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APRESENTAÇÃO CRÍTICA DE TRABALHOS ORIGINAIS PUBLICADOS EM REVISTAS INTERNACIONAIS APÓS O DOUTORAMENTO

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INTRODUÇÃO

Neste trabalho, apresentarei uma descrição da maioria dos meus trabalhos publicados em revistas internacionais com referee após a conclusão do meu doutoramento. Para isto deverei dividir, para fins de organização, em três partes a presente discussão. Isto se deve ao fato de que tenho desenvolvido trabalhos em basicamente três linhas de pesquisa. A saber: (i) teoria quântica de campos em duas dimensões espaciais, onde tenho me dedicado ao estudo de teorias de calibre que, via de regra são exatas nesta dimensão. (ii) teoria quântica de campos em três dimensões, em particular em modelos com interação de origem topológica, e anyons em geral. (iii) mecânica quântica, estudando classes de potenciais exatamente solúveis e aproximações analíticas para potenciais que não se incluam nestas categorias. Finalmente deverei apresentar, sempre que possível, um breve resumo dos planos presentes e futuros com respeito ao trabalhos nestas e outras linhas de pesquisa, incluindo aí alguns trabalhos submetidos para publicação. Observo ainda que os trabalhos em cada área serão apresentados em ordem cronológica, além disso ao final reproduziremos os originais dos trabalhos na sequência apresentada.



(i) *Teoria Quântica de Campos em Duas Dimensões*

1) *Mapping correlation functions of the Schwinger and the Axial models* [**Journal of Physics G**, vol. 19 (1993) 1127].

Desenvolvemos neste trabalho um estudo comparativo entre dois modelos muito conhecidos em $(1+1)D$, a saber, o modelo de Schwinger [1] e o modelo axial, também conhecido como modelo de Rothe e Stamatescu [2]. Neste estudo mostramos haver um mapeamento não-trivial entre as funções de correlação dos dois modelos [3]. Este mapeamento foi observado inicialmente entre as suas anomalias, o que foi demonstrado tanto em cálculo perturbativo quanto via método de point-splitting e de integral funcional via método de Fujikawa. Além disso, verificou-se em seguida que as funções de correlação fermiônicas 1PI a um loop eram mapeáveis. Observou-se através do cálculo não-perturbativo via integral funcional que para que exista realmente um mapeamento das funções de correlação, é necessário que o termo de fonte do funcional gerador seja não-local, permitindo ligar um modelo que não tem divergência infravermelha [2] a um que a tem [1], gerando assim este mapeamento não-trivial.

2) *Comparison of the anomalous and non-anomalous generalized Schwinger models via functional formalism* [**International Journal of Modern Physics A**, vol. 9 (1994) 2229].

Ainda estudando modelos bidimensionais, publicamos um trabalho [4] no qual, além de se fazer a quantização via integral funcional do modelo de Schwinger generalizado em suas versões anômala e não-anômala, propusemos um termo de Wess-Zumino generalizado, o qual demonstrou possuir características muito interessantes como o aparecimento de massas dependentes do calibre, e que parte delas desaparecem quando se utiliza um calibre conveniente, além disso ele demonstrou ser compatível com o esperado de um bom termo



de Wess-Zumino quando aplicado aos modelos de Schwinger vetorial e axial, uma vez que se obtiveram resultados compatíveis com aqueles presentes na literatura. Verificamos recentemente [5] que este novo Wess-Zumino é aquele responsável pelo restabelecimento da invariância de calibre de modelos anômalos de calibre com massas dependentes do calibre, os quais apareceram recentemente na literatura [6][7]. Abaixo apresentaremos uma versão resumida desta demonstração.

O modelo proposto por Basseto e Griguolo é definido através da seguinte densidade de lagrangiana:

$$L = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \bar{\psi}\gamma^\mu \left(i\partial_\mu + e \left(\frac{1+r\gamma_5}{2} \right) A_\mu \right) \psi, \quad (1)$$

onde r é um parâmetro real arbitrário que interpola entre o modelo com acoplamento vetorial ($r = 0$) e o do modelo de Schwinger quiral ($r^2 = 1$). Todavia, a fim de trabalhar com um modelo mais geral [4][8][9], usaremos a densidade de lagrangiana abaixo:

$$L = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \bar{\psi}\gamma^\mu (i\partial_\mu + (e_R P_+ + e_L P_-) A_\mu) \psi, \quad (2)$$

onde $P_\pm \equiv (1 \pm \gamma_5)/2$. É fácil ver que este contém o anterior como caso particular.

Em termos das componentes longitudinal e transversal do campo de calibre, temos a seguinte densidade de lagrangiana efetiva de ordem superior descrevendo o modelo,

$$L_{eff} = \bar{\zeta} (i\gamma_\mu \partial^\mu) \zeta + \frac{1}{2e^2} \chi \square^2 \chi + \frac{1}{2\pi e^2} (g_1^2 - M^2) \eta \square \eta + \frac{1}{2\pi e^2} (g_2^2 + M^2) \chi \square \chi - \frac{g_1 g_2}{\pi e^2} \eta \square \chi \quad (3)$$

com $M^2 \equiv ae^2$, sendo a o parâmetro arbitrário de regularização. Além disso pode-se verificar que a densidade de lagrangian acima não é invariante de calibre. Introduzindo então um termo de Wess-Zumino generalizado como proposto neste trabalho, dado por

$$L_{GWZ} = \frac{1}{2\pi e^2} (g_1^2 - M^2) \theta \square \theta + \frac{1}{\pi e^2} (g_1^2 - M^2) \theta \square \eta - \frac{g_1 g_2}{\pi e^2} \theta \square \chi - \frac{1}{2\alpha e^2} \eta \square^2 \eta$$

$$-\frac{1}{2\alpha e^2}\theta\Box^2\theta - \frac{1}{\alpha e^2}\theta\Box\eta, \quad (4)$$

fixando o calibre através da introdução do termo $-\frac{1}{2\beta}\partial_\mu\theta\partial^\mu\theta$, de forma que podemos escrever a densidade de lagrangiana resultante na forma $L = \frac{1}{2}\rho^T \mathcal{M}\rho$, onde $\rho^T \equiv (\chi, \eta, \theta)$, com $\mathcal{M}$ sendo

$$\mathcal{M} = \frac{\Box}{e^2} \begin{pmatrix} \left(\Box + \frac{(g_2^2 + M^2)}{\pi}\right) & -\frac{g_1 g_2}{\pi} & -\frac{g_1 g_2}{\pi} \\ -\frac{g_1 g_2}{\pi} & -\left(\frac{\Box}{\alpha} + \frac{(M^2 - g_1^2)}{\pi}\right) & -\left(\frac{\Box}{\alpha} + \frac{(M^2 - g_1^2)}{\pi}\right) \\ -\frac{g_1 g_2}{\pi} & -\left(\frac{\Box}{\alpha} + \frac{(M^2 - g_1^2)}{\pi}\right) & -\left(\frac{\Box}{\alpha} + \left(\frac{(M^2 - g_1^2)}{\pi} - \frac{e^2}{\beta}\right)\right) \end{pmatrix}. \quad (5)$$

Além disso os elementos da matriz inversa de $\mathcal{M}$ nos fornecem as funções de correlação de dois pontos $\langle \chi\chi \rangle$, $\langle \chi\eta \rangle$, etc... . Usando então estes tijolos básicos, podemos construir o propagador do fóton na representação de momentum:

$$\langle A_\mu A_\nu \rangle = -\frac{1}{e^2} \left[k_\mu k_\nu \langle \eta\eta \rangle + \tilde{k}_\mu k_\nu \langle \chi\eta \rangle + k_\mu \tilde{k}_\nu \langle \eta\chi \rangle + \tilde{k}_\mu \tilde{k}_\nu \langle \chi\chi \rangle \right], \quad (6)$$

que neste caso nos leva a:

$$\begin{aligned} D_{\mu\nu}(k) = & \frac{1}{(k^2 - m_+^2)(k^2 - m_-^2)} \left\{ \left[-k^2 g_{\mu\nu} - (\alpha - 1) k_\mu k_\nu \right] + \right. \\ & + \frac{\alpha}{\pi} \left[\pi s^2 g_{\mu\nu} + \frac{k_\mu k_\nu}{k^2} \left(-\pi s^2 - \frac{\pi(k^2 - r^2)\beta s^2}{e^2} + \frac{\pi\beta k^2(k^2 - r^2)}{\alpha e^2} + \frac{\beta g_1^2 g_2^2}{\pi e^2} \right) + \right. \\ & \left. \left. + g_1 g_2 \left(\frac{\tilde{k}_\mu k_\nu + k_\mu \tilde{k}_\nu}{k^2} \right) \right] \right\}, \quad (7) \end{aligned}$$

sendo as massas dadas por

$$m_\pm^2 \equiv \frac{1}{2\pi} \left[\pm \left(\alpha (M^2 - g_1^2) + g_2^2 + M^2 \right) + \right]$$

$$- \left(M^4 (\alpha - 1)^2 - 2M^2 (\alpha - 1) (\alpha g_1^2 + g_2^2) + (\alpha g_1^2 - g_2^2)^2 \right)^{\frac{1}{2}} \Big]. \quad (8)$$

Por outro lado, podemos calcular o propagador invariante de calibre [10], para um campo definido como $A_\mu^I \equiv A_\mu + \frac{1}{e} \partial_\mu \theta$. Isto nos levará então ao seguinte propagador:

$$\begin{aligned} \langle A_\mu^I A_\nu^I \rangle = & \frac{1}{(k^2 - m_1^2)(k^2 - m_2^2)} \left\{ \left[-k^2 g_{\mu\nu} - (\alpha - 1) k_\mu k_\nu \right] + \right. \\ & \left. \frac{\alpha}{\pi} \left[\pi s^2 g_{\mu\nu} + -\pi s^2 \frac{k_\mu k_\nu}{k^2} + g_1 g_2 \left(\frac{\tilde{k}_\mu k_\nu + k_\mu \tilde{k}_\nu}{k^2} \right) \right] \right\} = D_{\mu\nu}(k)_{(\beta=0)}. \end{aligned} \quad (9)$$

Finalmente, fazendo as particularizações dos nossos parâmetros para fins de comparação com aqueles calculados por Basseto e Griguolo, $g_1 \equiv \frac{re}{2}$, $g_2 \equiv \frac{e}{2}$, e $a \rightarrow \frac{a}{4}$, recuperamos os resultados que aparecem em seu trabalho [7]. Demonstramos assim que o termo de Wess-Zumino generalizado discutido em [4], é exatamente aquele necessário para a recuperação da invariância de calibre perdida devido à anomalia na corrente de calibre do modelo proposto por Basseto and Griguolo.

3) *Hamiltonian quantization of the non-anomalous generalized Schwinger model* [Zeitschrift für Physik C, vol. 66 (1995) 517].

Em uma colaboração com o professor César Natividade da UNESP, foram desenvolvidos ainda dois outros trabalhos em teoria quântica de campos em (1+1)D [17][18]. No primeiro discutimos a quantização a la Dirac do modelo de Schwinger generalizado anômalo e não-anômalo, mostrando ainda que esta quantização deve ser feita com muito cuidado a fim de se evitarem certas ambiguidades, as quais tem aparecido de forma errônea na literatura. Entre outras coisas mostramos neste trabalho que na quantização de Dirac, feita no caso particular do modelo de Schwinger quiral por Falck e Kramer [19], havia uma certa imprecisão na medida que, tendo o modelo dois vínculos de primeira classe, haviam sido escolhidas duas condições de calibre. Entretanto os autores não verificaram a consistência temporal destes vínculos. Felizmente no caso tratado por eles o

segundo vínculo imposto diferia do advindo da consistência temporal do primeiro por uma derivada total, preservando as equações de Euler-Lagrange, e consequentemente gerando os parênteses de Dirac corretos para o modelo. Entretanto frisamos neste trabalho que em geral este procedimento é extremamente perigoso, uma vez que os vínculos escolhidos podem não ter nenhuma relação com aqueles provenientes da consistência temporal dos demais.

4) *Chiral Schwinger model with a Podolsky term at finite temperature* [Zeitschrift für Physik C, vol. 67 (1995) 687].

Neste trabalho, agora colaborando também com o professor Henrique Boschi Filho da UFRJ, discutimos a quantização via integral de trajetória de um modelo de Schwinger quiral com termo de Podolsky. Procurávamos neste trabalho verificar se o termo de Podolsky poderia ser capaz de eliminar a anomalia de calibre, presente no modelo de Schwinger quiral, e por outro lado ver também se o acoplamento quiral poderia ser útil na eliminação do táquion que em geral surge associado a uma teoria com derivadas de ordem superior, como é o caso do termo de Podolsky.

Verificamos então que o que ocorria no caso não-singular, ou seja aquele cujo parâmetro arbitrário de regularização a é diferente de um, é que o táquion permanecia presente na teoria, e apenas havia uma renormalização finita da massa gerada dinamicamente. Todavia vimos com grata surpresa que no caso singular ($a = 1$), de fato havia o desaparecimento do táquion do espectro assintótico da teoria o que, até onde sabíamos, era talvez o primeiro caso de um modelo com derivadas de ordem superior sem táquions.

5) *Axial-Schwinger model* [International Journal of Modern Physics A, vol. 11 (1996) 2931].

Recentemente desenvolvemos em colaboração com os professores João Barcelos e Ricardo Amorim da UFRJ, um trabalho no qual estudamos o efeito de uma interação entre férmions de Dirac e fótons com um campo pseudo-escalar; neste trabalho mostramos



o aparecimento de polos massivos adicionais no espectro assintótico da teoria, abrindo a possibilidade da existência de efeitos novos e não-triviais em modelos com interações mistas em (1+1)D [12]. Seguindo nesta linha de interações combinadas, em uma colaboração envolvendo os pesquisadores Cesar Natividade da UNESP, Henrique Boschi Filho da UFRJ e ainda R. P. G. Amaral e L. V. Belvedere da UFF, realizamos estudos a respeito do efeito da auto-interação quártica de férmions na massa gerada dinamicamente para os fótons em modelos com simetria de calibre em duas dimensões, tanto do ponto de vista do formalismo de integral funcional [13], quanto do formalismo operatorial, nos casos abeliano e não-abeliano [14].

Vejamos por exemplo o caso do modelo com acoplamento vetorial, que para férmions de massa zero, no espaço Euclideano e no setor topologicamente trivial, tem como densidade de lagrangiana:

$$\mathcal{L} = \bar{\psi}(i\partial - e\mathcal{A})\psi - \frac{g^2}{2}(\bar{\psi}\gamma_\mu\psi)^2 + \frac{1}{4}(F_{\mu\nu})^2 + J_\mu A_\mu + \bar{\theta}\psi + \bar{\psi}\theta, \quad (10)$$

onde $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Usando a identidade

$$\exp\left\{+\frac{g^2}{2}\int d^2x(\bar{\psi}\gamma_\mu\psi)^2\right\} = \frac{1}{\mathcal{N}} \int \mathcal{D}B_\mu \exp\left\{-\int d^2x\left[\frac{e^2}{2}(B_\mu)^2 - eg\bar{\psi}\gamma_\mu B_\mu\psi\right]\right\} \quad (11)$$

torna-se (no calibre de Lorentz)

$$\mathcal{L} = \bar{\psi}(i\partial - e(\mathcal{A} + g\mathcal{B}))\psi + \frac{e^2}{2}B_\mu B^\mu + \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\alpha}(\partial_\mu A^\mu)^2 + \text{source terms}. \quad (12)$$

Fazendo agora a transformação $A_\mu = \bar{A}_\mu - \bar{B}_\mu$, onde $\bar{B}_\mu = gB_\mu$, a densidade de lagrangiana se torna

$$\begin{aligned} \bar{\mathcal{L}} = & \bar{\psi}(i\partial + e\bar{\mathcal{A}})\psi + \frac{e^2}{2g^2}\bar{B}_\mu\bar{B}^\mu + \frac{1}{4}\bar{F}_{\mu\nu}\bar{F}^{\mu\nu} + \frac{1}{4}\bar{M}_{\mu\nu}\bar{M}^{\mu\nu} - \frac{1}{2}\bar{F}_{\mu\nu}\bar{M}^{\mu\nu} - \frac{1}{2\alpha}(\partial_\mu\bar{A}^\mu)^2 \\ & - \frac{1}{2\alpha}(\partial_\mu\bar{B}^\mu)^2 + \frac{1}{\alpha}(\partial_\mu\bar{A}^\mu)(\partial_\nu\bar{B}^\nu) + \text{source terms}, \end{aligned} \quad (13)$$

onde $\bar{M}_{\mu\nu} = \partial_\mu\bar{B}_\nu - \partial_\nu\bar{B}_\mu$. Para descoplar os férmions e os campos de calibre $\bar{A}_\mu$ e $\bar{B}_\mu$ fazemos a decomposição

$$e\bar{A}_\mu = \partial_\mu\eta_A + \epsilon_{\mu\nu}\partial^\nu\chi_A; \quad e\bar{B}_\mu = \partial_\mu\eta_B + \epsilon_{\mu\nu}\partial^\nu\chi_B \quad (14)$$

e realizamos a transformação

$$\psi(x) = \exp \{-i\eta_A + \gamma_5 \chi_A\} \psi'(x); \quad \bar{\psi}(x) = \bar{\psi}'(x) \exp \{i\eta_A + \gamma_5 \chi_A\} . \quad (15)$$

O Jacobiano não trivial correspondente é dado por

$$J_F = \exp \left\{ + \frac{\lambda}{2\pi} \int d^2x [\chi_A \square \chi_A] \right\} , \quad (16)$$

onde λ é o parâmetro arbitrário de regularização. Aqui vamos nos restringir ao caso em que $\lambda = 1$. Desta forma obtemos que

$$\begin{aligned} \mathcal{L}_{eff} = & i\bar{\psi}' \not{\partial} \psi' + \frac{1}{2e^2} [(\square \chi_A)^2 + (\square \chi_B)^2] - \frac{1}{e^2} \square \chi_A \square \chi_B + \frac{1}{2g^2} [(\partial_\mu \eta_B)^2 + (\partial_\mu \chi_B)^2] \\ & + \frac{1}{2\pi} (\partial_\mu \chi_A)^2 - \frac{1}{2\alpha e^2} [(\square \eta_A)^2 + (\square \eta_B)^2] + \frac{1}{\alpha e^2} \square \eta_A \square \eta_B \\ & + \frac{1}{e} J_\mu [\partial^\mu (\eta_A - \eta_B) + \epsilon_{\mu\nu} \partial^\nu (\chi_A - \chi_B)] + \bar{\theta} e^{-i\eta_A + \gamma_5 \chi_A} \psi' + \bar{\psi}' e^{i\eta_A + \gamma_5 \chi_A} \theta . \end{aligned} \quad (17)$$

Fazendo agora uma mudança de escala tal que

$$\chi_A \rightarrow a \bar{\chi}_A; \quad \chi_B \rightarrow \bar{\chi}_B; \quad a = \sqrt{\frac{\pi}{g^2}} \quad (18)$$

e as transformações nos campos

$$\begin{pmatrix} \bar{\chi}_A \\ \bar{\chi}_B \end{pmatrix} = \begin{pmatrix} \cos \phi & \sin \phi \\ -\sin \phi & \cos \phi \end{pmatrix} \begin{pmatrix} \chi'_A \\ \chi'_B \end{pmatrix}; \quad (19)$$

$$\begin{pmatrix} \eta_A \\ \eta_B \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \eta'_A \\ \eta'_B \end{pmatrix}, \quad (20)$$

a densidade de lagrangiana (17) desacopla e encontramos que $\tan \phi = 1/a$, de modo que

$$\begin{aligned} \mathcal{L}'_{eff} = & i\bar{\psi}' \not{\partial} \psi' + \frac{1}{2g^2} [(\partial_\mu \chi'_A)^2 + (\partial_\mu \chi'_B)^2 + (\partial_\mu \eta'_B)^2] \\ & + \frac{a^2 + 1}{2e^2} (\square \chi'_A)^2 - \frac{1}{2\alpha e^2} (\square \eta'_A)^2 + \frac{1}{e} J^\mu [\partial_\mu \eta'_A + \sqrt{a^2 + 1} \epsilon_{\mu\nu} \partial^\nu \chi'_A] \\ & + \bar{\theta} e^{-i(\eta'_A + \eta'_B) + \gamma_5 a (\cos \phi \chi'_A + \sin \phi \chi'_B)} \psi' + \bar{\psi}' e^{i(\eta'_A + \eta'_B) + \gamma_5 a (\cos \phi \chi'_A + \sin \phi \chi'_B)} \theta . \end{aligned} \quad (21)$$

A partir desta densidade de lagrangiana podemos determinar os propagadores fundamentais da teoria, por exemplo para o propagador do fóton teremos:

$$\mathcal{D}_{\mu\nu}(k) = \frac{\delta^2 Z}{\delta J_\mu \delta J_\nu} \Big|_{J=0} = -\frac{1}{e^2} \left[k_\mu k_\nu \mathcal{D}_{\eta'_A} + (a^2 + 1) \epsilon_{\mu\sigma} \epsilon_{\nu\rho} k^\sigma k^\rho \mathcal{D}_{\chi'_A} \right]. \quad (22)$$

Usando a identidade bidimensional $\epsilon_{\mu\sigma} \epsilon_{\nu\rho} = \delta_{\mu\nu} \delta_{\sigma\rho} - \delta_{\mu\rho} \delta_{\sigma\nu}$ obtemos explicitamente que

$$\mathcal{D}_{\mu\nu}(k) = -\frac{i}{k^2 - m^2} \left\{ g_{\mu\nu} - \left[1 - \frac{\alpha(k^2 + m^2)}{k^2} \right] \frac{k_\mu k_\nu}{k^2} \right\}, \quad (23)$$

onde a massa gerada dinamicamente será dada por

$$m^2 = \frac{e^2}{\pi + g^2}. \quad (24)$$

Aqui vemos o efeito da auto-interação quártica de férmions sobre a massa gerada dinamicamente m , ao mantermos fixa a constante de calibre e ao variarmos parâmetro do acoplamento quártico g . Assim quando o acoplamento g vai a zero, recupera-se a massa usual do modelo de Schwinger [1] [15] [16], e para um acoplamento forte verifica-se que

$$m^2 \sim \left(\frac{e}{g}\right)^2; \quad (g \rightarrow \infty), \quad (25)$$

mostrando uma dualidade entre a constante do acoplamento quártico e a carga elétrica, no sentido em que o modelo com acoplamento forte tem suas funções de correlação iguais às do modelo de Schwinger com carga elétrica pequena. No momento estamos realizando cálculos no sentido de estender este estudo para três dimensões.

(ii) *Teoria Quântica de Campos em Três Dimensões*

6) *Free relativistic anyon with canonical spin algebra* [Physics Letters B, vol. 343 (1995) 225].

Discutimos também o problema da construção de hamiltonianas/lagrangianas covariantes de Lorentz para anyons livres. Como primeiro cálculo neste tema, e em colaboração com o professor Denis Dalmazi da UNESP, mostramos a inconsistência de um modelo recentemente proposto na literatura [22][23][24], além de verificarmos que ao se eliminarem as inconsistências recai-se em um modelo já estudado anteriormente [25], o qual não atinge o objetivo de se ter um modelo covariante de Lorentz para anyons livres. Em seguida, procuramos desenvolver uma resposta positiva à esta questão, tendo isto sido feito através de um cálculo independente do modelo, através da imposição direta dos requisitos desejados ao modelo no sistema de vínculos de segunda classe, os quais reduzem convenientemente o espaço de fase inicial do modelo mantendo a invariância de Lorentz, e ainda mantendo a álgebra correta para o spin do anyon [26]. Todavia neste procedimento foi quebrada a invariância de Poincaré, a qual é essencial a fim de dar um conteúdo físico adequado ao modelo. Estamos agora procurando restabelecer a simetria de translação do modelo, recuperando assim a de Poincaré. Estamos também realizando estudos no sentido de entender a possível origem do zitterbewegung nestes modelos e sua relação com o aparecimento da álgebra de spin correta mantendo o caráter de partícula livre no modelo, e sem perder a de invariância de Poincaré.

7) *Consistent higher derivative quantum field theory: A model without tachyons and ghosts* [Modern Physics Letters A, vol. 11 (1996) 775].

Em uma colaboração com o professor Marcelo Hott da UNESP, desenvolvemos alguns cálculos de quantização do modelo de Maxwell-Chern-Simons com termo de Podolsky [20].



Este trabalho tinha o objetivo inicial de verificar se o termo topológico de Chern-Simons era capaz de eliminar o táquion existente na teoria eletromagnética de Podolsky. Todavia verificamos que isto não ocorre, o que nos estimulou a procurar alguma generalização do termo topológico de Chern-Simons que fizesse este trabalho. Novamente trabalhando com o professor César Natividade, foi encontrado tal termo generalizado, o qual denominamos de termo de Chern-Simons de ordem superior por se tratar de um termo com derivadas de ordem superior a duas. Além disso verificamos que ainda era possível eliminar os fantasmas do modelo, desde que restringissemos ainda mais o espaço dos parâmetros do mesmo [21]. Posteriormente verificamos que este mecanismo pode ser estendido a dimensões ímpares em geral, e que nestas dimensões teremos uma hierarquia infinita de densidades de lagrangiana, todas elas relacionadas a uma densidade de lagrangiana original sem derivadas de ordem superior. Esta relação se dá via transformações duais entre os campos, relação esta que é tal que dadas as funções de correlação de uma delas, obtem-se as demais através das relação de dualidade.

(iii) Mecânica Quântica

8) *The partition function for an anyon-like oscillator* [Journal of Physics A, vol. 28 (1995) L7].

Ainda em relação ao tema de anyons realizamos, mas agora a nível de mecânica quântica, em colaboração com os professores Henrique Boschi Filho e Carlos Farina da UFRJ cálculos com o intuito de determinar a função de partição de osciladores sujeitos a condições de contorno anyonicas [27]. Este é um estudo bastante interessante, na medida em que tal função de partição para anyons é desconhecida na literatura. Sendo importante lembrar que o estudo de anyons tem crescido enormemente nos últimos anos, devido à sua importância na descrição do efeito Hall quântico e possível aplicação à supercondutividade



de altas temperaturas.

Todavia deve-se notar que, a nível relativístico, os anyons devem existir apenas em três dimensões e no espaço de Minkowsky devido à sua origem topológica, o que mostra que o caso estudado por nós por se tratar de um problema unidimensional, possivelmente estará mais facilmente relacionado com algum tipo de paraestatística, ou mesmo com as chamadas deformações quânticas.

No momento estamos estudando a possibilidade de estender tal análise ao problema de uma partícula na presença de um solenóide (cyon), sujeita a tais condições de contorno generalizadas, de modo a procurar entender melhor a relação entre tais sistemas.

9) *Conditionally exactly soluble class of quantum potentials* [Physical Review A, vol. 47 (1993) R2435].

10) *so(2,1) Lie algebra and the Green's functions for the conditionally exactly solvable potentials* [Physical Review A, vol. 50 (1994) 2915].

11) *Expanding the class of conditionally exactly solvable potentials* [Physical Review A, vol. 50 (1994) 4369].

Nesta seqüência de trabalhos demonstramos a existência de uma nova classe de potenciais que possuem solução exata, desde que alguns de seus parâmetros sejam fixados em valores bem específicos. Estes potenciais foram batizados de condicionalmente solúveis [28], sendo que posteriormente foi discutida por E. Papp [29], e independentemente por um grupo de indianos [30] a existência de supersimetria nestes potenciais, além de calcular funções β para estes potenciais [31]. Mais recentemente o mesmo autor estudou deformações quânticas destes potenciais [32]. Em seguida realizamos um trabalho no qual mostramos ser possível generalizar uma técnica que era aplicada aos potenciais exatamente solúveis [33], de modo a mostrar que os potenciais condicionalmente solúveis encontrados em [28] possuíam simetria dinâmica determinada pelo grupo $SO(2,1)$ [34]. Posteriormente, orientando um estudante alemão da Universidade de Leipzig no programa de intercâmbio do



IAESTE, mostramos a existência de potenciais desta classe também em duas dimensões espaciais [35]. Estes últimos se caracterizam por apresentar singularidades do potencial ao longo de uma linha semi-infinita, podendo servir de modelo para o potencial definido por cordas de Dirac. Trabalhando com um aluno de iniciação científica mostramos ainda que existem potenciais desta classe dependentes de funções hiperbólicas que, a exemplo de potenciais como o de Morse, Wood-Saxon, Rosen-Morse entre outros, poderiam a princípio ser utilizados em modelos atômicos ou nucleares [36]. Infelizmente porém, quando estávamos finalizando estes cálculos, tomamos conhecimento de um trabalho [37] no qual tal possibilidade foi analisada. Ainda mais recentemente, verificamos que C. Grosche [38][39], calculou as funções de Green via integral de trajetória para os potenciais desta classe.

12) Approximate analytic expression for the eigenenergies of the anharmonic oscillator
 $V(x) = Ax^6 + Bx^2$ [Physical Review A, vol. 51 (1995) 3480].

Em uma outra linha de trabalho na mecânica quântica não-relativística, temos desenvolvido um método para obter expressões analíticas aproximadas para os auto-estados de energia de potenciais anarmônicos. Para isto aproveitamos soluções particulares destes potenciais, classificados como quase-exatamente solúveis [40], e em seguida buscamos através de métodos de interpolação determinar uma solução analítica que seja compatível com aquelas soluções exatas, as quais somente são válidas para certas relações entre os parâmetros do potencial [40]. Neste trabalho inicial foi discutido o potencial $V(x) = Ax^6 + Bx^2$ ($A > 0$), encontrando uma solução do tipo $E_n = \sum a_n X^n$, com $X \equiv \frac{B}{\sqrt{A}}$, onde os três primeiros termos da série foram determinados [41] [42], e tivemos um resultado com boa aproximação para uma região de validade para os parâmetros A e B razoavelmente grande. Em seguida procuramos aumentar a estatística, melhorando a precisão e aumentando a região de validade para os parâmetros, além de incluir um termo de barreira centrífuga no potencial acima descrito, este trabalho foi realizado em colaboração com o professor Antonio Soares de Castro da UNESP e um estudante de



iniciação científica da FAPESP [43]. No momento estamos trabalhando na extensão desta técnica a potenciais com outros termos anarmônicos, além de sua aplicação em problemas de dois ou mais anyons na presença de campos eletromagnéticos externos.

13) Strong-Weak coupling duality symmetry in quantum mechanics [Anais do “XXI International colloquium on group theoretical methods in physics” - Goslar - Alemanha] (no prelo)

Neste trabalho discutimos uma dualidade existente na equação de Schroedinger, a qual através de uma transformação de escala relaciona um potencial com acoplamento forte a um com acoplamento fraco. Utilizamos ainda esta simetria a fim de obter soluções exatas para o espectro de energia de potenciais unidimensionais do tipo potência ($V(x) = ax^N$, $N = 4, 6, 8, \dots$). Estamos ainda realizando cálculos com o intuito de obter soluções analíticas aproximadas para o espectro de potenciais polinomiais, uma vez que esta simetria nos permite reduzir consideravelmente o número de parâmetros livres na expressão da energia. É importante frisar que esta dualidade é bem geral, de forma que poderemos aplicá-la na obtenção de expressões analíticas aproximadas para o espectro de problemas como a interação entre anyons, átomos de Rydberg em campos eletromagnéticos intensos, entre outros.



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Trabalhos em:

Teoria Quântica de Campos em Duas
Dimensões

Mapping correlation functions of Schwinger and axial models

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Abstract. We show that all Green's functions of the Schwinger and axial models can be obtained one from the other. In particular, we show that the two models have the same chiral anomaly. Finally it is demonstrated that the Schwinger model can keep gauge invariance for an arbitrary mass, dispensing with an additional gauge group integration.

1. Introduction

Since the Schwinger model (SM) was introduced [1], many other two-dimensional exactly solvable models have appeared, such as the Thirring model [2], describing fermions with quartic self-interaction, the axial model (AM) [3], which describes a massive pseudoscalar field interacting with massless fermions, the chiral-Schwinger model [4], etc. The great appeal of these models in $1+1$ dimensions is that they provide the possibility of studying problems such as confinement, asymptotic freedom, renormalizability, etc., in models which can be solved exactly.

Some years ago Rothe and Stamatescu [3] presented the axial model, solving it in the operational formalism. Many other works have appeared since then [5–12], discussing several aspects of the model. In this paper we intend to show that there is a mapping between the correlation functions of this model and that of the Schwinger one. However, it is important to remark that this mapping is not trivial, in the sense that they are equal models. In fact, as we will see later, the rules to pass from one model to the other one are such that the infrared singularity appearing in the SM appears only after the application of the rule, because the AM model is not infrared divergent. The material is organized as follows: in section 2 we relate the Lagrangian density of the SM with that of a pseudoscalar field with higher-derivative terms interacting with massless fermions, establishing their connection and demonstrating the relation between the boson self-energies of the SM and of the AM, which appears as an observation in the original work of Rothe and Stamatescu [3]; in section 3, we obtain the anomaly of the SM from that calculated for the AM, deriving the rules to pass from AM Green's function to that of the SM; then in section 4 we show the relation of the effective Lagrangian densities of the models by solving exactly the SM; section 5 is devoted to see the relation via the method of point-splitting; in section 6 it is shown that the SM has a bosonic effective action that is manifestly gauge invariant, with an arbitrary mass for the gauge boson. This is done without integrating in the gauge volume, as in Harada-Tsutsui's work [13], and this is interpreted. Finally, in section 7 we present our conclusions.

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2. The SM as a pseudoscalar model

The Lagrangian density of the SM is given by

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - e\gamma^\mu A_\mu)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + J_\mu A^\mu + \bar{\psi}\theta + \bar{\theta}\psi \quad (1)$$

where we have included the field sources, necessary to obtain correlation functions in the functional formalism. Throughout the work we use

$$\{\gamma^\mu, \gamma^\nu\} = 2\delta^{\mu\nu} \quad (2)$$

and in two dimensions and in Euclidean space it is valid that

$$\gamma_5 = i\gamma^0\gamma^1 \quad \gamma_5\gamma^\mu = i\epsilon^{\mu\nu}\gamma_\nu \quad \epsilon^{01} = +1. \quad (3)$$

Now we can use the vectorial identity, valid in 2D:

$$eA_\mu = g[\partial_\mu\eta + \epsilon_{\mu\nu}\partial^\nu\chi]. \quad (4)$$

In this section we will choose $\eta = 0$, as can always be done in the Lorentz gauge [14]. In section 4 we shall see that, in fact, this corresponds to the use of identity (4), putting at the end the gauge parameter α equal to zero (Landau gauge). Now, it is easy to rewrite the Lagrangian density as

$$\mathcal{L} = \bar{\psi}\gamma^\mu(i\partial_\mu - ig\gamma_5\partial_\mu\chi)\psi + (g^2/2e^2)\chi\Box\chi + (g/e)\epsilon_{\mu\nu}J^\mu\partial^\nu\chi + \bar{\psi}\theta + \bar{\theta}\psi \quad (5)$$

where we have used $\epsilon_{\mu\nu}\gamma_\mu\partial_\nu = i\gamma_\nu\gamma_5\partial_\mu$, and

$$\epsilon^{\mu\nu}\epsilon^{\sigma\rho} = -\delta^{\mu\sigma}\delta^{\nu\rho} + \delta^{\mu\rho}\delta^{\nu\sigma}. \quad (6)$$

The bosonic correlation functions are obtained by taking functional derivatives with respect to the source $J^\mu(x)$. To each derivative in this source we obtain a term of the type

$$\begin{aligned} (1/i)\delta Z/\delta J_\mu(x) &= N^{-1} \int \mathcal{D}\bar{\psi} \mathcal{D}\psi \mathcal{D}\chi ((g/e)\epsilon^{\mu\nu}\partial_\nu)\chi^{(x)} \exp(i\bar{S}_\kappa) \\ &= (g/e)\epsilon^{\mu\nu}\partial_\nu (1/i)\delta \bar{Z}/\delta K(x) \end{aligned} \quad (7)$$

where

$$\bar{Z}[\bar{\theta}, \theta, K] = N^{-1} \int \mathcal{D}\bar{\psi} \mathcal{D}\psi \mathcal{D}\chi \exp(i\bar{S}_\kappa) \quad (8a)$$

and $\bar{S}_\kappa$ comes from:

$$\bar{S}_\kappa = \bar{\psi}\gamma^\mu(i\partial_\mu - ig_0\gamma_5\partial_\mu\chi)\psi + (g^2/2e^2)\chi\Box\chi + \bar{\psi}\theta + \bar{\theta}\psi + K\chi. \quad (8b)$$

On the other hand, the Lagrangian density of the AM is given by

$$\mathcal{L}_{AM} = \bar{\psi}\gamma^\mu(i\partial_\mu - ig_0\gamma_5\partial_\mu\phi)\psi + \frac{1}{2}(\partial_\mu\phi)^2 - (m_0^2/2)\phi^2 + \bar{\psi}\theta + \bar{\theta}\psi + K\phi. \quad (9)$$

We can see that the only differences between (8a) and (9) are the corresponding free bosonic Lagrangian densities.

Using the rule (7), we have for the photon propagator:

$$\begin{aligned} D^{\mu\sigma}(x-y) &\equiv (1/i^2)\delta^2 Z/\delta J_\mu(x)\delta J_\sigma(y)|_{J=0} \\ &= (g/e)^2\epsilon^{\mu\nu}\epsilon^{\sigma\rho}\partial_\nu\partial_\rho(1/i^2)\delta^2 \bar{Z}/\delta K(x)\delta K(y)|_{K=0} \end{aligned} \quad (10a)$$

and so

$$D^{\mu\sigma}(x-y) = (g/e)^2\epsilon^{\mu\nu}\epsilon^{\sigma\rho}\partial_\nu\partial_\rho D(x-y). \quad (10b)$$

In momentum space, the free propagator $D_\kappa^0(k)$ for the boson χ in $\mathcal{L}_\kappa$ is given by

$$D_\kappa^0(k) = (e/g)^2/k^4. \quad (11)$$

Remembering that the equation (10b) is also valid for the case of the free photon propagators $D_{\mu\sigma}^0(x-y)$ and $D^0(x-y)$, we can substitute (11) in (10b) and, with the help of formula (6), obtain the free photon propagator,

$$D_{\mu\sigma}^0(k) = (1/k^2)(-g_{\mu\sigma} + k_\mu k_\sigma/k^2) \quad (12)$$

which corresponds to the expected results in the Landau gauge.

Now, we will use (10b) to find the relation between boson self-energies of the SM and AM, as noticed by Rothe and Stamatescu [3], although there it appears without demonstration. The complete propagators for these models are

$$\begin{aligned} D_{\mu\sigma}(x-y) &= \text{wavy line} + \text{wavy line} \text{---} \text{circle} \text{---} \text{wavy line} + \text{wavy line} \text{---} \text{circle} \text{---} \text{circle} \text{---} \text{wavy line} + \dots \\ &= D_{\mu\sigma}^0/(1 - \Pi^{\rho\nu} D_{\rho\nu}^0) \end{aligned} \quad (13a)$$

for the field A_μ in the Schwinger model, and

$$D(x-y) = D^0/(1 - \Pi D^0) \quad (13b)$$

for χ in $\mathcal{L}_\kappa$ (8b). Using (10a), and remembering that this relation is valid also for free propagators, and making straightforward calculations, we obtain:

$$\Pi D^0 D_{\mu\sigma}^0 = (g/e)^2\epsilon^{\mu\nu}\epsilon^{\sigma\rho}\partial_\nu\partial_\rho \Pi^{\mu\sigma} D^0 D_{\mu\sigma}^0 \quad (14)$$

so that, in momentum space we have

$$\Pi(k) = (g/e)^2\epsilon^{\mu\nu}\epsilon^{\sigma\rho}k_\nu k_\rho \Pi^{\mu\sigma} \quad (15)$$

which, up to the multiplicative factor $(g/e)^2$, is equal to that which appears in the original work in the AM [3]. Besides, we observe that this relation is valid only when we are working in the Landau gauge ($\alpha = 0$). It is interesting to note that now we can obtain similar relations for other Green's functions as, for example, the four-photon scattering amplitude. Finally, we observe that the above demonstration, made for the field χ , can easily be repeated for the field ϕ of the AM, because it does not depend on the explicit form of the propagator $D(x-y)$ used.

Through the use of (13a), the full photon propagator can be obtained [3],

$$D_{\mu\nu}(k) = (k^2 - m_c^2)^{-1}(-g_{\mu\nu} + k_\mu k_\nu/k^2) \quad \text{with} \quad m_c^2 = e^2/\pi. \quad (16)$$

3. The anomalies

We observe that the Lagrangian density (8b) differs from that of the AM (9) only through the form of the free boson propagator, and that in two dimensions the anomaly appears in the polarization tensor, which depends only on the fermionic loop. It is not difficult to conclude that these models have the same anomaly. If this is true, we can obtain the anomaly of the SM from that of the AM. The anomaly of the AM is expressed as [12]

$$\langle \partial_\mu j_{AM}^{\mu,5}(x) \rangle = (ag/\pi) \square \phi \quad (17)$$

where a is an arbitrary regularization parameter; now using (4), we have that

$$e\epsilon^{\rho\mu}\partial_\rho A_\mu = g\square\chi \quad (18)$$

so that, if we identify the field ϕ of the AM with the field χ of the SM, we get

$$\langle \partial_\mu j_{AM}^{\mu,5}(x) \rangle = (ae/2\pi)\epsilon^{\rho\mu}F_{\rho\mu} = \langle \partial_\mu j_{SM}^{\mu,5}(x) \rangle. \quad (19)$$

So, as we have proved that the two models have the same anomaly, we are stimulated to ask if the Feynman graphs of the two models are also related. Let us start calculating the electron self-energy to 1-loop

$$\text{Diagram: fermion line with a wavy photon loop} = e^2 \int (d^2k/4\pi^2) \gamma^\mu \frac{1}{\gamma^\sigma(p_\sigma + k_\sigma)} \gamma^\nu D_{\mu\nu} \quad (20)$$

where $D_{\mu\nu}$ is the complete photon propagator, which is given in (16). Substituting (12) in (19), we obtain that

$$\text{Diagram: fermion line with a wavy photon loop} \equiv \sum_{SM} (p) = e^2 \int (d^2k/4\pi^2) \left(\frac{1}{\gamma^\sigma(p_\sigma + k_\sigma)} \right) (k^2 - m_e^2)^{-1} \quad (21)$$

for the case of the massive AM, we have also:

$$\text{Diagram: fermion line with a dashed boson loop} = -g^2 \int (d^2k/4\pi^2) \left(\frac{1}{\gamma^\sigma(p_\sigma + k_\sigma)} \right) D(k) \equiv \sum_{AM} (p)$$

with $D(k)$ being the propagator of the AM, given by

$$D(k) = Z_\phi(k^2 - m_\phi^2)^{-1} \quad (22)$$

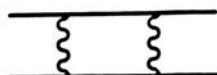
so that

$$\sum_{SM} (p) = -Z_\phi^{-1}(e/g)^2 \sum_{AM} (p). \quad (23)$$

Provided that the masses are identified by making use of the arbitrary parameter a in the AM, we come to

$$m_e^2 = m_\phi^2 \Rightarrow a = \pi(1 - m_0^2/e^2)/g^2. \quad (24)$$

It is also not difficult to show that for the one-particle irreducible graphic of the 4-point Green function:



$$\equiv \Gamma(p_1, p_2)$$

the relation is given by:

$$\Gamma_{\text{SM}}(p_1, p_2) = +Z_\phi^{-2}(e/g)^4 \Gamma_{\text{AM}}(p_1, p_2). \quad (25)$$

In general, the 1PI graphics of n -point Green's functions will be related, at least at 1-loop level, by:

$$\Gamma_{\text{SM}}^{(n)} = (-Z_\phi^{-1/2} e/g)^n \Gamma_{\text{AM}}^{(n)}. \quad (26)$$

For graphics with photon external legs, there will exist similar relations to that in equation (10a), by repeated use of the rule (10b).

4. Non-perturbative approach

In this section we intend to show the relation between these models directly by solving exactly the SM in an arbitrary gauge of Lorentz type, and then show that in a particular gauge the correlation functions can be related. For this we will use chiral rotation as a decoupling transformation [15, 16]. So we start from the Lagrangian density

$$\mathcal{L} = \bar{\psi} \gamma_\mu (i\partial_\mu - eA_\mu) \psi - \frac{1}{4} F_{\mu\nu} F_{\mu\nu} - 1/(2\alpha) (\partial_\mu A_\mu)^2 + J_\mu A_\mu + \bar{\psi} \theta + \bar{\theta} \psi \quad (27)$$

and using the identities (3) and (4) we get

$$\begin{aligned} \mathcal{L} = & \bar{\psi} \gamma_\mu (i\partial_\mu - g\partial_\mu \eta - ig\gamma_5 \partial_\mu \chi) \psi + (g^2/2e^2) \chi \square^2 \chi - (e^2/2\alpha) \eta \square^2 \eta \\ & + (g/e) J_\mu [\partial_\mu \eta + \epsilon_{\mu\nu} \partial_\nu \chi] + \bar{\psi} \theta + \bar{\theta} \psi. \end{aligned} \quad (28)$$

Making now the transformation in the Euclidean generating functional:

$$\psi = \exp(-ig\eta + g\gamma_5 \chi) \psi' \equiv U_5 \psi' \quad (29a)$$

$$\bar{\psi} = \bar{\psi}' \exp(ig\eta + g\gamma_5 \chi) \equiv \bar{\psi}' \bar{U}_5 \quad (29b)$$

we obtain

$$Z = \int \mathcal{D}\bar{\psi}' \mathcal{D}\psi' \mathcal{D}\eta \mathcal{D}\chi J_F \exp \left[- \int d^2x \mathcal{L}' \right] \quad (30)$$

where

$$\begin{aligned} \mathcal{L}' = & \bar{\psi}' (i\gamma_\mu \partial_\mu) \psi' + (g^2/2e^2) \chi \square^2 \chi - (g^2/2\alpha e^2) \eta \square^2 \eta + (g/e) J_\mu [\partial_\mu \eta + \epsilon_{\mu\nu} \partial_\nu \chi] \\ & + \bar{\psi}' \bar{U}_5 \theta + \bar{\theta} U_5 \psi'. \end{aligned} \quad (31)$$

As the Jacobian from the fermionic chiral transformation is not trivial, it is necessary to compute it. In order to calculate it, we will follow Banerjee [7] and Roskies and Schaposnik

[15]. This consists essentially of using Fujikawa's procedure [17], so that the infinitesimal fermionic Jacobian comes from:

$$\mathcal{D}\bar{\psi}\mathcal{D}\psi = J_F \mathcal{D}\bar{\psi}'\mathcal{D}\psi'$$

with

$$J_F = \exp \left[- \int d^2x \sum g \chi(x) \phi_m^+(x) \gamma_5 \phi_m(x) \right] \quad (32)$$

where ϕ_m is a complete set of normalized eigenvectors of a given operator, with which we can expand the fermion fields. As is well known, this is an ill defined quantity, so we regularize it in order to suppress large eigenvalues. With this in mind we replace the singular part of the Jacobian by

$$\lim_{M \rightarrow \infty} \sum_m \phi_m^+(x) \gamma_5 \exp(-\lambda_m^2/M^2) \phi_m(x) \quad (33)$$

with λ_m being the eigenvalues of the regulator in the basis ϕ_m . In general we can choose the regularization operator as

$$D \equiv i\gamma_\mu \partial_\mu - g a \gamma_\mu A_\mu \quad (34)$$

and a is the regularization arbitrary parameter. Using it we can rewrite (33) as

$$\lim_{M \rightarrow \infty} \sum_m \phi_m^+(x) \gamma_5 \exp(-D^2/M^2) \phi_m(x) \quad (35)$$

and after some manipulations [17], we obtain

$$J_F = \exp \left[- \int d^2x \frac{a g^2}{2\pi} \chi \square \chi \right] \quad (36)$$

where it was necessary to use the method of [15] in order to compute the Jacobian for the finite transformation. Now we can write the effective Lagrangian density:

$$\begin{aligned} \mathcal{L}' = & \bar{\psi}'(i\gamma_\mu \partial_\mu) \psi' + (g^2/2e^2) \chi [\square^2 + m_e^2] \chi - (g^2/2\alpha e^2) \eta \square^2 \eta + (g/e) J_\mu [\partial_\mu \eta + \epsilon_{\mu\nu} \partial_\nu \chi] \\ & + \bar{\psi}' \bar{U}_5 \theta + \bar{\theta} U_5 \psi' \end{aligned} \quad (37)$$

with $m_e^2 = a e^2 / \pi$. At this point we can introduce a new field in order to get rid of the higher derivatives. For this we perform the transformation

$$\chi = (e/\sqrt{\square}) \varphi \quad (38)$$

so that

$$\begin{aligned} \mathcal{L}_{\text{eff}} = & \bar{\psi}'(i\gamma_\mu \partial_\mu) \psi' + (g^2/2) \varphi [\square + m_e^2] \varphi - (g^2/2\alpha e^2) \eta \square^2 \eta \\ & + (g/e) J_\mu [\partial_\mu \eta + e \epsilon_{\mu\nu} (\partial_\nu / \sqrt{\square}) \varphi] + \bar{\psi}' \bar{U}_5 \theta + \bar{\theta} U_5 \psi'. \end{aligned} \quad (39)$$

It is easy to see now that, up to the source terms and a gauge fixing, we can identify this effective Lagrangian density with that of the axial model, provided that we impose

the evenness of the masses through the regularization arbitrary parameter, as is stated in (24). At this point it is important to remember the existence of an infrared singularity in the SM, as can be seen from the free part of (37). However, the AM does not have such singularity; this enlightens the rule taken by the non-local transformation (38), and it transfers this singularity from the propagator to the sources, as can be seen from (39), in the sense that when the functional derivatives are taken in order to get the correlation functions, this singularity will come from the mapping rule. As an example, we obtain the relation between the two-point Green function of the two models; from (39) we find in the Landau gauge:

$$D_{SM}^{\mu\nu}(x-y) = Z_\phi^{-1} \epsilon_{\mu\lambda} \epsilon_{\nu\rho} \left(\frac{\partial_\lambda \partial_\rho}{\square} \right) D_{AM}(x-y) \quad (40)$$

where the masses were identified through (24). Similarly, one can calculate the other relations between correlation functions of the two models by repeated use of the functional derivatives with respect to the sources.

The bosonic propagator in a general Lorentz-type gauge is easily computed by doing analogous calculations, giving for the full photon propagator:

$$D_{\mu\nu}(k) = (g/e)^2 [k_\mu k_\nu D_\eta(k) + e^2 \epsilon_{\mu\lambda} \epsilon_{\nu\sigma} (k_\lambda k_\sigma / k^2) D_\varphi(k)] \quad (41)$$

where the propagators of the fields η and φ are given respectively by

$$D_\eta(k) = -(g/e)^2 (\alpha/k^4) \quad D_\varphi(k) = (1/g^2) [1/(k^2 - m_c^2)] \quad (42)$$

and consequently we obtain

$$D_{\mu\nu}(k) = (1/(k^2 - m_c^2)) \{-g_{\mu\nu} + [1 - \alpha(1 - (m_c^2/k^2))](k_\mu k_\nu / k^2)\}. \quad (43)$$

From the above Green function we see easily that for the Landau gauge ($\alpha = 0$) the propagator (22) is re-obtained, as we have asserted before. Similarly we could work with the fermionic correlation functions.

5. Point-splitting

Here we discuss the relation of the anomaly of these models via the point-splitting method. For this we remember firstly that the gauge current and the chiral one, in Euclidean two-dimensional space, are related through

$$j_5^\mu(x) = i\epsilon^{\mu\nu} j^\nu(x) \quad (44)$$

where we defined $j^\mu(x) \equiv \bar{\psi}(x)\gamma^\mu\psi(x)$ and $j_5^\mu(x) \equiv \bar{\psi}(x)\gamma^\mu\gamma_5\psi(x)$. So, we only need to calculate the vectorial current and, from it, obtain the chiral one.

In the point-splitting method we can define the vectorial current as [18]:

$$j^\mu(x) \equiv \frac{1}{2} \left[\lim_{\epsilon \rightarrow 0^+} j^\mu(x, \epsilon, a) + \lim_{\epsilon \rightarrow 0^-} j^\mu(x, \epsilon, a) \right] \quad (45a)$$

and

$$j^\mu(x, \epsilon, a) \equiv \bar{\psi}(x - \epsilon/2)\gamma^\mu\psi(x + \epsilon/2) \exp \left\{ -ie(2a + 1) \int_{x-\epsilon/2}^{x+\epsilon/2} dz^\nu A^\nu(z) \right\} \quad (45b)$$

where a is the regularization parameter. In principle the above current will be gauge invariant only in the particular case when $a = -1$; below we will see that the introduction of a new gauge-dependent external field—the Wess–Zumino case—restores the gauge invariance for any value of a .

The average vectorial current is then written as

$$\langle j_\mu(x, \varepsilon, a) \rangle = \text{Tr} \left\{ \gamma_\mu G_+(x + \varepsilon/2, x - \varepsilon/2, A_\mu) \exp \left[-ie(2a + 1) \int_{x-\varepsilon/2}^{x+\varepsilon/2} dz^\nu A^\nu(z) \right] \right\} \quad (46)$$

where $G_+(x + \varepsilon/2, x - \varepsilon/2, A_\mu)$ is the fermionic Green function in presence of the external field $A_\mu(x)$, and is given by

$$G_+(x + \varepsilon/2, x - \varepsilon/2, A_\mu) = \exp[-ie[\chi(x + \varepsilon/2) - \chi(x - \varepsilon/2)]] G_+(\varepsilon) \quad (47a)$$

and $G_+(\varepsilon)$ is given by

$$G_+(\varepsilon) = (-i/2\pi) \frac{\gamma^\mu \varepsilon^\mu}{\varepsilon^2}. \quad (47b)$$

Expanding the expression in powers of ε , we obtain

$$\langle j_\mu(x, \varepsilon, a) \rangle = (-i/2\pi) \text{Tr} \left\{ \gamma_\mu [1 + ie\gamma_5(\partial_\sigma \chi)\varepsilon_\sigma] \frac{\gamma^\nu \varepsilon^\nu}{\varepsilon^2} [1 - ie(2a + 1)\varepsilon^\rho A_\rho] + O(\varepsilon^2) \right\}. \quad (48)$$

Taking the symmetric limit defined in (44a), and after straightforward calculations, we see that the current defined in (44b) is given by

$$\langle j^\mu(x) \rangle = (e/\pi) [\partial_\mu \partial_\nu / \square - a \delta_{\mu\nu}] A_\nu(x). \quad (49)$$

Now, with this result at hand we can compute the divergence of the currents:

$$\langle \partial_\mu j_\mu(x) \rangle = (e/\pi)(1 + a) \partial_\mu A_\mu(x) \quad (50a)$$

$$\langle \partial_\mu j_{\mu,5}(x) \rangle = -(iea/\pi) \epsilon_{\mu\nu} \partial_\mu A_\nu(x). \quad (50b)$$

For $a = -1$, there is no gauge anomaly; all anomaly is concentrated in the chiral current.

With the above results we discuss the relation with the AM. For this we use the decomposition of the gauge field in equation (4), and obtain:

$$\langle \partial_\mu j_{SM}^\mu(x) \rangle = (g/\pi)(1 + a) \square \eta \quad (51a)$$

and

$$\langle \partial_\mu j_{SM}^{\mu,5}(x) \rangle = (-iag/\pi) \square \chi. \quad (51b)$$

The divergence of the chiral current is exactly that for the AM, as can be seen in the work of Banerjee [7] and, for $a = 1$, in the original work of Rothe and Stamatescu [3]. Besides, for a special gauge choice ($\eta = 0$), as is made in [14] the vector current vanishes, as in the

case of the AM. This result corroborates and clarifies the observation in Banerjee's work, that there is a 'striking resemblance between the two theories'.

However, it could be argued that the arbitrary parameter appearing in the anomaly of the AM would not appear in SM because of gauge invariance. This argument is not valid because a Wess–Zumino field can be added, without changing physics, and the gauge invariance restored for any value of α [13].

In the point-splitting method this can be made by a suitable redefinition of the vectorial current,

$$j^\mu(x, \varepsilon, a) \equiv \bar{\psi}(x - \varepsilon/2) \gamma^\mu \psi(x + \varepsilon/2) \times \exp \left\{ ie \left[-(2a + 1) \int_{x-\varepsilon/2}^{x+\varepsilon/2} dz^\nu A^\nu(z) - 2(a + 1) \int_{x-\varepsilon/2}^{x+\varepsilon/2} dz^\nu \partial^\nu \theta(z) \right] \right\} \quad (52)$$

where the fields $A_\mu(x)$ and $\theta(x)$ change by a gauge transformation as:

$$A_\mu(x) \rightarrow A_\mu(x) + \partial_\mu \alpha(x) \quad \theta(x) \rightarrow \theta(x) - \alpha(x). \quad (53)$$

With this definition we see that the gauge invariance is restored for the vectorial current.

6. The effective action

In this section, we calculate the photon effective action for the SM in the trivial topological sector, showing that it can keep its mass not fixed (through an arbitrary parameter) and simultaneously preserving gauge invariance. Usually it is argued that this symmetry can be preserved only for the mass obtained originally by Schwinger [1] ($m_c^2 = e^2/\pi$). Recently, however, Harada and Tsutsui [13] showed that, if we perform the integration in the gauge volume, this gauge symmetry is kept for an arbitrary mass ($m_c^2 = ae^2/\pi$). Here, we intend to re-obtain the result of Harada and Tsutsui [13] without performing such integration.

Starting from the Lagrangian density (5) without the sources, and doing analogous calculations to those which lead to (37), we find,

$$\mathcal{L}' = \bar{\psi}'(i\gamma_\mu \partial_\mu) \psi' + (g^2/2e^2) \chi [\Box^2 + m_c^2 \Box] \chi. \quad (54)$$

Now, integrating over the fermion fields, and inverting the transformation (4) for the field χ ,

$$\chi = (e/2g) \epsilon_{\mu\nu} \left(\frac{F^{\mu\nu}}{\Box} \right) \quad (55)$$

we obtain finally the following effective bosonic effective action:

$$S_{\text{eff}} = -\frac{1}{4} \int d^2x [F_{\mu\nu} \Box^{-1} (\Box + m_c^2) F^{\mu\nu}] \quad (56)$$

where $m_c^2 = ae^2/\pi$. This expression is identical to that obtained by Harada and Tsutsui for this model, but has appeared in a completely different way, without using any gauge group integration at any stage. This is an intriguing result, because it is generally assumed

that the SM is gauge invariant only when the parameter a is made equal to one, except if one introduces the Wess–Zumino term via gauge volume integration. Besides it is not difficult to verify that the above procedure does not lead to an analogous result for the chiral Schwinger model (CSM) [4]. In fact, we think that this occurs because, when people are dealing with such models, the gauge-fixing term that comes from the Faddeev–Popov trick is not considered, and the absence of this term corresponds, in practice, to a particular choice of the gauge-fixing parameter α . Had we used this term, the longitudinal part of the gauge field would appear in the SM and so the effective Lagrangian density would be not gauge invariant. On the other hand, this part of the gauge field appears quite naturally in the CSM, and this prevents the gauge invariance of the effective Lagrangian density.

In our opinion, this makes necessary the consideration of the gauge-fixing term from the beginning in the Lagrangian density. Consequently, we would have a modified Wess–Zumino term. This conjecture is presently under investigation.

7. Conclusions

We have proved that the Green functions of SM and of the AM can be mapped, and this was made through the rules necessary to go from one model to the other and that their effective Lagrangians are identical, provided that the masses of the models are identified. Similarly, other models, such as the CSM [4] and the generalized SM [19], may have their ‘partner models’ with Green functions related. In this sense they would be redundant models.

Besides, as a by-product of our calculations, we re-derived the effective Lagrangian of the SM obtained in [11] without any gauge volume integration, and discussed the possible explanation of this intriguing result.

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COMPARISON OF THE ANOMALOUS AND NONANOMALOUS GENERALIZED SCHWINGER MODELS VIA THE FUNCTIONAL FORMALISM

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We calculate the Green functions of the two versions of the generalized Schwinger model, the anomalous and the nonanomalous one, in their higher order Lagrangian density form. Furthermore, it is shown through a sequence of transformations that the bosonized Lagrangian density is equivalent to the former, at least for the bosonic correlation functions. The introduction of the sources from the beginning, leading to a gauge-invariant source term, is also considered. It is verified that the two models have the same correlation functions only if the gauge-invariant sector is taken into account. Finally, there is presented a generalization of the Wess–Zumino term, and its physical consequences are studied, in particular the appearance of gauge-dependent massive excitations.

1. Introduction

For a long time,^{1,2} quantum field theory has been widely investigated in two-dimensional space-time, essentially with the hope that one will get a deeper understanding of the quantum field mechanisms and that it may be translated in some way to higher dimensions. Indeed, this has been done principally through the quest for models in which some types of interesting mechanisms work. This is exactly the case with the chiral Schwinger model,³ which was introduced by Jackiw and Rajaraman as an anomalous gauge model that, despite this feature, is unitary and renormalizable. The essential idea underlying the interest in this specific model was that of obtaining a model which dispenses with the Higgs field. This happens because the model presents a dynamical mass generation and concomitantly does not depend also on a “closing of family” in order to prevent the anomaly and preserve the consistency of the theory. This is focused on the nonappearance of the top

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particle. Later it was discovered that the model could be thought of as a gauge-fixed version of a nonanomalous model, provided that its functional measure was correctly manipulated.^{4,5} Since then, a lot of papers have appeared discussing a variety of its peculiarities and, in particular, the fact that the Wess-Zumino (WZ) term does not change the physical content of the model.⁶⁻¹³ More recently, people have been interested in a model that contains as particular cases the old vector Schwinger model (VSM), the axial Schwinger model (ASM) and the chiral Schwinger model (CSM).¹⁵⁻²⁰

This last model, the generalized Schwinger one (GSM), has been treated perturbatively by Chanowitz,¹⁵ who discussed some ambiguities in the specification of the anomaly, and the WZ term was also considered. Subsequently Miyake and Shizuya¹⁶ worked on the fermion content of the model, and Boyanovsky, Schimdt and Golterman⁹ used it as a starting point to make a comparison among the VSM and CSM properties. After that, Wotzasek and Naon¹⁷ studied its bosonized version through the Dirac algorithm and also the case of massive fermions. Furthermore, Shin Lee and Lee¹⁸ studied the problem of their currents and energy-momentum tensor, and Alonso, Cortes and Rivas¹⁹ used it to discuss the regularization ambiguity in Fujikawa's regularization.²² More recently, Dias and Linhares²⁰ presented the point-splitting computation of the fermionic determinant, and discussed the various classes of regularization. Furthermore, this author,²¹ starting from a gauge principle, obtained gauged bosonized theories, applying it for the GSM and also in the case of $2+1$ dimensions.

However, as far as we know, most of these papers discussed the models through their bosonized version, because in the functional formalism there is a possibility of rewriting locally the nonlocal gauge field effective Lagrangian density. Meanwhile, as we will see below, this nonlocal Lagrangian density can also be traded by a local one through the use of the decomposition property of the gauge field in two dimensions; but, in this case, one is faced with a higher order theory. Here we intend to study the generalized model in this version, showing the equivalence with the bosonized one, and discussing both the anomalous and the nonanomalous one. In this last case we will introduce two types of gauge-fixing. They contain as particular cases those appearing in the literature. We will discuss also the gauge-invariant correlation functions, showing that they correspond to the anomalous ones as observed by Girotti and Rothe,¹² this time using a modified source term in the generating functional as suggested by Linhares, Rothe and Rothe.¹⁰

All the computations will be done in the functional integral formalism; and, as the GSM, like the CSM, has a singular point for the regularization parameter ($M^2 = g_1^2$ in the GSM and $a = 1$ in the CSM), this case will be treated in a separate way. The arbitrary parameter case ($M^2 \neq g_1^2$) will be solved through field transformations that become singular when $M^2 = g_1^2$, making clear, in this formalism, the origin of the problem with this parameter choice. Then, in the case $M^2 = g_1^2$, we will use another approach in order to solve it.

In the last section of the article, we will argue about the introduction of a novel and generalized WZ term. After its presentation by using the method of Harada and Tsutsui,⁵ we will investigate the physical consequences of its postulation. In doing so, for the sake of simplicity, we will consider the cases of the VSM, ASM and CSM. The first two are used to get some confidence in the mechanism because of their well-stated physical properties. Finally, we construct one possible realization of the WZ term for the CSM and find out a surprising feature, that in this case there are two dynamically generated masses of the gauge field, and that one of them disappears when an appropriate gauge is used.

Another remarkable consequence of using generalized WZ terms is that, even when the gauge-invariant correlation functions are used, some arbitrariness does persist in their definitions. This arbitrariness could be used, for example, to overcome the misbehavior of the gauge propagator at low coupling constants, which has generated some controversy.^{7,12} In fact it appears somewhat strange that, differently from all other gauge theories, the nonanomalous CSM, and by extension the nonanomalous GSM, could be the only ones having their gauge propagators with the same form, no matter what gauge-fixing condition is being used.

This paper is organized as follows. In Sec. 2 is presented the model in two versions, the anomalous and the nonanomalous one. Also, we prove the equivalence of the bosonized and higher order versions of the model, at least for the bosonic correlation functions. In Secs. 3 and 4 we study the cases of $M^2 \neq g_1^2$ and $M^2 = g_1^2$ respectively. Section 5 is devoted to the generalized WZ term and its peculiarities. Finally, in Sec. 6 we make our final remarks.

2. The Model and Its Versions

The Lagrangian density for the GSM is given by

$$\mathcal{L} = \bar{\psi} \gamma_\mu (i\partial^\mu + e_R P_+ A^\mu + e_L P_- A^\mu) \psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + J_\mu A^\mu + \bar{\psi} \zeta + \bar{\zeta} \psi, \quad (1)$$

with the source terms included and $P_\pm \equiv (1/2)(1 \pm \gamma_5)$. The fermion and gauge fields can be decoupled through a suitable transformation,

$$\psi = \exp[-i\gamma_5(g_1\eta + g_2\chi) + i(g_2\eta + g_1\chi)]\psi' \equiv U_5\psi', \quad (2a)$$

$$\bar{\psi} = \bar{\psi}' \exp[-i\gamma_5(g_1\eta + g_2\chi) - i(g_2\eta + g_1\chi)]\psi' \equiv \bar{\psi}' \bar{U}_5, \quad (2b)$$

where η and χ are identified with the longitudinal and transverse components of the gauge field A_μ , as we will see below. Using (2) and also the identity $\gamma_\mu \epsilon^{\mu\sigma} = -\gamma^\sigma \gamma_5$, valid in the two-dimensional Minkowski space-time, one decouples the classical Lagrangian density, giving

$$\mathcal{L} = \bar{\psi}' i\gamma_\mu \partial^\mu \psi' - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + J_\mu A^\mu + \bar{\psi}' \bar{U}_5 \zeta + \bar{\zeta} U_5 \psi'. \quad (3)$$

Besides, as a consequence of the noninvariance of the functional measure under chiral transformations, there is an additional effective term in the Lagrangian density, coming from the regularized Jacobian.²² Taking this into account, we have an effective higher order Lagrangian density given by

$$\begin{aligned}\mathcal{L} = & \bar{\psi}'(i\gamma_\mu\partial^\mu)\psi' + \frac{1}{2e^2}\chi\Box\chi + \frac{1}{2\pi e^2}(g_1^2 - M^2)\eta\Box\eta \\ & + \frac{1}{2\pi e^2}(g_2^2 + M^2)\chi\Box\chi - \frac{g_1g_2}{\pi e^2}\eta\Box\chi \\ & + \frac{1}{e}J_\mu(\partial^\mu\eta + \epsilon^{\mu\sigma}\partial_\sigma\chi) + \bar{\psi}'\bar{U}_5\zeta + \bar{\zeta}U_5\psi',\end{aligned}\quad (4)$$

where $M^2 \equiv ae^2$, a being the arbitrary regularization parameter, $g_1 \equiv (e_R - e_L)/2$, $g_2 \equiv (e_R + e_L)/2$, and $\eta(x)$ and $\chi(x)$ are, respectively, the longitudinal and transversal components of the gauge field, which in 1 + 1 dimensions can always be decomposed as

$$eA_\mu = \partial_\mu\eta + \epsilon^{\mu\sigma}\partial_\sigma\chi. \quad (5)$$

From the above identity one can get two others,

$$\eta = e\partial \cdot A/\Box, \quad \chi = -e\bar{\partial} \cdot A/\Box, \quad (6)$$

where we have defined $\partial \cdot A \equiv \partial_\mu A^\mu$ and $\bar{\partial} \cdot A \equiv \epsilon^{\rho\sigma}\partial_\sigma A_\rho$. Using these identities and an additional one,

$$\epsilon^{\mu\rho}\epsilon^{\nu\sigma} = g^{\mu\sigma}g^{\rho\nu} - g^{\mu\nu}g^{\rho\sigma}, \quad (7)$$

all valid in two dimensions, we get for the effective Lagrangian density the expression⁹

$$\begin{aligned}\mathcal{L} = & \bar{\psi}'(i\gamma_\mu\partial^\mu)\psi' \\ & + \frac{1}{2\pi}A_\mu\left[(M^2 + g_2^2)g^{\mu\nu} - \frac{(g_1^2 + g_2^2)\partial^\mu\partial^\nu}{\Box} + \frac{g_1g_2(\partial^\mu\bar{\partial}^\nu + \bar{\partial}^\mu\partial^\nu)}{\Box}\right]A_\nu \\ & + J_\mu A^\mu + \bar{\psi}'\bar{U}_5\zeta + \bar{\zeta}U_5\psi'.\end{aligned}\quad (8)$$

This nonlocal version can be rewritten in another local form, by introducing an auxiliary field. Besides we can discard the sources and integrate out the fermions, obtaining the bosonized Lagrangian density⁹

$$\mathcal{L}_B = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{\sqrt{\pi}}(g_1g^{\mu\nu} - g_2\epsilon^{\mu\nu})\partial_\nu\phi A_\mu + \frac{M^2}{2\pi}A_\mu A^\mu - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \quad (9)$$

or, in terms of both A_μ component fields,

$$\begin{aligned}\mathcal{L}_B = & \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2e^2}(\Box\chi)^2 - \frac{1}{e\sqrt{\pi}}(g_1\partial^\mu\eta - g_2\partial^\mu\chi)\partial_\mu\phi \\ & + \frac{M^2}{2\pi e^2}(\partial_\mu\eta\partial^\mu\eta - \partial_\mu\chi\partial^\mu\chi).\end{aligned}\quad (10)$$

It is not difficult to see that these two local formulations — given by Eqs. (4) and (9) — for the GSM are equivalent. This can be seen by making a transformation in the bosonization field $\phi(x)$:

$$\phi(x) = \phi'(x) - \frac{g_1}{e\sqrt{\pi}} \eta(x) + \frac{g_2}{e\sqrt{\pi}} \chi(x). \quad (11)$$

Furthermore, this transformation has a trivial Jacobian for the functional measure. So, integrating over the $\phi'(x)$ field we get the same Lagrangian density as that in the expression (4), apart from a trivial redefinition of the generating functional normalization. Thus we have shown that the two local versions, one based on a higher order Lagrangian density [Eq. (4)] and the other with the usual derivative order but with the introduction of an additional field [Eq. (9)], are equivalent, because in the transformation necessary for linking these expressions the fields $\eta(x)$ and $\chi(x)$ remain unchanged in form, and they are the ones which appear in the correlation functions, as can be seen from the generating functional.

As far as we know, almost all the works that discuss the CSM³⁻¹⁴ and the GSM¹⁶⁻²⁰ are based on the bosonized version [Eq. (9)]; the only exception is the work of Chanowitz,¹⁵ which is carried out in the fermionic version through perturbative methods. Here, we will study the higher order version of the GSM [Eq. (4)], in its anomalous and nonanomalous formulations, and take the appropriate limits to check our results against those in the literature.

The nonanomalous formulation of the Lagrangian density (4) can be easily obtained through the substitution of the longitudinal component of the photon field by a gauge-invariant combination. To do this we introduce a compensating field $\theta(x)$, the WZ one, so that

$$A_\mu \rightarrow A_\mu + \frac{1}{e} \partial_\mu \Lambda, \quad \eta \rightarrow \eta + \Lambda, \quad \theta \rightarrow \theta - \Lambda. \quad (13)$$

Then substituting $\eta(x)$ by $\eta(x) + \theta(x)$ in the expression (4), we get the nonanomalous higher order formulation of the GSM,

$$\begin{aligned} \mathcal{L} = & \bar{\psi}'(i\gamma_\mu \partial^\mu) \psi' + \frac{1}{2e^2} \chi \square^2 \chi + \frac{1}{2\pi e^2} (g_1^2 - M^2)(\eta + \theta) \square (\eta + \theta) \\ & + \frac{1}{2\pi e^2} (g_2^2 + M^2) \chi \square \chi - \frac{g_1 g_2}{\pi e^2} (\eta + \theta) \square \chi, \end{aligned} \quad (14)$$

where we have omitted the source terms for future convenience.

3. The $M^2 \neq g_1^2$ Case

First of all we solve the anomalous GSM through the use of appropriate field transformations, to get a decoupled Lagrangian density. It is not difficult to see that making the transformations below in Eq. (4),

$$\eta = \eta' + \left(\frac{g_1 g_2}{g_1^2 - M^2} \right) \chi', \quad \chi = \chi', \quad (15)$$

one obtains

$$\begin{aligned} \mathcal{L}_{\text{AGSM}} = & \bar{\psi}'(i\gamma_\mu \partial^\mu)\psi' + \frac{1}{2e^2} \chi' \square (\square + m^2) \chi' + \frac{1}{2\pi e^2} (g_1^2 - M^2) \eta' \square \eta' \\ & + \frac{1}{e} J^\mu \left\{ \partial_\mu \left[\eta' + \left(\frac{g_1 g_2}{g_1^2 - M^2} \right) \chi' \right] + \epsilon_{\mu\rho} \partial^\rho \chi' \right\} + \bar{\psi}' \bar{U}'_5 \zeta + \bar{\zeta} U'_5 \psi' \end{aligned} \quad (16)$$

where

$$m^2 = \frac{(g_1 g_2)^2 - (g_1^2 - M^2)(g_2^2 + M^2)}{\pi(M^2 - g_1^2)} \quad (17)$$

is the dynamically generated mass of this model, and we have made the transformation in the source terms also. As the sources generate the correlation functions of the original fields, all we need now is to perform the usual functional derivatives with respect to these sources and, using the free propagators of the transformed fields, get the required Green functions. At this point one can see why the case $M^2 = g_1^2$ ($a = 1$ in the CSM) must be treated separately, as observed in the Dirac quantization for the CSM;⁶ in the present approach this is easily seen because the transformation (25) becomes singular for this case, so one should substitute this condition in the Lagrangian density and, only then, search for the decoupling transformations. As an example of a Green function we compute the photon propagator, which comes from

$$D_{\mu\nu}(x-y) = - \frac{\delta^2 Z}{\delta J_\mu(x) \delta J_\nu(y)} \Big|_{J=0}. \quad (18)$$

This way we get the propagator in the momentum representation,

$$D_{\mu\nu}(k) = -\frac{1}{e^2} \left\{ k_\mu k_\nu D_{\eta'} + \left[k_\mu k_\nu + \left(\frac{g_1 g_2}{g_1^2 - M^2} \right)^2 \bar{k}_\mu \bar{k}_\nu D_{\chi'} \right] \right\}, \quad (19)$$

from which, after the substitution of the free propagators of η' and χ' , we obtain

$$\begin{aligned} D_{\mu\nu}(k) = & \left(\frac{i}{(k^2 - m^2)} \right) \left(-g_{\mu\nu} + \left(\frac{1}{M^2 - g_1^2} \right) \right. \\ & \times \left. \left\{ k_\mu k_\nu \left[\pi - \frac{1}{k^2} (g_1^2 + g_2^2) \right] + \frac{g_1 g_2}{k^2} (k_\mu \bar{k}_\nu + \bar{k}_\mu k_\nu) \right\} \right), \end{aligned} \quad (20)$$

which is in agreement with that obtained in the literature for the anomalous GSM through the Dirac formalism.¹⁷

Now we will solve the nonanomalous GSM, by performing decoupling field transformations. For this we need to break the gauge symmetry. This is done through the introduction of a gauge-fixing (GF) term in the Lagrangian density. In this work we will use the two GF's

$$\mathcal{L}_{\text{GF}}^{(i)} = -\frac{1}{2\alpha} (\partial \cdot A + \beta \bar{\partial} \cdot A)^2 = -\frac{1}{2\alpha e^2} (\square \eta - \beta \square \chi)^2, \quad (21a)$$

$$\mathcal{L}_{\text{GF}}^{(\text{ii})} = -\frac{1}{2\alpha} \partial_\mu \theta \partial^\mu \theta, \quad (21b)$$

where α and β are gauge parameters.

Using $\mathcal{L}_{\text{GF}}^{(\text{i})}$, we decouple the Lagrangian density (14) by performing the following transformations in the fields:

$$\eta = \eta' + \beta \chi', \quad \chi = \chi', \quad \theta = \theta' + \left(\frac{g_1 g_2}{g_1^2 - M^2} - \beta \right) \chi', \quad (22)$$

so that we obtain the decoupled Lagrangian density

$$\begin{aligned} \mathcal{L} = \mathcal{L}_{\text{GSM}} + \mathcal{L}_{\text{GF}}^{(\text{i})} = & \bar{\psi}' (i\gamma_\mu \partial^\mu) \psi' + \frac{1}{2e^2} \chi' \square (\square + m^2) \chi' \\ & + \frac{1}{2\pi e^2} (g_1^2 - M^2) \theta' \square \theta' - \frac{1}{2\alpha e^2} \eta' \square^2 \eta' \\ & + \frac{1}{e} J^\mu [\partial_\mu (\eta' + \beta \chi') + \epsilon_{\mu\rho} \partial^\rho \chi'] + \bar{\psi}' \bar{U}'_5 \zeta + \bar{\zeta} U'_5 \psi'. \end{aligned} \quad (23)$$

Taking the functional derivatives with respect to the source J_μ , we are led to the expression for the photon propagator,

$$D_{\mu\nu}(k) = -\frac{1}{e^2} [k_\mu k_\nu D_{\eta'} + (\beta k_\mu + \bar{k}_\mu)(\beta k_\nu + \bar{k}_\nu) D_{\chi'}], \quad (24)$$

where $\bar{k}_\mu \equiv \epsilon_{\mu\rho} k^\rho$. Using now the free propagators for the fields η' and χ' , we get

$$D_{\mu\nu}(k) = -\frac{i}{k^2 - m^2} \left[g_{\mu\nu} - \left(1 + \beta^2 - \alpha + \frac{\alpha m^2}{k^2} \right) \frac{k_\mu k_\nu}{k^2} - \frac{\beta(k_\mu \bar{k}_\nu + \bar{k}_\mu k_\nu)}{k^2} \right]. \quad (25)$$

This last expression, as far as we know, has not been previously obtained for the GSM. It can be checked through its particular cases, specifically that of the CSM. For $\beta = 0$, $M^2 = ae^2$, $g_1 = -g_2 = e$, it corresponds to the case (ii) in the work of Harada and Tsutsui,¹³ and for $\beta = (a-1)^{-1}$ to their case (iii).

Now if we use the second GF condition, $\mathcal{L}_{\text{GF}}^{(\text{ii})}$, the decoupling transformations are

$$\eta = \eta' + \left(\frac{g_1 g_2}{g_1^2 - M^2} \right) \chi' - \theta', \quad \chi = \chi', \quad \theta = \theta', \quad (26)$$

also valid only for the $M^2 \neq g_1^2$ case, as can be easily seen. The transformed Lagrangian density in this case is

$$\begin{aligned} \mathcal{L} = \mathcal{L}_{\text{GSM}} + \mathcal{L}_{\text{GF}}^{(\text{ii})} = & \frac{1}{2e^2} \chi' \square (\square + m^2) \chi' + \frac{1}{2\pi e^2} (g_1^2 - M^2) \eta' \square \eta' \\ & + \frac{1}{2\alpha} \theta' \square \theta' + \frac{1}{e} J^\mu \left\{ \partial_\mu \left[\eta' + \left(\frac{g_1 g_2}{g_1^2 - M^2} \right) \chi' - \theta' \right] + \epsilon_{\mu\rho} \partial^\rho \chi' \right\} \\ & + \bar{\psi}' \bar{U}'_5 \zeta + \bar{\zeta} U'_5 \psi', \end{aligned} \quad (27)$$

leading us to the photon propagator given by

$$D_{\mu\nu}(k) = \frac{i}{k^2 - m^2} \left[-g_{\mu\nu} + \left(\frac{1}{M^2 - g_1^2} \right) \left(k_\mu k_\nu \left\{ \frac{1}{e^2} [\pi e^2 - \alpha(M^2 - g_1^2)] \right. \right. \right. \\ \left. \left. \left. + \frac{1}{k^2} \left[\frac{m^2 \alpha}{e^2} (M^2 - g_1^2) - (g_1^2 + g_2^2) \right] \right\} + \frac{g_1 g_2}{k^2} (k_\mu \bar{k}_\nu + \bar{k}_\mu k_\nu) \right) \right]. \quad (28)$$

This case can also be compared with the results of the CSM by particularizing it. For the case $\alpha = 0$ the anomalous result is recovered, in agreement with the results for the CSM.^{7,13} Besides, it is possible to eliminate the misbehaved term in the coupling constant ($1/e^2$).

4. The $M^2 = g_1^2$ Case

Now we are going to treat the case where $M^2 = g_1^2$. For this we will develop an alternative way to calculate the correlation functions, without the need to obtain the decoupling transformations. In this approach we rewrite the bosonic part of the Lagrangian density for the nonanomalous GSM [Eq. (14)] in a matrix form,

$$\mathcal{L} = \frac{1}{2} \rho^T \mathbb{M} \rho, \quad (29)$$

where $\rho = \begin{pmatrix} \chi \\ \eta \\ \theta \end{pmatrix}$, and $\mathbb{M}$ will differ from one gauge-fixing to another. Then we can compute the correlation functions by inverting $\mathbb{M}$, obtaining

$$\mathbb{M}^{-1} = \begin{pmatrix} \langle \chi\chi \rangle & \langle \chi\eta \rangle & \langle \chi\theta \rangle \\ \langle \eta\chi \rangle & \langle \eta\eta \rangle & \langle \eta\theta \rangle \\ \langle \theta\chi \rangle & \langle \theta\eta \rangle & \langle \theta\theta \rangle \end{pmatrix}, \quad (30)$$

where the elements $\langle \chi\chi \rangle$, $\langle \chi\eta \rangle$, etc., are the two-point correlation functions of their respective fields, from which one can write down the others as, for example, that for the photon propagator:

$$\langle A_\mu A_\nu \rangle = -\frac{1}{e^2} (k_\mu k_\nu \langle \eta\eta \rangle + k_\mu \bar{k}_\nu \langle \eta\chi \rangle + \bar{k}_\mu k_\nu \langle \chi\eta \rangle + \bar{k}_\mu \bar{k}_\nu \langle \chi\chi \rangle). \quad (31)$$

It is important to note that, in the construction of the matrix $\mathbb{M}$, we must symmetrize it with respect to the principal diagonal, as a consequence of the hermiticity requirement.

One more time we will start by solving the anomalous version. Here the matrix $\mathbb{M}$ is written as

$$\mathbb{M} = \frac{\square}{e^2} \begin{pmatrix} \square + \frac{g_1^2 + g_2^2}{\pi} & -\frac{g_1 g_2}{\pi} \\ -\frac{g_1 g_2}{\pi} & 0 \end{pmatrix}. \quad (32)$$

Inverting it, collecting the terms $\langle \eta\eta \rangle$, $\langle \chi\eta \rangle$, etc., and substituting them in (31), we get

$$D_{\mu\nu}(k) = \frac{\pi}{g_1 g_2 k^2} \left\{ k_\mu k_\nu \left[\frac{\pi k^2}{g_1 g_2} - \frac{g_1^2 + g_2^2}{g_1 g_2} \right] - (k_\mu \bar{k}_\nu + \bar{k}_\mu k_\nu) \right\}, \quad (33)$$

in agreement with the result of the calculations on the anomalous GSM appearing in Ref. 17, where the Dirac quantization method was used. In particular it is seen that this case does not generate mass dynamically.

Now we apply the approach sketched above to the case of the nonanomalous model in the gauges imposed through $\mathcal{L}_{\text{GF}}^{(i)}$ and $\mathcal{L}_{\text{GF}}^{(ii)}$ respectively. In the first case $\mathbb{M}$ is given by

$$\mathbb{M} = \frac{\square}{e^2} \begin{pmatrix} (1 - \beta^2/\alpha)\square + (g_1^2 + g_2^2)/\pi & (\beta/\alpha)\square - g_1 g_2/\pi & -g_1 g_2/\pi \\ (\beta/\alpha)\square - g_1 g_2/\pi & -(1/\alpha)\square & 0 \\ -g_1 g_2/\pi & 0 & 0 \end{pmatrix}. \quad (34)$$

Substituting the correlation functions $\langle \eta\eta \rangle$, $\langle \eta\chi \rangle$, etc., in the expression (27) and using the relation $\bar{k}_\mu \bar{k}_\nu = -g_{\mu\nu} k^2 + k_\mu k_\nu$, we find that

$$\langle A_\mu A_\nu \rangle = \frac{\alpha k_\mu k_\nu}{k^4}. \quad (35)$$

Surprisingly this shows us that one could gauge away this correlation function in the Landau gauge ($\alpha = 0$). In a certain sense, this reinforces the idea of using gauge-independent correlation functions, as we will see more below.

In the second case the matrix $\mathbb{M}$ is

$$\mathbb{M} = \frac{\square}{e^2} \begin{pmatrix} \square + (g_1^2 + g_2^2) & -g_1 g_2/2\pi e^2 & -g_1 g_2/2\pi e^2 \\ -g_1 g_2/2\pi e^2 & 0 & 0 \\ -g_1 g_2/2\pi e^2 & 0 & 1/2\alpha e^2 \end{pmatrix} \quad (36)$$

and the photon propagator is

$$\begin{aligned} \langle A_\mu A_\nu \rangle = & \pi \frac{k_\mu k_\nu}{k^2} \left[k^2 - (g_1^2 + g_2^2) + \alpha \frac{g_1 g_2}{\sqrt{\pi} e^2} \right] \\ & - \frac{\pi}{g_1 g_2} \left(\frac{k_\mu \bar{k}_\nu + \bar{k}_\mu k_\nu}{k^2} \right). \end{aligned} \quad (37)$$

Again, we find a nasty behavior in the coupling constant that cannot be eliminated when $\alpha = 0$. This time, however, the propagator cannot be eliminated through a convenient choice of the gauge parameter, as in the previous case.

From the results obtained above, one can see that the gauge-dependent correlation function $\langle A_\mu A_\nu \rangle$ changes drastically from one gauge to another. Now we will

discuss the case where the source terms include the WZ field.¹⁰ In the bosonic case the source term becomes

$$J_\mu \left(A^\mu + \frac{1}{e} \partial^\mu \theta \right) \equiv J_\mu A_\mu^I, \quad (38)$$

where A_μ^I is a gauge-invariant field. Now, we can see that the propagators in the above gauges, or in any other one, will be the same as that of the anomalous model, as previously observed through different approaches in the CSM.¹² As an example we compute the gauge-invariant photon propagator in the $M^2 = g_1^2$ case, so obtaining

$$\begin{aligned} D_{\mu\nu}^I(k) &= \langle A_\mu^I A_\nu^I \rangle = \langle A_\mu A_\nu \rangle - \frac{1}{e^2} k_\mu k_\nu (\langle \eta \theta \rangle + \langle \theta \eta \rangle + \langle \theta \theta \rangle) \\ &\quad - \frac{1}{e^2} (k_\mu \bar{k}_\nu \langle \theta \chi \rangle + \bar{k}_\mu k_\nu \langle \chi \theta \rangle). \end{aligned} \quad (39)$$

Then, after straightforward calculations we get the same propagator as in the anomalous case, appearing in (33). It is not difficult to convince ourselves, after a little experimentation, that this will happen with every gauge-invariant correlation function.

5. On a Generalized Wess–Zumino Term

In this section we shall look for the possibility of introducing a generalized WZ term. For this we will proceed analogously to the approach of Harada and Tsutsui.⁵ They showed in their paper that, if one deals carefully with the functional measure, that term appears quite naturally. Their starting point was the integral over only one representative of each class of the gauge field configurations in the vacuum-to-vacuum amplitude. This was done by using the Faddeev–Popov trick and then “unfixing” the gauge, going to an integral over all the gauge field configurations (equivalent or not). Let us now see how to do it in our case. First, we define $\Delta_f[A]$ by

$$\Delta_f[A] \int dg \delta(f[A^g]) = 1, \quad (40)$$

where g is an invariant measure on the gauge symmetry group G . Then, inserting (40) into the vacuum-to-vacuum amplitude we find that

$$Z = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} dA_\mu \Delta_f[A] dg \delta(f[A^g]) \exp[iI(\psi, \bar{\psi}, A)]. \quad (41)$$

Performing now a transformation in the integration field $A_\mu \rightarrow A_\mu^{g^{-1}}$, using the invariance of the measure dA_μ and of $\Delta_f[A]$, we obtain

$$Z = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} dA_\mu dg \delta(f[A^g]) \exp[iI(\psi, \bar{\psi}, A^{g^{-1}})], \quad (42)$$

where $\mathcal{D}A_\mu \equiv dA_\mu \Delta_f[A]$. Now, transforming the fermionic fields like

$$\psi_L \rightarrow \psi_L^g = g^{-1} \psi_L, \quad \bar{\psi}_L \rightarrow \bar{\psi}_L^g = \bar{\psi}_L g, \quad (43)$$

and remembering that the fermionic measure is not invariant (for anomalous models) under this transformation, we finally obtain

$$Z = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \mathcal{D}A_\mu dg \delta(f[A^g]) \exp\{i[I(\psi, \bar{\psi}, A) + \alpha(A, g^{-1})]\}, \quad (44)$$

where we have used $I(\psi^g, \bar{\psi}^g, A^{g^{-1}}) = I(\psi, \bar{\psi}, A)$. This expression, in contrast with (41), is totally gauge-invariant. This way we have "unfixed" the gauge defined by $f[A] = 0$. As an example we will treat the case of the Schwinger model in a Lorentz-type generalized WZ term. This is done to illustrate the differences between the conventional WZ term and the generalized one, in a very-well-established model.

Let us begin by writing down the Lagrangian density for the bosonized vector Schwinger model (VSM),

$$\begin{aligned} \mathcal{L}_{\text{VSM}} = & \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + e \epsilon^{\mu\nu} \partial_\mu \phi A_\nu \\ & + \frac{ae^2}{2} A_\mu A^\mu - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}, \end{aligned} \quad (45)$$

where we choose $g_1 = 0$, $g_2 = -\sqrt{\pi}e$ and $M^2 = \pi ae^2$ in Eq. (9). This model is invariant under the gauge transformations $\delta\phi = 0$ and $\delta A_\mu = -(1/e)\partial_\mu \lambda$, provided that the arbitrary parameter is fixed such that $a = 0$. On the other hand this parameter can be kept arbitrary if there is introduced a conventional WZ term,⁵

$$\mathcal{L}_{\text{WZ}} = \frac{a}{2} \partial_\mu \theta \partial^\mu \theta + ae \partial_\mu \theta A^\mu. \quad (46)$$

In this case the mass of the model is computed as $m^2 = (a+1)e^2/\pi$, and this was done without losing the gauge invariance. Furthermore, the usual mass e^2/π is recovered when a is chosen to be equal to zero. In this case, the WZ term vanishes, surviving an infinity that can be absorbed in the functional normalization.

Now we will introduce a generalized WZ term of the Lorentz type, and study its consequences:

$$\begin{aligned} \mathcal{L}_{\text{GWZ}} = & -\frac{1}{2\alpha} (\partial_\mu A^\mu)^2 + \frac{a}{2} \partial_\mu \theta \partial^\mu \theta + ae \partial_\mu \theta A^\mu \\ & - \frac{1}{2\alpha e^2} (\Box\theta)^2 - \frac{1}{\alpha e} \Box\theta \partial_\mu A^\mu. \end{aligned} \quad (47)$$

Imposing a gauge-fixing term of the type (ii) (with a gauge parameter γ), we get

$$\mathbb{M} = \frac{\Box}{e^2} \begin{pmatrix} \Box + (a+1)e^2/\pi & 0 & 0 \\ 0 & -\Box/\alpha - ae^2/\pi & -\Box/\alpha - ae^2/\pi \\ 0 & -\Box/\alpha - ae^2/\pi & -\Box/\alpha - (a/\pi - \gamma^{-1})e^2 \end{pmatrix}. \quad (48)$$

Inverting the above matrix and computing the gauge-invariant photon propagator, we find that

$$D_{\mu\nu}^I(k) = \frac{i}{(k^2 - m_f^2)} \left[-g_{\mu\nu} + \frac{k_\mu k_\nu}{k^2} \left(1 - \alpha \frac{k^2 - m_f^2}{k^2 - m_D^2} \right) \right], \quad (49)$$

where $m_f^2 = (a+1)e^2/\pi$ and $m_D^2 = \alpha ae^2/\pi$. As can be seen from the above, the more remarkable feature of the generalized WZ term is that of appearing gauge-dependent massive excitations like m_D . The excessive mass can be removed by choosing the gauge in a suitable way. This can be done by working in the Landau gauge ($\alpha = 0$), leading to the correct propagator in this gauge. Furthermore, the case $a = 0$ (the case where the usual WZ term vanishes) brings us to the usual propagator in a Lorentz-type gauge,

$$D_{\mu\nu}^I(k)|_{a=0} = \frac{i}{k^2 - e^2/\pi} \left\{ -g_{\mu\nu} + \frac{k_\mu k_\nu}{k^2} \left[1 - \alpha \left(1 - \frac{e^2}{k^2\pi} \right) \right] \right\}. \quad (50a)$$

This shows us that the generalized WZ term does not change the photon propagator when we use its gauge-invariant version $D_{\mu\nu}^I$, as should be expected from a well-behaved WZ term. It is interesting to observe that one could work in another gauge, where the two masses are equal. This could be achieved by choosing $\alpha = (a+1)/a$. In this gauge, however, the case $a = 0$ is no more accessible, and $D_{\mu\nu}^I(k)$ is written as

$$D_{\mu\nu}^I(k)|_{\alpha=(a+1)/a} = \frac{i}{k^2 - m_f^2} \left(-g_{\mu\nu} + \frac{1}{a} \frac{k_\mu k_\nu}{k^2} \right). \quad (50b)$$

A similar study can be made in the case of the axial Schwinger model, finding analogous results. Also, in this model the generalized WZ term gives rise to gauge-dependent masses, which disappear in the Landau gauge, recovering the usual photon propagator.

Now we are going to treat the case of the CSM. As is well known, this model is an anomalous one and, when it is added with the usual WZ term, we get its gauge-invariant counterpart. However, this model, in contrast with the VSM, does not have its mass fixed through the requirement of gauge invariance in both cases. In the anomalous case, there is no available choice of the parameter a such that the anomaly is canceled, and in the second case the gauge invariance is kept for an arbitrary value of a .

The case of the CSM in a Lorentz-type generalized WZ term, when written in the form (29), can be obtained from the GSM with the charges $g_1 = e$ and $g_2 = -e$, giving the matrix

$$M = \frac{\square}{e^2} \begin{pmatrix} \square + (M^2 + e^2)/\pi & e^2/\pi & e^2/\pi \\ e^2/\pi & -\square/\alpha + (e^2 - M^2)/\pi & -\square/\alpha + (e^2 - M^2)/\pi \\ e^2/\pi & -\square/\alpha + (e^2 - M^2)/\pi & -\square/\alpha + [(1 + \pi\gamma^{-1})e^2 - M^2]/\pi \end{pmatrix}, \quad (51)$$

whose inversion gives us the following elements in the momentum representation:

$$\langle \chi \chi \rangle = \frac{e^2 [\pi k^2 - \alpha (M^2 - e^2)]}{\pi k^2 (k^2 - m_f^2) (k^2 - m_D^2)}, \quad (52a)$$

$$\langle \chi \eta \rangle = \langle \eta \chi \rangle = \frac{\alpha e^4}{\pi k^2 (k^2 - m_f^2) (k^2 - m_D^2)}, \quad \langle \chi \theta \rangle = \langle \theta \chi \rangle = 0, \quad (52b)$$

$$\begin{aligned} \langle \eta \eta \rangle = & -\frac{1}{\pi k^2 (k^2 - m_f^2) (k^2 - m_D^2)} \\ & \times \left(-\pi \gamma k^4 + \{ \alpha [\pi e^2 + \gamma (M^2 - e^2)] + \gamma (M^2 + e^2) \} k^2 \right. \\ & \left. - \alpha \left[\frac{\gamma M^4}{\pi} + e^2 (M^2 + e^2) \right] \right), \end{aligned} \quad (52c)$$

$$\begin{aligned} \langle \eta \theta \rangle = \langle \theta \eta \rangle = & \frac{\gamma}{\pi k^2 (k^2 - m_f^2) (k^2 - m_D^2)} \\ & \times \left\{ -\pi k^4 + k^2 [M^2 (\alpha + 1) - e^2 (\alpha - 1)] - \frac{\alpha M^4}{\pi} \right\}, \end{aligned} \quad (52d)$$

$$\begin{aligned} \langle \theta \theta \rangle = & \frac{\gamma}{\pi k^2 (k^2 - m_f^2) (k^2 - m_D^2)} \\ & \times \left\{ \pi k^4 - k^2 [M^2 (\alpha + 1) - e^2 (\alpha - 1)] + \frac{\alpha M^4}{\pi} \right\}, \end{aligned} \quad (52e)$$

where we have defined

$$m_f^2 \equiv \frac{1}{2\pi} \{ [(M^4 + e^4)(\alpha - 1)^2 - 2M^2 e^2 (\alpha^2 - 1)]^{1/2} + M^2 (\alpha + 1) - e^2 (\alpha - 1) \}, \quad (53a)$$

$$m_D^2 = -\frac{1}{2\pi} \{ [(M^4 + e^4)(\alpha - 1)^2 - 2M^2 e^2 (\alpha^2 - 1)]^{1/2} - M^2 (\alpha + 1) + e^2 (\alpha - 1) \}. \quad (53b)$$

From the above elements we can compute, for example, the gauge-dependent two-point Green function (31):

$$\begin{aligned} D_{\mu\nu}(k) = & -\frac{1}{(k^2 - m_f^2) (k^2 - m_D^2)} \left[-g_{\mu\nu} \left[k^2 - \frac{\alpha (M^2 - e^2)}{\pi} \right] \right. \\ & + \frac{\alpha e^2 (k_\mu \bar{k}_\nu + \bar{k}_\mu k_\nu)}{\pi k^2} + \frac{k_\mu k_\nu}{\pi e^2 k^2} \\ & \left. \times (\pi \gamma k^4 + \{ (1 - \alpha) \pi e^2 - \gamma [\alpha (M^2 - e^2) + (M^2 + e^2)] \} k^2 + \alpha M^4 \gamma) \right]. \end{aligned} \quad (54)$$

Alternatively we can calculate the gauge-invariant correlation function (39), obtaining

$$D_{\mu\nu}^I(k) = -\frac{1}{(k^2 - m_J^2)(k^2 - m_D^2)} \left\{ -g_{\mu\nu} \left[k^2 - \frac{\alpha(M^2 - e^2)}{\pi} \right] + \frac{\alpha e^2(k_\mu \bar{k}_\nu + \bar{k}_\mu k_\nu)}{\pi k^2} - \frac{k_\mu k_\nu}{\pi k^2} [(\alpha - 1)\pi k^2 - 2\alpha e^2] \right\}. \quad (55)$$

Once more the Landau gauge is that in which one of the masses vanishes, for $\alpha = 0$, $m_D^2 = 0$ and $m_J^2 = (M^2 + e^2)/\pi$. In this convenient gauge, the photon propagator $D_{\mu\nu}^I(k)$ is equal to that of the VSM with the same mass for the gauge boson, and this is a very interesting feature, because by the use of the usual WZ term the propagators of the two models are equal but not the masses. Using the generalized WZ term, we find that the masses are equal too.

As can be obtained from (17), the anomalous model (or even the nonanomalous one with the usual WZ term) has an arbitrary mass, $m^2 = M^4/\pi(M^2 - e^2)$. Besides, unfortunately, the introduction of a generalized WZ term does not eliminate the arbitrariness of the mass for the gauge boson of the model, similarly to what happened in the cases of the VSM and ASM, as we have seen in this work.

Another important observation is that now the gauge-invariant propagator $D_{\mu\nu}^I$ maintains some arbitrariness in the parameter α , although the parameter γ is eliminated. So, this propagator, differently from the case of the usual WZ term,¹² keeps its arbitrariness through the parameter α . This feature opens the possibility of using a generalized WZ term of the θ type (ii), fixing a gauge where the bad behavior of the CSM for a vanishing coupling constant (the term $1/e^2$ in the propagator)³ is gauged away. Moreover these models have the characteristic that the parameter α cannot be arbitrarily fixed, at least when one wishes to keep unitarity.

As in the case of the usual WZ term, we must study it in a situation where $M^2 = g_1^2$, or $M^2 = e^2$, because we are now treating the special case of the CSM. From the above discussion one could expect that this case would lead no more to a physically different content for the model (in the case of the usual WZ term, it has not a massive excitation). This is so because now we see that the physical mass of the model, which appears when one works in the Landau gauge, does not have a similar dependence on the regularizing parameter.

Let us see if this indeed happens. For this we take the matrix (51) and impose the condition $M^2 = e^2$. Then we invert it, obtaining the two-point correlation functions $\langle \chi\chi \rangle$, $\langle \chi\eta \rangle$, etc. It is easy to see that these expressions match that in (52) when $M^2 = e^2$ is substituted in the latter. Consequently the masses are now written as

$$m_J^2 = \frac{e^2}{\pi} (\sqrt{1 - \alpha} + 1), \quad (56a)$$

$$m_D^2 = -\frac{e^2}{\pi} (\sqrt{1 - \alpha} - 1), \quad (56b)$$

from which we see that when $\alpha = 0$, $m_D^2 = 0$ and $m_f^2 = 2e^2/\pi$, as supposed above. Moreover, in this simple case it becomes clearer that, in order to keep the theory unitary, one must restrict the range of validity for the parameter α ($0 \leq \alpha \leq 1$).

6. Final Remarks

In this work we have discussed the higher order version of the GSM in its anomalous and nonanomalous versions. For this we used the path integral formalism for the quantization, and inside this framework two methods were employed, one looking for decoupling transformations and then doing the functional derivatives with respect to the sources to get the correlation functions, and the other calculating the elementary two-point correlation functions and then computing the full correlation functions of the gauge field. This was performed both in the case where $M^2 \neq g_1^2$ and in the singular case where $M^2 = g_1^2$ ($a = 1$ for the CSM). The corresponding Green functions were, when available, compared with those in the literature. Moreover we verified the equality of the gauge-invariant correlation functions with that of the anomalous model as stated by Girotti and Rothe in the case of the CSM.¹²

Furthermore there was introduced a generalized WZ term in an analogous fashion to the approach of Harada and Tsutsui,⁵ whose consequences were studied in the particular cases of the VSM, ASM and CSM. It was observed that this term has as a fundamental feature the introduction of further massive excitations, and that these have a gauge dependence through the gauge parameters. Another interesting characteristic is that some of these gauge-dependent masses disappear in an appropriately chosen gauge. This is well exemplified in the VSM and in the ASM.

As a consequence of applying this generalized WZ term to the case of the CSM, we saw that it makes the mass of this model equal to that appearing in the VSM, a surprising result. However, the most challenging goal, that of eliminating the mass ambiguity, was not reached also through the use of this novel WZ term. Furthermore the confirmation or not of their characteristics deserves a further study on the constraint structure of the theory with the generalized WZ term. This should be done in order to conclude if this new WZ term, as the usual one,^{8,9} does not alter the physical content of the model, or if it corresponds to a new model, in the sense that it has a completely different Hilbert space and particle spectrum. We expect to report on this in the near future.

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Hamiltonian quantization of the non-anomalous generalized Schwinger model

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Abstract. We quantize a generalized version of the Schwinger model, where the two chiral sectors couples with different strengths to the $U(1)$ gauge field. Starting from a theory which includes a generalized Wess-Zumino term, we obtain the equal time commutation relation for physical fields, both the singular and non-singular cases are considered. The photon propagators are also computed in their gauge dependent and invariant versions.

consistency of the model and understand the aspects of its quantum dynamics, we must submit it to different quantization situations, $a > (g_-)^2$ and $a = (g_-)^2$ which are equivalents to $a > 1$ and $a = 1$ in original CSM. An interesting aspect of quantization appears when we compare our results in the case $a = (g_-)^2$. We observe that apparently this limit has not been studied in a completely carefull way in [5]. We develop this point taking into account the time evolution of all constraints as considered by Dirac procedure [10].

1 In the last years, the problem of quantizing two dimensional anomalous gauge theories has been shown one of most interesting and attractive subjects from a theoretical point of view. In particular, the question of gauge noninvariance or gauge-invariance of the Chiral Schwinger Model (CSM) [1] has been extensively discussed [2–5]. The original version of this model shows to be consistent and unitary, despite the formulation be anomalous. However, a gauge-invariant formulation of CSM is possible by adding a Wess-Zumino (WZ) term to the original bosonized action as shown by the Faddeev and Shatashvili [2].

A generalized version of the CSM [6], which is based on the fact that left and right couplings of the anomalous theories are present, has given a new insight to this subject [6]. In a very interesting way, some authors [7] obtained a modified version of the Wess-Zumino term to add the original action of this model. By using functional methods, the quantization was performed and some properties of convergency of propagators have been discussed [8].

In a recent paper [9], it was discussed the Hamiltonian quantization of the anomalous version of this generalized Schwinger model. As far we know, a study of the Dirac brackets quantization [10] for this non-anomalous generalized Schwinger model GSM has not been considered yet. Here, we intend to complete this discussion by extending its non-anomalous case. In order to check the

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In order to get a deeper understanding of the problem, let us remember how to obtain the WZ term for the generalized model [7]. Starting from the Lagrangian density

$$\mathcal{L} = \bar{\Psi} \gamma^\mu \{ i \partial_\mu + e_L (1 - \gamma_5) + e_R (1 + \gamma_5) A_\mu \} \Psi - \frac{1}{4} (F_{\mu\nu})^2, \quad (2.1)$$

where we consider [12].

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \quad (2.2)$$

The path integral bosonization of this model has been calculated for massive case [9]. In the massless case the Lagrangian reads

$$\begin{aligned} \mathcal{L}_{GSM} = & -\frac{1}{4} (F_{\mu\nu})^2 + \frac{1}{2} (\partial_\mu \Phi)^2 - e(g_+ e^{\mu\nu} + g_- g^{\mu\nu}) \\ & \times (\partial_\mu \Phi) A_\nu + a \frac{e^2}{2} (A_\mu)^2, \end{aligned} \quad (2.3)$$

where we defined

$$\begin{aligned} e_+ &= \frac{e_R + e_L}{\sqrt{\pi}} = e g_+ \\ e^2 &= \frac{1}{2} (e_+^2 + e_-^2). \end{aligned} \quad (2.4)$$

The explicit mass term for the gauge field arises from a regularization ambiguity of the Fujikawa Jacobian. The parameter a reflect the arbitrariness in regularizing the currents in this procedure [6]. By inserting into the Lagrangian density (2.3) the well known decomposition for A_μ field

$$eA_\mu = \partial_\mu \eta + \varepsilon_{\mu\nu} \partial^\nu \chi \quad (2.5)$$

we obtain

$$\begin{aligned} \mathcal{L}_{GSM} = & \frac{1}{2e^2} \chi \square \square \chi + \frac{1}{2} (\partial_\mu \Phi)^2 - g_- (\partial_\mu \Phi) (\partial^\mu \eta) \\ & - g_+ (\partial_\mu \Phi) (\partial^\mu \chi) + \frac{a}{2} (\chi \square \chi - \eta \square \eta). \end{aligned} \quad (2.6)$$

Let us decouple the fields η and χ from Φ through the following transformation

$$\Phi' = \Phi - g_+ \chi - g_- \eta, \quad (2.7)$$

which leads to

$$\begin{aligned} \mathcal{L}_{GSM} = & \frac{1}{2e^2} \chi \square \square \chi + \frac{1}{2} (\partial_\mu \Phi')^2 - \frac{1}{2} [(g_+)^2 + a] (\partial_\mu \chi)^2 \\ & - (g_+ g_-) (\partial_\mu \eta) (\partial^\mu \chi) + \frac{1}{2} [(g_-)^2 + a] (\partial_\mu \eta)^2. \end{aligned} \quad (2.8)$$

Now, we can introduce a new set of gauge transformations

$$\begin{aligned} \eta & \rightarrow \eta + A \\ \theta & \rightarrow \theta - A, \end{aligned} \quad (2.9)$$

so that $\eta + \theta$ remains invariant. It is convenient now to isolate the θ -dependent terms, which will be related with the WZ term. After a straightforward calculus the last two terms in (2.8) can be written as

$$\begin{aligned} & - (g_+ g_-) (\partial_\mu \eta) (\partial^\mu \chi) - \frac{1}{2} [(g_-)^2 - a] (\partial_\mu \eta)^2 \\ & + \{ - (g_+ g_-) (\partial_\mu \theta) (\partial^\mu \chi) + \frac{1}{2} [(g_-)^2 - a] (\partial_\mu \theta)^2 \\ & - [(g_-)^2 - a] (\partial_\mu \theta) (\partial^\mu \eta) \}. \end{aligned} \quad (2.10)$$

By using the decomposition of A_μ field given by (2.5) we can write the WZ term for the generalized CSM,

$$\begin{aligned} \mathcal{L}_{WZ} = & - \frac{1}{2} [(g_-)^2 - a] (\partial_\mu \theta)^2 + e\theta \{ [(g_-)^2 - a] \\ & \times \partial_\mu A^\mu + g_+ g_- \varepsilon^{\mu\nu} \partial_\mu A_\nu \}. \end{aligned} \quad (2.11)$$

It is easy to verify the CSM limit when $g_+ = -g_- = 1$.

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Case 1: $a > (g_-)^2$

Starting from Lagrangians (2.3) and (2.11) we arrive at a gauge invariant version of generalized Schwinger model,

i.e.,

$$\mathcal{L}_I = \mathcal{L}_{GSM} + \mathcal{L}_{WZ}, \quad (3.1)$$

whose the corresponding canonical Hamiltonian is

$$\begin{aligned} \mathcal{H}_c = & \frac{1}{2} (\pi_1)^2 + \frac{1}{2(a - (g_-)^2)} (\pi_\theta)^2 + \frac{1}{2} (\pi_\phi)^2 \\ & + \pi_1 \partial_1 A_0 + e(g_- A_0 + g_+ A_1) \pi_\phi \\ & + \frac{1}{2} (\partial_1 \Phi)^2 + \frac{1}{2} (a - (g_-)^2) (\partial_1 \theta)^2 \\ & + \frac{e}{(a - (g_-)^2)} ((a - (g_-)^2) A_0 + (g_- g_+) A_1) \pi_\theta \\ & - e(g_+ A_0 + g_- A_1) \partial_1 \Phi + e \partial_1 \theta \\ & \times ((a - (g_-)^2) A_1 - (g_- g_+) A_0) \\ & + \frac{e^2}{2} \left[\frac{(g_+ g_-)^2 + (a - (g_-)^2)(g_+)^2 + a(a - (g_-)^2)}{(a - (g_-)^2)} \right] A_1^2. \end{aligned} \quad (3.2)$$

observe that the limit $g_+ = -g_- = 1$ corresponds to the CSM Hamiltonian.

So, by using of the Dirac formalism [10] to quantize the theory for $a > (g_-)^2$, the primary constraint is

$$\Omega_1 = \pi_0 \approx 0, \quad (3.3)$$

which corresponds to a local $U(1)$ invariance of the gauge theory. By using the consistency condition in the Dirac method [10] we find the corresponding Gauss law,

$$\Omega_2 = \partial_1 \pi_1 + e(g_- \pi_\phi + \pi_\theta) - e(g_+ \partial_1 \Phi + (g_+ g_-) \partial_1 \theta) \approx 0, \quad (3.4)$$

which are of first class as we have expected. At this point one can choose two different ways of treating the model. It can be done a gauge covariant discussion by using first class constraints to define the physical states [11], or tracing a more amenable way, by fixing the gauge through additional second class constraints [10]. We are going to choose one which eliminates the degree of freedom of the θ field [13]

$$\Omega_3 = \partial_1 \theta \approx 0, \quad (3.5)$$

such that its time evolution leads to

$$\Omega_4 = \pi_\theta + e((a - (g_-)^2) A_0 + (g_+ g_-) A_1) \approx 0. \quad (3.6)$$

Inserting Ω_3 and Ω_4 in Ω_2 and the Hamiltonian we find respectively

$$\begin{aligned} \Omega_2 = & \partial_1 \pi_1 + e g_- \pi_\phi - e g_+ \partial_1 \Phi \\ & - e^2 ((a - (g_-)^2) A_0 + (g_+ g_-) A_1) \approx 0, \end{aligned} \quad (3.7)$$

$$\begin{aligned} \mathcal{H}_c = & \frac{1}{2} (\pi_1)^2 + \frac{1}{2} (\pi_\phi)^2 - \pi_1 \partial_1 A_0 \\ & + e(g_- A_0 + g_+ A_1) \pi_\phi - e(g_+ A_0 + g_- A_1) \partial_1 \Phi \end{aligned}$$

$$\begin{aligned} & \frac{i}{2}(\partial_1 \Phi)^2 + \frac{e^2}{(a - (g_-)^2)} \\ & \times ((a - (g_-)^2)A_0 + (g_- g_+)A_1)^2 \\ & + \frac{e^2}{2} \left[\frac{(g_+ g_-)^2 + (a - (g_-)^2)(g_+)^2 + a(a - (g_-)^2)}{(a - (g_-)^2)} \right] A_1^2. \end{aligned} \quad (3.8)$$

Therefore, the set of second-class constraints is

$$\begin{aligned} \Omega_1 &= \pi_0 \approx 0 \\ \Omega_2 &= \partial \pi_1 + eg_- \pi_\phi - eg_+ \partial_1 \Phi \\ &\quad - e^2((a - (g_-)^2)A_0 + (g_+ g_-)A_1) \approx 0 \\ \Omega_3 &= \partial_1 \theta \approx 0 \\ \Omega_4 &= \pi_\theta + e((a - (g_-)^2)A_0 + (g_+ g_-)A_1) \approx 0. \end{aligned} \quad (3.9)$$

Now, with the gauge completely fixed we can building up the Poisson brackets algebra, so

$$\begin{aligned} \{\Omega_1(x_1), \Omega_4(y_1)\} &= -\{\Omega_4(x_1), \Omega_1(y_1)\} \\ &= -e[a - (g_-)^2]\delta(x_1 - y_1) \\ \{\Omega_2(x_1), \Omega_3(y_1)\} &= -\{\Omega_3(x_1), \Omega_2(y_1)\} \\ &= -e\partial_x \delta(x_1 - y_1) \\ \{\Omega_3(x_1), \Omega_4(y_1)\} &= -\{\Omega_4(x_1), \Omega_3(y_1)\} \\ &= e\partial_x \delta(x_1 - y_1). \end{aligned} \quad (3.10)$$

Therefore, the constraint matrix is

$$C_{ab}(x, y) = \begin{pmatrix} 0 & 0 & 0 & -e(a - (g_-)^2) \\ 0 & 0 & -e\partial_x & 0 \\ 0 & +e\partial_x & 0 & +e\partial_x \\ +e^2(a - (g_-)^2) & 0 & -\partial_x & 0 \end{pmatrix} \delta(x - y)$$

and its inverse

$$C_{ab}^{-1}(x, y) = \begin{pmatrix} 0 & \frac{-1}{e^2(a - (g_-)^2)} & 0 & \frac{1}{e(a - (g_-)^2)} \\ \frac{-1}{e^2(a - (g_-)^2)} & 0 & \frac{1}{e\partial_x} & 0 \\ 0 & \frac{-1}{e\partial_x} & 0 & 0 \\ \frac{-1}{e^2(a - (g_-)^2)} & 0 & 0 & 0 \end{pmatrix} \delta(x - y).$$

Before going on to obtain the Dirac brackets, it is convenient to reduce the number of variables of theory. It is possible choosing a set of independent variables. From (3.7) we conclude that

$$\begin{aligned} A_0(x) &= \frac{1}{e^2(a - (g_-)^2)} \\ &\times (\partial_1 \pi_1 + eg_- \pi_\phi - eg_+ \partial_1 \phi - e^2(g_- g_+)A_1), \end{aligned} \quad (3.11)$$

thus we can choose Φ , π_ϕ and A_1 , π_1 as our set of independent variables. It is important to mention that one could keep all fields until to the end and only choosing to eliminate one of them at the final of the calculation. From here we find the following canonical Dirac brackets

$$\begin{aligned} \{A_1(x), \pi_1(y)\}_D &= \delta(x_1 - y_1) \\ \{\phi(x), \pi_\phi(y)\}_D &= \delta(x_1 - y_1), \end{aligned} \quad (3.12)$$

and the quantized theory is obtained replacing

$$\{A(x), B(y)\}_D \rightarrow -i[A(x), B(y)]_{x_0=y_0}, \quad (3.13)$$

which agrees with result previously obtained in [13] when studying the CSM case.

Starting from Heisenberg equations of motion for A_1 and Φ fields we obtain

$$\square \Phi(x) - e(g_+ \varepsilon^{\mu\nu} - g_- g^{\mu\nu})(\partial_\mu A_\nu) = 0, \quad (3.14)$$

and

$$\partial_\mu F^{\mu\nu} + e^2 a A^\nu - e(g_+ \varepsilon^{\mu\nu} - g_- g^{\mu\nu})(\partial_\mu \Phi) = 0. \quad (3.15)$$

From (3.14)–(3.15) we find that

$$\partial_\mu A^\mu - \frac{g_- g_+}{[a - (g_-)^2]} \varepsilon^{\mu\nu} \partial_\mu A_\nu = 0, \quad (3.16)$$

which correspond to the motion of CSM in the limit $g_- = g_+ = -1$.

Case 2: $a = (g_-)^2$

Considering the Lagrangians (2.3) and (2.11) for GSM with WZ term in the above limit we arrive to

$$\begin{aligned} \mathcal{L}_{GSM} &= \frac{1}{2}(\dot{A}_1 - A'_0)^2 + \frac{1}{2}(\dot{\Phi} - \Phi')^2 - eg_+(\dot{\Phi}A_1 - \Phi'A_0) \\ &\quad - eg_-(\dot{\Phi}A_0 - \Phi'A_1) + \frac{e^2(g_-)^2}{2}(A_0^2 - A_1^2), \end{aligned} \quad (3.17)$$

and

$$\mathcal{L}_{WZ} = e(g_- g_+)(\partial'_0 A_0 - \partial'_1 A_1). \quad (3.18)$$

The corresponding canonical Hamiltonian is

$$\begin{aligned}\mathcal{H}_c = & \frac{1}{2}(\pi_1)^2 + \frac{1}{2}(\pi_\phi)^2 - \pi_1 \partial_1 A_0 + e(g_- A_0 + g_+ A_1) \pi_\phi \\ & + \frac{1}{2}(\partial_1 \Phi)^2 - e(g_+ A_0 + g_- A_1) \\ & \times \partial_1 \Phi + e^2(g_- g_+) A_1 A_0 \\ & - e(g_+ g_-) \partial_1 \theta A_0 + \frac{e^2}{2}((g_+)^2 + (g_-)^2)(A_1)^2,\end{aligned}\quad (3.19)$$

such that the primary constraints are

$$\begin{aligned}\chi_1 &= \pi_0 \approx 0 \\ \chi_2 &= \pi_\theta + e(g_+ g_-) A_1 \approx 0.\end{aligned}\quad (3.20)$$

By using the primary Hamiltonian and consistency condition we obtain new constraints

$$\begin{aligned}\chi_3 &= \partial_1 \pi_1 - e g_- \pi_\phi - e^2(g_- g_+) A_1 \\ &+ e(g_- g_+) \partial_1 \theta + e g_+ \partial_1 \Phi \approx 0,\end{aligned}\quad (3.21)$$

where its time evolution leads to

$$\chi_4 = e(g_- g_+) \pi_1 \approx 0. \quad (3.22)$$

Since χ_3 and χ_4 are second class, their time derivatives do not lead to new constraints but only fix Lagrange multipliers. Therefore we have two sets of constraints given by (3.20) and (3.21)–(3.22), being the first one first class. At this point one sees why the case $a = (g_-)^2$ must be treated separately, and not as a limit of the general case. As happens in the CSM [14], this case has twice the number of primary constraints of the previous one.

In order to attend the general theory of constrained systems [15] we fix the gauge for the first set of constraints and, consequently apply the consistency condition to verify the existence of new constraints.

Here, we are going to choose the same condition as in the case $a > (g_-)^2$, i.e.

$$\chi_5 = \alpha \partial_1 \theta \approx 0, \quad (3.23)$$

where now α is an arbitrary parameter. The time evolution of χ_5 is given by

$$\begin{aligned}\dot{\chi}_5 &= \{\chi_5, \mathcal{H}_p\} = \{\alpha \partial_1 \theta, \lambda_2 \pi_\theta\} \\ &= \alpha \partial_1 \lambda_2 \approx 0,\end{aligned}\quad (3.24)$$

where λ_2 is

$$\begin{aligned}\lambda_2 &= \frac{-1}{e(g_- g_+)}(e g_+ \pi_\phi + e^2((g_-)^2 + (g_+)^2) A_1 \\ &+ e^2(g_- g_+) A_0 - e g_- \partial \Phi) \approx 0,\end{aligned}\quad (3.25)$$

which substituting in (3.24) gives

$$\begin{aligned}\chi^6 &= \frac{-\alpha}{e(g_+ g_-)}(e g_+ \partial_1 \pi_\phi + e^2((g_-)^2 + (g_+)^2) \partial A_1 \\ &+ e^2(g_- g_+) \partial A_0 - e g_- \square \Phi) \approx 0,\end{aligned}\quad (3.26)$$

which coincides with result used in [5] except by a total derivative. At this point, it is important to mention that this constraint was obtained in order to satisfy the consistency condition and not as an arbitrary choice. In this way, we believe that the result of [5], for the CSM, was not obtained in a totally careful manner.

So, the constraints matrix can be constructed through the set given by (3.19), (3.21)–(3.23) and (3.26),

$$M_{ab}(x, y) = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \delta(x - y)$$

where,

$$A(x, y) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -e^3(g_- g_+)^2 \\ 0 & +e^3(g_- g_+)^2 & 0 \end{pmatrix}$$

$$B(x, y) = \begin{pmatrix} -e\alpha\partial_x & 0 & 0 \\ 2e\alpha\left(\frac{g_-}{g_+}\right)\partial_x^2 & 0 & e(g_- g_+)\partial_x \\ -e^2\alpha(g_+^2 + g_-^2)\partial_x & 0 & -e^2(g_- g_+)^2 \end{pmatrix}$$

$$C(x, y) = \begin{pmatrix} +e\alpha\partial_x & 2e\alpha\left(\frac{g_-}{g_+}\right)\partial_x^2 & -e^2\alpha(g_+^2 + g_-^2)\partial_x \\ 0 & 0 & 0 \\ 0 & -e(g_- g_+)\partial_x & e^2(g_- g_+)^2 \end{pmatrix}$$

$$D(x, y) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\alpha\partial_x \\ 0 & \alpha\partial_x & 0 \end{pmatrix},$$

and the inverse $M_{ab}^{-1}(x, y)$ is

$$M_{ab}^{-1}(x, y) = \begin{pmatrix} E & F \\ G & H \end{pmatrix} \delta(x - y)$$

such that,

$$E(x, y) = \begin{pmatrix} 0 & \frac{g_+^2 + g_-^2}{e^2(g_- g_+)^2} & \frac{2\partial_x}{e^3(g_+^3 g_-)} \\ -\frac{g_+^2 + g_-^2}{e^2(g_- g_+)^2} & 0 & \frac{1}{e^3(g_- g_+)^2} \\ -\frac{2\partial_x}{e^3(g_+^3 g_-)} & -\frac{1}{e^3(g_- g_+)^2} & 0 \end{pmatrix}$$

$$F(x, y) = \begin{pmatrix} \frac{1}{e\alpha\partial_x} & \frac{g_+^2 + g_-^2}{e\alpha(g_- g_+)} & 0 \\ 0 & -\frac{1}{e\alpha\partial_x} & 0 \\ 0 & -\frac{1}{e^2\alpha(g_+ g_-)} & 0 \end{pmatrix}$$

$$G(x, y) = \begin{pmatrix} -\frac{1}{e\alpha\partial_x} & 0 & 0 \\ -\frac{g_+^2 + g_-^2}{e\alpha(g_-g_+)} & \frac{1}{e\alpha\partial_x} & \frac{1}{e^2\alpha(g_+g_-)} \\ 0 & 0 & 0 \end{pmatrix}$$

$$H(x, y) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\alpha\partial_x} \\ 0 & -\frac{1}{\alpha\partial_x} & 0 \end{pmatrix}.$$

After some calculation we obtain the non-null Dirac brackets

$$\{\Phi(x_1), \pi_\phi(y_1)\}_D = \delta(x_1 - y_1)$$

$$\{A_1(x_1), \pi_\phi(y_1)\}_D = -\frac{1}{eg_-} \partial_x \delta(x_1 - y_1)$$

$$\{A_1(x_1), \phi(y_1)\}_D = +\frac{1}{eg_+} \delta(x_1 - y_1), \quad (3.27)$$

which coincides with the result previously obtained in the anomalous case [9]. Note that for this choice of independent variables, there is none dependence of the Dirac brackets in the gauge parameter α .

4

For sake of completeness, we will now proceed to the calculation of the photon propagator in the above non-covariant gauge ($\partial_1 \theta = 0$). This will be done in the two physically different versions of the model, $a \neq (g_-)^2$ and $a = (g_-)^2$. Moreover we will present the usual propagator and also the gauge-invariant one [8, 14].

In fact we will observe an interesting feature of this propagator, it will be seen that in the Landau gauge ($\alpha = 0$) it becomes equal to the photon propagator in a covariant gauge ($\partial_\mu \theta = 0$).

In the process of obtaining the propagator we will use a method developed in [8]. In that approach one uses the vector field decomposition law in two dimensions (2.5), and write down the Lagrangian density in the following form:

$$\mathcal{L} = \rho^T \mathbf{M} \rho, \quad (4.1)$$

where $\rho^T \equiv (\Phi, \chi, \eta, \theta)$. Then once one invert the matrix, one gets

$$\mathbf{M}^{-1} = \begin{pmatrix} \langle \Phi\Phi \rangle & \langle \Phi\chi \rangle & \langle \Phi\eta \rangle & \langle \Phi\theta \rangle \\ \langle \chi\Phi \rangle & \langle \chi\chi \rangle & \langle \chi\eta \rangle & \langle \chi\theta \rangle \\ \langle \eta\Phi \rangle & \langle \eta\chi \rangle & \langle \eta\eta \rangle & \langle \eta\theta \rangle \\ \langle \theta\Phi \rangle & \langle \theta\chi \rangle & \langle \theta\eta \rangle & \langle \theta\theta \rangle \end{pmatrix}, \quad (4.2)$$

where the elements $\langle \Phi\Phi \rangle$, $\langle \Phi\chi \rangle$, etc..., are the two-point correlation functions of their respective fields, from which one can compute the remaining. For instance the

photon propagator can be obtained from:

$$\langle A_\mu A_\nu \rangle = -\frac{1}{e^2} [-g_{\mu\nu} k^2 \langle \chi\chi \rangle + (\langle \eta\eta \rangle + \langle \chi\chi \rangle) k_\mu k_\nu + (k_\mu \tilde{k}_\nu + \tilde{k}_\mu k_\nu) \langle \chi\eta \rangle], \quad (4.3)$$

where $\tilde{k}_\mu \equiv \epsilon_{\mu\sigma} k^\sigma$, and we made use of the identity $\epsilon_{\mu\rho} \epsilon_{\nu\sigma} k^\rho k^\sigma = -g_{\mu\nu} k^2 + k_\mu k_\nu$. Due to the hermiticity requirement, it is necessary that $\mathbf{M}$ be a symmetric matrix.

Let us begin with the non-singular case ($a \neq (g_-)^2$). In this case the elements of the inverse matrix $\mathbf{M}^{-1}$ are

$$\begin{aligned} \langle \Phi\Phi \rangle &= -\frac{1}{(a - (g_-)^2)(k^2 - m^2)k^2 k_1^2} \\ &\times [\alpha(g_-)^2(a - (g_-)^2)k^4 - ak^2 k_1^2 \\ &- \alpha ae^2(g_-)^2 k^2(a + (g_+)^2 - (g_-)^2) + a^2 e^2 k_1^2]; \end{aligned} \quad (4.3a)$$

$$\langle \Phi\chi \rangle = \frac{ae^2(g_+)^2 k_1^2}{(a - (g_-)^2)(k^2 - m^2)k^2 k_1^2}; \quad (4.3b)$$

$$\begin{aligned} \langle \Phi\eta \rangle &= \langle \eta\Phi \rangle = -\frac{1}{(a - (g_-)^2)(k^2 - m^2)k^2 k_1^2} \\ &\times [\alpha(g_-)(a - (g_-)^2)k^4 - (g_-)k^2 k_1^2 \\ &- \alpha ae^2(g_-)(a + (g_+)^2 - (g_-)^2)k^2 + ae^2(g_-)k_1^2]; \end{aligned} \quad (4.3c)$$

$$\begin{aligned} \langle \Phi\theta \rangle &= \langle \theta\Phi \rangle = \frac{1}{(a - (g_-)^2)(k^2 - m^2)k^2 k_1^2} \\ &\times [\alpha(g_-) + (a - (g_-)^2)k^4 + \alpha(g_-) \\ &\times ae^2(a + (g_+)^2 - (g_-)^2)k^2]; \end{aligned} \quad (4.3d)$$

$$\langle \chi\chi \rangle = \frac{e^2}{(k^2 - m^2)k^2}; \quad \langle \chi\theta \rangle = \langle \theta\chi \rangle = 0, \quad (4.3e)$$

$$\begin{aligned} \langle \eta\eta \rangle &= -\frac{1}{(a - (g_-)^2)(k^2 - m^2)k^2 k_1^2} \\ &\times [\alpha(a - (g_-)^2)k^4 - k^2 k_1^2 \\ &- \alpha ae^2(a + (g_+)^2 - (g_-)^2)k^2 + (a + (g_+)e^2 k_1^2]; \end{aligned} \quad (4.3f)$$

$$\begin{aligned} \langle \eta\theta \rangle &= \langle \theta\eta \rangle = -\langle \theta\theta \rangle = \frac{\alpha}{(a - (g_-)^2)(k^2 - m^2)k_1^2} \\ &\times [(a - (g_-)^2)k^2 - ae^2(a + (g_+)^2 - (g_-)^2)], \end{aligned} \quad (4.3g)$$

where

$$m^2 \equiv \frac{ae^2(a + (g_+)^2 - (g_-)^2)}{(a - (g_-)^2)}. \quad (4.4)$$

which is in accordance with the mass obtained in the anomalous model [9]. Substituting the above set of correlation functions in that of the photon propagator (4.3),

one gets

$$D_{\mu\nu}(k) = \frac{1}{(k^2 - m^2)} \left\{ g_{\mu\nu} + \frac{k_\mu k_\nu}{k^2} \left[\frac{\alpha k^4}{e^2 k_1^2} - \frac{k^2}{e^2(a - (g_-)^2)} - 1 \right] - \frac{\alpha(a + (g_+)^2 - (g_-)^2) k^2}{(a - (g_-)^2) k_1^2} + \frac{(a + (g_+)^2)}{(a - (g_-)^2)} \right\} - \frac{g_+ g_-}{(a - (g_-)^2)} \left(\frac{k_\mu \tilde{k}_\nu + \tilde{k}_\mu k_\nu}{k^2} \right). \quad (4.5)$$

It is easy to verify that in the Landau gauge ($\alpha = 0$) the non-covariance of the propagator disappears, and it can be seen that it becomes equal to the propagator in a covariant gauge ($\partial_\mu \theta = 0$) [8].

Besides one can also obtain the propagator for a gauge invariant field [8, 14], $A_\mu^I \equiv A_\mu + \frac{1}{e} \partial_\mu \theta$, so that

$$D_{\mu\nu}^I \equiv \langle A_\mu^I A_\nu^I \rangle = D_{\mu\nu} - \frac{1}{e^2} [k_\mu k_\nu (2\langle \eta 0 \rangle + \langle 0 0 \rangle) + (k_\mu \tilde{k}_\nu + \tilde{k}_\mu k_\nu) \langle \chi 0 \rangle], \quad (4.6)$$

whose calculation leaves us with

$$D_{\mu\nu}^I(k) = \frac{1}{(k^2 - m^2)} \left\{ g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \left[1 + \frac{k^2}{(a - (g_-)^2) e^2} - \frac{(a + (g_+)^2)}{(a - (g_-)^2)} \right] - \frac{g_+ g_-}{(a - (g_-)^2)} \left(\frac{k_\mu \tilde{k}_\nu + \tilde{k}_\mu k_\nu}{k^2} \right) \right\}. \quad (4.7)$$

From the above one can notice that $D_{\mu\nu}^I = D_{\mu\nu}(\alpha = 0)$, so that the Landau gauge is such that the propagator of A_μ and A_μ^I are equal, and this explain the relation between $\partial_1 \theta = 0$ and $\partial_\mu \theta = 0$ gauges, when one choose this value for the gauge parameter.

Finally one can proceed in analogous way to obtain the photon propagator in the singular case ($a = (g_-)^2$), so obtaining

$$D_{\mu\nu}(k) = \frac{k_\mu k_\nu}{k^2} \left[\frac{k^2}{(g_+ g_-) e^2} - \frac{((g_+)^2 + (g_-)^2)}{(g_+ g_-)} \right] + \alpha \left(\frac{k}{k_1} \right)^2 + \frac{1}{(g_+ g_-)} \left[\frac{k_\mu \tilde{k}_\nu + \tilde{k}_\mu k_\nu}{k^2} \right]. \quad (4.8)$$

Once again the gauge-invariant photon propagator is equal to the above one in the Landau gauge, which happens to be equal to that of the anomalous model [8, 9].

5

In order to finish our analysis of the canonical quantization of GSM in a gauge-invariant version, we would like

to emphasize the discussion about the gauge fixing problem. Here, we adopted the condition $\partial_1 \theta = 0$ to fix the gauge in two different physical situations $a > (g_-)^2$ and $a = (g_-)^2$ corresponding to $a > 1$ and $a = 1$ in the usual CSM. Although this gauge eliminates the WZ terms of action, the commutation relations coincides with that of CSM one. On the other hand, we observe in [5] that gauge fixing for the case $a = 1$ was considered in a not completely carefull way. In fact, the authors have not considered the time evolution of the constraint

$$\Omega = \partial_1 \theta \approx 0$$

in order to satisfy the consistency condition. The reason that the naive choice in [5] does not affect the Poisson brackets is that it differs from the correct one by a derivative which, in last sense, preserve the Euler-Lagrange equations. However this is a quite casual feature, since the authors have not used the standard procedure in order to perform the Dirac method. In general, it could use any other constraints without relation with correct ones, coming from the consistency condition. Finally we notice that the propagator (4.8) corresponds to the usual CSM propagator when the appropriate limits are taken ($g_+ = g_- = 1$) [1].

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Chiral Schwinger model with a Podolsky term at finite temperature

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Abstract. We study an exactly solvable two-dimensional model which mimics the basic features of the standard model. This model combines chiral coupling with an infrared behavior which resembles low energy QCD. This is done by adding a Podolsky higher-order derivative term in the gauge field to the Lagrangian of the usual chiral Schwinger model. We adopt a finite temperature regularization procedure in order to calculate the non-trivial fermionic Jacobian and obtain the photon and fermion propagators, first at zero temperature and then at finite temperature in the imaginary and real time formalisms. Both singular and non-singular cases, corresponding to the choice of the regularization parameter, are treated. In the nonsingular case there is a tachyonic mode as usual in a higher order derivative theory, however in the singular case there is no tachyonic excitation in the spectrum.

1 Introduction

Higher order derivatives Lagrangians have always been the subject of some interest, in spite of its inherent difficulties. Historically, we can mention the pioneering work of Ostrogradskii [1] who attempted to generalize the variational Hamilton's principle in order to include this class of Lagrangians.

In the first years of the called Classical Theory of Fields, the works of Weyl and Eddington [2] with the study of a generalization of the Einstein Theory of Relativity and Bopp and Podolsky [3] who suggested to modify the Maxwell theory of electromagnetism in order to avoid the divergences of the self energy of the punctual charge, have began the discussions about higher order derivatives theories.

Despite of the above examples, these kind of theories were almost always classified as purely academic. However, since advent of supersymmetry and string theories, this concept has been changing. In the first case, we can

mention the superfield formulation which is replete of higher order derivative terms [4]. In the other case, there are important examples where higher order derivatives are present, for instance, the action proposed by Polyakov [5], which has a term proportional to the extrinsic curvature of the world sheet. Also in gauge theories it has been argued that at large distances (strong coupling) the Yang-Mills theory can be approximated by a Lagrangian containing the second derivatives of the field strength tensor [6]. This fact reveals that the gluon propagator could have an infrared asymptotic behavior like $(k^2)^{-2}$, $k^2 \rightarrow 0$. It is also important to remark that the inclusion of higher order derivative terms in supersymmetric theories has been shown to be a powerful regularizing mechanism [7]. We call attention for the fact that the dimensional regularization is not available in this case since this later method breaks the supersymmetry.

All above examples, showed that there was an increasing interest in the last years for this subject. Actually, we can cite many works which have been appearing and showing how to deal with such systems in a consistent way [8,9].

In order to enrich the understanding of such theories there have been given, recently, a special attention to study some systems in two dimensional space-time, containing higher order derivatives. It is a well known fact that, for instance, in the Schwinger model [10], the theory exhibits interesting aspects like charge screening, fermion confinement, asymptotic freedom and a nontrivial vacuum structure [11]. The combination of features of the Schwinger model containing higher order terms have given new informations about mass generation, axial anomaly, behavior of propagator and also on the characteristics cited before [12,13]. In [12], the problem of quantizing a system where fermions and gauge field are coupled through a higher order derivative was considered. On the other hand, in [13] a Podolsky term was added to the original Lagrangian of the Schwinger model. In both cases, new conclusions were obtained about the usual Schwinger model. In particular, the role of the tachyonic excitation which naturally appears in such theories.

and we obtain⁸

$$[d\bar{\psi}][\psi] = [d\bar{\psi}'][\psi'] \exp \left\{ 2i \int d^2x \epsilon(x) I(x) \right\}, \quad (13)$$

where

$$I(x) = \sum_n \phi_n^\dagger(x) \gamma_5 \phi_n(x). \quad (14)$$

The quantities $\phi_n(x)$ form a complete and orthonormal set of eigenfunctions of some Hermitian operator O_H . To determine the anomaly, it is then necessary to perform the calculation of the sum above. However, it is well known that sums like $\sum_n \phi_n^\dagger(x) \Gamma_{\mu\nu\dots} \phi(y)$, where $\Gamma_{\mu\nu\dots}$ are product of gamma matrices, are divergent when $x = y$. The way found by Fujikawa to regularize these sums is to introduce an exponential factor in order to avoid contributions of big eigenvalues. Concentrating on our particular case, this is done in the following way:

$$\begin{aligned} I(x) &= \lim_{M \rightarrow \infty} \sum_n \phi_n^\dagger(x) \gamma_5 \phi_n(x) e^{-\lambda_n^2/M^2} \\ &= \lim_{M \rightarrow \infty} \sum_n \phi_n^\dagger(x) \gamma_5 e^{-O_H^2/M^2} \phi_n(x) \\ &= \lim_{\substack{M \rightarrow \infty \\ x=y}} \left(\gamma_5 e^{-O_H^2/M^2} \right)_{ij} \sum_n \phi_{nj}(x) \phi_{ni}^\dagger(y) \\ &= \lim_{\substack{M \rightarrow \infty \\ x=y}} \left(\gamma_5 e^{-O_H^2/M^2} \right)_{ij} \delta_{ij} \delta^{(2)}(x-y) \\ &= \lim_{\substack{M \rightarrow \infty \\ x=y}} \text{tr} \gamma_5 e^{-O_H^2/M^2} \delta^{(2)}(x-y). \end{aligned} \quad (15)$$

According to the Fujikawa method, the operator that has to be used to regularize the sum is the one that appears in the theory. In our case, we have

$$\tilde{D} = \gamma^\mu (\partial_\mu - ieA_\mu - ig_0 \gamma_5 \partial_\mu \phi). \quad (16)$$

The point is that this operator is not Hermitian. An usual the procedure used in literature is to go to the Euclidean space. We do this by letting $x^0 \rightarrow -ix_4$, $\partial_0 \rightarrow i\partial_4$, $\gamma^0 \rightarrow i\gamma_4$, $\gamma^\mu \partial_\mu \rightarrow -\gamma_\mu \partial_\mu$. In this space, all gamma matrices are anti-Hermitians.^c So, the terms $\gamma_\mu \partial_\mu$ and $ie\gamma_\mu A_\mu$ are Hermitians, but $ig_0 \gamma_\mu \gamma_5 \partial_\mu \phi$ is not. Consequently, the operator

$$\tilde{D}_E = \gamma_\mu (\partial_\mu - ieA_\mu - ig_0 \gamma_5 \partial_\mu \phi) \quad (17)$$

obtained from

$$\tilde{D} \longrightarrow -\tilde{D}_E \quad (18)$$

^cIn the Euclidean space, our initial convention and notation have to be changed to $\{\gamma_\mu, \gamma_\nu\} = -2\delta_{\mu\nu}$, $\gamma_\mu \gamma_\nu = -\delta_{\mu\nu} - i\epsilon_{\mu\nu} \gamma_5$, $\gamma_5 \gamma_\mu = i\epsilon_{\mu\nu} \gamma_\nu$, $\gamma_5 = i\gamma_1 \gamma_4$, $\epsilon_{14} = +1$ etc.

In order to give continuity to this program it would be interesting to discuss an anomalous model in the presence of higher order derivative terms. One of the most important anomalous systems in two dimensional space-time discussed in the literature is the chiral Schwinger model, proposed by Jackiw and Rajaraman [14]. Its main characteristic is to raise the possibility of a mass generation mechanism, that could mimic the electroweak interactions in two dimensional space-time without needing a Higgs field. Another interesting aspect is the dependence of an arbitrary parameter in the mass of the model, due to the breakdown of gauge symmetry at the quantum level. As a consequence, the lost gauge invariance cannot be restored by fixing this parameter. Despite the absence of this symmetry, the model is unitary and consistent. On the other side, it has been shown that the inclusion of a Wess-Zumino term can restore the gauge invariance [15].

The idea of introducing higher order derivatives terms in this model will permit us to clear new questions like the relation between the axial anomaly and the tachyonic excitation, for $a > 1$ and the singular case $a = 1$. We can investigate if it is possible to cancel the anomaly for some choice of parameters. Another point it is related to tachyon mode. It is a well known fact [9,13] that the propagator of the Podolsky theory is proportional to $(k^2)^{-2}$ and admits a massless photon excitation ($k^2 = 0$) and a tachyonic ($k^2 < 0$) mode. Therefore, it would be interesting to analyze the behavior of the propagator of the resulting theory (chiral Schwinger- Podolsky) in two different cases: $a > 1$ and $a = 1$. Finally, we believe that our analysis would not be complete without the study of the same propagators in the finite temperature regime. At first, we are interested in understanding the same aspects early mentioned at zero temperature. Another point is to investigate the possibility of some critical value for the temperature to restore the symmetry.

Another motivation for the study of this model at finite temperature is related to the recent interest on the high temperature behavior of the standard model [16]. This is so because of the arguments listed above, *i. e.*, the facts that the chiral Schwinger model mimics the electroweak interactions in two space-time dimensions and the zero-momentum behavior of QCD by the Podolsky term. So we can investigate the thermal behavior of an exactly soluble model which we expect to resemble in some sense the standard model. This discussion is also closely related to the calculations of hard thermal loops in QCD [17], since these are done in the high temperature limit. In fact, the zero-momentum behavior of the Podolsky term gives also a glance at the nonperturbative effects of the soft thermal loops [18].

This paper is organized as follows. In section 2, we will use the path integral formalism to introduce the model and discuss its anomalous chiral fermionic Jacobian in a finite temperature approach [19,20]. As in the usual CSM, the fermionic measure is noninvariant under chiral rotations, so that we expect that this Jacobian will give rise to dynamically generated masses for the gauge field and these masses are compared with the one of the usual CSM. Then, in section 3, we construct the zero temperature propagators of the model, *i.e.*, the gauge field propagator in the nonsingular and singular cases and the rel-

evant fermionic propagator. In section 4 we extend these calculations to the finite temperature case in two frameworks: imaginary time and thermofield dynamics and in section 5 we present our conclusions.

2 The model and Fujikawa's finite temperature procedure

Let us start, writing the partition function, in two dimensional Euclidean space-time with signature $(+, +)$

$$Z(\beta) = N(\beta) \int_{\text{antiperiod.}} D\bar{\psi} D\psi \int_{\text{period.}} DA_\mu \exp \left\{ \int_\beta d^2x \mathcal{L}' \right\}, \quad (2.1)$$

where we use the convention $\int_\beta d^2x = \int_0^\beta dx_0 \int dx_1$, β^{-1} being proportional to the temperature and the Lagrangian density for CSM with the Podolsky term is given by

$$\mathcal{L}' = -\frac{1}{4} F_{\mu\nu} F_{\mu\nu} - b^2 \partial_\nu F_{\mu\nu} \partial_\sigma F_{\mu\sigma} + i\bar{\psi} \not{D} \psi, \quad (2.2)$$

where $b > 0$ is the Podolsky coupling constant, $F_{\mu\nu}$ is the usual antisymmetric tensor for the gauge field

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

and the Dirac operator which couples left fermions to the gauge field is given by

$$\not{D} = \gamma_\mu (\partial_\mu - ie(1 - \gamma_5) A_\mu). \quad (2.3)$$

As we are discussing the finite temperature theory corresponding to Lagrangian (2.2) we demand that the bosonic and fermionic fields obey the usual periodic and antiperiodic conditions in the imaginary time formalism. This is by no means restrictive since we may left, at any stage of the calculations $\beta \rightarrow \infty$, getting the zero temperature model, without periodic or antiperiodic conditions.

To compute the anomalous divergence of the axial-vector current, let us restrict our discussion to the fermionic part of the partition function (2.1),

$$Z_F(\beta) = N(\beta) \int_{\text{antiperiod.}} D\bar{\psi} D\psi \times \exp \left\{ \int_\beta d^2x i\bar{\psi} \gamma_\mu [\partial_\mu - ie(1 - \gamma_5) A_\mu] \psi \right\} \quad (2.4)$$

and implement the infinitesimal chiral transformations in the fermion fields

$$\psi = [1 + ie(1 - \gamma_5)\epsilon(x)]\psi' \quad (2.5a)$$

$$\bar{\psi} = \bar{\psi}' [1 - ie(1 + \gamma_5)\epsilon(x)]. \quad (2.5b)$$

As is well known, these operations lead to a change in fermionic measure of the partition function $D\bar{\psi} D\psi = D\bar{\psi}' D\psi' J_\beta(\epsilon(x))$, where $J_\beta(\epsilon(x))$ is a non-trivial Jacobian. In order to discuss the dynamical mass generation through obtaining this Jacobian at finite temperature we use the Fujikawa procedure [20] with the operator [21]

$$\not{D} = \gamma_\mu (\partial_\mu - ie\tilde{\epsilon} A_\mu + ie\eta\gamma_5 A_\mu), \quad (2.6)$$

being η and ξ real parameters. Now, following Ref. [22] we find that the anomalous Jacobian is given by

$$J_\rho(x) = \exp \left\{ -\frac{ie^2}{\pi} \int_\mu d^2x \epsilon(x) (\eta g_{\mu\nu} + \xi \epsilon_{\mu\nu}) \partial_\mu A_\nu \right\}. \quad (2.7)$$

which implies an anomalous divergence for the axial-vector current

$$\partial_\mu [\bar{\psi} \gamma_\mu (1 - \gamma_5) \psi] = \frac{e}{\pi} (\eta g_{\mu\nu} + \xi \epsilon_{\mu\nu}) \partial_\mu A_\nu, \quad (2.8)$$

which is explicit β -independent. To find the dynamically generated mass for CSM with the Podolsky term we write the boson field as

$$eA_\mu = \partial_\mu \sigma + v_{\mu\nu} \partial_\nu \rho \quad (2.9)$$

and iterate the infinitesimal Jacobian (2.7) to find

$$J_\rho(x) = \exp \left\{ -\frac{i}{2\pi} \int_\mu d^2x [\eta \sigma \square \sigma - \xi \rho \square \rho - (\eta - \xi) \sigma \square \rho] \right\}. \quad (2.10)$$

Substituting this result into the partition function (2.1) we arrive at

$$Z(\beta) = N(\beta) \int_{\text{antiperiod.}} D\bar{\psi} D\psi \int_{\text{period.}} D\rho D\sigma \exp \left\{ \int_\mu d^2x \mathcal{L}'_{\text{eff}} \right\}. \quad (2.11)$$

such that, the effective Lagrangian is given by

$$\mathcal{L}'_{\text{eff}} = \frac{1}{2e^2} \rho \square \square \rho + \frac{b^2}{e^2} (\square \partial_\mu \rho)^2 + i\bar{\psi}' \gamma_\mu \partial_\mu \psi' - \frac{1}{4\pi} [\eta \sigma \square \sigma - \xi \rho \square \rho - (\eta - \xi) \sigma \square \rho]. \quad (2.12)$$

Now, using the Euler-Lagrange equations for higher-order derivative systems [4]

$$\frac{\partial \mathcal{L}'}{\partial \rho} - \partial_\mu \frac{\partial \mathcal{L}'}{\partial (\partial_\mu \rho)} + \partial_\mu \partial_\nu \frac{\partial \mathcal{L}'}{\partial (\partial_\mu \partial_\nu \rho)} - \partial_\mu \partial_\nu \partial_\lambda \frac{\partial \mathcal{L}'}{\partial (\partial_\mu \partial_\nu \partial_\lambda \rho)} = 0 \quad (2.13)$$

we get the equation of motion for the $\rho(x)$ field

$$\left(\square - 2b^2 \square \square + \frac{e^2}{4\pi} \frac{(\eta + \xi)^2}{\eta} \right) \square \rho = 0. \quad (2.14)$$

This equation can be rewritten as

$$\square (\square + \mu^2) (\square - M_t^2) \rho(x) = 0, \quad (2.15)$$

and the field $\rho(x)$ can be expressed as a linear combination of three other fields: a massless free excitation, a pseudo-scalar field with mass

$$\mu^2 = \frac{1}{4b^2} [(1 + 8b^2 \mu_0^2)^{1/2} - 1] \quad (2.16)$$

and a tachyonic mode with mass (note the sign for M_t^2 in (2.15))

$$M_t^2 = \frac{1}{4b^2} [(1 + 8b^2 \mu_0^2)^{1/2} + 1] \quad (2.17)$$

where

$$\mu_0^2 = \frac{e^2 (\eta + \xi)^2}{4\pi \eta}. \quad (2.18)$$

These masses are β -independent. Taking $b \rightarrow 0$ we find that μ^2 coincides with zero temperature result of Jackiw and Rajaraman [14] (identifying $2a^2/(a-1) = (\eta + \xi)^2/\eta$) and also the Podolsky tachyon M_t^2 vanishes in this limit.

As we can see, the higher order derivative term is not sufficient to restore the symmetry which was destroyed at quantum level. So, the basic properties of the usual CSM persist despite the additive Podolsky term. In fact, its usual mass was redefined by the Podolsky coupling b and we get also a tachyonic mode M_t^2 .

On the other hand, the temperature independence of the anomalous divergence of the axial-vector current and the dynamically generated mass for the vector-meson shows the absence of a critical temperature in the CSM plus the Podolsky term. The key fact for this behavior is that these quantities come from the regularization of ultraviolet divergences which dominate over the temperature effects of the heat bath.

3 Propagators at zero temperature

As we mentioned before, the zero temperature discussion follows from the finite temperature one letting β approaches infinity. Now, we are going to calculate the zero temperature propagators for the theory described above in this limit. With this procedure, we intend to check our previous calculation and give a more detailed description of the model. For this, we will start by making a suitable transformation to decouple the fields $\sigma(x)$ from $\rho(x)$ in the effective Lagrangian density (2.12):

$$\sigma'(x) = \sigma(x) - \alpha \rho(x). \quad (3.1)$$

The decoupling is obtained by choosing $\alpha = (\eta - \xi)/2\eta$, so that we find (in Minkowsky space-time with signature $(-, +)$)

$$\begin{aligned} \mathcal{L}'_{\text{eff}} = & \frac{1}{2e^2} \rho \square \left(\square - b^2 \square \square - \frac{e^2}{\pi} \frac{(\eta + \xi)^2}{4\eta} \right) \rho \\ & + i\bar{\psi}' \gamma_\mu \partial_\mu \psi' - \frac{1}{2\pi} \eta \sigma' \square \sigma' \\ & + \frac{1}{e} J^\mu \left[\partial_\mu \sigma' + \frac{\eta - \xi}{2\eta} \partial_\mu \rho + e^{\mu\lambda} \partial_\lambda \rho \right] + \bar{\partial} U_5 \psi' + \bar{\psi}' U_5 \partial. \end{aligned} \quad (3.2)$$

The last three terms of the above equation come from the change in the sources and we have defined

$$U_5 = \exp \{ (1 + \gamma_5) (\sigma - \rho) \}; \quad \bar{U}_5 = \exp \{ (1 - \gamma_5) (\sigma - \rho) \}. \quad (3.3)$$

Now, using the fact that the photon propagator can be obtained from the functional derivative,

$$\mathcal{D}_{\mu\nu} = \frac{\delta^2 \mathcal{L} [J_\mu]}{\delta J^\mu \delta J^\nu} \Big|_{J=J=0}, \quad (3.4)$$

where $\mathcal{Z}[J^\mu]$ is the generating functional for the Lagrangian (3.2), rotated to Minkowsky space-time, we have

$$\mathcal{Z}_{\mu\nu}(k) = -\frac{i}{e^2} \left\langle \left[k_\mu \left(\sigma' + \frac{\eta + \xi}{2\eta} \rho \right) + v_{\mu\lambda} k^\lambda \rho \right] \times \left[k_\nu \left(\sigma' + \frac{\eta + \xi}{2\eta} \rho \right) + v_{\nu\lambda} k^\lambda \rho \right] \right\rangle. \quad (3.5)$$

After a straightforward calculation [23] and the use of the two dimensional identity, $v_{\mu\lambda} v_{\nu\lambda} k^\lambda k^\lambda = g_{\mu\nu} k^2 - k_\mu k_\nu$, we arrive at

$$\begin{aligned} \mathcal{Z}_{\mu\nu}(k) = & -i \frac{k^2}{e^2} g_{\mu\nu} \langle \rho \rho \rangle - i \frac{k_\mu k_\nu}{e^2} \\ & \times \left[\left(1 - \frac{(\eta - \xi)^2}{4\eta^2} \right) \langle \rho \rho \rangle - \langle \sigma' \sigma' \rangle \right] \\ & - i \frac{\eta - \xi}{2\eta} \left(\frac{k_\mu \tilde{k}_\nu + \tilde{k}_\mu k_\nu}{e^2} \right) \langle \rho \rho \rangle, \end{aligned} \quad (3.6)$$

in which we defined $\tilde{k}_\mu = v_{\mu\lambda} k^\lambda$ while $\langle \rho \rho \rangle$ and $\langle \sigma' \sigma' \rangle$ are the elementary two-point correlation functions. These can be calculated from the effective Lagrangian density (3.2) giving

$$\langle \rho \rho \rangle = \frac{e^2}{k^2(k^2 + 2b^2 k^4 - \mu_0^2)} \quad (3.7)$$

and

$$\langle \sigma' \sigma' \rangle = \frac{\pi}{\eta k^2}. \quad (3.8)$$

Substituting these results in (3.6) we get the full propagator for the photon field, corresponding to the Lagrangian density (2.2)

$$\begin{aligned} \mathcal{Z}_{\mu\nu}(k) = & -\frac{i}{k^2 + 2b^2 k^4 - \mu_0^2} \left\{ g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right. \\ & \times \left[\left(1 - \frac{(\eta - \xi)^2}{4\eta^2} \right) - \frac{\pi}{\eta e^2} (k^2 + 2b^2 k^4 - \mu_0^2) \right] \\ & \left. + \frac{\eta - \xi}{2\eta} \left(\frac{k_\mu \tilde{k}_\nu + \tilde{k}_\mu k_\nu}{k^2} \right) \right\}. \end{aligned} \quad (3.9)$$

The poles of the above propagator coincide with the masses given by (2.16) and (2.17), as was expected. Also, this propagator reduces to the well known Jackiw-Rajaraman result [14], when we take the vanishing limit $b \rightarrow 0$, which cancels the higher derivative Podolsky term.

On the other hand, there is an important particular case of the CSM for which, in Jackiw-Rajaraman notation, $a = 1$ and in ours notation it can be expressed as $\eta = (a - 1)/2 = 0$ and $\xi = (a + 1)/2 = 1$, simultaneously. This particular case seems to be singular since it can not be analyzed as the limit of the above discussed one [23,24]. This can be easily seen from the transformation (3.1), which is singular at this point in the parameters space.

In this case we will use an approach developed in [23] where it is not necessary to make any decoupling transformation. In essence, we write down the Lagrangian density in a matrix form like $\mathcal{L} = (1/2) V^T M V$, where

$V^T \equiv (\rho\sigma)$ and then invert the matrix and collect the elementary correlation functions $\langle \rho\rho \rangle$, $\langle \rho\sigma \rangle$, etc, in order to compute more complex correlation functions as the photon propagator $\langle A_\mu A_\nu \rangle$.

In this singular case ($\eta = 0$, $\xi = 1$ in (2.12)) the inverse matrix in the momentum representation is given by:

$$\tilde{M}^{-1} = \frac{\pi^2}{e^2 k^2} \begin{pmatrix} 0 & \frac{e^2}{\pi} \\ \frac{e^2}{\pi} & -k^2 - 2b^2 k^4 + \frac{e^2}{2\pi} \end{pmatrix}, \quad (3.10)$$

where the symmetry of the matrix in relation to the principal diagonal is a direct consequence of the Hermiticity of the model [23] so we get that

$$\langle \sigma\sigma \rangle = 0; \quad \langle \sigma\rho \rangle = \langle \rho\sigma \rangle = \frac{\pi}{k^2}; \quad (3.10a)$$

$$\langle \rho\rho \rangle = -\frac{\pi^2}{e^2 k^2} \left(k^2 + 2b^2 k^4 - \frac{e^2}{2\pi} \right). \quad (3.10b)$$

Now, remembering that the photon propagator come from the two-point correlation function $\langle A_\mu A_\nu \rangle$ and also that in two dimensions is always possible to decouple a vector field into its longitudinal and transverse parts as we did in Eq. (2.9), one gets in momentum representation that

$$\begin{aligned} \langle A_\mu A_\nu \rangle = & -\frac{1}{e^2} [k_\mu k_\nu \langle \rho\rho \rangle + (k_\mu \tilde{k}_\nu + \tilde{k}_\mu k_\nu) \\ & \times \langle \sigma\rho \rangle + \tilde{k}_\mu \tilde{k}_\nu \langle \sigma\sigma \rangle]. \end{aligned} \quad (3.11)$$

Then it is easy to see that the full photon propagator in the singular case will be

$$\mathcal{Z}_{\mu\nu}(k) = \frac{\pi^2}{e^4} \frac{k_\mu k_\nu}{k^2} \left(k^2 + 2b^2 k^4 - \frac{e^2}{2\pi} \right) - \frac{\pi}{e^2 k^2} (k_\mu \tilde{k}_\nu + \tilde{k}_\mu k_\nu), \quad (3.12)$$

which in the limit of $b \rightarrow 0$ recovers the usual expression for the singular case of the CSM [23,24]. Note that we got a model were no tachyonic excitation appears. As far as we know, it is the first case of a higher-order derivative model without tachyons.

Let us now obtain some fermionic correlation functions. Since we are dealing with a chiral model we use the decoupling fermionic transformations (2.5a,b), in order to calculate the right and left fermion components

$$\psi_R = P_+ \psi \rightarrow \psi_R = \psi'_R, \quad (3.13a)$$

$$\psi_L = P_- \psi \rightarrow \psi_L = \exp[2ie(\sigma - \rho)] \psi'_L \equiv U(\sigma, \rho) \psi'_L. \quad (3.13b)$$

From the above equation its easy to see that

$$\langle \bar{\psi}_R \psi_R \rangle = \langle \bar{\psi}'_R \psi'_R \rangle = S_{0,R}^F(x_1, x_2) \quad (3.14a)$$

and

$$\begin{aligned} \langle \bar{\psi}_L(x_1) \psi_L(x_2) \rangle = & S_{0,L}^F(x_1, x_2) \langle \bar{U}(\sigma(x_1), \\ & \times \rho(x_1)) U(\sigma(x_2), \rho(x_2)) \rangle, \end{aligned} \quad (3.14b)$$

$$\langle \bar{U}U \rangle = \int D\sigma D\rho \exp \{ 2ie[(\rho(x_1) - \rho(x_2)) - (\sigma(x_1) - \sigma(x_2))] \} \exp \{ \int d^2x \mathcal{L}(\sigma, \rho) \} \quad (3.14c)$$

and $S_{0,R}^F(x_1, x_2)$ and $S_{0,L}^F(x_1, x_2)$ are the free propagators for the right and left fermions, respectively. Now, using the bosonic decoupling transformation (3.1), (3.14c) can be written as

$$\langle \bar{U}U \rangle = I_\rho(x_1, x_2) I_\sigma(x_1, x_2), \quad (3.15a)$$

where

$$I_\rho(x_1, x_2) = \int D\rho \exp \left\{ ie \frac{(\eta - \xi)}{\eta} (\rho(x_1) - \rho(x_2)) \right\} \times \exp \left\{ \frac{1}{2} \int d^2x \rho \mathcal{C}_\rho \rho \right\}, \quad (3.15b)$$

and

$$I_\sigma(x_1, x_2) = \int D\sigma' \exp \{ 2ie(\sigma'(x_1) - \sigma'(x_2)) \} \times \exp \left\{ \frac{1}{2} \int d^2x \sigma' \mathcal{C}_\sigma \sigma' \right\}, \quad (3.15c)$$

whose operators $\mathcal{C}_\rho$ and $\mathcal{C}_\sigma$ are respectively that giving the following equations for their propagators:

$$\mathcal{C}_\rho \mathcal{J}_\rho \equiv \left[\frac{1}{e} \square (\square + 2b^2 \square^2 + \mu_0^2) \right] \mathcal{J}_\rho(x_1, x_2) = -\delta^2(x_1 - x_2), \quad (3.16a)$$

$$\mathcal{C}_\sigma \mathcal{J}_\sigma \equiv -\frac{1}{2\pi\eta} \square \mathcal{J}_\sigma(x_1 - x_2) = -\delta^2(x_1 - x_2). \quad (3.16b)$$

Furthermore, we can also rewrite the functional integrals I_ρ and I_σ in a form more suitable for doing the integration,

$$I_\rho(x_1, x_2) = \int D\rho \exp \left\{ \int d^2x \left[\frac{1}{2} \rho \mathcal{C}_\rho \rho - J \rho \right] \right\} = \exp \left\{ -\frac{1}{2} \int d^2x d^2y J(x) \mathcal{J}_\rho J(y) \right\}, \quad (3.17a)$$

and analogously

$$I_\sigma(x_1, x_2) = \exp \left\{ -\frac{1}{2} \int d^2x d^2y K(x) \mathcal{J}_\sigma K(y) \right\}, \quad (3.17b)$$

where we defined that

$$J(x) \equiv -ie \frac{(\eta + \xi)}{\eta} [\delta(x - x_1) - \delta(x - x_2)], \quad (3.18a)$$

$$K(x) \equiv 2ie[\delta(x - x_1) - \delta(x - x_2)]. \quad (3.18b)$$

Substituting these "sources" in the integrals I_ρ and I_σ , and after straightforward calculations one obtains that

$$\langle \bar{\psi}_L \psi_L \rangle = S_{0,L}^F(x_1 - x_2) \exp \left\{ 4e^2 \left[\left(\frac{\eta + \xi}{2\eta} \right)^2 \times [\mathcal{J}_\rho(x_1 - x_2) - \mathcal{J}_\rho(0)] + \mathcal{J}_\sigma(x_1 - x_2) - \mathcal{J}_\sigma(0) \right] \right\}, \quad (3.19)$$

which after introducing a renormalization constant such that $\psi_L^R = Z^{1/2} \psi_L$, with

$$Z_{\frac{1}{2}} \equiv \exp \left\{ -4e^2 \left[\left(\frac{\eta + \xi}{2\eta} \right)^2 \mathcal{D}_\rho(0) + \mathcal{D}_\sigma(0) \right] \right\},$$

lead us to the renormalized correlation function:

$$\langle \bar{\psi}_L^R \psi_L^R \rangle = S_{0,L}^F(x_1 - x_2) \exp \left\{ 4e^2 \times \left[\left(\frac{\eta + \xi}{2\eta} \right)^2 \mathcal{D}_\rho(x_1 - x_2) + \mathcal{D}_\sigma(x_1 - x_2) \right] \right\}, \quad (3.20)$$

where the propagators $\mathcal{D}_\rho(x_1 - x_2)$ and $\mathcal{D}_\sigma(x_1 - x_2)$ are, respectively, the coordinate space propagators corresponding to that appearing in (3.7) and (3.8) in the momenta space.

4 Finite temperature propagators

Now, we are going to extend the previous discussion on the propagators of the model to the finite temperature case. First, we use the imaginary time formalism, where the fields satisfy the periodic and antiperiodic relations [25]

$$\psi(\tau, x) = -\psi(\tau + i\beta, x) \quad (4.1a)$$

$$\bar{\psi}(\tau, x) = -\bar{\psi}(\tau + i\beta, x) \quad (4.1b)$$

$$A_\mu(\tau, x) = A_\mu(\tau + i\beta, x), \quad (4.1c)$$

where $\tau = -ix_0$. The propagators at finite temperature, in the imaginary time formalism, in momentum space can be constructed from those at the zero temperature simply rotating to Euclidean time τ , taking the energies k_0 as the Matsubara frequencies [25]

$$\omega_n = \frac{2n\pi}{\beta} \quad (n = 0, \pm 1, \pm 2, \dots) \quad (\text{bosons}) \quad (4.2a)$$

$$v_n = \frac{(2n+1)\pi}{\beta} \quad (n = 0, \pm 1, \pm 2, \dots) \quad (\text{fermions}) \quad (4.2b)$$

and substituting the integrals over k_0 into sum in the modes ω_n or v_n . With these prescriptions we find that the propagators (3.9), (3.12) and (3.20) become, respectively

$$\mathcal{Z}_{\mu\nu}(\omega_n, k_1) = -\frac{1}{k_1^2 + \omega_n^2 + 2b^2(k_1^2 + \omega_n^2)^2 - \mu_0^2} \times \left\{ \delta_{\mu\nu} - \frac{k_\mu k_\nu}{k_1^2 + \omega_n^2} \left[1 - \frac{(\eta - \xi)^2}{4\eta^2} - \frac{\pi}{\eta e^2} (k_1^2 + \omega_n^2 + 2b^2(k_1^2 + \omega_n^2)^2 - \mu_0^2) \right] + \frac{\eta - \xi}{2\eta} \left(\frac{k_\mu \bar{k}_\nu + \bar{k}_\mu k_\nu}{k_1^2 + \omega_n^2} \right) \right\}, \quad (4.3)$$

for the usual boson propagator ($\eta \neq 0$),

$$\mathcal{Z}_{\mu\nu}(\omega_n, k_1) = \frac{-\pi^2}{e^4} \frac{k_\mu k_\nu}{k_1^2 + \omega_n^2} \left[k_1^2 + \omega_n^2 2b^2(k_1^2 + \omega_n^2)^2 - \frac{e^2}{2\pi} \right] - \frac{\pi}{e^2(k_1^2 + \omega_n^2)} (k_\mu \bar{k}_\nu + \bar{k}_\mu k_\nu), \quad (4.4)$$

for the boson propagator in the singular case $\eta = 0$, $\xi = 1$ and for the fermionic propagator of the renormalized left

fermions we find

$$\langle \bar{\psi}_L^R \psi_L^R \rangle = S_{0,L}^F(x_1, x_2) \exp \left\{ \frac{4e^2}{\beta} \sum_n \int \frac{dk_1}{2\pi} \exp \{ -ik(x-y) \} \right. \\ \left. \times \left[\left(\frac{\eta + \xi}{2\eta} \right)^2 \frac{e^2}{k^2(k^2 + 2b^2k^4 - \mu_0^2)} + \frac{\pi}{\eta k^2} \right] \right\} \quad (4.5)$$

where $k_0 = \omega_n = 2\pi n/\beta$ as in the case for a boson field since this propagator has a fermionic structure given by $S_{0,L}^F$ times a bosonic correction which carries all its temperature dependence. This behavior is also present in other two dimensional models at finite temperature [26] and will be also encountered in the real time formalism that we discuss below.

Thermofield dynamics. Let us now reconstruct the propagators for the CSM plus the Podolsky term in a real time formalism. As is well known, this implies in a complex contour path integral or, alternatively, in doubling the Hilbert space with a vacuum which is β -dependent [27]. The later formalism, which we will follow, is called thermofield dynamics and was introduced by Takahashi and Umezawa many years ago [28]. It has been applied successfully to the two dimensional Thirring and Schwinger models [29]. In this formalism, every field is accompanied by a thermal ghost, which does not appear at final asymptotic states but contribute to internal lines. The Lagrangian (2.2) can be rewritten here as (Minkowsky space-time)

$$\mathcal{L}_{TFD} = \sum_{r=1,2} \tau_{rr} \left\{ -\frac{1}{4} F_{\mu\nu}^r F^{\mu\nu r} - b^2 \partial_\nu F^{\mu\nu r} \partial^\mu F_{\mu\sigma}^r + i \bar{\psi}^r \not{D} \psi^r \right\}, \quad (4.6a)$$

with

$$\not{D} = \gamma^\mu (\partial_\mu - ie(1 - \gamma_5) A_\mu^r), \quad (4.6b)$$

where $\tau_{rs} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ gives the double Hilbert space structure. The fields with $r=1$ are the conventional zero-temperature ones

$$A_\mu^r(x) = A_\mu(x); \quad \psi^r(x) = \psi(x); \quad \bar{\psi}^r(x) = \bar{\psi}(x), \quad (r=1) \quad (4.7)$$

while for $r=2$ we have the respective ghost fields which satisfy

$$A_\mu^r(x) = -\tilde{A}_\mu^T(x), \quad \psi^r(x) = i(\tilde{\psi}^T)^T(x); \\ \bar{\psi}^r(x) = i(\tilde{\psi}^T)^T(x), \quad (r=2) \quad (4.8)$$

where T denotes matrix transposition and the tilde indicate that these are independent fields from its counterparts (4.7). The propagators in this formalism carry the temperature dependence through the matrices

$$U^B = \begin{pmatrix} \cosh \theta_B(\beta) & \sinh \theta_B(\beta) \\ \sinh \theta_B(\beta) & \cosh \theta_B(\beta) \end{pmatrix}; \\ \sinh^2 \theta_B(\beta) = [\exp(\beta|u|) - 1]^{-1}, \quad (4.9)$$

for bosons, where u is the relative velocity of the thermal fluid in which the model is suppose to be immersed in. For

fermions, we have

$$U^F = \begin{pmatrix} \cos \theta_F(\beta) & \sin \theta_F(\beta) \\ -\sin \theta_F(\beta) & \cos \theta_F(\beta) \end{pmatrix}; \\ \sin^2 \theta_F(\beta) = [\exp(\beta|u|) + 1]^{-1}. \quad (4.10)$$

Using this formalism to reconstruct the propagators for the CSM plus the Podolsky term we find for the boson propagator in the nonsingular case ($\eta \neq 0$)

$$\mathcal{D}_{\mu\nu}^{rs}(k) = \sum_{i=1,2} U_{ii}^B \left\{ -\frac{i\tau_{ii}}{k^2 + 2b^2k^4 - \mu_0^2 + i\epsilon\tau_{ii}} \times \left[g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2 + i\epsilon\tau_{ii}} \left(1 - \frac{(\eta - \xi)^2}{4\eta^2} - \frac{\pi}{\eta e^2} (k^2 + 2b^2k^4 - \mu_0^2) \right) \right. \right. \\ \left. \left. + \frac{\eta - \xi}{2\eta} \left(\frac{k_\mu \tilde{k}_\nu + \tilde{k}_\mu k_\nu}{k^2 + i\epsilon\tau_{ii}} \right) \right] \right\} U_{is}^B. \quad (4.11)$$

For the singular case corresponding to (3.12), here we find

$$\mathcal{D}_{\mu\nu}^{rs}(k) = \sum_{i=1,2} U_{ii}^B \left\{ -\frac{i\pi^2 \tau_{ii}}{e^4} \frac{k_\mu k_\nu}{k^2 + i\epsilon\tau_{ii}} \left(k^2 + 2b^2k^4 - \frac{e^2}{2\pi} \right) - \frac{\pi}{e^2(k^2 + i\epsilon\tau_{ii})} (k_\mu \tilde{k}_\nu + \tilde{k}_\mu k_\nu) \right\} U_{is}^B \quad (4.12)$$

Now, for the fermionic propagator $\langle \bar{\psi}_L^R \psi_L^R \rangle$, we have

$$\langle \bar{\psi}_L^R \psi_L^R \rangle^{rs} = S_{0,L}^{Frs}(x_1, x_2) \\ \times \exp \left\{ 4e^2 \int \frac{d^2k}{(2\pi)^2} \exp \{ -ik(x-y) \} \sum_{i=1,2} U_{ii}^B \left[\left(\frac{\eta + \xi}{2\eta} \right)^2 \right. \right. \\ \left. \left. \times \frac{-i\tau_{ii}e^2}{(k^2 + i\epsilon\tau_{ii})(k^2 + i\epsilon\tau_{ii} + 2b^2k^4 - \mu_0^2)} + \frac{-i\tau_{ii}\pi}{\eta(k^2 + i\epsilon\tau_{ii})} \right] U_{is}^B \right\} \quad (4.13)$$

Note that the above expressions give the propagators for both the physical and thermal ghost fields, since they carry the r and s indices. For the physical fields one has only to choose $r=s=1$ to get its real time representation explicitly. In particular, (4.13), for $r=s=1$, is the real time counterpart of (4.5) and the bosonic finite temperature structure is also present here, as can be seen through the presence of the bosonic matrices U^B instead of the fermionic U^F ones.

Note also that in the zero-temperature limit $\beta \rightarrow \infty$ the thermal matrices become $U^B = U^F = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and the ghost fields decouple from the ordinary ones and we regain the results of section 3. The same occurs for the propagators in the imaginary time formalism, discussed before.

5 Conclusions

We have discussed an extension of the usual CSM with higher order derivatives by adding to its Lagrangian a Podolsky term. We have seen that this results in a model which resembles the original CSM but with a renormalized mass (2.16) and a tachyonic mode (2.17) (in the non-singular case), which is a signature of the Podolsky

term. This model has the interesting feature that it mimics in some sense the standard model. As the propagators obtained resemble the ones of low energy QCD some further study to search for the possibility of describing bound states from this approach is also desirable.

We also verify that in the singular case ($\eta = 0$, $\xi = 1$) the tachyonic mode disappears. This is a quite interesting feature, because in general higher-order derivatives models present tachyons. However, in this case no massive boson is generated. This strongly indicates that dynamical mass generation and the presence of tachyons in higher-order derivative theories are strictly correlated.

The results first obtained from a Fujikawa's analysis were then confirmed by the careful calculation of the relevant propagators of the theory done in sections 3 and 4. Also, its finite temperature discussion had proven that, analogous to the usual CSM [22], the present model does not possess a phase transition at any finite temperature. Anyway, from the propagators at finite temperature one can explore its high and low temperature limits comparing these results with the ones expected for the standard model. In connection with this, it is also possible to infer the hard and soft thermal loops in this model, since it is exactly solvable and this may be of some help for the corresponding studies in four-dimensional QCD.

Despite the exact solvability of the non-singular model, it cannot be accepted as a physical theory due to the presence of tachyons. However, recently, it has been shown that theories with higher order derivatives can be physically consistent if one treats them as effective field theories [9]. This analysis opens a window on the possible interpretations of such theories as the one discussed in this paper.

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AXIAL-SCHWINGER MODEL

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We consider an extension of the axial model where local gauge symmetries are taken into account. The result is a mixing of the axial and Schwinger models. The anomaly of the axial current is calculated by means of the Fujikawa path integral technique and the model is also solved. Besides the well-known features of the particular models (axial and Schwinger) an effective interaction of scalar and gauge fields via a topological current is obtained. This term is responsible for the appearance of massive poles in the propagators that are different from those of both models.

1. Introduction

The axial model corresponds to a theory where a real scalar field interacts with a fermionic axial current via a derivative coupling. It was introduced by Rothe and Stamatescu almost twenty years ago¹ and has been studied since then on its more diverse aspects.²⁻⁶ Its Lagrangian density reads^a

$$\mathcal{L} = i \bar{\psi} \gamma^\mu (\partial_\mu + ig_0 \gamma_5 \partial_\mu \phi) \psi + \mathcal{L}_\phi, \quad (1)$$

where

$$\mathcal{L}_\phi = \frac{1}{2} (\partial_\mu \phi \partial^\mu \phi - m_0^2 \phi^2). \quad (2)$$

We easily observe that it exhibits global U(1) and chiral symmetries. In consequence, from the Noether's theorem, both vector and axial currents are conserved. The two main features of this model are

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^aWe adopt throughout this paper the following convention and notation: $\eta_{\mu\nu} = \text{diag}(+1, -1)$, $\epsilon^{01} = +1$, $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$, $\gamma^\mu = \gamma^0 \gamma^{\mu\dagger} \gamma^0$ ($\gamma^{0\dagger} = \gamma^0$, $\gamma^{1\dagger} = -\gamma^1$), $\gamma_5 = \gamma^0 \gamma^1$ ($\gamma_5^\dagger = \gamma_5$), $\gamma^\mu \gamma^\nu = \eta^{\mu\nu} + \epsilon^{\mu\nu} \gamma_5$, $\gamma_5 \gamma^\mu = \epsilon^{\mu\nu} \gamma_\nu$ (these last two are only true in two dimensions).



(i) The divergence of the axial current is anomalous:

$$\partial_\mu j_5^\mu = \frac{i g_0}{\pi} \square \phi, \quad (3)$$

where j_5^μ is defined as

$$j_5^\mu = \bar{\psi} \gamma^\mu \gamma_5 \psi. \quad (4)$$

(ii) The mass m_0 is renormalized to

$$m^2 = \frac{m_0^2}{1 - \frac{g_0^2}{\pi}}. \quad (5)$$

This model does not have the corresponding local U(1) and chiral symmetries. Apparently, one possibility of having them would be to consider the coupling of the axial current with a gauge field A_μ , namely, $g_0 \bar{\psi} \gamma^\mu \psi A_\mu$, and taking $A_\mu = \partial_\mu \phi$ as a particular gauge choice. However, this cannot be true. A simple argument against this procedure is that the coupling term above does not have the correct dimension.

The consistent way of implementing local gauge transformations in the axial model is actually by means of a gauge field, but taken independently of $\partial_\mu \phi$. The general Lagrangian density then reads

$$\mathcal{L} = i \bar{\psi} \not{D} \psi - g_0 \bar{\psi} \gamma^\mu \gamma_5 \psi \partial_\mu \phi + \mathcal{L}_\phi + \mathcal{L}_A, \quad (6)$$

where $\mathcal{L}_A$ is the well-known kinetic gauge field term

$$\mathcal{L}_A = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}, \quad (7)$$

and the covariant derivative reads

$$D_\mu = \partial_\mu - ie A_\mu. \quad (8)$$

We list below the mass dimension of the quantities that appear in the Lagrangian (6):

$$[\psi] = \frac{1}{2}, \quad [\phi] = 0, \quad [A_\mu] = 0, \quad [e] = 1, \quad [g_0] = 0, \quad [m_0] = 1. \quad (9)$$

The model described by (6) is a mixing of the axial model, described by expressions (1) and (2), and the Schwinger model.⁷ The purpose of this work is to study the features of such an extended model, which will be called "axial-Schwinger model" (ASM). It might be opportune to make some general comments about it first:

(i) The derivative that is acting on the scalar field cannot be replaced by a covariant one. This is so because the scalar field is real and consequently does not couple to the electromagnetic one (the gauge transformation of ϕ is zero).

(ii) Inadvertently, we could think of making $g_0 = e$ to eliminate the second term of (6) by a gauge transformation of A_μ . As was previously said, this cannot be done since g_0 and e do not have the same mass dimension.

(iii) Another wrong reasoning would be the trial to obtain the scalar term $-\frac{1}{2} \partial_\mu \phi \partial^\mu \phi$ from the gauge field one $-\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$ and vice versa by considering that in two dimensions the gauge field can always be written as

$$A_\mu = \partial_\mu \sigma + \epsilon_{\mu\nu} \partial^\nu \rho. \quad (10)$$

Also here, this would not be successful. σ and ρ have the mass dimension -1 and, consequently, cannot be related to ϕ . In fact, in the case of replacing A_μ given by (10) into (7), we would obtain terms in $\square \sigma \square \sigma$, $\square \rho \square \rho$ etc.

So we conclude that Lagrangian (6) cannot be simplified. It has to be used in the way it appears. It is important to emphasize that it is not a trivial generalization of the axial and Schwinger models, even though both are particular cases. We shall see, for example, that the anomaly of the axial current of the ASM is more general and contains crossed terms of both sectors. However, mass poles of the ASM propagators are not the same as the usual models when taken separately. This is so because after solving the ASM, the scalar field effectively interacts with the gauge field via a topological current, which will generate two massive modes depending on the parameters of the two models.

Our paper is organized as follows. In Sec. 2 we calculate the anomaly of the axial current of the ASM by means of the Fujikawa path integral technique.⁸ The model is solved by integrating on the fermionic fields in Sec. 3. Using this effective Lagrangian density, propagators are calculated in Sec. 4, where one identifies the mass poles. Some final remarks are left for Sec. 5.

2. Axial Current Anomaly

To calculate the anomaly of the axial current we make use of the Fujikawa technique.⁸ We begin by writing down the general expression of the vacuum functional

$$Z = N \int [d\bar{\psi}][d\psi][d\phi][dA] \exp\{iS\}, \quad (11)$$

where S is the action corresponding to the Lagrangian density (6) plus some gauge-fixing term.^b The chiral anomaly arises in the path integral formalism from the fact that the measure $[d\bar{\psi}][d\psi]$ is not invariant under chiral gauge transformations. In the infinitesimal case, these transformations read

$$\begin{aligned} \psi'(x) &= [1 + i\epsilon(x)\gamma_5] \psi(x), \\ \bar{\psi}'(x) &= \bar{\psi}(x) [1 + i\epsilon(x)\gamma_5], \end{aligned} \quad (12)$$

^bIt is understood that we have functionally integrated on ghost fields that come from the Faddeev-Popov or, more generically, from the BFM or BV method.

is still not Hermitian.

There is a simple way of circumventing this problem. It consists in taking a kind of analytical extension of the field ϕ when we go to the Euclidean space.^{1,9} So, instead of the operator (17), we use the Hermitian

$$\tilde{D}_H = \gamma_\mu (\partial_\mu - ie A_\mu + g_0 \gamma_5 \partial \tilde{\phi}), \quad (19)$$

where it was taken

$$\phi \rightarrow i\tilde{\phi}. \quad (20)$$

We shall use $\tilde{D}_H$ as the regulating operator of the Fujikawa technique. It is then just a matter of algebraic calculation to obtain

$$I(x) = \frac{1}{2\pi} (g_0 \square \tilde{\phi} - 2ie g_0 A_\mu \partial_\mu \tilde{\phi} + e \epsilon_{\mu\nu} \partial_\mu A_\nu). \quad (21)$$

Rotating back to the Minkowski space and not forgetting to make $i\tilde{\phi} \rightarrow \phi$, we have

$$I(x) = \frac{1}{2\pi} (ig_0 \square \phi + 2eg_0 A^\mu \partial_\mu \phi + e \epsilon^{\mu\nu} \partial_\mu A_\nu). \quad (22)$$

With this result the measure changes to

$$[d\tilde{v}][d\psi] = [d\tilde{v}'][d\psi'] \exp \left\{ \frac{i}{\pi} \int d^2x \epsilon(x) (ig_0 \square \phi + 2eg_0 A^\mu \partial_\mu \phi + e \epsilon^{\mu\nu} \partial_\mu A_\nu) \right\}, \quad (23)$$

which implies through

$$\lim_{\epsilon \rightarrow 0} \frac{\delta Z}{\delta \epsilon(x)} = 0 \quad (24)$$

that

$$\partial_\mu j_5^\mu = -\frac{1}{2\pi} (e \epsilon_{\mu\nu} F^{\mu\nu} + 2ig_0 \square \phi + 4eg_0 A^\mu \partial_\mu \phi). \quad (25)$$

We notice that when we take $e = 0$, we obtain the anomaly of the axial current of the Rothe and Stamatescu's model. The corresponding anomaly of the Schwinger model is obtained by taking $g_0 = 0$. We emphasize the presence of a mixing term with eg_0 coupling. This leads to an effective interaction between A_μ and ϕ . We will discuss its consequences with more details in the following sections.

3. Path Integral Solution of the Model

In both the Schwinger and axial models there are effective theories that can be obtained by integrating on the fermionic fields. Let us see what kind of effective theory can be obtained here. Considering the fermionic part of the Lagrangian (6) and using A_μ given by (10), we have

$$\begin{aligned} \mathcal{L}_F &= i \bar{\psi} \gamma^\mu (\partial_\mu - ie \partial_\mu \sigma - ie \epsilon_{\mu\nu} \partial^\nu \rho) \psi + g_0 \bar{\psi} \gamma_5 \gamma^\mu \psi \partial_\mu \phi \\ &= i \bar{\psi} \gamma^\mu (\partial_\mu - ie \partial_\mu \sigma - i \gamma_5 \partial_\mu \xi) \psi. \end{aligned} \quad (26)$$

where

$$\xi = e\rho - g_0\phi. \quad (27)$$

We notice that if we choose the gauge transformation

$$\psi = e^{i\gamma_5\xi + ie\sigma}\chi, \quad (28)$$

we will obtain that the couplings of the fermionic current with σ and ξ disappear. Doing this iteratively by means of infinitesimal gauge transformations we get for the measure^{3,4,8}

$$[d\bar{\psi}][d\psi] = [d\bar{\chi}][d\chi] \exp \left[-\frac{i}{2\pi} \int d^2x (\epsilon^2 A_\mu A^\mu - 2eg_0\epsilon^{\mu\nu}\partial_\nu\phi A_\mu - g_0^2\partial_\mu\phi\partial^\mu\phi) \right], \quad (29)$$

where we have used again (10). The vacuum functional then turns out to be

$$Z = N \int [d\bar{\chi}][d\chi][dA][d\phi] \exp \left\{ i \int d^2x \left[i\bar{\chi}\not{\partial}\chi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{\epsilon^2}{2\pi}A_\mu A^\mu + \frac{1}{2} \left(1 - \frac{g_0^2}{\pi} \right) \partial_\mu\phi\partial^\mu\phi - m_0^2\phi^2 - \frac{eg_0}{\pi}\epsilon^{\mu\nu}\partial_\nu\phi A_\mu + \mathcal{L}_{GF} \right] \right\}. \quad (30)$$

As one observes, the resulting fermionic field does not interact with A_μ and ϕ anymore. It may disappear from the theory by integrating over it and absorbing the result into the renormalization factor N . The so-obtained effective Lagrangian contains the well-known results of the Schwinger and the axial models in the respective limits, that is to say, the photon field acquires a mass given by $e/\sqrt{\pi}$ when $g_0 = 0$ and the mass m_0 of the scalar field is renormalized to $m_0(1 - g_0^2/\pi)^{-1/2}$ if $e = 0$. The new point here is that the scalar field that initially did not interact with the gauge field, does now via the current

$$\mathcal{J}^\mu = \frac{eg_0}{\pi}\epsilon^{\mu\nu}\partial_\nu\phi. \quad (31)$$

As can be trivially verified, $\mathcal{J}^\mu$ is conserved independently of the equation of motion and it is not associated to any symmetry of action. These facts give a topological character to it. Due to the boundary conditions we are using in the evaluation of most of the quantities throughout this work (fields vanishing at the spatial infinity), there is only a null topological charge associated to $\mathcal{J}^\mu$.

4. The ASM Propagators

In this section we calculate the propagators in order to identify their massive poles. We first consider the case with $m_0 = 0$, where it is possible to easily decouple the scalar and gauge fields by just making convenient field transformations. The effective Lagrangian density of (30) then becomes

$$\begin{aligned} \mathcal{L}_{\text{eff}} = & -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{\epsilon^2}{2\pi}A_\mu A^\mu + \frac{1}{2} \left(1 - \frac{g_0^2}{\pi} \right) \partial_\mu\phi\partial^\mu\phi \\ & - \frac{eg_0}{\pi}\epsilon^{\mu\nu}\partial_\nu\phi A_\mu - \frac{1}{2\alpha}(\partial_\mu A^\mu)^2 + J_\mu A^\mu + K\phi, \end{aligned} \quad (32)$$

where we have taken the gauge-fixing Lagrangian as $-(\partial_\mu A^\mu)/2\alpha$. Note that we have also included external sources, which will be used in order to generate the corresponding correlation functions through functional derivatives. It is again convenient to use the decomposition of the vector field in its transverse and longitudinal components given at (10). This will lead us to the Lagrangian density

$$\mathcal{L}_{\text{eff}} = \frac{1}{2} \rho \square^2 \rho + \frac{e^2}{2\pi} \rho \square \rho - \frac{1}{2\alpha} \sigma \square^2 \sigma - \frac{\beta}{2} \phi \square \phi - \frac{e g_0}{\pi} \phi \square \rho + J_\mu A^\mu + K \phi, \quad (33)$$

with $\beta = 1 - g_0^2/\pi$, and where one can see the coupling term $\phi \square \rho$, which comes from the topological coupling $\mathcal{J}_\mu A^\mu$. In this case one can decouple the Lagrangian density by performing the field transformations

$$\begin{aligned} \phi &= \tilde{\phi} - \frac{g_0}{\beta\pi} \tilde{\rho}, \\ \rho &= \tilde{\rho}, \\ \sigma &= \tilde{\sigma}, \end{aligned} \quad (34)$$

which leave us with

$$\begin{aligned} \tilde{\mathcal{L}}_{\text{eff}} &= \frac{1}{2} \tilde{\rho} \square \left[\square + \left(\frac{e^2}{\pi} + \frac{g_0^2 e^2}{\beta\pi^2} \right) \right] \tilde{\rho} - \frac{1}{2\alpha} \tilde{\sigma} \square^2 \tilde{\sigma} + \frac{\beta}{2} \tilde{\phi} \square \tilde{\phi} \\ &\quad + J_\mu (\partial_\mu \tilde{\sigma} + \epsilon_{\mu\nu} \partial^\nu \tilde{\rho}) + K \tilde{\phi}. \end{aligned} \quad (35)$$

Going back with this result to the expression of the generating functional, we directly obtain the photon propagator

$$D_{\mu\nu}(k) = -k_\mu k_\nu \langle \tilde{\sigma} \tilde{\sigma} \rangle - \epsilon_{\mu\lambda} \epsilon_{\nu\alpha} k^\lambda k^\alpha \langle \tilde{\rho} \tilde{\rho} \rangle, \quad (36)$$

where $\langle \tilde{\sigma} \tilde{\sigma} \rangle$ and $\langle \tilde{\rho} \tilde{\rho} \rangle$ are the corresponding propagators for $\tilde{\sigma}$ and $\tilde{\rho}$:

$$\begin{aligned} \langle \tilde{\sigma} \tilde{\sigma} \rangle &= -\frac{\alpha}{k^4}, \\ \langle \tilde{\rho} \tilde{\rho} \rangle &= \frac{1}{k^2 (k^2 - m_\beta^2)}. \end{aligned} \quad (37)$$

The pole m_β^2 is the effective mass coming from the coupling due to the topological current, and is given by

$$m_\beta^2 = \frac{e^2}{\pi - g_0^2}. \quad (38)$$

Note that in the limit where $g_0 = 0$, we recall the expected Schwinger mass. We observe that the propagator of ϕ does not contain any massive pole.

We now discuss the case with $m_0 \neq 0$. The decoupling transformation is not so simple. We are going to use the approach discussed in Ref. 10. We rewrite the effective Lagrangian density as

$$\mathcal{L}_{\text{eff}} = \frac{1}{2} \theta^T \mathcal{M} \theta, \quad (39)$$

where $\theta^T \equiv (\rho, \phi, \sigma)$ and the matrix $\mathcal{M}$ is given by

$$\mathcal{M} = \begin{pmatrix} \square \left(\square + \frac{e^2}{\pi} \right) & -\frac{eg_0}{\pi} \square & 0 \\ -\frac{eg_0}{\pi} \square & -\beta \square - m_0^2 & 0 \\ 0 & 0 & -\frac{\square^2}{\alpha} \end{pmatrix}, \quad (40)$$

which can be inverted, giving us the correlation functions $\langle \sigma \sigma \rangle$, $\langle \phi \phi \rangle$, $\langle \rho \rho \rangle$, $\langle \rho \phi \rangle$ etc. defined in

$$\tilde{\mathcal{M}}^{-1} = \begin{pmatrix} \langle \rho \rho \rangle & \langle \rho \phi \rangle & \langle \rho \sigma \rangle \\ \langle \phi \rho \rangle & \langle \phi \phi \rangle & \langle \phi \sigma \rangle \\ \langle \sigma \rho \rangle & \langle \sigma \phi \rangle & \langle \sigma \sigma \rangle \end{pmatrix}, \quad (41)$$

where in the momentum representation, the nonvanishing ones are given by

$$\begin{aligned} \langle \rho \rho \rangle &= \frac{\beta k^2 - m_0^2}{\beta k^2 (k^2 - m_+^2) (k^2 - m_-^2)}, \\ \langle \rho \phi \rangle &= -\frac{g_0}{\pi \beta (k^2 - m_+^2) (k^2 - m_-^2)}, \\ \langle \sigma \sigma \rangle &= -\frac{\alpha}{\pi^2 \beta k^4} [\pi (\beta k^2 - m_0^2) (\pi k^2 - e^2) - g_0^2 e^2 k^2], \\ \langle \phi \phi \rangle &= \frac{\pi k^2 - e^2}{\pi \beta (k^2 - m_+^2) (k^2 - m_-^2)}. \end{aligned} \quad (42)$$

In the above relations, $\langle \phi \phi \rangle$ is the propagator of the scalar field. The propagator of the photon field is

$$\begin{aligned} D_{\mu\nu}(k) &= k_\mu k_\nu \langle \sigma \sigma \rangle + \epsilon_{\nu\lambda} k_\mu k^\lambda \langle \sigma \rho \rangle + \epsilon_{\mu\lambda} k^\lambda k_\nu \langle \rho \sigma \rangle + (g_{\mu\nu} k^2 - k_\mu k_\nu) \langle \rho \rho \rangle, \\ &= -\frac{1}{(k^2 - m_+^2) (k^2 - m_-^2)} \left\{ \left(k^2 - \frac{m_0^2}{\beta} \right) \left[g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right] \right. \\ &\quad \left. - \frac{\alpha k_\mu k_\nu}{\pi^2 \beta k^6} [\pi (\beta k^2 - m_0^2) (\pi k^2 - e^2) - g_0^2 e^2 k^2] \right\}. \end{aligned} \quad (43)$$

We observe that there are two nontrivial massive poles at

$$m_\pm^2 = \frac{1}{2\pi(\pi - g_0^2)} \left[\pi^2 m_0^2 + \pi e^2 \pm \sqrt{\pi^2 (\pi m_0^2 + e^2)^2 - 4\pi^2 (\pi - g_0) m_0^2 e^2} \right]. \quad (44)$$

Note that these two expressions reduce to the previous case when one takes $m_0 = 0$, as it should be. Furthermore, in the limits where $e = 0$ and $g_0 = 0$ we re-obtain the usual renormalized mass appearing in the axial model and the dynamical mass of the Schwinger model, respectively.

At this point it is important to stress that it should be necessary to perform a careful analysis over the parameters space in order to define the region without tachyons. This fact could lead us to think on restrictions over the range of the coupling constants of the ASM. However it is important to remember that, due to the fact that the axial field ϕ does not have any gauge symmetry, it will emerge an arbitrary renormalization parameter a^{11} such that, instead of $\beta = 1 - g_0^2/\pi$, we would have $\beta_a = 1 - a g_0^2/\pi$, and the restrictions due to the physical criteria would fall upon this parameter, leaving the coupling constants e and g_0 free.

5. Conclusion

In this work we have considered the quantization of an extended axial model that contains the Schwinger and the axial models as convenient limits, but bearing in mind that it is not a trivial superposition of them. By using the Fujikawa prescription, it was possible to show that the axial current is actually anomalous. This anomaly has the usual contributions from the axial model sector as well as from the Schwinger model, but it contains also a crossed term which is a new feature of the considered model, expressing the fact that at the quantum level there is an interference of both sectors.

This fact also comes true when the model is solved by functionally integrating on the fermionic sector. The effective Lagrangian which survives presents a coupling between the vector field and the scalar one by means of a topological current. We have calculated the propagators and identified the mass poles.

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Trabalhos em:

Teoria Quântica de Campos em Três
Dimensões

A free relativistic anyon with canonical spin algebra

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Abstract

We discuss a relativistic free particle with fractional spin in 2+1 dimensions, where the dual spin components satisfy the canonical angular momentum algebra $\{S_\mu, S_\nu\} = \epsilon_{\mu\nu\gamma} S^\gamma$. It is shown that it is a general consequence of these features that the Poincaré invariance is broken down to the Lorentz one, so indicating that it is not possible to keep simultaneously the free nature of the anyon and the translational invariance.

1. Introduction

In the last few years the study of particles with generalized statistics, the so called anyons [1], has been a growing field of interest. This is mainly due to the possibility of explaining very important physical phenomena, the fractional quantum Hall effect and the high temperature superconductors [2,3].

However, most of the work on this matter was done through nonrelativistic models, or by using the statistical field A_μ of the topological Chern-Simons term [4]. Notwithstanding, some authors have been looking for the answer to the question whether this statistical field not only changes the statistics, but also endows the anyons with interaction. One has been looking in particular, into the possibility of constructing a point-particle model for anyons, without the need of using any statistical field [5–8]. This quest should necessarily begin by studying the representations of the Lorentz group in 2+1 dimensions [5–7].

The Lorentz group in 2+1 dimensions corresponds to the $SO(2,1)$ group whose irreducible unitary representations, are all labeled by a set of half-integer and integer numbers [5]. Thus if we are interested in a Lorentz covariant description of particles with fractional spin, we have to deal in principle with an infinite dimensional representation of the Lorentz generators. On the other hand, Jackiw and Nair have shown in [6] that the anyon possess just one physical polarization state for a given sign of the energy, and they have obtained an induced representation for the Poincaré group on those physical polarization states. What is actually important for us is that, in the induced representation, the spin components commute ($[S_\mu, S_\nu] = 0$) and therefore do not satisfy the usual angular momentum algebra. Nevertheless, some point-particle models corresponding to the induced representation have appeared in the literature [7–9]. However, as far as we know, none of the models discussed up to now has been able to get two ingredients holding simultaneously, namely the free nature of the relativistic anyon and the canonical spin algebra $\{S_\mu, S_\nu\} = \epsilon_{\mu\nu\gamma} S^\gamma$ for the spin components [8]. In this work we intend to

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explore the consequences of imposing these features simultaneously. In the next section we show explicitly how this can be carried through in a particular point-particle model. In Section 3, we perform a model independent analysis, then in Section 4 we draw some conclusions and summarize the work.

2. A model for a free particle with fractional spin

The Poincaré algebra in 2+1 dimensions is given by:

$$\begin{aligned} [J_\mu, J_\nu] &= \epsilon_{\mu\nu\gamma} J^\gamma, \\ [J_\mu, P_\nu] &= \epsilon_{\mu\nu\gamma} P^\gamma, \\ [P_\mu, P_\nu] &= 0, \end{aligned} \quad (1)$$

where J_μ are the dual components of the total angular momentum tensor, i.e. $J_\mu = \epsilon_{\mu\nu\gamma} J^{\nu\gamma}$ and we are using $\eta_{\mu\nu} = \text{diag}(-, +, +)$, $\epsilon_{012} = +1$.

Let us suppose now that we have a relativistic model for a point-particle whose total angular momentum tensor and momenta components are given in the phase space by

$$J_{\mu\nu} = x_\mu p_\nu - p_\mu x_\nu + n_\mu \pi_\nu - \pi_\mu n_\nu, \quad P_\mu = p_\mu. \quad (2)$$

Assuming that the vectors x_μ , p_μ , n_μ , π_μ satisfy the canonical Poisson brackets, $\{x_\mu, p_\nu\} = \eta_{\mu\nu} = \{n_\mu, \pi_\nu\}$, with the remaining brackets vanishing, it is easy to derive the Poincaré algebra in terms Poisson brackets. Note that the variables (n_μ, π_μ) are just introduced to describe the spin of the particle $S_\mu = \epsilon_{\mu\nu\beta} n^\nu \pi^\beta$, which satisfies $\{S_\mu, S_\nu\} = \epsilon_{\mu\nu\gamma} S^\gamma$. The Poincaré algebra (1) possess two quadratic Casimir invariants which can be represented in our model by p^2 and $S \cdot p$, they will be used in the form of constraints to specify the mass and helicity of the particle respectively:

$$\phi_1 = p^2 + m^2 \approx 0, \quad (3)$$

$$\phi_2 = S \cdot p + \alpha m \approx 0, \quad (4)$$

where α is an arbitrary real constant. In analogous fashion to the $\alpha = 1/2$ case [6], Eq. (4) will play the role (for arbitrary α) of the Dirac equation [10.6]. We expect that the constraints ϕ_1, ϕ_2 be first class, in order to specify the physical states which belong to

the Poincaré representation specified by α and m , such that the hamiltonian of the model can be assumed to be of the form

$$H = e \phi_1 - \sigma \phi_2 - \lambda^a \chi_a, \quad (5)$$

where e , σ and λ^a ($a = 1, \dots, 4$) are the lagrange multipliers, whereas χ_a represent four second class constraints. The reason why we need exactly four second class constraints comes from a simple counting of degrees of freedom as follows; by assuming that the spin components S_μ and the momenta p_μ are parallel, one can show for arbitrary α that $g = 2$, where g is the gyromagnetic constant [11]. This result agrees with previous calculations [12] in field theory. If this is indeed the case, the spin of a particle in 2+1 dimensions is entirely specified by the helicity $S \cdot p$ which is fixed in our model by the constraint ϕ_2 and since ϕ_2 is assumed to be a first class constraint, it will eliminate two degrees of freedom of the extra six variables (n_μ, π_μ) introduced to describe the spin degrees of freedom. So we have to further impose four second class constraints in order to eliminate the remaining variables. It can be checked that the easiest way to assure that S_μ and p_μ be parallel is to impose the constraints [8,9]

$$\chi_1 = p \cdot n = 0, \quad \chi_2 = p \cdot \pi = 0, \quad (6)$$

these constraints imply

$$S_\mu = \frac{(S \cdot p)}{p^2} p_\mu. \quad (7)$$

Due to the fact that (6) does not depend on x_μ , after quantization the commutation relations of p_μ would not be changed by such constraints and we would have $[S_\mu, S_\nu] = 0$ as a consequence of (7). In order to recover the usual spin algebra one must further impose x_μ dependent constraints, for instance,

$$\chi_3 = x \cdot n = 0, \quad \chi_4 = x \cdot \pi = 0. \quad (8)$$

The constraints (6) and (8) form a set of four independent second class constraints which can be eliminated by a Dirac bracket $\{, \}_D$ leading to

$$\{\pi_\mu, \pi_\nu\}_D = 0 = \{n_\mu, n_\nu\}_D,$$

$$\{n_\mu, \pi_\nu\}_D = \eta_{\mu\nu}.$$

$$\begin{aligned}
\{p_\mu, n_\nu\}_D &= \frac{[(\pi \cdot n)n_\mu - n^2 \pi_\mu] p_\nu}{S^2}, \\
\{p_\mu, \pi_\nu\}_D &= \frac{[\pi^2 n_\mu - (\pi \cdot n) \pi_\mu] p_\nu}{S^2}, \\
\{x_\mu, \pi_\nu\}_D &= \frac{[\pi^2 n_\mu - (\pi \cdot n) \pi_\mu] x_\nu}{S^2}, \\
\{x_\mu, x_\nu\}_D &= \frac{x^2}{S^2} \epsilon_{\mu\nu\gamma} S^\gamma, \\
\{p_\mu, p_\nu\}_D &= \frac{p^2}{S^2} \epsilon_{\mu\nu\gamma} S^\gamma, \\
\{x_\mu, p_\nu\}_D &= \frac{[(x \cdot p) \epsilon_{\mu\nu\gamma} S^\gamma + S_\mu S_\nu]}{S^2}. \quad (9)
\end{aligned}$$

Although rather complicated, the above brackets have some remarkable features. Firstly, the Dirac brackets involving only the extra variables (n_μ, π_μ) are exactly equal to the canonical Poisson brackets, therefore we recover the canonical spin algebra that we were searching, i.e., one checks that

$$\{S_\mu, S_\nu\}_D = \epsilon_{\mu\nu\gamma} S^\gamma. \quad (10)$$

Furthermore, using (9) we deduce (for future comparison):

$$\{x_\mu, S_\nu\}_D = \epsilon_{\mu\nu\gamma} x^\gamma, \quad \{p_\mu, S_\nu\}_D = \epsilon_{\mu\nu\gamma} p^\gamma. \quad (11)$$

By comparing (11) with the Poincaré algebra (1) we notice that S_μ behave like the total angular momentum J_μ and (11) is nothing but the Lorentz transformation law for the vectors x_μ and p_μ . Indeed, since the constraints (6) imply $\epsilon_{\mu\nu\gamma} S^\nu p^\gamma = 0$, we analogously have from (8) that $\epsilon_{\mu\nu\gamma} S^\nu x^\gamma = 0$ and therefore the canonical angular momentum $L_\mu = \epsilon_{\mu\nu\gamma} x^\nu p^\gamma$ vanishes and we can identify S_μ with J_μ . Thus, the brackets (10), (11) and the fact that $\{p_\mu, p_\nu\}_D \neq 0$ (see (9)) are simply telling us that the Poincaré algebra (1) has been broken down to the Lorentz algebra due to the lack of translational invariance of the constraints (8). In the next section we will show that the identification between S_μ and J_μ and the absence of translational invariance are not peculiar features of this model but a consequence of the spin algebra and the parallelness of S_μ and p_μ . Another important characteristic of the brackets (9) to be noticed, is that they guarantee the conservation of p_μ and S_μ ($\dot{p}_\mu = \{p_\mu, H\}_D = 0 = \dot{S}_\mu$) as well as their gauge invariance. These conservation laws can be used to show that $\ddot{x}_\mu = 0$ and therefore the

model under study really corresponds to a free particle, although the Dirac brackets involving x_μ and p_μ are quite complicated. So, we were able to reach the goal of obtaining a free relativistic anyon with canonical spin algebra.

3. A model independent analysis

In this section, we will work from a more general point of view without specifying how the phase space (x_μ, p_ν) has to be extended to include extra variables to describe the spin degree of freedom, and that is precisely what we mean by a model independent analysis. Let us take our hamiltonian to be of the form (5) but, since we are not going to specify the extended phase space, we have, instead of four, N second class constraints χ^A ($A = 1, \dots, N$) where N will of course depend on the number of variables of the extended phase space. After the elimination of the constraints χ^A by some Dirac bracket³ $\{\cdot, \cdot\}^*$, the hamiltonian becomes simply

$$H = e(p^2 + m^2) + \sigma(S \cdot p + \alpha m). \quad (12)$$

Now we would like to examine the equations of motion which come from H . For a generic phase space variable q we have,

$$\dot{q} = \{q, H\}^* \equiv \{q, H\} - f_{AB} \{q, \chi^A\} \{\chi^B, H\} \quad (13)$$

where f_{AB} is some suitable function of the phase space variables which certainly depends on specific features of the extended phase space (see footnote). By using the fact that in general we have $\{\chi^A, \phi_i\} = c_i^{Aj} \phi_j + d_{iB}^A \chi^B$, where the functions c_i^{Aj}, d_{iB}^A might depend on the details of the model studied, it is not difficult to derive that, in a typical gauge where e and σ are set to a constant,

$$\ddot{x}_\mu = \sigma^2 \epsilon_{\mu\nu\gamma} p^\nu S^\gamma + h_\mu^i \phi_i \quad (14)$$

where h_μ^i are functions of the phase space variables which certainly depend on specific features of the extended phase space. Since $\varphi_\mu = \epsilon_{\mu\nu\gamma} p^\nu S^\gamma$ are clearly

³ In this section we relax the definition of the Dirac bracket in order to include brackets used to eliminate second class constraints which are not linearly independent.

independent of the first class constraints ϕ_i we come to the important conclusion that, independent of the specific details of the model, p^μ and S^μ must be parallel, i.e. $\varphi_{\mu} = 0$, in order that the particle be free ($\ddot{x}_\mu = 0$) and besides, we have to assume that the model proposed is such that the second term on the r.h.s. of (14) must independently vanish.⁴ Since ϕ_i ($i = 1, 2$) are the only first class constraints of the theory the parallelness of p_μ and S_μ must be a consequence of some of the second class constraints χ^A . Thus, we can use from now on the identification (7) strongly. Furthermore, if we now impose that the particle besides being free possess spin components S_μ which are gauge invariant⁵, i.e. $\{S_\mu, \phi_i\}^* = 0$ and satisfy the canonical spin algebra,

$$\{S_\mu, S_\nu\}^* = \epsilon_{\mu\nu\gamma} S^\gamma, \quad (15)$$

we consequently have from (7) that

$$\{S_\mu, p_\nu\}^* = \epsilon_{\mu\nu\gamma} p^\gamma. \quad (16)$$

From (16) and (7) we obtain

$$\{p_\mu, p_\nu\}^* = \frac{p^2}{S^2} \epsilon_{\mu\nu\gamma} S^\gamma, \quad (17)$$

now by using that p_μ is a vector, i.e. $\{p_\mu, J_\nu\}^* = \epsilon_{\mu\nu\gamma} p^\gamma$ where $J_\mu = \epsilon_{\mu\nu\gamma} x^\nu p^\gamma + S_\mu$, we can deduce the following expression

$$\epsilon_{\nu\gamma\beta} p^\beta \{p_\mu, x^\gamma\}^* = \frac{p^2}{S^2} ((S \cdot x) \eta_{\mu\nu} - x_\mu S_\nu), \quad (18)$$

from which we have $x_\mu = (S \cdot x / S \cdot p) p_\mu$ and thus $L_\mu = \epsilon_{\mu\nu\gamma} x^\nu p^\gamma = 0$ identically, consequently J_μ and S_μ can be identified. Therefore it seems that the only way out for the spin components of a free fractional spinning particle to satisfy the usual angular momentum algebra is to identify them with the total angular momentum J_μ which will certainly satisfy such algebra. Thus, we have seen that the identification between S_μ and J_μ as well as the lack of translational invariance (see (17)) are not special features of the last section model but a more general consequence of the

canonical spin algebra. Lorentz covariance and the requirement that the particle be free. Besides the above general features, we can also almost fully determine $\{x_\mu, x_\nu\}^*$ and $\{x_\mu, p_\nu\}^*$ from our basic assumptions as follows. First notice that from the identification between S_μ and J_μ we have for the vector x_μ :

$$\{x_\mu, S_\nu\}^* = \epsilon_{\mu\nu\gamma} S^\gamma, \quad (19)$$

since x_μ and p_μ are parallel like p_μ and S_μ we can write $x_\mu = (S \cdot x / S^2) S_\mu$, thus by using (19) and the spin algebra (15) we obtain (compare with (9))

$$\{x_\mu, x_\nu\}^* = \frac{x^2}{S^2} \epsilon_{\mu\nu\gamma} S^\gamma. \quad (20)$$

In order to fix $\{x_\mu, p_\nu\}^*$ we have to work a little more. By use of the equality $\{x_\gamma, (S \cdot p) S_\mu\}^* = \{x_\gamma, S^2 p_\mu\}^*$ one can deduce

$$\{x_\gamma, p_\nu\}^* \mathcal{P}_\mu^\nu = \left(\frac{S \cdot p}{S^2} \right) \epsilon_{\gamma\mu\beta} x^\beta, \quad (21)$$

where $\mathcal{P}_\mu^\nu = \delta_\mu^\nu - S^\nu S_\mu / S^2$ is a projection operator. Since $\{x_\mu, p_\nu\}^*$ is a second rank tensor with 9 components, we can always decompose it in terms of a 3 components vector c_μ and a 6 components symmetric tensor $N_{\alpha\beta}$ without loss of generality as follows

$$\{x_\gamma, p_\nu\}^* = \epsilon_{\gamma\nu\mu} c^\mu + N_{\gamma\nu}. \quad (22)$$

The quantities c^μ and $N_{\gamma\nu}$ can be partially determined from (21) and the parallelness of x^μ, p^μ and S^μ such that we end up, after some algebra, with

$$\begin{aligned} \{x_\mu, p_\nu\}^* &= \frac{(p \cdot x)}{S^2} \epsilon_{\mu\nu\gamma} S^\gamma + \frac{2 b_\mu S_\nu}{S^2} \\ &\quad - \frac{(S \cdot b) S_\mu S_\nu}{(S^2)^2}, \end{aligned} \quad (23)$$

where $b_\mu \equiv N_{\mu\nu} S^\nu$.

Since $S_\nu \mathcal{P}_\mu^\nu = 0$, we will have no further information about b_μ from Eq. (21), but substituting (23) into (18) we obtain $\epsilon_{\mu\nu\gamma} p^\nu b^\gamma = 0$ and therefore $b_\gamma = f S_\gamma$ where f is an arbitrary function of the extended phase space variables. Back in (23) we get

$$\{x_\mu, p_\nu\}^* = \frac{(p \cdot x)}{S^2} \epsilon_{\mu\nu\gamma} S^\gamma + f \frac{S_\mu S_\nu}{S^2}. \quad (24)$$

Although we cannot fully determine the function f we can obtain further constraints on it. Namely, all Jacobi

⁴ The reader can check that in order to have $h_\mu^i = 0$ it is sufficient, although not necessary, that $c_i^A = 0$.

⁵ Note that this is a more severe requirement than to simply impose that S_μ be observables.

identities involving x_μ , p_ν and S_γ are satisfied if and only if $\{f, S_\gamma\}^* = 0$, as can be explicitly checked. Moreover in order that the equations of motion for x_μ be reproduced it is necessary that $f = 1$ on shell. Since there are many candidates for f which satisfy those constraints like, e.g., $f = S \cdot p/\alpha m$; p^2/m^2 ; etc. We think that the specific form of f depends on the specific model analysed, i.e. depends on how the phase space has been extended.

After having obtained (20), (24) and comparing with the results of the model presented in section two, we see that all the brackets involving x_μ , p_μ and S_γ completely agree if we take $f = 1$ which is the simplest solution to our constraints. Therefore we are now able to recognize that the complicated brackets (9) are not just an unpleasant feature of that specific model but a quite general characteristic of the particles with fractional spin whose spin components satisfy the canonical algebra. It is also clear that such complications will go through the quantization of the model.

We conclude that for a free fractional spinning particle with the canonical spin algebra, the constraints of the theory must guarantee that $L_\mu = 0$ and $S_\mu \propto p_\mu$, and in this case we will be unavoidably faced with the rather complicated phase space structure given in (17), (20), and (24).

In finishing this section we give a simpler way to derive such results, although less general, where we implement the above physical conditions, i.e. the parallelness of S_μ , p_μ and x_μ using the constraints below

$$\begin{aligned}\varphi_\mu &= \epsilon_{\mu\nu\gamma} p^\nu S^\gamma = 0, \\ \tilde{\varphi}_\mu &= \epsilon_{\mu\nu\gamma} x^\nu S^\gamma = 0.\end{aligned}\quad (25)$$

It is easy to show that those constraints are all second class, thus they could in principle be eliminated by a Dirac bracket, however this is not possible [13] since the matrix of the Poisson brackets between the constraints has no inverse due to the fact that not all constraints φ_μ and $\tilde{\varphi}_\mu$ are linearly independent. Nevertheless we can eliminate them interactively by defining a kind of Dirac bracket as follows: [13] first we eliminate, e.g., φ_μ through the bracket $\{, \}^\dagger$,

$$\begin{aligned}\{A, B\}^\dagger &= \{A, B\} \\ &+ \frac{p_\gamma}{p^2(S \cdot p)} \epsilon^{\gamma\delta\beta} \{A, \varphi_\delta\} \{\varphi_\beta, B\}.\end{aligned}\quad (26)$$

and finally we eliminate $\tilde{\varphi}_\mu$ through $\{, \}^*$:

$$\begin{aligned}\{A, B\}^* &= \{A, B\}^\dagger \\ &- \frac{p^2}{S^2(S \cdot p)} p_\gamma \epsilon_{\mu\nu\gamma} \{A, \tilde{\varphi}_\mu\}^\dagger \{\tilde{\varphi}_\nu, B\}^\dagger.\end{aligned}\quad (27)$$

The reader can check that after the elimination of φ_μ and $\tilde{\varphi}_\mu$, the brackets involving S_μ , p_μ and x_μ will match with the general results obtained in this section for the case $f = 1$.

4. Conclusion and summary

Based on some mild assumptions like Lorentz covariance and the fact that a fractional spinning particle in $2 + 1$ dimensions obeys, analogous to the spin $1/2$ and spin 1 particles [6], an equation of the form $S \cdot P + \alpha m = 0$, we have shown in this work that some peculiar features of some point-particle models recently proposed in the literature can be understood from a rather general point of view. The first of such features is that, irrespective of the spin algebra, we can only have a free ($\ddot{x}_\mu = 0$) fractional spinning particle whenever spin and momentum are parallel, as an example of such rule we can take two models defined in [8]; for the first model, defined in Eqs. (2.12)–(2.14) of [8], S_μ and p_μ are parallel whereas this is not the case for the second one, worked out in the fourth section of the same reference, and indeed we have respectively $\ddot{x}_\mu = 0$ and $\ddot{x}_\mu \neq 0$. Furthermore, if the particle besides of being free is required to have spin components which are gauge invariant and satisfy the canonical angular momentum algebra, both in the strong sense, then the algebra of the quantities S_μ , p_ν , and the coordinates x_μ can be, as we have shown, almost completely fixed independently of the details of the extended phase space introduced to describe the spin degree of freedom. It has been shown in particular that our basic assumptions lead to theories where the Poincaré symmetry is broken down to the Lorentz one and the spin components will necessarily play the role of the Lorentz generators. These general predictions have been confirmed in an example given in the first section of this article. Moreover we have also seen that the canonical structure of the variables x_μ and p_ν will be quite complicated which will clearly pose some difficulties in the quantization program.

Finally, it should be noticed that it is not yet clear in the literature (see [6] and comments in [7] and references therein) whether fractional spinning particles in $2 + 1$ dimensions can be really formulated as free relativistic field theories and, although our assumptions can be softened, we think that the rather complicated canonical structure and the lack of translational invariance found here can be seen as indications of this fact.

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CONSISTENT HIGHER DERIVATIVE QUANTUM FIELD THEORY: A MODEL WITHOUT TACHYONS AND GHOSTS

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We present a higher derivative gauge theory in $(2 + 1)$ dimensions which can have its parameters suitably tuned in order to become a consistent quantum field theory, in the sense that both tachyons and ghosts are absent from the particle spectrum of the theory.

The interest in studying classical systems depending on higher order derivative Lagrangian densities has begun with Weyl and Eddington,¹ who attempted to add squared terms of the Ricci tensor to the original Einstein-Hilbert Lagrangian density, in order to extend the general relativity theory. Also, Bopp² and Podolsky³ proposed an extended electromagnetism by adding to the Lagrangian density of Maxwell, a quadratic term with derivatives of the field strength. This was done in order to avoid infinite self-energies of point particles.

Next Pais and Uhlenbeck⁴ investigated whether the use of higher-order derivative Lagrangian might lead to the elimination of the divergent quantities that plague quantum field theory. They conclude that, in general, it is not possible to reconcile finiteness, positivity of energy and causality. Therefore, states with negative norm and possibly unitarity violation are unavoidable in higher-order theories. These facts provide strong arguments to search for a consistent higher-order derivative model.

Furthermore, due to other nice features like for instance a better uv behavior and less problems with renormalizability when compared with the usual theories, the higher order derivative models have attracted much attention in the last three decades⁵ — since the advent of supersymmetry⁶ it has been shown that higher derivative terms appear quite naturally in the superfield formulation of supersymmetry,⁷ and also in the context of string theories as proposed by Polyakov.⁸ On the other hand, higher derivative terms can be used as regularizing procedure, and indeed this is the only known method of regularization which preserves the supersymmetry.⁹

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Though most part of those problems were discussed in $(3+1)$ dimensions,¹⁰ some investigations in $(1+1)$ quantum electrodynamics have been done recently; in particular, the physical content of the Schwinger model with higher order derivative couplings, both in the path integral formalism and in the operator one.¹¹

On the other hand, in the past years models at three spacetime dimensions have attracted a lot of attention.¹² This is due to the possibility of describing the d -Einstein gravity with topological mass term. Also in $(2+1)$ dimensions it is possible to mimic other physical events like high temperature superconductivity,¹³ the fractional Hall effect,¹⁴ and even of a dynamical mass generation mechanism. In fact, recently a model with higher derivatives in three dimensions was proposed in which an extended electromagnetism by combining a Maxwell-Podolsky model with a Chern-Simons term was studied.¹⁵ Hott and de Souza Dutra¹⁵ tried to tune the parameters of the theory conveniently to avoid unphysical excitations, but were unsuccessful.

However, as mentioned in the Sezgin and Nieuwenhuizen's paper,¹⁶ we can search for a higher-order term which cancels the effect of the Podolsky factor. In fact, these authors studied a class of R^2 -like actions for the gravity without the presence of tachyons and ghosts, by adding counterterms into the original Einstein theory.

From that last attempt of getting a consistent higher derivative quantum field theory, we introduce here a further higher-order term, besides the Podolsky one, a kind of "higher derivative Chern-Simons" term. This has proven to be the necessary ingredient to avoid tachyons, through a convenient restriction on the range of validity in the parameter space. After that we make an analysis of the imaginary part of the residues of the poles in the photon propagator. This is done in order to eliminate the ghosts from the spectrum of particles of the theory, and produces further constraints in the range of parameters which keep the theory consistent. This interesting result suggests the possibility of quantizing in a consistent way other classes of models, as for instance the higher order derivative gravity.¹⁷

In order to check the consistency of the model, its canonical quantization was performed by using the Dirac procedure of quantizing constrained higher derivative theories.¹⁸ So we conclude from the calculations done in this work that it is possible to construct a consistent quantum field theory with higher derivative terms in $(2+1)$ dimensions, by restricting the range of the parameters in the Lagrangian density.

Let us begin by writing the Lagrangian density of the model:

$$\mathcal{L} = -\frac{aF_{\mu\nu}^2}{4} + b\partial_\mu F^{\mu\lambda}\partial^\nu F_{\nu\lambda} + \theta_1\varepsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho + \theta_2 g^{\sigma\delta}\varepsilon^{\mu\nu\rho}\partial_\sigma A_\mu\partial_\nu\partial_\delta A_\rho, \quad (1)$$

where a , b , θ_1 and θ_2 are the arbitrary parameters which permit us to take the many limits in the theory.

The propagator in the Lorentz gauge, and in the momentum representation, is given by

$$D_{\mu\nu}(k) = \frac{1}{(\theta_2 k^2 + 2\theta_1)^2 - (a - 2bk^2)^2 k^2} \left\{ (a - 2bk^2)g_{\mu\nu} + \frac{\alpha}{k^4} \left[(a - 2bk^2) \left(\left(a - \frac{1}{\alpha} \right) k^2 - 2bk^4 \right) - (\theta_2 k^2 + 2\theta_1)^2 \right] - i(\theta_2 k^2 + 2\theta_1) \frac{\varepsilon_{\mu\nu\theta} k^\theta}{k^2} \right\}. \quad (2)$$

where α is the gauge parameter.

The above propagator has its poles defined through the following third degree equation

$$x^3 + a_1 x^2 + a_2 x + a_3 = 0, \quad (x \equiv k^2), \quad (3)$$

with $a_1 \equiv -\frac{(\theta_2^2 + 4ab)}{4b^2}$, $a_2 \equiv \frac{(a^2 - 4\theta_1\theta_2)}{4b^2}$ and $a_3 \equiv -\frac{\theta_1^2}{4b^2}$.

The above third degree equation has a discriminant D such that, if $D \geq 0$, we fall into unphysical situations. The only physical possibility would be $D < 0$, where one defines $D \equiv Q^3 + R^2$, with

$$Q \equiv \frac{(3a_2 - a_1^2)}{9}, \quad R \equiv \frac{(9a_1 a_2 - 27a_3 - 2a_1^3)}{54}. \quad (4)$$

We have the solution of the above polynomial equation given by ($D < 0$):

$$x_0 = m_0^2 = 2\rho^{1/3} \cos\left(\frac{\alpha}{3}\right) - \frac{a_1}{3}, \quad (5)$$

$$x_{\pm} = m_{\pm}^2 = -\rho^{1/3} \left[\cos\left(\frac{\alpha}{3}\right) \pm \sqrt{3} \sin\left(\frac{\alpha}{3}\right) \right] - \frac{a_1}{3},$$

where $\rho = \sqrt{R^2 + L^2}$, with $L \equiv \sqrt{|D|}$ and $\alpha = \tan^{-1}(L/R)$. Since we want to eliminate the possibility of the existence of tachyons and ghosts, we impose that these three squared masses be real, positive and different among themselves.

In order to make evident the elimination of the tachyonic mode, let us consider for simplicity a case where the usual Chern-Simons term does not exist ($\theta_1 = 0$). In this case, Eq. (3) looks like:

$$x(x^2 + a_1 x + a_2) = 0, \quad (6)$$

so, the poles will be given by

$$m_0^2 = 0, \quad m_{\pm}^2 = \frac{1}{2} \left[-a_1 \pm \sqrt{a_1^2 - 4a_2} \right]. \quad (7)$$

The imposition that the squared masses be positive, leads us to restrict the validity region of the parameter θ_2 ,

$$\theta_2^2 > -8ab, \quad (8)$$

as usual one takes $a = 1$ and $b = -c^2$, where c is an arbitrary constant, we have $\theta_2^2 > 8c^2$.

Let us now discuss the behavior of the residues of the poles of the propagator (2) (in the case where $\theta_1 = 0$), so that we can obtain further restrictions on the parameters of the present theory and thus avoiding unphysical excitations like ghosts.

To achieve this goal, we saturate the propagator with external conserved currents J_μ and J_μ^* , and then looking at the imaginary part of the residues at the poles we obtain:

- (i) The first case is that of the zero-mass pole:

$$\text{Im Res}[J_\mu D^{\mu\nu}(k)J_\nu^*]_{k^2=0} = -\frac{a|J_2|^2}{4b^2m_1^2m_2^2}, \quad (9)$$

where $m_{1,2}$ are identified with $m_{+,-}$ respectively. Note that we worked in a reference frame where $k_\mu = (m, m, 0)$, and that $J_\mu = (cJ_0, \mathbf{J})$ so that, imposing time-likeness in the current, we obtain $|J_2|^2 < 0$. Consequently we conclude that the above pole corresponds to a dynamical excitation with zero mass.

- (ii) For the pole at $k^2 = m_1^2$ we get

$$\text{Im Res}[J_\mu D^{\mu\nu}(k)J_\nu^*]_{k^2=m_1^2} = \frac{1}{4b^2m_1^2(m_1^2 - m_2^2)} [(a - 2b^2m_1^2)|J_1|^2], \quad (10)$$

again working in a particular frame ($k_\mu = (m_1, 0, 0)$), which implies from the transversality condition that $J_0 = 0$, and from compatibility with the case where $m_1 = 0$, we get $J_1 = J_2$. Taking care of these conditions we obtain that

$$\text{Im Res}[J_\mu D^{\mu\nu}(k)J_\nu^*]_{k^2=m_1^2} = -\frac{(a - 2b^2m_1^2)}{2b^2m_2^2(m_1^2 - m_2^2)} |J_1|^2. \quad (11)$$

Observe that once this result is positive we have a dynamical massive particle.

- (iii) In a completely analogous fashion, we obtain for the pole at $k^2 = m_2^2$:

$$\text{Im Res}[J_\mu D^{\mu\nu}(k)J_\nu^*]_{k^2=m_2^2} = -\frac{(a - 2b^2m_2^2)}{2b^2m_2^2(m_2^2 - m_1^2)} |J_1|^2. \quad (12)$$

On the other hand, it is not difficult to verify that $m_1 > m_2$ from Eq. (7). Using this and imposing the positivity of the imaginary part of the residues at the poles $k^2 = m_1^2$ and $k^2 = m_2^2$, we obtain the following restrictions on the range of the theory parameters:

$$\frac{a}{2m_2^2} \geq b^2 \geq \frac{a}{2m_1^2}. \quad (13)$$

consequently there is a region in the space of parameters, where it is possible to have well-behaved excitations.

Before going further, it is necessary to make some comments on the results given in Eqs. (8) and (13). In the Maxwell theory, the presence of states with negative norm can also occur. A way to overcome this problem is to impose the Gupta-Bleuler condition for the positive frequency part of Lorentz gauge

$$\partial \cdot A^{(+)}|\Psi\rangle = 0. \quad (14)$$

such that the energy for the physical states is always positive-definite.

In the present problem, we deal with a model where, in principle, the propagator given by Eq. (2) contains a mixing of positive and negative norm states so that our result appears contradictory. However, following the same strategy used in the case of the electrodynamics, we can build up a generalized gauge condition in order to avoid the unphysical content of the theory.

Observing the equation of motion appearing in (24) below, one can impose a generalized Lorentz gauge condition with positive frequencies

$$(a - 2b^2\Box)\partial \cdot A^{(+)}|\Psi\rangle = 0. \quad (15)$$

Hence, we can write this condition in terms of the annihilation operators by making a Fourier decomposition for the field^a A_μ

$$A_\mu = \frac{1}{(2\pi)^2} \int \frac{d^2k}{2\omega} \sum_{\lambda=0}^2 [a_\lambda(k) \varepsilon_\mu^\lambda(k) e^{-ik \cdot x} + \text{h.c.}], \quad (16)$$

where $\varepsilon_\mu^\lambda(k)$ is the polarization vector. In this way we get,

$$(a - 2b^2\Box)\partial \cdot A^{(+)} = \frac{-i}{(2\pi)^2} \int \frac{d^2k}{2\omega} \sum_{\lambda=0}^2 (a + 2b^2k^2) a_\lambda(k) (k \cdot \varepsilon^\lambda(k)) e^{-ik \cdot x}, \quad (17)$$

since $k \cdot \varepsilon^\lambda(k) = 0$, ($\lambda = 1, 2$) and $k_\mu = (m, 0, 0)$, we obtain the following constraint:

$$(a + 2b^2m_1^2)a_0(m_1)|\Psi\rangle = 0, \quad (18)$$

which for values inside the region where $\frac{a}{2m_1^2} \leq b^2 \leq \frac{a}{2m_2^2}$, implies

$$a_0(m_1)|\Psi\rangle = 0. \quad (19)$$

Therefore, in the above range it is possible to eliminate one degree of freedom from A_μ , leaving behind only the transverse components. Consequently, terms proportional to

$$\sum_{\lambda=1,2} \langle \Psi | a_\lambda^\dagger(k) a_\lambda(k) | \Psi \rangle, \quad (20)$$

will always be positive-definite.

^aFor $\frac{a}{2m_1^2} \leq b^2 \leq \frac{a}{2m_2^2}$ and $\theta_1 \equiv 0$, the field A_μ satisfies $\Box A_\mu(x) = 0$.

On the other hand, since the energy-momentum tensor of the present model is just Maxwell-Podolsky, we can write its zero component as

$$T_{00} = \frac{a}{2}(\mathbf{E}^2 + \mathbf{B}^2) + 2b^2 \left[(E_i \Box E_i + B_i \Box B_i) + \frac{1}{2}(\nabla \cdot \mathbf{E})^2 - \frac{1}{2}((\partial_t \mathbf{E})^2 - \nabla \cdot \mathbf{B})^2 \right] \quad (21)$$

where $E^i = F^{0i}$ and $B^i = \frac{1}{2}\varepsilon^{ijk}F_{jk}$ are the electric and magnetic fields respectively.

Now, as mentioned before, in the range under question the A_μ field depends only on the transverse components $A_i = A_T$ and $\partial_i A_T = 0$. Then we note that at this same limit $\Box A_i = 0$, which implies disregarding the terms $(\nabla \cdot \mathbf{E})^2$ and $E_i \Box E_i$ in the above expression.

Writing down the $\mathbf{B}$ field as a function of the gauge field, it is simple to show, for the same reasons, that the other two terms which come from Podolsky theory should be canceled. Therefore the contribution of the Podolsky term for the energy has been skipped due to the transversality condition, which is at last a consequence of the introduction of the higher-order Chern-Simons term into the model.

Consequently the energy of the system is just due to the Maxwell term

$$\mathcal{U} = \int dx T_{00} = \int dx \frac{a}{2}(\mathbf{E}^2 + \mathbf{B}^2), \quad (22)$$

which is clearly proportional to the positive sum given at Eq. (20). The above result is not surprising since in the ordinary Maxwell-Chern-Simons theory the energy spectrum is given only by the Maxwell term. Here the role of higher-order Chern-Simons term is, at last, to restrict the phase space through the transversality condition (15), which cancels the contribution of the Podolsky terms in the energy-momentum tensor.

To make it complete, it is important to mention that the above result has been obtained when we consider the range $\frac{a}{2m_1^2} \leq b^2 \leq \frac{a}{2m_2^2}$. For any other regions the presence of ghosts is unavoidable.

Now, we can pass to the canonical quantization of the above described theory with $a = 1$, $b = -c^2$ and θ_2 obeying the inequality (8).

Initially we make a functional variation of the Lagrangian density (1),²⁰ from which we obtain the equation of motion,

$$(1 - 2c^2 \Box) \partial_\mu F^{\mu\nu} + \theta_1 \varepsilon^{\mu\nu\alpha} \partial_\mu A_\alpha - 2\theta_2 \varepsilon^{\mu\nu\beta} \partial_\mu \Box A_\beta = 0, \quad (23)$$

besides, the momenta are given by

$$\pi^\nu = -F^{0\nu} - 2c^2(\partial^k \partial^\lambda F_{0\lambda} \delta_k^\nu - \partial^0 \partial^\lambda F_\lambda^\nu) + \theta_1 \varepsilon^{\beta\nu} A_\beta + 2\theta_2 \varepsilon^{\beta\mu\nu} \partial_\mu \partial^0 A_\beta \quad (24)$$

and

$$s^\nu = 2c^2(\partial_\mu F^{\mu\nu} - \delta_0^\nu \partial_\rho F^{0\rho}) + \theta_2 \varepsilon^{\beta\nu} \partial_0 A_\beta, \quad (25)$$

as $\varepsilon^{\beta\nu} \equiv \varepsilon^{0\beta\nu}$, it is easy to see that we have the following primary constraint,

$$\Omega_1 = s^0 \approx 0. \quad (26)$$

Now using the momenta (9) and (10) one can construct the canonical Hamiltonian density $\mathcal{H}_c = \pi_\mu \dot{\tilde{A}}^\mu + s_\mu \cdot \dot{\tilde{A}}^\mu - \mathcal{L}$ (with $\tilde{A} \equiv \cdot A$), such that

$$\begin{aligned} \mathcal{H}_c = & \pi_i \dot{\tilde{A}}^i - \frac{1}{4c^2} s_i s^i + s_i \partial_k F^{ik} - \tilde{A}_0 \partial_k s^k + \frac{1}{2} (\tilde{A}_i - \partial_i A_0)^2 \\ & - c^2 (\partial_i \tilde{A}^i - \nabla^2 A_0)^2 + \frac{1}{4} F^{ij} F_{ij} - 2\theta_1 \varepsilon_{ki} A^i \tilde{A}^k - 2\theta_1 \varepsilon_{ij} A_0 \partial^i A^j \\ & - \theta_2 \varepsilon_{ij} \tilde{A}^j \left(\frac{1}{(2c^2)} s^i + \partial_k F^{ik} + \partial^i \tilde{A}_0 + \theta_2 \varepsilon_{ik} \tilde{A}^k \right) \\ & - 2\theta_2 (\varepsilon_{sl} \partial^k A^s \partial_k \tilde{A}^l + \varepsilon_{ki} \tilde{A}_0 \partial^k A^i + \varepsilon_{lj} \partial_k A_0 \partial^k \partial^l A^j). \end{aligned} \quad (27)$$

Imposing now the temporal consistency of the constraints, one obtains the following secondary constraints:

$$\begin{aligned} \Omega_1 \equiv \Omega_2 = & \partial_i s^i - \pi^0 - \theta_2 \varepsilon_{ij} \partial^i (A^j + 2\tilde{A}^j) \approx 0, \\ \Omega_2 \equiv \Omega_3 = & \partial_k \pi^k + 2\theta_1 \varepsilon_{ij} \partial^j A^i \approx 0. \end{aligned} \quad (28)$$

As should be expected, the consistency of the constraints leads to the "Gauss law". It is also easy to see that we have a theory with first-class constraints. Now we should choose a convenient gauge fixing. As it is typical of models with higher order derivatives, we will choose a higher order gauge fixing,¹⁷

$$\Upsilon_1 = (1 - 2c^2 \square) \partial_k A^k \approx 0, \quad (29)$$

whose consistency generates the following constraints:

$$\Upsilon_2 = A^0 + \mathcal{P} \varepsilon_{ij} \partial^i A^j \approx 0, \quad (30)$$

$$\Upsilon_3 = \tilde{A}^0 + \mathcal{P} \varepsilon_{ij} \partial^i \tilde{A}^j \approx 0, \quad (31)$$

where $\mathcal{P} \equiv 2(\theta_1 - \theta_2 \square)(1 - 2c^2 \square)^{-1} \nabla^2$. Now, we note that the set Ω_i, Υ_j is of second-class. So, after some algebra we obtain the nonvanishing Poisson brackets:

$$\begin{aligned} \{\Omega_1(\mathbf{x}), \Upsilon_3(\mathbf{y})\} &= -\delta^2(\mathbf{x} - \mathbf{y}), \\ \{\Omega_2(\mathbf{x}), \Upsilon_2(\mathbf{y})\} &= \delta^2(\mathbf{x} - \mathbf{y}), \\ \{\Omega_3(\mathbf{x}), \Upsilon_1(\mathbf{y})\} &= (1 - 2c^2 \nabla^2) \nabla^2 \delta^2(\mathbf{x} - \mathbf{y}). \end{aligned} \quad (32)$$

Now we can construct the Dirac matrix and its inverse, so obtaining the nonvanishing Dirac brackets which, using the quantization rule of exchanging the Dirac brackets by their corresponding commutators ($\{, \}^* \rightarrow \frac{1}{i}[,]$), give us the following commutators:

$$\begin{aligned} [A_i(\mathbf{x}), \pi_j(\mathbf{y})] &= \frac{1}{i} [\delta_{ij} \delta^2(\mathbf{x} - \mathbf{y}) - (1 - 2c^2 \nabla^2) \partial_i \partial_j G(\mathbf{x} - \mathbf{y})], \\ [\tilde{A}_i(\mathbf{x}), s_j(\mathbf{y})] &= \frac{1}{i} [\delta_{ij} \delta^2(\mathbf{x} - \mathbf{y})], \\ [\pi_i(\mathbf{x}), \pi_j(\mathbf{y})] &= \frac{1}{i} [(1 - 2c^2 \nabla^2) (2\theta_1 \varepsilon_{ki} \partial_j - \theta_2 \varepsilon_{kj} \partial_i) \partial^k \delta^2(\mathbf{x} - \mathbf{y})], \end{aligned} \quad (33)$$

where $G(\mathbf{x} - \mathbf{y})$ obeys the equation:

$$(1 - 2c^2 \nabla^2) \nabla^2 G(\mathbf{x} - \mathbf{y}) = \delta^2(\mathbf{x} - \mathbf{y}). \quad (34)$$

By inspecting the commutation relations (24), we observe that the two first ones correspond to that of the usual Podolsky theory.²² The last one comes from the Chern-Simons term and its generalization. Taking the limit where $\theta_2 = 0$ and also $c^2 = 0$, we recover the usual Maxwell-Chern-Simons theory. The choice of the gauge condition (20) was suggested due to some reasons, the most important one was obviously that of simplifying the calculations of the theory. Besides, it satisfies some important physical requirements like preservation under dynamical evolution of the system and Lorentz covariance.

The next steps of our study of such higher derivative model should be that of verifying if the unitarity and causality found above, persist when one introduces interaction with matter fields. This should be done through a careful analysis of the asymptotic behavior of scattering amplitudes,²¹ and also from the study of the radiative corrections to see if they do not reintroduce the tachyons or the ghosts. Furthermore it should be quite interesting to verify if the interactions of this theory with matter fields are able to produce bound states responsible for the appearance of phenomena like superconductivity.²³

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LETTER TO THE EDITOR

The partition function for an anyon-like oscillator

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Abstract. We compute the partition function of an anyon-like harmonic oscillator. The well known results for both the bosonic and fermionic oscillators are then re-obtained as particular cases of our function. The technique we employ is a non-relativistic version of the Green function method used in the computation of one-loop effective actions of quantum field theory.

Partition functions give the statistical behaviour of a system of particles in thermal equilibrium with each other and with a thermal bath. In general, the search for these functions is not an easy task although for some particular systems there are well known results. Naturally, many techniques are available for performing this task. In particular, Gibbons [1] used the fact that the partition functions for the bosonic and fermionic oscillators could be written as determinants of the relevant operator, with periodic and antiperiodic boundary conditions, respectively, in order to compute them using the ζ -function method.

Specifically speaking, it follows directly from the definition of a partition function that, for any bosonic system, we can write

$$\begin{aligned} Z^B(\beta) &= \text{Tr} e^{-\beta H} \\ &= \int dx_0 \langle x_0 | e^{-\beta H} | x_0 \rangle \\ &= \int dx_0 K(x_0, x_0; \tau = -i\hbar\beta) \end{aligned} \quad (1)$$

where $K(x, y; \tau)$ is the usual Feynman propagator. For the bosonic (harmonic) oscillator, we substitute its path integral representation and obtain

$$\begin{aligned} Z^B(\beta) &= \int dx_0 \int_{x(0)=x_0}^{x(\beta)=x_0} [Dx] \exp \left(-\frac{1}{2} \int_0^\beta x(\tau)(\omega^2 - \partial^2)x(\tau) d\tau \right) \\ &= \int_{x(0)=x(\beta)} [Dx] \exp \left(-\frac{1}{2} \int_0^\beta x(\tau)(\omega^2 - \partial^2)x(\tau) d\tau \right) \\ &= \det^{-1/2}(\omega^2 - \partial^2)|_{\text{periodic}}. \end{aligned} \quad (2)$$

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Analogously, using standard Grassmann variables, it can be shown that the partition function for the (second-order) fermionic oscillator is given by [1, 2]

$$Z^F(\beta) = \det^1(\omega^2 - \partial_t^2)|_{\text{antiperiodic}}. \quad (3)$$

Here, we will in fact generalize these results by making analogous calculations, but this time we shall impose a generalized boundary condition which contains the particular cases of the periodic and antiperiodic conditions discussed in Gibbon's paper. Besides, we shall use an alternative technique which is the non-relativistic version of the Green function method for computing effective actions in quantum field theory [3, 4].

Suppose then that we want to compute the following determinant:

$$\exp[\Gamma_s^\theta(\omega)] = \det^s(\omega^2 + \partial_t^2)_\theta \equiv \det^s(L)_\theta \quad (4)$$

where L acts on functions that satisfy some given boundary condition specified by the label θ . The power of the determinant given by the parameter s is left completely arbitrary to take into account the cases that interpolate between the fermionic and the bosonic oscillators. Hence, playing with the boundary condition and the parameter s , we can pass continuously from the bosonic to the fermionic oscillator. This is why we refer to this determinant as the partition function of an 'anyon-like' oscillator.

Following a method usually employed in quantum field theory, we write

$$\begin{aligned} \frac{\partial}{\partial \omega} \Gamma_s^\theta(\omega) &= 2s\omega \text{Tr}(L^{-1})_\theta \\ &= 2s\omega \int_0^\tau G_\omega^\theta(t, t) dt \end{aligned} \quad (5)$$

where the Green function $G_\omega^\theta(t, t')$ satisfies

$$(\omega^2 + \partial_t^2)G_\omega^\theta(t, t') = \delta(t - t') \quad (6)$$

as well as some boundary condition (to be given in a moment). Integrating the above equation, we obtain

$$\Gamma_s^\theta(\omega) - \Gamma_s^\theta(0) = 2s \int_0^\omega d\omega' \omega' \int_0^\tau dt G_{\omega'}^\theta(t, t). \quad (7)$$

Since our purpose here is to use a generalized boundary condition, in the sense that the periodic and antiperiodic cases will appear as particular cases, and in order to make connection with the behaviour of correlation functions of anyon-like systems, it is natural to impose the following θ -dependent condition:

$$G_\omega^\theta(t + \tau, t') = e^{-i\theta} G_\omega^\theta(t, t'). \quad (8)$$

It is clear that this boundary condition becomes periodic for $\theta = 0$ and antiperiodic for $\theta = \pi$. Depending on these conditions and the value of the parameter s , which can be thought of as a 'statistical' parameter, this determinant will be mapped into different partition functions. As these particular cases are related to bosonic and fermionic systems, this condition of general periodicity could, in principle, be related to particles whose statistics interpolate bosons and fermions, i.e. anyons [5].

It is straightforward to construct the Green function $G_\omega^\theta(t - t')$. Using basically the same technique that Kleinert [6] employed for the simpler cases of periodic and antiperiodic boundary conditions, it can be shown that (see the appendix)

$$G_\omega^\theta(t - t') = \frac{e^{-i\theta/2}}{4\omega} \left[\frac{e^{i\omega(t-t'-\tau/2)}}{\sin(\frac{1}{2}\omega\tau + \theta)} + \frac{e^{-i\omega(t-t'-\tau/2)}}{\sin(\frac{1}{2}\omega\tau - \theta)} \right] \quad t - t' \in [0, \tau). \quad (9)$$

Substituting this Green function into equation (7), for the interval $[0, \tau)$ and with $t = t'$, we obtain

$$\begin{aligned} \Gamma_s^\theta(\omega) - \Gamma_s^\theta(0) &= 2s \int_0^\tau dt \int_0^\omega d\omega' \omega' \left\{ \frac{e^{-i\theta/2}}{4\omega'} \left[\frac{e^{-i\omega'\tau/2}}{\sin(\frac{1}{2}\omega'\tau + \theta)} + \frac{e^{i\omega'\tau/2}}{\sin(\frac{1}{2}\omega'\tau - \theta)} \right] \right\} \\ &= \ln \{ e^{i\theta} [-1 + e^{-i(\theta+\omega\tau)}] [1 - e^{-i(\theta-\omega\tau)}] \}^s. \end{aligned} \quad (10)$$

Recalling equation (4), we see that the exponential of $\Gamma_s^\theta(\omega)$ is the desired determinant. Identifying $\tau = -i\beta$ ($\hbar = 1$), taking $\theta = 0$ (periodic boundary condition) and $s = -\frac{1}{2}$, this determinant reduces, apart from a constant factor $\exp[\Gamma_s^\theta(0)]$, which hereafter we call C , to the partition function for a bosonic oscillator [1, 6, 7], namely

$$Z^B(\beta) = \exp[\Gamma_{-1/2}^0(\omega)] = \frac{C}{2 \sinh(\omega\beta/2)}. \quad (11)$$

Note that any thermodynamical quantity which can be obtained from the partition function does not depend on C .

Analogously, for a fermionic oscillator, we just make $\theta = \pi$ (antiperiodic boundary condition) and $s = +1$, so that $\exp[\Gamma_s^\theta(\omega)]$ reduces to the following partition function

$$Z^{F(2)}(\beta) = \exp[\Gamma_{+1}^\pi(\omega)] = 4C \cosh^2(\omega\beta/2). \quad (12)$$

This result corresponds to the partition function for a second-order fermionic oscillator. One can check this by explicitly calculating the eigenvalues for the Finkelstein-Villasante Grassmann oscillator (with $N = 2$) [8] and then finding its partition function by summing the trace of $\exp(-\beta E_n)$. This result differs from the one given by Gibbons [1] in the quadratic power of $\cosh(\omega\beta/2)$, once he considered an equivalent linear Grassmann oscillator opposed to the quadratic case discussed here. This linear case can also be obtained from our discussion if we take, from the very beginning, the determinant of $L^{1/2}$ instead of L , so that we find

$$Z^{F(1)}(\beta) = \exp[\Gamma_{+1}^\pi(\omega)]_{\text{Linear}} = 2C \cosh(\omega\beta/2) \quad (13)$$

which is the well known partition function for the linear fermionic oscillator [1, 6, 7].

For the general case, the partition function reads:

$$\begin{aligned} Z_s^\theta(\beta) &\equiv \exp[\Gamma_s^\theta(\omega)] \\ &= [e^{-i\theta} (-1 + e^{-i(\theta+\omega\tau)}) (1 - e^{-i(\theta-\omega\tau)})]^s \\ &= 4^s [\cosh^2 \frac{1}{2}\omega\beta - \cos^2 \frac{1}{2}\theta]^s \end{aligned} \quad (14)$$

where for simplicity we put $C = 1$. Naturally, the above calculated partition functions are particular cases of equation (14).

Note that, in equation (14), we have left the statistics parameter s free. In fact, it may be a function of the periodicity parameter θ , interpolating between $s(\theta = 0) = -\frac{1}{2}$ (bosonic case) and $s(\theta = \pi) = +1$ (fermionic case), as for example $s(\theta) = -\frac{1}{2} + \frac{3}{2}f(\theta)$, where $f(\theta)$ may be a function which satisfies $f(0) = 0$ and $f(\pi) = +1$. We wonder whether this factor can be obtained from the functional integration of a generalized variable, with an arbitrary commutation relation, interpolating the cases of bosonic (c -number) and Grassmannian variables, as a kind of q -deformed calculation [9, 10].

Another possible interpretation for this generalized partition function is in relation to parasytems [11], where there is a parameter for which convenient limits reproduce the bosonic and fermionic oscillators [12]. However, the connection between these systems with the present calculations (if any) seems to be non-trivial and deserves further study. This will be discussed elsewhere.

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Appendix

In this appendix we construct the Green function (9). The following discussion is similar to that found in Kleinert [6]. In fact, we generalize Kleinert's calculations for the cases where the boundary conditions are neither periodic nor antiperiodic, but anyon-like.

The spectral representation for $G_\omega^\theta(t)$ is given by

$$\begin{aligned} G_\omega^\theta(t) &= \frac{1}{\tau} \sum_{m=-\infty}^{\infty} \frac{e^{-i\omega_m^\theta t}}{\omega^2 - \omega_m^{\theta 2}} \\ &= \frac{1}{\tau} \sum_{m=-\infty}^{\infty} \left\{ \frac{e^{-i\omega_m^\theta t}}{2i\omega} \left[\frac{i}{\omega - \omega_m^\theta} + \frac{i}{\omega + \omega_m^\theta} \right] \right\} \\ &= \frac{1}{2\omega i} \{G_-^\theta(t) + G_+^\theta(t)\} \end{aligned} \quad (15)$$

where $\omega_m^\theta = (2\pi m + \theta)/\tau$ and we have identified the spectral representations of $G_\pm^\theta(t)$, which are, respectively, the Green functions associated with the first-order operators $(i\partial_t \pm \omega)$ also satisfying the generalized periodic boundary condition (8).

Using the Poisson summation formula [13]

$$\sum_{m=-\infty}^{\infty} f(m) = \int_{-\infty}^{\infty} d\mu \sum_{n=-\infty}^{\infty} e^{2\pi i \mu n} f(\mu) \quad (16)$$

we can write for $G_\pm^\theta(t)$

$$\begin{aligned} G_\pm^\theta(t) &= \frac{i}{\tau} \sum_{m=-\infty}^{\infty} \frac{e^{-i\omega_m^\theta t}}{\omega \pm \omega_m^\theta} \\ &= \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} e^{-i(\omega' t - n\omega' \tau + n\theta)} \left(\frac{i}{\omega \pm \omega'} \right). \end{aligned} \quad (17)$$

Identifying

$$G_{\pm}(t) = \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} e^{-i\omega't} \left(\frac{i}{\omega \pm \omega'} \right) \quad (18)$$

as the Green functions associated with the first-order operators $(i\partial_t \pm \omega)$, but this time valid for an infinite time interval, we may cast (17) in the form

$$G_{\pm}^{\theta}(t) = \sum_{n=-\infty}^{\infty} e^{-in\theta} G_{\pm}(t - n\tau). \quad (19)$$

Hence, to obtain $G_{\pm}^{\theta}(t)$, we need first to compute $G_{\pm}(t)$. By residue calculations this can be made after setting $\omega \rightarrow \omega - i\eta$. With this prescription, it can easily be shown that

$$G_{\pm}(t) = -\Theta(\mp t) e^{\pm i\omega t} \quad (20)$$

where $\Theta(t)$ is the usual Heaviside step function. Substituting (20) into (19), we obtain

$$G_{\pm}^{\theta}(t) = - \sum_{n=-\infty}^{\infty} e^{-in\theta} \Theta(\mp t \pm n\tau) e^{\pm i\omega(t-n\tau)}. \quad (21)$$

Since this expression has a period τ , we can restrict ourselves to the interval $t \in [0, \tau)$. Hence, the sum appearing in the right-hand side of (21) can be obtained easily, yielding, for $G_{+}^{\theta}(t)$,

$$\begin{aligned} G_{+}^{\theta}(t) &= -e^{i\omega t} \sum_{n=1}^{+\infty} e^{-in(\omega\tau+\theta)} \\ &= -e^{i\omega t} \{e^{-i(\omega\tau+\theta)} + e^{-2i(\omega\tau+\theta)} + \dots\} \\ &= -\frac{e^{i\omega t} e^{-i(\omega\tau+\theta)}}{1 - e^{-i(\omega\tau+\theta)}}. \end{aligned} \quad (22)$$

Analogously, for $G_{-}^{\theta}(t)$, we have

$$\begin{aligned} G_{-}^{\theta}(t) &= -e^{-i\omega t} \sum_{n=0}^{-\infty} e^{-in(\omega\tau-\theta)} \\ &= -e^{-i\omega t} \{1 + e^{-i(\omega\tau-\theta)} + e^{-2i(\omega\tau-\theta)} + \dots\} \\ &= -\frac{e^{-i\omega t}}{1 - e^{-i(\omega\tau-\theta)}}. \end{aligned} \quad (23)$$

Of course, outside this interval, the result can be obtained by periodicity. Substituting (22) and (23) into (15) and rewriting t as $t - t'$, we finally obtain

$$G_{\omega}^{\theta}(t - t') = \frac{e^{-i\theta/2}}{4\omega} \left[\frac{e^{i\omega(t-t'-\tau/2)}}{\sin(\frac{1}{2}\omega\tau + \theta)} + \frac{e^{-i\omega(t-t'-\tau/2)}}{\sin(\frac{1}{2}\omega\tau - \theta)} \right] \quad t - t' \in [0, \tau). \quad (24)$$

This Green function satisfies equations (6) and (8). This formula generalizes the previously known particular cases of periodic ($\theta = 0$) and antiperiodic ($\theta = \pi$) boundary conditions, for which we find the well known results in accordance with the literature [6].

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Conditionally exactly soluble class of quantum potentials

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We present a different class of quantum-mechanical potentials. These are midway between the exactly solvable potentials and the quasiexactly ones. Their fundamental feature is that one can find the entire s -wave spectrum of a given potential, provided that some of its parameters are conveniently fixed.

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Since the appearance of quantum mechanics, there has been continual searching for exactly soluble (ES) potentials [1]. This relies on the importance of such solutions in many branches of physics, and on the fact that these solutions can be used as a basis to perform perturbative calculations in nonexact potentials. Until recently it was thought that there would exist only two classes of potentials in quantum mechanics, namely, the exactly soluble and the nonexactly soluble ones. However, about a decade ago [2], another class was discovered and, since then, has been widely discussed in its various aspects [3–7]. This class, the so-called quasiexactly soluble (QES) potentials, is characterized by the fact that it is only possible to have an exact finite number of their energy eigenstates: the remaining eigenstates can only be obtained through numerical calculations, as with the nonexactly soluble potentials. The importance in the study of these potentials, apart from intrinsic academic interest, rests on the possibility of using their solutions to test the quality of numerical methods and in the possible existence of real physical systems that they could represent.

Here we intend to report the finding of another class of potentials, what we call the conditionally exactly soluble (CES) potentials, because its principal feature is that of having exact solutions only when certain conditions are made, namely, that some of the parameters of the potential be fixed to a very specific value.

This class of potentials is positioned between the ES and the QES potentials. All of their energy levels can be exactly obtained, as with the ES potentials, but their parameters cannot be arbitrarily chosen, as with the QES potentials. The first feature put them in conditions to be used in perturbative calculations, having the advantage that, for the potentials belonging approximately to their form, they take into account the anharmonicity of the potential, unlike the traditional perturbative method where the wave functions of the harmonic oscillator or any other exact potential with a more appropriate form are commonly used. The second feature put the CES potentials in a more suitable situation, when compared with the Q potentials, in order to testify about the quality of a given numerical approach. Furthermore, we will see that one could expect some fashion of reality in at least one of these potentials.

The way to obtain such potentials as we use is to look

for a mapping between them and a driven harmonic oscillator. This is done by performing nonlinear coordinate transformations, and then requiring that a certain term vanish. We start with the Schrödinger equation for a given potential $V(r)$, perform the variable transformation $r=f(u)$, and redefine the wave function as

$$\psi(r,t) = \sqrt{f'(u(r))} \chi(r,t), \quad (1)$$

where the prime denotes differentiation with respect to the variable u . This new wave function obeys the transformed Schrödinger equation

$$\left\{ -\frac{\hbar^2}{2\mu} \frac{\partial^2}{\partial u^2} + V_T(u) \right\} \chi(u) = E_T \chi(u), \quad (2)$$

where μ is the mass, and E_T is some constant resulting from the transformation that takes the role of the “energy” in the new equation. Furthermore, it is straightforward to show that

$$V_T(u) - E_T = (f'(u))^2 [V(f(u)) - E] + \Delta V(u), \quad (3a)$$

with

$$\Delta V(u) = \frac{\hbar^2}{\mu} \left[-\frac{1}{4} \frac{f'''(u)}{f'(u)} + \frac{3}{8} \left(\frac{f''(u)}{f'(u)} \right)^2 \right]. \quad (3b)$$

This type of approach has been used extensively in the path-integral method of quantization [8]. In fact, the above procedure was also used in the Schrödinger picture to relate different power-law potentials [9], and also to relate spherically symmetric potentials at different space dimensions [10]. However, for transformations where $f(u)=u^\alpha$, with α being real, $\Delta V(u)$ will always produce a u^{-2} term, which should be removed in order to get a driven harmonic oscillator potential in the new variable. This is the origin, for the cases here considered, of the very particular fixing of one of the potential parameters. At this point it is interesting to make some considerations about the boundary conditions after the transformation $r=f(u)$. The principal care to be taken is with respect to the normalizability of the wave function, in order to maintain the normalization of the final wave function after returning to the physical variable.

Let us now look at the first two representatives of this new class of potentials. First of all we see that an exten-

sion of our original Coulomb potential belongs to it. This potential is

$$V(r) = \frac{A}{r} + \frac{B}{r^{1/2}} + \frac{G_0}{r^2}. \quad (4)$$

The transformation is the same one that links the Coulomb potential with a three-dimensional harmonic oscillator $f(u) = u^2$ [8]. In this case the expression (2) reads

$$V_T(u) - E_T = -4Eu^2 + 4Bu + 4A + \left[4G_0 + \frac{3\hbar^2}{8\mu} \right] u^{-2}, \quad (5)$$

where we can identify E_T with $4A$, the frequency of the oscillator $\omega = \sqrt{-8E/\mu}$, and the driving force is $4B$. In order to get an exact solution one can follow one of two possible ways. The first one is the elimination of the driving force ($B=0$), leading to the solution of the usual Coulomb potential. The second way is that of imposing the elimination of the centrifugal-barrier-type term; this is obtained by imposing that $G_0 = -3\hbar^2/32\mu$. This last case led us to the first CES potential presented in this work. It is interesting to observe that for great distances it behaves like an s -wave Coulomb potential.

However, before obtaining the solution of the problem, let us make a further transformation to map the CES potential into the harmonic oscillator. Using the translation $v = u - B/(2E)$, we get

$$V_T(v) - E_T = -4Ev^2 + 4A + B^2/E. \quad (6)$$

Now using the solution for the spectrum of the harmonic oscillator, we get the following equation for the spectrum of the CES potential:

$$E_T = (n + 1/2)\hbar\omega, \quad (7a)$$

so that

$$4A + B^2/E_n = (n + 1/2)\hbar\sqrt{-8E_n/\mu}, \quad (7b)$$

which after simple manipulations leads us to the equation for E_n

$$\hbar^2(n + 1/2)^2 E_n^3 + 2\mu A^2 E_n^2 + \mu AB^2 E_n + \mu B^4/8 = 0. \quad (8)$$

On the other hand, using the solutions of the eigenstates of the harmonic oscillator, returning to the variable r , and using (1), we obtain the eigenfunctions

$$\psi_n(r) = (2^{(n-1)}n!)^{-1/2} \left[\frac{r\beta_n^2}{\pi} \right]^{1/4} H_n \left[\beta_n \left(r^{1/2} - \frac{B}{2E_n} \right) \right] \times \exp \left[-\frac{\beta_n^2}{2} \left(r^{1/2} - \frac{B}{2E_n} \right)^2 \right], \quad (9)$$

where we defined

$$\beta_n \equiv \left[-\frac{8\mu E_n}{\hbar^2} \right]^{1/4}.$$

From the three solutions of the spectrum equation (8) we

shall discard two. This can be done by a simple physical criterion. As can be seen from the form of the potential, we see that, when $B=0$, the energy spectrum must be reduced to one of the Coulomb type. Using this criterion one gets the solution

$$E_n = [R + (Q^3 + R^2)^{1/2}]^{1/3} + [R - (Q^3 + R^2)^{1/2}]^{1/3} - \frac{a_1}{3}, \quad (10a)$$

where

$$Q \equiv (3a_2 - a_1^2)/9, \quad R \equiv (9a_1a_2 - 27a_3 - 2a_1^3)/54, \quad (10b)$$

and

$$a_1 = \frac{2\mu A^2}{\hbar^2(n + 1/2)^2}, \quad a_2 = \frac{\mu AB^2}{\hbar^2(n + 1/2)^2}, \quad (10c)$$

$$a_3 = \frac{\mu B^4}{8\hbar^2(n + 1/2)^2}.$$

The second example of a CES potential, as far as we know, does not have in some limit any other well-known ES potential, as happened in the previous example. The potential appears like

$$V(r) = Ar^{2/3} + \frac{B}{r^{2/3}} + \frac{g_0}{r^2}, \quad (11)$$

where g_0 is chosen to be equal to $-5\hbar^2/72\mu$, in a completely analogous fashion to the previous case. It is remarkable to observe that a nonexact version of this potential was considered recently [11], in connection with an effective quark-antiquark potential model for heavy and light mesons. In fact, it is easy to see that for high radial quantum numbers the above CES potential will have its spectrum be closer and closer to the nonexact case appearing in [11]. The transformation function in this case is $f(u) = u^{3/2}$, Eq. (3) looks like

$$V_T(u) - E_T = (9A/4)u^2 - (9E/4)u + 9B/4, \quad (12)$$

and now we identify E_T with $9B/4$, $\omega = \sqrt{9A/2\mu}$, and the driving force is represented by $-9E/4$. Using a procedure analogous to the previous case we see that this time the eigenfunctions are

$$\psi_n(r) = (2^{(n+1)}n!/3)^{-1/2} \left[\frac{r^{2/3}\beta^2}{\pi} \right]^{1/4} \times H_n \left[\beta \left[r^{2/3} - \frac{E_n}{2A} \right] \right] \times \exp \left[-\frac{\beta^2}{2} \left[r^{2/3} - \frac{E_n}{2A} \right]^2 \right], \quad (13)$$

with $\beta \equiv (9\mu A/2\hbar^2)^{1/4}$. The equation for the spectrum is simply

$$E_n^2 = \frac{16A}{9} [(n + 1/2)\sqrt{9A/2\mu} + 9B/4], \quad (14a)$$

with the solutions

$$E_n = \pm \frac{4\sqrt{A}}{3} [(n+1/2)\sqrt{9A/2\mu} + 9B/4]^{1/2}. \quad (14b)$$

It is not difficult to convince oneself that, for the cases where the parameter B is positive, the positive solution is quite good, but if B is negative one can see that there will exist prohibited quantum numbers. This can be seen by imposing that E_n must be a real quantity, so

$$n \geq \text{Int} \left[\left[\frac{9B^2}{8A} \right]^{1/2} - \frac{1}{2} \right], \quad (15)$$

where $\text{Int}[\]$ stands for the first integer after the value of its argument. This, however, does not say that there are not such energy levels, but that perhaps they should be searched for in the negative solutions in (14b). It is only a matter of experimentation to verify that, for instance, with $n=1$ the negative solution looks like a ground-state eigenfunction, because it does not have any node.

The problem is that this potential, in contrast with the

former, does not have some type of limit as a guide for deciding about what solution to use. This problem is under investigation and we intend to report on it in a further publication.

As should be expected, these two cases can be mapped one onto another through a suitable transformation: in the first one, for example, it would be $f(u)=u^{4/3}$. Some extensions can be thought of as looking for supersymmetric partners; one can find other CES representative potentials.

We can also study the possibility of applying the second potential presented above to the case of quark-antiquark potential models. These and other problems related to the CES potential are presently under investigation, and we expect to report on them in the near future.

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so(2,1) Lie algebra and the Green's functions for the conditionally exactly solvable potentials

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The Green's functions of the recently discovered conditionally exactly solvable potentials are computed. This is done through the use of a second-order differential realization of the so(2,1) Lie algebra. So we present the dynamical symmetry underlying the solvability of such potentials and show that they belong to a general class of solvable and partially solvable potentials.

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I. INTRODUCTION

The so(2,1) Lie algebra [isomorphic to su(1,1) and sl(2,R)] is known to generate a family of exactly solvable (ES) potentials, among them the harmonic oscillator, the Coulomb and Morse potentials [1]. This was presented in a unified way using a direct construction of the so(2,1) generators [2] or with the Milshtein-Strakhovenko method [3,4]. So it seemed that, at least, the one-dimensional so(2,1) class of potentials was exhausted. In fact, the so(2,1) Lie algebra can also be used to describe two, three, and higher dimensional systems [5].

Recently, however, it was pointed out that there is a special class of exactly solvable potentials which can be mapped into a harmonic oscillator although with a restriction in some of its parameters. They were called conditionally exactly solvable (CES) potentials. They were discussed independently by Stillinger [6] and de Souza Dutra [7]. Probably due to historical reasons, Stillinger has not perceived that the potentials he was studying were only the first representatives of a broader class. In fact, two-dimensional examples were recently added to this family [8] and also some other one-dimensional ones are under investigation [9]. Furthermore, a supersymmetric origin for these potentials was discussed in Ref. [10].

What distinguishes the potentials belonging to the CES class, is that it is only possible to get exact solutions when specific conditions are provided. Namely, the exact solvability is only possible when some of the potential parameters hold fixed to a very special value [6-10].

Until 1979, as far as we know, only two classes of

quantum-mechanical potentials were known, the exact and the nonexact ones. But in that year, Flessas [11] presented examples of potentials that could be solved analytically only when some constraints among the potential parameters were obeyed. Besides only part of their spectra could be obtained exactly, the remaining should be obtained by standard numerical calculations. This new class of potentials were called quasiexact solvable potentials (QES) or partially algebrized [11-15]. This last name was due to the observed fact that it was only possible to generate algebraically the exact part of its spectrum [13-15].

Now a natural question is that of wondering if there is a dynamical algebra behind the exact solvability of the CES class of potentials. In fact, we are going to show that the algebraic method used before was not quite as general as one could have thought and we will improve this method in order to enlarge as wide as possible the class of dynamically generated so(2,1) potentials. The result is that the method that we are going to obtain can generate both ES, CES, and QES potentials. This is possible through a redefinition of the resolvent operator with an arbitrary scaling and the consequent relaxing of a condition related to the fact that the Hamiltonian of a nonrelativistic system usually does not possess first-order space derivatives.

Furthermore, we hope it is possible to do a systematization of the quest for new CES potentials through the use of dynamical algebras. For this we will improve the method developed in Ref. 4, and also get the Green's functions for these CES potentials.

This work is organized as follows: in the second section we perform the necessary improvements in the approach used in Ref. [4], in order to deal with CES potentials. In the third section we obtain the CES from the so(2,1) Lie algebra in a systematized way. Section four is devoted to obtaining the Green's function of the CES potentials and in the last section we do our final considerations.

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II. THE ALGEBRAIC METHOD AND THE $so(2,1)$ GENERATORS

The imposition of a dynamical algebra on a quantum system, corresponds to saying that its Hamiltonian must be written as a combination of the corresponding generators. In the present case we use the $so(2,1)$ Lie algebra, whose generators satisfy the relations,

$$[T_1, T_2] = -iT_1, \quad [T_2, T_3] = -iT_3, \quad [T_1, T_3] = -iT_2. \quad (2.1)$$

Besides we will use a second-order differential operator realization, in such a way that the Hamiltonian shall be constructed as a linear combination of the generators. In general, second-order generators obeying (2.1) can be written [4] as

$$T_1 = \alpha_2 x^{2-j} \frac{d^2}{dx^2} + \alpha_1 x^{1-j} \frac{d}{dx} + \alpha_0 x^{-j}, \quad (2.2a)$$

$$T_2 = -\frac{i}{j} x \frac{d}{dx} - i\beta, \quad (2.2b)$$

$$T_3 = \lambda x^j, \quad (2.2c)$$

where $\beta = (\frac{1}{2j}) \left(\frac{\alpha_1}{\alpha_2} + j - 1 \right)$ and $\lambda = -(2\alpha_2 j^2)^{-1}$. Naturally the parameters α_i and j ($\neq 0$), will be chosen conveniently to construct a desired Hamiltonian. In fact a usual Hamiltonian does not possess a first-order space derivative so in practice we will set $\alpha_1 = 0$. However, in the following for generality, we left α_1 free but at the end of the calculations the above restriction will be applied.

Performing a point canonical transformation $x = F(u)$, the generators (2.2) become [4]

$$T_1 = \alpha_2 \frac{F(u)^{2-j}}{\dot{F}(u)^2} \frac{d^2}{du^2} + \frac{F(u)^{1-j}}{\dot{F}(u)} \left(\alpha_1 - \alpha_2 \frac{\ddot{F}(u)F(u)}{\dot{F}(u)^2} \right) \frac{d}{du} + \alpha_0 F(u)^{-j}, \quad (2.3a)$$

$$T_2 = -\frac{i}{j} \frac{F(u)}{\dot{F}(u)} \frac{d}{du} - i\beta, \quad (2.3b)$$

$$T_3 = \lambda F(u)^j, \quad (2.3c)$$

where the dot stands for derivatives in variable u . At this point, one can see the appearance of a linear derivative term in the generator T_1 (even when $\alpha_1 = 0$). As in general the Hamiltonian for an one-dimensional Schrödinger equation, does not have such a term, it is necessary to eliminate it. So we started with a system free of first-order space derivatives, implement a canonical transformation which brings up such a term and then we need a second transformation to end up with a Hamiltonian again without first-order space derivatives. For this we begin by remembering that the Green's function obeys

the nonhomogeneous equation:

$$(\mathcal{H} - \mathcal{E}) G_{\mathcal{E}}(x, x') = \delta(x - x'); \quad (2.4)$$

and consequently, due to the form of the generators T_i , it is necessary to introduce the resolvent operator [4],

$$\lambda = \frac{F(u)^{2-j}}{\dot{F}(u)^2} (\mathcal{H} - \mathcal{E}) = g_0 + g_1 T_1 + g_2 T_2 + g_3 T_3, \quad (2.5)$$

where g_i ($i = 0, 1, 2, 3$) are arbitrary constants, that will be fixed in each case. From this resolvent operator we can apply the algebraic method and find the corresponding Green's function [4]

$$G_{\mathcal{E}}(u, u') = -j \sum_{\nu} e^{2\pi i \nu} (4k\lambda)^{\nu+1} \times [\dot{F}(u)\dot{F}(u')]^{-1/2} [F(u)F(u')]^{\frac{i\nu+1}{2}} \times \sum_{n=0}^{\infty} \frac{n! L_n^{\nu}[4k\lambda F(u)^j] L_n^{\nu}[4k\lambda F(u')^j]}{\Gamma(n+\nu+1) [g_0 + k(\nu+1+2n)]} \times \exp\{-2k\lambda[F(u)^j + F(u')^j]\}, \quad (2.6)$$

where $L_n^{\nu}(z)$ are the Laguerre's polynomials [16] and

$$k = \sqrt{\frac{g_1 g_3}{2}}, \quad \nu = \pm \frac{1}{j} \sqrt{\left(1 - \frac{\alpha_1}{\alpha_2}\right)^2 - \frac{4\alpha_0}{\alpha_2}}. \quad (2.7)$$

Note that the poles of the Green's function determine the energy spectrum through

$$g_0 + k(\nu+1+2n) = 0 \quad (n = 0, 1, 2, \dots), \quad (2.8)$$

once the energy $\mathcal{E}$ is related to these parameters.

Here, the idea is to redefine the Green's function, and also the resolvent operator, in such a way as to eliminate the first-order derivative term in the generator T_1 which appears in Eq. (2.5). From this point we change the approach presented in Ref. [4]. The easiest way of discarding the linear derivative term in (2.3a) is to choose

$$\alpha_1 = \alpha_2 \frac{\ddot{F}(u)F(u)}{\dot{F}(u)^2},$$

but this will restrict undesirably the class of potentials that could be generated by the algebraic method. A simple way to circumvent this restriction is to introduce an arbitrary scaling in the Green's function,

$$G_{\mathcal{E}}(x, x') = e^{h(x) + h(x')} S_{\mathcal{E}}(x, x'), \quad (2.9)$$

so that we can choose $h(x)$ in order that $S_{\mathcal{E}}(x, x')$ obeys an equation where only second-order derivatives appear. Consequently we have that

$$\bar{\Lambda} = (g_0 + g_1 T_1 + g_2 T_2 + g_3 T_3) e^{h(x) + h(x')} = e^{h(x) + h(x')} (g_0 + g_1 \tilde{T}_1 + g_2 \tilde{T}_2 + g_3 \tilde{T}_3). \quad (2.10)$$

Under this change the generators become

$$\hat{T}_1 = a_2 \frac{d^2}{du^2} + [2 a_2 \dot{h}(u) + a_1] \frac{d}{du} + A_0, \quad (2.11a)$$

$$\hat{T}_2 = -\frac{i F(u)}{j \dot{F}(u)} \frac{d}{du} - i \left(\beta + \frac{1}{j} \frac{F(u)}{\dot{F}(u)} \dot{h}(u) \right), \quad (2.11b)$$

$$\hat{T}_3 = \lambda F(u)^j, \quad (2.11c)$$

where we defined

$$a_2 \equiv \alpha_2 \frac{F(u)^{2-j}}{\dot{F}(u)^2}, \quad (2.12a)$$

$$a_1 \equiv \frac{F(u)^{1-j}}{\dot{F}(u)} \left(\alpha_1 - \alpha_2 \frac{\ddot{F}(u) F(u)}{\dot{F}(u)^2} \right),$$

and

$$A_0 \equiv a_2 [\ddot{h}(u) + \dot{h}(u)^2] + a_1 \dot{h}(u) + \alpha_0 F(u)^j. \quad (2.12b)$$

Finally, we can choose $h(u)$ in order to get a vanishing first-order derivative term in the generator $\hat{T}_1$, so we get

$$\dot{h}(u) = -\frac{\dot{F}(u)}{2 \alpha_2 F(u)} \left(\alpha_1 - \alpha_2 \frac{\ddot{F}(u) F(u)}{\dot{F}(u)^2} \right), \quad (2.13)$$

whose solution is given by

$$h(u) = \frac{1}{2} \ln \left(\frac{\dot{F}(u)}{F(u)^{\frac{\alpha_1}{\alpha_2}}} \right). \quad (2.14)$$

Substituting this $h(u)$ in Eq. (2.12b) we get

$$\begin{aligned} A_0 = & \frac{\alpha_2}{2} \left[\frac{\dot{F}(u)}{\dot{F}(u)} - \frac{1}{2} \left(\frac{\ddot{F}(u)}{\dot{F}(u)} \right)^2 - \frac{3 \alpha_1}{\alpha_2} \frac{\ddot{F}(u)}{F(u)} \right. \\ & \left. + \frac{\alpha_1}{\alpha_2} \left(1 + \frac{\alpha_1}{2 \alpha_2} \right) \left(\frac{\dot{F}(u)}{\dot{F}(u)} \right)^2 \right] \\ & - \frac{\alpha_1}{2} \left(\frac{\ddot{F}(u)}{\dot{F}(u)} - \frac{\alpha_1}{\alpha_2} \frac{\dot{F}(u)}{F(u)} \right) + \alpha_0 F(u)^j. \end{aligned} \quad (2.15)$$

From now on we will make some particularizations on the generator (2.11a) and the resolvent operator $\hat{\Lambda}$, Eq. (2.10). Without any loss of generality we take $\alpha_1 = g_2 = 0$ and $g_1 = 1$. The first choice reflects the fact that an usual nonrelativistic Hamiltonian of a general form does not possess first-order space derivatives and the second is a normalization constant. In this case we obtain

$$\hat{\Lambda} = e^{h(u) + h(u')} \left(\frac{F(u)^{2-j}}{\dot{F}(u)^2} \right) [\mathcal{H} - \mathcal{E}], \quad (2.16)$$

where

$$\begin{aligned} \mathcal{H} - \mathcal{E} = & -\frac{\hbar^2}{2 \mu} \frac{d^2}{du^2} + \Delta V(u) \\ & + \left(\frac{\dot{F}(u)}{F(u)} \right)^2 [\alpha_0 + g_0 F(u)^j - g_3 \lambda F(u)^{2j}], \end{aligned} \quad (2.17)$$

with

$$\Delta V(u) = \frac{\hbar^2}{\mu} \left[-\frac{1}{4} \frac{\ddot{F}(u)}{\dot{F}(u)} + \frac{3}{8} \left(\frac{\ddot{F}(u)}{\dot{F}(u)} \right)^2 \right]. \quad (2.18)$$

Now we are prepared to generate the one-dimensional family of so(2,1) potentials including the CES class. This is going to be done in the next section.

III. CES POTENTIALS FROM so(2,1) LIE ALGEBRA

In this section we will obtain a family of one-dimensional so(2,1) potentials, with the method developed above. This family includes the classes of ES, QES, and CES potentials. The first two were extensively studied in the literature [1-4,11-15] and we will concentrate on the CES potentials [6,7] which is our main interest here. So we are going to see how the so(2,1) Lie algebra can generate this new class of potentials. With this in mind, we redefine the arbitrary function $F(u)$ through a translation

$$F(u) = f(u) + a, \quad (3.1)$$

where a is an arbitrary constant. It is easy to see that apart from the kinetic term, the Hamiltonian in Eq. (2.17) has a potential of the form

$$\begin{aligned} V(u) = & \mathcal{E} + \Delta V(u) \\ & + \left(\frac{\dot{f}(u)}{f(u) + a} \right)^2 \{ \alpha_0 + g_0 [f(u) + a]^j \\ & + g_3 \lambda [f(u) + a]^{2j} \}, \end{aligned} \quad (3.2)$$

with $\Delta V(u)$ remaining unchanged in its form, because $\dot{F}(u) = \dot{f}(u)$. The above expression for the potential $V(u)$ can accommodate a large family of potentials including ES, QES, and CES. In fact, its form is quite general and for practical purposes we will make some restrictions on it, in order to elucidate its analysis. Remembering that the potentials discussed in Ref. [7] were mapped into a harmonic oscillator, and also that for the harmonic oscillator, the constant j is taken to be equal to 2 [4], we make this choice in Eq. (3.2), obtaining

$$\begin{aligned} V(u) = & \mathcal{E} + \Delta V(u) + \dot{f}(u)^2 \left[\alpha + 2\sigma f(u) \right. \\ & \left. + g_3 \lambda f(u)^2 + \frac{\alpha_0}{[f(u) + a]^2} \right] \end{aligned} \quad (3.3)$$

where we defined $\alpha \equiv g_0 + \lambda g_3 a^2$ and $\sigma \equiv \lambda a g_3$. Note that with the choice $j = 2$ we can only describe potentials which could be mapped into a harmonic oscillator. This can easily be seen taking $f(u) = u - a$ which turns $V(u)$ into a simple harmonic oscillator plus a centrifugal barrier. Other choices for j would lead to different potentials. For example, taking $j = 1$ and $f(u) = u - a$ in Eq. (3.2) we get the Coulomb potential plus a centrifugal barrier [4].

As we are looking for exact potentials, the potential $V(u)$ in (3.3) cannot be energy dependent, contrary to what happens with the quasisexactly solvable potentials [11–15]. So, at least one of the terms in (3.3) must be a constant. This is necessary to cancel the explicit dependence of $V(u)$ in the energy.

This condition is expressed through the following nonlinear differential equations

$$\sigma \dot{f}(u)^2 f(u) = \gamma_1^2, \tag{3.4a}$$

$$-\lambda g_3 \dot{f}(u)^2 f(u)^2 = \gamma_2^2, \tag{3.4b}$$

$$\frac{\alpha_0 \dot{f}(u)^2}{[f(u) + a]^2} = \gamma_3^2, \tag{3.4c}$$

$$\frac{3 \hbar^2}{8 \mu} \left(\frac{\ddot{f}(u)}{\dot{f}(u)} \right)^2 = \gamma_4^2, \tag{3.4d}$$

$$\frac{\hbar^2}{4 \mu} \frac{\ddot{f}(u)}{\dot{f}(u)} = \gamma_5^2, \tag{3.4e}$$

$$\frac{\hbar^2}{\mu} \left[-\frac{1}{4} \frac{\ddot{f}(u)}{\dot{f}(u)} + \frac{3}{8} \left(\frac{\ddot{f}(u)}{\dot{f}(u)} \right)^2 \right] = \gamma_6^2. \tag{3.4f}$$

The solution of each one of the above equations will give us a transformation linking the harmonic oscillator with a particular potential. In some cases it will be an exactly solvable, in other ones it will be a CES and may also be a QES one.

The first three equations have the solutions:

$$f(u) = u^{2/3}, \tag{3.5a}$$

$$f(u) = u^{1/2}, \tag{3.5b}$$

$$f(u) = e^c u - a, \tag{3.5c}$$

where we choose the constants conveniently, without any loss of generality. The third transformation (3.5c), leads us to the Morse potential, as described in Ref. [4]. The other two lead us, respectively, to the two CES studied formerly [6,7]. The first transformation gives the following potential,

$$V^{(2/3)}(u) = -\frac{5\hbar^2}{72\mu} \frac{1}{u^2} + \mathcal{E} + \frac{4}{9} \left[2\sigma + \lambda g_3 u^{\frac{2}{3}} + \frac{\alpha}{u^{\frac{2}{3}}} + \frac{\alpha_0}{u^{\frac{2}{3}} \left(u^{\frac{2}{3}} + a \right)^2} \right]. \tag{3.6}$$

So, we must take $\mathcal{E} = -8\sigma/9$, in order to keep $V^{(2/3)}(u)$ energy independent. This general potential is, in fact, not exactly solvable since the presence of the term with coefficient α_0 depends on the energy $\mathcal{E}$. In order to get a complete solution we must take $\alpha_0 = 0$ rendering this QES potential a CES one, since the strength of the centrifugal barrier has a very special value of $-5\hbar^2/72\mu$ [7]. Substituting these choices in Eq. (2.8) we find the energy spectrum ($\nu = \pm 1/2$)

$$\mathcal{E} = \pm \frac{8}{9} \sqrt{\alpha + \sqrt{\frac{g_3}{2}} \left(n + \frac{1}{2} \right)} \quad (n = 0, 1, 2, \dots), \tag{3.7}$$

in accordance with previous results [6,7]. The second transformation, Eq. (3.5b), gives the potential

$$V^{(1/2)}(u) = -\frac{3\hbar^2}{32\mu} \frac{1}{u^2} + \mathcal{E} + \frac{1}{4} \left[g_3 \lambda + \frac{2\sigma}{u^{1/2}} + \frac{\alpha}{u} + \frac{\alpha_0}{u(u^{1/2} + a)} \right], \tag{3.8}$$

with the same choices as before. Again the term with α_0 brings this potential a QES one. Taking $\alpha_0 = 0$ it becomes a CES one with the strength of the centrifugal barrier fixed to $-3\hbar^2/32\mu$ [7]. The energy spectrum is given by Eq. (2.8):

$$\alpha + 4\mathcal{E}a + \left(-\frac{2\mathcal{E}}{\lambda} \right)^{1/2} \left(n + \frac{1}{2} \right) = 0 \quad (n = 0, 1, 2, \dots), \tag{3.9}$$

which is easily shown to be

$$\frac{2\mathcal{E}^3}{\lambda} \left(n + \frac{1}{2} \right)^{1/2} + \alpha \mathcal{E}^2 + \frac{\alpha \sigma^2 \mathcal{E}}{2} + \frac{\sigma^4}{16} = 0 \quad (n = 0, 1, 2, \dots), \tag{3.10}$$

as was expected [6,7].

The remaining three conditions (3.4d)–(3.4f) lead to the following transformations:

$$f(u) = ke^{\gamma u} \tag{3.11a}$$

for both (3.4d) and (3.4e), and

$$f(u) = tgh(\gamma u) \tag{3.11b}$$

corresponding to (3.4f). The transformation (3.11a) leads to the following potential ($\alpha_0 = 0$):

$$V_{\mathcal{E}}(u) = \mathcal{E} + \frac{\hbar^2 \gamma^2}{8\mu} + k^2 \gamma^2 g_3 \lambda e^{2\gamma u} [k^2 e^{2\gamma u} + 2kae^{\gamma u} + a^2], \tag{3.11c}$$

which is a QES potential since $\mathcal{E} = -\hbar^2\gamma^2/8\mu$ and the constant γ appears in all the potential terms leaving $V_{\mathcal{E}}(u)$ energy dependent. Notice that no choice for the other parameters in (3.12) can remedy this situation.

The last transformation (3.11b) leads also to a QES potential because the same relation between energy and parameters is present here, as can easily be verified with its substitution in the general potential, Eq. (3.2). Many other QES potentials can also be generated by taking for example $j \neq 2$ in (3.2), but we will concentrate on the CES ones since for this class we can find the analytical

solution for the entire spectrum. Let us now discuss the Green's function for the CES found above.

IV. GREEN'S FUNCTION FOR THE CES POTENTIALS

Now, once we have established the one-dimensional so(2,1) CES potentials we are going to present their Green's function. Applying the rescaling (2.9) and the shift (3.1) into Eq. (2.6) we have the Green's function for an arbitrary so(2,1) potential ($\alpha_1 = 0$):

$$G_{\mathcal{E}}(u, u') = -j \sum_{\nu} e^{2\pi i \nu} (4k\lambda)^{\nu+1} \{ [f(u) + a][f(u') + a] \}^{\frac{j\nu+1}{2}} \times \sum_{n=0}^{\infty} \frac{n! L_n^{\nu} \{ 4k\lambda [f(u) + a]^j \} L_n^{\nu} \{ 4k\lambda [f(u') + a]^j \}}{\Gamma(n + \nu + 1) [g_0 + k(\nu + 1 + 2n)]} \exp(-2k\lambda \{ [f(u) + a]^j + [f(u') + a]^j \}). \quad (4.1)$$

This general form can be restricted to the cases discussed in the preceding section. For $V^{(2/3)}(u)$, Eq. (3.6), with $f(u)$ given by (3.5a) we find ($\alpha_0 = 0$; $j = 2$; $k = \sqrt{g_3/2}$)

$$G_{\mathcal{E}}^{(2/3)}(u, u') = -2 \sum_{\nu=-1/2}^{+1/2} e^{2\pi i \nu} (4k\lambda)^{\nu+1} [(u^{2/3} + a)(u'^{2/3} + a)]^{\frac{2\nu+1}{2}} \times \sum_{n=0}^{\infty} \frac{n! L_n^{\nu} [4k\lambda(u^{2/3} + a)^2] L_n^{\nu} [4k\lambda(u'^{2/3} + a)^2]}{\Gamma(n + \nu + 1) [\alpha - a\sigma + k(\nu + 1 + 2n)]} \exp\{-2k\lambda[(u^{2/3} + a)^2 + (u'^{2/3} + a)^2]\}. \quad (4.2)$$

From this Green's function it is easy to obtain the wave functions (up to a phase) which satisfy the Schrödinger equation for that potential:

$$\Psi_{n,\nu}^{(2/3)}(u) = \sqrt{\frac{2n!}{\Gamma(n + \nu + 1)}} (4k\lambda)^{\frac{\nu+1}{2}} (u^{2/3} + a)^{\frac{2\nu+1}{2}} L_n^{\nu} [4k\lambda(u^{2/3} + a)^2] \exp\{-2k\lambda(u^{2/3} + a)^2\}. \quad (4.3)$$

Note that $n (= 0, 1, 2, \dots)$ is the usual principal quantum number and $\nu (= \pm 1/2)$ defines the parity of the solution. Owing to the relation between Laguerre and Hermite polynomials [16], one can see that this solution coincides with the previous results for this potential appearing in the literature [6,7].

An analogous situation occurs for the potential $V^{(1/2)}(u)$, Eq. (3.8). With the transformation (3.5b) we find its Green's function ($\alpha_0 = 0$, $j = 2$, $k = \sqrt{g_3/2}$)

$$G_{\mathcal{E}}^{(1/2)}(u, u') = -2 \sum_{\nu=-1/2}^{+1/2} e^{2\pi i \nu} (4k\lambda)^{\nu+1} [(u^{1/2} + a)(u'^{1/2} + a)]^{\frac{2\nu+1}{2}} \times \sum_{n=0}^{\infty} \frac{n! L_n^{\nu} [4k\lambda(u^{1/2} + a)^2] L_n^{\nu} [4k\lambda(u'^{1/2} + a)^2]}{\Gamma(n + \nu + 1) [\alpha - a\sigma + k(\nu + 1 + 2n)]} \exp\{-2k\lambda[(u^{1/2} + a)^2 + (u'^{1/2} + a)^2]\}, \quad (4.4)$$

and the corresponding wave functions

$$\Psi_{n,\nu}^{(1/2)}(u) = \sqrt{\frac{2n!}{\Gamma(n + \nu + 1)}} (4k\lambda)^{\frac{\nu+1}{2}} (u^{1/2} + a)^{\frac{2\nu+1}{2}} \times L_n^{\nu} [4k\lambda(u^{1/2} + a)^2] \times \exp\{-2k\lambda(u^{1/2} + a)^2\}, \quad (4.5)$$

in accordance, as before, with the literature [6,7]. We limit ourselves in discussing the Green's function for the CES potentials since it is well known that it is meaningless to construct wave functions for QES potentials.

only part of its spectrum is exactly solvable. The other exactly solvable potentials were discussed before [4].

V. CONCLUSIONS

In this paper, we have shown that the class of potentials which can be described by the so(2,1) Lie algebra can be enlarged, adding to it the CES potentials to the previously known ES and QES ones. The approach used here led to obtaining the Green's function for these po-

functions in accordance with the literature [6,7].

So far, we have only considered one-dimensional systems but the technique employed here can be extended to systems in two, three, or arbitrary dimensions with symmetries described by direct sums of the $so(2,1)$ Lie algebra, as is well known for the ES and QES potentials [4,5,15]. This is presently under investigation, and we hope to report on it in the near future.

As a final comment let us mention that the discussion presented here about the $so(2,1)$ potentials is restricted to the ones for which the Hamiltonian can be written in terms of the second-order generators of $so(2,1)$ Lie algebra. This essentially takes care of the potentials which can be mapped into a harmonic oscillator. However, another approach based on first-order generators is also known

and in this case the Hamiltonian is related to the Casimir operator of the $so(2,1)$ Lie algebra. This second approach leads to another family of $so(2,1)$ potentials which are deformable into Posch-Teller potentials [17]. At present it is not clear how the discussion of this paper can be applied to this second family of $so(2,1)$ potentials.

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Expanding the class of conditionally exactly solvable potentials

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Recently a class of quantum-mechanical potentials was presented that is characterized by the fact that they are exactly solvable only when some of their parameters are fixed to a convenient value, so they were christened as conditionally exactly solvable potentials. Here we intend to expand this class by introducing examples in two dimensions. As a byproduct of our search, we found also another exactly solvable potential.

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Recently the existence of a whole class of exactly solvable potentials in quantum mechanics was discovered [1]. Furthermore it was observed that these potentials obey a supersymmetric algebra [2]. In fact the potentials discussed in Ref. [1] were also discussed independently in a previous work of Stillinger [3]. In that article the author did not perceive that such potentials were only the first representatives of an entire new class of potentials. What distinguishes these potentials from the usual exactly solvable potentials is the fact that at least one of their parameters must be fixed to a specific value, because if you do not do that the potentials no longer have exact solutions. On the other hand, they cannot be classified among the so called quasi exactly solvable potentials [4–10], because these have only a finite number of exactly solvable energy levels, and this happens only when the potential parameters obey some constraint equations, so that only some of the parameters are free, and the others are fixed in terms of them. The continuous interest in obtaining and classifying quantum-mechanical potentials is because of their importance as a basis for expanding more realistic cases, and also for use of their solutions to test numerical calculations. Another possibility is that of trying to use them as a kind of theoretical laboratory, searching for new interesting physical features. Some of the examples that we will study below have, in fact, the interesting characteristic of representing a kind of semi-infinite charged string, because, as can be seen in Fig. 1, the potential is infinitely attractive along the x semiaxis. Here in this work we intend to expand the class of conditionally exactly solvable (CES) potentials. This is going to be done through the presentation of two examples in two dimensions. As a byproduct of our search for CES potentials, we found another exactly solvable potential, whose solution will also be presented.

The potentials that will be considered here are

$$V_I(x,y) = \left[\frac{1}{x^2 + y^2} \right] \left[\frac{A}{\arctan(y/x)} + \frac{B}{[\arctan(y/x)]^{1/2}} + \frac{G_0}{[\arctan(y/x)]^2} + C \right] + D(x^2 + y^2) \quad (1a)$$

and

$$V_{II}(x,y) = \left[\frac{1}{x^2 + y^2} \right] \times \left[A \left[\arctan \frac{y}{x} \right]^{2/3} + \frac{B}{[\arctan(y/x)]^{2/3}} + \frac{g_0}{[\arctan(y/x)]^2} + C \right] \quad (1b)$$

where the function $\arctan(y/x)$ is defined in the principal branch. It can be observed that the first case above has an exact solution for any value of G_0 if $B=0$. Otherwise the case $B \neq 0$ has an exact solution for $G_0 = -3\hbar^2/32\mu$, and only for a restricted region of the space (x and y both bigger or both less than zero).

The second potential has exact solutions only when $g_0 = -5\hbar^2/32\mu$. In this case the solution is valid along all the x - y plane. It is not difficult to verify that these potentials are singular along the positive semiaxis x , as can be seen from Fig. 1. So they represent a kind of “charged string.”

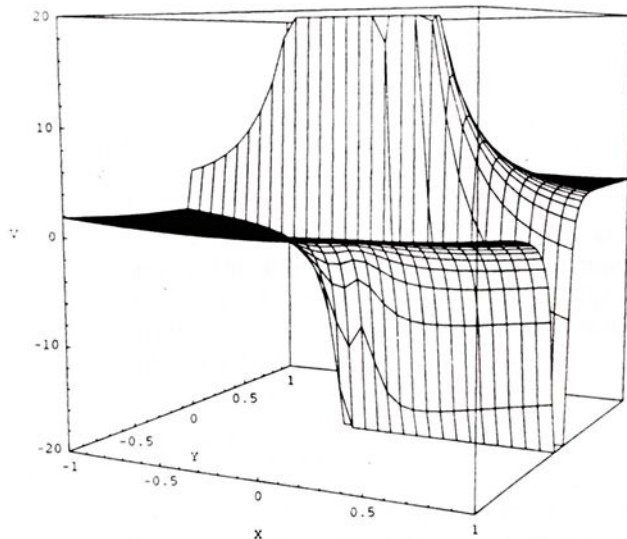


FIG. 1. Plot of the potential CES II with unitary parameters.

The solution for these potentials can be obtained through a suitable change of coordinates:

$$x = u \cos(v), \quad y = u \sin(v).$$

After this change of variables, and their separation, the Schrödinger equation can be split into two others. In the variable u we have that

$$\left\{ -\frac{\hbar^2}{2\mu} \frac{d^2}{du^2} + V_e(u) - \mathcal{E} \right\} \Phi(u) = 0, \quad (2a)$$

where

$$V_e(u) = Du^2 + \frac{(C + K - \hbar^2/8\mu)}{u^2}, \quad (2b)$$

with K being the separation constant and μ the mass of the particle. The subscript e in the potential stands for its exact solvability. In the variable v we have

$$\left\{ -\frac{\hbar^2}{2\mu} \frac{d^2}{dv^2} + V_{\text{CES}}(v) - K \right\} \chi(v) = 0, \quad (3)$$

with $V_{\text{CES}}(v)$ having two possibilities:

$$V_{\text{CES}}^I(v) = \frac{A}{v} + \frac{B}{v^{1/2}} + \frac{G_0}{v^2}, \quad (4a)$$

and

$$V_{\text{CES}}^{II}(v) = Av^{2/3} + \frac{B}{v^{2/3}} + \frac{g_0}{v^2}. \quad (4b)$$

The wave function can be obtained by returning $\Psi(u, v) = u^{-1/2} \Phi(u) \chi(v)$ to the original variables.

At this point, we shall observe that the transformation used leads us to a mapping into an angle variable $v = \arctan(y/x)$, which can, in principle, create problems

like multivaluedness of the wave function. However, one can see that this transformation is typical of a stereographic mapping, in this case of $\mathbb{R}^2$ onto S^2 . This kind of compactification can be done provided that one has suitable asymptotic conditions [11]. In fact, this implies that one will have a finite wave function throughout variation of the variables, and particularly that at infinity it should vanish. This imposes the condition that the parameter D be nonzero, because along the positive x semiaxis the wave function will be singular at infinity if $D=0$. Furthermore, there is no periodicity of the wave function in the variables x and y , as we will see later.

So we see that the original potentials were, in some cases, reduced to one-dimensional CES potentials, whose solutions were obtained previously. Now we proceed to the solution of the above mentioned possibilities.

Case I for $B=0$ and G_0 arbitrary. The first example is that of an exactly solvable potential. In this case the potential $V_I(x, y)$ is mapped into a harmonic oscillator with centrifugal barrier in the variable u , and a Coulomb plus centrifugal barrier in the variable v . So we only need to identify the corresponding parameters, substitute them in the equation for the "energy," which in the transformed equation is the parameter K , and solve it for the original energy

$$\mathcal{E}_{n,m}^I(B=0) = \left[4CD - \frac{8\mu A^2 D}{\hbar^2 [2m+1+a_1]^2} \right]^{1/2} + \hbar \left[\frac{2D}{\mu} \right]^{1/2} (2n+1), \quad (5)$$

where $a_1 \equiv (1 + 8\mu G_0/\hbar^2)^{1/2}$. It can be seen that the above spectrum is a kind of mixing of the Coulomb and harmonic oscillator spectra. The normalized wave functions are

$$\begin{aligned} \psi_{n,m}^{I,B=0}(x,y) = & \left[\frac{32\mu D}{\hbar^2} \right]^{1/8} \left[\frac{2\Gamma(n+1)}{\Gamma(a_1+n+1)} \right]^{1/2} \left[\left[\frac{2\mu D}{\hbar^2} \right]^{1/2} (x^2+y^2) \right]^{a_1/2} \left[\arctan \frac{y}{x} \right]^{1/4} \\ & \times \left[\frac{\Gamma(m+1)}{\Gamma(a_1+m+1)} \right]^{1/2} \left[\left[\frac{-8\mu K}{\hbar^2} \right]^{1/2} \arctan \frac{y}{x} \right]^{(2a_1+1)/4} L_n^{a_1} \left[\left[\frac{2\mu D}{\hbar^2} \right]^{1/2} (x^2+y^2) \right] \\ & \times L_m^{a_1} \left[\left[\frac{-8\mu K}{\hbar^2} \right]^{1/2} \arctan \frac{y}{x} \right] \exp \left\{ - \left[\frac{\mu D}{2\hbar^2} \right]^{1/2} (x^2+y^2) - \left[\frac{-2\mu K}{\hbar^2} \right]^{1/2} \arctan \frac{y}{x} \right\}, \end{aligned} \quad (6)$$

where

$$a_2 \equiv \left[\frac{\mu}{2\hbar^2 D} \right]^{1/2} \mathcal{E}_{n,m}^I - (2n+1), \quad (7a)$$

$$K \equiv \frac{1}{4C} \left[\mathcal{E}_{n,m}^I - \hbar \left[\frac{2D}{\mu} \right]^{1/2} (2n+1) \right]^2 - C, \quad (7b)$$

and $L_n^a(x, y)$ is the generalized Laguerre polynomial. It is remarkable that, as can be seen from Eq. (5), the domain of validity of the parameter C is defined through the equation

$$C \geq \frac{2\mu A^2}{(2m+a_1+1)\hbar^2}, \quad (8)$$

in order to keep the energies real. So, as the left hand side of the above equation has its minimum value when $m=0$ (n arbitrary), we must have $C \geq 2\mu A^2/(a_1+1)\hbar^2$. Apparently if C is less than this we should impose the condition that

$$m \geq \text{Int} \left[\frac{\mu A^2}{\hbar^2 C} - \frac{1}{2}(a_1+1) \right], \quad (9)$$

where $\text{Int}()$ stands for taking the first integer greater than or equal to the argument. This is a strange feature because, in this case, the ground state would have nodes looking like an excited state. This characteristic should

be verified through numerical calculation. Perhaps we would need some kind of analytical continuation for the energy in such cases. Now, the next case.

Case I for $B \neq 0$ and $G_0 = -3\hbar^2/32\mu$. In this case we have our first example of a two-dimensional CES potential. After repeating the previous procedure we get the following equation for the energy eigenstates:

$$\hbar^2(m + \frac{1}{2})K_m^3 + 2\mu A^2 K_m^2 + \mu AB^2 K_m + \frac{\mu B^4}{8} = 0, \quad (10)$$

with

$$K_m = \frac{1}{4D} \left[\mathcal{G}_{n,m}^I - \left[\frac{2\hbar^2 D}{\mu} \right]^{1/2} (2n+1) \right]^2 - C. \quad (11)$$

As we are looking for $\mathcal{G}_{n,m}^I$, we see that it is necessary to solve a third order polynomial equation for K_m , and then use this solution in the expression for the energy

$$\mathcal{G}_{n,m}^I = \pm 2\sqrt{D} (K_m + C)^{1/2} + \left[\frac{2\hbar^2 D}{\mu} \right]^{1/2} (2n+1); \quad (12)$$

once more we have a limitation on the parameter C . The solution of the equation for the parameter K_m is obtained following the procedure appearing in Ref. [1]. This gives us

$$K_m = (R + D)^{1/3} + (R - D)^{1/3} - \frac{a_1}{3}, \quad (13a)$$

with $D \equiv \sqrt{Q^3 + R^2}$, $Q \equiv (3a_2 - a_1^2)/9$, $R \equiv (9a_1 a_2 - 27a_3 - 2a_1^3)/54$, and

$$a_1 \equiv \frac{2\mu A^2}{\hbar^2(m + \frac{1}{2})^2}, \quad a_2 \equiv \frac{\mu AB^2}{\hbar^2(m + \frac{1}{2})^2}, \quad (13b)$$

$$a_3 \equiv \frac{\mu B^4}{8\hbar^2(m + \frac{1}{2})^2}.$$

The wave function is given by

$$\begin{aligned} \psi_{n,m}^I(x,y) = & \left[\frac{32\mu D}{\hbar^2} \right]^{1/8} \left[\frac{\Gamma(n+1)}{2^{m+1} m! \Gamma(a+n+1)} \right]^{1/2} \left[\left[\frac{2\mu D}{\hbar^2} \right]^{1/2} (x^2 + y^2) \right]^{a/2} \left[\frac{\beta^2}{\hbar} \arctan \frac{y}{x} \right]^{1/4} \\ & \times L_n^a \left[\left[\frac{2\mu D}{\hbar^2} \right]^{1/2} (x^2 + y^2) \right] H_m \left[\beta \left[\sqrt{\arctan(y/x)} - \frac{B}{2K_m} \right] \right] \\ & \times \exp \left\{ - \left[\frac{\mu D}{2\hbar^2} \right]^{1/2} (x^2 + y^2) - \frac{\beta^2}{2} \left[\sqrt{\arctan(y/x)} - \frac{B}{2K_m} \right]^2 \right\}, \end{aligned} \quad (14)$$

where we defined

$$a \equiv \left[\frac{\mu}{2\hbar^2 D} \right]^{1/2} \mathcal{G}_{n,m}^I - (2n+1),$$

and

$$\beta \equiv \left[\frac{8\mu K_m}{\hbar^2} \right]^{1/4},$$

and $H_m(x,y)$ is the Hermite polynomial.

It is easy to verify that the wave function, being proportional to $\arctan(y/x)$, vanishes along the positive x semiaxis. This should be expected due to the singularity of the potential in this region. Moreover, it is easy to see from the wave function (14) that, as observed previously when we presented the potential, one must restrict the region of space such that x and y are both positive or both negative. Furthermore, the inverse tangent function is defined in the principal branch, as said previously. Finally we treat the second case.

Case II ($g_0 = -5\hbar^2/72\mu$). Now we present the last example of a two-dimensional CES potential that will be discussed in this work. If we repeat once more the same procedure we get the following equation for the energy in this last case:

$$\begin{aligned} & \pm \frac{4}{3} \sqrt{A} \left[(m + 1/2) \hbar \left[\frac{9A}{2\mu} \right]^{1/2} + \frac{9B}{4} \right]^{1/2} \\ & = \frac{1}{4C} \left[\mathcal{G}_{n,m}^{II} - \left[\frac{2\hbar^2 D}{\mu} \right]^{1/2} (2n+1) \right]^2 - C, \end{aligned} \quad (15a)$$

with the solution

$$\begin{aligned} \mathcal{G}_{n,m}^{II} = & \left\{ \pm \frac{16}{3} D \sqrt{A} \left[m + \frac{1}{2} \right] \left[\frac{9\hbar^2 A}{2\mu} \right]^{1/2} + \frac{9B}{4} \right\}^{1/2} \\ & + 4CD \left\{ + \left[\frac{2\hbar^2 D}{\mu} \right]^{1/2} (2n+1) \right\}. \end{aligned} \quad (15b)$$

This time we need to do a careful analysis of the range of validity for the potential parameters. In the case of the parameter B , we have the restriction

$$B \geq - \left[\frac{2\hbar^2 A}{9\mu} \right]^{1/2}, \quad (16)$$

in order to include $m=0$ in the solution. Besides, the parameter C is also restricted by

$$C \geq \mp \frac{4\sqrt{A}}{3} \left[(m + \frac{1}{2}) \left[\frac{9\hbar^2 A}{2\mu} \right]^{1/2} + \frac{9B}{4} \right]^{1/2}. \quad (17)$$

So we see that the negative solution (positive in the above equation) introduces a limit on the number of bound states. The positive case (negative above) is such that we only need impose that C is bigger than the left hand side

of Eq. (17) with $m=0$.

After this brief analysis of the region of validity of the potential parameters, we present the corresponding wave functions:

$$\begin{aligned} \psi_{n,m}^{\text{II}}(x,y) = & \left[\frac{32\mu D}{\hbar^2} \right]^{1/8} \left[\frac{3\Gamma(n+1)}{2^{m+1}m!\Gamma(a+n+1)} \right]^{1/2} \left[\left[\frac{2\mu D}{\hbar^2} \right]^{1/2} (x^2+y^2) \right]^{a/2} \left[\left[\arctan \frac{y}{x} \right]^{2/3} \beta^2 \right]^{1/4} \\ & \times L_n^a \left[\left[\frac{2\mu D}{\hbar^2} \right]^{1/2} (x^2+y^2) \right] H_m \left\{ \beta \left[\left[\arctan \frac{y}{x} \right]^{2/3} - \frac{K}{2A} \right] \right\} \\ & \times \exp \left\{ - \left[\frac{\mu D}{2\hbar^2} \right]^{1/2} (x^2+y^2) - \frac{\beta^2}{2} \left[\left[\arctan \frac{y}{x} \right]^{2/3} - \frac{K}{2A} \right]^2 \right\}, \end{aligned} \quad (18)$$

where

$$a \equiv \left[\frac{\mu}{2\hbar^2 D} \right]^{1/2} \mathcal{E}_{n,m}^{\text{II}} - (2n+1)$$

and

$$\beta \equiv \left[\frac{9\mu A}{2\hbar^2} \right]^{1/4}.$$

This finishes the presentation of examples of CES potentials, now in two-dimensional space. The principal feature presented by these potentials, besides belonging to the CES class of potentials, is that a stringlike singularity of the potential appears. This is corroborated by the fact that the probability of finding the particle over this singularity is zero. This accounts for the interest in doing further investigations of the characteristics of such poten-

tials. It would be interesting to study the behavior of their corresponding coherent states, and also see what happens with wave packets around the linear singularity of the potential.

Furthermore, in analogy with what has been done with the exact potentials [12,13], and also with the quasisexact ones [8,9], it is our intention to look for the dynamical algebra behind the solvability of the CES potentials. This study is presently under development and we hope to report on it in the near future.

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Approximate analytic expression for the eigenenergies of the anharmonic oscillator

$$V(x) = Ax^6 + Bx^2$$

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An approximate analytic expression for the eigenenergies of the anharmonic oscillator $V(x) = Ax^6 + Bx^2$ is introduced, starting from particular analytic solutions which are valid when certain relations between the parameters A and B are held.

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I. INTRODUCTION

A problem that has been challenging physicists and mathematicians for years is the search for analytic solutions for anharmonic oscillators. As it is well known, the general solution for this problem has not yet been found; however, particular analytic solutions have been discovered when the potential parameters obey certain relations. It is important to note that these particular analytic solutions do not cover the entire spectrum of the problem even in the case of simpler examples. Potentials with these characteristics are then called partially algebraized or quasi-exactly-solvable [1-21].

Since the complete spectra of anharmonic oscillators are not analytically known it is usual to implement numerical methods to find the energy eigenvalues. This is quite satisfactory for a particular potential with all its parameters fixed to certain numerical values. The price paid in this approach is that one cannot in general write a closed expression for the relevant functions related to the solutions of the problem as the eigenenergies or wave functions even for a more restricted class of potentials.

Clearly there is a gap between these two approaches and in this paper we want to make some effort in the direction of connecting them. To do this we will start with the partial analytic solutions known for the double-well potential $V(x) = Ax^6 + Bx^2$ ($A > 0, B < 0$) including their energy spectrum but only in a recurrence relation form and then obtain its numerical counterpart. From these numerical data we establish relations between the energy E and the parameters A and B .

Apart from its intrinsic interest, the double-well potential also plays an important role in the quantum study of

the tunneling time problem [22], in spectra of molecules such as ammonia and hydrogen-bonded solids [23], and in field theory [24]. In fact, the double-well potential that will be focused on in this work might be used as a potential model for quark confinement in quantum chromodynamics [25]. On the other hand, by performing suitable point-canonical transformations, it is possible to show that there is a mapping from the potential of Rydberg atoms in uniform magnetic fields into that of some double-well potentials [26]. Both these problems are generally studied in terms of standard perturbative approaches. It should be very profitable to obtain some analytical information, even though approximately, like that proposed here.

This paper is organized as follows. In Sec. II we review the algebraic approach for the potential $V(x) = Ax^6 + Bx^2$ and determine the energy relations for the first analytic eigenfunctions. In Sec. III we take these energy relations, which are valid when A and B satisfy a constraint relation, and interpolate them numerically. Then we obtain an expression for the energy as a function of A , B and n , the principal quantum number, for $n \leq 3$. In Sec. IV we show that these results can be extended to higher excited states and in Sec. V we present our final considerations.

II. ALGEBRAIC APPROACH

In this section we review the analytic properties of the Schrödinger equation

$$-\frac{1}{2}\psi'' + V\psi = E\psi \quad (1)$$

for the sextic anharmonic potential [7,21]

$$V(x) = Ax^6 + Bx^2, \quad A > 0, \quad (2)$$

obtaining explicitly the expressions for the energy corresponding to the ground and lowest excited states. As is

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well known, the Schrödinger equation for this potential can be solved by writing the wave function as

$$\Psi(x) = \tilde{\Psi}(x) e^{-\sqrt{2A} x^4/4}, \quad (3)$$

where the remaining function $\tilde{\Psi}(x)$ must be a polynomial of definite parity:

$$\tilde{\Psi}_{\text{even}}(x) = \sum_{j=0,2,\dots}^N a_j x^j \quad (4)$$

or

$$\tilde{\Psi}_{\text{odd}}(x) = \sum_{j=1,3,\dots}^N a_j x^j. \quad (5)$$

The exact solutions are only possible when certain relations among the coefficients a_j 's are satisfied, which can be written as recurrence relations

$$a_{j+2} = -2 \frac{E a_j + (N+2-j)\sqrt{2A} a_{j-2}}{(j+2)(j+1)}, \quad (6)$$

where E is the energy eigenvalue. Note that for $j' > N$ all $a_{j'} = 0$. This equation can be used to obtain the energy equation

$$a_N E - 2\sqrt{2A} a_{N-2} = 0. \quad (7)$$

These restrictions imply that the potential parameters A and B must be related by

$$B = -\sqrt{2A} (N + \frac{3}{2}), \quad (8)$$

in order to ensure exact solvability. One should observe that the parameter B is always negative. Therefore, we would not expect any kind of harmonic oscillator limit, in the case of arbitrary parameters, when A goes to zero. In order to take into account the case where B is positive, we suppose that some sort of analytical continuation should be done. Indeed, a particular case of the above mentioned potential has been studied, regarding problems related to analytical continuations of their parameters by Bender and Turbinger [27]. Moreover some discontinuities in the eigenenergy spectra of some polynomial potentials have been reported by Panday and Varma [28]. Thus it is quite clear that any limit and extrapolations should be done very carefully. In fact, the potential $V(x) = Ax^6 + Bx^2$ belongs to a general family of quasi-exactly-solvable potentials [7,21] for which partial analytic solutions are known when its parameters satisfy constraint conditions. Choosing a certain value for N , we fix also the number of exact levels that can be found in this approach: $1 + N/2$ levels when N is even and $(1 + N)/2$ levels when N is odd.

The expressions for the energy eigenvalues for the first values of N , corresponding to the potential (2), may be expressed as

$$N = 0 \text{ or } 1, \quad 0 = E, \quad (9)$$

$$N = 2, \quad 0 = E^2 - 2\sqrt{2A}, \quad (10)$$

$$N = 3, \quad 0 = E^2 - 6\sqrt{2A}, \quad (11)$$

$$N = 4, \quad 0 = E(E^2 - 16\sqrt{2A}), \quad (12)$$

$$N = 5, \quad 0 = E(E^2 - 32\sqrt{2A}), \quad (13)$$

$$N = 6, \quad 0 = E^4 - 60E^2\sqrt{2A} + 360A, \quad (14)$$

$$N = 7, \quad 0 = E^4 - 100E^2\sqrt{2A} + 1512A, \quad (15)$$

$$N = 8, \quad 0 = E(E^4 - 160E^2\sqrt{2A} + 5888A), \quad (16)$$

$$N = 9, \quad 0 = E(E^4 - 240E^2\sqrt{2A} + 16128A), \quad (17)$$

$$N = 10, \quad 0 = E^6 - 350E^4\sqrt{2A} + 41432E^2A - 162000\sqrt{2A}^{3/2}, \quad (18)$$

$$N = 11, \quad 0 = E^6 - 490E^4\sqrt{2A} + 91160E^2A - 831600\sqrt{2A}^{3/2}. \quad (19)$$

This method may be applied for an arbitrary N , but in practice this is difficult to handle. In this analysis, we first confine ourselves to $N \leq 11$ with the advantage of finding algebraic equations for E which can be reduced to the third power (for the energy squared) so that their exact roots may be easily obtained. Finally, we obtain the complete dependence of E on the parameters A , B , and n . In Sec. IV we extend this discussion to $N = 20$ and 30, when only numerical solutions are possible.

III. ANALYTIC-NUMERICAL APPROACH

In order to find an analytic expression which gives approximate values for the eigenenergies for arbitrary parameters [not only for those satisfying (8)] we use a numerical method of adjustment and interpolation starting with the analytical eigenenergies for parameters which satisfy (8). These two variables are then fitted through a polynomial of arbitrary degree which best fit resulted in one of third degree:

$$E = a + bB + cB^2 + dB^3.$$

Noting that the coefficients a , b , c , and d must depend on the parameter A and the principal quantum number n , we write

$$E_n = a_n(A) + b_n(A)B + c_n(A)B^2 + d_n(A)B^3. \quad (20)$$

The next step is to find out these dependences explicitly. The best fit in this case is obtained by single power functions for each coefficient which read

$$a_n(A) = \alpha_n A^{1/4}, \quad (21)$$

$$b_n(A) = \beta_n A^{-1/4}, \quad (22)$$

$$c_n(A) = \gamma_n A^{-3/4}, \quad (23)$$

$$d_n(A) = \delta_n A^{-5/4}. \quad (24)$$

Keeping Eqs. (20) and (21)–(24) in mind, it is possible to state that the energy can be written as a function of parameters A and B and the quantum number n :

$$E_n = A^{1/4} \left[\alpha_n + \beta_n \frac{B}{\sqrt{A}} + \gamma_n \left(\frac{B}{\sqrt{A}} \right)^2 + \delta_n \left(\frac{B}{\sqrt{A}} \right)^3 \right]. \quad (25)$$

The behavior of the coefficients α , β , γ , and δ against n results in the polynomials

$$\alpha_n = 0.2310 + 2.995n - 1.466n^2 + 0.4490n^3, \quad (26)$$

$$\beta_n = (-1.224 + 9.067n - 4.919n^2 + 0.9516n^3) \times 10^{-1}, \quad (27)$$

$$\gamma_n = (-11.00 - 3.057n + 9.876n^2 - 2.562n^3) \times 10^{-2}, \quad (28)$$

$$\delta_n = (-1.778 - 1.953n + 4.176n^2 - 1.073n^3) \times 10^{-3}. \quad (29)$$

These results were obtained for range of values $0.01 \leq A \leq 100$ and $-12.5 \leq \frac{B}{\sqrt{2A}} \leq -1.5$ (and $0 \leq N \leq 11$). The expression (25) is in excellent agreement with the energy levels obtained analytically for $n \leq 3$. In these cases the error was always less than 1%, except for $n = 2$, which presents a maximum discrepancy that is still smaller than 4%. This can be seen from the Table I, where we show the energy eigenvalues, exact and approximate, for the case $A = 1$.

IV. EXTENSION TO HIGHER EXCITED STATES

So far we have only considered the energy for the ground state and the first three excited states. This was so because we have chosen the order N of the polynomials (4) and (5) to be no higher than 11. In that case, the solutions for the energy equation were obtained exactly. With that choice we have varied the parameters A and B through large ranges. As a matter of fact, the choice $N = 11$ leads to six exact levels, but as we take the safe limit of four values of the eigenenergies to be interpolated, we have found the expression (24) for $n \leq 3$ only.

In order to study the behavior of higher excited states we increase N for specific values of the parameters A and B , satisfying relation (8). Choosing $N = 20$ and applying Eq. (6) iteratively, similarly to what has been done for Eqs. (10)–(19), we find that the energy-squared levels $y = E^2$ obeys the equation (for $\sqrt{2A} = 10^{-4}$)

$$\sqrt{y} (y^5 - 0.44y^4 + 0.605299y^3 - 2.99171 \times 10^{-3}y^2 + 4.58305 \times 10^{-5}y - 1.25083 \times 10^{-7}) = 0, \quad (30)$$

which leads to the energy eigenvalues shown in Table II, with the corresponding principal quantum number n .

The best fitting for these points leads to the equation

$$E_n^{N=20} |_{\sqrt{2A}=10^{-4}} = -0.482986 + 7.44313 \times 10^{-2}n - 3.91992 \times 10^{-3}n^2 + 1.30664 \times 10^{-4}n^3. \quad (31)$$

Analogously, for $N = 30$ we find ($\sqrt{2A} = 10^{-10}$) a polynomial equation of eighth power for the energy squared, which leads to the eigenvalues also shown in Table II. These data lead to the equation

$$E_n^{N=30} |_{\sqrt{2A}=10^{-10}} = -8.95776 \times 10^{-4} + 9.16173 \times 10^{-5}n - 3.18989 \times 10^{-6}n^2 + 7.08864 \times 10^{-8}n^3. \quad (32)$$

TABLE I. Comparison between the exact and approximate results for the eigenenergies with $A = 1$. The precision of the approximate solutions is measured by $\delta \equiv (E_{\text{exact}} - E_{\text{approx}})/E_{\text{exact}}$.

B	E_{exact}	E_{approx}	δ
$n=0$			
-2.1213	0.0000	0.0037	
-4.9497	-1.6818	-1.6722	5.7×10^{-3}
-7.7782	-4.7568	-4.7557	-2.3×10^{-4}
-10.6066	-8.9651	-8.9482	-1.9×10^{-3}
-13.4350	-14.0098	-14.0272	1.2×10^{-3}
-16.2635	-19.7569	-19.7516	2.6×10^{-4}
$n=1$			
-3.5355	0.0000	-0.0016	
-6.3640	-2.9130	-2.9076	-1.8×10^{-3}
-9.1924	-6.7272	-6.7324	8.2×10^{-4}
-12.0208	-11.3917	-11.3914	-3.0×10^{-5}
-14.8492	-16.8011	-16.7985	-1.5×10^{-4}
-17.6777	-22.8680	-22.8690	5.0×10^{-4}
$n=2$			
-4.9497	1.6818	1.6682	-8.1×10^{-3}
-7.7782	0.0000	0.0548	
-10.6066	-2.1164	-2.1984	3.9×10^{-2}
-13.4350	-5.4771	-5.4223	-1.0×10^{-2}
-16.2635	-9.9345	-9.9482	1.4×10^{-3}
$n=3$			
-6.3640	2.9130	2.9101	-9.8×10^{-4}
-9.1924	0.0000	0.0111	
-12.0208	-3.4134	-3.4302	4.9×10^{-3}
-14.8492	-7.5588	-7.5477	-1.5×10^{-3}
-17.6777	-12.4725	-12.4754	2.3×10^{-3}

Note that Eqs. (31) and (32) have the same form as Eq. (25), since all of these expressions are given by polynomials of third degree in n . One should also note that these equations have a scaling symmetry, since if we choose, for example, $\sqrt{2A} = 1$, we find that the energy eigenvalues of Table II are only shifted by the factors 10^2 and 10^5 for $N = 20$ and 30 , respectively. This scaling symmetry is a consequence of the fact that the energy eigenvalues are proportional to $A^{1/4}$, as could be observed in Eqs. (10)–(19) or (25) as well. This exact dependence was also observed by Bender and Turbinger [27] by applying a variational method. Note that the reflection symmetry between the negative and positive energy eigenval-

TABLE II. Eigenenergies for $N = 20$ and $N = 30$.

E_n	n
$N=20$	
-4.783810×10^{-1}	0
-3.564750×10^{-1}	2
-2.446070×10^{-1}	4
-1.442440×10^{-1}	6
-5.878460×10^{-2}	8
0.000000	10
5.878460×10^{-2}	12
1.442440×10^{-1}	14
2.446070×10^{-1}	16
3.564750×10^{-1}	18
4.783810×10^{-1}	20
$N=30$	
-8.84041×10^{-4}	0
-7.32793×10^{-4}	2
-5.89470×10^{-4}	4
-4.54646×10^{-4}	6
-3.29087×10^{-4}	8
-2.13924×10^{-4}	10
-1.11289×10^{-4}	12
-2.93390×10^{-5}	14
2.933904×10^{-5}	16
1.112891×10^{-4}	18
2.139249×10^{-4}	20
3.290873×10^{-4}	22
4.546464×10^{-4}	24
5.894709×10^{-4}	26
7.327931×10^{-4}	28
8.840412×10^{-4}	30

ues appearing in Table II are due in part to the method employed here and also to the quasixact symmetry of potential (2).
Analyzing the behavior of the higher excited states discussed above for $N = 20$ and 30 , we expect that their properties will also be accompanied by situations with arbitrary N . In practice, however, as N increases, the power of the polynomial equation from which we find the energy eigenvalues also increases, leading to technical difficulties in calculating its roots. In fact, the levels are not equally spaced and as the number of roots increases, one can find that these differences may be of many orders of magnitude, for which a numerical approach requires rapidly increasing precision. It is important to remark that the behavior of E_n , as always being a polynomial of third degree in n , appears to be a kind of “exact behavior” in the case studied, because it persists when the number of analytically obtained excited states increases.

V. CONCLUSIONS

Some final considerations are now in order. First, we observe that the above developed method, apart from the obvious advantage of producing an approximate energy spectrum for arbitrary values (restricted to a given region of validity) of the potentials parameters, permits at least in principle an approximate analytical expression also for the wave function. This can be achieved by truncating the series appearing in the wave function, defined through the expressions (2)–(4), in a term such that

$$j = R\left(-\left(\frac{B}{\sqrt{2A}} + \frac{3}{2}\right)\right), \tag{33}$$

where $R(x)$ rounds x to the nearest integer. In the case where the parameters A and B are such that $j = N$, we get the exact solutions. In the remaining cases we have an approximation for the wave function, which in turn permits us to have some information about the probabilities of a given process, not only about its eigenenergies. There is another approach that gives an analytical approximation for the wave function introduced by Chhajlany and Malnev [18] and improved by Fernández. [29] It is based on an approach where exact solutions of a potential of a higher order are obtained and then, taking convenient limits, one can restrict the power of this potential to the desired one of interest, for which no exact solution could be found. It would be very interesting to compare these two approaches.

We intend to extend the approach developed here in order to include other polynomial potentials. Furthermore, we are also improving the statistics with more points, for obtaining the full analytical energy expression for greater values of the principal quantum number n . These and other questions are under study and we expect to report on them in the near future.

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Strong-Weak Coupling Duality in Quantum Mechanics

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Abstract

We present a strong-weak coupling duality for quantum mechanical potentials. Similarly to what happens in quantum field theory, it relates two problems with inverse couplings, leading to a mapping of the strong coupling regime into the weak one, giving information from the nonperturbative region of the parameters space. It can be used to solve exactly power-type potentials and to extract deep information about the energy spectra of polynomial ones.

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Although the solutions of the Schroedinger equation has been studied since its origin, some of its important features were discovered along the time as for example the Aharonov-Bohm effect [1], the study of quantum nondemolition measurements [2], coherent states for general potentials [3], and the geometrical Berry's phase [4]. Another interesting feature, is the existence of a number of analytical solutions for certain polynomial potentials, when some relations among the potential parameters hold [5]. Moreover some very important applications demand for solutions of anharmonic oscillators, as happens with problems related to tunneling phenomena [6]. Furthermore they can also be used in order to study models analogous to some quantum field theory ones [7].

In this letter we present a very simple and, at same time, very powerful duality for potentials in quantum mechanics. It is capable of defining the analytical dependence of the energy spectrum on the potential parameters, for a large class of potentials. In fact it is just a matter of experimentation to confirm the validity of this feature for potentials like the harmonic oscillator, the Coulomb potential and other exactly solvable potentials. The principal goal of this letter is to show that potentials which can not be treated through perturbation theory are connected with others which can, letting one to use this connection to get the energy in an strong coupling regime through perturbative calculations in the dual potential. This is very similar to the recently studied electric-magnetic duality in $N = 2$ supersymmetric Yang-Mills theory [8]. In addition we get the exact behavior of the energy spectra of the general power potential $V(x) = ax^N$ [9]. We also discuss its application in polynomial potentials [10], [11], [14].

So let us begin by setting the problem in a general way. We start with the one-dimensional time-independent Schroedinger equation

$$-\frac{\hbar^2}{2\mu} \frac{d^2\psi(x)}{dx^2} + V(x)\psi(x) = E\psi(x), \quad (1)$$

perform a dilatation in the spatial coordinate $x \rightarrow gx$, so that

$$-\frac{\hbar^2}{2\mu} \frac{d^2\psi(x)}{dx^2} + U(x)\psi(x) = \mathcal{E}\psi(x), \quad (2)$$

with $U(x) \equiv g^2V(gx)$ and $\mathcal{E} \equiv g^2E$.

As an example we consider the polynomial potential $V(x) = ax^6 + bx^4 + cx^2$ [10]. In this case one has:

$$U(x) = \tilde{a}x^6 + \tilde{b}x^4 + \tilde{c}x^2, \quad (\tilde{a} \equiv ag^8, \tilde{b} \equiv bg^6, \tilde{c} \equiv cg^4), \quad (3)$$

so that, if a is great compared with c , it is not possible to apply the usual perturbation theory, and even if one use the asymptotic approach like that in [10], valid for great values of the principal quantum number, the problem for the low values of n is still

maintained. If however one chooses a conveniently small g , it is possible to invert the situation, obtaining a perturbative potential where $\tilde{c}$ is much greater than $\tilde{a}$. In fact, for a given set of potential parameters, a family of potentials have their energy fixed due to this symmetry. As an example one can compare two numerical results: $(a = 100, b = 1, c = 1)$ with $E_0 = 13.042\,036\,330$, and $(a = 10^{-6}, b = 10^{-6}, c = 10^{-4})$, obtained from the first setting $g = 10^{-1}$, with $E_0 = 0.130\,420\,363$, which is in accordance with the respective numerical result. Other important situations can be treated in a quite similar way, as for instance the problem of atoms in strong magnetic fields [12], interaction of anyons [13], etc.. In all these cases, one can make a connection between the weak and the strong coupling regime.

Let us now see that this quite simple idea is almost powerful enough to determine by itself the energy spectra for the power type potentials. In this case $V(x) = ax^N$, and after making the above transformations and imposing the covariance of the energy, one gets

$$E_n = k_n^{(N)} \left(\frac{\hbar^2}{\mu} \right)^{\frac{N}{N+2}} a^{\frac{2}{2+N}}, \quad (4)$$

where $k_n^{(N)}$ is a factor which can not be determined by the duality, and that for $N = 2$ (harmonic oscillator) will be equal to $n + \frac{1}{2}$. For the other potentials it is only necessary determine it for a single value of a ($a = 1$ for example) and the remaining will stay fixed. One can take for instance the cases with $N = 4$ and $N = 6$. In these cases the first five levels are such that one get the results in table I.

Now if one wishes to obtain the value of the energy of any level, with other values of a and μ (and using the correct value of $\hbar$) is just a question of substitution and evaluation of Eq.(4), using the constants obtained in table I. Furthermore one can also obtain exactly the wave function by solving the Schroedinger equation, getting an infinite series depending on the energy. For $N = 6$, one has

$$\Psi(x) \equiv \sum_{n=0}^{\infty} a_n x^n e^{-\frac{\gamma x^4}{4}} \quad (5)$$

where $\gamma \equiv \sqrt{2a}$ and the coefficients a_n should be computed through the recurrence equation $a_{n+2} = \frac{-[2Ea_n - (2n-1)\gamma a_{n-2}]}{(n+1)(n+2)}$. So, substituting the exact energy obtained above, one gets the corresponding exact wave function.

One can do a comparison between the energy obtained with the method developed above and that calculated through numerical calculations for some different values of the potential parameter a where, for sake of simplicity, we used $\hbar = \mu = 1$. In doing so one obtain the figures appearing in table II. Note that the only difference, when it exists, is in the last decimal due to the necessary approximation. Nevertheless, if one calculates with even more precision one sees that this is only a consequence of the approximation.



As our next application of this duality, we consider a polynomial potential like $V(x) = a x^N + b x^M + c x^L$ (M and L lesser than N). As for large values of x , the N -power term dominates, the energy will have its dependence factored and the dependence on the remaining parameters is such that one can write down a series invariant under this symmetry. So we will have something like

$$E_n = a^{\frac{2}{2+N}} \sum_{m=0}^{\infty} b_m x^m, \quad (6)$$

where $b_0 = k_n^{(N)} \left(\frac{\hbar^2}{\mu} \right)^{\frac{N}{N+2}}$, and x must be invariant under this symmetry, so we conclude that $x = a^\alpha b^\beta c^\gamma$, with the coefficients α, β, γ obeying the equation $\alpha(N+2) + \beta(M+2) + \gamma(L+2) = 0$, in order to keep the symmetry. In fact one can easily verify that the polynomial potentials treated in [10] and [14] have a series where the corresponding expanding variable obeys this equation, so preserving the symmetry outlined here. However in both cases the series diverges when a vanishes, but this feature can be an artifact of the series expansion. One could for example in the potential with $N = 6, M = 2, L = 0$, try two non-singular functions: $E_n = [\alpha_n^2 a^{\frac{1}{2}} + \beta_n^2 b]^{\frac{1}{2}}$ or $E_n = [\alpha_n^4 a + \beta_n^4 b^2]^{\frac{1}{4}}$, which have as expansion series, at a given order, singular expressions like that in (6). Furthermore, it can be verified that they represent upper and lower approximations for the exact levels respectively, with errors below ten percent. Now we are looking for more precise analytically approximated energies.

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TABLE I: k_n^N coefficients for N equal to 4 and 6, for the first five energy levels.

TABLE I		
n	$k_n^{(4)}$	$k_n^{(6)}$
0	0.66798625	0.68070361
1	2.39038996	2.56845109
2	4.69361420	5.38898250
3	7.34950780	8.88132184
4	10.24505604	12.80383861

TABLE II: Comparison among the energies obtained from equation (4) and from numerical calculations for $N = 6$, $a = 2$ and $a = \pi$.

TABLE II				
n	$E_n(a = 2)$	$E_{numerical}$	$E_n(a = \pi)$	$E_{numerical}$
0	0.80949758	0.80949758	0.90624478	0.90624478
1	3.05442031	3.05442030	3.41946977	3.41946976
2	6.40861633	6.408621632	7.17454298	7.17454297
3	10.56173112	10.56173111	11.82401784	11.82401785
4	15.22641597	15.22641598	17.04620314	17.04620315



