable solutions are found. Certainly, the study of possible interaction terms between ELKO and matter fields within the scope of Friedmann-Robertson-Walker backgrounds is far from trivial. The associated dynamical system is quite involved, and extracting relevant physical information is a rather difficult task. Interestingly enough, we have found some conditions on the β and δ parameters under which the system presents stability. Specifically, for the interactions B, C and E of the section 4 were found fixed stable points in order to alleviate the cosmic coincidence problem. For the interactions C and E it was found that the range of δ is related to the matter density parameter Ω_m according to $\frac{3}{2}\Omega_m < \delta < \frac{3}{2}$. Such constrain on the δ parameter, when satisfied by the potential, opens the possibility to apply the ELKO field as a candidate to dark energy in the universe, and so explain the present phase of acceleration of the universe through the decay of the ELKO field into matter.

Acknowledgments

SHP is grateful to CNPq — Conselho Nacional de Desenvolvimento Científico e Tecnológico, Brazilian research agency, for the financial support, process number 477872/2010-7. JMHS thanks to CNPq (482043/2011-3; 308623/2012-6) for financial support.

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A Dynamical system equations

Here we briefly present the deduction of the dynamical system equations, namely eqs. (2.17)–(2.19). The main goal is to clarify the appearance of the missing factor 2 in equation of x' from Basak et al. [73].

By taking eqs. (2.9), the derivative $\dot{\rho_\phi} = \rho'_\phi H$ and $\rho'_\phi = X' + \tilde{V}'$ into (2.13) we arrive at

$$X' + 6X + \tilde{V}' = -\frac{Q}{H}\,. \tag{A.1}$$

Taking the derivative $'$ of x^2 from (2.14) we have

$$X' = 6xx'\frac{H^2}{\kappa^2} + 6x^2\frac{H'H}{\kappa^2}\,. \tag{A.2}$$

Using $\frac{H'}{H} = \frac{\dot{H}}{H^2} = -\epsilon$ into (A.2) and then substituting into (A.1) it is possible, after rearranging terms, to get

$$x' = (\epsilon - 3)x - \frac{\lambda}{2H}\frac{y^2}{x} - \frac{Q_1}{x}\,, \tag{A.3}$$

where $\lambda = \frac{\dot{\tilde{V}}}{\tilde{V}}$ and $Q_1 = \frac{\kappa^2 Q}{6H^3}$.

The expression for v' can be derived in a similar manner. By using (2.12) we have

$$\rho'_m + 3\gamma\rho_m = \frac{Q}{H}\,. \tag{A.4}$$

Taking the derivative $'$ of v^2 from (2.14) we have

$$\rho'_m = 6vv'\frac{H^2}{\kappa^2} + 2\rho_m\frac{H'}{H}\,. \tag{A.5}$$

As before, we write H' in terms of ϵ and then substitute the result into (A.4). Doing some simple manipulations it is possible find that

$$v' = \left(\epsilon - \frac{3}{2}\gamma\right)v + \frac{Q_1}{v}\,. \tag{A.6}$$

Finally, the expression for y' can be obtained by taking the derivative $'$ of y^2 from (2.14) and using the above definitions for λ and ϵ, and also that $\dot{\tilde{V}} = \tilde{V}'H$. After rearrange the terms we have

$$y' = \left(\epsilon + \frac{\lambda}{2H}\right)y\,. \tag{A.7}$$

References

[1] Supernova Search Team collaboration, A.G. Riess et al., *Observational evidence from supernovae for an accelerating universe and a cosmological constant*, *Astron. J.* **116** (1998) 1009 [astro-ph/9805201] [INSPIRE].

[2] Supernova Cosmology Project collaboration, S. Perlmutter et al., *Measurements of* Ω *and* Λ *from 42 high redshift supernovae*, *Astrophys. J.* **517** (1999) 565 [astro-ph/9812133] [INSPIRE].

[3] SNLS collaboration, P. Astier et al., *The Supernova Legacy Survey: measurement of* Ω_M, Ω_Λ *and* w *from the first year data set*, Astron. Astrophys. **447** (2006) 31 [astro-ph/0510447] [INSPIRE].

[4] A.G. Riess et al., *New Hubble Space Telescope discoveries of type Ia supernovae at* $z \geq 1$*: narrowing constraints on the early behavior of dark energy*, Astrophys. J. **659** (2007) 98 [astro-ph/0611572] [INSPIRE].

[5] V. Sahni and A.A. Starobinsky, *The case for a positive cosmological* Λ*-term*, Int. J. Mod. Phys. **D 9** (2000) 373 [astro-ph/9904398] [INSPIRE].

[6] T. Padmanabhan, *Cosmological constant: the weight of the vacuum*, Phys. Rept. **380** (2003) 235 [hep-th/0212290] [INSPIRE].

[7] P.J.E. Peebles and B. Ratra, *The cosmological constant and dark energy*, Rev. Mod. Phys. **75** (2003) 559 [astro-ph/0207347] [INSPIRE].

[8] J.A.S. Lima, *Alternative dark energy models: an overview*, Braz. J. Phys. **34** (2004) 194 [astro-ph/0402109] [INSPIRE].

[9] V. Sahni and A. Starobinsky, *Reconstructing dark energy*, Int. J. Mod. Phys. **D 15** (2006) 2105 [astro-ph/0610026] [INSPIRE].

[10] M. Davis, G. Efstathiou, C.S. Frenk and S.D.M. White, *The evolution of large scale structure in a universe dominated by cold dark matter*, Astrophys. J. **292** (1985) 371 [INSPIRE].

[11] G. Bertone et al., *Particle dark matter: observations, models and searches*, Cambridge University Press, Cambridge U.K. (2010) [INSPIRE].

[12] Planck collaboration, P.A.R. Ade et al., *Planck 2013 results. XVI. Cosmological parameters*, arXiv:1303.5076 [INSPIRE].

[13] S. Weinberg, *The cosmological constant problem*, Rev. Mod. Phys. **61** (1989) 1 [INSPIRE].

[14] D. Pavón, *Nonequilibrium fluctuations in cosmic vacuum decay*, Phys. Rev. **D 43** (1991) 375 [INSPIRE].

[15] J.C. Carvalho, J.A.S. Lima and I. Waga, *Cosmological consequences of a time-dependent* Λ *term*, Phys. Rev. **D 46** (1992) 2404 [INSPIRE].

[16] J.A.S. Lima and M. Trodden, *Decaying vacuum energy and deflationary cosmology in open and closed universes*, Phys. Rev. **D 53** (1996) 4280 [astro-ph/9508049] [INSPIRE].

[17] S. Carneiro and J.A.S. Lima, *Time dependent cosmological term and holography*, Int. J. Mod. Phys. **A 20** (2005) 2465 [gr-qc/0405141] [INSPIRE].

[18] I.L. Shapiro, J. Solà and H. Štefančić, *Running* G *and* Λ *at low energies from physics at* M_X*: possible cosmological and astrophysical implications*, JCAP **01** (2005) 012 [hep-ph/0410095] [INSPIRE].

[19] E. Elizalde, S. Nojiri, S.D. Odintsov and P. Wang, *Dark energy: vacuum fluctuations, the effective phantom phase and holography*, Phys. Rev. **D 71** (2005) 103504 [hep-th/0502082] [INSPIRE].

[20] J.S. Alcaniz and J.A.S. Lima, *Interpreting cosmological vacuum decay*, Phys. Rev. **D 72** (2005) 063516 [astro-ph/0507372] [INSPIRE].

[21] J.V. Cunha, J.S. Alcaniz and J.A.S. Lima, *Cosmological constraints on Chaplygin gas dark energy from galaxy clusters X-ray and supernova data*, Phys. Rev. **D 69** (2004) 083501 [astro-ph/0306319] [INSPIRE].

[22] F.C. Carvalho, J.S. Alcaniz, J.A.S. Lima and R. Silva, *Scalar-field-dominated cosmology with a transient accelerating phase*, Phys. Rev. Lett. **97** (2006) 081301 [astro-ph/0608439] [INSPIRE].

[23] L.R. Abramo and N. Pinto-Neto, *Stability of phantom k-essence theories*, *Phys. Rev.* **D 73** (2006) 063522 [astro-ph/0511562] [INSPIRE].

[24] Y.M. Cho and Y.Y. Keum, *Dilatonic dark matter and unified cosmology: a new paradigm*, *Class. Quant. Grav.* **15** (1998) 907 [INSPIRE].

[25] R. Dick, *The dilaton as a candidate for dark matter*, *Rev. Mex. Fis.* **41** (1996) 791.

[26] T. Matos, F.S. Guzmán and L.A. Ureña-López, *Scalar field as dark matter in the universe*, *Class. Quant. Grav.* **17** (2000) 1707 [astro-ph/9908152] [INSPIRE].

[27] T. Matos and F.S. Guzmán, *Scalar fields as dark matter in spiral galaxies*, *Class. Quant. Grav.* **17** (2000) L9 [gr-qc/9810028] [INSPIRE].

[28] A. Suárez, V.H. Robles and T. Matos, *A review on the scalar field/Bose-Einstein condensate dark matter model*, *Astrophys. Space Sci. Proc.* **38** (2014) 107 [arXiv:1302.0903] [INSPIRE].

[29] T. Matos, A. Vázquez-González and J. Magaña, *ϕ^2 as dark matter*, *Mon. Not. Roy. Astron. Soc.* **393** (2009) 1359 [arXiv:0806.0683] [INSPIRE].

[30] E.J. Copeland, M. Sami and S. Tsujikawa, *Dynamics of dark energy*, *Int. J. Mod. Phys.* **D 15** (2006) 1753 [hep-th/0603057] [INSPIRE].

[31] L.A. Ureña-López, *Bose-Einstein condensation of relativistic scalar field dark matter*, *JCAP* **01** (2009) 014 [arXiv:0806.3093] [INSPIRE].

[32] L.A. Ureña-López and M.J. Reyes-Ibarra, *On the dynamics of a quadratic scalar field potential*, *Int. J. Mod. Phys.* **D 18** (2009) 621 [arXiv:0709.3996] [INSPIRE].

[33] J. Magaña, T. Matos, A. Suárez and F.J. Sánchez-Salcedo, *Structure formation with scalar field dark matter: the field approach*, *JCAP* **10** (2012) 003 [arXiv:1204.5255] [INSPIRE].

[34] V.H. Robles and T. Matos, *Flat central density profile and constant DM surface density in galaxies from scalar field dark matter*, *Mon. Not. Roy. Astron. Soc.* **422** (2012) 282 [arXiv:1201.3032] [INSPIRE].

[35] C.G. Böhmer and T. Harko, *Can dark matter be a Bose-Einstein condensate?*, *JCAP* **06** (2007) 025 [arXiv:0705.4158] [INSPIRE].

[36] V.H. Robles and T. Matos, *Exact solution to finite temperature SFDM: natural cores without feedback*, *Astrophys. J.* **763** (2013) 19 [arXiv:1207.5858] [INSPIRE].

[37] S. Das, P.S. Corasaniti and J. Khoury, *Super-acceleration as signature of dark sector interaction*, *Phys. Rev.* **D 73** (2006) 083509 [astro-ph/0510628] [INSPIRE].

[38] C. Feng, B. Wang, Y. Gong and R.-K. Su, *Testing the viability of the interacting holographic dark energy model by using combined observational constraints*, *JCAP* **09** (2007) 005 [arXiv:0706.4033] [INSPIRE].

[39] R. Mainini and S. Bonometto, *Mass functions in coupled dark energy models*, *Phys. Rev.* **D 74** (2006) 043504 [astro-ph/0605621] [INSPIRE].

[40] G.M. Kremer, *Dark energy interacting with neutrinos and dark matter: a phenomenological theory*, *Gen. Rel. Grav.* **39** (2007) 965 [arXiv:0704.0371] [INSPIRE].

[41] C.G. Böhmer, G. Caldera-Cabral, R. Lazkoz and R. Maartens, *Dynamics of dark energy with a coupling to dark matter*, *Phys. Rev.* **D 78** (2008) 023505 [arXiv:0801.1565] [INSPIRE].

[42] W. Zimdahl, D. Pavón and L.P. Chimento, *Interacting quintessence*, *Phys. Lett.* **B 521** (2001) 133 [astro-ph/0105479] [INSPIRE].

[43] L.P. Chimento, A.S. Jakubi, D. Pavón and W. Zimdahl, *Interacting quintessence solution to the coincidence problem*, *Phys. Rev.* **D 67** (2003) 083513 [astro-ph/0303145] [INSPIRE].

[44] S. del Campo, R. Herrera and D. Pavón, *Late universe expansion dominated by domain walls and dissipative dark matter*, *Phys. Rev.* **D 70** (2004) 043540 [astro-ph/0407047] [INSPIRE].

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[45] D. Pavon and W. Zimdahl, *Holographic dark energy and cosmic coincidence*, *Phys. Lett.* **B 628** (2005) 206 [gr-qc/0505020] [INSPIRE].

[46] B. Wang, Y.-g. Gong and E. Abdalla, *Transition of the dark energy equation of state in an interacting holographic dark energy model*, *Phys. Lett.* **B 624** (2005) 141 [hep-th/0506069] [INSPIRE].

[47] B. Wang, C.-Y. Lin and E. Abdalla, *Constraints on the interacting holographic dark energy model*, *Phys. Lett.* **B 637** (2006) 357 [hep-th/0509107] [INSPIRE].

[48] B. Wang, J. Zang, C.-Y. Lin, E. Abdalla and S. Micheletti, *Interacting dark energy and dark matter: observational constraints from cosmological parameters*, *Nucl. Phys.* **B 778** (2007) 69 [astro-ph/0607126] [INSPIRE].

[49] E. Abdalla, L.R.W. Abramo, L. Sodré Jr. and B. Wang, *Signature of the interaction between dark energy and dark matter in galaxy clusters*, *Phys. Lett.* **B 673** (2009) 107 [arXiv:0710.1198] [INSPIRE].

[50] B. Wang, C.-Y. Lin, D. Pavón and E. Abdalla, *Thermodynamical description of the interaction between dark energy and dark matter*, *Phys. Lett.* **B 662** (2008) 1 [arXiv:0711.2214] [INSPIRE].

[51] D. Pavón and B. Wang, *Le Chatelier-Braun principle in cosmological physics*, *Gen. Rel. Grav.* **41** (2009) 1 [arXiv:0712.0565] [INSPIRE].

[52] P. Wang and X.-H. Meng, *Can vacuum decay in our universe?*, *Class. Quant. Grav.* **22** (2005) 283 [astro-ph/0408495] [INSPIRE].

[53] J.F. Jesus, *Constraints from the old quasar APM 08279+5255 on two classes of $\Lambda(t)$-cosmologies*, *Gen. Rel. Grav.* **40** (2008) 791 [astro-ph/0603142] [INSPIRE].

[54] Z.-K. Guo, N. Ohta and S. Tsujikawa, *Probing the coupling between dark components of the universe*, *Phys. Rev.* **D 76** (2007) 023508 [astro-ph/0702015] [INSPIRE].

[55] M. Quartin, M.O. Calvão, S.E. Jorás, R.R.R. Reis and I. Waga, *Dark interactions and cosmological fine-tuning*, *JCAP* **05** (2008) 007 [arXiv:0802.0546] [INSPIRE].

[56] S.H. Pereira and J.F. Jesus, *Can dark matter decay in dark energy?*, *Phys. Rev.* **D 79** (2009) 043517 [arXiv:0811.0099] [INSPIRE].

[57] D.V. Ahluwalia and D. Grumiller, *Dark matter: a spin one half fermion field with mass dimension one?*, *Phys. Rev.* **D 72** (2005) 067701 [hep-th/0410192] [INSPIRE].

[58] D.V. Ahluwalia and D. Grumiller, *Spin half fermions with mass dimension one: theory, phenomenology and dark matter*, *JCAP* **07** (2005) 012 [hep-th/0412080] [INSPIRE].

[59] D.V. Ahluwalia, *On a local mass dimension one Fermi field of spin one-half and the theoretical crevice that allows it*, arXiv:1305.7509 [INSPIRE].

[60] C.G. Böhmer, *The Einstein-Cartan-Elko system*, *Annalen Phys.* **16** (2007) 38 [gr-qc/0607088] [INSPIRE].

[61] C.G. Böhmer and J. Burnett, *Dark spinors with torsion in cosmology*, *Phys. Rev.* **D 78** (2008) 104001 [arXiv:0809.0469] [INSPIRE].

[62] C.G. Böhmer and D.F. Mota, *CMB anisotropies and inflation from non-standard spinors*, *Phys. Lett.* **B 663** (2008) 168 [arXiv:0710.2003] [INSPIRE].

[63] L. Fabbri, *The most general cosmological dynamics for ELKO matter fields*, *Phys. Lett.* **B 704** (2011) 255 [arXiv:1011.1637] [INSPIRE].

[64] C.G. Böhmer, *The Einstein-Elko system — can dark matter drive inflation?*, *Annalen Phys.* **16** (2007) 325 [gr-qc/0701087] [INSPIRE].

[65] C.G. Böhmer, *Dark spinor inflation: theory primer and dynamics*, *Phys. Rev.* **D 77** (2008) 123535 [arXiv:0804.0616] [INSPIRE].

[66] C.G. Böhmer, J. Burnett, D.F. Mota and D.J. Shaw, *Dark spinor models in gravitation and cosmology*, *JHEP* **07** (2010) 053 [arXiv:1003.3858] [INSPIRE].

[67] D. Gredat and S. Shankaranarayanan, *Modified scalar and tensor spectra in spinor driven inflation*, *JCAP* **01** (2010) 008 [arXiv:0807.3336] [INSPIRE].

[68] A. Basak and J.R. Bhatt, *Lorentz invariant dark-spinor and inflation*, *JCAP* **06** (2011) 011 [arXiv:1104.4574] [INSPIRE].

[69] C.G. Böhmer and J. Burnett, *Dark energy with dark spinors*, *Mod. Phys. Lett.* **A 25** (2010) 101 [arXiv:0906.1351] [INSPIRE].

[70] J.M. Hoff da Silva and S.H. Pereira, *Exact solutions to Elko spinors in spatially flat Friedmann-Robertson-Walker spacetimes*, *JCAP* **03** (2014) 009 [arXiv:1401.3252] [INSPIRE].

[71] H. Wei, *Spinor dark energy and cosmological coincidence problem*, *Phys. Lett.* **B 695** (2011) 307 [arXiv:1002.4230] [INSPIRE].

[72] H.M. Sadjadi, *On coincidence problem and attractor solutions in ELKO dark energy model*, *Gen. Rel. Grav.* **44** (2012) 2329 [arXiv:1109.1961] [INSPIRE].

[73] A. Basak, J.R. Bhatt, S. Shankaranarayanan and K.V. Prasantha Varma, *Attractor behaviour in ELKO cosmology*, *JCAP* **04** (2013) 025 [arXiv:1212.3445] [INSPIRE].

[74] A.R. Liddle and D.H. Lyth, *Cosmological inflation and large scale structure*, Cambridge University Press, Cambridge U.K. (2000) [INSPIRE].

[75] S.H. Strogatz, *Nonlinear dynamics and chaos*, Perseus Books Publishing, Massachusetts U.S.A. (1994).

4.2 Modelo Sigma não-linear para Espinores Escuros

No início da década de noventa, houve certo esforço para se definir o que poderia ser uma formulação de modelo sigma não linear para férmions, culminando em um esforço eminentemente geométrico no tratamento da álgebra de Clifford e seus vínculos. Aparte todo o contexto aplicativo de modelos sigma (usuais) em teoria de cordas, há um apelo estético inegável quando se relaciona uma formulação de modelo sigma em um *target space* arbitrário com uma teoria de Utiyama-Yang-Mills. Essa relação mostra, dentre outras coisas, como noções típicas de geometria diferencial são traduzidas para as teorias não-abelianas e vice-versa.

Como brevemente mencionado, as tentativas anteriores de construção de modelos sigma para férmions levaram a cenários que tratavam eminentemente de geometrizações compatíveis com a álgebra de Clifford. A razão para um tal desvio da formulação de modelos sigma usuais é a equação de movimento para espinores de Dirac ser de primeira ordem. Obviamente, para espinores de massa um a construção mais parecida com a usual pode ser realizada. Entretanto, o caráter anticomutante dos objetos fermiônicos em questão sugere a introdução de termos de torção no *target space*. Uma separação da geometria de tal espaço em parte simétrica e anti-simétrica nos habilita o início da construção que culmina na introdução formal de produtos tensoriais e exteriores isomorfos aos usuais, mas cuja atuação é mais eficaz no caso em questão. Ao final da formalização termos de torção contribuem para a densidade lagrangiana total.

Como uma aplicação do formalismo desenvolvido, estudamos implicações cosmológicas advindas da construção do modelo sigma, em geral, atentando para os aspectos referentes à presença da torsão em particular. É mostrado como o modelo sigma pode engendrar o aparecimento de uma constante cosmológica efetiva (e variável) cujo sinal pode variar com o tempo, levando de um universo desacelerado para um acelerado. Aqui, embora o foco central do trabalho seja a formalização do modelo sigma para tais férmions, é relevante ter-se encontrado um comportamento rico também do ponto de vista cosmológico.

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March 2016

EPL, **113** (2016) 60001 www.epljournal.org
doi: 10.1209/0295-5075/113/60001

A framework to a mass-dimension-one fermionic sigma model

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received 14 January 2016; accepted in final form 24 March 2016
published online 5 April 2016

PACS 03.65.Fd – Algebraic methods
PACS 11.10.-z – Field theory
PACS 03.50.Kk – Other special classical field theories

Abstract – In this paper a mass-dimension-one fermionic sigma model, realized by the eigenspinors of the charge conjugation operator with dual helicity (Elko spinors), is developed. Such spinors are chosen as a specific realization of mass-dimension-one spinors, wherein the non-commutative fermionic feature is ruled by torsion. Moreover, we analyse Elko spinors as a source of matter in a background in expansion and we have found that such kind of mass-dimension-one fermions can serve not only as dark matter but they also induce an effective cosmological constant.

Introduction. – The concept of non-linear sigma models has been extensively studied in the scientific literature in a broader context. The usefulness of such framework is widely evinced from the symmetries and interconnections among different areas that it allows. Usually, apart from outstanding applications in string theory (see [1] and references therein), non-linear sigma models have served to evidence the interplay between target spaces and Utiyama-Yang-Mills theories [2].

A more subtle issue concerning sigma models is the underlying framework to relativistic fermions. The first-order Dirac equation is certainly an additional element of difficulty. The description of the physical reality based upon matter (fermionic fields) and its interactions mediated by Abelian and non-Abelian gauge theories, however, has motivated the study of possible sigma models in the broader context of spinor theory [3]. In ref. [3], by exploring the so-called bispinor geometry associated to the bispinor algebra, it was shown that for a particular class of spinors —whose density and pseudo-scalar density are non-null— the geometry of the physical observables space is given by a three-dimensional hyperbolic Robertson-Walker space.

Nearly ten years ago, the systematic study of Majorana spinors has been leading to the appreciation of mass-dimension-one spinor fields [4,5], called Elko[1]. This field satisfies the Klein-Gordon, but not the Dirac equation. As a crucial property, these spinors are neutral, under local gauge interactions, by means of the requirement that they are eigenspinors of the charge conjugation operator. In fact, Elko interactions, with matter and gauge fields of the standard model, are suppressed by at least one order of magnitude regarding the unification scale, providing an *ab initio* origin of "darkness" of dark matter. In other words, interactions of Elko spinor fields are constrained to the Higgs field and gravitons, supplying a prominent direction towards physics beyond the Standard Model.

The theoretical formulation of completeness for these spinors is encoded in the (sub)groups (of the Lorentz group) which retain the underlying relativistic structure [6]. Starting from the usual concept of Dirac spinors as elements that carry the representation $(1/2,0)\oplus(0,1/2)$ of the Lorentz group, one can relate the different sectors of the representation space by means of the parity operator P. In the context of the full Lorentz group, P is a discrete symmetry and its implementation in a given spinorial formulation culminates in the standard Dirac dynamical equation [7]. Nevertheless, the two parts of the representation space can be also related through the so-called "Pauli matrices magic" [8] without reference to any discrete symmetry. In this context, the resulting dynamics is not provided by the Dirac equation, but rather by the Klein-Gordon equation only. Working out the particularities of such spinor field in the second quantisation program, a violation of the full Lorentz symmetry appears in the spin sums. Interestingly enough, it is possible to show

[1]Eigenspinors of the charge conjugation operator with dual helicity [4].

that the spin sums, and therefore the whole formulation, is invariant under SIM(2) group transformations [6]: precisely the subgroups of the Lorentz group obtained by removing the discrete symmetries [9]. Additionally, Elko can be experimentally produced by Higgs interactions [10,11].

In the standard model of particle physics, all regarded spinors are Dirac, Weyl or Majorana ones. Such spinors obey a first-order derivative field equation. This characteristic implies a quantum propagator that, for large momentum, is proportional to p^{-1}. This asymptotic behaviour of the associated propagator results, among other things, in the fact that mass dimension must be 3/2. Now, the unique kinematic operator that is satisfied by Elko is the Klein-Gordon equation, a second-order derivative field equation. For this case, for a large momentum, the quantum propagator is proportional to $1/p^2$, contradicting the previous case. The Klein-Gordon operator is the proper kinetic operator for Elko fields. We are, therefore, led to conclude that the mass dimension of the Elko field is *one*, rather than 3/2, as it would be usually expected for a fermionic field.

All the above-mentioned physical aspects of these fields (mass dimension one, neutrality, spin 1/2) enable Elko spinors to be dark-matter candidates, constructed from the very first principles. Hence the systematic investigation of these dark-matter candidates, slightly and safely departing from the usual quantum description has led us to the analysis of the mass-dimension-one fermionic sigma model presented here.

In this paper we construct and investigate a non-linear sigma model associated to mass-dimension-one spinor fields. Recent formal results point to the fact that there are many kinds of such spinors [12]. However, to fix ideas, we shall report on Elko spinors, which are prototypes of mass-dimension-one fermionic fields. By sigma model we mean the mapping of the Minkowski space into a complete target space, performed by spinor fields, and its relationship with Utiyama-Yang-Mills theory. In this vein, bearing in mind the Grassmannian character of the spinor variable, we further endow the target space with torsion. We organise this paper as follows: in the next section we depict the general set up of the non-linear sigma model and its relation to non-Abelian gauge theories, also endowed to torsion in order to encompass the non-commutative fermionic aspect. Moving forward, in the third section we construct a representative sigma model for mass-dimension-one fermions. In the final section we conclude. Whenever it is possible, we provide some starting points to the application of the general formulation in cosmology.

The general setup with torsion. – This section shall present a straightforward generalisation, regarding the interplay between non-linear sigma models and non-Abelian theories [13], encoding torsion terms. As remarked in the Introduction, we envision further applications to a specific, although essentially fermionic, case.

We start by depicting the general aspects, not particularising to mass-dimension-one fermionic fields immediately. Let $\{\xi_i\}$ be the canonical basis of a natural inertial frame in the target space Σ, and $\{\mathrm{d}\xi^i\}$ its dual basis. It is possible to split the target space geometry by defining an effective metric $g \in \sec(T_p\Sigma)^* \times \sec(T_p\Sigma)^*$, where, as usual, p is an arbitrary point belonging to Σ, and $\sec T_p\Sigma$ is a section of the tangent bundle of Σ at p, such that given $\phi = \varphi^i\xi_i$, it yields

$$g(\varphi,\varphi) = [g_{mn}\mathrm{d}\xi^m \otimes \mathrm{d}\xi^n + \gamma_{mn}\mathrm{d}\xi^m \wedge \mathrm{d}\xi^n](\varphi^i\xi_i, \varphi^j\xi_j), \tag{1}$$

being $\otimes$ and $\wedge$ the tensor and the exterior product, respectively. The geometrical splitting is, indeed, fulfilled by eq. (1). A direct computation of (1), taking into account the anti-symmetry relation between the two products, leads to

$$g(\varphi,\varphi) = g_{ij}\varphi^i\varphi^j + \frac{1}{2}\tilde{\gamma}_{ij}\varphi^i\varphi^j, \tag{2}$$

where $\tilde{\gamma}_{ij} = \gamma_{ij} - \gamma_{ji}$. By writing a given product as its commuting and anti-commuting counterparts, *i.e.* $\varphi^i\varphi^j = \frac{1}{2}[\varphi^i,\varphi^j] + \frac{1}{2}\{\varphi^i,\varphi^j\}$, it yields

$$g(\varphi,\varphi) = \frac{1}{2}g_{ij}\{\varphi^i,\varphi^j\} + \frac{1}{4}\tilde{\gamma}_{ij}[\varphi^i,\varphi^j]. \tag{3}$$

It is worth mentioning that a thorough classification of spinors based upon bilinear covariants in a space-time wherein the metric has both symmetric and anti-symmetric parts has been accomplished in [14].

A complete sigma model, in the sense of eq. (3), can be thus studied, by means of the free Lagrangian

$$\mathcal{L} = \frac{1}{2}g(\partial_\mu\varphi,\partial^\mu\varphi) = \frac{1}{4}g_{ij}\{\partial_\mu\varphi^i,\partial^\mu\varphi^j\} + \frac{1}{8}\tilde{\gamma}_{ij}[\partial_\mu\varphi^i,\partial^\mu\varphi^j], \tag{4}$$

where Greek indexes stand for space-time coordinates. Obviously, in the usual commutative case, it yields $[\partial_\mu\varphi^i,\partial^\mu\varphi^j] = 0$, hence we are simply left with $\mathcal{L} = \frac{1}{2}g_{ij}\partial_\mu\varphi^i\partial^\mu\varphi^j$. In this last case, the connection with Utiyama-Yang-Mills theories is determined by the requirement $\frac{\delta\mathcal{L}}{\delta\varphi_i} = 0$ (under space-time volume integration). In this vein, the Christoffel symbols $\Gamma^i_{jk}(\varphi)$ are automatically generated, in terms of which the connection $A^{\,i}_{\mu k}(\varphi) = \Gamma^i_{jk}(\varphi)\partial_\mu\varphi^j$ is identified. The equation of motion, then, reads

$$D^{\,i}_{\mu\ j}(\partial^\mu\varphi^j) = 0, \tag{5}$$

where the covariant derivative is given by $D^{\,i}_{\mu\ j} = \partial_\mu\delta^i_j + A^{\,i}_{\mu j}$. Besides, the contraction of the Riemann curvature tensor R^i_{jkl} with $\partial_\mu\varphi^k\partial_\nu\varphi^l$ leads to the non-Abelian field strength

$$R^i_{jkl}(\partial_\mu\varphi^k)(\partial_\nu\varphi^l) = \partial_\mu A^{\,i}_{\nu j} - \partial_\nu A^{\,i}_{\mu j} + ([A_\mu, A_\nu])^i{}_j. \tag{6}$$

Returning to the complete case, including the non-commutative sector, the functional variation of the Lagrangian leads to

$$\partial_\mu \partial^\mu \varphi^m + \tilde{\Gamma}^m_{ij}(\varphi)\partial_\mu \varphi^i \partial^\mu \varphi^j = 0, \tag{7}$$

with

$$\tilde{\Gamma}^m_{ij}(\varphi) = \Gamma^m_{ij}(\varphi) + \Lambda^m_{ij}(\varphi), \tag{8}$$

where $\Lambda^m_{ij}(\varphi)$ is defined as

$$\Lambda^m_{ij}(\varphi) \equiv \frac{1}{2} g^{mk}\Big(\partial_i g_{jk} - \partial_j g_{ik} - \frac{1}{2}\partial_k \tilde{\gamma}_{ij}\Big), \tag{9}$$

and we have the explicit contribution of the torsion terms. Moreover, the general target space curvature tensor is given by

$$\tilde{R}^i_{jkl} = R^i_{jkl} + \partial_{[k}\Lambda^i_{jl]} + (\Gamma^i_{m[k} + \Lambda^i_{m[k})\Lambda^m_{jl]} + \Lambda^i_{m[k}\Gamma^m_{jl]}. \tag{10}$$

Notice that if $\tilde{\gamma}_{ij} = 0$, and requiring $\left[\partial_\mu \varphi^i, \partial^\mu \varphi^j\right] = 0$, (culminating with $\Lambda^m_{ij}(\varphi) = 0$), eqs. (7) and (10) reduce to the usual case, as expected. Besides, as a matter of fact, it is not trivial (and, perhaps, not even insightful) to find the Yang-Mills counterpart of the geometric quantities as in (6), for the case at hand. The important aspect to be stressed here is that, when commutativity is lifted in the target space, torsion terms are generated.

In order to envisage the implementation of an application, we depict some cosmological implications of the sigma model. We shall briefly present the main equations that must be considered, given a specific form for the target space fields. In a curved background the action containing the corresponding contribution of the sigma model Lagrangian (4) reads

$$S = \int \sqrt{-\bar{g}}\,\mathrm{d}^4x \left[-\frac{R}{2\kappa} + \frac{1}{2}h_{ij}\partial_\mu \varphi^i \partial_\nu \varphi^j \bar{g}^{\mu\nu} - W(\varphi)\right], \tag{11}$$

where R is the Ricci scalar, $\kappa = 8\pi G$ and $h_{ij} = g_{ij} + \frac{1}{2}\tilde{\gamma}_{ij}$ can be obtained from the symmetric and anti-symmetric property of (3). The space-time metric is represented here by $\bar{g}^{\mu\nu}$ and $W(\varphi)$ stands for a self-interacting potential.

The variation of action (11), with respect to the metric $\bar{g}^{\mu\nu}$, leads to the Einstein equations

$$R_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}R = 2\kappa T_{\mu\nu}, \tag{12}$$

where $R_{\mu\nu}$ is the Ricci tensor, and $T_{\mu\nu}$ is the canonical energy-momentum tensor corresponding to the matter content:

$$T_{\mu\nu} = h_{ij}\partial_\mu \varphi^i \partial_\nu \varphi^j - \bar{g}_{\mu\nu}\left[\frac{1}{2}h_{ij}\partial_\alpha \varphi^i \partial_\beta \varphi^j \bar{g}^{\alpha\beta} - W(\varphi)\right]. \tag{13}$$

Variation with respect to φ^k leads to the equation of motion for the fields,

$$\frac{1}{\sqrt{-\bar{g}}}\partial^\mu(\sqrt{-\bar{g}}\partial_\mu \varphi^m) + \tilde{\Gamma}^m_{ij}(\partial_\mu \varphi^i \partial^\mu \varphi^j) + (\partial_k W)g^{km} = 0, \tag{14}$$

with $\tilde{\Gamma}^m_{ij}$ given by (8).

It is straightforward to check that (14) reduces to (7), when the space-time metric $\bar{g}^{\mu\nu}$ is the Minkowski flat metric and the potential is null. By specifying the fields φ^i, and the potential $W(\varphi)$, we can obtain the Friedmann equations from (12). An emergent universe, supported by a non-linear sigma model without torsion, was studied in [15]. In the context approached here, by taking advantage of the fact that h_{ij} encodes torsion terms, it is quite plausible that the dynamics of the evolution will be changed.

Building up the sigma model. – Part of the structure of Elko spinors, λ, is built upon the requirement $C\lambda = \pm\lambda$, being C the charge conjugation operator. There exists the self-conjugated spinors λ^S_α ($C\lambda^S_\alpha = +\lambda^S_\alpha$) and the anti–self-conjugated spinors λ^A_α ($C\lambda^A_\alpha = -\lambda^A_\alpha$). Moreover, a quite judicious analysis shows that the right dual to λ (from the relativistic point of view) reads $\overset{\neg}{\lambda}_\alpha = \pm i[\lambda_\beta]^\dagger\gamma^0$ [16], where the labels α and β denote different types of spinors and, clearly, the dual relation stands for both self- and anti–self-conjugated spinors. As a last necessary remark, we remember that there are four different Elko spinors: two of them corresponding to different states for the self-conjugated case, and similarly to the anti–self-conjugated case [4].

It is possible to provide a particular basis adapted to eigenspinors of the charge conjugation operator, related to the Majorana representation. Alternatively, we can use the chiral representation, paradigmatically explored in all the literature of Elko. Whatever the basis is chosen, it is worth to mention that our approach is basis independent. It is useful for our purposes to construct all the possible spinors as follows: let c^i and d_j be c-numbers and write $\lambda = c^i\lambda_i$ and $\overset{\neg}{\lambda} = d_j \overset{\neg j}{\lambda}$. Now let us decompose λ (and $\overset{\neg}{\lambda}$) in terms of the usual canonical basis (and its corresponding dual basis) as

$$\lambda_i = \begin{pmatrix} \lambda^1_i \\ \lambda^2_i \\ \lambda^3_i \\ \lambda^4_i \end{pmatrix} = \lambda^1_i \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \cdots + \lambda^4_i \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \tag{15}$$

$$\overset{\neg j}{\lambda} = \left(\overset{\neg j}{\lambda}_1, \overset{\neg j}{\lambda}_2, \overset{\neg j}{\lambda}_3, \overset{\neg j}{\lambda}_4\right) = \overset{\neg j}{\lambda}_1 (1,\ 0,\ 0,\ 0) + \cdots + \overset{\neg j}{\lambda}_4 (0,\ 0,\ 0,\ 1). \tag{16}$$

Hence, by denoting the element basis by $\{\xi_a\}$ (and the corresponding dual by $\{\overset{\neg b}{\xi}\}$), we have $\lambda = c^i\lambda^a_i\xi_a$ and

$\overset{\neg}{\lambda} = d_j \overset{\neg j}{\lambda_b} \overset{\neg b}{\xi}$, with $\overset{\neg b}{\xi}(\xi_a) = \delta^b_a$. In order to properly implement the sigma model to the case at hand, it is necessary to modify the tensor and exterior products, encompassing the fermionic character of the fields.

We start by defining the product $\tilde{\otimes}$ in the following way: let $\mathbb{S}$ be the complex vector space generated by all the finite linear combinations of the usual Cartesian products $(v_i, \overset{\neg i}{v})$, where v_i and $\overset{\neg i}{v}$ are spanned by the respective canonical basis. Besides, take $\mathbb{I}$ as the subspace of $\mathbb{S}$ generated by

$$(v_i + u_i, \overset{\neg i}{w}) - (v_i, \overset{\neg i}{v}) - (u_i, \overset{\neg i}{v}), \tag{17}$$

$$(v_i, \overset{\neg i}{w} + \overset{\neg i}{u}) - (v_i, \overset{\neg i}{u}), \tag{18}$$

$$(kv_i, l\, \overset{\neg i}{w}) - kl(v_i, \overset{\neg i}{w}), \tag{19}$$

being k, l c-numbers and v_i, u_i, and so on, spanned by means of the canonical basis (similarly for the dual case). The product $\tilde{\otimes}$ is defined by conjugating elements in the space $\mathbb{S}/\mathbb{I}$, and, therefore, eqs. (17)–(19) ensure bilinearity. Hereupon, we shall pinpoint some important remarks in order to clarify the relevant properties of $\tilde{\otimes}$. Obviously, a basis of $\mathbb{S}/\mathbb{I}$ is given by $\{\xi_i \tilde{\otimes} \overset{\neg i}{\xi}\}$. We define the action of $\tilde{\otimes}$ as

$$(v \tilde{\otimes} \overset{\neg}{w})(\overset{\neg}{u}, x) = (v^i \xi_i \tilde{\otimes} \overset{\neg}{w}_j \overset{\neg j}{\xi})(\overset{\neg}{u}_k \overset{\neg k}{\xi}, x^l \xi_l) = v^i\, \overset{\neg}{w}_j \overset{\neg}{u}_i\, x^j, \tag{20}$$

where $\overset{\neg}{w}_j$ and $\overset{\neg}{u}_j$ are just coefficients and, more importantly, we have defined the action on the basis as $(\xi_i \tilde{\otimes} \overset{\neg j}{\xi})(\overset{\neg k}{\xi}, \xi_l) = \overset{\neg k}{\xi}(\xi_i)\, \overset{\neg j}{\xi}(\xi_l) = \delta^k_i \delta^j_l$. It is important to stress that the product $\tilde{\otimes}$ is unique, up to isomorphisms[2]. Finally, let $\{\xi_i\} \cup \{\xi_i \tilde{\otimes} \overset{\neg j}{\xi}\}$ over the field $\mathbb{C}$ be the basis of $\tilde{\mathbb{T}}$ and $\tilde{\mathbb{I}}$ the bilateral ideal generated by $\xi_i \tilde{\otimes} \overset{\neg i}{\xi}$. The product $\tilde{\wedge}$ acting on $\tilde{\mathbb{T}}/\tilde{\mathbb{I}}$ related to $\tilde{\otimes}$ is, as expected, given by $\xi_m \tilde{\wedge} \overset{\neg n}{\xi} = \frac{1}{2}(\xi_m \tilde{\otimes} \overset{\neg n}{\xi} - \xi_n \tilde{\otimes} \overset{\neg m}{\xi})$.

We are now in the position to implement the splitting of eq. (1), although this time endowed with the tilde products. Hence it yields

$$\mathcal{L} = \frac{1}{2} g^m_n \xi_m \tilde{\otimes} \overset{\neg n}{\xi} (\partial^\mu \overset{\neg}{\lambda}, \partial_\mu \lambda) + \frac{1}{2} \gamma^m_n \xi_m \tilde{\wedge} \overset{\neg n}{\xi} (\partial^\mu \overset{\neg}{\lambda}, \partial_\mu \lambda). \tag{21}$$

According to our previous construction, the fermionic decomposition along with the bilinearity of tilde products can be used in a fairly direct fashion. It is necessary, however, to call attention to the fact that the Fermi-Dirac statistics is a key feature of the Elko formulation [4], which one cannot preclude. Therefore, in the adopted decomposition $\lambda = c^i \lambda^a_i \xi_a$ and $\overset{\neg}{\lambda} = d_j \overset{\neg j}{\lambda_b} \overset{\neg b}{\xi}$ the terms λ^a_i and $\overset{\neg j}{\lambda_b}$ are understood as Grassmannian variables[3]. Taking advantage of these remarks, the Lagrangian (21) yields

$$\mathcal{L} = \frac{1}{2} g^i_j d_a c^b \partial^\mu \overset{\neg a}{\lambda_i} \partial_\mu \lambda^j_b + \frac{1}{4} \tilde{\gamma}^i_j d_a c^b \partial^\mu \overset{\neg a}{\lambda_i} \partial_\mu \lambda^j_b. \tag{22}$$

It is important to stress that the formal structure of the Elko spinors is constructed taking advantage of the spinor rest frame [4]. Therefore, it is quite conceivable to add a mass term in the above Lagrangian. Therefore, it reads

$$\begin{aligned} \mathcal{L} = {} & \frac{1}{2} g^i_j d_a c^b \partial^\mu \overset{\neg a}{\lambda_i} \partial_\mu \lambda^j_b + \frac{1}{4} \tilde{\gamma}^i_j d_a c^b \partial^\mu \overset{\neg a}{\lambda_i} \partial_\mu \lambda^j_b \\ & + m^2 d_j c^i \overset{\neg j}{\lambda_a} \lambda^a_i. \end{aligned} \tag{23}$$

It serves as the starting point for the formulation of a mass-dimension-one fermionic sigma model. Generally speaking, the target space is undertaken as a coset space of the isometry by the isotropy group. In relevant cases torsion can be added to such coset spaces [17]. The Lagrangian (23) can be adjusted to encompass these situations, $\{\xi_a\}$ and $\{\overset{\neg b}{\xi}\}$ being the bases in the fiber bundle formulation, accordingly. From the Lagrangian density (23), it is interesting to note that the torsion terms do affect the spin current density. In fact, the appearance of $\tilde{\gamma}^i_j$ terms in the expression below makes this point explicit:

$$S^\mu_{\alpha\beta} = -\frac{1}{2} d_a c^b \left(g^i_j + \frac{1}{2} \tilde{\gamma}^i_j \right) \left[\partial^\mu \overset{\neg a}{\lambda_i} \frac{\delta \lambda^j_b}{\delta \omega^{\alpha\beta}} + \frac{\delta \overset{\neg a}{\lambda_i}}{\delta \omega_{\alpha\beta}} \partial^\mu \lambda^j_b \right]. \tag{24}$$

We shall finalize by considering the Elko Lagrangian (23) as the source of matter, in a curved expanding background. Some care is necessary, in order to correctly write the corresponding action in such a case. First, we define the fields $\overset{\neg}{\phi}_i = d_a \overset{\neg a}{\lambda_i}$, $\phi^j = c^b \lambda^j_b$, and introduce the covariant derivatives, by $\nabla_\mu \overset{\neg}{\phi}_i = \partial_\mu \overset{\neg}{\phi}_i + \overset{\neg}{\phi}_i \Gamma_\mu$, and $\nabla_\mu \phi^j = \partial_\mu \phi^j - \Gamma_\mu \phi^j$, where Γ_μ stands for the spin connections coupling Elko spinors to the background metric. The action in a curved background reads

$$S = \int \sqrt{-\bar{g}}\, \mathrm{d}^4 x \left[-\frac{R}{2\kappa} + \frac{1}{2} h^i_j \nabla_\mu \overset{\neg}{\phi}_i \nabla_\nu \phi^j \bar{g}^{\mu\nu} - W(\overset{\neg}{\phi}, \phi) \right], \tag{25}$$

where $h^i_j = g^i_j + \frac{1}{2} \tilde{\gamma}^i_j$. The equations of motion for the fields follow directly, by taking the variation with respect to $\overset{\neg}{\phi}_i$ and ϕ^j. Equation (25) may serve as the starting point to apply the formulation presented here in the cosmological context. In particular, the quartic interaction appearing due to the (spin connection) torsion contribution (as in the usual fermionic case) is generated.

In order to show an explicit contribution coming from the symmetric and anti-symmetric part of the sigma-model target space into the cosmological equations, let us take for simplicity a set of constant Elko spinor fields, such that $\partial_\mu \overset{\neg}{\lambda} = 0 = \partial^\mu \lambda$, and the potential as the quadratic

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[2]From a complementary point of view, the defined product $\tilde{\otimes}$ is just an isomorphism of the usual tensor product $\otimes$.

[3]Roughly speaking this is the attempt to reproduce quantum features of the field by absorbing the creation/annihilation operators into the expansion coefficients.

one in the form $W = \frac{1}{2}m^2 \stackrel{\neg}{\lambda} \lambda$. We can choose to write h^i_j in the following simple form:

$$h^i_j = \begin{pmatrix} g & \gamma & \gamma & \gamma \\ -\gamma & g & \gamma & \gamma \\ -\gamma & -\gamma & g & \gamma \\ -\gamma & -\gamma & -\gamma & g \end{pmatrix}, \tag{26}$$

with g and γ representing the symmetric and anti-symmetric components of h^i_j. It may sound as an oversimplification; however (as we shall see), this particularization leads to a relevant physical consequence. In a flat, homogeneous and isotropic FRW metric, $\bar{g}_{\mu\nu} = \mathrm{diag}[1, -a(t)^2, -a(t)^2, -a(t)^2]$, the spin connections Γ_μ can be determined as $\Gamma_0 = 0$ and $\Gamma_k = -\frac{\dot{a}}{2}\gamma^0\gamma^k$, where γ^μ are the standard Dirac matrices and the dot indicates the time derivative. Notice that, even for constant spinor fields, the spin connection term couples to the metric through the term $\stackrel{\neg}{\phi}_i \Gamma_\mu\Gamma^\mu\phi^j$, with $\Gamma_\mu\Gamma^\mu = -\frac{3}{4}\frac{\dot{a}^2}{a^2}\mathbb{I}$ and, therefore, contributes with a time-dependent term. The action (25) can be written as

$$S = \int \sqrt{-\bar{g}}\mathrm{d}^4x\left[-\frac{R}{2\kappa} + \Lambda_S(t) + \Lambda_A(t) - \Lambda_m\right]. \tag{27}$$

The Λ_m term comes from the potential part, namely $\Lambda_m = \frac{1}{2}m^2 \stackrel{\neg}{\lambda} \lambda$, and represents a cosmological constant term, with a dependence on the mass of the Elko spinor field. The terms $\Lambda_S(t)$ and $\Lambda_A(t)$ stand for the symmetric and anti-symmetric contributions coming from the sigma model, respectively, and can be written as $\Lambda_S(t) = \frac{3}{8}gC(d_a, c^b, \stackrel{\neg}{\lambda}{}^a_i, \lambda^i_b)H(t)^2$ and $\Lambda_A(t) = \frac{3}{16}\gamma C(d_a, c^b, \stackrel{\neg}{\lambda}{}^a_i, \lambda^i_b)H(t)^2$, where $C(d_a, c^b, \stackrel{\neg}{\lambda}{}^a_i, \lambda^i_b)$ is a constant and $H(t) = \dot{a}/a$ is the Hubble parameter. They act as an effective time-varying cosmological constant, $\Lambda_{eff}(t) = \Lambda_S(t) + \Lambda_A(t) - \Lambda_m$. At early times of the cosmological evolution, when $\Lambda_S(t) + \Lambda_A(t) > \Lambda_m$, the positive contribution from Λ_{eff} acts as an attractive gravitational field in the geometric side of the Einstein equation, leading to a decelerating universe. As the universe expands, $H(t)$ decreases. When the condition $\Lambda_S(t) + \Lambda_A(t) < \Lambda_m$ is reached the universe turns to be dominated by a negative Λ_{eff}, which implies a repulsive gravitational force, driven by a constant cosmological term Λ_m, in perfect agreement with the ΛCDM model. It is straightforward to see that the constant parameters could be adjusted, in order to reproduce the transition from a decelerated to an accelerated expansion of the universe. A much richer scenery concerns the study of Elko spinor fields dynamic coupled to the gravitational field [18].

Concluding remarks. – We analyzed and studied a mass-dimension-one fermionic sigma model, realized by Elko spinors. A non-commutative fermionic feature was introduced by the prominent role of torsion. The effective connection (8) that rules the Euler-Lagrange equations (7) is defined with respect to the anti-symmetric part of the metric in the target space. Thereat, Elko spinors play the role of a source of matter in an expanding background.

By the very nature of mass-dimension-one fermions, the study of the action (25) concerning the sigma model for Elko spinors can provide further insights into the dark-matter problem. However, it is also interesting to pursue questions concerning the sigma model itself. For instance, the study of the non-minimal coupling between the Riemann tensor and four Elkos, as in the standard realization of supersymmetric sigma models. Moreover, the usual $N = 1$ ($D = 4$) supersymmetric sigma-model case also imposes the necessity of a Kaehlerian target space. Hence the investigation of what type of geometric condition may arise from the extension of the present work to the supersymmetric case is also in order. We shall delve these questions in the future.

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The authors express their gratitude to professor José Abdalla Helayel-Neto for the privilege of his comments and appreciation of the manuscript. JMHdS acknowledges CNPq grants No. 308623/2012-6 and No. 445385/2014-6 for partial support. RdR is grateful for the CNPq grants No. 303293/2015-2 and No. 473326/2013-2, FAPESP grant No. 2015/10270-0, and for the INFN grant "Classification of Spinors", which has provided partial support. SHP acknowledges CNPq grant No. 304297/2015-1.

REFERENCES

[1] Tseytlin A. A., *Int. J. Mod. Phys. A*, **4** (1989) 1257.

[2] Gell-Mann M. and Levy M., *Nuovo Cimento*, **16** (1960) 705.

[3] Crawford J. P., *J. Math. Phys.*, **31** (1990) 1991.

[4] Ahluwalia D. V. and Grumiller D., *JCAP*, **07** (2005) 012.

[5] Ahluwalia D. V., Lee C. Y. and Schritt D., *Phys. Rev. D*, **83** (2011) 065017.

[6] Ahluwalia D. V. and Horvath S. P., *JHEP*, **11** (2010) 078.

[7] Sperança L. D., *Int. J. Mod. Phys. D*, **23** (2014) 1444003.

[8] Ramond P., *Field Theory: A Modern Primer, second edition* (Westview Press) 2001.

[9] Cohen A. G. and Glashow S. L., *Phys. Rev. Lett.*, **97** (2006) 021601.

[10] Dias M., de Campos F. and Hoff da Silva J. M., *Phys. Lett. B*, **706** (2012) 352.

[11] Agarwal B., Jain P., Mitra S., Nayak A. C. and
140 Verma R. K., *Phys. Rev. D*, **92** (2015) 075027.

[12] Coronado Villalobos C. H., Hoff da Silva J. M. and da Rocha R., *Eur. Phys. J. C*, **75** (2015) 266.

[13] Mukhi S., *Classical and Quantum Theory of Supersymmetric σ-Models*, in *Superstrings, Unified Theories*

and Cosmology, Proceedings of the Summer Workshop in High Energy Physics and Cosmology, Trieste, Italy, 30 June–15 August 1986, edited by Furlan G., Jengo R., Pati J. C., Sciama D. W., Sezgin E. and Shafi Q., *The ICTP Series in Theoretical Physics*, Vol. **3** (World Scientific Pub. Co., Teaneck, NJ) 1987, ISBN 9971-50-271-2, pp. 53–70.

[14] Abłamowicz R., Gonçalves I. and da Rocha R., *J. Math. Phys.*, **55** (2014) 103501.

[15] Beeshan A., Chervon S. V. and Maharaj S. D., *Class. Quantum Grav.*, **26** (2009) 075017.

[16] Ahluwalia D. V., Lee C. Y. and Schritt D., *Phys. Lett. B*, **687** (2010) 248.

[17] Batakis N. A., Farakos K., Kapetanakis D., Koutsoumbas G. and Zoupanos G., *Phys. Lett. B*, **220** (1989) 513.

[18] Böhmer C. G., Burnett J., Mota D. F. and Shaw D. J., *JHEP*, **07** (2010) 053.

Capítulo 5

Considerações Finais

Ao longo de nossa exposição, percorremos três áreas de atuação que dimensionam, em certo sentido, o trabalho que realizamos em teoria espinorial. A ideia foi seguir um molde expositivo que abarcasse nosso intento de estudar desde o embasamento algébrico formal até consequências físicas, inclusive algumas passíveis de observação.

Antes porém de realizarmos um desfecho aos temas aqui tratados, gostaríamos de reservar parte das considerações finais para uma crítica ao trabalho. Vamos à ela. A ramificação do trabalho exposto, contemplando três subareas de atuação, explicita as diversas possibilidades de atuação nessa área de pesquisa. Entretanto, é necessário que se diga, a polivalência também traz em seu bojo aspectos não tão salutares. Assim, por vezes deixamos de lado uma exploração continuada de certos tópicos passíveis de análise ulterior. Para efeitos de argumento, vejamos alguns exemplos concretos em casa subarea abordada.

No que concerne à teoria algébrica de espinores, há ainda espaço para formalizações adicionais no espaço de configurações fermiônicos. Ainda, notemos que boa parte dos resultados obtidos nessa temática se valeram da classificação algébrica baseada nos bilineares covariantes. Seria bastante relevante se tentar uma generalização consistente dessa classificação, mas levando em conta segunda quantização. Um tal programa apontaria de modo taxativo para possíveis desdobramentos em teoria quântica de campos.

Um viés de trabalho formal em teoria de campos contemplando espinores de dimensão canônica de massa um é a adequação, ou generalização, do princípio de Cluster (desenvolvido por Weinberg) para os Elkos. A extensão apresenta-se factível, mas não foi levada a termo. Esse programa certamente colocaria a linha de pesquisa em um quadro de trabalho rígido, beneficiando a própria área em si. Ainda no que concerne a aspectos voltados à teoria de campos, desta vez com relação à busca por sinais em aceleradores, já há disponível na base do LHC um programa que testa modelos de partículas passíveis de descoberta no acelerador. Esse programa ajudaria a colocar um limite nos parâmetros de interação do Elko, mas sua implementação é sutil e demanda certo trabalho. No entanto é mais uma característica que merece atenção.

No escopo da cosmologia, várias aplicações do Elko poderiam ter sido estudadas para sua viabilização (ou o descarte) como candidato a matéria escura. Para sermos específicos, notemos que nem mesmo a tentativa de reprodução de curvas de rotação anômalas de certas galáxias espirais foram estudadas tendo o Elko como fonte. Desse modo, embora parte da literatura evolua para esse fim, ainda há muito o que se fazer no sentido de se viabilizar completamente o Elko como candidato principal a matéria escura.

Essa espécie de listagem de possíveis trabalhos a serem realizados dos parágrafos anteriores, muito embora possa ser utilizada como perspectiva na área, entra aqui como crítica ao trabalho que estamos desenvolvendo. Em cada uma dessas subareas existem trabalhos importantes que precisam ser abordados, levando-nos à conclusão de que a concentração de esforços em uma dessas áreas urge ser realizada. É nesse sentido que acreditamos ser a polivalência danosa ao nosso próprio trabalho.

Feita e embasada a crítica, faz-se igualmente importante nesse capítulo final a observação de um ponto, de cunho mais pessoal, que pretende imprimir à tese um tom de desfecho. Espinores escuros ou, antes, a proposta teórica dos mesmos traz a necessidade de revisitação de inúmeros conceitos, diversas teorias, e variadas aplicações para seu estudo. Essa abordagem (a revisitação) permite que olhemos os conceitos já bem estabelecidos de um modo mais crítico, atentando para o que é, e o que não é, essencial. Nesse contexto, todo o escopo teórico/fenomenológico abordado nessa tese pode ser abarcado entendendo-se os espinores escuros como ferramenta para uma melhor compreensão dos conceitos já bem estabelecidos. Essa característica pode, por alguns, ser interpretada como algo menor, de importância reduzida, ou ainda colocada como parte de um programa pouco ambicioso. Mas ainda que essa última interpretação possa ser pertinente, eis um questionamento igualmente pertinente: quantos programas de trabalho com o qual nos envolvemos nos permite, em um cotidiano acadêmico azafamado como o que vivemos, o ensejo de se ter contato (reiterado) com muitos dos pilares da física moderna e parte de sua fundamentação matemática?