

Friedmann–Lemaître–Robertson–Walker braneworlds



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ABSTRACT

We study the cosmological evolution with nonsingular branes generated by a bulk scalar field coupled to gravity. The specific setup investigated leads to branes with a time-dependent warp factor. We calculate the effective Hubble parameter and the effective scale factor for the FLRW branes obtained solutions. The spatially dependent branes solutions also were found.

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1. Introduction

Braneworld models were introduced as a genuine branch of research in high energy physics since the outstanding works presented in Refs. [1–3]. After those works, several authors tried to construct models compatible with the Friedmann–Lemaître–Robertson–Walker (FLRW) cosmology. In fact, the modeling of realistic braneworld scenarios from the large scale physics point of view must include, at least in some level, the description encoded in the cosmological standard model. The attempts put forward to this program can be roughly separated into two categories: one dealing with infinitely thin branes, in which the extra dimensional effects enters as corrections to the Einstein equations via the presence of the Weyl tensor (the so-called ‘dark fluid’) and quadratic contributions to the stress tensor, and another one, whose modeling arrive at thick branes described by one or more bulk scalar fields. We refer the reader to the references [4] and [5] for quite complete reviews about thin and thick braneworld models and cosmological consequences, respectively. In the former approach, the corrections to the gravitational equations comes from the Gauss–Codazzi procedure and, potentially, all the relevant aspects of the four-dimensional cosmology are revisited [6]. It turns out, however, that an infinitely thin brane seems to be only an approximation of the more realistic case, at best. In fact, by keeping in mind the simple fact that at very short scales a classic gravitational theory must be replaced by its quantum counterpart (whatever it is), it is mandatory some thickness to the brane itself.

Unfortunately, in the thick brane context it is not possible, at the best of our knowledge, to apply the Gauss–Codazzi for-

malism. The reason is that it is not clear what (if any) are the Israel–Darmois junction conditions in a thick brane context, for early attempts see [7]. The junction conditions are at the heart of the projection procedure, and its lack makes the whole program fall apart. In this vein, the investigation of a five-dimensional thick braneworld setup whose four-dimensional part describes a FLRW universe is, indeed, in order. There are, nevertheless, only a few works addressing this crucial point. The cumbersome algebraic task inevitably present in this endeavor can be attributed as a cause for such. In Refs. [8,9] a more or less recent attempt to find out thick braneworld scenarios whose four dimensional part describes relevant cosmology, was taken into account. The general approach used in both cases take advantage of a well known functional form to the spacial part of the warp factor, as the one presented in Ref. [10]. While this is a clever starting point, it would be desirable to have a more powerful method arriving at relevant cosmological setups. In this paper we address ourselves to this task, starting from a five dimensional scalar field whose solution describes a FLRW brane, i.e., a braneworld whose four-dimensional part is given by a FLRW universe. The main braneworld characteristic, the warp factor, is also taken into account in the solution. Part of the solutions obtained via a given ansatz in Refs. [8,9] are recovered here as particular cases of a broad approach.

As a general procedure, we shall use a separable warp factor function. This might be seen as an oversimplification. In fact, there is an interesting discussion about non separable warp factors generating traveling-like defects in five dimensional space-times [11]. We stress, however, that in order to reproduce four-dimensional cosmological solutions (which is not the aim of Ref. [11]) separable warp factors are indeed in order. Besides, the adoption of a separable warp factor (in addition to being mathematically consistent, although not the most general possibility) is in line with the hierarchy problem approach via thick branes [12].

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We organize this paper as follows. In Section 2, we present the mathematical preliminaries supporting the metric ansatz with the four-dimensional background metric given by the Friedmann–Lemaître–Robertson–Walker. Then we derive gravitational equations and the expressions for the scalar field and its potential. In Section 3, we find the time-dependent FLRW brane solutions. We solve the field equations for two cases with respect to spatial curvature, i.e., $k = 0$ and $k \neq 0$. Going further we determine the spatial part of the set of equations in Section 4. In Section 5 we present the effective Hubble parameter as well as the effective scale factor for all the possibilities found previously. In the final section we conclude.

2. Field equations

The assumption of isotropy and homogeneity implies the large scale geometry described by a metric of the form

$$ds^2 = -dt^2 + a^2(t) \left[\frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \right], \quad (1)$$

in a synchronized coordinate system (a suitable set of coordinates called comoving coordinates). The comoving observers, also called Hubble observers, are the ones located at spacelike hypersurfaces accompanying the cosmic fluid, which is at rest with respect to such hypersurfaces. Here $a(t)$ is an arbitrary function of the cosmic time called scale factor and $k = 0, \pm 1$, denotes the spatial curvature of the universe for Minkowski, Riemann and Lobachevsky geometry, respectively.

Let us consider 5D spacetimes for which the metric takes the following form

$$ds^2 = a^2(t, y) \left\{ -dt^2 + u^2(t) \left[\frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \right] \right\} + b^2(t, y) dy^2, \quad (2)$$

where the background metric in 4D is given by the Friedmann–Lemaître–Robertson–Walker line element (1). The metric signature is given by $(- + + +)$. The function $a(t, y)$ is an warp factor with time and extra dimension dependence, while $u(t)$ performs the usual scale factor for an homogeneous and isotropic universe. The function $u(t)$ can, indeed, be absorbed in a rescaling of the time coordinate. For the sake of exposition, however, we shall keep it explicitly along the text in order to be in touch with the FLRW cosmological model. The function $b(t, y)$ shows the dynamics of the extra dimension at different times and positions in the bulk.

Let us consider the 5D action in the presence of a bulk scalar field with the potential $V(\phi)$ minimally coupled to the gravitational sector

$$S = \int d^5x \sqrt{-g} \left\{ 2M^3 R - \frac{1}{2} g^{MN} \nabla_M \phi \nabla_N \phi - V(\phi) \right\}, \quad (3)$$

where M is the Planck mass and R is the five-dimensional Ricci scalar. In general we suppose that the scalar field ϕ depends only on time and the extra dimension y .

The Einstein equations read

$$R_{MN} - \frac{1}{2} g_{MN} R = \frac{1}{4M^3} T_{MN}, \quad (4)$$

and the energy momentum-tensor T_{MN} for the scalar field $\phi(t, y)$ is

$$T_{MN} = \nabla_M \phi \nabla_N \phi - g_{MN} \left(\frac{1}{2} g^{AB} \nabla_A \phi \nabla_B \phi + V(\phi) \right). \quad (5)$$

The time–time component of the field equations for the space–time under consideration is given by

$$3 \left[\frac{1}{a^2} \frac{\dot{a}^2}{a^2} - \frac{1}{b^2} \left(\frac{a''}{a} + \frac{a'^2}{a^2} \right) + \frac{1}{b^2} \frac{a'}{a} \frac{b'}{b} \right. \\ \left. + \frac{1}{a^2} \left(2 \frac{\dot{a}}{a} \frac{\dot{u}}{u} + \frac{\dot{a}}{a} \frac{\dot{b}}{b} + \frac{\dot{u}^2}{u^2} + \frac{\dot{b}}{b} \frac{\dot{u}}{u} + \frac{k}{u^2} \right) \right] \\ = \frac{1}{2b^2} \dot{\phi}^2 + \frac{1}{2a^2} \dot{\phi}^2 + V(\phi), \quad (6)$$

whilst the space components give

$$\frac{1}{a^2} \left[2 \frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} + \frac{\dot{a}}{a} \frac{\dot{b}}{b} + 4 \frac{\dot{a}}{a} \frac{\dot{u}}{u} + 2 \frac{\dot{b}}{b} \frac{\dot{u}}{u} + \frac{\ddot{b}}{b} + 2 \frac{\ddot{u}}{u} + \frac{\dot{u}^2}{u^2} + \frac{k}{u^2} \right] \\ - \frac{3}{b^2} \left[\frac{a''}{a} + \frac{a'^2}{a^2} - \frac{a'}{a} \frac{b'}{b} \right] = \frac{1}{2b^2} \dot{\phi}^2 - \frac{1}{2a^2} \dot{\phi}^2 + V(\phi). \quad (7)$$

The extra dimensional part contributes with

$$\left[\frac{a'}{a} \frac{\dot{a}}{a} + \frac{a'}{a} \frac{\dot{b}}{b} - \frac{\dot{a}'}{a} \right] = \frac{1}{3} \dot{\phi} \phi', \quad (8)$$

$$3 \left[2 \frac{a'^2}{a^2} \frac{1}{b^2} - \frac{\ddot{a}}{a} \frac{1}{a^2} - 3 \frac{\dot{a}}{a} \frac{\dot{u}}{u} \frac{1}{a^2} - \frac{\dot{u}^2}{u^2} \frac{1}{a^2} - \frac{\ddot{u}}{u} \frac{1}{a^2} - \frac{k}{a^2 u^2} \right] \\ = \frac{1}{2a^2} \dot{\phi}^2 + \frac{1}{2b^2} \dot{\phi}^2 - V(\phi). \quad (9)$$

Finally, the scalar equation of motion reads

$$\frac{1}{b^2} \phi'' - \frac{1}{a^2} \ddot{\phi} + \frac{4}{b^2} \frac{a'}{a} \phi' - \frac{2}{a^2} \frac{\dot{a}}{a} \dot{\phi} - \frac{1}{a^2} \frac{\dot{b}}{b} \dot{\phi} - \frac{3}{a^2} \frac{\dot{u}}{u} \dot{\phi} - \frac{1}{b^2} \frac{b'}{b} \phi' \\ - \frac{dV}{d\phi} = 0, \quad (10)$$

where a dot denotes a derivative with respect to t , and a prime represents a derivative with respect to the extra dimension y .

By combining Eqs. (6)–(7) and (9), we arrive at convenient expressions for $\dot{\phi}$, ϕ' and $V(\phi)$:

$$\dot{\phi}^2 = 2 \left[2 \frac{\dot{a}^2}{a^2} - \frac{\ddot{a}}{a} + \frac{\dot{a}}{a} \frac{\dot{b}}{b} + \frac{\dot{a}}{a} \frac{\dot{u}}{u} + \frac{1}{2} \frac{\dot{b}}{b} \frac{\dot{u}}{u} - \frac{1}{2} \frac{\ddot{b}}{b} - \frac{\ddot{u}}{u} + \frac{\dot{u}^2}{u^2} + \frac{k}{u^2} \right], \quad (11)$$

$$\phi'^2 = b^2 \left[\frac{1}{a^2} \left(-\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} - 5 \frac{\dot{a}}{a} \frac{\dot{u}}{u} + \frac{\dot{a}}{a} \frac{\dot{b}}{b} + 2 \frac{\dot{b}}{b} \frac{\dot{u}}{u} - 2 \frac{\ddot{u}}{u^2} - \frac{\ddot{u}}{u} \right. \right. \\ \left. \left. + \frac{\ddot{b}}{b} - \frac{2k}{u^2} \right) \right] - 3 \left(-\frac{a'^2}{a^2} + \frac{a''}{a} + \frac{a'}{a} \frac{b'}{b} \right), \quad (12)$$

$$V(\phi) = \frac{3}{2} \left[\frac{1}{a^2} \left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + 5 \frac{\dot{a}}{a} \frac{\dot{u}}{u} + \frac{\dot{a}}{a} \frac{\dot{b}}{b} + 2 \frac{\ddot{u}}{u^2} + \frac{\ddot{u}}{u} + \frac{\dot{b}}{b} \frac{\dot{u}}{u} + \frac{2k}{u^2} \right) \right. \\ \left. - \frac{3}{b^2} \frac{a'^2}{a^2} - \frac{1}{b^2} \frac{a''}{a} + \frac{1}{b^2} \frac{a'}{a} \frac{b'}{b} \right]. \quad (13)$$

In what follows we shall depict several important cases resulting from some relevant particularizations. Let us assume that $\phi(t, y) \equiv \phi(y)$ and $b(t, y) \equiv 1$. Thus the G^0_5 component of the field equation becomes

$$\frac{a'}{a} \frac{\dot{a}}{a} - \frac{\dot{a}'}{a} = 0, \quad (14)$$

implying the possibility of spatial and temporal separation, i.e., $a(y, t) \equiv \alpha(y)\beta(t)$. Therefore, the equations (10)–(13) take the form

$$\phi'' + 4\left(\frac{\alpha'}{\alpha}\right)\phi' - \frac{dV}{d\phi} = 0, \quad (15)$$

$$\frac{\dot{\phi}^2}{2} = \left[2\left(\frac{\dot{\beta}}{\beta}\right)^2 - \left(\frac{\ddot{\beta}}{\beta}\right) + \left(\frac{\dot{\beta}}{\beta}\right)\left(\frac{\dot{u}}{u}\right) - \left(\frac{\ddot{u}}{u}\right) + \left(\frac{\dot{u}}{u}\right)^2 + \left(\frac{k}{u^2}\right) \right] \equiv 0, \quad (16)$$

$$\begin{aligned} \phi'^2 &= \frac{1}{\alpha^2 \beta^2} \left[-\left(\frac{\ddot{\beta}}{\beta}\right) - \left(\frac{\dot{\beta}}{\beta}\right)^2 - 5\left(\frac{\dot{\beta}}{\beta}\right)\left(\frac{\dot{u}}{u}\right) - 2\left(\frac{\dot{u}}{u}\right)^2 - \left(\frac{\ddot{u}}{u}\right) - \left(\frac{2k}{u^2}\right) \right] - 3 \left[\left(\frac{\alpha''}{\alpha}\right) - \left(\frac{\alpha'}{\alpha}\right)^2 \right] \\ &= -\frac{\Delta}{\alpha^2} - 3 \left[\left(\frac{\alpha''}{\alpha}\right) - \left(\frac{\alpha'}{\alpha}\right)^2 \right], \end{aligned} \quad (17)$$

$$\begin{aligned} \frac{2}{3}V(\phi) &= \frac{1}{\alpha^2 \beta^2} \left[\left(\frac{\ddot{\beta}}{\beta}\right) + \left(\frac{\dot{\beta}}{\beta}\right)^2 + 5\left(\frac{\dot{\beta}}{\beta}\right)\left(\frac{\dot{u}}{u}\right) + 2\left(\frac{\dot{u}}{u}\right)^2 + \left(\frac{\ddot{u}}{u}\right) + \left(\frac{2k}{u^2}\right) \right] - 3 \left(\frac{\alpha'}{\alpha} \right)^2 - \left(\frac{\alpha''}{\alpha} \right) \\ &= \frac{\Sigma}{\alpha^2} - 3 \left(\frac{\alpha'}{\alpha} \right)^2 - \left(\frac{\alpha''}{\alpha} \right). \end{aligned} \quad (18)$$

At this point we need to consider separately some different regimes of the above equations. Afterwards, we solve the time-dependent part of the solutions and, then, the spatial part is analyzed.

3. Time-dependent part of the solutions

From now on we shall evince all the relevant solutions, solving the time dependent part of the solutions, after what we concentrate in the spacial part of the solutions.

3.1. The case $\Delta \equiv \Sigma \equiv 0$ with $k = 0$ and $k \neq 0$

Within the specifications outlined in this subsection epigraph, the time dependence of the equations (16)–(18) must vanish.¹ Thus, the relevant equations read

$$\left(\frac{\ddot{\beta}}{\beta}\right) - 2\left(\frac{\dot{\beta}}{\beta}\right)^2 - \left(\frac{\dot{\beta}}{\beta}\right)\left(\frac{\dot{u}}{u}\right) + \left(\frac{\ddot{u}}{u}\right) - \left(\frac{\dot{u}}{u}\right)^2 - \left(\frac{k}{u^2}\right) = 0, \quad (19)$$

$$\begin{aligned} \left(\frac{\ddot{\beta}}{\beta}\right) + \left(\frac{\dot{\beta}}{\beta}\right)^2 + 5\left(\frac{\dot{\beta}}{\beta}\right)\left(\frac{\dot{u}}{u}\right) + 2\left(\frac{\dot{u}}{u}\right)^2 + \left(\frac{\ddot{u}}{u}\right) + \left(\frac{2k}{u^2}\right) \\ = -\Delta \beta^2 = 0, \end{aligned} \quad (20)$$

$$\begin{aligned} \left(\frac{\ddot{\beta}}{\beta}\right) + \left(\frac{\dot{\beta}}{\beta}\right)^2 + 5\left(\frac{\dot{\beta}}{\beta}\right)\left(\frac{\dot{u}}{u}\right) + 2\left(\frac{\dot{u}}{u}\right)^2 + \left(\frac{\ddot{u}}{u}\right) + \left(\frac{2k}{u^2}\right) \\ = \Sigma \beta^2 = 0. \end{aligned} \quad (21)$$

¹ Notice that as an equation on the spatial variable, Eq. (15) shall be faced in the next Section.

When $k = 0$, Eqs. (19)–(21) constraint $\beta(t)$ and $u(t)$ as

$$\beta(t) = \frac{C}{u(t)}, \quad (22)$$

where C is an arbitrary constant of integration.

It is interesting to notice that the set of equations (19)–(21) with $k \neq 0$ and $u = 1$, recover the result obtained by [8], whose solution is

$$\beta(t) = \beta_0 e^{\pm \sqrt{-k}t}. \quad (23)$$

On the other hand, for the case in which $k \neq 0$ e $u \neq 1$ and using the following redefinition

$$u(t) = f(t)\beta(t) \quad (24)$$

where $f(t)$ is an arbitrary function, we have

$$\beta(t) = \frac{1}{\sqrt{f(t)}} \left[C \pm \sqrt{-k} \int_0^t \frac{dt'}{\sqrt{f(t')}} \right]. \quad (25)$$

3.2. The case $\Delta = -\Sigma$ with $k = 0$ and $k \neq 0$

Now, for the case specified here, the set of equations to be solved is

$$\left(\frac{\ddot{\beta}}{\beta}\right) - 2\left(\frac{\dot{\beta}}{\beta}\right)^2 - \left(\frac{\dot{\beta}}{\beta}\right)\left(\frac{\dot{u}}{u}\right) + \left(\frac{\ddot{u}}{u}\right) - \left(\frac{\dot{u}}{u}\right)^2 - \left(\frac{k}{u^2}\right) = 0, \quad (26)$$

$$\begin{aligned} \left(\frac{\ddot{\beta}}{\beta}\right) + \left(\frac{\dot{\beta}}{\beta}\right)^2 + 5\left(\frac{\dot{\beta}}{\beta}\right)\left(\frac{\dot{u}}{u}\right) + 2\left(\frac{\dot{u}}{u}\right)^2 + \left(\frac{\ddot{u}}{u}\right) + \left(\frac{2k}{u^2}\right) \\ = -\Delta \beta^2, \end{aligned} \quad (27)$$

$$\begin{aligned} \left(\frac{\ddot{\beta}}{\beta}\right) + \left(\frac{\dot{\beta}}{\beta}\right)^2 + 5\left(\frac{\dot{\beta}}{\beta}\right)\left(\frac{\dot{u}}{u}\right) + 2\left(\frac{\dot{u}}{u}\right)^2 + \left(\frac{\ddot{u}}{u}\right) + \left(\frac{2k}{u^2}\right) \\ = \Sigma \beta^2. \end{aligned} \quad (28)$$

All together, these equations implies $\Delta = -\Sigma$ and therefore the solution is given by

$$\beta(t) = \left[u(t) \left(C_1 \pm \sqrt{\frac{\Sigma}{3}} \int_0^t \frac{dt'}{u(t')} \right) \right]^{-1}. \quad (29)$$

Note that as far as $k = 0$ and $u(t) = 1$, we recover result obtained in [8], i.e.,

$$\beta(t) \propto \frac{1}{t},$$

as expected.

For $k \neq 0$, the solution is given by

$$\begin{aligned} \beta(t) &= -\frac{1}{u(t)} \sqrt{\frac{3}{\Sigma}} \cot \left[C_1 \sqrt{3k} - \sqrt{k} \int_0^t \frac{dt'}{u(t')} \right] \\ &\times \sqrt{k + k \tan^2 \left[C_1 \sqrt{3k} - \sqrt{k} \int_0^t \frac{dt'}{u(t')} \right]}. \end{aligned} \quad (30)$$

Making use of the usual trigonometric relation $\sec^2 x = 1 + \tan^2 x$, we have

$$\beta(t) = -\frac{1}{u(t)} \sqrt{\frac{3}{\Sigma}} \cot \left[C_1 \sqrt{3k} - \sqrt{k} \int_0^t \frac{dt'}{u(t')} \right] \times \sqrt{k \sec^2 \left[C_1 \sqrt{3k} - \sqrt{k} \int_0^t \frac{dt'}{u(t')} \right]}, \quad (31)$$

or simply

$$\beta(t) = -\frac{1}{u(t)} \sqrt{\frac{3k}{\Sigma}} \frac{1}{\sin \left(C_1 \sqrt{3k} - \sqrt{k} \int_0^t \frac{dt'}{u(t')} \right)}. \quad (32)$$

Thus Eq. (30) can be rewritten as

$$\beta(t) = -\frac{1}{u(t)} \sqrt{\frac{3k}{\Sigma}} \sec \left(\sqrt{k} \int_0^t \frac{dt'}{u(t')} \right), \quad (33)$$

where use was made of

$\pm \sin x = \pm \cos(\pi/2 - x)$ with

$$x = -\sqrt{k} \int_0^t \frac{dt'}{u(t')} \text{ and } C_1 \sqrt{3k} = \pi/2. \quad (34)$$

Note that, similarly to the case of Eq. (23), one can reproduce the results obtained in [8], in which the solutions given in (33), for $k = 1$, $k = -1$ (both for $u = 1$) are respectively

$$\beta(t) \propto \sec(t), \quad (35)$$

and

$$\beta(t) \propto \operatorname{sech}(t). \quad (36)$$

We shall investigate the cosmological outputs of the obtained solutions in Section 5.

4. Spatial-dependent part of the solutions

Now we shall consider the spatial part of the equations (15), (17) and² (18):

$$\phi'' + 4 \left(\frac{\alpha'}{\alpha} \right) \phi' - \frac{dV}{d\phi} = 0, \quad (37)$$

$$\phi'^2 = -\frac{\Delta}{\alpha^2} - 3 \left[\left(\frac{\alpha''}{\alpha} \right) - \left(\frac{\alpha'}{\alpha} \right)^2 \right], \quad (38)$$

$$\frac{2}{3} V(\phi) = \frac{\Sigma}{\alpha^2} - 3 \left(\frac{\alpha'}{\alpha} \right)^2 - \left(\frac{\alpha''}{\alpha} \right). \quad (39)$$

There are two interesting cases, concerning the separation constants which we are going to investigate in detail. The first one is given by the vanishing of both.

4.1. The case $\Delta = \Sigma = 0$

For better dealing with the system of equations we use the re-definition,

$$\alpha(y) = e^{A(y)}. \quad (40)$$

In the light of (40), the Equations (38)–(37) become

$$\frac{1}{3} \phi'^2 = -A'', \quad \frac{2}{3} V(\phi) = -A'' - 4(A')^2, \quad (41)$$

and

$$\phi'' + 4A'\phi' = \frac{dV}{d\phi}. \quad (42)$$

Assuming, for ulterior convenience, that

$$\phi' = rW_\phi(\phi), \quad A' = sW(\phi), \quad (43)$$

and substituting the equations (43) in (42) we find $s = -1/3$. Therefore the potential acquires the form

$$V(\phi) = r^2 \left[\frac{1}{2} (W_\phi)^2 - \frac{2}{3} W(\phi)^2 \right]. \quad (44)$$

The derivative of Eq. (44) with respect to ϕ , V_ϕ , reads

$$\begin{aligned} V_\phi &= r^2 \left[W_{\phi\phi} W_\phi - \frac{4}{3} W_\phi W(\phi) \right] \\ &= r^2 \left[\frac{d}{d\phi} \left(\frac{W_\phi^2}{2} \right) - \frac{4}{3} \frac{d}{d\phi} \left(\frac{W(\phi)^2}{2} \right) \right] \\ &= \frac{d}{d\phi} \left[r^2 \left(\frac{W_\phi^2}{2} - \frac{2}{3} W(\phi)^2 \right) \right]. \end{aligned} \quad (45)$$

Therefore the potential itself can be expressed as

$$V(\phi) = \frac{W_\phi^2}{2} - \frac{2}{3} W(\phi)^2, \quad (46)$$

and the Eqs. (43) become

$$\phi' = W_\phi(\phi), \quad A' = -\frac{1}{3} W(\phi), \quad (47)$$

with

$$A(y) = -\frac{1}{3} \int W[\phi(y)] dy. \quad (48)$$

Now we turn ourselves to a different arrangement of the separation constants.

4.2. The case $\Delta = -\Sigma \neq 0$

Making use of the expression (40) in (38)–(37), one gets

$$\phi'^2 = \Sigma e^{-2A} - 3A'', \quad \frac{2}{3} V(\phi) = \Sigma e^{-2A} - 4(A')^2 - A'', \quad (49)$$

and

$$\phi'' + 4A'\phi' = \frac{dV}{d\phi}. \quad (50)$$

Combining the equations (49), we obtain

$$V(\phi) = 3[A'' - 2(A')^2] + \frac{3}{2} \phi'^2. \quad (51)$$

By assuming [13] that

$$\phi' = aW_{1\phi}(\phi), \quad A' = bW_2(\phi), \quad (52)$$

$$\phi'' = a^2 W_{1\phi\phi} W_{1\phi}, \quad A'' = ab W_{2\phi} W_{1\phi}, \quad (53)$$

and substituting Eqs. (52) and (53) into (51), we have

$$V(\phi) = 3[ab W_{2\phi} W_{1\phi} - 2b^2 W_2^2] + \frac{3}{2} a^2 W_{1\phi}^2. \quad (54)$$

Taking the derivative of (54) with respect to ϕ one gets

² Notice that the equation (16) has not spatial-dependence.

$$\begin{aligned} \frac{dV}{d\phi} = & 3[abW_{2\phi\phi}W_{1\phi} + abW_{1\phi\phi}W_{2\phi} - 4b^2W_2W_{2\phi}] \\ & + 3a^2W_{1\phi}W_{1\phi\phi}, \end{aligned} \quad (55)$$

and inserting (55) in the scalar field equation (50), one arrives at the following consistence equation for the superpotential

$$a[4bW_2 - 3bW_{2\phi\phi} - 2aW_{1\phi\phi}]W_{1\phi} = 3b[aW_{1\phi\phi} - 4bW_2]W_{2\phi}, \quad (56)$$

where $a = 1$ and $b = 1/3$.

Now we define the following quantities

$$W_1 = W + \lambda Z \quad \text{and} \quad W_2 = W + \sigma Z, \quad (57)$$

from which one can see that some terms can be written as a total derivative, and the equation (56) takes the form

$$\begin{aligned} \frac{d}{d\phi} \left[\frac{2}{3}W^2 + \frac{1}{2}W_\phi^2 - \frac{2}{3}\lambda\sigma Z^2 + \frac{\alpha}{2}(2\sigma - \lambda)Z_\phi^2 + \frac{4}{3}\sigma^2 Z^2 \right. \\ \left. + \sigma W_\phi Z_\phi + \frac{4}{3}\sigma W Z \right] + \frac{4}{3}(\sigma - \lambda)W Z_\phi = 0. \end{aligned} \quad (58)$$

In order to deal with a concrete and exact case, we assume that

$$W = C_1 + C_2 Z + C_3 Z_{\phi\phi}. \quad (59)$$

For simplicity, some authors consider $Z(\phi) = W(\phi)$ [13,14]. Here we shall consider a more general case, substituting (59) into Eq. (58). This procedure leads to

$$\begin{aligned} \frac{2}{3}C_1^2 - C_4 + \frac{4}{3}C_1C_3Z_{\phi\phi} + \frac{2}{3}C_3Z_{\phi\phi}^2 + \frac{1}{2}C_3^2Z_{\phi\phi\phi}^2 \\ + Z_\phi Z_{\phi\phi\phi}(C_2C_3 + C_3\beta) \\ + Z_\phi^2 \left(\frac{C_2^2}{2} - \frac{2C_3\lambda}{3} - \frac{\lambda^2}{2} + C_2\sigma + \frac{2}{3}C_3\sigma + \lambda\sigma \right) \\ + Z^2 \left(\frac{2C_2^2}{3} - \frac{2C_2\lambda}{3} + 2C_2\sigma - \frac{2\lambda\sigma}{3} + \frac{4\sigma^2}{3} \right) \\ + Z \left[\frac{4C_1C_2}{3} - \frac{4C_1\lambda}{3} + \frac{8C_1\sigma}{3} + Z_{\phi\phi} \left(\frac{4C_2C_3}{3} + \frac{4C_3\sigma}{3} \right) \right] \\ = 0. \end{aligned} \quad (60)$$

In order to keep some similarity with the well known literature, we look for a solution as the one presented in Ref. [15], for instance. We choose thereof

$$Z(\phi) = Z_0 \cos(v\phi + s). \quad (61)$$

In the light of Eq. (60), gathering Z , Z_ϕ^2 and Z^2 terms together, we have

$$\begin{aligned} \frac{2}{3}C_1^2 - C_4 + \frac{4Z}{3} \left(C_2 - C_3v^2 - \lambda + 2\sigma \right) C_1 + Z_\phi^2 \left[\frac{C_2^2}{3} + \frac{C_3v^4}{2} \right. \\ \left. - \frac{2C_3\lambda}{3} - \frac{\lambda^2}{2} + C_2\sigma + \frac{2C_3\sigma}{3} + \lambda\sigma - v^2(C_2C_3 + C_3\sigma) \right] \\ + Z^2 \left[\frac{2C_2^2}{3} + \frac{2C_3v^4}{3} - \frac{2C_2\lambda}{3} + 2C_2\sigma - \frac{2\lambda\sigma}{3} + \frac{4\sigma^2}{3} \right. \\ \left. - v^2 \left(\frac{4C_2C_3}{3} + \frac{4C_3\sigma}{3} \right) \right] = 0. \end{aligned} \quad (62)$$

By taking the Z coefficient equal to zero

$$C_2 - C_3v^2 - \lambda + 2\sigma = 0, \quad (63)$$

we get as the solution for v , the following expression

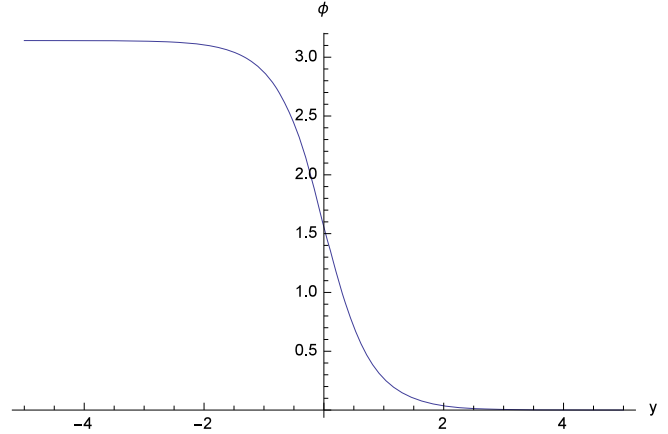


Fig. 1. Behavior of $\phi(y)$ as a function of y for $\Delta \neq \Sigma$, $\lambda = 2$ and $\sigma = 1$.

$$v = \sqrt{\frac{C_2 - \lambda + 2\sigma}{C_3}}, \quad (64)$$

where $C_1 = \pm \sqrt{3/2C_4}$ and $C_2 = C_3 + \lambda - 2\sigma$ if $v = 1$, in order to full accomplish the consistence constraint. In this vein, the expression (59) for W , becomes

$$W = C_1 + (\lambda - 2\sigma) \cos \phi, \quad (65)$$

with the choice $Z_0 = 1$ and $s = 0$ in equation (61).

Using Eq. (65) into (52) and (53), we obtain

$$\phi' = 2(\sigma - \lambda) \sin \phi, \quad (66)$$

and

$$A' = \frac{1}{3}[C_1 + (\lambda - \sigma) \cos \phi]. \quad (67)$$

Therefore, the solutions for ϕ and A , are respectively given by

$$\phi(y) = 2 \operatorname{arccot}[e^{2y(\lambda - \sigma)}], \quad (68)$$

and

$$A(y) = \frac{1}{3}y(C_1 + \lambda - \sigma) + \frac{1}{6} \ln(1 + e^{4y(\lambda - \sigma)}). \quad (69)$$

By means of Eqs. (68) and (69) for the expression (51), we obtain the following shape for the potential

$$V(\phi) = -\frac{1}{3}(\lambda - \sigma)^2 \left\{ -11 + 13 \cos[4 \operatorname{arccot}[\cot(\phi/2)]] \right\}. \quad (70)$$

In Figs. 1 and 2 we depict the profiles of $\phi(y)$ and $\exp[A(y)]$ in the relevant range where the scalar field is also varying. Before to delve into the effective quantities study, we remark by passing that, despite the rather non-trivial functional form of the obtained solutions, the resulting spacetime is after all well behaved. In fact, all the Kretschmann scalars associated to the solutions are finite.

5. Effective Hubble parameter and scale factor

In this section, from the found solutions for $\beta(t)$, we obtain the effective Hubble parameter as well as the effective scale factor. The Hubble parameter, $H = \dot{a}/a$, is used to measure the expansion rate of the universe. The time elapsed in this scenario is the so called proper time or cosmic time.

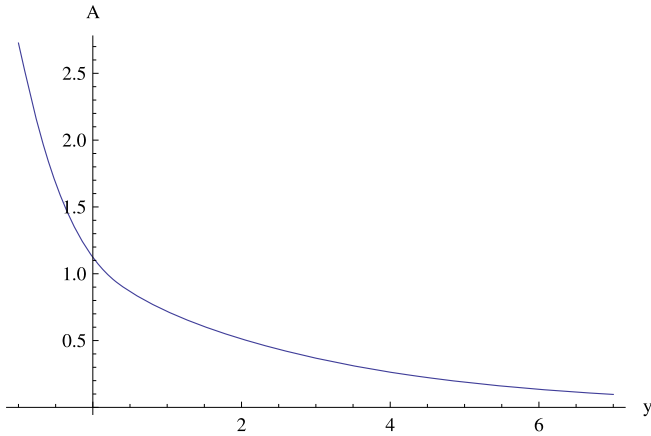


Fig. 2. The exponential of the warp factor function $A(y)$ as a function of y for $\Delta \neq \Sigma$, $\lambda = 2$, $\sigma = 1$ and $C_1 = -1$.

In view of the transformation

$$d\tau = \beta(t)dt, \quad (71)$$

it is possible to construct the effective Hubble parameter

$$H_{eff} = \frac{1}{\beta(t)} \frac{d}{dt} [\ln(\beta(t)u(t))] = \frac{1}{a_{eff}} \frac{da_{eff}}{d\tau} = \frac{\dot{a}_{eff}}{a_{eff}}, \quad (72)$$

where

$$a_{eff}(t) = \beta(t)u(t). \quad (73)$$

5.1. The case $\Delta = \Sigma = 0$

Consider for a while the case that $k = 0$, so that

$$\beta(t) = \frac{C}{u(t)} \quad (74)$$

and therefore

$$H_{eff}(\tau) = 0 \quad \text{and} \quad a_{eff} = \text{constant}. \quad (75)$$

This case leads, then, to the typical Hubble parameter describing a static universe, sometimes called Einstein's universe.

Within this case ($\Delta = \Sigma = 0$) but now with $k \neq 0$, the solution is given by

$$\beta(t) = \frac{1}{\sqrt{f(t)}} \left[C \pm \frac{\sqrt{-k}}{\sqrt{f(t)}} \int_0^t \frac{dt'}{\sqrt{f(t')}} \right]. \quad (76)$$

Hence we have

$$d\tau = \beta(t)dt = \frac{dt}{\sqrt{f(t)}} \left[C \pm \frac{\sqrt{-k}}{\sqrt{f(t)}} \int_0^t \frac{dt'}{\sqrt{f(t')}} \right] = dz [C \pm \sqrt{-k}z], \quad (77)$$

where we defined that $dz = dt/\sqrt{f(t)}$. Thus, we find the expression for the cosmic time τ as given by

$$\tau = Cz \pm z^2 \sqrt{-k}/2. \quad (78)$$

Choosing, for simplicity, $C = 0$ and using $f(t) \equiv e^{-2at}$, we have $z = (1/a)e^{at}$ and, consequently

$$\tau = \pm \frac{\sqrt{-k}}{2} \frac{1}{a^2} e^{2at}. \quad (79)$$

Therefore

$$t = \frac{1}{2a} \ln \left(\pm \frac{2a^2 \tau}{\sqrt{-k}} \right), \quad (80)$$

and it is possible to see that returning to Eq. (76) one arrives at

$$\beta(t) = \pm \frac{\sqrt{-k}}{a} e^{2at}. \quad (81)$$

Finally, substituting the expressions (81) and (80) into (72), one can verify that the effective Hubble parameter decays with the inverse of τ , while that scale factor is linearly growing and real as far as $k < 0$.

$$H_{eff}(\tau) = \frac{1}{\tau} \quad \text{and} \quad a_{eff}(\tau) = 2\sqrt{-k} \tau. \quad (82)$$

It is reasonable that the behavior of $H_{eff}(\tau)$ is the one expected both for the matter and radiation dominated phase of the universe. Unfortunately, however, the corresponding scale factor grows much faster than it should in a realistic scenario.

5.2. The case $\Delta = -\Sigma \neq 0$

Now let us consider the case in which $\Delta = -\Sigma \neq 0$ both for $k = 0$ and $k \neq 0$. Firstly, we analyze the simplest case with $k = 0$, leading to

$$\beta(t) = \frac{1}{u(t)} \frac{1}{C_1 \pm \sqrt{\frac{\Sigma}{3}} \int_0^t \frac{dt'}{u(t')}}, \quad (83)$$

and from Eq. (67),

$$d\tau = \beta(t)dt = \frac{dt}{u(t)(C_1 \pm bz)} \rightarrow \tau = \mp \frac{1}{b} \ln(C_1 \pm bz) \quad (84)$$

where $b = \sqrt{\Sigma/3}$ and $z = \int_0^t dt'/u(t')$. Inserting Eqs. (83) and (84) into the expression (72), one gets

$$\begin{aligned} H_{eff}(t) &= u(t)(C_1 \pm bz) \frac{d}{dt} \left[\ln \left(\frac{1}{(C_1 \pm bz)} \right) \right] = \\ &= u(t)(C_1 \pm bz) \frac{d}{dt} (\pm bz) \\ &= u(t)(C_1 \pm bz) \pm b \frac{1}{u(t)(C_1 \pm bz)} \\ &= \pm b, \end{aligned} \quad (85)$$

and therefore

$$H_{eff}(\tau) = \pm \sqrt{\frac{\Sigma}{3}}, \quad (86)$$

whose integration leads to

$$a_{eff}(\tau) = a_0 \exp \left(\pm \sqrt{\frac{\Sigma}{3}} \tau \right). \quad (87)$$

We can note that the above solution to $a_{eff}(\tau)$ with $k = 0$ is similar to the solution found for an usual universe (without any brane), and dominated by the vacuum energy, i.e.,

$$a(t) \propto \exp \left(\pm \sqrt{\frac{\Lambda}{3}} t \right). \quad (88)$$

where $\Lambda > 0$ is the cosmological constant. This fact is more important that it may sound. In fact, in the context of thick braneworlds, as discussed in the Introduction, there is no how to directly relate

the four-dimensional cosmological constant with some counter-part quantity in five dimensions, or even some property of the brane modeling. In this approach, however, we see that the separation constant (necessary to solve the gravitational system endowed with a brane) takes the place of the four-dimension cosmological constant. The solution given in (87) could, indeed, represent the current phase of accelerated expansion of our universe in a Λ CDM (Lambda Cold Dark Matter) model.

Finally, for the case in which $k \neq 0$, and remembering that

$$\beta(t) = -\frac{1}{u(t)} \sqrt{\frac{3k}{\Sigma}} \sec(\sqrt{k}z), \quad (89)$$

where $z \equiv \int_0^t \frac{dt'}{u(t')}$, it can be seen that

$$d\tau = \beta(t)dt = -\sqrt{\frac{3k}{\Sigma}} \sec(\sqrt{k}z) dz. \quad (90)$$

By taking $dz = dt/u(t)$ and integrating the above expression, we obtain

$$\tau = -\sqrt{\frac{3k}{\Sigma}} \ln \left| \sec(\sqrt{k}z) + \tan(\sqrt{k}z) \right|. \quad (91)$$

Now, by inverting the Eq. (91) for z as a function of τ , one gets

$$z = \frac{2}{\sqrt{k}} \arccos \left\{ \frac{[1 + \exp(\tau \sqrt{\Sigma/3})]^{1/2}}{\sqrt{2}} \right\}. \quad (92)$$

The Hubble parameter, then, reads

$$H_{eff} = \frac{1}{\beta(t)} \frac{d}{dt} [\ln(\beta(t)u(t))] = \frac{d}{d\tau} \left[\ln \left(-\sqrt{\frac{3k}{\Sigma}} \sec(\sqrt{k}z) \right) \right]. \quad (93)$$

In order to reach Eq. (93) we have used $d\tau = \beta(t)dt$ in the first equality. Substituting Eq. (91) in (92) one arrives at

$$H_{eff}(\tau) = \frac{d}{d\tau} \left[\ln \left(-\sqrt{\frac{3k}{\Sigma}} \sec \left(2 \arccos \left\{ \frac{[1 + \exp(\tau \sqrt{\Sigma/3})]^{1/2}}{\sqrt{2}} \right\} \right) \right) \right], \quad (94)$$

and therefore

$$H_{eff}(\tau) = \sqrt{\frac{\Sigma}{3}} \tanh \left[\sqrt{\frac{\Sigma}{3}} \tau \right]. \quad (95)$$

By means of (95) the effective scale factor is given by

$$a_{eff}(\tau) = \ln \left[\cosh \sqrt{\frac{\Sigma}{3}} \tau \right]. \quad (96)$$

The above Hubble parameter, despite the fact that it presents an expected behavior at large values of τ , is again too fast at lower values of τ .

6. Concluding remarks

We have investigated exact solutions for a FLRW braneworld, whose brane is performed by a bulk scalar field. The general idea was to find out explicit solutions which could, at least in some regime, to perform the large scale dynamics of the universe. In the course of our analysis a plenty of possibilities had appeared. Among them, we believe we pay attention to the most physically appealing cases.

In some aspects, the physical outputs can model a specific era of the known universe, as in the case represented by Eq. (82) in which the matter and radiation phases can be reached. By the same token, in the specific $k = 0$, $\Delta \neq -\Sigma$ case, the separation constant Σ can mimic a four-dimensional cosmological constant for a de-Sitter-like universe. Therefore, the late-time acceleration can be modeled without regarding to any type of dark energy.

Currently we are delving into the possibility to describe more aspects of the cosmic evolution. To accomplish that, more bulk scalar fields, as well as different potentials may be in order. We shall postpone these generalizations for a future work.

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