Fermion localization on a split brane

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In this work we analyze the localization of fermions on a brane embedded in five-dimensional, warped and nonwarped, space-time. In both cases we use the same nonlinear theoretical model with a non-polynomial potential featuring a self-interacting scalar field whose minimum energy solution is a soliton (a kink) which can be continuously deformed into a two-kink. Thus a single brane splits into two branes. The behavior of spin $1/2$ fermions wave functions on the split brane depends on the coupling of fermions to the scalar field and on the geometry of the space-time.

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I. INTRODUCTION

The idea that our Universe can be realized inside a domain wall embedded in a five-dimensional world [1] has provided many creative ways to solve the hierarchy of interactions problem [2,3] and the cosmological constant problem [4] in worlds with large extra dimensions, without resorting to the compactification of the extra dimension. It has also been shown [5] that the effective gravitational potential between two particles recovers the Newtonian behavior, since one has localization of gravitons on a thin brane in five-dimensional space-time with a warped geometry and the cosmological constant is related to the brane tension. Localization of matter (spin-zero, spin-$1/2$ and spin-$3/2$) in the Randall-Sundrum framework was shown to be possible, under certain conditions over the brane tension [6]. The localization of spin $1/2$ fermions on thin branes is due to a soliton, via the mechanism created by Jackiw and Rebbi [7] to demonstrate the phenomenon of fermion charge fractionization. By its turn, the domain wall which we would live in, according to the scenario proposed by Rubakov and Shaposhnikov [1], is the topological defect provided by the very same soliton.

The Rubakov-Shaposhnikov scenario [1] has been extended to five-dimensional warped space-time by means of a self-interacting scalar field, or a set of (self-)interacting scalar fields also coupled to gravity [8]. In those nonlinear models, also inspired on previous studies on the stabilization of gravity fluctuations on domain walls in supergravity theories [9], the thick brane solutions are minimum energy configurations (in many cases they are Bogomol'nyi-Prasad-Sommerfield (BPS) solutions) that separates the space in two patches with a peculiar warp factor whose asymptotic behavior is an anti-de Sitter (AdS_5) space.

In some models the minimum energy configurations are wide solitons [10–12], also called double-wall. Double-wall configurations also appear as solutions, degenerate in energy [13], of a model with two interacting scalar fields [14,15], and they have been called degenerate Bloch branes. The model used in [14] also comprises critical Bloch branes [13] which are analogues of the extreme domain walls described in [9]. Such a variety of solutions [16] is due to an arbitrary constant of integration of the equations of motion. The continuous deformation from a single brane (domain wall) to a double-brane, up to an extreme one, by varying the constant of integration, is similar to the phenomenon of brane splitting discussed in [17]. Such phenomenon might also be described by effective nonpolynomial potentials [18] constructed from the model with two scalar fields [13,14,16] with the advantage of having a model whose minimum energy solutions are obtained analytically and are the BPS ones.

The localization of fermions on a double-wall (double-brane) in warped space-time has been studied in [19] and in [20]. It has been shown that the double-wall can localize massless fermions. Notwithstanding, a close inspection on the behavior of the zero mode eigenfunction in those cases reveals that it is peaked just in the region between the branes and has tails inside the branes, then the possibility of detecting massless fermions inside the brane is relatively small. We notice that this behavior is due to the Yukawa coupling, $\phi\bar{\Psi}\Psi$, of fermions to the soliton. The zero mode eigenfunction should be peaked around the location of the core of the brane, in order to have a high probability to find the massless fermion inside the brane. Thus, when there is a brane splitting, the peak of the zero mode eigenfunction either should be stuck to one of the branes or it would follow the trend of the brane, that is, it also should split.

In this work we show that the first behavior can be achieved by adding a convenient five-dimensional mass term to the simplest Yukawa coupling while the second behavior is obtained by coupling the fermion to the scalar field in a way that is reminiscent from $N = 1$ supersymmetry (SUSY), namely $W_{\phi\phi}\bar{\Psi}\Psi$, where $W_{\phi\phi}$ is the 2nd derivative of the superpotential of the field theory model with respect to the field taken at the BPS configuration.

We develop the calculations by using one of the models obtained in [18], but they could also be done by using any

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other nonlinear model which admits two-kink solutions for the scalar field. We also analyze the localization of massless and massive fermions in the Rubakov-Shaposhnikov framework [1], that is, in a nonwarped space-time. In the latter case one clearly sees that the coupling $\phi\bar{\Psi}\Psi$ provides an effective single well potential where the fermions would be trapped in, but that well is just in the region between the branes. On the other hand, the constant five-dimensional mass added to the Yukawa coupling provides a well potential whose bottom is located at the core of one of the branes, and the coupling $W_{\phi\phi}\bar{\Psi}\Psi$ provides a double-well potential with wells at the core of the branes. In the case of flat space-time, the Numerov method is used to obtain the massive localized modes. We emphasize the SUSY inspired coupling, with which there is room to prepare a chiral mixed state of quasidegenerate massless and tiny mass fermion states. We also show that such a mixed state describes the tunneling of a fermion between the two branes, with time of tunneling inversely proportional to the tiny mass which, in its turn, decreases as the distance between the walls increases.

We would like to warn the reader that the brane splitting considered here is not a dynamic process; in fact, the distance between the walls is one of the parameters of the nonlinear potential and the tunneling of fermions is analyzed at a fixed distance between the walls.

In the next section we show the main features of the nonlinear model coupled to gravity in five-dimensional space-time [18]. In the third section we are concerned about the localization of massless fermions on a split brane in warped space-time. In the fourth section we deal with the localization and tunneling of massive fermions in flat five-dimensional space-time. A few remarks on fermion localization on double-walls are left to the last section.

II. THE MODEL, THE DOUBLE-BRANE AND THE WARP FACTOR

The model we are going to deal with includes a self-interacting scalar field coupled minimally to gravity with one extra space dimension, denoted by r. The action that leads to Euler-Lagrange for the scalar field and Einstein equations is given by

$$S = \int d^4xdy\sqrt{|g|}\left(-\frac{1}{4}R + \frac{1}{2}g_{ab}\partial^a\phi\partial^b\phi - V(\phi)\right), \quad (1)$$

where $g \equiv \mathrm{Det}(g_{ab})$ and the metric is

$$ds^2 = g_{ab}dx^a dx^b = e^{2A(r)}\eta_{\mu\nu}dx^\mu dx^\nu - dr^2, \quad a, b = 0, \ldots, 4, \quad (2)$$

where $\eta_{\mu\nu}$ is the Minkowski metric and $e^{2A(r)}$ is the warp factor, which is supposed to depend only on the extra dimension r. The Greek indices run from 0 to 3.

We consider that the potential $V(\phi)$ can be written as

$$V(\phi) = \frac{1}{2}\left(\frac{dW(\phi)}{d\phi}\right)^2 - \frac{4}{3}(W(\phi))^2. \quad (3)$$

In this case the BPS solution of the following first-order differential equations

$$\frac{d\phi}{dr} = \pm\frac{dW(\phi)}{d\phi} = \pm W_\phi \quad \text{and} \quad \frac{dA}{dr} = \mp\frac{2}{3}W(\phi) \quad (4)$$

are also solutions of the second-order differential equations of motion in the static limit.

Here we take the superpotential $W(\phi)$:

$$W(\phi) = 2\mu\left[\phi\left(\frac{\phi^2}{3} + f^2 + \frac{b}{2}\sqrt{\phi^2 + f^2}\right) + \frac{bf^2}{2}\sinh^{-1}\left(\frac{\phi}{f}\right)\right], \quad (5)$$

with $f = \sqrt{b^2 - a^2}$ and $b < -a < 0$, we have

$$\frac{dW(\phi)}{d\phi} = W_\phi(\phi) = 2\mu(\phi^2 + f^2 + b\sqrt{\phi^2 + f^2}), \quad (6)$$

and the solutions of the first-order differential Eqs. (4) can be found straightforwardly. The classical configuration for the scalar field is given by

$$\phi = \pm a\frac{\sinh(2\mu a r)}{\cosh(2\mu a r) - b/f}, \quad (7)$$

where the (lower)upper sign stands for (anti-)kink configuration. Reference [18] has more details on the soliton profile, as well as on the behavior of the warp factor in this model. As has been noted in the second paper of Ref. [16], the solution (7) can be conveniently written as

$$\phi = \pm\frac{a}{2}[\tanh(\mu a(r + L)) + \tanh(\mu a(r - L))], \quad (8)$$

where we have identified $b = -a\coth(2\mu a L)$. The above expression can be seen as a merging of two kinks, namely $\phi_\pm = \frac{a}{2}[\pm 1 + \tanh(\mu a(r \mp L))]$, which are localized at $r = -L$ and $r = +L$.

Concerning the behavior of the warp factor, one can see that it separates the space along the extra dimension in two similar regions, whose asymptotic behavior is an AdS_5 space and, as L increases, a double-brane structure is triggered at a specific value of L. Such a double-brane structure is evident from the peaks of the Ricci scalar $R = -(8d^2A/dr^2 + 20(dA/dr)^2)$ just at the points where the core of the branes are located (see Fig. 1). In the limit $L \to \infty$ the branes are infinitely separated from each other, that is an extreme brane. Such an extreme case is discussed in Ref. [13,21] where the brane is called a *critical Bloch brane*.

It worth mentioning that the potential $V(\phi)$ in this model is unbounded from below. This is a peculiarity of almost all models for thick branes whose potential presents the

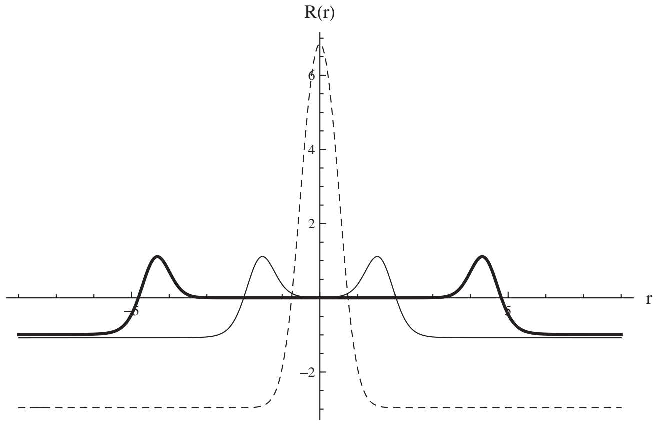

FIG. 1. Ricci scalar for $L = 0.01$ (dashed line), $L = 1.6$ (thin solid line) and $L = 4.5$ (thick solid line) evidences the formation of a double-wall structure as L increases.

structure in Eq. (3). Perhaps, the only exception is the oscillating potential introduced in the seventh paper in Ref. [8]. That is the very high price one pays in order to have exact minimum energy solutions. The stability of the minimum energy solutions against linear small perturbations for such kind of models is done in most of the papers in Ref. [8] and in [18] for the model considered here. One finds two coupled differential equations, one for the fluctuations of the metric and the other one for the fluctuation of the scalar field. Those equations are difficult to be decoupled. Nevertheless, the equation for the fluctuation of the metric can be decoupled in the case of transverse and traceless modes and it can be shown that the massless mode of the fluctuations of the metric are localized on the brane. Moreover, in most of the cases, the only localized modes are the massless ones, for the metric and the scalar field, and there is no room for tachyon models.

Thus one can say that, up to small fluctuations, such a system is very robust, even when one considers the influence of fermion fields. As explained in the next sections, one looks for fermion physical states localized inside the brane. Those physical states could be thought as the seeds for the realization of the observable matter in the Universe.

III. LOCALIZATION OF FERMIONS: WARPED SPACE-TIME

In this section we analyze the localization of fermions on the brane described in the previous section by also taking into account the curvature of the space. The problem is approached by searching normalized solutions of the Dirac equation for fermions coupled with the scalar field by a general Yukawa coupling $\bar{\Psi}F(\phi)\Psi$, where $F(\phi)$ is a functional of the field $\phi(r)$ taken at the classical solution. The functional form of $F(\phi)$, as well as the dependence of the metric on r, is crucial for the possible localization of fermions on the brane, as we discuss below.

The fermion action defined as

$$S_{\text{fermion}} = \int d^5x\sqrt{-g}(\bar{\Psi}i\Gamma^a D_a\Psi - \bar{\Psi}F(\phi)\Psi), \qquad (9)$$

leads to the Dirac equation

$$[i\Gamma^a D_a - F(\phi)]\Psi = 0. \qquad (10)$$

The gamma matrices satisfy the algebra $\{\Gamma^a, \Gamma^b\} = 2g^{ab}$, and can be defined in the irreducible representation as $\Gamma^\mu = e^{-A(y)}\gamma^\mu$, $\Gamma^5 = -i\gamma^5$ in terms of the 4×4 gamma matrices, γ^a, in flat space-time. By means of the definition of the covariant derivative $D_a = (\partial_a + \omega_a)$ and the metric in (2), one finds that the only nonvanishing components of the connection ω_a are $\omega_\mu = \frac{1}{2}e^A(\partial_r A)\gamma_\mu\gamma_5$ and that the Dirac equation turns out to be written as

$$\{i\gamma^\mu\partial_\mu + e^{A(r)}\gamma^5[\partial_r + 2\partial_r A(r)] - e^{A(r)}F(\phi)\}\Psi(x,r) = 0. \qquad (11)$$

To ease the separation of variables one can resort to the chiral decomposition (x stands for the four space-time coordinates)

$$\Psi(x,r) = \sum_n \psi_{Ln}(x)\alpha_{Ln}(r) + \sum_n \psi_{Rn}(x)\alpha_{Rn}(r), \qquad (12)$$

where $\psi_{L(R)n}(x)$ are the chiral modes which satisfy $\gamma^5\psi_{Ln}(x) = -\psi_{Ln}(x)$, $\gamma^5\psi_{Rn}(x) = \psi_{Rn}(x)$ and also the four-dimensional massive Dirac equations $i\gamma^\mu\partial_\mu\psi_{L(R)n} = m_n\psi_{R(L)n}$. Then, we find the following differential equations for the r-dependent scalar parts of the spinor:

$$\alpha'_{Rn} + [2A' - F(\phi)]\alpha_{Rn} = -m_n e^{-A}\alpha_{Ln}, \qquad (13)$$

and

$$\alpha'_{Ln} + [2A' + F(\phi)]\alpha_{Ln} = m_n e^{-A}\alpha_{Rn}, \qquad (14)$$

where the prime stands for the first-derivative with respect to r. The equations above can be put in a more familiar form, namely,

$$R'_n - F(\phi)R_n = -m_n e^{-A}L_n \quad \text{and}$$
$$L'_n + F(\phi)L_n = m_n e^{-A}R_n, \qquad (15)$$

by using the redefinitions $\alpha_{Rn}(r) = e^{-2A(r)}R_n(r)$ and $\alpha_{Ln}(r) = e^{-2A(r)}L_n(r)$.

In the brane world scenario massive ($m_n \neq 0$) as well as massless ($m_n = 0$) fermions might be localized inside the brane, depending on the functional coupling $F(\phi)$. The issue of localization of massive modes on the brane is not going to be discussed in this section, even though, we would like to remark that the analysis of the possible massive spectrum of bound states is not as simple as the search for localized massless modes [21,22]. Moreover, normalizable $\alpha_{R(L)n}(r)$-functions are guaranteed if they satisfy the orthonormalization relations

$$\int_{-\infty}^{\infty} e^{3A(r)}\alpha_{Ln}(r)\alpha_{Lm}(r)dr = \int_{-\infty}^{\infty} e^{3A(r)}\alpha_{Rn}(r)\alpha_{Rm}(r)dr = \delta_{nm},$$
$$\int_{-\infty}^{\infty} e^{3A(r)}\alpha_{Ln}(r)\alpha_{Rm}(r)dr = 0. \tag{16}$$

From these relations one can note that the warp factor is crucial to determine the localization of whatever mode.

Particularly, for the massless mode one finds that

$$\alpha_{R_0}(r) = N_{R_0}\exp[-2A(r) + \int^r F(r')dr']$$
$$\alpha_{L_0}(r) = N_{L_0}\exp[-2A(r) - \int^r F(r')dr'], \tag{17}$$

where we have used $F(r) = F(\phi(r))$ and $N_{R(L)_0}$ are constants of normalization that are found according to the normalization relations

$$|N_{R_0}|^2 \int_{-\infty}^{\infty} e^{-A(r)+2\int^r F(r')dr'}dr = |N_{L_0}|^2 \int_{-\infty}^{\infty} e^{-A(r)-2\int^r F(r')dr'}dr = 1. \tag{18}$$

One can note that the normalization conditions are determined by the asymptotic behavior of the integrands in the expressions above and that the factor $\exp[-A(r)]$ goes asymptotically to infinity, hence one has to choose $F(\phi(r))$ in such a way that the decreasing of $\exp[\pm\int^r F(r')dr']$ is faster than the increasing of $\exp[-A(r)]$, but this choice does not guarantee the normalization of both chiralities, because of the different signs, $\pm$, in the exponent, that is, if $\alpha_{R_0}(r)$ is normalizable, $\alpha_{L_0}(r)$ is not and vice-versa.

Moreover, in the model considered here the brane splits into two and one wants to know what should be the behavior of the massless mode under such a splitting. Naturally, such a behavior depends on the functional $F(\phi(r))$, for which there is no *a priori* prescription, except for the normalization conditions imposed over the r-dependent part of the wave functions as remarked above. We have analyzed the behavior of the massless chiral modes for two different cases, namely, $F(\phi) = \eta\phi(r)$, $F(\phi) = -\eta W_{\phi\phi}$, where $\eta > 0$ is a coupling constant, and $W_{\phi\phi}$ is the second-derivative of the superpotential (5) with respect to ϕ taken at the kink configuration in (7) with $\mu = a = 1$.

The first case is the simplest Yukawa coupling of fermions to a scalar field which provides a localized massless left-handed mode. In this case, $\alpha_{L_0}(r)$ does not follow the brane splitting, that is, while the double-wall is formed, the peak of $\alpha_{L_0}(r)$ is midway between the two walls, there being a small probability density to find such a mode on the core of the branes. The behavior of non-normalized $\alpha_{L_0}(r)$ is shown in the first frame of Fig. 2. In order to have the peak $\alpha_{L_0}(r)$ just in the core of at least one of the branes, such that matter should be realized inside that brane, one has to add a five-dimensional constant mass to the simplest Yukawa coupling, namely, $F(\phi) = \eta(\phi(r) + M)$. That mass term has to be chosen in such a way that the peak of $\alpha_{L_0}(r)$ follows the core of that brane and that the normalization of $\alpha_{L_0}(r)$ is not spoiled. An adequate choice is $M = \frac{a}{2}\tanh(2\mu a L)$. From the second frame of Fig. 2 one

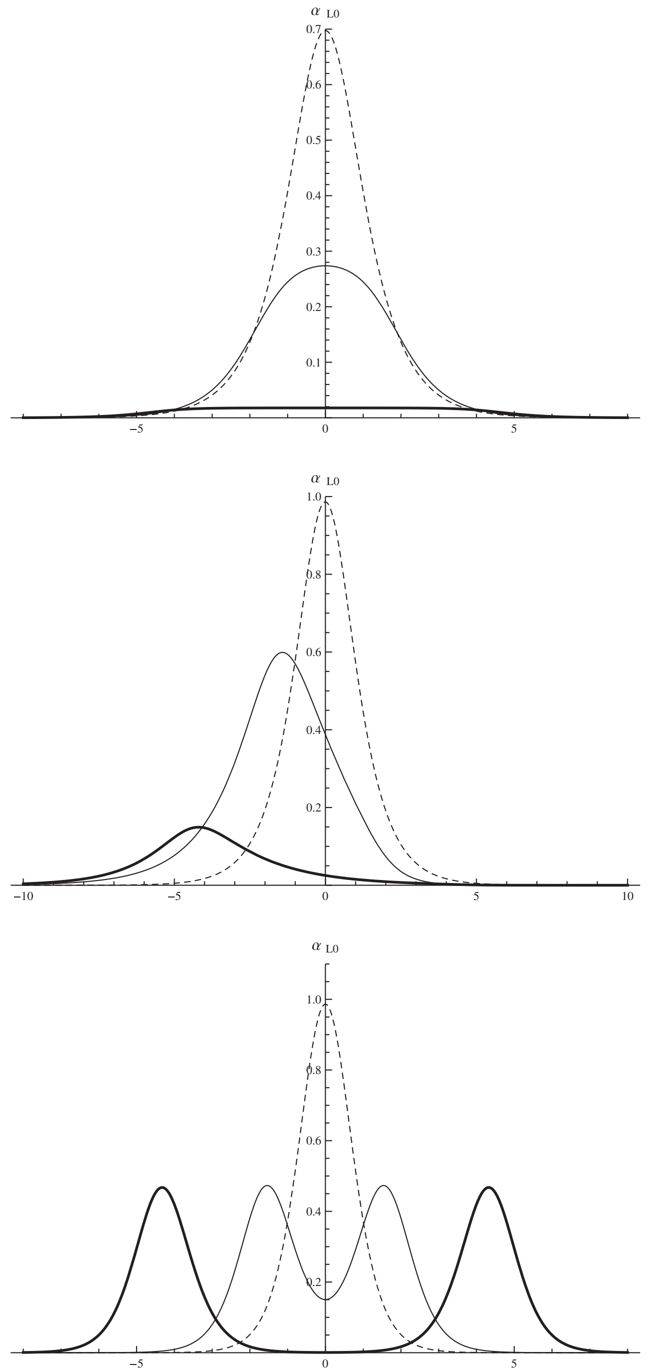

FIG. 2. $\alpha_{L_0}(r)$ (warped geometry) in the cases $F(\phi) = \phi(r)$ (upper panel), $F(\phi) = \phi(r) + M$ (middle panel) and $F(\phi) = -W_{\phi\phi}$ (lower panel), for $L = 0.01$ (dashed line), $L = 1.6$ (thin solid line) and $L = 4.5$ (thick solid line).

can see that $\alpha_{L_0}(r)$ is peaked at the point where the core of the brane to the left is located. By inverting the sign of M one gets the peak of $\alpha_{L_0}(r)$ located at the core of the brane to the right. This five-dimensional mass can assume slightly different values for different fermion flavors in order to have distinguishable fermions confined at different points inside the same brane, as it is done in [23].

However, by choosing one of the branes as the preferable one where the Universe would be realized, it naturally breaks the symmetry under a r-coordinate inversion. One way to keep such a symmetry is by realizing the localization of both branes. One can achieve that by taking the second functional coupling mentioned above, namely, $F(\phi) = -\eta W_{\phi\phi}$. In this case the $\alpha_{L_0}(r)$ presents two peaks, each one located on the core of each one of the branes. Moreover, as can be seen from the third frame in Fig. 2, $\alpha_{L_0}(r)$ smoothly decreases on the bulk between the two branes as L increases, signalizing a small probability for the massless left-handed mode to be found between the branes. Moreover, we have found that such a coupling comes from an underlying $N = 1$ supersymmetry, which can be appreciated in the framework presented in the next section, namely, when the brane is embedded in a five-dimensional flat space-time.

IV. LOCALIZATION OF FERMIONS: FLAT SPACE-TIME

In this section we adopt the usual analysis to find fermion bound states under the action of the scalar field. Particularly, we focus on the scenario proposed in [1], that is, a brane (or domain wall) immersed in a five-dimensional flat space-time. Now, one no longer has the inconvenience to work with a field theory model whose potential is unbounded from below. In fact, the nonlinear field theoretical potential $V(\phi) = W_\phi^2/2$ the same BPS solution in (7).

The action for the fermion field is given in (9) with $g_{ab} = \eta_{ab}$ the Minkowskian metric, $D_a \equiv \partial_a$ and the irreducible form of the gamma matrices, $\Gamma^\mu = \gamma^\mu$, is going to be used. The chiral decomposition (12) can also be used to separate the four space-time variables, x^μ, from the variable r. Now, the r-dependent functions appearing in the chiral decomposition obey the following equations:

$$\alpha'_{Rn} - F(\phi)\alpha_{Rn} = -m_n\alpha_{Ln}, \tag{19}$$

and

$$\alpha'_{Ln} + F(\phi)\alpha_{Ln} = m_n\alpha_{Rn}. \tag{20}$$

Particularly, for the massless mode one finds

$$\begin{aligned} \alpha_{R_0}(r) &= N_{R_0}\exp\Big[+\int^r F(r')dr'\Big] \\ \alpha_{L_0}(r) &= N_{L_0}\exp\Big[-\int^r F(r')dr'\Big]. \end{aligned} \tag{21}$$

Now, the normalization of the massless modes depends on the asymptotic behavior of $\int^r F(r')dr'$ only, hence one usually has a unpaired (isolated) chiral (left-handed or right-handed) zero mode, which is the main feature for having fermion number fractionization, as shown in [7].

We again analyze the behavior of the massless modes by setting $F(\phi) = \eta\phi(r)$, $F(\phi) = \eta(\phi(r) + M)$ and $F(\phi) = -\eta W_{\phi\phi}$, with $\eta > 0$. In the first case we have

$$\begin{aligned} &\alpha_{R_0}(r) = 0 \quad \text{and} \\ &\alpha_{L_0}(r) = N_{L_0}(\text{Cosh}2L + \text{Cosh}2r)^{-\eta/2}. \end{aligned} \tag{22}$$

As in the previous section, the function $\alpha_{L_0}(r)$ is symmetric in r and has a peak at $r = 0$, hence the massless left-handed mode is localized in the region between the branes with a very small probability density at the core of the branes. The localization of massless fermions is achieved with the Yukawa coupling of the second type, where $M = \frac{1}{2}\tanh 2L$. As one can check, $\alpha_{L_0}(r)$ is given by

$$\alpha_{L_0}(r) = N_{L_0}e^{-\eta M}(\text{Cosh}2L + \text{Cosh}2r)^{-\eta/2}. \tag{23}$$

For $F(\phi) = -\eta W_{\phi\phi}$ one finds

$$\begin{aligned} &\alpha_{R_0}(r) = 0 \quad \text{and} \\ &\alpha_{L_0}(r) = N_{L_0}(\text{sech}^2(r + L) + \text{sech}^2(r - L))^\eta. \end{aligned} \tag{24}$$

From Fig. 3 we can note that $\alpha_{L_0}(r)$ is symmetric in r, has no nodes and exhibits peaks at the core of the branes. One can also note that the probability density to find the left-handed massless mode in the midway between the branes decreases as L increases.

As one knows, Eqs. (19) and (20) can be decoupled to 2 s-order differential equations, namely,

$$\begin{aligned} -\alpha''_{Rn} + U_R(r)\alpha_{Rn} &= m_n^2\alpha_{Rn}, \\ -\alpha''_{Ln} + U_L(r)\alpha_{Ln} &= m_n^2\alpha_{Ln}, \end{aligned} \tag{25}$$

where $U_R(r) = F^2 + F'$ and $U_L(r) = F^2 - F'$. The equations above are time-independent Schrödinger equations, whose corresponding Hamiltonians are superpartners of each other, that is, one has a quantum mechanics supersymmetry. This is formally true whatever the functional coupling $F(\phi)$ is. Such a nice feature has allowed us to find that in the case $F(\phi) = \eta(\phi(r) + \frac{1}{2}\tanh 2L)$ the potentials $U_R(r)$ and $U_L(r)$ do not support any localized states which would correspond to a massive fermion. This is because $U_R(r)$, in this case, presents a barrier where $U_L(r)$ is a well potential, that is, around $r = -L$. Moreover, from supersymmetric quantum mechanics one can show that any localized state of $U_R(r)$ would imply into a localized state of $U_L(r)$, but the zero mode which belongs only to $U_L(r)$. We have also found that in the case $F(\phi) = \eta\phi(r)$, both $U_R(r)$ and $U_L(r)$ can support massive localized states, depending on the value of L. The profiles of those states,

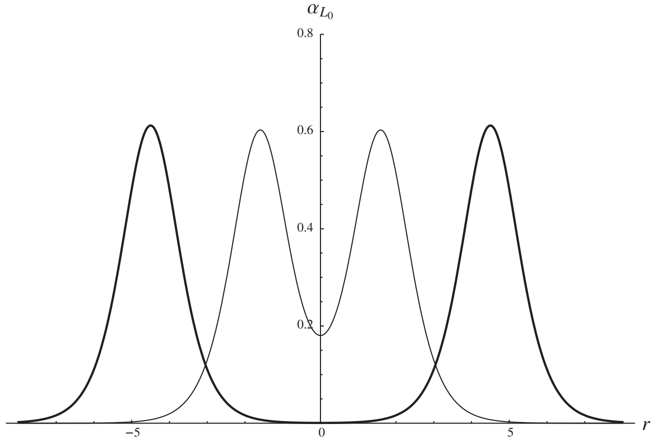

FIG. 3. $\alpha_{L_0}(r)$ (flat space-time) in the case $F(\phi) = -W_{\phi\phi}$, for $L = 1.6$ (thin solid line) and $L = 4.5$ (thick solid line).

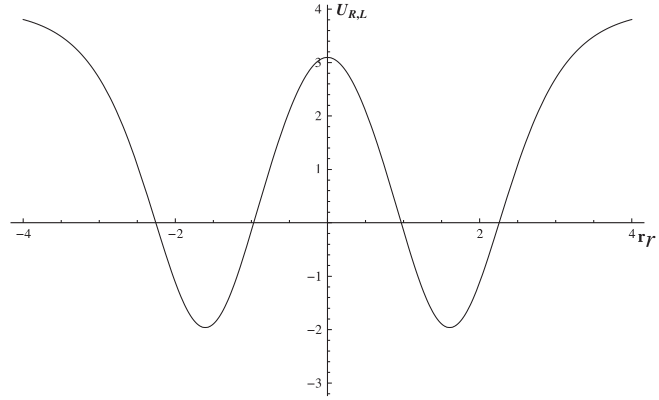

FIG. 4. Effective potentials of Eq. (24) with $L = 1.5$. $U_L(r)$ (solid line), $U_R(r)$ (dashed line).

as well as the energy (m_n^2) have been obtained by means of the Numerov method for values of $L \geq 3$. Furthermore, since the effective potentials are even in r, those states have well-defined parity. From that we have found that, except for α_{L1} and α_{R2}, all the other states have a prominent probability density in the bulk midway between the two branes and that all the states, except for α_{R1}, have prominent probability density around $r = \pm L$, that is, where the cores of the branes are located.

In the case $F(\phi) = -\eta W_{\phi\phi}$ one finds that $U_R(r) = (\eta W_{\phi\phi})^2 - \eta W'_{\phi\phi}$ and $U_L(r) = (\eta W_{\phi\phi})^2 + \eta W'_{\phi\phi}$, and that the supersymmetry at the quantum mechanics level seems to be a reflection of a supersymmetry at the fundamental level. In other words, one can note that the r-dependent part of the excitations of the scalar field (*branons*), picked up to quadratic terms on the fundamental Lagrangian density in flat space-time $\mathcal{L} = \frac{1}{2} \times (\eta_{ab}\partial^a\phi\partial^b\phi - W_\phi^2)$, obeys a time-dependent Schrödinger equation similar to the one obeyed by $\alpha_{Ln}(r)$ with $\eta = 1$, that is, with an effective potential given by $U_{\mathrm{eff}}(r) = (W_{\phi\phi})^2 + W'_{\phi\phi}$, resulting identical mass spectrum for *branons* (bosonic excitations) and fermions.

Figure 4 shows the form of the potentials $U_R(r)$ and $U_L(r)$ for a specific value of L and $\eta = 1$. For values of L close to zero, $U_L(r)$ is a single well potential, which starts to be deformed into a double-well potential as L approaches to a critical value L_c, that is determined by the condition $U''_L(r = 0) = 0$; for $L \gtrsim L_c$, dimples are observed around $r = \pm L$, and a remarkable double-well is observed for the value of $L = l$ corresponding to $U_L(r = 0) = 0$, signalizing the possible entrapment of a massive state, besides the, always present, massless one. This possible appearance of a massive bound state can also be seen from the behavior of $U_R(r)$, which is a single well potential whose deepness and width increase as L increases.

One can also observe that $U_L(r) = 2(2 - 3\mathrm{sech}^2 r)$ and $U_R(r) = 2(2 - \mathrm{sech}^2 r)$ for $L = 0$ and $\eta = 1$; such that the first potential admits two bound states and the later admits only one bound state. The fundamental state of $U_L(r)$ for $L = 0$ and $\eta = 1$ is $\alpha_{L_0}(r) \simeq \mathrm{sech}^2 r$, whereas the first excited state is $\alpha_{L_1}(r) \simeq \mathrm{sech} r \tanh r$ and the fundamental state of $U_R(r)$ for $L = 0$ and $\eta = 1$ is $\alpha_{R_1}(r) \simeq \mathrm{sech} r$. Moreover, from the expression (24) with $\eta = 1$, one can construct an antisymmetric function, $\alpha_{L_1}(r) \sim \mathrm{sech}^2(r + L) - \mathrm{sech}^2(r - L)$, as an approximate expression for the first excited state of $U_L(r)$ when $L \gg l$. This approximation for the first excited state is commonly used to approach the discrete spectrum of double-well potentials [24,25].

We have used the results described above, together with the Numerov method to analyze the behavior of $\alpha_{L_1}(r)$, $\alpha_{R_1}(r)$ and the eigenvalue m_1^2 at intermediary values of L. Those behaviors are shown in Figs. 5 and 6. From them one can see that $\alpha_{R_1}(r)$ is mainly distributed on the region between the branes, hence there is a small probability for the massive right-handed mode being observed inside the wells where the universe(s) would be realized, whereas the probability density associated with $\alpha_{L_1}(r)$ is pronounced just on the core of the branes. In summary, at least one massive left-handed mode is localized on the branes. The eigenvalue m_1^2 decreases smoothly as L increases; that is an expected result when one is dealing with double-well potentials in nonrelativistic quantum mechanics. Thus, one can construct a mixed left-handed massive state with both the ground state and the first excited state, which are quasidegenerate for very large values of L, namely,

$$\Psi_{L,\mathrm{mix}}(t, r) = N(\alpha_{L_0}(r) + \alpha_{L_1}(r)e^{-im_1 t})\chi_L, \qquad (26)$$

where χ_L is a constant spinor which satisfies $\gamma^5\chi_L = -\chi_L$. We have tried to be cautious when proposing this mixed state, since we are assuming that there is a rest

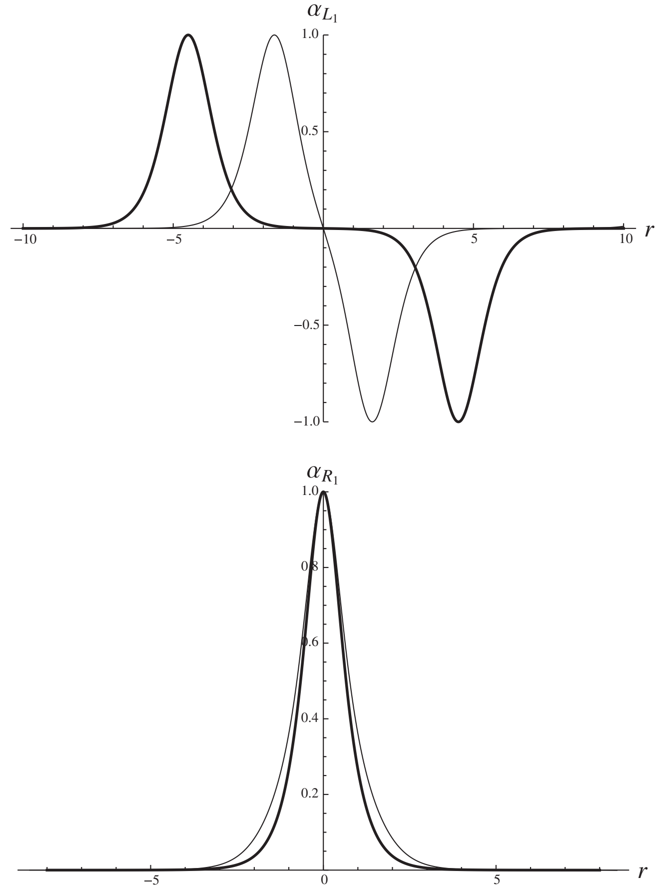

FIG. 5. $\alpha_{L1}(r)$ (upper) and $\alpha_{R1}(r)$ (lower) in the case $F(\phi) = -W_{\phi\phi}$, for $L = 1.6$ (thin solid line) and $L = 4.5$ (thick solid line), in flat space-time.

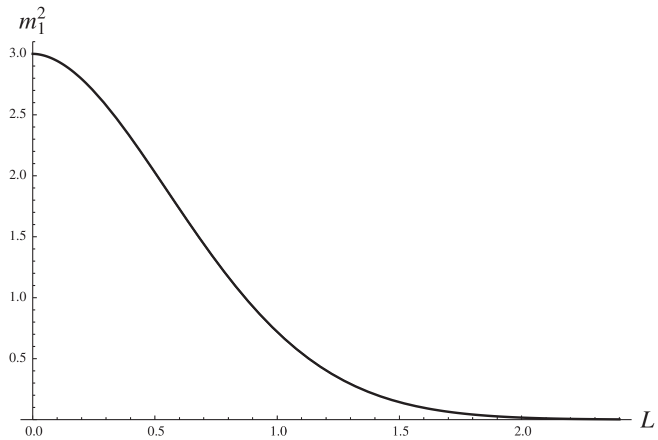

FIG. 6. The eigenvalue of the first excited state in (25) against L.

reference frame for the particle in such a mixed state. With this reasoning, the Dirac equation $i\gamma^\mu \partial_\mu \psi_{L,\text{mix}} = m_n \psi_R$ is satisfied, since $m_1 \gamma^0 \chi_L e^{-im_1 t} = m_1 \chi_R e^{-im_1 t}$ ($\gamma^5 \chi_R = \chi_R$) and there is no right-handed massless state, neither inside nor outside the branes. The probability density associated with this mixed state is given by

$$\rho(r,t) = |N|^2[\alpha_{L_0}(r)^2 + \alpha_{L_1}(r)^2 + 2\alpha_{L_0}(r)\alpha_{L_1}(r) \times \cos(m_1 t)], \tag{27}$$

which is an oscillating probability density with period of oscillation $T = h/m_1 c^2$, such that a tiny mass implies a long tunneling time. In this scenario the fermion tunnels from one brane to the other, likely found on both branes, but not simultaneously. As L increases, other massive localized states can be realized inside the branes. In fact, we have found numerically that there is room to one more massive state in the present case. The appearance of a tower of localized massive states is very dependent on the deepness and width of the double-well effective potential $U_L(r)$, that is ultimately dependent on the superpotential $W(\phi)$ whose classical solution is a kink that can be continuously deformed into a two-kink solution. In the next section we comment on the construction of such models.

V. CONCLUSIONS

We have studied the mechanism that leads to the localization of massless fermions on a split brane in the cases of warped and flat geometries. The brane is immersed in a five-dimensional space-time and is defined by the behavior of a scalar field coupled with gravity in the case of warped space-time. The nonpolynomial potential of the self-interacting scalar field which generates the split brane was introduced before in Ref. [18], but any other model which has deformable solitons as minimal energy configurations could be used as well; for example, a convenient ϕ^6 polynomial potential [26].

The case of flat geometry is more manageable than the case of warped geometry, because we obtain the localized modes in a simple way and because the possible entrapment of massive fermions can be easily seen. Moreover, in the Rubakov-Shaposhnikov framework, one can see that the functional coupling $F(\phi) = -W_{\phi\phi}$ leads to a SUSY at the quantum mechanics level, which is a reflection of a SUSY at a fundamental level, that is between fermions and *branons* (the excitations of the brane). Such a coupling has been shown to provide a convenient behavior for the massless fermion wave functions; the ground state wave function follows the brane splitting, i.e., as the brane splits into two branes the wave function is also split, with peaks at the cores of the two branes.

We have also shown that the simplest Yukawa coupling of fermions to scalar fields, $F(\phi) = \eta\phi(r)$, also provides localization of massless fermions. The corresponding massless fermion wave function present a peak just between the two branes, such that the observation of massless fermions is suppressed inside the branes. In order to localize massless fermions on one of the branes a five-dimensional constant and uniform mass M has to be added to the simplest Yukawa coupling. We have found that, under the conditions that the wave function is normalizable and that the peak of the wave function is located at the core of one of the branes, $M = \frac{1}{2}\tanh 2L$. Based on SUSY at the quantum mechanics level, we have also found that such a coupling does not support localized states associated to massive fermions.

Notwithstanding, in the case $F(\phi) = -W_{\phi\phi}$, massive fermions can be localized on the branes in flat space. We have analyzed the behavior of massive states and have found that a massive fermion can eventually tunnel between the branes. Such a tunneling lasts until the branes are infinitely separated from each other. It can also be observed that the number of massive localized states depends on the deepness and width of the quantum mechanics effective potentials, which are defined by the field theory model one chooses to deal with.

Other models, whose classical solution exhibit a two-kink profile (split brane) can afford, in the case of flat space-time, a tower of localized massive fermions states. A class of such models, called deformed models, have been proposed [27] as deformation of known models. We notice that those models can also be constructed from the deformation of zero mode excitations of well-known models whose classical solutions are single kinks. Our hope is that at least one of those models will provide a potential $V(\phi) = \frac{1}{2}W_\phi^2 - \frac{4}{3}W^2$, in a warped space-time scenario, which is bound from below. This is still under analysis and will be reported later elsewhere.

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