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On the integrable perturbations of the Camassa–Holm equation

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We present an investigation of the nonlinear partial differential equations (PDE) which are asymptotically representable as a linear combination of the equations from the Camassa–Holm hierarchy. For this purpose we use the infinitesimal transformations of dependent and independent variables of the original PDE. This approach is helpful for the analysis of the systems of the PDE which can be asymptotically represented as the evolution equations of polynomial structure. © *2000 American Institute of Physics.* [S0022-2488(00)02605-0]

I. INTRODUCTION

The Camassa–Holm equation (sometimes also called Fuchssteiner–Fokas–Camassa–Holm equation) has been found as an equation describing the propagation of long-waves in shallow-water[1,2] when higher-order terms are taken into account. A remarkable fact is that, like, say, KdV, it is an integrable equation.[1,3] This equation reads

$$m_{t_1}(u)=\mathrm{CH}^{(0)}(u)=-2ku_x-um_x-2u_xm,\quad m=u-u_{xx}. \tag{1}$$

Many of its properties have been studied recently in Refs. 3–9. Being a completely integrable equation, a hierarchy in the Lax sense can be constructed.[4] We will use the following symbolic representation for this hierarchy:

$$m_{t_m}(u)=\mathrm{CH}^{(m)}(u),\quad m=u-u_{xx}. \tag{2}$$

The simplest equations of this hierarchy are

$$m_{t_{-1}}=\mathrm{CH}^{(-1)}=-m_x-\frac{1}{2}(\partial_x^3-\partial_x)\frac{1}{\sqrt{m+k}}, \tag{3}$$

$$m_{t_2}=\mathrm{CH}^{(2)}=-Vm_x-2kV_x-2V_xm,\quad m=u-u_{xx}, \tag{4}$$

where V is defined by the following equation:

$$2ku_x+2u_xm+um_x+(\partial_x^3-\partial_x)V=0.$$

On the other hand, Eq. (1) has been obtained as the leading order term in the expansion of the shallow water system in the powers of small parameters characterizing scales in amplitude and wavelength. Thus, in a physical sense, it is only an approximate equation. We are thus taken naturally to consider the next terms in the perturbative expansions. These terms appear in the CH as the corrections of small order. This leads us to the study the perturbed CH equation.

A rather classical question to ask about the perturbations of an integrable equation is the determination of the perturbations that do not destroy the integrability of the system. Among them are the perturbations which are a superposition of the higher order equations of the corresponding hierarchy. However, we should recall that the equation under consideration is to describe a physical system. Therefore it comes from the perturbative series in small parameter ϵ which has to be truncated at a certain order ϵ^N. We are, thus, taken to consider as equivalent (up to this order) all equations which can be obtained one from the other by infinitesimal transformations up to the same order ϵ^N. We say then that the two equations are asymptotically equivalent.[10,11] If an equation is asymptotically equivalent to an integrable equation we say that it is asymptotically integrable.[12] In the same order of ideas we say that a perturbation of a completely integrable equation is asymptotically integrable, if the resulting equation is asymptotically integrable. With this in mind, the relevant question becomes: what are the perturbations of the CH equation that are asymptotically integrable?. This is the problem we investigate in this paper.

The analogous problem to the one formulated above has been considered for a number of integrable evolution-type equations such as the nonlinear Schrödinger equation (NLS),[11–15] Korteweg–de Vries equation (KdV),[10,11,16] Kadomtsev–Petviashvili equation (KP),[12] and the Burgers equation.[17] However, the CH equation is of the nonevolutionary type. This introduces a considerable complication in the problem and will call for several adaptations of the techniques used previously.

The paper is organized as follows. In Sec. II we remind the definition of infinitesimal transformation and define the class of nonlinear PDE asymptotically equivalent to the superposition of the equations from the CH-hierarchy (asymptotic integrability). In Sec. III we give the multiscale decomposition of the equations considered in Sec. II and consider the first obstacle to the asymptotic integrability of the CH-type equation with perturbations, based on the symmetry approach.[12,18] Then we discuss different types of infinitesimal transformations (transformations of dependent and independent variables). In Sec. IV we formulate the definition of an approximate symmetry for the nonintegrable PDE of the evolutionary type. Finally, we present the general conclusions.

II. INFINITESIMAL TRANSFORMATIONS

We say that a PDE,

$$m_\tau(v)=P(v,\epsilon),\quad \epsilon\ll 1,\quad m(v)=v-v_{xx} \tag{5}$$

is asymptotically equivalent to the CH-hierarchy up to the order ϵ^N, iff

(i) after the expansion in powers of small parameter ϵ it becomes

$$m_\tau(v)=\sum_k a_k(\epsilon)\mathrm{CH}^{(k)}(v)+Q(\epsilon,v),\quad Q=\sum_{m=M_0}^{M}\epsilon^m Q_m(v) \tag{6}$$

and

(ii) under the infinitesimal transformations of dependent and/or independent variables of the general form,

$$v(\xi,\tau_n)=u(x,t_n)+\epsilon_0 F_0(x,t_n,\epsilon), \tag{7}$$

$$\xi=x+\epsilon_1 F_1(x,t_n,\epsilon), \tag{8}$$

$$\tau_n=t_n+\epsilon_{n+1}F_{n+1}(x,t_n,\epsilon),\quad n=1,2,\ldots,\quad \epsilon_k=\epsilon_k(\epsilon)\ll 1, \tag{9}$$

Eq. (6) gets the following structure:

$$m_t(u)=\sum_k a_k(\epsilon)\mathrm{CH}^{(k)}(u)+O(\epsilon^{(N)}),\quad \partial_t\equiv\sum_k a_k(\epsilon)\partial_{t_k}=\partial_\tau. \tag{10}$$

For example, transformations (7)–(9) reduce the equation

$$m_{\tau}(v)+2kv_{\xi}+vm_{\xi}(v)+2v_{\xi}m(v)-\mathcal{Q}=0,\ \ m(v)=v-v_{\xi\xi} \tag{11}$$

with perturbations $\mathcal{Q}$ in first order in ϵ_k,

$$\mathcal{Q}\equiv\sum_k \epsilon_k P_k,\ \ F_k=F_k(\xi,\tau_k), \tag{12}$$

$$P_0=2kF_{0\xi}+F_{0\tau_1}-F_{0\xi\xi\tau_1}+(3F_0v)_{\xi}-(2F_{0\xi}v_{\xi})_{\xi}-F_0v_{\xi\xi\xi}-F_{0\xi\xi\xi}v, \tag{13}$$

$$\begin{aligned}P_1=&-2kv_{\xi}F_{1\xi}-F_{1\tau_1}v_{\xi}-3vv_{\xi}F_{1\xi}+6v_{\xi}v_{\xi\xi}F_{1\xi}+2v_{\xi\xi}F_{1\xi\tau}+2v_{\xi}^2F_{1\xi\xi}\\&+v_{\xi\tau}F_{1\xi\xi}+3vv_{\xi\xi}F_{1\xi\xi}+2v_{\xi\xi\tau_1}F_{1\xi}+v_{\xi}F_{1\xi\xi\tau_1}+v_{\xi\xi\xi}F_{1\tau_1}+3vv_{\xi\xi\xi}F_{1\xi}+vv_{\xi}F_{1\xi\xi\xi},\end{aligned} \tag{14}$$

$$\begin{aligned}P_2=&-2kF_{2\xi}v_{\tau_1}-v_{\tau_1}F_{2\tau_1}-3vv_{\tau_1}F_{2\xi}+4v_{\xi}F_{2\xi}v_{\xi\tau_1}+2v_{\xi\tau_1}F_{2\xi\tau_1}+2F_{2\xi}v_{\xi\tau_1\tau_1}+2v_{\tau_1}F_{2\xi}v_{\xi\xi}\\&+v_{\tau_1\tau_1}F_{2\xi\xi}+2v_{\tau_1}v_{\xi}F_{2\xi\xi}+3vv_{\xi\tau_1}F_{2\xi\xi}+F_{2\tau_1}v_{\xi\xi\tau_1}+3vF_{2\xi}v_{\xi\xi\tau_1}+v_{\tau}F_{2\xi\xi\tau_1}+vv_{\tau_1}F_{2\xi\xi\xi},\end{aligned} \tag{15}$$

$$\begin{aligned}P_k=&-2kF_{k\xi}v_{\tau_k}-v_{\tau_k}F_{k\tau_1}-3vv_{\tau_k}F_{k\xi}+4v_{\xi}F_{k\xi}v_{\xi\tau_k}+2v_{\xi\tau_k}F_{k\xi\tau_1}+2F_{k\xi}v_{\xi\tau_1\tau_k}\\&+2v_{\tau_k}F_{k\xi}v_{\xi\xi}+v_{\tau_1\tau_k}F_{k\xi\xi}+2v_{\tau_k}v_{\xi}F_{k\xi\xi}+3vv_{\xi\tau_k}F_{k\xi\xi}+F_{k\tau_1}v_{\xi\xi\tau_k}+3vF_{k\xi}v_{\xi\xi\tau_k}\\&+v_{\tau_k}F_{k\xi\xi\tau_1}+vv_{\tau_k}F_{k\xi\xi\xi},\quad k>2,\end{aligned} \tag{16}$$

to the CH equation up to the order $O(\epsilon^2)$.

In what follows we discuss briefly some features associated with the infinitesimal transformations (7)–(9). For simplicity let us consider the first order infinitesimal transformations (7)–(9) with F_k not depending on $\epsilon_j(j=0,1,...)$, that is $F_k=F_k(x,t_n)$. First of all note that the transformation associated with v, Eq. (7), takes a predominant position since it allows us to eliminate from Eq. (6) the perturbations of the form,

$$\mathcal{Q}=\epsilon_0((1-\partial_{\xi}^2)\partial_{\tau_1}F_0(\xi,\tau_1)+G(F_0(\xi,\tau_1),v)), \tag{17}$$

where G is the nonlinear part of the correction and its structure is defined by the operators $\mathrm{CH}^{(n)}$ from Eq. (6) and does not contain the τ derivative of the function v. So if one considers the function F_0 of the form,

$$F_0(v)=f(m(v))+L(v), \tag{18}$$

where f is a local function of m and its ξ-derivatives and L is a linear differential operator in ξ with constant coefficients then the first term in the perturbation (17) has the form,

$$\sum_k\left((1-\partial_{\xi}^2)\frac{\partial f(m(v))}{\partial m_k(v)}+\frac{\partial L(m(v))}{\partial m_k(v)}\right)\partial_{\tau_1}m_k(v),\ \ m_k(v)\equiv\frac{\partial^k m(v)}{\partial\xi^k},$$

and the time derivatives can be eliminated from the perturbation of this order by using the main order of Eq. (6).

As a result, transformation (7) allows us to treat *local* perturbations in Eq. (6) which are polynomials in v and its ξ derivatives. We wish to stress that only the infinitesimal transformation (7) possesses this property whereas the other ones (8), (9) after substitution into Eq. (6) lead to the t derivatives of the function u which can be eliminated only by introducing a nonlocal operator $(1-\partial_{xx})^{-1}$. This happens due to the fact that the CH is a nonevolutionary type equation which differs from other integrable systems like KdV, NLS, and so on.

Now let us discuss the relations between the solution $u(x,t)$ and $v(\xi,\tau)$ of Eqs. (10) and (6). One can construct without any problem the solution v of Eq. (6) which comes from the given solution u of Eq. (10) by performing the direct calculations through formulas (7)–(9). But some times one needs to reverse the infinitesimal transformations, which is not a trivial problem. An interesting case of this kind is related with the initial value problem for Eq. (6). For example, let us consider a function $v_0(\xi)=v(\xi,\tau)|_{\tau=0}$ and using this we try to construct the corresponding initial data $u_0(x)=u(x,t)|_{t=0}$ for Eq. (10). By doing this we relate the initial value problems of Eqs. (6) and (10). To perform this algorithm first of all one needs to solve the system of equations for function u_0 and variable x,

$$v_0(x+\epsilon_1 F_1(x,0))=u_0(x)+\epsilon_0 F_0(x), \quad \xi=x+\epsilon_1 F_1(x,0) \tag{19}$$

(one should remember that the functions F_j are the given functions of arguments) to find $u_0(x)$. It is not difficult to do if all the functions F_k in Eqs. (7)–(9) are given functions of the independent variables x, t_n and do not depend explicitly on the solution u itself. In this case the perturbation in Eq. (6) has inhomogeneous, linear, and nonlinear terms with variable coefficients. But of particular interest are the situations when the functions F_k's are the functions of u and its derivatives and do not depend on independent variables explicitly. In this case Eq. (6) has the form of a nonlinear equation with constant coefficients and Eq. (19) for determining u_0 is a differential equation (algebraic in particular cases) on the function u_0 and can be integrated both numerically and by the perturbation method, as far as the small parameter ϵ is involved in these equations. To clarify the last statement let us note that Eq. (19) can be expanded in powers of the parameters ϵ_k up to the first order

$$v_0(x)-u_0(x)-\epsilon_0 F_0(x,0)+\epsilon_1 \partial_x v_0(x) F_1(x,0)+O(\epsilon_j^2)=0$$

so that one can look for the solution $u_0(x)$ of the form,

$$u_0(x)=v_0(x)+\sum_{j=0}^{1}\sum_{k>0}\epsilon_j^k u_{jk}(x).$$

We will come back to the problem of inversion of the infinitesimal transformations (7)–(9) in Sec. III B.

Now let us come to the general algorithm to construct the solutions of Eq. (6) with an arbitrary perturbation which comes from the physical point of view.

Let us assume for simplicity that the equation under consideration has the form (6) with first order perturbation of the form $\mathcal{Q}=\epsilon\hat{P}(\xi,\tau_1)$. To construct the solutions of these equations up to the order ϵ one needs to consider the transformations (7)–(9) which reduces this equation to the form (10) up to the order ϵ. To do this let us substitute Eqs. (7)–(9) into (6), then the first order correction has the following form $\epsilon(-P(F_k(x,t),u)+\hat{P}(x,t))$. To eliminate it one needs to solve the equation

$$\hat{P}(x,t)=P(F_k(x,t),u) \tag{20}$$

with respect to the functions F_k.

This equation is a PDE which can be solved, generally speaking, numerically for any given solution u of Eq. (10). The only requirement is that the functions F_k should be restricted for all values of the parameters ξ and τ. Only under these circumstances the transformations (7)–(9) can be considered as the infinitesimal ones. There are many types of the infinitesimal transformations which remove the correction of the above form and one can choose a suitable one according to the convenience. After this the solution of the original equation can be calculated by using Eqs. (7)–(9) for any given solution u of the transformed Eq. (10).

This general discussion of the first order perturbations can be straightforwardly extended to the perturbations of higher order.

III. MULTISCALE DECOMPOSITION AND ASYMPTOTICALLY EQUIVALENT EQUATIONS

In the previous section we have discussed the possibility for Eq. (6) to be considered as asymptotically equivalent to the CH-hierarchy (10). However, we have not specified the role of the small parameter ϵ there. Of particular interest is Eq. (6) with the parameter ϵ which appears from the rescaling of v and x (multiscale decomposition). For example, CH can be derived from the shallow water equation by introducing the small scales in x and u. So we wish to investigate system (6) under scaling. In order to do this let us substitute the rescaled variables,

$$u \to \delta_1 u, \quad v \to \delta_1 v, \quad \xi \to \delta_2, \quad \tau \to \tau/\delta_2, \tag{21}$$

where δ_k, $k=1,2$ are small parameters, into systems (2) and (6) and expand the equations in powers of these parameters. For convenience let us choose $\delta_1 = \epsilon^2$, $\delta_2 = \epsilon$. The main advantage of this rescaling is that it allows us to rewrite system (2) in an evolutionary form represented by the infinite series

$$u_{t_k} = \sum_{m \geqslant 0} \epsilon^m \mathrm{CH}_m^{(k)}, \tag{22}$$

whereas Eq. (6) transforms to

$$v_\tau = \sum_k \sum_{m \geqslant 0} \alpha_k \mathrm{CH}_m^{(k)} + Q(\epsilon, v). \tag{23}$$

We say that Eq. (23) is equivalent to the hierarchy of CH given by the above Eq. (22) up to the order ϵ^N if there exists infinitesimal transformations of the form (7)–(9) such that Eq. (23) can be brought to the form,

$$u_t = \sum_k \sum_{m \geqslant 0} \epsilon^k a_k \mathrm{CH}_m^{(k)} + O(\epsilon^{N+1}). \tag{24}$$

In the following we write down a few equations explicitly which comes from the CH hierarchy under the scaling (21) (up to the order ϵ^6),

$$\begin{aligned} u_{t_1} = & -2ku_x - \epsilon^2(3uu_x + 2ku_{xxx}) - \epsilon^4(7u_x u_{xx} + 2uu_{xxx} + 2ku_{xxxxx}) \\ & - \epsilon^6(2ku_7 + 23u_{xx}u_{xxx} + 11u_x u_{xxxx} + 2uu_{xxxxx}) + O(\epsilon^8), \end{aligned} \tag{25}$$

$$\begin{aligned} u_{t_3} = & -u_x - \frac{u_x}{2k^{3/2}} - \epsilon^2\left(\frac{u_{xxx}}{2k^{3/2}} + \frac{3uu_x}{4k^{5/2}}\right) - \epsilon^4\left(\frac{15u^2u_x}{16k^{7/2}} + \frac{3u_x u_{xx}}{4k^{5/2}} + \frac{3uu_{xxx}}{4k^{5/2}}\right) \\ & + \epsilon^6\left(\frac{35u^3u_x}{32k^{9/2}} + \frac{15uu_x u_{xx}}{8k^{7/2}} + \frac{15u^2u_{xxx}}{16^{7/2}} + \frac{3u_{xx}u_{xxx}}{4k^{5/2}}\right) + O(\epsilon^8), \end{aligned} \tag{26}$$

$$\begin{aligned} u_{t_{-2}} = & -4k^2u_x - k\epsilon^2(12uu_x + 8ku^{xxx}) \\ & - \epsilon^4\left(\frac{15u^2u_x}{2} + 48ku_x u_{xx} + 18kuu_{xxx} + 12k^2u_5\right) \\ & - \epsilon^6\left(16k^2u_7 + \frac{35u_x^3}{2} + 55uu_x u_{xx} + 10u^2u_{xxx} + 202ku_{xx}u_{xxx}\right. \\ & \left. + 108ku_x u_{xxxx} + 26kuu_{xxxxx}\right) + O(\epsilon^8). \end{aligned} \tag{27}$$

We wish to mention that only the above equations from the CH-hierarchy have a completely local polynomial structure under the multiscale transformation, whereas all other equations contain the nonlocal terms.

With these ideas in mind let us now proceed to investigate Eq. (23) with corrections by using the concept of asymptotic integrability of the PDE of the evolutionary type which was discussed in detail in Ref. 12. Even though the main ideas of our algorithm are the same, we find certain important differences related with the particular structure of the CH-hierarchy.

A. Asymptotically commuting flows and obstacles to the integrability

To begin with let us recall some facts from the symmetry approach to the integrability of the nonlinear PDE in $(1+1)$-dimensions.[12,18] Suppose that the evolutions in t_1 and t_2 are described by the equations,

$$u_{t_1}=F(\mathbf{u}),\quad u_{t_2}=G(\mathbf{u}), \tag{28}$$

where $F(\mathbf{u})$ and $G(\mathbf{u})$ are functions of u and its x-derivatives, are considered as commuting flows if $u_{t_1t_2}=u_{t_2t_1}$. In other words,

$$\mathbf{K}(F,G)\equiv\sum_k\left(\frac{\partial F}{\partial u_k}\partial_x^kG-\frac{\partial G}{\partial u_k}\partial_x^kF\right)=0. \tag{29}$$

Analogously if G and F are represented in the form of a series in small parameter ϵ,

$$F=\sum_k\epsilon^kF_k,\quad G=\sum_k\epsilon^kG_k, \tag{30}$$

then we can call these flows are commuting up to the order ϵ^N when the commutator (29) is of the order $O(\epsilon^{(N+1)})$: $\mathbf{K}(F,G)=O(\epsilon^{(N+1)})$. It is evident that the infinitesimal transformations (7)–(9) do not disturb the integrability.

In the following we focus our attention on evolutionary type equations having local polynomial type perturbations, i.e., the functions F_k and G_k in (30) have local polynomial structure in u and its x derivatives.

We wish to construct a *general* polynomial type Eqs. (28)–(30) which would *completely* commute with CH. If it so then the equation we have constructed can said to be *a local polynomial symmetry* for the CH-equation. By the direct commutation one can find this symmetry of the form (we give several terms)

$$\begin{aligned}u_t=&\alpha_{01}u_x+\epsilon^2(\alpha_{21}u_{xxx}+\alpha_{22}uu_x)+\epsilon^4(\alpha_{41}u_5+\alpha_{42}uu_{xxx}+\alpha_{43}u_xu_{xx}+\alpha_{44}u^2u_x)\\&+\epsilon^6(\alpha_{61}u_7+\alpha_{62}uu_5+\alpha_{63}u_xu_4+\alpha_{64}u_{xx}u_{xxx}+\alpha_{65}u^2u_{xxx}+\alpha_{66}uu_xu_{xx}+\alpha_{67}u_x^3+\alpha_{68}u^3u_x)\\&+\epsilon^8(\alpha_{81}u_9+\alpha_{82}uu_7+\alpha_{83}u_xu_6+\alpha_{84}u_{xx}u_5+\alpha_{85}u_{xxx}u_4+\alpha_{86}u^2u_5+\alpha_{87}uu_xu_4+\alpha_{88}uu_{xx}u_{xxx}\\&+\alpha_{89}u_x^2u_{xxx}+\alpha_{8(10)}u_xu_{xx}^2+\alpha_{8(11)}u^3u_{xxx}+\alpha_{8(12)}u^2u_xu_{xx}+\alpha_{8(13)}uu_x^3+\alpha_{8(14)}u^4u_x)+O(\epsilon^{10}),\end{aligned} \tag{31}$$

where all coefficients α_{nj} are fixed in terms of the coefficients α_{k1} $(k\leqslant n)$ which are left arbitrary. We represent them in the following form:

$$\alpha_{22}=(1/2k)(3,0,0,0),\quad \alpha_{42}=(1/2k)(-3,5,0,0),\quad \alpha_{43}=(1/2k)(-3,10,0,0),$$

$$\alpha_{44}=(15/8k^2)(-1,1,0,0),\quad \alpha_{62}=(1/2k)(0,-5,7,0),\quad \alpha_{63}=(1/2k)(0,-10,21,0),$$

$$\alpha_{64}=(1/2k)(3,-15,35,0),\quad \alpha_{65}=(5/8k^2)(3,-10,7,0),$$

$$\alpha_{66}=(5/4k^2)(3,-17,14,0),\quad \alpha_{67}=(35/8k^2)(0,-1,1,0),$$

$$\alpha_{68}=(35/16k^3)(1,-2,1,0),\quad \alpha_{82}=(1/2k)(0,0,-7,9),$$

$$\alpha_{83}=(3/2k)(0,0,-7,12),\quad \alpha_{84}=(1/2k)(0,5,-42,84),$$

$$\alpha_{85}=(1/k)(0,5,-28,63),\quad \alpha_{86}=(7/8k^2)(0,5,-14,9),$$

$$\alpha_{87}=(7/4k^2)(0,10,-37,27),\quad \alpha_{88}=(-15/4k^2)(1,-8,28,-21),$$

$$\alpha_{89}=(21/8k^2)(0,5,-28,23),\quad \alpha_{810}=(1/8k^2)(-15,155,-791,651),$$

$$\alpha_{811}=(35/16k^3)(-1,5,-7,3),\quad \alpha_{812}=(105/16k^3)(-1,8,-13,6),$$

$$\alpha_{813}=(315/16k^3)(0,1,-2,1),\quad \alpha_{814}=(315/128k^4)(-1,3,-3,1), \tag{32}$$

where we adopted the designation

$$(a_1,a_2,a_3,a_4)=(a_1\alpha_{21}+a_2\alpha_{41}+a_3\alpha_{61}+a_4\alpha_{81}).$$

It follows from the above form of corrections that if the first nontrivial term in the symmetry (31) is of the order ϵ^N, then all higher order terms cannot be equal to zero. From this it follows the most important property of the symmetry (31); this symmetry is represented by an *infinite series* in powers of the parameter ϵ.

Now let us construct a differential equation which has a general polynomial structure and asymptotically equivalent to the general symmetry (31) of the CH. For this purpose let us consider an infinitesimal transformation of the form,

$$\begin{aligned} v=u&+\epsilon^2(a_{21}u^2+a_{22}u_2+a_{23}\partial_x^{-1}(u)u_1)+\epsilon^3(a_{31}uu_1+a_{32}u_3)\\ &+\epsilon^4(a_{41}u^3+a_{42}u_1^2+a_{43}uu_2+a_{44}u_4+a_{45}u_3\partial_x^{-1}u+a_{46}u_2\partial_x^{-1}u^2\\ &+a_{47}uu_1\partial_x^{-1}u+a_{48}u_1\partial_x^{-1}u^2)+\cdots \end{aligned} \tag{33}$$

which transforms Eq. (23) into an another polynomial equation. If this equation coincides with the symmetry (31) up to the order ϵ^N, then the original Eq. (23) is asymptotically equivalent to the CH-symmetry up to the order ϵ^N. We suppose that the transformation (33) covers all possible transformations (7)–(9), which are related with the perturbations of the local polynomial type, but we leave the proof of this statement beyond the scope of this paper. Also we consider only perturbations up to the order ϵ^6. Let us substitute the transformation (33) into Eq. (23) and take the coefficients a_{kn} in formula (33) from the condition that the final transformed equation coincides with the symmetry (31). This can be done if the original Eq. (23) has the form,

$$\begin{aligned} v_t=\alpha_{01}v_x&+\epsilon^2(\alpha_{21}v_{xxx}+\alpha_{22}vv_x)+\epsilon^4(\alpha_{41}v_5+\beta_{42}vv_{xxx}+\beta_{43}v_xv_{xx}+\beta_{44}v^2v_x)\\ &+\epsilon^5\beta_{51}(v_{xx}^2+v_xv_{xxx})+\epsilon^6(\alpha_{61}u_7+\beta_{62}vv_5+\beta_{63}v_xv_4+\beta_{64}v_{xx}v_{xxx}\\ &\beta_{65}v^2v_{xxx}+\beta_{66}vv_xv_{xx}+\beta_{67}v_x^3+\beta_{68}v^3v_x)+O(\epsilon^7). \end{aligned} \tag{34}$$

It is worth noting that the order ϵ^5 appears in the above Eq. (34), while it does not exist in the symmetry (31). The correction up to the order ϵ^4 does not give any obstacle to the integrability while the correction of the order ϵ^6 requires the coefficients β_{kn} to satisfy the following relation:

$$3(70\alpha_{61}-k(85\beta_{62}-30\beta_{63}+12\beta_{64})+k^2(16\beta_{65}+18\beta_{66}-48\beta_{67})-24k^3\beta_{68})\alpha_{21}$$
$$-(300\alpha_{41}-335k\alpha_{41}\beta_{42}+k^2(-12\beta_{43}^2+60\beta_{42}\beta_{43}+8\beta_{42}^2+120\alpha_{41}\beta_{44})$$
$$+k^3(24\beta_{43}\beta_{44}+8\beta_{42}\beta_{44}))=0. \tag{35}$$

The above Eq. (35) represents an obstacle to the integrability for Eq. (34) up to the order ϵ^6. It is the same as the one derived in Refs. 10 and 16.

B. Different types of infinitesimal transformations

In the previous discussion we have considered an equation of the general form (34) which is reducible to an integrable equation up to the order ϵ^6 through an infinitesimal *gauge transformation* (7), (33). However, we have already seen in Sec. II. [vide Eq. (19)] that it is not so easy to reverse this transformation, if required. That is why for the practical purpose of constructing solutions to the equation with perturbation it can be useful to consider infinitesimal transformation of the independent variables as far as each of Eqs. (7)–(9) has its own distinct structures.

In this section we demonstrate the advantage of the infinitesimal transformation (8) by considering a simple example.

Let us consider an equation of the form,

$$v_t=\sum_k \mathrm{CH}_k^{(0)}(v)-\mathcal{Q}(\epsilon,v),$$

$$\mathcal{Q}=\epsilon^5(9a_{31}v_2^2-6ka_{32}v_2^2+9a_{31}v_xv_3-6ka_{32}v_xv_3)+\epsilon^6(3a_{41}v_x^3-12ka_{41}v_2v_3).$$

The perturbation $\mathcal{Q}$ can be treated by two manners; either through the transformation

$$\text{(i)}\quad v=u+\epsilon^3(a_{31}u_{xxx}+a_{32}uu_x)+\epsilon^4a_{41}u_x^2,\quad \xi=x,\quad \tau=t,\quad v=v(\xi,\tau),\quad u=u(x,t), \tag{36}$$

or through the transformation,

$$\text{(ii)}\quad v(\xi,\tau)=u(x,t),\quad \xi=x+\epsilon^2b_{21}u+\epsilon^2b_{31}u_\xi,\quad \tau=t,\quad v=v(\xi,\tau),\quad u=u(x,t), \tag{37}$$

where

$$b_{21}=\frac{1}{2k}(3a_{31}-2ka_{32}),\quad b_{31}=-a_{41}.$$

Each of these transformations leads to CH up to the order ϵ^6. Of course it is more convenient to consider Eq. (36) if one needs to construct the solution $v(\xi,\tau)$ related with given $u(x,t)$. However, if one needs to reverse the infinitesimal transformation and find function u which is related with the given v (for instance to solve the initial value problem) the second transformation (37) is more preferable since it involves only one differentiation with respect to x. However, we will not consider an explicit example for this kind of solutions here.

IV. GENERALIZATIONS OF THE ASYMPTOTIC INTEGRABILITY

Generally speaking, the asymptotic integrability is not related with complete integrability. For instance, it is possible to consider the equation of the general form, which does not possess an exact symmetry but can possess an approximate symmetry,

$$u_{t_j}=\alpha_{01}^{(j)}u_1+\epsilon(\alpha_{11}^{(j)}u_2+\alpha_{12}^{(j)}u^2)+\epsilon^2(\alpha_{21}^{(j)}u_3+\alpha_{22}^{(j)}uu_1)$$
$$+\epsilon^3(\alpha_{31}^{(j)}u_4+\alpha_{32}^{(j)}u_1^2+\alpha_{33}^{(j)}uu_2+\alpha_{34}^{(j)}u^3)+\epsilon^4(\alpha_{41}^{(j)}u_5+\alpha_{42}^{(j)}uu_3+\alpha_{43}^{(j)}u_1u_2+\alpha_{44}^{(j)}u^2u_1)$$
$$+\epsilon^5(\alpha_{51}^{(j)}u_6+\alpha_{52}^{(j)}u_2^2+\alpha_{53}^{(j)}u_1u_3+\alpha_{54}^{(j)}uu_4+\alpha_{55}^{(j)}uu_1^2+\alpha_{56}^{(j)}u^2u_2+\alpha_{57}^{(j)}u^4)+\epsilon^6(\alpha_{61}^{(j)}u_7$$
$$+\alpha_{62}^{(j)}uu_5+\alpha_{63}^{(j)}u_1u_4+\alpha_{64}^{(j)}u_2u_3+\alpha_{65}^{(j)}u^2u_3+\alpha_{66}^{(j)}uu_1u_2+\alpha_{67}^{(j)}u_1^3+\alpha_{68}^{(j)}u^3u_1). \tag{38}$$

To find an approximate symmetry, let us consider the commutator of two equations of the above form. It is represented by the infinite series in powers of ϵ. One can find the restrictions on the coefficients of Eq. (38) from the conditions that this commutator is of the order $\epsilon^{(N+1)}$. Direct calculations give the following results. The commutator is of the order ϵ^3 if

$$-2\alpha_{12}^{(j)}\alpha_{11}^{(i)}+2\alpha_{11}^{(j)}\alpha_{12}^{(i)}=0,$$

of the order ϵ^4 if

$$-2\alpha_{22}^{(j)}\alpha_{11}^{(i)}-6\alpha_{12}^{(j)}\alpha_{21}^{(i)}+6\alpha_{21}^{(j)}\alpha_{12}^{(i)}+2\alpha_{11}^{(j)}\alpha_{22}^{(i)}=0,$$

$$\alpha_{22}^{(j)}\alpha_{12}^{(i)}-\alpha_{12}^{(j)}\alpha_{22}^{(i)}=0,$$

of the order ϵ^5 if

$$-2\alpha_{32}^{(j)}\alpha_{11}^{(i)}-3\alpha_{22}^{(j)}\alpha_{21}^{(i)}-6\alpha_{12}^{(j)}\alpha_{31}^{(i)}+6\alpha_{31}^{(j)}\alpha_{12}^{(i)}+3\alpha_{21}^{(j)}\alpha_{22}^{(i)}+2\alpha_{11}^{(j)}\alpha_{32}^{(i)}=0,$$

$$-2\alpha_{33}^{(j)}\alpha_{11}^{(i)}-3\alpha_{22}^{(j)}\alpha_{21}^{(i)}-8\alpha_{12}^{(j)}\alpha_{31}^{(i)}+8\alpha_{31}^{(j)}\alpha_{12}^{(i)}+3\alpha_{21}^{(j)}\alpha_{22}^{(i)}+2\alpha_{11}^{(j)}\alpha_{33}^{(i)}=0,$$

$$\alpha_{33}^{(j)}\alpha_{12}^{(i)}-\alpha_{12}^{(j)}\alpha_{33}^{(i)}=0,$$

$$-6\alpha_{34}^{(j)}\alpha_{11}^{(i)}+2\alpha_{32}^{(j)}\alpha_{12}^{(i)}+2\alpha_{33}^{(j)}\alpha_{12}^{(i)}-2\alpha_{12}^{(j)}\alpha_{32}^{(i)}-2\alpha_{12}^{(j)}\alpha_{33}^{(i)}+6\alpha_{11}^{(j)}\alpha_{34}^{(i)}=0,$$

$$\alpha_{34}^{(j)}\alpha_{12}^{(i)}-\alpha_{12}^{(j)}\alpha_{34}^{(i)}=0,$$

of the order ϵ^6 if

$$-2\alpha_{43}^{(j)}\alpha_{11}^{(i)}-6\alpha_{32}^{(j)}\alpha_{21}^{(i)}-3\alpha_{33}^{(j)}\alpha_{21}^{(i)}-10\alpha_{22}^{(j)}\alpha_{31}^{(i)}-20\alpha_{12}^{(j)}\alpha_{41}^{(i)}+20\alpha_{41}^{(j)}\alpha_{12}^{(i)}$$
$$+10\alpha_{31}^{(j)}\alpha_{22}^{(i)}+6\alpha_{21}^{(j)}\alpha_{32}^{(i)}+3\alpha_{21}^{(j)}\alpha_{33}^{(i)}+2\alpha_{11}^{(j)}\alpha_{43}^{(i)}=0,$$

$$-2\alpha_{42}^{(j)}\alpha_{11}^{(i)}-3\alpha_{33}^{(j)}\alpha_{21}^{(i)}-4\alpha_{22}^{(j)}\alpha_{31}^{(i)}-10\alpha_{12}^{(j)}\alpha_{41}^{(i)}+10\alpha_{41}^{(j)}\alpha_{12}^{(i)}+4\alpha_{31}^{(j)}\alpha_{22}^{(i)}+3\alpha_{21}^{(j)}\alpha_{33}^{(i)}$$
$$+2\alpha_{11}^{(j)}\alpha_{42}^{(i)}=0,$$

$$\alpha_{42}^{(j)}\alpha_{12}^{(i)}-\alpha_{12}^{(j)}\alpha_{42}^{(i)}=0,$$

$$-2\alpha_{44}^{(j)}\alpha_{11}^{(i)}-6\alpha_{34}^{(j)}\alpha_{21}^{(i)}+2\alpha_{43}^{(j)}\alpha_{12}^{(i)}+\alpha_{32}^{(j)}\alpha_{22}^{(i)}-\alpha_{22}^{(j)}\alpha_{32}^{(i)}-2\alpha_{12}^{(j)}\alpha_{43}^{(i)}+6\alpha_{21}^{(j)}\alpha_{34}^{(i)}+2\alpha_{11}^{(j)}\alpha_{44}^{(i)}=0,$$

$$-4\alpha_{44}^{(j)}\alpha_{11}^{(i)}-18\alpha_{34}^{(j)}\alpha_{21}^{(i)}+2\alpha_{43}^{(j)}\alpha_{12}^{(i)}+6\alpha_{42}^{(j)}\alpha_{12}^{(i)}+2\alpha_{33}^{(j)}\alpha_{22}^{(i)}-2\alpha_{22}^{(j)}\alpha_{33}^{(i)}$$
$$-2\alpha_{12}^{(j)}\alpha_{43}^{(i)}-6\alpha_{12}^{(j)}\alpha_{42}^{(i)}+18\alpha_{21}^{(j)}\alpha_{34}^{(i)}+4\alpha_{11}^{(j)}\alpha_{44}^{(i)}=0,$$

$$2\alpha_{44}^{(j)}\alpha_{12}^{(i)}-\alpha_{34}^{(j)}\alpha_{22}^{(i)}+\alpha_{22}^{(j)}\alpha_{34}^{(i)}-2\alpha_{12}^{(j)}\alpha_{44}^{(i)}=0,$$

and so on. Equation (38), with different i, j, whose coefficients satisfy the above conditions up to the order ϵ^N are said to be *commuting up to the order* ϵ^N. Analogously one can consider a class of equations, equivalent to the given one, as it was done in Sec. II for the CH-type equations. We are not considering this question here. The important fact is that none of Eq. (38) with different j does necessarily belong to a completely integrable hierarchy. There is no regular way to construct, in general, the solutions of this type of equations except by the perturbation method.

V. CONCLUDING REMARKS

The PDE of the polynomial structure (38) can be considered as the perturbations of the KdV- or CH-type equations. The obstacles to integrability do not depend on what kind of the integrable equation we are considering. But these equations have quite different solutions, for example, cuspon and peakon of CH. And both of these equations possess soliton solutions. We have seen that the CH hierarchy under the rescaling (21) gets the structure of the evolution equations of the polynomial type which are *not scale invariant and are represented as infinite series in powers of* ϵ [see Eqs. (25)–(27)]. This is the main point of the CH hierarchy.

We have considered a class of equations which is asymptotically equivalent to the CH-hierarchy up to the order ϵ^6.

We have discussed the advantages of different types of the infinitesimal transformations (7)–(9). We have shown that along with gauge transformation (7) the transformation of independent variables are useful in the cases when one needs to reverse the infinitesimal transformations (for example, for solving the initial value problem).

Generally speaking, the *asymptotic* integrability is not related with *complete* integrability. One can consider the approximate symmetries of the nonintegrable Eq. (38) and definition of the equation which is asymptotically equivalent to the given (nonintegrable) one.

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