



Generalized semiclassical orthogonal polynomials on the unit circle: A Riemann–Hilbert perspective

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Abstract

In this work we show how to get advantage from the Riemann–Hilbert analysis in order to obtain first and second order differential equations for the orthogonal polynomials and associated functions with a weight on the unit circle. We deduce properties for the recurrence relation coefficients from differential properties of the weight. We take the so called generalized modified Jacobi and Bessel weights as a case study.

Keywords Orthogonal polynomials on the unit circle · Riemann–Hilbert Problems · Modified Jacobi weight · Modified Bessel weight · Recurrence relations

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1 Introduction

The purpose of this work is to explore the connection between the theories of the Riemann–Hilbert problem and of the orthogonal polynomials on the unit circle defined in terms of Hölder type weights. Furthermore, we show how Riemann–Hilbert analysis could help in deriving analytical properties for sequences of orthogonal polynomials associated with specific weight functions.

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A weight function, w , on the unit circle, $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, is called semiclassical if it satisfies the equation

$$\frac{d}{d\theta} \left(A(e^{i\theta}) w(\theta) \right) = B(e^{i\theta}) w(\theta), \quad \theta \in [0, 2\pi], \quad (1)$$

where A and B are Laurent polynomials and $A(e^{i\theta}) = 0$ at the singular points of $1/w$. This definition is due to Magnus [24] in the case of the unit circle. For the case of the real line, we refer to [23] and the references therein. The (1) will be called a Pearson-type differential equation. In [6] the Laguerre–Hahn families on the unit circle are treated, which include the semiclassical weight functions as particular cases.

If we allow in (1) that the B can be taken as an entire function, we call w a *generalized semiclassical weight*.

For orthogonality on the unit circle, the classification problem of semiclassical orthogonal polynomials was studied and divided into classes in [8], considering linear functionals that satisfy a Pearson-type equation. This approach can be adapted to (1) considering that, if $\deg A = p$ and $\max\{p-1, \deg((p-1)A(z) + iB(z))\} = q$, then w belongs to the class (p, q) of semiclassical weight functions.

A well known example of a semiclassical weight function on the unit circle is

$$w(\theta) = e^{\ell \cos(\theta)}, \quad 0 \leq \theta \leq 2\pi, \quad \text{with} \quad \ell > 0,$$

see [21]. The associated sequence of orthogonal monic polynomials, $\{\Phi_n\}_{n \geq 0}$, on the unit circle satisfies the structure relation

$$\Phi'_n(z) = n\Phi_{n-1}(z) + \frac{\ell \kappa_{n-2}^2}{2 \kappa_n^2} \Phi_{n-2}(z), \quad n \geq 2. \quad (2)$$

From this structure relation, it is possible to show the nonlinear difference equation

$$\alpha_n(\ell) + \alpha_{n-2}(\ell) = -\frac{2n}{\ell} \frac{\alpha_{n-1}(\ell)}{1 - \alpha_{n-1}^2(\ell)}, \quad n \geq 2. \quad (3)$$

where $\alpha_{n-1} = -\overline{\Phi_n(0)}$ are known as Verblunsky coefficients. This equation was presented by Periwal and Shevitz [26]. It corresponds to a discrete Painlevé dP_{II} equation, see [29]. Studies on semiclassical weight functions on the unit circle can also be found in [7, 24] as well as in [19, 22]. In the last two papers the authors derive raising and lowering operators and find second order differential equations for the orthogonal polynomials on the unit circle associated with semiclassical weights. In [19] it is stated a Riemann–Hilbert problem which is different to the one we present here, but the authors did not take advantage of it to derive differential properties for the polynomials and functions of second kind related with semiclassical weights.

The Riemann–Hilbert method is of great importance in the theory of boundary value problems for holomorphic functions. The main objective is to find a function that is holomorphic in a certain region, taking into account certain jump relations between the values of its limits over the points of a given contour. This approach to

orthogonal polynomials was formulated by Fokas, Its, and Kitaev in [18], and a method for calculating asymptotics with Riemann–Hilbert problems was presented by Deift and Zhou in [17], first in the context of integrable systems. The most comprehensive application to orthogonal polynomials occurred in 1999, thanks to the work of Deift, Kriecherbauer, McLaughlin, Venakides, and Zhou, see [12, 15, 16]. These discoveries were strongly influenced by the relationship between orthogonal polynomials and the theory of random matrices [11, 14].

This technique has been widely used in the theory of orthogonal polynomials to obtain asymptotic and differential properties. In this work, we derive first and second order differential equations for generalized semiclassical orthogonal polynomials on the unit circle, using the Riemann–Hilbert techniques. We also show that some of the first-order differential equations are equivalent to structure relations found in [3]. More details about the Riemann–Hilbert Problem can be found in [1, 2, 13]. Here we follow the Riemann–Hilbert Problem stated in [13] (see also [25]) where there appears two systems of associated functions that, so far as we know, have not been interpreted in terms of their differential properties. For more recent results on Riemann–Hilbert theory related with matrix orthogonality, see [4, 5, 9, 10].

This work is structured as follows. In Section 2, we start with the definition of orthogonality on the unit circle and present the Szegő recurrence satisfied by these polynomials. Next we state the Riemann–Hilbert problem and a technical lemma necessary for the development of the work. In Section 3 we study the analytic properties of the structure matrices associated with the generalized semiclassical weights. We construct first and second order differential operators for the generalized semiclassical orthogonal polynomials on the unit circle and associated function as well.

Furthermore, we determine, for the generalized semiclassical weights, the zero curvature formulas and find difference equations satisfied by Verblunsky coefficients. We end this work, with Section 4, where by using the Riemann–Hilbert approach, we determine the structure matrices associated with the generalized modified, Bessel and Jacobi, weight functions, and construct the first and second order differential operators for orthogonal polynomials and associated functions with these two considered weight functions on the unit circle.

2 Riemann–Hilbert problem

Let μ be a positive measure on the unit circle $\mathbb{T}$, absolutely continuous, with respect to the Lebesgue measure, i.e.

$$d\mu(z) = \frac{v(z)}{iz} dz, \quad \text{with} \quad w(\theta) = v(e^{i\theta}), \quad \theta \in [0, 2\pi].$$

The corresponding sequence of monic orthogonal polynomials, $\{\Phi_n\}_{n \geq 0}$, satisfies

$$\int_{\mathbb{T}} \Phi_n(z) \overline{\Phi_m(z)} d\mu(z) = \kappa_n^{-2} \delta_{n,m}, \quad n, m \in \mathbb{N} \cup \{0\},$$

with $\kappa_n > 0$, or equivalently,

$$\int_{\mathbb{T}} \Phi_n(z) \overline{\Phi_m(z)} \frac{\nu(z)}{iz} dz = \kappa_n^{-2} \delta_{n,m}, \quad n, m \in \mathbb{N} \cup \{0\}. \quad (4)$$

Studies about orthogonal polynomials on the unit circle can be found in [21, 28]. These polynomials satisfy the following relation, known as Szegő recurrence, see [28],

$$\Phi_n(z) = z\Phi_{n-1}(z) - \bar{\alpha}_{n-1}\Phi_{n-1}^*(z), \quad n \in \mathbb{N}, \quad (5)$$

where $\Phi_0(z) = 1$, $\Phi_n^*(z) = z^n \overline{\Phi_n(1/\bar{z})}$ is a reciprocal polynomial and $\alpha_{n-1} = -\Phi_n(0)$, are the Verblunsky coefficients. Note that if the Verblunsky coefficients are known, then the relation (5) generates the sequence of polynomials $\{\Phi_n\}_{n \geq 0}$.

In terms of the powers of z , and the sequence of polynomials $\{\Phi_n\}_{n \geq 0}$ and $\{\Phi_n^*\}_{n \geq 0}$, the orthogonality relations (4) read as

$$\begin{aligned} \int_{\mathbb{T}} z^{-k} \Phi_n(z) d\mu(z) &= 0, \quad k = 0, 1, \dots, n-1 \quad \text{and} \quad \int_{\mathbb{T}} z^{-n} \Phi_n(z) d\mu(z) = \kappa_n^{-2}, \\ \int_{\mathbb{T}} z^{-k} \Phi_n^*(z) d\mu(z) &= 0, \quad k = 1, 2, \dots, n \quad \text{and} \quad \int_{\mathbb{T}} \Phi_n^*(z) d\mu(z) = \kappa_n^{-2}, \end{aligned}$$

or, equivalently, as

$$\int_{\mathbb{T}} \Phi_n(z) \frac{\nu(z)}{z^j} dz = 0, \quad \int_{\mathbb{T}} \Phi_n^*(z) \frac{\nu(z)}{z^{j+1}} dz = 0, \quad j = 1, \dots, n, \quad (6)$$

$$\int_{\mathbb{T}} \Phi_n(z) \frac{\nu(z)}{iz^{n+1}} dz = \kappa_n^{-2}, \quad \int_{\mathbb{T}} \Phi_n^*(z) \frac{\nu(z)}{iz} dz = \kappa_n^{-2}, \quad n \in \mathbb{N}. \quad (7)$$

From Szegő recurrence, we have the following equations

$$\begin{aligned} \Phi_{n+1}^*(z) &= \Phi_n^*(z) - \alpha_n z \Phi_n(z), & n \geq 0, \\ \Phi_{n+1}^*(z) &= (1 - |\alpha_n|^2) \Phi_n^*(z) - \alpha_n \Phi_{n+1}(z), & n \geq 0, \end{aligned} \quad (8)$$

and $\kappa_n^2 / \kappa_{n+1}^2 = 1 - |\alpha_n|^2$, see [28]. We denote the monic orthogonal polynomials on the unit circle as

$$\Phi_n(z) = z^n + \Phi_1^n z^{n-1} + \Phi_2^n z^{n-2} + \dots + \Phi_{n-1}^n z + \Phi_n^n, \quad n \geq 0. \quad (9)$$

Applying the Szegő recurrence (5), it is easy to see that the coefficients Φ_1^n in (9) are given in terms of the Verblunsky coefficients as

$$\Phi_1^n = \sum_{j=0}^{n-1} \bar{\alpha}_j \alpha_{j-1}, \quad n \geq 1 \quad \text{with} \quad \alpha_{-1} = -1.$$

Let ν be a Hölder type weight on the unit circle $\mathbb{T}$, oriented in the positive direction. Then, it can be seen (cf. [13, 25]), that the unique solution of the Riemann–Hilbert problem, which consists in the determination of a 2×2 matrix complex function such that:

(RH1) Y_n is holomorphic in $\mathbb{C} \setminus \mathbb{T}$;

(RH2) Y_n satisfies the jump condition for all $t \in \mathbb{T}$,

$$(Y_n)_+(t) = (Y_n)_-(t) \begin{bmatrix} 1 & \nu(t)/t^n \\ 0 & 1 \end{bmatrix},$$

where Y_n has continuous boundary values $Y_+(z)$ as z approaches the unit circle from the inside, and $Y_-(z)$ from the outside;

(RH3) Y_n has the following asymptotic behavior at infinity

$$Y_n(z) = \left(I_2 + O(z^{-1}) \right) \begin{bmatrix} z^n & 0 \\ 0 & z^{-n} \end{bmatrix} \quad \text{as} \quad z \rightarrow \infty$$

(the I_2 is the order 2 identity matrix) is given by

$$Y_n(z) = \begin{bmatrix} \Phi_n(z) & \frac{1}{2\pi} \int_{\mathbb{T}} \frac{\Phi_n(t) \nu(t)}{t-z} \frac{1}{i t^n} dt \\ -2\pi \kappa_{n-1}^2 \Phi_{n-1}^*(z) & -\kappa_{n-1}^2 \int_{\mathbb{T}} \frac{\Phi_{n-1}^*(t) \nu(t)}{t-z} \frac{1}{i t^n} dt \end{bmatrix}, \quad n \in \mathbb{N}.$$

Moreover, it is also proved that $\det Y_n = 1$, $n \in \mathbb{N}$.

We will write, Y_n , in the form

$$Y_n(z) = \begin{bmatrix} \Phi_n(z) & G_n(z) \\ -b_{n-1} \Phi_{n-1}^*(z) & -b_{n-1} G_{n-1}^*(z) \end{bmatrix}, \quad (10)$$

where

$$b_{n-1} = 2\pi \kappa_{n-1}^2, \quad n \in \mathbb{N}, \quad (11)$$

and

$$G_n(z) = \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{\Phi_n(t) \nu(t)}{t-z} \frac{1}{t^n} dt, \quad G_{n-1}^*(z) = \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{\Phi_{n-1}^*(t) \nu(t)}{t-z} \frac{1}{t^n} dt, \quad (12)$$

will be called associated functions with the orthogonal polynomials, Φ_n and weight function ν on the unit circle. Again, since $\det Y_n = 1$, we can write

$$Y_n^{-1}(z) = \begin{bmatrix} -b_{n-1} G_{n-1}^*(z) & -G_n(z) \\ b_{n-1} \Phi_{n-1}^*(z) & \Phi_n(z) \end{bmatrix}. \quad (13)$$

Now, we will work on the associated functions G_n and G_{n-1}^* .

Proposition 2.1 Let $d\mu(z) = \frac{v(z)}{iz} dz$ be a measure defined on the unit circle. Then their sequences of associated functions defined by (12) satisfies the following recurrence relations,

$$G_n(z) = G_{n-1}(z) - \bar{\alpha}_{n-1} G_{n-1}^*(z), \quad n \in \mathbb{N}, \quad (14)$$

$$zG_n^*(z) = G_{n-1}^*(z) - \alpha_{n-1} G_{n-1}(z), \quad n \in \mathbb{N}. \quad (15)$$

Proof To show the (14) we multiply (5) in t variable by $\frac{v(t)}{2\pi i t^{n+1}(t-z)}$ and then take integral in $\mathbb{T}$ with respect to the variable t , i.e.

$$\frac{1}{2\pi i} \int_{\mathbb{T}} \frac{\Phi_{n+1}(t)v(t)}{t^{n+1}(t-z)} dt = \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{t\Phi_n(t)v(t)}{t^{n+1}(t-z)} dt - \bar{\alpha}_n \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{\Phi_n^*(t)v(t)}{t^{n+1}(t-z)} dt$$

which is equivalent to (14). To show (15), we proceed in the same way as before, now using the relation (8) in t variable and multiplying by $\frac{v(t)}{2\pi i t^n(t-z)}$, i.e.

$$\frac{1}{2\pi i} \int_{\mathbb{T}} \frac{\Phi_n^*(t)v(t)}{t^n(t-z)} dt = \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{t\Phi_{n-1}(t)v(t)}{t^n(t-z)} dt - \alpha_{n-1} \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{t\Phi_{n-1}(t)v(t)}{t^n(t-z)} dt.$$

Now, taking into account that

$$\begin{aligned} \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{\Phi_n^*(t)v(t)}{t^n(t-z)} dt &= \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{t-z+z}{t-z} \frac{\Phi_n^*(t)v(t)}{t^{n+1}} dt \\ &= \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{\Phi_n^*(t)v(t)}{t^{n+1}} dt + \frac{z}{2\pi i} \int_{\mathbb{T}} \frac{\Phi_n^*(t)v(t)}{t^{n+1}(t-z)} dt \quad \text{and by (6)} \\ &= \frac{z}{2\pi i} \int_{\mathbb{T}} \frac{\Phi_n^*(t)v(t)}{t^{n+1}(t-z)} dt. \end{aligned}$$

So, we obtain (15). $\square$

Using the orthogonality properties, in the next result we present the asymptotic expansion of G_n and G_{n-1}^* about ∞ .

Lemma 2.1 Let $d\mu(z) = \frac{v(z)}{iz} dz$ be a measure defined on the unit circle. Then,

$$G_n(0) = \frac{1}{b_n}, \quad G_{n-1}^*(0) = \frac{\alpha_{n-1}}{b_{n-1}}, \quad (16)$$

and

$$G_n(z) = -\frac{\bar{\alpha}_n}{b_n} \frac{1}{z^{n+1}} + \left(\frac{\bar{\alpha}_n}{b_n} \Phi_1^{n+1} - \frac{\bar{\alpha}_{n+1}}{b_{n+1}} \right) \frac{1}{z^{n+2}} + \dots, \quad n \geq 0, \quad (17)$$

$$G_{n-1}^*(z) = \frac{1}{b_{n-1}} \left(-\frac{1}{z^n} + \frac{\Phi_1^n}{z^{n+1}} - \frac{\Phi_1^n \Phi_1^{n+1} - \Phi_2^{n+1}}{z^{n+2}} + \dots \right), \quad n \geq 1. \quad (18)$$

Proof Taking $z = 0$ in (12), we have

$$G_n(0) = \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{\Phi_n(t)}{t^{n+1}} v(t) dt.$$

Using (7) and (11) we get the first identity in (16). Now, taking $z = 0$ in the second identity in (16) and using (8) together with the orthogonality properties, we get

$$G_{n-1}^*(0) = \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{\Phi_n^*(t) + \alpha_{n-1}t \Phi_{n-1}(t)}{t^{n+1}} v(t) dt.$$

Therefore, using (7), (6), and (11) we get the second identity in (16). To prove (18), we take into account the second identity in (12) in order to get

$$\begin{aligned} G_{n-1}^*(z) &= \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{\Phi_{n-1}^*(t)v(t)}{t^n(t-z)} dt \\ &= - \sum_{j=0}^{\infty} \frac{1}{z^{j+1}} \frac{1}{2\pi} \int_{\mathbb{T}} t^j \Phi_{n-1}^*(t) \frac{v(t)}{i t^n} dt \quad \text{and by (6)} \\ &= - \sum_{j=0}^{\infty} \frac{1}{z^{j+n}} \frac{1}{2\pi} \int_{\mathbb{T}} t^j \Phi_{n-1}^*(t) \frac{v(t)}{i t} dt. \end{aligned}$$

Now, by (7) we get that the coefficient of the $(-n)$ -th z power is $-b_{n-1}^{-1}$. To determine the coefficient of the $(-n-1)$ -th z power, we begin by identify

$$\int_{\mathbb{T}} \Phi_{n-1}^*(t) \frac{v(t)}{i} dt = \int_{\mathbb{T}} t^n \overline{\Phi_{n-1}(t)} \frac{v(t)}{i t} dt.$$

Hence plugging into the orthogonal relation the expression of Φ_n given in (9)

$$\int_{\mathbb{T}} \Phi_n(t) \overline{\Phi_{n-1}(t)} \frac{v(t)}{i t} dt = \int_{\mathbb{T}} (t^n + \Phi_1^n t^{n-1} + \Phi_2^n t^{n-2} + \dots) \overline{\Phi_{n-1}(t)} \frac{v(t)}{i t} dt = 0$$

we arrive, taking into account the orthogonality condition (4), to the equation

$$\int_{\mathbb{T}} t^n \overline{\Phi_{n-1}(t)} \frac{v(t)}{i t} dt + \Phi_1^n \int_{\mathbb{T}} t^{n-1} \overline{\Phi_{n-1}(t)} \frac{v(t)}{i t} dt = 0,$$

and so the coefficient of the $(-n-1)$ -th z power is $\Phi_1^n b_{n-1}^{-1}$. To determine the third coefficient,

$$\int_{\mathbb{T}} t \Phi_{n-1}^*(t) \frac{v(t)}{i} dt = \int_{\mathbb{T}} t^{n+1} \overline{\Phi_{n-1}(t)} \frac{v(t)}{i t} dt.$$

Again, using the same argument, i.e.,

$$\int_{\mathbb{T}} \Phi_{n+1}(t) \overline{\Phi_{n-1}(t)} \frac{v(t)}{it} dt = \int_{\mathbb{T}} (t^{n+1} + \Phi_1^{n+1} t^n + \Phi_2^{n+1} t^{n-1} + \dots) \overline{\Phi_{n-1}(t)} \frac{v(t)}{it} dt = 0,$$

we arrive, taking into account the orthogonality condition (4), to the equation

$$\int_{\mathbb{T}} t^{n+1} \overline{\Phi_{n-1}(t)} \frac{v(t)}{it} dt = -\Phi_1^{n+1} \int_{\mathbb{T}} t^n \overline{\Phi_{n-1}(t)} \frac{v(t)}{it} dt - \Phi_2^{n+1} \int_{\mathbb{T}} t^{n-1} \overline{\Phi_{n-1}(t)} \frac{v(t)}{it} dt$$

and so the coefficient of the $(-n-2)$ -th z power is $(\Phi_1^{n+1} \Phi_1^n - \Phi_2^{n+1}) b_{n-1}^{-1}$. Therefore, in general, the coefficients of the z powers in the expansion (18) are given by

$$\int_{\mathbb{T}} t^{n+j} \overline{\Phi_{n-1}(t)} \frac{v(t)}{it} dt = -\sum_{k=1}^{j+1} \Phi_k^{n+j} \int_{\mathbb{T}} t^{n-k+j} \overline{\Phi_{n-1}(t)} \frac{v(t)}{it} dt, \quad j \in \mathbb{N} \cup \{0\}.$$

From (12), we have

$$\begin{aligned} G_n(z) &= \frac{1}{2\pi} \int_{\mathbb{T}} \frac{\Phi_n(t) v(t)}{it^n(t-z)} dt = -\sum_{j=0}^{\infty} \frac{1}{z^{j+1}} \frac{1}{2\pi} \int_{\mathbb{T}} t^j \Phi_n(t) \frac{v(t)}{it^n} dt \\ &= -\sum_{j=0}^{\infty} \frac{1}{z^{n+j+1}} \frac{1}{2\pi} \int_{\mathbb{T}} t^{j+1} \Phi_n(t) \frac{v(t)}{it} dt. \end{aligned}$$

For the first coefficients, we observe that

$$\int_{\mathbb{T}} t \Phi_n(t) \frac{v(t)}{it} dt = \frac{\bar{\alpha}_n}{\kappa_n^2}. \quad (19)$$

In fact, using the Szegő recurrence (5), we obtain

$$\int_{\mathbb{T}} t \Phi_n(t) \frac{v(t)}{it} dt = \int_{\mathbb{T}} \Phi_{n+1}(t) \frac{v(t)}{it} dt + \bar{\alpha}_n \int_{\mathbb{T}} \Phi_n^*(t) \frac{v(t)}{it} dt,$$

and taking into account (6) and (7), we arrive to (19). Now, proceeding as before departing from

$$\int_{\mathbb{T}} t^2 \Phi_n(t) \frac{v(t)}{it} dt = \int_{\mathbb{T}} t \Phi_{n+1}(t) \frac{v(t)}{it} dt + \bar{\alpha}_n \int_{\mathbb{T}} t \Phi_n^*(t) \frac{v(t)}{it} dt,$$

and using the previous calculations, we get

$$\int_{\mathbb{T}} t^2 \Phi_n(t) \frac{v(t)}{it} dt = \frac{\bar{\alpha}_{n+1}}{\kappa_{n+1}^2} - \Phi_1^{n+1} \frac{\bar{\alpha}_n}{\kappa_n^2}.$$

Therefore, in general, we have the relation

$$\int_{\mathbb{T}} t^{j+1} \Phi_n(t) \frac{\nu(t)}{i t} dt = \frac{\bar{\alpha}_{n+j}}{\kappa_{n+j}^2} - \bar{\alpha}_n \int_{\mathbb{T}} t^{n+j} \overline{\Phi_n(t)} \frac{\nu(t)}{i t} dt.$$

In this way, we get the asymptotic expansion for G_n at infinity (17). $\square$

Now, we apply the Riemann–Hilbert problem to derive recurrence relations for orthogonal polynomials and associated functions.

Proposition 2.2 *Let Y_n be the solution of the Riemann–Hilbert problem, then for all $n \in \mathbb{N}$,*

$$Y_{n+1}(z) \begin{bmatrix} 1 & 0 \\ 0 & z \end{bmatrix} = T_n(z) Y_n(z), \quad \text{where} \quad T_n(z) = \begin{bmatrix} z + \bar{\alpha}_n \alpha_{n-1} & \bar{\alpha}_n / b_n \\ \alpha_{n-1} b_n & 1 \end{bmatrix} \quad (20)$$

which is called the transfer matrix.

Proof The matrices Y_n and Y_n^{-1} are holomorphic at $\mathbb{C} \setminus \mathbb{T}$, then the matrix function

$$T_n(z) = Y_{n+1}(z) \begin{bmatrix} 1 & 0 \\ 0 & z \end{bmatrix} Y_n^{-1}(z)$$

is holomorphic at $\mathbb{C} \setminus \mathbb{T}$. However, T_n has no jump on the unit circle $\mathbb{T}$. In fact, we have

$$\begin{aligned} (T_n)_+(z) &= (Y_{n+1})_+(z) \begin{bmatrix} 1 & 0 \\ 0 & z \end{bmatrix} (Y_n^{-1})_+(z) \\ &= (Y_{n+1})_-(z) \begin{bmatrix} 1 & \nu(z)/z^{n+1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & z \end{bmatrix} \begin{bmatrix} 1 & -\nu(z)/z^n \\ 0 & 1 \end{bmatrix} (Y_n^{-1})_-(z) \\ &= (Y_{n+1})_-(z) \begin{bmatrix} 1 & 0 \\ 0 & z \end{bmatrix} (Y_n^{-1})_-(z) = (T_n)_-(z), \quad z \in \mathbb{T}. \end{aligned}$$

Therefore, T_n is holomorphic in $\mathbb{C}$ and it is an entire function. Moreover, from the representation of Y_n and Y_n^{-1} given by (10) and (13), respectively, for $z \in \mathbb{C} \setminus \mathbb{T}$, we obtain

$$\begin{aligned} T_n(z) &= \begin{bmatrix} \Phi_{n+1}(z) & G_{n+1}(z) \\ -b_n \Phi_n^*(z) & -b_n G_n^*(z) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & z \end{bmatrix} \begin{bmatrix} -b_{n-1} G_{n-1}^*(z) & -G_n(z) \\ b_{n-1} \Phi_{n-1}^*(z) & \Phi_n(z) \end{bmatrix} \\ &= \begin{bmatrix} b_{n-1}(z \Phi_{n-1}^*(z) G_{n+1}(z) - \Phi_{n+1}(z) G_{n-1}^*(z)) & -\Phi_{n+1}(z) G_n(z) + z \Phi_n(z) G_{n+1}(z) \\ b_n b_{n-1} (\Phi_n^*(z) G_{n-1}^*(z) - z \Phi_{n-1}^*(z) G_n^*(z)) & b_n (\Phi_n^*(z) G_n(z) - z \Phi_n(z) G_n^*(z)) \end{bmatrix}. \end{aligned}$$

Now, using (17) and (18), and taking into account the behavior at infinity, and using Liouville’s Theorem, we get the explicit expression for T_n in (20). $\square$

As a direct consequence from (20) we obtain the recurrence relations for the entries in Y_n .

Corollary 2.1 *Let Y_n be the solution of the Riemann–Hilbert problem, then for all $n \in \mathbb{N}$,*

$$\begin{aligned}\Phi_{n+1}(z) &= (z + \bar{\alpha}_n \alpha_{n-1}) \Phi_n(z) - \frac{b_{n-1}}{b_n} \bar{\alpha}_n \Phi_{n-1}^*(z), \\ -b_n \Phi_n^*(z) &= \alpha_{n-1} b_n \Phi_n(z) - b_{n-1} \Phi_{n-1}^*(z), \\ z G_{n+1}(z) &= (z + \bar{\alpha}_n \alpha_{n-1}) G_n(z) - \frac{b_{n-1}}{b_n} \bar{\alpha}_n G_{n-1}^*(z), \\ -b_n z G_n^*(z) &= \alpha_{n-1} b_n G_n(z) - b_{n-1} G_{n-1}^*(z).\end{aligned}$$

3 Fundamental matrices from Riemann–Hilbert problem

We introduce the constant jump fundamental matrix, Z_n , which will be instrumental in what follows

$$Z_n(z) = Y_n(z) C_n(z), \quad \text{where} \quad C_n(z) = \begin{bmatrix} \frac{z^{-n/2}}{(v(z))^{-1/2}} & 0 \\ 0 & \frac{z^{n/2}}{(v(z))^{1/2}} \end{bmatrix}, \quad n \in \mathbb{N}. \quad (21)$$

Proposition 3.1 *Let Y_n be the solution of the Riemann–Hilbert problem, then*

$$\sqrt{z} Z_{n+1}(z) = T_n(z) Z_n(z), \quad n \in \mathbb{N},$$

where T_n is defined in (20).

Proof It follows from direct computations, using (20) and (21). $\square$

Now, we introduce the structure matrix, given in terms of the logarithmic derivatives of the fundamental matrix defined in (21)

$$M_n(z) = Z'_n(z) Z_n^{-1}(z), \quad n \in \mathbb{N}. \quad (22)$$

We study the analytic properties of the functions just defined, for the case of *generalized semiclassical weights* v , i.e. weights v satisfying

$$zA(z)v'(z) = q(z)v(z), \quad (23)$$

with A and q are a given polynomial and an entire function, respectively, and the differential equation is taken off the set of the zeros of the polynomial $zA(z)$, that will be called $\mathcal{Z}_A$.

Note that, when $q(z) = B(z) - zA'(z)$, where B is a polynomial, w satisfies (1) and so is a semiclassical weight.

Proposition 3.2 *Let $v(z)$ be a generalized semiclassical weight (cf. (23)), then the matrix function Z_n defined by (21) is such that:*

- i) Z_n is holomorphic in $\mathbb{C} \setminus (\mathbb{T} \cup \mathcal{Z}_A)$;
- ii) Z_n satisfies the constant jump condition

$$(Z_n)_+(t) = (Z_n)_-(t) \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad t \in \mathbb{T};$$

- iii) Z_n has the following asymptotic behavior at infinity

$$Z_n(z) = \left(I_2 + O\left(\frac{1}{z}\right) \right) \begin{bmatrix} z^{-n/2} v^{-1/2}(z) & 0 \\ 0 & z^{n/2} v^{1/2}(z) \end{bmatrix} \quad \text{as } z \rightarrow \infty.$$

Proof i) Since C_n is a matrix of entire functions in $\mathbb{C} \setminus (\mathbb{T} \cup \mathcal{Z}_A)$ and Y_n is a holomorphic matrix in $\mathbb{C} \setminus \mathbb{T}$, then Z_n is holomorphic at $\mathbb{C} \setminus (\mathbb{T} \cup \mathcal{Z}_A)$.

- ii) For $t \in \mathbb{T}$, we have

$$\begin{aligned} (Z_n)_+(t) &= (Y_n)_+(t) \begin{bmatrix} \frac{t^{-n/2}}{v^{-1/2}} & 0 \\ 0 & \frac{t^{n/2}}{v^{1/2}} \end{bmatrix} = (Y_n)_-(t) \begin{bmatrix} 1 & \frac{v(t)}{t^n} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{t^{-n/2}}{v^{-1/2}} & 0 \\ 0 & \frac{t^{n/2}}{v^{1/2}} \end{bmatrix} \\ &= (Y_n)_-(t) \begin{bmatrix} \frac{t^{-n/2}}{v^{-1/2}} & \frac{v^{1/2}}{t^{n/2}} \\ 0 & \frac{t^{n/2}}{v^{1/2}} \end{bmatrix} = (Y_n)_-(t) \begin{bmatrix} \frac{t^{-n/2}}{v^{-1/2}} & 0 \\ 0 & \frac{t^{n/2}}{v^{1/2}} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \\ &= (Z_n)_-(t) \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}. \end{aligned}$$

- iii) It follows from the asymptotic behavior of the matrix Y_n at infinity. □

Proposition 3.3 *Let $v(z)$ be a generalized semiclassical weight. Then, the corresponding structure matrix M_n , $n \in \mathbb{N}$, given in (22), is a holomorphic function in $\mathbb{C} \setminus \mathcal{Z}_A$.*

Proof Since Z_n is holomorphic in $\mathbb{C} \setminus (\mathbb{T} \cup \mathcal{Z}_A)$, then $M_n(z) = Z'_n(z)Z_n^{-1}(z)$ is also holomorphic in $\mathbb{C} \setminus (\mathbb{T} \cup \mathcal{Z}_A)$. Due to the fact that Z_n has a constant jump on $\mathbb{T} \cup \mathcal{Z}_A$, the matrix function Z'_n has the same constant jump on $\mathbb{T} \cup \mathcal{Z}_A$, so the matrix M_n has no jump on $\mathbb{T} \cup \mathcal{Z}_A$, and it follows that in each element of $\mathcal{Z}_A$, the functions M_n have an isolated singularity. □

The next proposition is very important for finding nonlinear difference equations satisfied by the Verblunsky coefficients.

Proposition 3.4 *Let $v(z)$ be a generalized semiclassical weight. Then, the corresponding structure matrix M_n , $n \in \mathbb{N}$, given in (22) satisfies*

$$T'_n - \frac{1}{2z} T_n = M_{n+1} T_n - T_n M_n, \quad n \in \mathbb{N}, \quad (24)$$

which is a zero curvature formula.

Proof From Proposition 3.1 we know that $\sqrt{z} Z_{n+1} = T_n Z_n$, then taking derivative and again using Proposition 3.1, we obtain

$$\left\{ T'_n - \frac{1}{2z} T_n + T_n M_n - M_{n+1} T_n \right\} Z_n = \mathbf{0}_2.$$

Therefore, since Z_n is invertible, the result follows. $\square$

We also have a kind of reciprocal of Proposition 3.4.

Proposition 3.5 *Let v be a Hölder type weight defined on $\mathbb{T}$, with sequences of transfer matrices $\{T_n\}_{n \in \mathbb{N}}$ and constant jump fundamental matrices $\{Z_n\}_{n \in \mathbb{N}}$. If M_n , $n \in \mathbb{N}$, satisfies (24), with $M_0(z) := Z'_0(z) Z_0^{-1}(z)$. Then, $\{M_n\}_{n \in \mathbb{N}}$ satisfies (22).*

Proof Multiplying (24) by Z_n , and taking into account that $\sqrt{z} Z_{n+1} = T_n Z_n$, we arrive to

$$\sqrt{z} (Z'_{n+1} - M_{n+1} Z_{n+1}) = T_n (Z'_n - M_n Z_n), \quad n \in \mathbb{N}.$$

Multiplying by $(\sqrt{z})^n$ and iterating the procedure we arrive to

$$(\sqrt{z})^{n+1} (Z'_{n+1} - M_{n+1} Z_{n+1}) = T_n T_{n-1} \cdots T_0 (Z'_0 - M_0 Z_0),$$

and taking into account our hypothesis we get that (22) takes place. $\square$

Next, we state a second curvature formula.

Proposition 3.6 *Let $v(z)$ be a generalized semiclassical weight on $\mathbb{T}$. Then, the corresponding structure matrix M_n , $n \in \mathbb{N}$, given in (22) satisfies*

$$M_{n+1} \left\{ T'_n - \frac{1}{2z} T_n \right\} + \left\{ T'_n - \frac{1}{2z} T_n \right\} M_n = M_{n+1}^2 T_n - T_n M_n^2.$$

Proof Multiplying on the left of (24) by M_{n+1} , we get

$$M_{n+1} T'_n - M_{n+1} \frac{1}{2z} T_n = M_{n+1}^2 T_n - M_{n+1} T_n M_n \quad (25)$$

and on the other hand, multiplying on the right by M_n we obtain

$$T'_n M_n - M_{n+1} \frac{1}{2z} T_n M_n = M_{n+1}^2 T_n M_n - M_{n+1} T_n M_n^2. \quad (26)$$

Adding (25) and (26), we get the desired equation. $\square$

Now, we will derive a second order differential operator for the sequence of matrix function $\{Y_n\}_{n \in \mathbb{N}}$ associated with a generalized semiclassical weight.

Proposition 3.7 *Let ν be a generalized semiclassical weight on $\mathbb{T}$. Then $\{Y_n\}_{n \in \mathbb{N}}$ satisfies*

$$Y_n'' + 2Y_n' C_n' C_n^{-1} + Y_n C_n'' C_n^{-1} = (M_n' + M_n^2) Y_n. \quad (27)$$

Proof Taking derivative on (22), i.e. $Z_n' Z_n^{-1} = M_n$ we arrive to

$$Z_n'' Z_n^{-1} - (Z_n' Z_n^{-1})(Z_n' Z_n^{-1}) = M_n'.$$

Again, using (22), we get

$$Z_n'' = (M_n' + M_n^2) Z_n. \quad (28)$$

By definition, $Z_n = Y_n C_n$, and so

$$Z_n' = Y_n' C_n + Y_n C_n', \quad Z_n'' = Y_n'' C_n + 2Y_n' C_n' + Y_n C_n''.$$

Ploughing this into (28) we get the desired result. $\square$

The second order differential equation in Proposition 3.7, given for $\{Y_n\}_{n \in \mathbb{N}}$, could trace back to a first order one for the $\{Z_n\}_{n \in \mathbb{N}}$, i.e. (22).

Proposition 3.8 *Let ν be a Hölder type weight defined on $\mathbb{T}$. If $\{Y_n\}_{n \in \mathbb{N}}$ satisfies (27), then $\{Z_n\}_{n \in \mathbb{N}}$ satisfies*

$$Z_n' = -T_n^{-1}(-z) \left\{ z((M_{n+1}' + M_{n+1}^2)T_n - T_n(M_n' + M_n^2)) + T_n' - \frac{3}{4z}T_n \right\} Z_n, \quad n \in \mathbb{N}. \quad (29)$$

Proof Multiplying equation (27) by C_n , and taking into account (21) we get that (28) takes place. Rewriting (28) in $n + 1$ we see that

$$Z_{n+1}'' = (M_{n+1}' + M_{n+1}^2) Z_{n+1}.$$

By Proposition 3.1 we know that $Z_{n+1} = z^{-1/2} T_n Z_n$, hence substituting this into the last equation, and making some simplifications we arrive to

$$\frac{3}{4} T_n Z_n - z(T_n' Z_n + T_n Z_n') + z^2(2T_n' Z_n' + T_n Z_n'') = z^2(M_{n+1}' + M_{n+1}^2) T_n Z_n.$$

Now, applying our hypothesis, i.e. $Z_n'' = (M_n' + M_n^2) Z_n$ we get

$$(-T_n + 2zT_n') Z_n' = \left\{ z((M_{n+1}' + M_{n+1}^2)T_n - T_n(M_n' + M_n^2)) + T_n' - \frac{3}{4z}T_n \right\} Z_n,$$

and taking into account that $-T_n(z) + 2zT_n'(z) = -T_n(-z)$ we arrive to (29). $\square$

4 Examples

As we have seen in Section 3 the matrix functions M_n are a key to understanding generalized semiclassical orthogonal polynomials. When A is equal to z or $z - 1$ the weight function is of the type

$$v(z) = (1 - z)^\alpha z^\beta e^{-\gamma/z} H(z),$$

for some constants $\alpha, \beta, \gamma \in \mathbb{C}$ and where H is an entire function.

Here we will take the generalized modified Bessel and generalized modified Jacobi as a case study.

4.1 Modified bessel

Here we consider the generalized semiclassical weight function

$$w(\theta) = e^{\ell \cos(\theta)} H(e^{i\theta}) \quad \text{and} \quad v(z) = e^{\ell(z+z^{-1})/2} H(z), \quad z = e^{i\theta}, \quad \theta \in [0, 2\pi],$$

where $\ell > 0$ is a real parameter. In the case when $H(z) = 1$, the Verblunsky coefficients are real and depending of ℓ . This is the weight function related to the modified Bessel polynomials [21].

We know from Proposition 3.2 that the matrix Z_n defined in (21) related to the generalized modified Bessel weight function is analytic in $\mathbb{C} \setminus (\mathbb{T} \cup \{0\})$, as $\mathcal{Z}_A = \{0\}$. Furthermore, from Proposition 3.3, we know that the corresponding structure matrix M_n , $n \in \mathbb{N}$, given in (22), is a holomorphic function in $\mathbb{C} \setminus \mathcal{Z}_A$.

Now we will study the analytic character for the structure matrix associated with orthogonal polynomials in relation to the *generalized modified Bessel* weight function.

Proposition 4.1 *Let $v(z) = e^{\ell(z+z^{-1})/2} H(z)$, where H is an entire function. Then, the corresponding structure matrix M_n , $n \in \mathbb{N}$, given in (22), is a holomorphic function in $\mathbb{C} \setminus \{0\}$ with a pole of order 2 at $z = 0$.*

Proof Note that

$$M_n(z) = Z'_n(z)Z_n^{-1}(z) = Y'_n(z)Y_n^{-1}(z) + Y_n(z)C'_n(z)C_n^{-1}(z)Y_n^{-1}(z).$$

Multiplying both sides of the last equation by z^2 , we get

$$z^2 M_n(z) = z^2 Y'_n(z)Y_n^{-1}(z) + z^2 Y_n(z)C'_n(z)C_n^{-1}(z)Y_n^{-1}(z).$$

Therefore, using (16), we obtain

$$\lim_{z \rightarrow 0} z^2 M_n(z) = \begin{bmatrix} \frac{\ell}{4}(b_{n-1}b_n^{-1} - \alpha_{n-1}^2) & -\frac{\ell}{2}b_n^{-1}\alpha_{n-1} \\ -\frac{\ell}{2}b_{n-1}\alpha_{n-1} & -\frac{\ell}{4}(b_{n-1}b_n^{-1} - \alpha_{n-1}^2) \end{bmatrix} \neq \mathbf{0}_2,$$

where $\mathbf{0}_2$ is the zero matrix. It follows that M_n has a pole of order two at $z = 0$. $\square$

In the hypothesis of the Proposition 4.1 we have that $\tilde{M}_n = z^2 M_n$, is, for each $n \in \mathbb{N}$, an entire matrix function. Now, we will explicitly determine the $\tilde{M}_n$ just defined for the case when $H(z) = 1$.

We know that $\tilde{M}_n(z) = z^2 Y'_n(z) Y_n^{-1}(z) + z^2 Y_n(z) C'_n(z) C_n^{-1}(z) Y_n^{-1}(z)$. From (10), (13), and (21), and using (17) and (18), we obtain the asymptotic behavior for $z \rightarrow \infty$,

$$\tilde{M}_n(z) = \begin{bmatrix} \frac{\ell z^2}{4} + \frac{nz}{2} - \frac{b_{n-1}b_n^{-1}\alpha_n\alpha_{n-2} + \ell + 4\Phi_1^n}{4} & \frac{\alpha_n}{b_n}(\frac{\ell z}{2} + n + 1 + \frac{\ell}{2}(\Phi_1^n - 1)) + \frac{\ell\alpha_{n-1}}{2b_{n+1}} \\ b_{n-1}((\frac{\ell z}{2} + n - 1 - \frac{\ell}{2}\Phi_1^n)\alpha_{n-2} + \overline{\Phi_1^{n-1}}) - \frac{\ell z^2}{4} - \frac{nz}{2} - \frac{b_{n-1}b_n^{-1}\alpha_n\alpha_{n-2} - \ell - 4\Phi_1^n}{4} \end{bmatrix} + O(z^{-1}).$$

Therefore, since $\tilde{M}_n$ is an entire matrix function, the Liouville Theorem implies that

$$\tilde{M}_n(z) = \begin{bmatrix} \frac{\ell z^2}{4} + \frac{nz}{2} - \frac{1}{4}(b_{n-1}b_n^{-1}\alpha_n\alpha_{n-2} + \ell + 4\Phi_1^n) & b_n^{-1}\frac{\ell}{2}\alpha_n z + b_n^{-1}\alpha_n(n + 1 + \frac{\ell}{2}(\Phi_1^n - 1)) + \frac{\ell}{2}b_{n+1}^{-1}\alpha_{n-1} \\ b_{n-1}(\frac{\ell}{2}\alpha_{n-2}z + b_{n-1}(n - 1 - \frac{\ell}{2}\Phi_1^n)\alpha_{n-2} + \overline{\Phi_1^{n-1}}) & -\frac{\ell}{4}z^2 - \frac{nz}{2} - \frac{1}{4}(b_{n-1}b_n^{-1}\alpha_n\alpha_{n-2} - \ell - 4\Phi_1^n) \end{bmatrix}. \quad (30)$$

Note that $\tilde{M}_n$ in (30) is a polynomial function of degree 2, that is, we can write it as $\tilde{M}_n(z) = \tilde{F}_n^2 z^2 + \tilde{F}_n^1 z + \tilde{F}_n^0$, where $\tilde{F}_n^2$, $\tilde{F}_n^1$, and $\tilde{F}_n^0$ are constants and $\lim_{z \rightarrow 0} \tilde{M}_n(z) = \tilde{F}_n^0$. Therefore, using the Lemma 2.1, we get

$$\tilde{F}_n^0 = \begin{bmatrix} \frac{\ell}{4}(b_{n-1}b_n^{-1} - \alpha_{n-1}^2) & -\frac{\ell}{2}b_n^{-1}\alpha_{n-1} \\ -\frac{\ell}{2}b_{n-1}\alpha_{n-1} & -\frac{\ell}{4}(b_{n-1}b_n^{-1} - \alpha_{n-1}^2) \end{bmatrix}.$$

Moreover, from (30), we have

$$\tilde{F}_n^1 = \begin{bmatrix} \frac{n}{2} & \frac{\ell}{2}b_n^{-1}\alpha_n \\ \frac{\ell}{2}b_{n-1}\alpha_{n-2} & -\frac{n}{2} \end{bmatrix} \quad \text{and} \quad \tilde{F}_n^2 = \begin{bmatrix} \frac{\ell}{4} & 0 \\ 0 & -\frac{\ell}{4} \end{bmatrix}.$$

Therefore, when $z \rightarrow 0$, we arrive to

$$\tilde{M}_n(z) = \begin{bmatrix} \frac{\ell}{4}z^2 + \frac{n}{2}z + \frac{\ell}{4}(b_{n-1}b_n^{-1} - \alpha_{n-1}^2) & -\frac{\ell}{2}b_n^{-1}(\alpha_{n-1} - \alpha_n z) \\ -\frac{\ell}{2}b_{n-1}(\alpha_{n-1} - \alpha_{n-2}z) & -\frac{\ell}{4}z^2 - \frac{n}{2}z - \frac{\ell}{4}(b_{n-1}b_n^{-1} - \alpha_{n-1}^2) \end{bmatrix}. \quad (31)$$

The zero curvature formula is important for finding nonlinear difference equations satisfied by Verblunsky coefficients. As a consequence of Proposition 3.4 we can see that the matrix function, $\tilde{M}_n$, defined in (31) satisfies

$$z^2 T'_n + T_n \tilde{M}_n - \frac{z}{2} T_n - \tilde{M}_{n+1} T_n = \mathbf{0}_2, \quad n \in \mathbb{N}. \quad (32)$$

In fact, we only have to multiply equation (24) by z^2 and using the fact that $\tilde{M}_n = z^2 M_n$, $n \in \mathbb{N}$. As an easy consequence of (32) we get that

$$\alpha_n(\ell) + \alpha_{n-2}(\ell) = -\frac{2n}{\ell} \frac{\alpha_{n-1}(\ell)}{1 - \alpha_{n-1}^2(\ell)}, \quad n \geq 2, \quad \ell \in \mathbb{R}. \quad (33)$$

The (33) corresponds to the discrete Painlevé equation (dP_{II}).

Now, we derive the differential properties for the functions Φ_n , Φ_n^* , G_n and G_n^* coming from the Riemann–Hilbert problem associated with the modified Bessel weight. In fact, multiplying by z^2 , equation (22) we obtain

$$z^2 Z'_n(z) = \tilde{M}_n(z) Z_n(z), \quad n \in \mathbb{N}.$$

Taking into account (21) we arrive to

$$z^2 Y'_n = \tilde{M}_n Y_n - z^2 Y_n C'_n C_n^{-1}, \quad n \in \mathbb{N}. \quad (34)$$

The first order matrix differential (34) splits into the following differential relations

$$\begin{aligned} z^2 \Phi'_n(z) &= \left\{ nz + \frac{\ell}{2} - \frac{\ell}{2} \alpha_{n-1}^2 \right\} \Phi_n(z) + \frac{\ell}{2} b_{n-1} b_n^{-1} (\alpha_{n-1} - \alpha_n z) \Phi_{n-1}^*(z), \\ z^2 G'_n(z) &= \left\{ \frac{\ell}{2} z^2 - \frac{\ell}{2} \alpha_{n-1}^2 \right\} G_n(z) + \frac{\ell}{2} b_{n-1} b_n^{-1} (\alpha_{n-1} - \alpha_n z) G_{n-1}^*(z), \\ z^2 (\Phi_{n-1}^*)'(z) &= \frac{\ell}{2} (\alpha_{n-1} - \alpha_{n-2} z) \Phi_n(z) + \left\{ -\frac{\ell}{2} z^2 + \frac{\ell}{2} \alpha_{n-1}^2 \right\} \Phi_{n-1}^*(z), \\ z^2 (G_{n-1}^*)'(z) &= \frac{\ell}{2} (\alpha_{n-1} - \alpha_{n-2} z) G_n(z) + \left\{ -nz - \frac{\ell}{2} + \frac{\ell}{2} \alpha_{n-1}^2 \right\} G_{n-1}^*(z). \end{aligned} \quad (35)$$

Note that with some manipulations, it can be shown that (35) is equivalent to the structure relation

$$z \Phi'_n(z) = n \Phi_n(z) + \frac{\ell}{2} \frac{\kappa_{n-1}^2}{\kappa_n^2} (\Phi_{n-1}(z) - \bar{\alpha}_n \Phi_{n-1}^*(z)) \quad (36)$$

presented in [3]. Furthermore, doing $z = 0$ in (36) and some more calculations, it can be shown that (35) coincides with (2).

Applying Proposition 3.7, we can derive second order differential equation for the matrix Y_n . In fact, we only have to multiply (27) by z^2 , to obtain

$$z^2 Y''_n + 2Y'_n(z^2 C'_n C_n^{-1}) + Y_n(z^2 C''_n C_n^{-1}) = \left(z^2 M'_n + \frac{\tilde{M}_n^2}{z^2} \right) Y_n.$$

We can see that $z^2 M'_n = \tilde{M}'_n - \frac{2}{z} \tilde{M}_n$ and

$$\tilde{M}_n = z^2 Z'_n Z_n^{-1} = z^2 (Y'_n Y_n^{-1} + Y_n C'_n C_n^{-1} Y_n^{-1}).$$

Substituting this into the last equation we arrive to

$$z^2 Y''_n + 2Y'_n(z^2 C'_n C_n^{-1} + z I_2) + Y_n(z^2 C''_n C_n^{-1} + 2z C'_n C_n^{-1}) = \left(\tilde{M}'_n + \frac{\tilde{M}_n^2}{z^2} \right) Y_n. \quad (37)$$

The second order matrix differential (37) splits into the following differential relations

$$\begin{aligned} z^2 \Phi''_n(z) + \left(\frac{\ell}{2} z^2 + (2-n)z - \frac{\ell}{2} \right) \Phi'_n(z) + \left(-\frac{\ell n}{2} z - \frac{\ell^2}{4} - n \right. \\ \left. - \frac{\ell^2}{4} ((1 - \alpha_{n-1}^2) \alpha_n \alpha_{n-2} - \alpha_{n-1}^2) \right) \Phi_n(z) = -\frac{\ell}{2} (1 - \alpha_{n-1}^2) \alpha_n \Phi_{n-1}^*(z), \\ z^2 G''_n(z) + \left(-\frac{\ell z^2}{2} + (n+2)z + \frac{\ell}{2} \right) G'_n(z) - \left(\ell \left(\frac{n}{2} + 1 \right) z + \frac{\ell^2}{4} \right. \\ \left. + \frac{\ell^2}{4} ((1 - \alpha_{n-1}^2) \alpha_n \alpha_{n-2} - \alpha_{n-1}^2) \right) G_n(z) = -\frac{\ell}{2} (1 - \alpha_{n-1}^2) \alpha_n G_{n-1}^*(z), \\ z^2 (\Phi_{n-1}^*)''(z) + \left(\frac{\ell z^2}{2} + (2-n)z - \frac{\ell}{2} \right) (\Phi_{n-1}^*)'(z) - \left(\ell \left(\frac{n}{2} - 1 \right) z + \frac{\ell^2}{4} \right. \\ \left. + \frac{\ell^2}{4} ((1 - \alpha_{n-1}^2) \alpha_n \alpha_{n-2} - \alpha_{n-1}^2) \right) \Phi_{n-1}^*(z) = -\frac{\ell}{2} \alpha_{n-2} \Phi_n(z), \\ z^2 (G_{n-1}^*)''(z) + \left(-\frac{\ell z^2}{2} + (n+2)z + \frac{\ell}{2} \right) (G_{n-1}^*)'(z) - \left(\frac{\ell n}{2} z + \frac{\ell^2}{4} - n \right. \\ \left. + \frac{\ell^2}{4} ((1 - \alpha_{n-1}^2) \alpha_n \alpha_{n-2} - \alpha_{n-1}^2) \right) G_{n-1}^*(z) = -\frac{\ell}{2} \alpha_{n-2} G_n(z). \end{aligned}$$

To get these second order differential relations we only have to substitute the representation of $\tilde{M}_n$ given in (31) into (37). In fact, from (31), we see that

$$\begin{aligned} \frac{\tilde{M}_n^2}{z^2} &= \left(\frac{\ell^2}{16} z^2 + \frac{\ell n}{4} z + \frac{\ell^2}{8} + \frac{n^2}{4} - \frac{\ell^2}{4} \alpha_{n-1}^2 + \frac{\ell^2}{4} (1 - \alpha_{n-1}^2) \alpha_{n-2} \alpha_n + \frac{\ell n}{4z} + \frac{\ell^2}{16z^2} \right) I_2, \\ \tilde{M}'_n &= \begin{bmatrix} \frac{\ell}{2} z + \frac{n}{2} & \frac{\ell}{2} b_n^{-1} \alpha_n \\ \frac{\ell}{2} b_{n-1} \alpha_{n-2} & -\frac{\ell}{2} z - \frac{n}{2} \end{bmatrix}. \end{aligned}$$

Furthermore, taking into account the definition of C_n in (21), we have

$$\begin{aligned} z^2 C_n'' C_n^{-1} + 2z C_n' C_n^{-1} &= \left(\frac{\ell^2}{16} z^2 - \frac{\ell n}{4} z - \frac{\ell^2}{8} + \frac{n^2}{4} + \frac{\ell n}{4z} + \frac{\ell^2}{16z^2} \right) \mathbf{I}_2 + \begin{bmatrix} \frac{\ell}{2} z - \frac{n}{2} & 0 \\ 0 & -\frac{\ell}{2} z + \frac{n}{2} \end{bmatrix}, \\ z^2 C_n' C_n^{-1} + z \mathbf{I}_2 &= \begin{bmatrix} \frac{\ell}{4} z^2 + (1 - \frac{n}{2})z - \frac{\ell}{4} & 0 \\ 0 & -\frac{\ell}{4} z^2 + (1 + \frac{n}{2})z + \frac{\ell}{4} \end{bmatrix}. \end{aligned}$$

By substituting all these matrices into (37), we obtain the desired equations.

4.2 Jacobi modified

We consider here a generalization of the semiclassical weight function, studied in [27], namely

$$w(\theta) = \tau(b) e^{-\eta\theta} \left(\sin^2(\theta/2) \right)^\lambda H(e^{i\theta}), \quad \theta \in [0, 2\pi],$$

where H is an entire function, and

$$\eta \in \mathbb{R}, \quad \lambda > -1/2, \quad b = \lambda + i\eta, \quad \text{and} \quad \tau(b) = \frac{e^{\pi\eta} 2^{b+\bar{b}} |\Gamma(b+1)|^2}{2\pi \Gamma(b+\bar{b}+1)}.$$

Since $z = e^{i\theta}$, we can write the weight function as

$$\nu(z) = \frac{\tau(b)}{2^{b+\bar{b}}} (-z)^{-\bar{b}} (1-z)^{b+\bar{b}} H(z), \quad z \in \mathbb{T}.$$

For simplicity, in this work we consider $\nu(z) = (-z)^{-\bar{b}} (1-z)^{b+\bar{b}} H(z)$, $z \in \mathbb{T}$, which coincides up to a constant with the above weight.

We will study the analytic character for the structure matrix associated with these *generalized modified Jacobi* orthogonal polynomials.

Proposition 4.2 *Let $\nu(z) = (-z)^{-\bar{b}} (1-z)^{b+\bar{b}} H(z)$, then the structure matrix M_n , $n \in \mathbb{N}$, is a holomorphic function in $\mathbb{C} \setminus \{0, 1\}$ with a simple pole at $z = 0$ and a removable singularity or a simple pole at $z = 1$.*

Proof From Proposition 3.3, it follows that M_n has isolated singularities at $z = 0$ and $z = 1$. Multiplying M_n by z we have

$$zM_n = zY_n' Y_n^{-1} + zY_n C_n' C_n^{-1} Y_n^{-1}.$$

Note that, using (16), we get

$$\lim_{z \rightarrow 0} zM_n(z) = \begin{bmatrix} -\frac{1}{2}(\bar{b}+n)(2|\alpha_{n-1}|^2 - 1) & -b_n^{-1}(\bar{b}+n)\bar{\alpha}_{n-1} \\ -b_{n-1}(\bar{b}+n)\alpha_{n-1} & \frac{1}{2}(\bar{b}+n)(2|\alpha_{n-1}|^2 - 1) \end{bmatrix} \neq \mathbf{0}_2.$$

Therefore, by direct calculation, we can state that M_n has a simple pole at $z = 0$.

To prove the statement related to the singularity at the point $z = 1$, it is sufficient to prove that $\lim_{z \rightarrow 1} (z - 1)^2 M(z) = \mathbf{0}_2$. At the point $z = 1$, using [20, Ch. 8.3] and [1, Lemma 7.2.2], we can state that

$$G_n(z) = \begin{cases} O(1), & b + \bar{b} = \beta + i\gamma, \quad \beta > 0, \\ O(\log(z - 1)), & b + \bar{b} = i\gamma, \\ O((z - 1)^\beta), & b + \bar{b} = \beta + i\gamma, \quad -1 < \beta < 0, \end{cases}$$

and analogously

$$G_n^*(z) = \begin{cases} O(1), & b + \bar{b} = \beta + i\gamma, \quad \beta > 0, \\ O(\log(z - 1)), & b + \bar{b} = i\gamma, \\ O((z - 1)^\beta), & b + \bar{b} = \beta + i\gamma, \quad -1 < \beta < 0. \end{cases}$$

It also follows that $G'_n(z) = \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{\Phi_n(t)}{(t - z)^2} \frac{v(t)}{t^n} dt$. So, taking into account that

$$\begin{aligned} (z - 1)G'_n(z) &= \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{(z - 1)\Phi_n(t)}{(t - z)^2} \frac{v(t)}{t^n} dt \\ &= -\frac{1}{2\pi i} \int_{\mathbb{T}} \frac{\Phi_n(t)}{t - z} \frac{v(t)}{t^n} dt + \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{(t - 1)\Phi_n(t)}{(t - z)^2} \frac{v(t)}{t^n} dt, \end{aligned}$$

and using integration by parts, and again the behavior of the Cauchy integral at the power singularity $(1 - z)^{b+\bar{b}}$, we get that

$$(z - 1)G'_n(z) = \begin{cases} O(1), & b + \bar{b} = \beta + i\gamma, \quad \beta > 0, \\ O(\log(z - 1)), & b + \bar{b} = i\gamma, \\ O((z - 1)^\beta), & b + \bar{b} = \beta + i\gamma, \quad -1 < \beta < 0. \end{cases}$$

Analogously,

$$(z - 1)G_n^{*'}(z) = \begin{cases} O(1), & b + \bar{b} = \beta + i\gamma, \quad \beta > 0, \\ O(\log(z - 1)), & b + \bar{b} = i\gamma, \\ O((z - 1)^\beta), & b + \bar{b} = \alpha + i\gamma, \quad -1 < \beta < 0. \end{cases}$$

So, it holds in any case that

$$\begin{aligned} \lim_{z \rightarrow 1} (z - 1)G_n(z) &= 0, & \lim_{z \rightarrow 1} (z - 1)G_n^*(z) &= 0, \\ \lim_{z \rightarrow 1} (z - 1)^2 G_n'(z) &= 0, & \lim_{z \rightarrow 1} (z - 1)^2 G_n^{*'}(z) &= 0, \end{aligned}$$

and, using that $\det Y_n = 1$, we arrive to

$$Y'_n(z) = \begin{bmatrix} O(1) & o(\frac{1}{(z-1)^2}) \\ O(1) & o(\frac{1}{(z-1)^2}) \end{bmatrix}, \quad Y_n^{-1}(z) = \begin{bmatrix} o((z-1)^{-1}) & o((z-1)^{-1}) \\ O(1) & O(1) \end{bmatrix}, \quad z \rightarrow 1.$$

From these we successively get that

$$\begin{aligned} & \lim_{z \rightarrow 1} (z-1)^2 (Y_n)' (Y_n)^{-1} \\ &= \lim_{z \rightarrow 1} (z-1)^2 \begin{bmatrix} o((z-1)^{-1}) + o((z-1)^{-2}) & o((z-1)^{-1}) + o((z-1)^{-2}) \\ o((z-1)^{-2}) + o((z-1)^{-1}) & o((z-1)^{-2}) + o((z-1)^{-1}) \end{bmatrix} \\ &= \lim_{z \rightarrow 1} (z-1)^2 \begin{bmatrix} o((z-1)^{-2}) & o((z-1)^{-2}) \\ o((z-1)^{-2}) & o((z-1)^{-2}) \end{bmatrix} = \mathbf{0}_2, \end{aligned}$$

and

$$\begin{aligned} & \lim_{z \rightarrow 1} (z-1)^2 Y_n C'_n C_n^{-1} Y_n^{-1} \\ &= \lim_{z \rightarrow 1} (z-1)^2 \begin{bmatrix} O(1) & o((z-1)^{-1}) \\ O(1) & o((z-1)^{-1}) \end{bmatrix} \begin{bmatrix} -\frac{n+\bar{b}}{2z} + \frac{b+\bar{b}}{2(z-1)} + O(1) & 0 \\ 0 & \frac{n+\bar{b}}{2z} - \frac{b+\bar{b}}{2(z-1)} + O(1) \end{bmatrix} \\ & \quad \begin{bmatrix} o((z-1)^{-1}) & o((z-1)^{-1}) \\ O(1) & O(1) \end{bmatrix} \\ &= \lim_{z \rightarrow 1} (z-1)^2 \begin{bmatrix} O((z-1)^{-1}) o((z-1)^{-1}) & O((z-1)^{-1}) o((z-1)^{-1}) \\ O((z-1)^{-1}) o((z-1)^{-1}) & O((z-1)^{-1}) o((z-1)^{-1}) \end{bmatrix} = \mathbf{0}_2. \end{aligned}$$

By combining previous results, we obtain $\lim_{z \rightarrow 1} (z-1)^2 M_n = \mathbf{0}_2$ as we wanted to prove. $\square$

In the hypothesis of the Proposition 4.2, we have that $\tilde{M}_n = z(1-z)M_n$ is an entire matrix function. Now, we will explicitly determine the $\tilde{M}_n$ just defined when $H(z) = 1$. We know that

$$z(1-z)M_n = z(1-z)Y'_n Y_n^{-1} + z(1-z)Y_n C'_n C_n^{-1} Y_n^{-1}.$$

From (10), (13), and (21), then by (17) and (18), we obtain the following asymptotic behavior about infinity,

$$\tilde{M}_n(z) = \begin{bmatrix} -\frac{(b+n)z + (\bar{b}-n)}{2} + \Phi_1^n & -(b+n+1)b_n^{-1}\bar{\alpha}_n \\ -(b+n-1)b_{n-1}\alpha_{n-2} & \frac{(b+n)z + (\bar{b}-n)}{2} - \Phi_1^n \end{bmatrix} + O(z^{-1}).$$

Using the Liouville Theorem, it follows that

$$\tilde{M}_n(z) = \begin{bmatrix} -\frac{1}{2}((b+n)z + (\bar{b}-n)) + \Phi_1^n & -(b+n+1)b_n^{-1}\bar{\alpha}_n \\ -(b+n-1)b_{n-1}\alpha_{n-2} & \frac{1}{2}((b+n)z + (\bar{b}-n)) - \Phi_1^n \end{bmatrix}. \quad (38)$$

Note that $\tilde{M}_n(z) = \tilde{F}_n^1 z + \tilde{F}_n^0$, where $\tilde{F}_n^1$ and $\tilde{F}_n^0$ are constants and $\lim_{z \rightarrow 0} z(1-z)M_n = \tilde{F}_n^0$. Therefore, applying the Lemma 2.1, we get

$$\tilde{F}_n^0 = \begin{bmatrix} -\frac{1}{2}(\bar{b} + n)(2|\alpha_{n-1}|^2 - 1) & -b_n^{-1}(\bar{b} + n)\bar{\alpha}_{n-1} \\ -b_{n-1}(\bar{b} + n)\alpha_{n-1} & \frac{1}{2}(\bar{b} + n)(2|\alpha_{n-1}|^2 - 1) \end{bmatrix}.$$

Moreover, from $\tilde{M}_n$ in (38), we have

$$\tilde{F}_n^1 = \begin{bmatrix} -\frac{1}{2}(b + n) & 0 \\ 0 & \frac{1}{2}(b + n) \end{bmatrix},$$

and so we arrive to

$$\tilde{M}_n(z) = \begin{bmatrix} -\frac{(b+n)z + (\bar{b}+n)(2|\alpha_{n-1}|^2 - 1)}{2} & -b_n^{-1}(\bar{b}+n)\bar{\alpha}_{n-1} \\ -b_{n-1}(\bar{b}+n)\alpha_{n-1} & \frac{(b+n)z + (\bar{b}+n)(2|\alpha_{n-1}|^2 - 1)}{2} \end{bmatrix}. \quad (39)$$

It is easy to see that the residue matrix of M_n at the pole $z = 1$ is given by

$$\lim_{z \rightarrow 1} (z-1)M_n(z) = \begin{bmatrix} \frac{(b+n) + (\bar{b}+n)(2|\alpha_{n-1}|^2 - 1)}{2} & b_n^{-1}(\bar{b}+n)\bar{\alpha}_{n-1} \\ b_{n-1}(\bar{b}+n)\alpha_{n-1} & -\frac{(b+n) + (\bar{b}+n)(2|\alpha_{n-1}|^2 - 1)}{2} \end{bmatrix}.$$

Moreover, comparing the (1, 1) and (1, 2) entries in the representation of $\tilde{M}_n$ given by (39) and (38), we obtain

$$\Phi_1^n = \bar{b} - (\bar{b} + n)|\alpha_{n-1}|^2 \quad \text{and} \quad \alpha_n = \frac{b + n}{\bar{b} + n + 1} \alpha_{n-1}, \quad n \in \mathbb{N}. \quad (40)$$

The expression (40) for the α_n was obtained in [27].

Applying Proposition 3.4, we get a zero curvature formula. In fact, multiplying (24) by $z(1-z)$, we arrive to

$$z(1-z)T'_n + T_n \tilde{M}_n - \frac{1-z}{2} T_n - \tilde{M}_{n+1} T_n = \mathbf{0}_2, \quad n \in \mathbb{N},$$

and continue as in Proposition 3.6 we get

$$\tilde{M}_{n+1} \left\{ z(1-z)T'_n - \frac{(1-z)}{2} T_n \right\} + \left\{ z(1-z)T'_n - \frac{(1-z)}{2} T_n \right\} \tilde{M}_n = \tilde{M}_{n+1}^2 T_n - T_n \tilde{M}_n^2.$$

We are interested in the differential equations fulfilled by the orthogonal polynomials on the unit circle. Here, we use the Riemann–Hilbert problem approach in order to derive these differential relations.

Multiplying (22) by $z(1-z)$ we get $z(1-z)Z'_n(z) = \tilde{M}_n Z_n(z)$, and from this, we get

$$z(1-z)Y'_n = \tilde{M}_n Y_n - z(1-z)Y_n C'_n C_n^{-1},$$

which entrywise reads as:

$$\begin{aligned} z(1-z)\Phi'_n(z) &= (-nz + (\bar{b} + n)(1 - |\alpha_{n-1}|^2))\Phi_n(z) + (\bar{b} + n)(1 - |\alpha_{n-1}|^2)\bar{\alpha}_{n-1}\Phi_{n-1}^*(z), \\ z(1-z)G'_n(z) &= (-bz - (\bar{b} + n)|\alpha_{n-1}|^2)G_n(z) + (\bar{b} + n)(1 - |\alpha_{n-1}|^2)\bar{\alpha}_{n-1}G_{n-1}^*(z), \\ z(1-z)(\Phi_{n-1}^*)'(z) &= (bz + (\bar{b} + n)|\alpha_{n-1}|^2)\Phi_{n-1}^*(z) + (\bar{b} + n)\alpha_{n-1}\Phi_n(z), \\ z(1-z)(G_{n-1}^*)'(z) &= (nz - (\bar{b} + n)(1 - |\alpha_{n-1}|^2))G_{n-1}^*(z) + (\bar{b} + n)\alpha_{n-1}G_n(z). \end{aligned} \quad (41)$$

Note that with some manipulations, it can be shown that (41) is equivalent to the structure relation

$$(z-1)\Phi'_n(z) = -(\bar{b} + n)(1 - |\alpha_{n-1}|^2)\Phi_{n-1}(z) + n\Phi_n(z)$$

presented in [3].

Applying Proposition 3.7, we can derive second order differential equation for the matrix Y_n . In fact, we only have to multiply (27) by $z(1-z)$, to obtain

$$z(1-z)Y''_n + 2Y'_n(z(1-z)C'_n C_n^{-1}) + Y_n(z(1-z)C''_n C_n^{-1}) = \left(z(1-z)M'_n + \frac{\tilde{M}_n^2}{z(1-z)}\right)Y_n.$$

We can see that $z(1-z)M'_n = \tilde{M}'_n - \frac{1}{z}\tilde{M}_n + \frac{1}{1-z}\tilde{M}_n$ and

$$\tilde{M}_n = z(1-z)Z'_n Z_n^{-1} = z(1-z)(Y'_n Y_n^{-1} + Y_n C'_n C_n^{-1} Y_n^{-1}).$$

Substituting this into the last equation we arrive to

$$\begin{aligned} z(1-z)Y''_n + Y'_n[2z(1-z)C'_n C_n^{-1} + (1-2z)I_2] \\ + Y_n[z(1-z)C''_n C_n^{-1} + (1-2z)C'_n C_n^{-1}] = \left(\tilde{M}'_n + \frac{\tilde{M}_n^2}{z(1-z)}\right)Y_n. \end{aligned} \quad (42)$$

Now, we will split (42) in four second order differential equations for $\Phi_n, \Phi_n^*, G_n, G_n^*$. From the representation of $\tilde{M}_n$ given in (39), we get

$$\begin{aligned} \tilde{M}_n^2 &= (z|\alpha_{n-1}|^2(n+b)(\bar{b}+n) + \frac{1}{4}(\bar{b}-z(n+b)+n)^2)I_2, \\ \tilde{M}'_n &= \begin{bmatrix} -\frac{n+b}{2} & 0 \\ 0 & \frac{n+b}{2} \end{bmatrix}, \end{aligned}$$

and by definition of C_n in (21), we have

$$\begin{aligned} 2z(1-z)C'_n C_n^{-1} + (1-2z)I_2 &= \begin{bmatrix} 1-n(1-z) - (2+b)z - \bar{b} & 0 \\ 0 & 1+n+(b-2-n)z + \bar{b} \end{bmatrix}, \\ z(1-z)C''_n C_n^{-1} + (1-2z)C'_n C_n^{-1} &= \begin{bmatrix} -\frac{1}{4}(n-b-2)(n-b) & 0 \\ 0 & -\frac{1}{4}(n-b)(n-b+2) \end{bmatrix} \\ &\quad + \frac{(\bar{b}+n)^2}{4z}I_2 + \frac{\operatorname{Re}(b)^2}{1-z}I_2. \end{aligned}$$

Moreover, it can be seen that

$$\frac{\tilde{M}_n^2}{z(1-z)} = \left(-\frac{1}{4}(n+b)^2 + \frac{(\bar{b}+n)^2}{4z} + \frac{-\operatorname{Im}(b)^2 + (b+n)(\bar{b}+n)|\alpha_{n-1}|^2}{1-z} \right) I_2$$

Now, from the second identity in (40) it can be proven that

$$(n+b)(\bar{b}+n)|\alpha_{n-1}|^2 = |b|^2,$$

and so, substituting all these matrices into (42), we obtain

$$\begin{aligned} z(1-z)\Phi''_n(z) + [(n-b-2)z + (1-n-\bar{b})]\Phi'_n(z) + n(1+b)\Phi_n(z) &= 0, \\ z(1-z)G''_n(z) + [(b-n-2)z + (1+n+\bar{b})]G'_n(z) + b(1+n)G_n(z) &= 0, \\ z(1-z)(\Phi_{n-1}^*)''(z) + ((n-b-2)z + (1-n-\bar{b}))(\Phi_{n-1}^*)'(z) + b(n-1)\Phi_{n-1}^*(z) &= 0, \\ z(1-z)(G_{n-1}^*)''(z) + ((b-n-2)z + (1+n+\bar{b}))(G_{n-1}^*)'(z) + n(b-1)G_{n-1}^*(z) &= 0, \end{aligned}$$

which are hypergeometric differential equations for Φ_n , Φ_n^* , G_n , and G_n^* .

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