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Câmpus de São José do Rio Preto

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Nilpotent Centers on Center Manifolds, Cyclicity of Hopf Centers and the Period function near a Persistent Polycycle

São José do Rio Preto
2024

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Orientador: Prof. Dr. Claudio Gomes Pessoa
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Dedico este trabalho a você.

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RESUMO

Este trabalho aborda três problemas relevantes e atuais da teoria qualitativa das equações diferenciais ordinárias. O primeiro trata-se do problema do centro em uma variedade central de sistemas diferenciais analíticos tridimensionais cuja parte linear do campo vetorial associado é $y\partial_x - \lambda z\partial_z$, para $\lambda \neq 0$. Obtemos uma forma normal formal para esses sistemas, alguns resultados sobre sua integrabilidade e estendemos, para este caso, algumas técnicas utilizadas para investigar centros nilpotentes no plano. Por exemplo, mostramos que centros nilpotentes de sistemas tridimensionais, em variedades centrais analíticas, são limites de centros do tipo Hopf. O segundo problema considerado é o problema da ciclicidade de pontos de Hopf para sistemas quadráticos em $\mathbb{R}^3$. Usando expansões de ordem superior dos coeficientes de Lyapunov, obtemos uma nova cota inferior para a ciclicidade destes pontos singulares. Mais precisamente, encontramos um exemplo com 13 ciclos limites bifurcando do ponto singular. No terceiro problema, consideramos famílias de campos vetoriais polinomiais planares que possuem um policiclo hiperbólico Γ persistente. Provamos, sob algumas condições genéricas, que bifurca exatamente um ciclo limite a partir de Γ. Neste caso, obtemos informações quantitativas sobre sua função período em relação aos parâmetros de perturbação.

Palavras-chave: Teoria Qualitativa, Problema do Centro Nilpotente, Ciclos Limite, Função Período, Policiclo.

ABSTRACT

This work deals with three current and relevant problems in the qualitative theory of differential equations. The first one is the center problem on a center manifold of differential systems such that the linear part of their associated vector field is $y\partial_x - \lambda z\partial_z$*, with* $\lambda \neq 0$*. We obtain a formal normal form for those systems, some results on their integrability and extended, for this case, some techniques employed to study nilpotent centers in the plane. For instance, we show that nilpotent centers of three-dimensional systems on analytic center manifolds are limits of Hopf type centers. The second problem we consider is the cyclicity problem for Hopf singular points in quadratic systems in* $\mathbb{R}^3$*. Using higher order expansions of the Lyapunov coefficients, we obtain a new lower bound for the cyclicity of such singular points. More precisely, we exhibit an example with 13 limit cycles bifurcating from the singular point. In the third problem, we consider families of polynomial planar vector fields having a hyperbolic persistent polycycle* Γ*. We prove, under generic conditions, that exactly one limit cycle from* Γ*. In this case, we obtain quantitative information on its associated period function with respect to the perturbation parameters.*

Keywords: *Qualitative Theory, Nilpotent Center Problem, Limit cycles, Period Function, Polycycle.*

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Summary

1 Introduction

The theory of differential equations is one of the most interesting and relevant fields in contemporary mathematics. Within its scope, we have several distinct problems, all equally intriguing and fruitful. From the early works of Poincaré [1] to the recent advancements by contemporary researchers, the sheer amount of theoretical results and practical applications is consistently increasing throghout the century. In this context, one's ambition to contribute is met with several difficulties. Picking a problem to tackle, studying the literature to understand what has already been done and discovering new results to have something to add to the extensive collective knowledge. These are not easy tasks and are often overlooked when we think about the role of the researcher.

Nonetheless, we took up on the challenge and the results of our attempt to contribute to our beloved scientific field are comprised into the present thesis. In the workings of producing a thesis, we find ourselves collaborating with different researchers, considering different approaches and tackling different problems. This makes for a rich experience which in our case are comprised into three different chapters that constitutes this thesis: *Nilpotent Centers on Center Manifolds in* $\mathbb{R}^3$, *Hopf singular points in* $\mathbb{R}^3$ *and the cyclicity problem* and *Bifurcation phenomena in time of differential equations.*

The object of our interest are the following differential equation

$$\dot{x} = F(x; \mu), \tag{1.1}$$

where $x \in \mathbb{R}^m$, $\mu \in \Lambda \subset \mathbb{R}^N$ and F is a well-behaved function. Also, x is often referred to as the spatial variables and μ is called the perturbation parameters. The qualitative theory of differential equations constitutes in obtaining information about the solutions of such equations without explicitly computing them. Among the different behaviors that the solutions can have, the one that we focus on is the periodic ones, i.e. the closed orbits. The general behavior of the solution curves in a neighborhood of a closed orbit is often the subject of relevant open problems. In the present thesis, we consider three different problems concerning closed orbits.

The first one is the *Nilpotent Center Problem.* In the plane ($m = 2$), a nilpotent singular point is a singular point of (1.1) such that the Jacobian matrix of F evaluated at the singular point has two zero eigenvalues but is not the null matrix. When this singular point is monodromic, the Nilpotent Center Problem consists in determining whether or not the orbits in a neighborhood of the nilpotent singular point are all closed. This problem is widely studied for the planar case. However, if we consider the same problem in $\mathbb{R}^3$, there are few works in the literature. This is precisely the subject of our first chapter. We consider (1.1) where $m = 3$, F is analytic and DF_p has the Jordan canonical form

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\lambda \end{pmatrix},$$

for p being a singular point, which is called *nilpotent singular point* in $\mathbb{R}^3$. These differential systems admit invariant two-dimensional manifolds (called *Center Manifolds*). Our goal is to develop tools to tackle the Nilpotent Center Problem on a center manifold of such systems. In this endeavor, we obtain a formal normal form for three-dimensional systems having a nilpotent singular point (see Theorem 2.29), some results on their integrability (for instance, Theorems 2.40 and 2.45) and extended some known techniques for the planar nilpotent center problem to the three-dimensional case: the Inverse Jacobi Multiplier method (see Theorem 2.56, which is an extension of the inverse integrating factor method described in [2]), and the fact that nilpotent centers of three-dimensional systems, on analytic center manifolds, are limits of Hopf-type centers (see Theorem 2.64, which is inspired by the main result of [3] and its previous versions for the planar scenario).

The problem of determining whether a singular point is a center or not was first considered for less degenerated types of singular points. More precisely, the *Hopf singular points*. These are the singular points of (1.1), with $m = 3$, for which the Jacobian matrix of F has two purely imaginary eigenvalues and one non-zero real one. There are several works on the center problem for Hopf singular points (see [4, 5, 6], for instance).

In the literature, it is widely known that the center problem is closely related to the cyclicity problem, i. e. the problem of determining how many isolated closed orbits (*limit cycles*) can arise when perturbing equation (1.1). The study of the cyclicity usually aims to obtain lower bounds on the number of limit cycles than can unfold, when varying values of μ in the parameter space Λ around a parameter μ_0 with a specific feature. In the second chapter of the present thesis, this feature is that we have a center on a center manifold for $\mu = \mu_0$ and our work consists in applying the most recent results of the literature to obtain new lower bounds on the cyclicity of Hopf singular points. As far as we know, we have successfully obtained the highest lower bound of the cyclicity of quadratic systems having a Hopf singular point (Theorem 3.11).

The last chapter of the present thesis concerns a different side of the study of closed orbits: their period. The *period function* of a periodic orbit often depends on the perturbation parameters $\mu \in \Lambda$. The classification of the behavior of the period function is the subject of several works in the literature (see , for instance, [7, 8]). We consider (1.1) with $m = 2$, $F(\cdot, \mu)$ polynomial and $F(x, \cdot)$ a smooth function such that, for $\mu = \mu_0$ the system admits a polycicle that maintains its topology for small perturbations on μ near μ_0. In this setting, we study the limit cycles that bifurcate from the persistent polycycle, i. e. the closed orbits that arise in a small neighborhood of the polycylce, seeking quantitative information about their period function. For this matter, we obtained two main results. In the first one, we prove that under some generic conditions on the return map, exactly one limit-cycle bifurcates from Γ and we determine the behavior of its associated period function with respect to the perturbation parameters (Theorem 4.10). The second one concerns the case where the aforementioned generic conditions are not satisfied, for which the behavior of the period function cannot be determined (Theorem 4.21).

We believe that this thesis present relevant new results in all three problems considered. Although the field of differential equations is vast and diverse, we hope that our work represents a meaningful contribution to its development. Hopefully, this work inspires the reader to carry on with their research and give their own contribution too.

The results of the present thesis are distributed in papers [9, 10, 11, 12, 13, 14].

5 Final Considerations

This thesis presents advancements in the qualitative theory of differential equations, focusing on three relevant problems.

Firstly, we address the nilpotent center problem for three-dimensional systems associated with vector fields with linear part $y\partial_x - \lambda z\partial_z$, with $\lambda \neq 0$. In our approach, we obtained a formal normal form for such systems, which allowed us to characterize the formally integrable systems. Furthermore, we proved that nilpotent systems having an analytical nilpotent center on a center manifold, with Andreev number 2, admit a formal inverse Jacobi multiplier. We also proved that nilpotent centers of three-dimensional systems, on analytic center manifolds, are limits of Hopf-type centers.

Secondly, we studied the cyclicity problem for Hopf singular points of quadratic three-dimensional systems. In this endeavor, we used the analysis of the higher-order terms of the Lyapunov coefficients to obtain new lower bounds for the cyclicity of the considered systems, including an example with 13 limit cycles bifurcating from the center, which, as far as we know, is the highest known lower bound on the cyclicity of Hopf singular points in quadratic systems.

Lastly, we investigated the period of the limit cycle bifurcating from a persistent polycycle Γ in families of polynomial planar vector fields depending on a perturbation parameter μ. We proved that, when some generic conditions on the return map depending on μ are satisfied, excatly one limit cycle bifurcates from Γ whose period function behaves as $\dfrac{1}{|1-r(\mu)|}$ where $r(\mu)$ is the graphic number associated to the polycycle Γ.

Overall, our work enhances our understanding of nilpotent centers, cyclicity and bifurcation phenomena. The results of this thesis are distributed in papers [9, 10, 11, 12, 13, 14].

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