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Equivalence proof of pure spinor and Ramond-Neveu-Schwarz superstring amplitudes

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Abstract: A new manifestly spacetime-supersymmetric prescription for superstring amplitude computations is given using the pure spinor formalism which does not contain subtleties from poles in the pure spinor ghosts. This super-Poincaré covariant prescription is related by a U(5)-covariant field redefinition to the Ramond-Neveu-Schwarz amplitude prescription where the pure spinor parameterizes the choice of SO(10)/U(5). For F-term scattering amplitudes which preserve a subset of the spacetime supersymmetries, the new pure spinor amplitude prescription reduces to the previous pure spinor prescription.

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Contents

1 Introduction

Most of our current understanding of superstring theory comes from the computation of perturbative superstring scattering amplitudes, and the two main formalisms for computing these amplitudes are the Ramond-Neveu-Schwarz (RNS) formalism and the pure spinor formalism. The RNS formalism has the advantage of being based on a geometrical formulation of the worldsheet theory on N=1 super-Riemann surfaces, but has the disadvantage that spacetime supersymmetry is not manifest. On the other hand, the pure spinor formalism has manifest spacetime supersymmetry which simplifies the computation of multiloop amplitudes, but there is no obvious geometric formulation of the worldsheet theory. All amplitudes that have been computed up to now using the two formalisms coincide, but there is no general proof of their equivalence. Furthermore, there are different types of subtleties in multiloop computations using the two formalisms, and an equivalence proof would need to relate these subtleties. The RNS amplitude prescription has subtleties coming from dependence on the location of picture-changing operators, and the pure spinor prescription has subtleties coming from poles when the pure spinor ghosts vanish.

In this paper, the two amplitude prescriptions will be proven to be equivalent by relating them (after Wick-rotation to SO(10) signature) to a U(5)-covariant prescription [8] which is a conformally invariant extension of the U(4)-covariant light-cone Green-Schwarz prescription. In this U(5)-covariant prescription, the ten components of the RNS spacetime vector variable ψ^m are related by a field redefinition to 5 components of the Green-Schwarz spacetime spinor variable θ^α and its conjugate momentum p_α. This prescription can be made SO(10)-covariant by adding bosonic variables λ^α satisfying the d=10 pure spinor constraint $\lambda\gamma^m\lambda = 0$ which parameterize the choice of SO(10)/U(5), and including the remaining 11 components of θ^α and p_α through the topological term $\oint \lambda^\alpha p_\alpha$ in the BRST operator so that the additional degrees of freedom do not modify the physical spectrum.

Using the field redefinition from the RNS variables, one obtains an extended pure spinor formalism which includes the original pure spinor variables $(x^m, \theta^\alpha, p_\alpha, \lambda^\alpha, w_\alpha)$ together with a set of (b, c) and $(\widetilde{\beta}, \widetilde{\gamma})$ worldsheet ghosts of conformal weight $(2, -1)$ and opposite statistics. The new pure spinor amplitude prescription involves picture-changing operators coming from the RNS picture-changing operators which resemble the composite B-ghost in the original pure spinor prescription [3] but contain an additional term. And the pure spinor

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vertex operators coming from the field redefinition of RNS vertex operators also contain an additional term related to the pure spinor antifield of ghost-number +2. For super-Yang-Mills states, the new vertex operator is

$$V = ce^{-\widetilde{\phi}}\lambda^\alpha A_\alpha(x,\theta) + ce^{-2\widetilde{\phi}}\lambda^\alpha\lambda^\beta A^*_{\alpha\beta}(x,\theta) \tag{1.1}$$

where $\widetilde{\phi}$ comes from the bosonic ghost $\widetilde{\gamma} = \eta e^{\widetilde{\phi}}$, and $\lambda^\alpha A_\alpha$ and $\lambda^\alpha\lambda^\beta A^*_{\alpha\beta}$ are the operators of ghost-number +1 and +2 in the massless cohomology of $\oint \lambda^\alpha d_\alpha$ [2].

For F-term scattering amplitudes which preserve a subset of spacetime supersymmetries, one can show that the additional terms in the picture-changing operator and vertex operator do not contribute and the path integral over the (b,c) and $(\widetilde{\beta},\widetilde{\gamma})$ ghosts cancel each other out, so the new prescription reduces to the original pure spinor amplitude prescription [3]. For D-term scattering amplitudes which do not preserve any spacetime supersymmetries, the additional terms contribute and the new prescription has no subtleties coming from poles in the pure spinor ghosts, but has subtleties like the RNS formalism coming from dependence on the location of the picture-changing operators. At two loops and below, it is easy to see that the subtleties can be resolved to obtain manifestly super-Poncaré covariant expressions as in [7]. So equivalence with the RNS prescription has been proven for all amplitudes which have been computed using the original pure spinor prescription, including F-terms at arbitrary loop such as [5] and D-terms up to two loops. For D-terms at more than two loops, one needs to use the new pure spinor prescription to obtain super-Poincaré covariant expressions and it is hoped that an approach similar to [4] in the RNS formalism will be possible for treating the subtleties coming from the choice of the locations of picture-changing operators in these multiloop amplitudes. Note that unlike in the RNS formalism, these subtleties affect manifest Lorentz invariance but do not affect manifest spacetime supersymmetry, and should not complicate the proof of perturbative finiteness of multiloop amplitudes.

2 U(5)-covariant field redefinition

In light-cone gauge, the SO(8) RNS spacetime vector ψ^J for $J = 1$ to 8 can be related to the SO(8) Green-Schwarz spacetime spinor θ^A for $A =$ to 8 by splitting ψ^J into 4_{+1} and $\overline{4}_{-1}$ representations of U(4) as (ψ^j,ψ_j) and bosonizing $\psi^j = e^{i\sigma_j}$ and $\psi_j = e^{-i\sigma_j}$ where $j = 1$ to 4. In terms of σ_j, θ^A is described by $\theta^j = e^{i\sigma_j - \frac{i}{2}\sum_{k=1}^4 \sigma_k}$ and $\theta_j = e^{-i\sigma_j + \frac{i}{2}\sum_{k=1}^4 \sigma_k}$ where (θ^j,θ_j) are $(4_{-1},\overline{4}_{+1})$ representations of U(4) [1]. This U(4)-covariant field redefinition can be generalized to a U(5)-covariant field redefinition by defining [8]

$$\theta^a = e^{\frac{1}{2}\phi}e^{i\sigma_a - \frac{i}{2}\sum_{b=1}^5 \sigma_b}, \quad p_a = e^{-\frac{1}{2}\phi}e^{-i\sigma_a + \frac{i}{2}\sum_{b=1}^5 \sigma_b} \tag{2.1}$$

for $a = 1$ to 5 where the d=10 RNS vector ψ^m for $m = 1$ to 10 splits into $(5_1,\overline{5}_{-1})$ representations of U(5) which are bosonized as $(\psi^a = e^{i\sigma_a},\psi_a = e^{-i\sigma_a})$, the (β,γ) RNS ghosts have been fermionized as $(\beta = \partial\xi e^{-\phi},\gamma = \eta e^{\phi})$, and θ^α and p_α for $\alpha = 1$ to 16 are d=10 spacetime spinors of opposite chirality which decompose under U(5) as $(1_{\frac{5}{2}},\overline{10}_{\frac{1}{2}},5_{-\frac{3}{2}})$ and $(\overline{5}_{\frac{3}{2}},10_{-\frac{1}{2}},1_{-\frac{5}{2}})$ with θ^a and p_a being the $5_{-\frac{3}{2}}$ and $\overline{5}_{\frac{3}{2}}$ representations. It will also be useful to define the chiral boson

$$\widetilde{\phi} = \frac{3}{2}\phi - \frac{i}{2}\sum_{b=1}^{5}\sigma_b \tag{2.2}$$

which has no singular OPE's with θ^a and p_a. In terms of $(\widetilde{\phi}, \theta^a, p_a, x^a, x_a, b, c, \eta, \xi)$, the RNS BRST operator is

$$Q_{\text{RNS}} = \oint [\eta e^{\widetilde{\phi}} p_a \partial x^a + \eta e^{2\widetilde{\phi}}(p^4)^a \partial x_a + b\eta\partial\eta e^{3\widetilde{\phi}} p^5 + cT + bc\partial c] \tag{2.3}$$

where $T = p_a\partial\theta^a + \partial x^a \partial x_a + b\partial c + \partial(bc) + \widetilde{\beta}\partial\widetilde{\gamma} + \partial(\widetilde{\beta}\widetilde{\gamma})$ and $(\widetilde{\beta} = \partial\xi e^{-\widetilde{\phi}}, \widetilde{\gamma} = \eta e^{\widetilde{\phi}})$ are chiral bosons of conformal weight $(+2, -1)$ with η defined to carry ghost number $+1$. Note that Q_{RNS} is manifestly invariant under U(5) and under the 6 spacetime supersymmetries generated by $q_a = \oint p_a$ and $q_+ = \oint(\theta^a \partial x_a + b\eta e^{\widetilde{\phi}})$ which transform as $\overline{5}_{\frac{3}{2}}$ and $1_{-\frac{5}{2}}$ representations.

Under this field redefinition, the RNS gluon vertex operator in the -1 picture $V = ce^{-\phi}\psi^m a_m(x)$ maps to $V = ce^{-\widetilde{\phi}}\theta^a a_a(x) + ce^{-2\widetilde{\phi}}(\theta^4)_a a^a(x)$, which generalizes to the U(5)-covariant superfield $V = ce^{-\widetilde{\phi}}A(x,\theta) + ce^{-2\widetilde{\phi}}A^*(x,\theta)$ for the super-Yang-Mills multiplet. $QV = 0$ implies that the U(5)-covariant superfields $A(x,\theta)$ and $A^*(x,\theta)$ satisfy

$$\partial_a\partial^a A = \partial_a\partial^a A^* = \left(\frac{\partial}{\partial\theta}\right)_a \partial^a A + \left[\left(\frac{\partial}{\partial\theta}\right)^4\right]^a \partial_a A^* = \left[\left(\frac{\partial}{\partial\theta}\right)^3\right]_{abc} A = \left[\left(\frac{\partial}{\partial\theta}\right)^2\right]_{[ab} \partial_{c]} A = 0, \tag{2.4}$$

which implies the component expansion

$$A = \rho_+(x) + \theta^a a_a(x) + \frac{1}{2}\theta^a\theta^b \xi_{ab}(x), \qquad A^* = \ldots + \frac{1}{2}(\theta^3)_{ab}\rho^{ab}(x) + (\theta^4)_a a^a(x) + (\theta^5)\xi^+(x), \tag{2.5}$$

where $\ldots$ can be gauged to zero and the 16 gluino polarizations are given by the linear combinations $(\xi^+, \xi_{ab} + \frac{1}{2}\epsilon_{abcde}\partial^c\rho^{de}, \partial^a\rho_+ - \partial_b\rho^{ab})$ with $\partial_{[a}\xi_{bc]} = \partial_a\xi^+ - \partial^b\xi_{ab} = 0$. Note that the terms in the vertex operator proportional to ξ^+ and ξ_{ab} describe 4 on-shell components and correspond to RNS vertex operators in the $-\frac{1}{2}$ picture, whereas the terms proportional to ρ_+ and ρ^{ab} describe the other 4 on-shell components and correspond to RNS vertex operators in the $-\frac{3}{2}$ picture. Since the choice of RNS picture is not Lorentz invariant, the resulting scattering amplitude is only Lorentz covariant up to surface terms coming from changing the locations of the picture-changing operators.

3 Relation with pure spinor formalism

Although the U(5)-covariant description of the previous section can be used to compute superstring amplitudes with 6 manifest spacetime supersymmetries, it is convenient to make all 16 supersymmetries manifest. The first step to covariantize the BRST operator and vertex operator of (2.3) and (2.5) is to introduce the 11 bosonic and fermionic variables $(\lambda^+, \lambda_{ab}; \theta^+, \theta_{ab})$ and their conjugate momenta $(w_+, w^{ab}; p_+, p^{ab})$ by adding to the BRST operator the topological term $\oint[\lambda^+ p_+ + \frac{1}{2}\lambda_{ab}p^{ab}]$ and performing the similarity transformation $Q \to e^R Q e^{-R}$ where $R = \oint c(w_+\partial\theta^+ + \frac{1}{2}w^{ab}\partial\theta_{ab})$. The resulting BRST operator is

$$Q = \oint[\lambda^+ p_+ + \frac{1}{2}\lambda_{ab}p^{ab} + \widetilde{\gamma} p_a \partial x^a + \eta e^{2\widetilde{\phi}}(p^4)^a \partial x_a + \eta\partial\eta e^{3\widetilde{\phi}} p^5 (b + w_\alpha \partial\theta^\alpha) + cT + bc\partial c] \tag{3.1}$$

where $T = p_\alpha \partial\theta^\alpha + \partial x^m \partial x_m + w_\alpha \partial\lambda^\alpha + b\partial c + \partial(bc) + \widetilde{\beta}\partial\widetilde{\gamma} + \partial(\widetilde{\beta}\widetilde{\gamma})$ is now Lorentz covariant, $\lambda^\alpha = (\lambda^+, \lambda_{ab}, -(8\lambda^+)^{-1}\epsilon^{abcde}\lambda_{bc}\lambda_{de})$ is a d=10 pure spinor satisfying $\lambda\gamma^m\lambda = 0$, and $w_\alpha = (w_+, w^{ab}, 0)$ is its conjugate momentum in the gauge $w_a = 0$.

The 6 manifest spacetime supersymmetry generators are $q_a = \oint p_a$ and $q_+ = \oint(p_+ + \partial x_a \theta^a + \widetilde{\gamma}(b + w_\alpha\partial\theta^\alpha))$, and it will be convenient to perform a similarity transformation by $R' = \oint \theta^+(\partial x_a\theta^a + \widetilde{\gamma}(b + w_\alpha \partial\theta^\alpha))$ so that $q_a = \oint(p_a + \theta^+\partial x_a)$ and $q_+ = \oint p_+$. It will also be convenient to shift $\widetilde{\phi} \to \widetilde{\phi} - \log\lambda^+$ and $w_+ \to w_+ + (\lambda^+)^{-1}\partial\widetilde{\phi}$ so that $e^{\widetilde{\phi}}$ does not transform under U(5) and carries ghost number +1. After this similarity transformation and shift,

$$\begin{aligned} Q = \oint[&\lambda^+(p_+ - \theta^a\partial x_a) + \frac{1}{2}\lambda_{ab}p^{ab} + \widetilde{\gamma}((\lambda^+)^{-1}p_a(\partial x^a + \theta^a\partial\theta^+) + b + w_\alpha\partial\theta^\alpha) \\ &+ \eta e^{2\widetilde{\phi}}(\lambda^+)^{-2}(p^4)^a\partial x_a + \eta\partial\eta e^{3\widetilde{\phi}}(\lambda^+)^{-3}p^5(b + w_\alpha\partial\theta^\alpha + (\lambda^+)^{-1}\partial\widetilde{\phi}\partial\theta^+) + cT + bc\partial c]. \end{aligned} \tag{3.2}$$

Although Q now has terms with $(\lambda^+)^{-1}$ dependence, one can avoid problems when $\lambda^+ = 0$ by requiring that all terms with $(\lambda^+)^{-1}$ dependence are spacetime supersymmetric under $q_+ = \oint p_+$, i.e. all terms with $(\lambda^+)^{-1}$ dependence must be independent of the θ^+ zero mode. This is necessary since $Q(\theta^+(\lambda^+)^{-1} + c\theta^+\partial\theta^+(\lambda^+)^{-2}) = 1$, so the cohomology would be trivial if terms were allowed to depend on both the θ^+ zero mode and $(\lambda^+)^{-1}$.

The next step in covariantizing the BRST operator is to perform a similarity transformation involving θ_{ab} which transforms (3.2) into the manifestly spacetime supersymmetric expression

$$\begin{aligned} Q = \oint[&\lambda^+ d_+ + \frac{1}{2}\lambda_{ab}d^{ab} + \lambda^a d_a + \widetilde{\gamma}((\lambda^+)^{-1}d_a\Pi^a + b + w_\alpha\partial\theta^\alpha) + cT + bc\partial c \\ &+ \eta e^{2\widetilde{\phi}}(\lambda^+)^{-3}(d^4)^a(\lambda^+\Pi_a + \lambda_{ab}\Pi^b) + \eta\partial\eta e^{3\widetilde{\phi}}(\lambda^+)^{-3}d^5(b + w_\alpha\partial\theta^\alpha + (\lambda^+)^{-1}\partial\widetilde{\phi}\partial\theta^+)] \end{aligned} \tag{3.3}$$

where (d_a, d^{ab}, d_+) are the U(5) components of $d_\alpha = p_\alpha - \frac{1}{2}\partial x_m(\gamma^m\theta)_\alpha - \frac{1}{8}(\theta\gamma_m\partial\theta)(\gamma^m\theta)_\alpha$, (Π^a, Π_a) are the U(5) components of $\Pi^m = \partial x^m + \frac{1}{2}\theta\gamma^m\partial\theta$, and (d_α, Π^m) are the Green-Schwarz-Siegel variables of [10] which commute with the 16 spacetime supersymmetry generators $q_\alpha = \oint[p_\alpha + \frac{1}{2}\partial x^m(\gamma_m\theta)_\alpha + \frac{1}{24}(\theta\gamma^m\partial\theta)(\gamma_m\theta)_\alpha]$.

Finally, as in the non-minimal pure spinor formalism, Q can be made manifestly Lorentz covariant by introducing a new pure spinor variable $\overline{\lambda}_\alpha$ and a constrained fermionic spinor r_α satisfying $\overline{\lambda}\gamma^m\overline{\lambda} = \overline{\lambda}\gamma^m r = 0$, together with their conjugate momenta $\overline{w}^\alpha$ and s^α, and adding the topological term $\oint \overline{w}^\alpha r_\alpha$ to the BRST operator so that the new variables decouple from the cohomology. After including non-minimal terms in Q depending on r_α so that the BRST operator remains nilpotent, one obtains the manifestly super-Poincaré invariant expression

$$
\begin{aligned}
Q = \oint [\lambda^\alpha d_\alpha + \overline{w}^\alpha r_\alpha + \widetilde{\gamma}\left(\Pi^m \overline{\Gamma}_m - \frac{(\lambda\gamma^{mn}r)}{4(\lambda\overline{\lambda})}\overline{\Gamma}_m\overline{\Gamma}_n + b + w_\alpha\partial\theta^\alpha + s^\alpha\partial\overline{\lambda}_\alpha\right) + cT + bc\partial c \\
+ \frac{(\lambda\gamma^{m_1\dots m_5}\lambda)}{120}(5\eta e^{2\widetilde{\phi}}\overline{\Gamma}_{m_1}\dots\overline{\Gamma}_{m_4}\Pi_{m_5} + \eta\partial\eta e^{3\widetilde{\phi}}\overline{\Gamma}_{m_1}\dots\overline{\Gamma}_{m_5}(b + s^\alpha\partial\overline{\lambda}_\alpha + (w_\alpha + \partial\widetilde{\phi}\frac{\overline{\lambda}_\alpha}{(\lambda\overline{\lambda})})\partial\theta^\alpha)]
\end{aligned}
\tag{3.4}
$$

where $\overline{\Gamma}_m = \frac{1}{2}(\lambda\overline{\lambda})^{-1}(\overline{\lambda}\gamma_m d - \frac{1}{8}(\lambda\overline{\lambda})^{-1}\overline{\lambda}\gamma_{mnp}r(w\gamma^{np}\lambda))$ is defined as in [6] and, to simplify the expression, w_α has been gauge-fixed such that $\overline{\lambda}\gamma^m w = 0$. Nilpotence of the BRST operator of (3.4) can be verified by noting that $Q = e^{R_1}e^{R_2}\oint(\lambda^\alpha d_\alpha + \overline{w}^\alpha r_\alpha + b\widetilde{\gamma})e^{-R_2}e^{-R_1}$ where

$$
\begin{aligned}
R_1 &= \oint c\left(\Pi^m\overline{\Gamma}_m - \frac{(\lambda\gamma^{mn}r)}{4(\lambda\overline{\lambda})}\overline{\Gamma}_m\overline{\Gamma}_n + w_\alpha\partial\theta^\alpha + s^\alpha\partial\overline{\lambda}_\alpha + \widetilde{\beta}\partial c\right), \\
R_2 &= \frac{1}{120}\oint \eta e^{2\widetilde{\phi}}(\lambda\gamma^{m_1\dots m_5}\lambda)\overline{\Gamma}_{m_1}\dots\overline{\Gamma}_{m_5}.
\end{aligned}
\tag{3.5}
$$

After including the new worldsheet variables, the massless U(5)-covariant vertex operator of (2.5) easily generalizes to the super-Poincaré covariant vertex operator

$$
V = ce^{-\widetilde{\phi}}A + ce^{-2\widetilde{\phi}}A^* \quad \text{where} \quad A = \lambda^\alpha A_\alpha(x,\theta), \quad A^* = \lambda^\alpha\lambda^\beta A^*_{\alpha\beta}(x,\theta), \tag{3.6}
$$

and $A_\alpha(x,\theta)$ and $A^*_{\alpha\beta}(x,\theta)$ are the super-Yang-Mills fields and antifields defined in [2]. The equation of motion $QV = 0$ implies that A and A^* satisfy

$$
\partial_m\partial^m A = \partial_m\partial^m A^* = D_m\partial^m A + \frac{1}{24}(\lambda\gamma^{m_1\dots m_5}\lambda)D_{m_1}\dots D_{m_4}\partial_{m_5}A^* = 0, \tag{3.7}
$$

$$
\lambda^\alpha D_\alpha A = \lambda^\alpha D_\alpha A^* = (\lambda\gamma^{m_1\dots m_5}\lambda)D_{m_1}D_{m_2}D_{m_3}A = (\lambda\gamma^{m_1\dots m_5}\lambda)D_{m_1}D_{m_2}\partial_{m_3}A = 0 \tag{3.8}
$$

where $D_m \equiv (\overline{\lambda}\gamma_m)^\alpha D_\alpha$ and $D_\alpha = \frac{\partial}{\partial\theta^\alpha} - \frac{1}{2}(\gamma^m\theta)_\alpha\partial_m$ is the supersymmetric derivative, which implies as in (2.5) that A and A^* describe an on-shell gluon and gluino.

4 Equivalence of amplitude prescriptions

After adding the non-minimal variables $(\lambda^+, \lambda_{ab}, \theta^+, \theta_{ab}, \overline{\lambda}_+, \overline{\lambda}^{ab}, r_+, r^{ab})$ and their conjugate momenta $(w_+, w^{ab}, p_+, p^{ab}, \overline{w}^+, \overline{w}_{ab}, s^+, s_{ab})$, the RNS N-point g-loop amplitude prescription is

$$
\mathcal{A} = \int d^{3g-3+N}\tau \langle \mathcal{N}\prod_{r=1}^{N}V_r(z_r)\prod_{I=1}^{3g-3+N}\int \mu_I b\prod_{j=1}^{2g-2-P}\{Q,\xi(y_j)\}\rangle \tag{4.1}
$$

where P is the sum of the pictures of the vertex operators, $\{Q,\xi(y_j)\}$ are the picture-raising operators, μ_I are Beltrami differentials associated with the $3g-3+N$ Teichmuller parameters τ_I, and

$$
\mathcal{N} = \exp\left[-Q\left(\theta^\alpha\overline{\lambda}_\alpha + \sum_{J=1}^{g}w_\alpha^{(J)}s^{(J)\alpha}\right)\right] = \exp\left[-\lambda^\alpha\overline{\lambda}_\alpha - \theta^\alpha r_\alpha - \sum_{J=1}^{g}(w_\alpha^{(J)}\overline{w}^{(J)\alpha} + d_\alpha^{(J)}s^{(J)\alpha})\right] \tag{4.2}
$$

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is a BRST-invariant regulator [3] which is needed to integrate over the zero modes of the non-minimal variables.

Using the BRST operator and vertex operator of (3.4) and (3.6), this amplitude can be computed in a manifestly spacetime supersymmetric manner where the picture-changing operator is

$$
\begin{aligned}
&\{Q,\xi\}= \\
&c\partial\xi+e^{\widetilde{\phi}}\left(b+\Pi^m\overline{\Gamma}_m-\frac{(\lambda\gamma^{mn}r)}{4(\lambda\overline{\lambda})}\overline{\Gamma}_m\overline{\Gamma}_n+w_\alpha\partial\theta^\alpha+s^\alpha\partial\overline{\lambda}_\alpha\right) \\
&+\frac{1}{120}(\lambda\gamma^{m_1\dots m_5}\lambda)(5e^{2\widetilde{\phi}}\overline{\Gamma}_{m_1}\dots\overline{\Gamma}_{m_4}\Pi_{m_5}+2\partial\eta e^{3\widetilde{\phi}}\overline{\Gamma}_{m_1}\dots\overline{\Gamma}_{m_5}(b+(w_\alpha+\partial\widetilde{\phi}\frac{\overline{\lambda}_\alpha}{(\lambda\overline{\lambda})})\partial\theta^\alpha+s^\alpha\partial\overline{\lambda}_\alpha)) \\
&+\frac{1}{120}\eta\partial[(\lambda\gamma^{m_1\dots m_5}\lambda)e^{3\widetilde{\phi}}\overline{\Gamma}_{m_1}\dots\overline{\Gamma}_{m_5}(b+(w_\alpha+\partial\widetilde{\phi}\frac{\overline{\lambda}_\alpha}{(\lambda\overline{\lambda})})\partial\theta^\alpha+s^\alpha\partial\overline{\lambda}_\alpha)].
\end{aligned}
\tag{4.3}
$$

One might worry that, as in the original pure spinor amplitude prescription, the factors of $(\lambda\overline{\lambda})^{-1}$ in $\overline{\Gamma}_m$ in the picture-changing operators might make the functional integral ill-defined near $(\lambda\overline{\lambda})=0$. Note that each factor of $(\lambda\overline{\lambda})^{-1}$ in $\overline{\Gamma}_m$ is accompanied by either a factor of $\overline{\lambda}_\alpha$ or r_α. So before including the factors of r_α coming from the regulator $\mathcal{N}$, all poles of order $(\overline{\lambda})^{-n}$ are multiplied by $(r)^n$.

However, Lorentz invariance of $\mathcal{A}$ in the RNS amplitude prescription implies that the integrand of $\mathcal{A}$ in (4.1) is independent of $\overline{\lambda}_\alpha$ up to possible surface terms. And because of the term $\int \overline{w}^\alpha r_\alpha$ in the BRST operator, independence of $\overline{\lambda}$ up to surface terms implies by BRST invariance that all factors of r in the integrand of $\mathcal{A}$ must be proportional to surface terms. So if surface terms can be ignored, there are no factors of r in the integrand of $\mathcal{A}$ which implies there are no poles when $(\lambda\overline{\lambda})\to 0$. The property that $\mathcal{A}$ in the pure spinor amplitude prescription is Lorentz invariant up to possible surface terms is analogous to the property that $\mathcal{A}$ in the RNS amplitude prescription is spacetime supersymmetric up to possible surface terms

To avoid problems when $(\lambda\overline{\lambda})\to 0$, the new pure spinor amplitude prescription of (4.1) must differ in 3 important ways from the original pure spinor amplitude prescription: the worldsheet variables include new "ghost" variables $(\widetilde{\beta},\widetilde{\gamma})$ and (b,c) of conformal weight $(2,-1)$; the super-Yang-Mills vertex operator depends on both the pure spinor superfields $A=\lambda^\alpha A_\alpha(x,\theta)$ and $A^*=\lambda^\alpha\lambda^\beta A^*_{\alpha\beta}(x,\theta)$; and the composite B ghost is replaced by insertion of the picture-changing operator of (4.3). Nevertheless, it will now be shown that the new prescription coincides with the original pure spinor prescription for F-term amplitudes in which at least one spacetime supersymmetry is preserved, i.e. when at least one of the 16 θ^α zero modes in the path integral comes from the regulator $\mathcal{N}$ and not from the vertex operators $\prod_{r=1}^N V_r$. To show this, first perform the similarity transformation by R_1 and R_2 of (3.5) which maps $Q\to Q'=\oint(\lambda^\alpha d_\alpha+\overline{w}^\alpha r_\alpha+b\widetilde{\gamma})$. After the similarity transformation,

$$
\int\mu b\to\int\mu(b+B+\widetilde{\beta}\partial c+\partial(\widetilde{\beta}c)) \quad \text{and} \quad \{Q,\xi\}\to be^{\phi}+\{Q',\frac{1}{120}e^{2\widetilde{\phi}}(\lambda\gamma^{m_1\dots m_5}\lambda)\overline{\Gamma}_{m_1}\dots\overline{\Gamma}_{m_5}\}
\tag{4.4}
$$

in the amplitude prescription of (4.1) where $B=\Pi^m\overline{\Gamma}_m-\frac{(\lambda\gamma^{mn}r)}{4(\lambda\overline{\lambda})}\overline{\Gamma}_m\overline{\Gamma}_n+w_\alpha\partial\theta^\alpha+s^\alpha\partial\overline{\lambda}_\alpha$ is the composite B ghost [6] in the original pure spinor formalism.

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Naively, the term $\{Q', \frac{1}{120} e^{2\widetilde{\phi}} (\lambda\gamma^{m_1 \ldots m_5}\lambda) \overline{\Gamma}_{m_1} \ldots \overline{\Gamma}_{m_5}\}$ in the picture-changing operator and the term $ce^{-2\widetilde{\phi}} \lambda^\alpha \lambda^\beta A^*_{\alpha\beta}(x,\theta) = [Q', c\partial c \partial \xi e^{-3\widetilde{\phi}} \lambda^\alpha \lambda^\beta A^*_{\alpha\beta}(x,\theta)]$ in the vertex operator can be dropped since they are BRST-trivial. However, one needs to be careful since BRST-trivial quantities $\{Q', \Lambda\}$ where Λ depends on $\overline{\lambda}_\alpha$ and all 16 θ^α zero modes may not decouple. For example, $\Lambda = (\lambda^3\theta^5)(\lambda\overline{\lambda} + r\theta)^{-1} \overline{\lambda}_\alpha \theta^\alpha = \ldots + (\lambda^3 \overline{\lambda} r^{10})(\lambda\overline{\lambda})^{-11}(\theta)^{16}$ satisfies $Q'(\Lambda) = (\lambda^3\theta^5)$ where $(\lambda^3\theta^5)$ denotes $(\theta\gamma_{mnp}\theta)(\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\lambda\gamma^p\theta)$. But $\langle \mathcal{N}(\lambda^3\theta^5)\rangle$ is non-zero, which implies that $Q'(\Lambda)$ does not decouple. Note that terms which explicitly depend on $\overline{\lambda}_\alpha$ are not Lorentz invariant in terms of the RNS variables and can have poles when $(\lambda\overline{\lambda}) \to 0$.

Fortunately, the above BRST-trivial terms in the picture-changing operator and vertex operator decouple for F-term amplitudes since Λ cannot depend on all 16 θ^α zero modes if at least one of the θ^α zero modes must come from $\mathcal{N}$. As discussed in [9], any θ^α zero mode coming from $\mathcal{N}$ of (4.2) is accompanied by r_α, which implies that F-term amplitudes cannot have terms proportional to $(\lambda\overline{\lambda})^{-11}$ which would make the functional integral ill-defined. After dropping the above BRST-trivial terms, the functional integral over the (b,c) and $(\widetilde{\beta}, \widetilde{\gamma})$ ghosts cancel each other out and one obtains the usual pure spinor amplitude prescription $\mathcal{A} = \int d^{3g-3+N}\tau \langle \mathcal{N} \prod_{r=1}^{N} \lambda^\alpha A_{r\alpha}(z_r) \prod_{I=1}^{3g-3+N} \int \mu_I B\rangle$. So for F-term amplitudes, the RNS prescription has been proven to agree with the original pure spinor amplitude prescription.

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References

[1] E. Witten, $D = 10$ *superstring theory*, *Prog. Math. Phys.* **9** (1983) 395 [INSPIRE].

[2] N. Berkovits, *Covariant quantization of the superparticle using pure spinors*, *JHEP* **09** (2001) 016 [hep-th/0105050] [INSPIRE].

[3] N. Berkovits, *Pure spinor formalism as an* $N = 2$ *topological string*, *JHEP* **10** (2005) 089 [hep-th/0509120] [INSPIRE].

[4] A. Sen and E. Witten, *Filling the gaps with PCO's*, *JHEP* **09** (2015) 004 [arXiv:1504.00609] [INSPIRE].

[5] H. Gomez and C.R. Mafra, *The closed-string 3-loop amplitude and S-duality*, *JHEP* **10** (2013) 217 [arXiv:1308.6567] [INSPIRE].

[6] N. Berkovits, *Dynamical twisting and the b ghost in the pure spinor formalism*, *JHEP* **06** (2013) 091 [arXiv:1305.0693] [INSPIRE].

[7] E. D'Hoker, C.R. Mafra, B. Pioline and O. Schlotterer, *Two-loop superstring five-point amplitudes. Part I. Construction via chiral splitting and pure spinors*, *JHEP* **08** (2020) 135 [arXiv:2006.05270] [INSPIRE].

[8] N. Berkovits, *Quantization of the superstring with manifest $U(5)$ superPoincare invariance*, *Phys. Lett. B* **457** (1999) 94 [hep-th/9902099] [INSPIRE].

[9] N. Berkovits and N. Nekrasov, *Multiloop superstring amplitudes from non-minimal pure spinor formalism*, *JHEP* **12** (2006) 029 [hep-th/0609012] [INSPIRE].

[10] W. Siegel, *Classical Superstring Mechanics*, *Nucl. Phys. B* **263** (1986) 93 [INSPIRE].