UNIVERSIDADE ESTADUAL PAULISTA
"JÚLIO DE MESQUITA FILHO"
Câmpus de São José do Rio Preto

Lucas Queiroz Arakaki

Nilpotent Centers on Center Manifolds, Cyclicity of Hopf Centers and the Period function near a Persistent Polycycle

São José do Rio Preto
2024

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Tese apresentada como parte dos requisitos para obtenção do título de Doutor em Matemática, junto ao Programa de Pós-Graduação em Matemática, do Instituto de Biociências, Letras e Ciências Exatas da Universidade Estadual Paulista "Júlio de Mesquita Filho", Câmpus de São José do Rio Preto.

Orientador: Prof. Dr. Claudio Gomes Pessoa
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Financiadora: FAPESP Processos 2019/13040-7 e 2021/14450-4

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Arakaki, Lucas Queiroz

Nilpotent Centers on Center Manifolds, Cyclicity of Hopf Centers and the Period function near a Persistent Polycycle / Lucas Queiroz Arakaki. -- São José do Rio Preto, 2024

112 p. : il., tabs.

Tese (doutorado) - Universidade Estadual Paulista (UNESP), Instituto de Biociências Letras e Ciências Exatas, São José do Rio Preto

Orientador: Claudio Gomes Pessoa

Coorientador: Jordi Villadelprat Yagüe

1. Teoria Qualitativa. 2. Problema do Centro Nilpotente. 3. Ciclos Limite. 4. Função Período. 5. Policiclo. I. Título.

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Comissão Examinadora

Prof. Dr. Claudio Gomes Pessoa
Orientador

Prof. Dr. Douglas Duarte Novaes
UNICAMP, Campinas

Prof. Dr. Paulo Ricardo da Silva
UNESP - São José do Rio Preto

Prof. Dr. Regilene D. S. Oliveira
ICMC - USP, São Carlos

Prof. Dr. Marcelo Messias
UNESP - Presidente Prudente

São José do Rio Preto
02 de agosto de 2024

Dedico este trabalho a você.

AGRADECIMENTOS

Agradeço a Deus pela oportunidade de viver, me aventurar e criar novas histórias.

Aos meus pais, Rosa e Paulo e minha irmã Fernanda, pelo apoio constante, pelo amor e pela confiança. Graças ao seu suporte, pude concluir o que comecei e seguir em frente, mesmo quando sentia que não podia. Sem vocês, eu não seria. Com vocês, acredito que posso ser tudo.

À Ju, minha companheira e melhor amiga, por ser uma luz na minha vida e meu porto seguro. Juntos somos mais fortes, e essa força me impulsionou a concluir esta etapa, principalmente nos momentos mais difíceis. Obrigado por segurar minha mão em qualquer caminho que eu siga.

À minha família, que sempre está na minha retaguarda. À Dê, pelas importantes palavras de apoio e pela amizade. Obrigado por me ajudar a manter minha criança interior viva.

Aos companheiros de IBILCE, nossa casa, parceiros de carreira que se tornam inspirações: Otávio, Kumon, Ana Lívia, Jarne, Ronísio, André e Willian. Ao Bononi, que está comigo desde o começo da graduação! Ao Bigode, companheiro musical e de chopp também.

À Jé, pelas conversas profundas e momentos que compartilhamos. Sei que sempre podemos contar um com o outro. Ao Fofo, companheiro que carrego com muito carinho no coração. Obrigado pelo apoio e pelos incentivos durante essa jornada. À Carol, pela amizade e parceria cuja importância é inestimável. Obrigado pela caminhada e companheirismo durante os momentos mais difíceis dos últimos anos.

Aos meus amigos de jogatina, que sobreviveram comigo à pandemia: Lola, Braia, Camila, Digo, Igor. Aos de RPG: Luan, Gabrix, Thaiane, Lady, Natália e Kelvin. O convívio virtual foi de suma importância para podermos atravessar o medo da melhor maneira que conseguimos. Ao Tynho e ao Markin, pelas amizades virtuais que se fizeram fundamentais.

Ao Gustavo e à Bruna, pelo acolhimento e por me receberem em sua casa. Obrigado pela “adoção” e por toda ajuda que me deram para me adaptar ao estrangeiro. À Joyce e ao Bruno, companheiros de aventura, agradeço por termos criado nossa rede de apoio e abrasileirar nossa Catalunha.

Als amics catalans: Eloi, Roger, Conrad, Gerard, Olga i Ohiane. Gràcies per l’acollida i per tots els grans moments que vam viure junts. A la Montse, per l’amistat que viatja països, platges i continents. A la Dolors per ensenyar-me molt més que una llengua, sinó una gran cultura de la qual estic enamorat. Als Castellers de Terrassa, la colla del meu cor.

Ao meu orientador, Claudio, por me acompanhar durante toda minha carreira acadêmica. Obrigado por sua serenidade e paciência. Al meu supervisor a Catalunya, Jordi i al David. Gràcies per l’oportunitat de treballar i aprendre no només nous conceptes matemàtiques, però també noves formes de treballar.

À banca examinadora, pela disponibilidade e pelas correções.

Aos funcionários e servidores do IBILCE, por todo o trabalho imprescindível que mantém este instituto que orgulhosamente chamamos de casa.

Agradeço à FAPESP pela concessão da bolsa de pesquisa, sob o processo número 2019/13040-7, Fundação de Amparo à Pesquisa do Estado de São Paulo (FAPESP) e também pela concessão da BEPE, sob o processo número 2021/14450-4.

RESUMO

Este trabalho aborda três problemas relevantes e atuais da teoria qualitativa das equações diferenciais ordinárias. O primeiro trata-se do problema do centro em uma variedade central de sistemas diferenciais analíticos tridimensionais cuja parte linear do campo vetorial associado é $y\partial_x - \lambda z\partial_z$, para $\lambda \neq 0$. Obtemos uma forma normal formal para esses sistemas, alguns resultados sobre sua integrabilidade e estendemos, para este caso, algumas técnicas utilizadas para investigar centros nilpotentes no plano. Por exemplo, mostramos que centros nilpotentes de sistemas tridimensionais, em variedades centrais analíticas, são limites de centros do tipo Hopf. O segundo problema considerado é o problema da ciclicidade de pontos de Hopf para sistemas quadráticos em $\mathbb{R}^3$. Usando expansões de ordem superior dos coeficientes de Lyapunov, obtemos uma nova cota inferior para a ciclicidade destes pontos singulares. Mais precisamente, encontramos um exemplo com 13 ciclos limites bifurcando do ponto singular. No terceiro problema, consideramos famílias de campos vetoriais polinomiais planares que possuem um policiclo hiperbólico Γ persistente. Provamos, sob algumas condições genéricas, que bifurca exatamente um ciclo limite a partir de Γ. Neste caso, obtemos informações quantitativas sobre sua função período em relação aos parâmetros de perturbação.

Palavras-chave: Teoria Qualitativa, Problema do Centro Nilpotente, Ciclos Limite, Função Período, Policiclo.

ABSTRACT

This work deals with three current and relevant problems in the qualitative theory of differential equations. The first one is the center problem on a center manifold of differential systems such that the linear part of their associated vector field is $y\partial_x - \lambda z\partial_z$*, with* $\lambda \neq 0$*. We obtain a formal normal form for those systems, some results on their integrability and extended, for this case, some techniques employed to study nilpotent centers in the plane. For instance, we show that nilpotent centers of three-dimensional systems on analytic center manifolds are limits of Hopf type centers. The second problem we consider is the cyclicity problem for Hopf singular points in quadratic systems in* $\mathbb{R}^3$*. Using higher order expansions of the Lyapunov coefficients, we obtain a new lower bound for the cyclicity of such singular points. More precisely, we exhibit an example with 13 limit cycles bifurcating from the singular point. In the third problem, we consider families of polynomial planar vector fields having a hyperbolic persistent polycycle* Γ*. We prove, under generic conditions, that exactly one limit cycle from* Γ*. In this case, we obtain quantitative information on its associated period function with respect to the perturbation parameters.*

Keywords: *Qualitative Theory, Nilpotent Center Problem, Limit cycles, Period Function, Polycycle.*

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Summary

1 Introduction

The theory of differential equations is one of the most interesting and relevant fields in contemporary mathematics. Within its scope, we have several distinct problems, all equally intriguing and fruitful. From the early works of Poincaré [1] to the recent advancements by contemporary researchers, the sheer amount of theoretical results and practical applications is consistently increasing throghout the century. In this context, one's ambition to contribute is met with several difficulties. Picking a problem to tackle, studying the literature to understand what has already been done and discovering new results to have something to add to the extensive collective knowledge. These are not easy tasks and are often overlooked when we think about the role of the researcher.

Nonetheless, we took up on the challenge and the results of our attempt to contribute to our beloved scientific field are comprised into the present thesis. In the workings of producing a thesis, we find ourselves collaborating with different researchers, considering different approaches and tackling different problems. This makes for a rich experience which in our case are comprised into three different chapters that constitutes this thesis: *Nilpotent Centers on Center Manifolds in* $\mathbb{R}^3$, *Hopf singular points in* $\mathbb{R}^3$ *and the cyclicity problem* and *Bifurcation phenomena in time of differential equations.*

The object of our interest are the following differential equation

$$\dot{x} = F(x; \mu), \tag{1.1}$$

where $x \in \mathbb{R}^m$, $\mu \in \Lambda \subset \mathbb{R}^N$ and F is a well-behaved function. Also, x is often referred to as the spatial variables and μ is called the perturbation parameters. The qualitative theory of differential equations constitutes in obtaining information about the solutions of such equations without explicitly computing them. Among the different behaviors that the solutions can have, the one that we focus on is the periodic ones, i.e. the closed orbits. The general behavior of the solution curves in a neighborhood of a closed orbit is often the subject of relevant open problems. In the present thesis, we consider three different problems concerning closed orbits.

The first one is the *Nilpotent Center Problem.* In the plane ($m = 2$), a nilpotent singular point is a singular point of (1.1) such that the Jacobian matrix of F evaluated at the singular point has two zero eigenvalues but is not the null matrix. When this singular point is monodromic, the Nilpotent Center Problem consists in determining whether or not the orbits in a neighborhood of the nilpotent singular point are all closed. This problem is widely studied for the planar case. However, if we consider the same problem in $\mathbb{R}^3$, there are few works in the literature. This is precisely the subject of our first chapter. We consider (1.1) where $m = 3$, F is analytic and DF_p has the Jordan canonical form

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\lambda \end{pmatrix},$$

for p being a singular point, which is called *nilpotent singular point* in $\mathbb{R}^3$. These differential systems admit invariant two-dimensional manifolds (called *Center Manifolds*). Our goal is to develop tools to tackle the Nilpotent Center Problem on a center manifold of such systems. In this endeavor, we obtain a formal normal form for three-dimensional systems having a nilpotent singular point (see Theorem 2.29), some results on their integrability (for instance, Theorems 2.40 and 2.45) and extended some known techniques for the planar nilpotent center problem to the three-dimensional case: the Inverse Jacobi Multiplier method (see Theorem 2.56, which is an extension of the inverse integrating factor method described in [2]), and the fact that nilpotent centers of three-dimensional systems, on analytic center manifolds, are limits of Hopf-type centers (see Theorem 2.64, which is inspired by the main result of [3] and its previous versions for the planar scenario).

The problem of determining whether a singular point is a center or not was first considered for less degenerated types of singular points. More precisely, the *Hopf singular points*. These are the singular points of (1.1), with $m = 3$, for which the Jacobian matrix of F has two purely imaginary eigenvalues and one non-zero real one. There are several works on the center problem for Hopf singular points (see [4, 5, 6], for instance).

In the literature, it is widely known that the center problem is closely related to the cyclicity problem, i. e. the problem of determining how many isolated closed orbits (*limit cycles*) can arise when perturbing equation (1.1). The study of the cyclicity usually aims to obtain lower bounds on the number of limit cycles than can unfold, when varying values of μ in the parameter space Λ around a parameter μ_0 with a specific feature. In the second chapter of the present thesis, this feature is that we have a center on a center manifold for $\mu = \mu_0$ and our work consists in applying the most recent results of the literature to obtain new lower bounds on the cyclicity of Hopf singular points. As far as we know, we have successfully obtained the highest lower bound of the cyclicity of quadratic systems having a Hopf singular point (Theorem 3.11).

The last chapter of the present thesis concerns a different side of the study of closed orbits: their period. The *period function* of a periodic orbit often depends on the perturbation parameters $\mu \in \Lambda$. The classification of the behavior of the period function is the subject of several works in the literature (see , for instance, [7, 8]). We consider (1.1) with $m = 2$, $F(\cdot, \mu)$ polynomial and $F(x, \cdot)$ a smooth function such that, for $\mu = \mu_0$ the system admits a polycicle that maintains its topology for small perturbations on μ near μ_0. In this setting, we study the limit cycles that bifurcate from the persistent polycycle, i. e. the closed orbits that arise in a small neighborhood of the polycylce, seeking quantitative information about their period function. For this matter, we obtained two main results. In the first one, we prove that under some generic conditions on the return map, exactly one limit-cycle bifurcates from Γ and we determine the behavior of its associated period function with respect to the perturbation parameters (Theorem 4.10). The second one concerns the case where the aforementioned generic conditions are not satisfied, for which the behavior of the period function cannot be determined (Theorem 4.21).

We believe that this thesis present relevant new results in all three problems considered. Although the field of differential equations is vast and diverse, we hope that our work represents a meaningful contribution to its development. Hopefully, this work inspires the reader to carry on with their research and give their own contribution too.

The results of the present thesis are distributed in papers [9, 10, 11, 12, 13, 14].

2 Nilpotent Centers on Center Manifolds in $\mathbb{R}^3$

Consider a planar vector field X defined in a open set $U \subset \mathbb{R}^2$. We say that a singular point p is a nilpotent singular point of X if the jacobian matrix DX_p has two zero eigenvalues but is not the null matrix. In other words, p is a nilpotent singular point of vector field X if there exists a linear change of coordinates such that, in those new coordinates, $DX_p = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$. We may assume, by means of translation, that $p = 0$. Thus, the associated differential system can be written as:

$$\begin{aligned} \dot{x} &= y + P(x, y), \\ \dot{y} &= Q(x, y), \end{aligned} \tag{2.1}$$

where $j^1P(0) = j^1Q(0) = 0$ (here $j^lF(0)$ denotes de l-jet of a function F at 0, i.e. the Taylor polynomial of degree l at 0). System (2.1) is widely studied in the literature. There are several possibilities of phase portraits of this system in a neighborhood of the origin. In fact, almost all possibilities are classified, except the monodromic case (see [15, Theorem 3.5, p. 116] and also [16, 17]). We recall the definition of monodromy [18].

Let p be a singular point of X and $\gamma = \{\gamma(t) : t \in \mathbb{R}\}$ be an orbit of X that tends to p when t tends to ∞. In this case, γ is a *characteristic orbit* if there exists $\lim_{t\to\infty} \gamma(t)/|\gamma(t)|$, where $|\cdot|$ is the Euclidean norm. We may also consider orbits tending to p as t tends to $-\infty$, but we can always change the sign of the parameter t in the system associated with X and assume that t tends to ∞. We have that p is a *monodromic singular point* of X if there is no characteristic orbit associated with it.

In the case which (2.1) is analytical, the only monodromic possibilities for the origin to be are the nilpotent center or the nilpotent focus. The problem of distinguishing between these characterizes the so-called *Nilpotent Center Problem* [19, 20].

In this work, we study analytic vector fields X in $\mathbb{R}^3$ having a singular point p such that DX_p has the following Jordan canonical form

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\lambda \end{pmatrix},$$

where $\lambda \neq 0$. We can assume that the singular point is located at the origin. The associated differential system can be written, by a linear change of variables, in the following form:

$$\begin{aligned} \dot{x} &= y + P(x, y, z), \\ \dot{y} &= \qquad Q(x, y, z), \\ \dot{z} &= -\lambda z + R(x, y, z), \end{aligned} \tag{2.2}$$

where P, Q, R are analytic functions and $j^1P(0) = j^1Q(0) = j^1R(0) = 0$. For such systems, the following theorem holds.

Theorem 2.1 (Center Manifold Theorem). *Consider system* (2.2)*. There exists an invariant bidimensional C^r-manifold tangent to the xy-plane at the origin for every $r \geqslant 1$.*

This theorem is proven in [21] and the manifolds described in its statement are called *Center Manifolds*. An extensive study on the center manifolds can be found in [22]. We remark that neither the analiticity nor the uniqueness of the center manifold is guaranteed by Theorem 2.1.

Since every center manifold is tangent to the xy-plane at the origin, it has a local parametrization as $z = h(x, y)$, where $h : U \subset \mathbb{R}^2 \to \mathbb{R}$ is a C^r-function with $j^1h(0, 0) = 0$. Substituting $z = h(x, y)$ in (2.2), we obtain a restricted bidimensional differential system given by

$$\begin{aligned} \dot{x} &= y + P(x, y, h(x, y)), \\ \dot{y} &= \quad\;\; Q(x, y, h(x, y)). \end{aligned} \tag{2.3}$$

This restricted system has a nilpotent singular point at the origin. Henceforth we say that a three-dimensional analytical vector field has a *nilpotent singular point* if its associated differential system can be written as (2.2).

In the plane, nilpotent singular points are widely studied in the literature [19, 15, 23]. One of the most important problems in the study of these types of singularities is the so-called *Nilpotent Center Problem* [19, 20] which consists of distinguishing whether the origin of (2.1) is a center or not.

The goal of this work is to study the Nilpotent Center Problem for system (2.2), that is, to determine whether the origin is a nilpotent center or a nilpotent focus on a center manifold. This problem is barely explored in the literature and, as far as we know, there are only two works on this subject [24, 25]. In this papers, the authors study particular families of system (2.2) and the main technique used to obtain the results consists in restricting the system to a center manifold using a truncated expression of its parametrization. This method is not very practical for computational purposes. Our approach seeks to study the nilpotent center problem without going through the restriction of the system to a center manifold.

2.1 Monodromy

2.1.1 Monodromy Criterion

The first step to study the Nilpotent Center Problem is to verify if the singular point is monodromic. For analytic system (2.1), the following result (see [15, Theorem 3.5, p. 116] and [17, Theorem 66, p. 357]) provides a monodromy criterion.

Theorem 2.2 (Andreev's Theorem). *Let X be the vector field associated to analytic system* (2.1) *and the origin be an isolated singular point. Let $y = F(x)$ be the solution of the equation $y + P(x, y) = 0$ in a neighborhood of $(0, 0)$ and consider $f(x) = Q(x, F(x))$ and $\Phi(x) = \mathrm{div}X|_{(x,F(x))}$. Then, we can write*

$$f(x) = ax^{\alpha} + O(x^{\alpha+1}),$$

$$\Phi(x) = bx^{\beta} + O(x^{\beta+1}).$$

The origin is monodromic if and only if $a < 0, \alpha = 2n - 1$ and one of the following conditions holds:

i) $\beta > n - 1$;

ii) $\beta = n - 1$ *and* $b^2 + 4an < 0$;

iii) $\Phi \equiv 0$.

The lack of analyticity of the restricted system (2.3) apparently poses an obstacle for us to use Theorem 2.2 to study the monodromy of the singularity on a center manifold. However, the proof presented in [19] uses the concept of the Poincaré index which is not dependent on the analyticity of the vector field. The other important step of the proof uses the *generalized polar coordinates*, first introduced by Lyapunov [26] (see also [27]), which also do not depend on the analyticity. Thus, by supposing $j^r f(0) \neq 0$, the same arguments can be made and we have a less restrictive version of Theorem 2.2. We now present a summary of the necessary aspects of Poincaré index (see [17, Chapter V, p. 193]) and generalized polar coordinates and then we reproduce the proof written in [19] under the hypothesis $j^r f(0) \neq 0$ for the sake of completeness.

Proposition 2.3. *Let C be a simple and closed curve of $\mathbb{R}^2$ and X a vector field defined in a open simply connected subset of $\mathbb{R}^2$ containing C. Let L be a straight line of the plane. Suppose there exist a finite number of points M_k, $k = 1, \ldots, n$, of C such that the vector $X(M_k)$ is parallel to L. Let M be a point that describes the curve in the counterclockwise sense and p (respectively q) the number of points M_k such that $X(M)$ passes through the direction L in the counterclockwise sense (resp. in the clockwise sense). Let $I_C = (p-q)/2$. Suppose C encloses a single singular point P of X and does not contain any singular points. If S is any other simple closed curve without singular points on it that only surrounds the singular point P, then $I_C = I_S$.*

Definition 2.4. Let P be a singular point for a vector field X. The *Poincaré index* of X at P is the quantity I_C computed for any simple closed curve C containing P in its interior such that there are no other singular points of X either inside or on C.

We recall that monodromic singular points have Poincaré index 1 [17, §11]. Now we introduce the generalized polar coordinates. Given n a positive integer, consider the following differential system

$$\frac{du}{d\theta} = -v; \qquad \frac{dv}{d\theta} = u^{2n-1},$$

with initial conditions $u(0) = 1$, $v(0) = 0$. Let $\operatorname{Cs}\theta$ and $\operatorname{Sn}\theta$ be the solutions of this Cauchy problem. We shall call these functions the *generalized trigonometric functions*. The following result present their most useful properties and its proof can be found in [26, 27, 19].

Proposition 2.5. *The following holds:*

a) $\operatorname{Cs}\theta$ *and* $\operatorname{Sn}\theta$ *are T-periodic functions, where*

$$T = 2\sqrt{\frac{\pi}{n}}\frac{\Gamma(\frac{1}{2n})}{\Gamma(\frac{n+1}{2n})};$$

b) $\mathrm{Cs}^{2n}\theta + n\mathrm{Sn}^2\theta = 1$;

c) $\mathrm{Cs}\,\theta$ *is an even function and* $\mathrm{Sn}\,\theta$ *is an odd function;*

d) $\mathrm{Cs}\left(\frac{T}{2}+\theta\right) = -\mathrm{Cs}\,\theta$ *and* $\mathrm{Sn}\left(\frac{T}{2}+\theta\right) = -\mathrm{Sn}\,\theta$;

e) $\mathrm{Cs}\left(\frac{T}{2}-\theta\right) = -\mathrm{Cs}\,\theta$ *and* $\mathrm{Sn}\left(\frac{T}{2}-\theta\right) = \mathrm{Sn}\,\theta$;

f) $\displaystyle\int_0^\theta \mathrm{Sn}\,\varphi\, \mathrm{Cs}^q\varphi\, d\varphi = -\frac{\mathrm{Cs}^{q+1}\theta}{q+1}$;

g) $\displaystyle\int_0^\theta \mathrm{Sn}^p\varphi\, \mathrm{Cs}^{2n-1}\varphi\, d\varphi = \frac{\mathrm{Sn}^{p+1}\theta}{p+1}$;

h) $\displaystyle\int_0^T \mathrm{Sn}^p\varphi\, \mathrm{Cs}^q\varphi\, d\varphi = 0$, *if either* p *or* q *are odd;*

i) $\displaystyle\int_0^T \mathrm{Sn}^p\varphi\, \mathrm{Cs}^q\varphi\, d\varphi = \frac{2}{\sqrt{n^{p+1}}}\frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2n}\right)}{\Gamma\left(\frac{p+1}{2}+\frac{q+1}{2n}\right)}$, *if both* p *and* q *are even;*

When working with planar vector fields having a nilpotent singular point, a useful change of variables based on the above defined functions is the following:

$$x = \rho\,\mathrm{Cs}\,\theta,\ y = \rho^n\,\mathrm{Sn}\,\theta.$$

We refer to this change of variables as the *generalized polar coordinates.* We now can prove the C^r version of Theorem 2.2.

Theorem 2.6 (C^r-Andreev's Theorem)**.** *Let* X *be the vector field associated to the* C^r*-system,* $r \geqslant 3$, *given by*

$$\begin{aligned} \dot{x} &= y + X_2(x,y), \\ \dot{y} &= Y_2(x,y), \end{aligned} \tag{2.4}$$

where $X_2, Y_2 \in C^r$, $j^1X_2(0) = j^1Y_2(0) = 0$ *and such that the origin is an isolated singular point. Let* $y = F(x)$ *be the solution of the equation* $y + X_2(x,y) = 0$ *through* $(0,0)$ *and consider* $f(x) = Y_2(x, F(x))$ *and* $\Phi(x) = \mathrm{div}X|_{(x,F(x))}$. *We can write*

$$f(x) = ax^\alpha + O(x^{\alpha+1}),$$

$$\Phi(x) = bx^\beta + O(x^{\beta+1}).$$

for $\alpha < r$. *Suppose that* $a \neq 0$, *then the origin is monodromic if and only if* $a < 0, \alpha = 2n-1$ *and one of the following conditions holds:*

i) $\beta > n-1$ *or* $j^r\Phi(0) \equiv 0$;

ii) $\beta = n-1$ *and* $b^2 + 4an < 0$;

Proof: Function F is well defined by the Implicit Function Theorem. By making the change of variables $x = x,\ y = v + F(x)$, system (2.4) becomes:

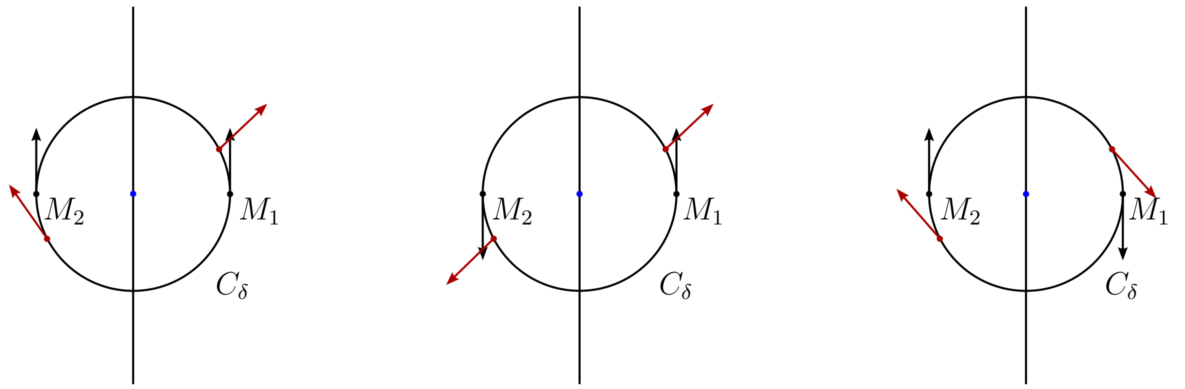
$$\begin{aligned} \dot{x} &= v + F(x) + X_2(x, v + F(x)) = v(1 + X_1(x,v)), \\ \dot{v} &= \dot{y} - F'(x)\dot{x} = f(x) + \Phi(x)v + Y_0(x,v)v^2. \end{aligned}$$

We distinguish two cases.

Case α even: By means of linear parametrization, we can assume $a = 1$. Consider the circumference $C_\delta = \{x^2 + v^2 = \delta^2\}$ with δ small enough for X to be well defined on C_δ. To compute the Poincaré index at the origin we consider the v-axis. The points $M_1 = (\delta, 0)$, $M_2 = (-\delta, 0)$, are such that $X(M_1) = (0, f(\delta))$ and $X(M_2) = (0, f(-\delta))$ are parallel to the v-axis, and are the only ones on C_δ with this property.

Since α is even, we have $f(\delta)f(-\delta) > 0$. Evaluating the vector field near M_1 and M_2 following the curve C_δ counterclockwise, we see that the vector field near M_1 ceases to be parallel to the v-axis to be in the quadrant $x > 0, v > 0$. Meanwhile near M_2, the vector field enter the quadrant $x < 0, v > 0$. In other words, following C_δ counterclockwise, the vector field turns clockwise near M_1 and counterclockwise near M_2. Thus, the Poincaré index at the origin is 0 and therefore it cannot be a monodromic singular point.

Case α odd: Considering C_δ once more, we apply the same argument as above to compute the Poincaré index at the origin. Now we have $f(\delta)f(-\delta) < 0$. If $a > 0$, then $f(\delta) > 0$ and as we evaluate vector field X along the curve C_δ counterclockwise it turns clockwise near M_1 and clockwise near M_2. Hence, Poincaré index at the origin is -1.



(a) α even (b) α odd with $a > 0$ (c) α odd with $a < 0$

Figura 2.1: Motion of vector field X along C_δ.

Now for $a < 0$, then $f(\delta) < 0$, and as we follow C_δ counterclockwise, the vector field turns counterclockwise near M_1 and counterclockwise near M_2. Therefore, Poincaré index at the origin is 1. We conclude that monodromy can only occur for $a < 0$.

Suppose $\alpha = 2n - 1$ and $a < 0$. Making the change of variables $x = (-\frac{1}{a})^{\frac{1}{2n-2}}\bar{x}$, $v = -(-\frac{1}{a})^{\frac{1}{2n-2}}\bar{y}$, and dropping the bars, yields the following system:

$$\begin{aligned} \dot{x} &= y(-1 + X_1(x, y)), \\ \dot{y} &= g(x) + \Phi(x)y + Y_0(x, y)y^2, \end{aligned} \tag{2.5}$$

where $g(x) = x^{2n-1} + O(x^{2n})$. We have three cases to consider:

Case $\beta < n - 1$: We now make the generalized polar coordinate change. System (2.5) becomes

$$\begin{aligned} \dot{\rho} &= b\mathrm{Cs}^{\beta}(\theta)\mathrm{Sn}^2(\theta)\rho^{\beta+1} + O(\rho^{\beta+2}), \\ \dot{\theta} &= b\mathrm{Cs}^{\beta+1}(\theta)\mathrm{Sn}(\theta)\rho^{\beta} + O(\rho^{\beta+1}). \end{aligned}$$

We have that $\theta = 0$ is a simple zero of $b\mathrm{Cs}^{\beta+1}(\theta)\mathrm{Sn}(\theta)\rho^{\beta}$, and thus there are solution curves of X that approaching or leaving the origin with concrete slope $\theta = 0$, i.e. we have characteristic orbits. The origin cannot be monodromic.

Case $\beta > n-1$ or $j^r\Phi(0) \equiv 0$: Under the generalized polar coordinates, system (2.5) is written as:

$$\frac{d\rho}{d\theta} = \frac{O(\rho^2)}{1+O(\rho)}.$$

Since the denominator does not vanish at $\rho = 0$, there are no arrival directions and we can define a first return function. Thus the origin is monodromic.

Case $\beta = n-1$: System (2.5) can be put in the form

$$\frac{d\rho}{d\theta} = \frac{b\rho \mathrm{Cs}^{n-1}(\theta)\mathrm{Sn}^2(\theta) + O(\rho^2)}{1 + b\mathrm{Cs}^n(\theta)\mathrm{Sn}(\theta) + O(\rho)}.$$

By statement (b) of Proposition 2.5, $1 + b\mathrm{Cs}^n(\theta)\mathrm{Sn}(\theta) = (\mathrm{Cs}^n(\theta))^2 + b\,\mathrm{Sn}(\theta)\mathrm{Cs}^n(\theta) + n\mathrm{Sn}^2(\theta)$. Hence the discriminant of the equation $1 + b\mathrm{Cs}^n(\theta)\mathrm{Sn}(\theta) = 0$ is given by $(b^2 - 4n)\mathrm{Sn}^2\theta$. Then, for $b^2 - 4n < 0$, $1 + b\mathrm{Cs}^{\beta+1}(\theta)\mathrm{Sn}(\theta)$ does not vanish, and therefore the origin is monodromic. For $b^2 - 4n > 0$, the expression $1 + b\mathrm{Cs}^n(\theta)\mathrm{Sn}(\theta)$ has simple zeros and so we have characteristic orbits.

For $b^2 - 4n = 0$, we make the directional blow-up $\bar{x} = x$, $\bar{y} = y/x^n$ in system (2.5) and conclude that the obtained singular point on its $\bar{y}$-axis is a saddle-node. Therefore system (2.5) has characteristic orbits associated with the origin, i.e. the origin is not monodromic. This concludes the proof. □

The above theorem provides a tool to study the monodromy of the origin for system (2.2) on a center manifold. In order to perform this study, we apply Theorem 2.6 to the restricted system (2.3) and check if the function f is not flat, since for any $r \geqslant 3$, there exists a center manifold such that (2.3) is a C^r-system.

The positive integer n in the statements of Theorems 2.2 and 2.6 plays an important role in the study of nilpotent monodromic singular points. So we define the *Andreev number* of a nilpotent singular point by the number n in function $f(x) = ax^{2n-1} + O(x^{2n})$ described in Theorems 2.2 and 2.6. In [2] the author proves that the Andreev number is invariant by analytical and formal orbital equivalence, i.e. via analytical and formal diffeomorphisms and time rescalings.

The following result is essentially a shortcut to verify monodromy for a given bidimensional system.

Lemma 2.7 (Monodromy criterion). *Consider a C^r-system* (2.1)*, the functions $F(x)$, $f(x)$, $\Phi(x)$ given in Theorem 2.6. Let $\tilde{a}$ be the coefficient of x^{2n-1} in the power series expansion of $f(x)$, $\tilde{b}$ the coefficient of x^{n-1} in the power series expansion of $\Phi(x)$ and $\Delta = \tilde{b}^2 + 4\tilde{a}n$. Suppose $j^{2n-2}f(0) = 0$ and $j^{n-2}\Phi(0) = 0$, then the singular point is monodromic with Andreev number n if and only if $\Delta < 0$.*

Proof: Note that $\Delta < 0$ gives us immediately the first monodromy condition, i.e. $\alpha = 2n-1$ and $a = \tilde{a} < 0$. If $\tilde{b} = 0$, then either $j^r\Phi(0) \equiv 0$ or $\Phi(x) \in O(x^n)$ and the monodromy condition (i) in Theorem 2.6 is satisfied. If $\tilde{b} \neq 0$, the condition (ii) in Theorem 2.6 is satisfied. Thus, the singular point is monodromic with Andreev number n. Conversely, suppose the singular point is monodromic with Andreev number n. Then $\tilde{a} < 0$ and one of the monodromy conditions (i) or (ii) in Theorem 2.6 must hold. If (i) holds, $\tilde{b}$ is null and $\tilde{a} < 0$ implies $\Delta < 0$. On the other hand, if (ii) is satisfied then Δ must be negative. Therefore $\Delta < 0$ regardless the monodromy condition. □

Armed with the above results, we can procedurally apply an algebraic algorithm to obtain conditions on the parameters of system (2.2) for the origin to be monodromic on a center manifold. This algorithm is described in the next section.

2.1.2 Monodromy classes and Andreev number

Consider the following representation of analytic system (2.2):

$$\begin{aligned}\dot{x} &= \quad y + \sum_{j+k+l\geqslant 2} a_{jkl}x^j y^k z^l,\\ \dot{y} &= \qquad\; \sum_{j+k+l\geqslant 2} b_{jkl}x^j y^k z^l,\\ \dot{z} &= -\lambda z + \sum_{j+k+l\geqslant 2} c_{jkl}x^j y^k z^l.\end{aligned} \tag{2.6}$$

By the Center Manifold Theorem 2.1, any center manifold for (2.6) has a local parametrization $z = h(x, y)$ for which $h \in C^r$ and $j^1h(0) = 0$. Being an invariant manifold, it must satisfy the following equation:

$$\frac{\partial h}{\partial x}(x,y)(y+P(x,y,h(x,y))) + \frac{\partial h}{\partial y}(x,y)(Q(x,y,h(x,y))) + \lambda h(x,y) - R(x,y,h(x,y)) = 0. \tag{2.7}$$

This equation allows us to obtain the Taylor polynomial of $h(x, y)$ to any finite order m. In order to achieve this, we write $h(x, y)$ as a formal series $h(x, y) = \sum_{j+k\geqslant 2} h_{jk}x^j y^k$ and substitute in (2.7). We expand the left-hand side of this equation in a power series of x, y and equate the coefficients to zero to obtain the coefficients h_{jk} given in terms of the coefficients $a_{jkl}, b_{jkl}, c_{jkl}$ from (2.6).

More precisely, we start by equating the coefficients of the homogeneous terms of degree 2, which is the lowest, to zero. Then, we determine h_{20}, h_{11}, h_{02}. The coefficients h_{jk} of homogeneous degree 3, i.e. such that $j + k = 3$, are given by the coefficients $a_{jkl}, b_{jkl}, c_{jkl}$ of system (2.6) and the coefficients h_{jk} of the previous homogeneous degree. We repeat this iterative process obtaining successively the coefficients h_{jk} of a homogeneous part once having computed the coefficients h_{jk} of the previous ones.

Once the coefficients h_{jk} are obtained up to a large enough homogeneous degree m, we have successfully retrieved the Taylor polynomial $j^m h(0) = \sum_{j+k=2}^{m} h_{jk}x^j y^k$. It is well known that any C^r center manifold, for $r \geqslant m$, has the same m-jet (see, for instance, [22]). Substituting $h(x, y) = j^m h(0) + O(\|x, y\|^{m+1})$ in (2.6) yields a restricted C^r-system (2.3) for which the m-jet of each component of the associated vector field is known.

We can proceed analogously to retrieve the m-jet of function $F(x)$ from Theorem 2.6. It is sufficient to consider the equation

$$F(x) + P(x, F(x), h(x, F(x))) = 0.$$

Consequently, determining $j^m F(0)$ also determines $j^m f(0)$ and $j^m\Phi(0)$. Hence, if there exists an $m \in \mathbb{N}$ such that $j^m f(0) \neq 0$, we can apply this process to study the monodromy using Theorem 2.6.

Applying the algorithm described above to $m = 3$, we obtain:

$$j^2h(0) = \frac{c_{200}}{\lambda}x^2 + \frac{(\lambda c_{110} - 2c_{200})}{\lambda^2}xy + \frac{(2c_{200} - \lambda c_{110} + c_{020}\lambda^2)}{\lambda^3}y^2.$$

Which yields the following:

$$\begin{aligned} j^3 f(0) &= b_{200}x^2 - \frac{(b_{110}a_{200}\lambda - b_{300}\lambda - b_{101}c_{200})\,x^3}{\lambda},\\ j^1\Phi(0) &= (2a_{200} + b_{110})x.\end{aligned} \tag{2.8}$$

The full expressions of $j^3h(0)$ and $j^3\Phi(0)$ are too extensive, so we omit them here. With the information given by (2.8), it is possible to conclude, by Theorem 2.6, that monodromy can only occur if $b_{200} = 0$. Furthermore, we have the necessary tools to classify the first family of systems having a monodromic singular point at the origin on a center manifold.

Proposition 2.8. *The origin is a nilpotent monodromic singular point with Andreev number* 2 *on a center manifold of system* (2.6) *if and only if* $b_{200} = 0$ *and*

$$\frac{b_{101}c_{200}}{\lambda} < -\frac{(2a_{200} - b_{110})^2}{8} - b_{300}. \tag{2.9}$$

Moreover, if $2a_{200} + b_{110} \neq 0$*, the restricted system satisfies the monodromy condition* $\beta = n - 1$ *in Theorem 2.6.*

Proof: The origin is monodromic only if $b_{200} = 0$. Thus, by Lemma 2.7, the necessary and sufficient condition for the Andreev number to be 2 is $\Delta < 0$, which by (2.8) becomes:

$$\frac{\left(4\,{a_{200}}^2 - 4\,b_{110}a_{200} + {b_{110}}^2 + 8\,b_{300}\right)\lambda + 8\,b_{101}c_{200}}{\lambda} < 0,$$

or equivalently

$$\frac{b_{101}c_{200}}{\lambda} < -\frac{(2a_{200} - b_{110})^2}{8} - b_{300}.$$

The second statement also follows from (2.8). □

We highlight the singular points with Andreev number 2 since, in the plane, there are interesting tools to study the Center Problem that only hold for this case. For instance, in [2], a method to solve the Center Problem for nilpotent singular points is presented which consists in finding a formal inverse integrating factor for such systems, i.e. a formal series $V(x, y)$ satisfying $XV = V\mathrm{div}X$. Throughout this thesis, we will refer to this method as the *inverse integrating factor method*. More precisely, the method is based on the following result:

Theorem 2.9 (Theorem 3 in [2])**.** *Consider analytic system* (2.1) *with a monodromic nilpotent singular point at the origin such that the Andreev number* n *is even. If* (2.1) *has a formal inverse integrating factor then the origin is a center. Moreover if* $n = 2$ *then the converse is also true.*

The case $n = 2$ was also studied in [28, 29] where the authors also found a canonical form for nilpotent monodromic singularities of third order in the plane. In sum, the nilpotent centers of planar analytic systems having Andreev number 2 are completely characterized by the existence of a formal inverse integrating factor. Proposition 2.8 characterizes the monodromic nilpotent singular points with Andreev number 2 in $\mathbb{R}^3$ and is a first step to extend these results to three-dimensional systems.

2.1.3 Some results for planar C^r systems with nilpotent singular points

We now prove some results for planar C^r-systems which are useful for the Nilpotent Center Problem in $\mathbb{R}^3$. First, we need the following definitions.

Definition 2.10. A polynomial $p \in \mathbb{R}[x,y]$ is a (t_1,t_2)*-quasi-homogeneous polynomial of weighted degree* k if $p(\lambda^{t_1}x, \lambda^{t_2}y) = \lambda^k p(x,y)$. A general expression for such polynomials is $p(x,y) = \sum_{t_1 i + t_2 j = k} a_{ij}x^i y^j$ where $a_{ij} \in \mathbb{R}$. The vector space of all (t_1,t_2)-quasi--homogeneous polynomial of weighted degree k is denoted by $\mathcal{P}_k^{(t_1,t_2)}$.

The next proposition is the C^r version of a result proven for the analytic case in [30].

Proposition 2.11. *Consider a C^r-system* (2.1)*. Let n be a positive integer such that $2n-1 < r$. The origin is a monodromic isolated singular point with Andreev number n if and only if there exists a local analytical change of variables that transforms* (2.1) *into the following form*

$$\begin{aligned} \dot{x} &= y + \mu x^n + \sum_{i+nj \geqslant n+1} \tilde{a}_{ij}x^i y^j + O(\|x,y\|^r), \\ \dot{y} &= -nx^{2n-1} + n\mu x^{n-1}y + \sum_{i+nj \geqslant 2n} \tilde{b}_{ij}x^i y^j + O(\|x,y\|^r). \end{aligned} \tag{2.10}$$

Proof: Note that the origin of system (2.10) is monodromic by Theorem 2.6. Conversely, considering the functions defined in Theorem 2.6, the change of variables $x \to x$, $y \to y - F(x)$, transforms system (2.1) into

$$\begin{aligned} \dot{x} &= y + y\tilde{P}(x,y), \\ \dot{y} &= f(x) + y\Phi(x) + y^2\tilde{Q}(x,y). \end{aligned} \tag{2.11}$$

By Theorem 2.6, since the origin is monodromic, the above system can be rewritten as

$$\begin{aligned} \dot{x} &= y + \textstyle\sum_{k \geqslant n+1} P_k(x,y) + O(\|x,y\|^r), \\ \dot{y} &= ax^{2n-1} + \tilde{b}x^{n-1}y + \textstyle\sum_{k \geqslant 2n} Q_k(x,y) + O(\|x,y\|^r), \end{aligned}$$

where $P_k(x,y), Q_k(x,y) \in \mathcal{P}_k^{(1,n)}$, $a \neq 0$, $\tilde{b} \in \mathbb{R}$ and $2n+1 < r$. Note that the origin has Andreev number n. By Lemma 2.7, we have $\Delta = \tilde{b}^2 + 4an < 0$. Define $A = \frac{\Delta}{4n^2}$. The mapping

$$\Psi(x,y) = \left(|A|^{\frac{1}{2(n-1)}}x, |A|^{\frac{1}{2(n-1)}} \left(y - \frac{\tilde{b}}{2n}x^n \right) \right),$$

is a diffeomorphism, and it transforms (2.11) into

$$\begin{aligned} \dot{x} &= y + \mu x^n + \textstyle\sum_{k \geqslant n+1} \tilde{P}_k(x,y) + O(\|x,y\|^r), \\ \dot{y} &= -nx^{2n-1} + n\mu x^{n-1}y + \textstyle\sum_{k \geqslant 2n} \tilde{Q}_k(x,y) + O(\|x,y\|^r), \end{aligned}$$

where $\tilde{P}_k(x,y), \tilde{Q}_k(x,y) \in \mathcal{P}_k^{(1,n)}$ and $\mu = \frac{\tilde{b}}{2n|A|^{\frac{1}{2}}}$. $\square$

Remark 2.12. The expression of μ given above yields the following implication: Consider system (2.1) with a monodromic nilpotent singular point. It satisfies the monodromy condition $\beta = n-1$ (item (ii) from Theorem 2.6) if and only if $\mu \neq 0$ in canonical form (2.10). In other words, the conditions (i) and (ii) in Theorem 2.6 are preserved in the canonical form (2.10).

We now turn our attention to systems in family (2.10). Applying the generalized polar coordinate change yields:

$$\begin{aligned}\dot\rho &= \rho^n\left(\mu\mathrm{Cs}^{n-1}\theta-(n-1)\mathrm{Cs}^{2n-1}\theta\,\mathrm{Sn}\,\theta+P(\rho,\theta)+O(\rho^r)\right),\\ \dot\theta &= -n\rho^{n-1}\left(1-(n-1)\mathrm{Sn}^2\theta+Q(\rho,\theta)+O(\rho^r)\right),\end{aligned}$$

where $P(\rho,\theta)$ and $Q(\rho,\theta)$ are given by:

$$\begin{aligned}P(\rho,\theta) &= \sum_{i+nj\geqslant n+1}\tilde a_{ij}\rho^{i+nj-n}\mathrm{Cs}^{i+2n-1}\theta\,\mathrm{Sn}^j\theta\\ &\quad+\sum_{i+nj\geqslant 2n}\tilde b_{ij}\rho^{i+nj-2n+1}\mathrm{Cs}^{i}\theta\,\mathrm{Sn}^{j+1}\theta,\\ Q(\rho,\theta) &= \sum_{i+nj\geqslant n+1}\tilde a_{ij}\rho^{i+nj-n}\mathrm{Cs}^{i}\theta\,\mathrm{Sn}^{j+1}\theta\\ &\quad-\sum_{i+nj\geqslant 2n}\frac{\tilde b_{ij}}{n}\rho^{i+nj-2n+1}\mathrm{Cs}^{i+1}\theta\,\mathrm{Sn}^{j}\theta.\end{aligned}$$

We then obtain the following differential equation:

$$\frac{d\rho}{d\theta}=R(\rho,\theta)=-\rho\frac{\left(\mu\mathrm{Cs}^{n-1}\theta-(n-1)\mathrm{Cs}^{2n-1}\theta\,\mathrm{Sn}\,\theta+P(\rho,\theta)+O(\rho^r)\right)}{n\left(1-(n-1)\mathrm{Sn}^2\theta+Q(\rho,\theta)+O(\rho^r)\right)}. \tag{2.12}$$

Using Proposition 2.5 item (b), $1-(n-1)\mathrm{Sn}^2\theta=\mathrm{Cs}^{2n}\theta+\mathrm{Sn}^2\theta$, and so the denominator of (2.12) does not vanish at $\rho=0$. Thus $R(\rho,\theta)$ is a C^r function in a neighborhood of $\rho=0$. Let $\tilde\rho(\theta,\rho_0)$ be the solution of (2.12) with initial condition $\tilde\rho(0,\rho_0)=\rho_0$. We can expand it as a Taylor polynomial, i.e.

$$\tilde\rho(\theta,\rho_0)=\sum_{k=1}^{r}v_k(\theta)\rho_0^k+O(\rho^r),$$

and define the so-called *focal values*.

Definition 2.13. Consider function $\tilde\rho(\theta,\rho_0)$ associated to equation (2.12) defined above. The *displacement function* is defined by

$$d(\rho_0)=\tilde\rho(T,\rho_0)-\rho_0.$$

The *focal values* are the coefficients of the Taylor expansion of $d(\rho_0)$ given by $v_1(T)-1$ and $v_k(T)$ for $k\geqslant 2$.

For the origin to be a center for system (2.10) all the focal values must be null. In the literature, for the non-degenerate singular point, it is well known that the first non-zero focal value is the coefficient of an odd power of ρ_0 (see [31, Lemma 5, page 243] or [32, Proposition 3.1.4, page 94]). In [29], the authors studied system (2.10) for $n=2$ and concluded that the first non-zero focal value is the coefficient of an even power of ρ_0. As a matter of fact, a more general result is true.

Proposition 2.14. *For system* (2.10) *with monodromic nilpotent singular points having Andreev number n, the first non-zero focal value is the coefficient of a power of ρ_0 whose parity is the same as the parity of n.*

A proof of this proposition for the analytic case can be found in [26, pages 50 to 54]. The C^r case also holds, and the key argument to prove Proposition 2.14 is that the differential equation (2.12) is invariant under the following transformations:

$$\rho \mapsto -\rho,\ \theta \mapsto \theta + \frac{T}{2},\ \text{for } n \text{ odd},$$

$$\rho \mapsto -\rho,\ \theta \mapsto -\left(\theta + \frac{T}{2}\right),\ \text{for } n \text{ even}.$$

It is sufficient to follow the steps of [31, Section 24 of Chapter IX] to prove the proposition. For the sake of completeness, we include the proof in Section A.2.

Lemma 2.15. *The coefficient v_1 in function $\tilde{\rho}(\theta, \rho_0)$ associated to equation* (2.12) *is given by:*

$$\exp\left(-\int_0^\theta \frac{\left(\mu Cs^{n-1}\varphi - (n-1)\,Cs^{2n-1}\varphi\, Sn\,\varphi\right)}{n\left(1-(n-1)Sn^2\varphi\right)} d\varphi\right). \tag{2.13}$$

Proof: Substituting the Taylor expansion of $\tilde{\rho}(\theta, \rho_0)$ in the equation (2.12) and comparing the coefficients of ρ_0 yields the result. □

Lemma 2.16. *Consider function $\tilde{\rho}(\theta, \rho_0)$ associated to equation* (2.12)*. Then $v_1(T) = 1$ for n even and $v_1(T) = e^{-\mu A^2}$ for n odd, where $A \neq 0$.*

Proof: Using the properties described in Proposition 2.5, we compute the expression (2.13) for $\theta = T$:

$$v_1(T) = \exp\left(-\mu \int_0^T \frac{\left(\mathrm{Cs}^{n-1}\theta\right)}{n\left(1-(n-1)\mathrm{Sn}^2\theta\right)} d\theta\right).$$

Let $G(\theta)$ be the integrand of the above expression. For n odd, $G(\theta)$ is a non-negative even function. Thus, the above integral is not zero, and then $v_1(T) = e^{-\mu A^2}$ for some $A \neq 0$. If n is even, $G(\theta)$ is an odd T-periodic function. Therefore $v_1(T) = 1$. □

Theorem 2.17. *Suppose that the C^r system* (2.1) *has nilpotent monodromic singular point with odd Andreev number n, with $2n - 1 < r$. If the system satisfies the monodromy condition $\beta = n - 1$ then the origin is a nilpotent focus.*

Proof: Consider system (2.1) under the hypothesis of Theorem 2.17. By Proposition 2.11, it can be written as system (2.10). By Remark 2.12, we have $\mu \neq 0$ in canonical form (2.10). Lemma 2.16 implies that $v_1(T) \neq 1$ for $\mu \neq 0$. Therefore, the origin must be a nilpotent focus. □

Theorem 2.17 is a C^r version of a result that can be found for analytic systems in [33, Proposition 6, page 6]. The proof presented in [33] is based in the normal form (2.10) and [34, Theorem 5, page 422]. This result can be immediately extended to the three-dimensional setting.

Corollary 2.18. *Suppose that the origin of system* (2.2) *is monodromic with odd Andreev number n and satisfies the monodromy condition $\beta = n - 1$. Then the origin is a nilpotent focus on a center manifold.*

Although the Andreev number is invariant by orbital equivalence (see [2] for a proof), the number β is not. The following system [23, Example 13],

$$\begin{aligned}\dot{x} &= y + a_1x^9 + a_2x^6y - a_1(54a_1^2 - 6a_2 + b_2)x^3y^2 + a_4y^3,\\ \dot{y} &= -x^{11} - 9a_1x^8y + b_2x^5y^2 + 3a_1(18a_1^2 + b_2)x^2y^3,\end{aligned}$$

has Andreev number $n = 6$ and $\beta = 14$. Performing an analytic change of variables, we can transform the above system into the form (2.11):

$$\begin{aligned}\dot{x} &= y + O(\|x, y\|^{13}),\\ \dot{y} &= -x^{11}(1 + O(x)) + yx^{11} + O(\|x, y\|^{13}),\end{aligned}$$

for which the Andreev number is $n = 6$ and $\beta = 11$. However, note that for both systems, the monodromy condition $\beta > n-1$, i.e. condition (ii) in Theorem 2.6, is satisfied. In fact, using the canonical form (2.11), it is possible to prove that the monodromy conditions (i) and (ii) in Theorem 2.6 are invariant by local diffeomorphisms. Before stating the result, we need the following definition of quasi-homogeneous vector fields.

Definition 2.19. A planar vector field $X : \mathbb{R}^2 \to \mathbb{R}^2$, $X = (X_1, X_2)$ is (t_1, t_2)-*quasi-homogeneous of weighted degree* k if $X_1(x, y) \in \mathcal{P}_{k+t_1}^{(t_1,t_2)}$ and $X_2(x, y) \in \mathcal{P}_{k+t_2}^{(t_1,t_2)}$. The vector space of all (t_1, t_2)-quasi-homogeneous vector fields of weighted degree k is denoted by $\mathcal{Q}_k^{(t_1,t_2)}$.

One can argue that a more natural definition would be taking $X_1, X_2 \in \mathcal{P}_k^{(t_1,t_2)}$. However, the above definition is better suited for our study and is also used in several papers, for instance [30, 2].

Remark 2.20. The vector spaces $\mathcal{P}_k^{(t_1,t_2)}$ and $\mathcal{Q}_k^{(t_1,t_2)}$ are well-behaved. Some of their properties that are useful in the understanding of the next proof are listed below:

- If $f \in \mathcal{P}_k^{(t_1,t_2)}$ then $\dfrac{\partial f}{\partial x} \in \mathcal{P}_{k-t_1}^{(t_1,t_2)}$ and $\dfrac{\partial f}{\partial y} \in \mathcal{P}_{k-t_2}^{(t_1,t_2)}$;
- If $f \in \mathcal{P}_{k_1}^{(t_1,t_2)}$ and $g \in \mathcal{P}_{k_2}^{(t_1,t_2)}$ then $fg \in \mathcal{P}_{k_1+k_2}^{(t_1,t_2)}$;
- If $\varphi \in \mathcal{Q}_{k_1}^{(t_1,t_2)}$ and $X \in \mathcal{Q}_{k_2}^{(t_1,t_2)}$ then $d\varphi \cdot X \in \mathcal{Q}_{k_1+k_2}^{(t_1,t_2)}$.

Proposition 2.21. *For* C^r *system* (2.1) *having an isolated monodromic nilpotent singular point at the origin, with Andreev number* n *such that* $2n-1 < r$*, the monodromy conditions (i) and (ii) in Theorem 2.6 are invariant by local diffeomorphisms.*

Proof: Let X and Y be vector fields having an isolated nilpotent singular point satisfying the monodromy conditions (i) and (ii) in Theorem 2.6, respectively. Suppose that there exists a local diffeomorphism $(\bar{x}, \bar{y}) = \varphi(x, y)$ that transforms X into Y. Without loss of generality, we can assume that the associated systems are in canonical form (2.11). We have that

$$d\varphi \cdot Y = X \circ \varphi. \tag{2.14}$$

We write $\varphi(x, y) = (\sum_{j+k\geqslant 1} a_{jk}x^jy^k, \sum_{j+k\geqslant 1} b_{jk}x^jy^k) + O(\|x, y\|^r)$. The homogeneous terms of degree 1 in both sides of (2.14) must satisfy the following equation:

$$\begin{pmatrix} a_{10}y \\ b_{10}y \end{pmatrix} = \begin{pmatrix} b_{10}x + b_{01}y \\ 0 \end{pmatrix}.$$

Thus, $a_{10} = b_{01}$ and $b_{10} = 0$. Note that a_{10} must not be zero or else φ is not a diffeomorphism in a neighborhood of the origin.

Now, we write $X = X_{n-1} + \bar{X}$, $Y = Y_{n-1} + \bar{Y}$ where $X_{n-1}, Y_{n-1} \in \mathcal{Q}_{n-1}^{(1,n)}$ and $\bar{X}, \bar{Y}$ are C^r vector fields containing the $(1,n)$-quasi-homogeneous terms of degree greater than $n-1$. We also write $\varphi = \sum_{k=-(n-1)}^{n-1} \varphi_k + \tilde{\varphi}$ where $\varphi_k \in \mathcal{Q}_k^{(1,n)}$ and $\tilde{\varphi}$ is a C^r map containing the $(1,n)$-quasi-homogeneous terms of degree greater than $n-1$. Note that, for $k = 1, \ldots, n-1$, φ_{-k} is given by:

$$\varphi_{-k}(x, y) = (0, b_{n-k,0} x^{n-k}) \text{ and } \varphi_0 = (a_{10} x, a_{10} y + b_{n0} x^n).$$

Since X and Y satisfy the monodromy conditions (i) and (ii) in Theorem 2.6, respectively, we have

$$X_{n-1} = \begin{pmatrix} y \\ a x^{2n-1} \end{pmatrix}, \; Y_{n-1} = \begin{pmatrix} y \\ A x^{2n-1} + B x^{n-1} y \end{pmatrix},$$

for some $a, A < 0$ and $B \neq 0$. Let $k \in \{1, \ldots, n-1\}$ be such that φ_{-k} is the first non-zero $(1,n)$-quasi-homogeneous term in φ, i.e. $\varphi = \varphi_{-k} + \cdots + \tilde{\varphi}$. Then the lowest $(1,n)$-quasi-homogeneous term in the expression $d\varphi \cdot Y$ is $d\varphi_{-k} \cdot Y_{n-1}$, whose weighted degree is $n-k-1$, i.e.

$$d\varphi_{-k} \cdot Y_{n-1} = \begin{pmatrix} 0 \\ (n-k) b_{n-k,0} x^{n-k-1} y \end{pmatrix}.$$

Now, the $(1,n)$-quasi-homogeneous term of weighted degree $n-k-1$ in $X \circ \varphi$ is given by $(b_{n-k,0} x^{n-k}, 0)^T$. Equating this expression to the one above, we obtain that $b_{n-k,0} = 0$, which means that $\varphi_{-k} \equiv 0$. But this is a contradiction. Thus, $\varphi = \sum_{k=0}^{n-1} \varphi_k + \tilde{\varphi}$.

We go back to (2.14) once more to complete the proof. The lowest $(1,n)$-quasi-homogeneous term in the expression $d\varphi \cdot Y$ is $d\varphi_0 \cdot Y_{n-1}$, whose weighted degree is $n-1$. Denoting by $(X \circ \varphi)_{n-1}$ the $(1,n)$-quasi-homogeneous term of degree $n-1$ in $X \circ \varphi$, it follows that $d\varphi_0 \cdot Y_{n-1} = (X \circ \varphi)_{n-1}$, that is

$$\begin{pmatrix} a_{10} y \\ A a_{10} x^{2n-1} + (n b_{n0} + a_{10} B) x^{n-1} y \end{pmatrix} = \begin{pmatrix} a_{10} y + b_{n0} x^n \\ a a_{10}^{2n-1} x^{2n-1} \end{pmatrix}.$$

For this equation to be satisfied, we must have $b_{n0} = 0$ and $a_{10} B = 0$. Since $a_{10} \neq 0 \neq B$, this is not possible. We conclude that if φ is a local diffeomorphism, it must preserve the monodromy conditions. $\square$

Remark 2.22. The above result holds for formal settings, that is, formal vector fields and formal changes of variables φ such that $d\varphi(0) \neq 0$.

2.2 Auxiliary Linear Operators

In this section, we prove auxiliary results regarding some useful linear operators that show up frequently when working with vector fields having nilpotent singular points. These particular linear operators provide some interesting results on normal forms and methods involving formal series.

We denote by $H_{(3,1)}^{(n)}$ the vector space of homogeneous polynomials of degree n in three variables. A basis for this vector space is given by the collection of all monomials $x^j y^k z^l$

for which $j,k,l \geqslant 0$ and $j+k+l=n$. Given a polynomial $p \in H^{(n)}_{(3,1)}$, let $\langle p \rangle$ be the vector subspace spanned by p. We consider the following linear operator:

$$\begin{aligned} T_n \colon H^{(n)}_{(3,1)} &\to \qquad H^{(n)}_{(3,1)} \\ p \quad &\mapsto y\frac{\partial p}{\partial x} - \lambda z\frac{\partial p}{\partial z}, \end{aligned}$$

where $\lambda \neq 0$.

Lemma 2.23. *The kernel of linear operator T_n is given by* $\ker T_n = \langle y^n \rangle$.

Proof: Clearly $T_n(y^n) = 0$. Let $p = \sum_{j+k+l=n} p_{j,k,l} x^j y^k z^l$. If $p \in \ker T_n$, then $T_n(p) = 0$, i.e.:

$$\sum_{j+k+l=n} ((j+1)p_{j+1,k-1,l} - \lambda l p_{j,k,l}) x^j y^k z^l = 0.$$

Consider $l \neq 0$. For $k=0$, the coefficient of $x^j z^l$ in the left-hand side of this equation is $-\lambda l p_{j,0,l}$ which implies that $p_{j,0,l} = 0$. For $k=1$, the equation yields $p_{j,1,l} = 0$. Proceeding step by step until index $k = n-1$, we conclude that $p_{j,k,l} = 0$. Now, for $l = 0$, the coefficient of monomial $x^j y^k$ in the left-hand side expression above is $(j+1)p_{j+1,k-1,0}$, which implies that $p_{j,k,0} = 0$ for $j > 0$. Thus $p = p_{0,n,0} y^n \in \langle y^n \rangle$. □

Lemma 2.24. *For every $q \in H^{(n)}_{(3,1)}$, there is a choice of $p \in H^{(n)}_{(3,1)}$ such that $T_n(p) + q \in \langle x^n \rangle$.*

Proof: It is sufficient to prove $H^{(n)}_{(3,1)} = \langle x^n \rangle \oplus \mathrm{Im} T_n$. By Lemma 2.23, the codimension of $\mathrm{Im} T_n$ is 1. Since $\langle x^n \rangle$ has dimension 1, we only need to prove that the intersection of both vector subspaces is $\{0\}$. We have $T_n(p) \in \langle x^n \rangle$ if and only if $y\frac{\partial p}{\partial x} - \lambda z \frac{\partial p}{\partial z} = \alpha x^n$, which is only possible when $\frac{\partial p}{\partial x} = \frac{\partial p}{\partial z} = \alpha = 0$. Therefore $T_n(p) \in \langle x^n \rangle$ if and only if $p \in \langle y^n \rangle = \ker T_n$. The result follows. □

Now, we turn our attentions to the following linear operator:

$$\begin{aligned} L_n \colon H^{(n)}_{(3,1)} &\to \qquad H^{(n)}_{(3,1)} \\ p \quad &\mapsto y\frac{\partial p}{\partial x} - \lambda z\frac{\partial p}{\partial z} + \lambda p, \end{aligned}$$

where $\lambda \neq 0$.

Lemma 2.25. *The kernel of linear operator L_n is given by* $\ker L_n = \langle y^{n-1} z \rangle$.

Proof: Trivially $L_n(y^{n-1}z) = 0$. Let $p = \sum_{j+k+l=n} p_{j,k,l} x^j y^k z^l$. If $p \in \ker L_n$, then $L_n(p) = 0$ which equates to:

$$\sum_{j+k+l=n} ((j+1)p_{j+1,k-1,l} - \lambda(l-1) p_{j,k,l}) x^j y^k z^l = 0.$$

Consider $l \neq 1$. For $k=0$, the coefficient of $x^j z^l$ in the left-hand side of this equation is $-\lambda(l-1) p_{j,0,l}$ which implies that $p_{j,0,l} = 0$. For $k=1$, the equation yields $p_{j,1,l} = 0$. Proceeding step by step until index $k = n-1$, we conclude that $p_{j,k,l} = 0$. Now, for $l = 1$, the coefficient of monomial $x^j y^k z$ in the left-hand side expression above is $(j+1)p_{j+1,k-1,1}$, which implies that $p_{j,k,1} = 0$ for $j > 0$. Thus $p = p_{0,n-1,1} y^{n-1} z \in \langle y^{n-1} z \rangle$. □

Lemma 2.26. *For every* $q \in H^{(n)}_{(3,1)}$, *there is a choice of* $p \in H^{(n)}_{(3,1)}$ *such that* $L_n(p) + q \in \langle x^{n-1}z \rangle$.

Proof: Again, we prove that $H^{(n)}_{(3,1)} = \langle x^{n-1}z \rangle \oplus \mathrm{Im}L_n$. Lemma 2.25 tells us that the codimension of $\mathrm{Im}L_n$ is 1. As in the proof of Lemma 2.24, it is enough to prove that the intersection of $\langle x^{n-1}z \rangle$ and $\mathrm{Im}L_n$ is $\{0\}$. We have that $L_n(p) \in \langle x^{n-1}z \rangle$ if and only if $y\dfrac{\partial p}{\partial x} - \lambda z\dfrac{\partial p}{\partial z} + \lambda p = \alpha x^{n-1}z$. The coefficient of $x^{n-1}z$ in the left-hand side of this equation is zero, therefore $\alpha = 0$. Hence $L_n(p) \in \langle x^{n-1}z \rangle$ if and only if $p \in \langle y^{n-1}z \rangle = \ker L_n$. The result holds. □

Remark 2.27. Note that if we consider similar operators

$$\tilde{T}_n : H^{(n)}_{(3,1)} \to H^{(n)}_{(3,1)},\ p \mapsto x\frac{\partial p}{\partial y} - \lambda z\frac{\partial p}{\partial z}, \text{and}$$

$$\tilde{L}_n : H^{(n)}_{(3,1)} \to H^{(n)}_{(3,1)},\ p \mapsto y\frac{\partial p}{\partial x} - \lambda z\frac{\partial p}{\partial z} + \lambda p,$$

just by exchanging x and y in the expressions above yields analogous results. More precisely, we have $\ker \tilde{T}_n = \langle x^n \rangle$, $\ker \tilde{L}_n = \langle x^{n-1}z \rangle$ and $H^{(n)}_{(3,1)} = \langle y^n \rangle \oplus \mathrm{Im}\tilde{T}_n = \langle y^{n-1}z \rangle \oplus \mathrm{Im}\tilde{L}_n$.

2.3 Normal Forms

2.3.1 Belitskii Normal Form

An important step in the study of vector fields having nilpotent singular points is to find normal forms for its associated differential system. We first look at the Normal Form theory to search the most practical normal forms for system (2.2). The following theorem, which is proven in [35, Theorem 5.1] gives us a starting point:

Theorem 2.28 (Belitskii Normal Form)**.** *Consider a formal system of differential equations of form*

$$\dot{\mathbf{x}} = A\mathbf{x} + X(\mathbf{x}),$$

where $\mathbf{x} \in \mathbb{R}^n$, $A \in M_n(\mathbb{R})$ *and* $j^1X(0) = 0$. *Then there exists a germ at* 0 *of diffeomorphism* $\mathbf{x} \to \mathbf{x} + \varphi(\mathbf{x})$, $j^1\varphi(0) = 0$, *that transforms the above system into the following:*

$$\dot{\mathbf{x}} = A\mathbf{x} + f(\mathbf{x}),$$

where $j^1 f(0) = 0$ *and*

$$A^t f(\mathbf{x}) - df(\mathbf{x})A^t\mathbf{x} = 0. \tag{2.15}$$

We apply Theorem 2.28 to system (2.2) and obtain its correspondent normal form. System (2.2) is written in the desired form $\dot{\mathbf{x}} = A\mathbf{x} + X(\mathbf{x})$ where $\mathbf{x} \in \mathbb{R}^3$ with

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\lambda \end{pmatrix}.$$

Writting function $f(\mathbf{x})$ in Theorem 2.28 as $f(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}), f_3(\mathbf{x}))^t$, we can solve the equation (2.15) to obtain its expression. Since

$$A^t f(\mathbf{x}) = \begin{pmatrix} 0 \\ f_1(\mathbf{x}) \\ -\lambda f_3(\mathbf{x}) \end{pmatrix}, \text{ and } df(\mathbf{x})A^t\mathbf{x} = \begin{pmatrix} x\dfrac{\partial f_1}{\partial y}(\mathbf{x}) - \lambda z\dfrac{\partial f_1}{\partial z}(\mathbf{x}) \\ x\dfrac{\partial f_2}{\partial y}(\mathbf{x}) - \lambda z\dfrac{\partial f_2}{\partial z}(\mathbf{x}) \\ x\dfrac{\partial f_3}{\partial y}(\mathbf{x}) - \lambda z\dfrac{\partial f_3}{\partial z}(\mathbf{x}) \end{pmatrix}.$$

equation (2.15) becomes

$$\begin{aligned} -x\frac{\partial f_1}{\partial y}(\mathbf{x}) + \lambda z\frac{\partial f_1}{\partial z}(\mathbf{x}) &= 0, \\ f_1(\mathbf{x}) - x\frac{\partial f_2}{\partial y}(\mathbf{x}) + \lambda z\frac{\partial f_2}{\partial z}(\mathbf{x}) &= 0, \\ -\lambda f_3(\mathbf{x}) - x\frac{\partial f_3}{\partial y}(\mathbf{x}) + \lambda z\frac{\partial f_3}{\partial z}(\mathbf{x}) &= 0. \end{aligned} \tag{2.16}$$

Let us write $f_i(\mathbf{x}) = \sum_{n\geqslant 2} f_i^n(\mathbf{x})$ where $f_i^n(\mathbf{x})$ are homogeneous polynomials of degree n for $i = 1, 2, 3$. The first equation in (2.16) implies

$$x\frac{\partial f_1^n}{\partial y}(\mathbf{x}) - \lambda z\frac{\partial f_1^n}{\partial z}(\mathbf{x}) = 0, \text{ for } n \geqslant 2.$$

Recalling Remark 2.27 and Lemma 2.23, by the above equation we obtain that for $n \geqslant 2$, f_1^n must be in the kernel of linear operator $\tilde{T}_n$, i. e. $f_1^n \in \langle x^n \rangle$. Thus, we conclude that $f_1(\mathbf{x}) = xP_1(x)$ with $P_1(0) = 0$.

Now, we turn our attention to the third equation in (2.16). It is equivalent to the following equation:

$$x\frac{\partial f_3^n}{\partial y}(\mathbf{x}) - \lambda z\frac{\partial f_3^n}{\partial z}(\mathbf{x}) + \lambda f_3^n(\mathbf{x}) = 0, \text{ for } n \geqslant 2,$$

which, by Lemma 2.25 and Remark 2.27, is equivallent to

$$f_3^n \in \ker \tilde{L}_n = \langle x^{n-1} z \rangle,$$

for $n \geqslant 2$. Hence, $f_3(\mathbf{x}) = zR_1(x)$ where $R_1(0) = 0$. It remains to find a more precise expression for function $f_2(\mathbf{x})$. The second equation in (2.16) implies

$$x\frac{\partial f_2^n}{\partial y}(\mathbf{x}) - \lambda z\frac{\partial f_2^n}{\partial z}(\mathbf{x}) = f_1^n(\mathbf{x}), \text{ for } n \geqslant 2.$$

To continue this investigation, it is convenient to write the functions $f_1(\mathbf{x}), f_2(\mathbf{x})$ as power series expansions: $f_i(\mathbf{x}) = \sum_{j+k+l\geqslant 2} F^i_{j,k,l} x^j y^k z^l$ for $i = 1, 2$. To simplify the arguments, we adopt the convention $F^i_{j,k,l} = 0$ if any of the indexes j, k, l is negative. Thus, the above equation is equivalent to

$$\sum_{j+k+l\geqslant 2} F^1_{j,k,l} x^j y^k z^l - \sum_{j+k+l\geqslant 2} kF^2_{j,k,l} x^{j+1} y^{k-1} z^l + \sum_{j+k+l\geqslant 2} \lambda l F^2_{j,k,l} x^j y^k z^l = 0,$$

which yields:

$$F^1_{j,k,l} - (k+1)F^2_{j-1,k+1,l} + \lambda l F^2_{j,k,l} = 0. \tag{2.17}$$

We know that $F^1_{j,k,l} = 0$ for $k^2 + l^2 \neq 0$. For $l \neq 0$, equation (2.17) implies that all coeficients $F^2_{j,k,l}$ are null. For $l = 0$, (2.17) becomes

$$F^1_{j,k,0} - (k+1)F^2_{j-1,k+1,0} = 0.$$

Thus, the only coefficients of $f_2(\mathbf{x})$ not necessarily zero are $F^2_{j,1,0}$ and $F^2_{j,0,0}$. In fact, by the above equation, the coefficients $F^2_{j,1,0}$ must satisfy $F^2_{j,1,0} = F^1_{j+1,0,0}$. Moreover, the coefficients $F^2_{j,0,0}$ have no restriction presented by equation (2.17). Therefore, $f_2(\mathbf{x}) = Q_2(x) + yP_1(x)$ where $j^1Q_2(0) = 0$. Hence, we have proven the following result.

Theorem 2.29 (Nilpotent Normal Form in $\mathbb{R}^3$)**.** *For system* (2.2) *having a nilpotent singular point at the origin, there exist a formal change of variables that transforms it into the formal normal form*

$$\begin{aligned}\dot{x} &= y + xP_1(x),\\ \dot{y} &= Q_2(x) + yP_1(x),\\ \dot{z} &= -\lambda z + zR_1(x).\end{aligned} \tag{2.18}$$

for which $P_1(0) = j^1Q_2(0) = R_1(0) = 0$.

However, we were not able to prove that the normal form (2.18) is analytic, i.e. that the series P_1, Q_2 and R_1 are convergent. That does not mean that this normal form is not useful. We remark that the above normal form has $z = 0$ as an invariant surface which is a center manifold and the first two components are decoupled from the third.

Remark 2.30. The Belitskii normal form for planar systems (2.1) having a nilpotent singular point is

$$\begin{aligned}\dot{x} &= y + xP_1(x),\\ \dot{y} &= Q_2(x) + yP_1(x),\end{aligned} \tag{2.19}$$

where $j^1Q_2(0) = P_1(0) = 0$ (see Theorem 1.8.6, page 37 in [36]), which is very similar to its three-dimensional counterpart. From this form it is possible to arrive at the Liénard Canonical Form

$$\begin{aligned}\dot{x} &= -y,\\ \dot{y} &= \tilde{Q}_2(x) + y\tilde{P}_1(x),\end{aligned}$$

where $j^1\tilde{Q}_2(0) = \tilde{P}_1(0) = 0$. In [20] it is proven that this form is actually analytic. In fact, in [20], the authors also prove that it is possible to do $\tilde{Q}_2(x) = x^{2n-1}$ and in this case, the origin is a center if and only it $\tilde{P}_1(x)$ is an odd function (see [37]).

Theorem 2.29 is our main result of this subsection and it is published in [12] and also appears with a proof in [9].

2.3.2 Normal Form up to order k

We follow closely the ideas presented in [38] to obtain a realizable normal form up to order k for system (2.2), that is, a normal form for which the terms of order less or equal to k are the simplest possible, for some $k \in \mathbb{N}$. The following results hold for any system in $\mathbb{R}^n$ of the form

$$\dot{\mathbf{x}} = A\mathbf{x} + f(\mathbf{x}),$$

where $\mathbf{x} \in \mathbb{R}^n$, $A \in M_n(\mathbb{R})$ and $j^1f(0) = 0$.

Definition 2.31. Let $A \in M_n(\mathbb{C})$ and $\sigma(A) = \{\lambda_1, \ldots, \lambda_n\}$ be its spectrum. The following equations are called *resonance conditions for* A:

$$\sum_{i=1}^{n} \lambda_i \alpha_i = \lambda_j, \ j = 1, \ldots, n,$$

where $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$. A monomial $a\mathbf{x}^\alpha \cdot e_j = a x_1^{\alpha_1} \ldots x_n^{\alpha_n} \cdot e_j$ is a *resonant monomial* when α satisfies the $j-$th resonance condition.

Theorem 2.32. *Consider system* $\dot{\mathbf{x}} = A\mathbf{x} + f(\mathbf{x})$ *and the decomposition* $A = D + N$ *of* A*, where* D *is a diagonal matrix and* N *is a nilpotent matrix such that* $DN = ND$*. Then for any* $k \in \mathbb{N}$*, there exists a finite sequence of change of variables* $\mathbf{x} = \mathbf{y} + \varphi(\mathbf{y})$ *that transforms the system into* $\dot{\mathbf{y}} = A\mathbf{y} + g(\mathbf{y})$*, where* $j^1 g(0) = 0$ *and* $j^k g(0)$ *consists of all resonant monomials up to order* k*.*

This theorem is Corollary 1.20 proven in Section 2.1 of [38]. For system (2.2), we have $DX_0 = D + N$ where

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\lambda \end{pmatrix}, \ N = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and $DN = ND = O$ where O denotes the null matrix. The spectrum of $A = DX_0$ is $\{0, 0, -\lambda\}$ and the resonance conditions are given by:

$$\begin{aligned} j = 1: &\quad -\lambda\alpha_3 = 0 \Rightarrow \alpha_3 = 0 \\ j = 2: &\quad -\lambda\alpha_3 = 0 \Rightarrow \alpha_3 = 0 \\ j = 3: &\quad -\lambda\alpha_3 + \lambda = 0 \Rightarrow \alpha_3 = 1. \end{aligned}$$

Thus, by the previous result it is possible to write system (2.2) in the form:

$$\begin{aligned} \dot{x} &= y + P(x, y) + r_1(x, y, z), \\ \dot{y} &= Q(x, y) + r_2(x, y, z), \\ \dot{z} &= -\lambda z + zR(x, y) + r_3(x, y, z), \end{aligned} \tag{2.20}$$

where $j^1 P(0) = j^1 Q(0) = R(0) = 0$ and $j^k r_i(0) = 0$ for $i = 1, 2, 3$.

The theory developed in [38] is presented for systems in $\mathbb{C}^n$. However, for vector fields X having nilpotent singular points, since the spectrum DX_0 is contained in the real line, all normal forms have real coefficients.

Theorem 2.33 (Normal Form up to order k)**.** *For* $k \in \mathbb{N}$*, there exists a finite sequence of near-identity change of variables that transforms system* (2.2) *into normal form* (2.20)*.*

2.3.3 Takens-like Normal Form

In the intent of proving that there is an analytical normal form for system (2.2), we follow the ideas presented in [20]. Using classical normal form theory, we exhibit yet another normal form for system (2.2). Unfortunately, we were not able to prove its analyticity.

Essentially, we apply an infinite succession of changes of variables of the form $\phi(\mathbf{x}) = \mathbf{x} + \phi^{(k)}(\mathbf{x})$ where $\mathbf{x} \in \mathbb{R}$ and $\phi^{(k)} \in H^{(k)}_{(3,3)}$, i.e. $\phi^{(k)}$ is a polynomial map with all its components being homogeneous of degree k.

Let X denote the vector field associated to system (2.2). The transformed vector field Y satisfies the following equation:

$$Y \circ \phi = d\phi \cdot X.$$

Denote by $X^{(k)}$ and $Y^{(k)}$ the homogeneous terms of degree k of X and Y and $\phi^{(k)} = (\phi_1, \phi_2, \phi_3)$. The homogeneous terms of degree k in the above equation are given by the following relationship:

$$\begin{pmatrix} \phi_2(x,y,z) \\ 0 \\ -\lambda\phi_3(x,y,z) \end{pmatrix} + Y^{(k)} = X^{(k)} + \begin{pmatrix} y\dfrac{\partial\phi_1}{\partial x} - \lambda z\dfrac{\partial\phi_1}{\partial z} \\ y\dfrac{\partial\phi_2}{\partial x} - \lambda z\dfrac{\partial\phi_2}{\partial z} \\ y\dfrac{\partial\phi_3}{\partial x} - \lambda z\dfrac{\partial\phi_3}{\partial z} \end{pmatrix}.$$

Which implies that

$$Y^{(k)} = X^{(k)} + \begin{pmatrix} -\phi_2 + T_k(\phi_1) \\ T_k(\phi_2) \\ L_k(\phi_3) \end{pmatrix}.$$

By Lemmas 2.24 and 2.26, we conclude that it is possible to choose $\phi^{(k)}$ such that

$$Y^{(k)} = \begin{pmatrix} a_k x^k \\ b_k x^k \\ c_k z x^{(k-1)} \end{pmatrix}.$$

Thus, the infinite succession of compositions of the changes of variables $\phi(\mathbf{x})$ yields a formal change of variables that transforms system (2.2) into the following formal normal form:

$$\begin{aligned} \dot{x} &= y + P_2(x), \\ \dot{y} &= Q_2(x), \\ \dot{z} &= -\lambda z + z R_1(x), \end{aligned}$$

where $j^1 P_2(0) = j^1 Q_2(0) = R_1(0) = 0$.

Remark 2.34. The first two components of the above normal form are also decoupled from the third as is normal form (2.18). We remark that the first two components are the Takens pre-normal form for the nilpotent singular point in the plane [20].

We note that the Takens-like normal form and the normal form up to order k presented here are not used to obtain the results of the thesis. However, we believe that it may be useful in future research.

2.4 Nilpotent singular points and Formal series

2.4.1 First Integrals

It is well-known that the center problem for systems having a Hopf singular point is equivalent to the existence of a first integral at the point [39, 4, 40]. In the nilpotent case, this equivalence does not hold. However, the study of the integrability problem has interesting applications to the Nilpotent Center Problem. This is the subject of this section.

Definition 2.35. Let X be a vector field defined on an open set $U \subset \mathbb{R}^k$ and $H : U \subset \mathbb{R}^k \to \mathbb{R}$ be a C^1-function not locally constant. If H satisfies

$$XH \equiv 0,$$

then H is a *first integral* for X and in this case X is *integrable.*

Remark 2.36. In the literature, one can find a different notion of integrability of a vector field, which is associated with the existence of $k-1$ independent first integrals. To us, this is the notion of *full integrability.* In the remainder of our text, we shall use the definition of integrability as in Definition 2.35 above.

In the remainder of this section, we use the following notation: given a function $F : \mathbb{R}^3 \to \mathbb{R}$, we denote by F_n the homogeneous part of degree n in its power series expansion.

Theorem 2.37. *Consider a vector field X associated to system* (2.2) *having a nilpotent singular point. Then there exists a formal series $H(x, y, z) = y^2 + \sum_{n\geqslant 3} H_n(x, y, z)$ such that $XH = \sum_{n\geqslant 4} \omega_n x^n$.*

Proof: The Lie derivative XH is given by:

$$\begin{aligned} XH &= (y + P_2 + P_3 + \dots)\left(\frac{\partial H_3}{\partial x} + \frac{\partial H_4}{\partial x} + \dots\right) \\ &\quad + (-\lambda z + R_2 + R_3 + \dots)\left(\frac{\partial H_3}{\partial z} + \frac{\partial H_4}{\partial z} + \dots\right) + \\ &\quad + (Q_2 + Q_3 + \dots)\left(2y + \frac{\partial H_3}{\partial y} + \frac{\partial H_4}{\partial y} + \dots\right). \end{aligned}$$

Note that $y\dfrac{\partial H_n}{\partial x} - \lambda z\dfrac{\partial H_n}{\partial z}$ is a homogeneous polynomial of degree n. Hence, rewriting the above expression by organizing the homogeneous terms of degree n in brackets, we have that

$$\begin{aligned} XH &= \left(y\frac{\partial H_3}{\partial x} - \lambda z\frac{\partial H_3}{\partial z} + 2yQ_2\right) + \left(y\frac{\partial H_4}{\partial x} - \lambda z\frac{\partial H_4}{\partial z} + F_4\right) + \dots = \\ &= \sum_{n\geqslant 3}\left(y\frac{\partial H_n}{\partial x} - \lambda z\frac{\partial H_n}{\partial z} + F_n\right) = \sum_{n\geqslant 3} T_n(H_n) + F_n, \end{aligned}$$

where $F_n \in H^{(n)}_{(3,1)}$ are obtained by the homogeneous parts of P, Q, R and H of degree less than n. More precisely,

$$F_n = 2yQ_{n-1} + \sum_{i+j\geqslant n+1}\left(P_i\frac{\partial H_j}{\partial x} + Q_i\frac{\partial H_j}{\partial y} + R_i\frac{\partial H_j}{\partial z}\right).$$

By Lemma 2.24, we can choose H_n such that $T_n(H_n) + F_n = \omega_n x^n$, $\omega_n \in \mathbb{R}$. Moreover, for $n = 3$, $F_3 = 2yQ_2 \notin \langle x^3\rangle$ which means that there is a choice of H_3 such that $T_3(H_3) = -2yQ_2$. Then, by making the suitable choices, $XH = \sum_{n\geqslant 4} \omega_n x^n$. □

Considering the normal form (2.18), the following result gives a characterization of the shape of a formal first integral.

Lemma 2.38. *If H is a formal first integral for the normal form* (2.18)*, then $\dfrac{\partial H}{\partial z} \equiv 0$, that is $H = H(x, y)$.*

Proof: Let $H(x,y,z) = \sum_{j+k+l\geqslant 1} p_{jkl}x^j y^k z^l$ be a formal first integral for (2.18). We assume $p_{jkl} = 0$ if any of the indexes j,k,l is negative. The following equation holds:

$$\frac{\partial H}{\partial x}(y + xP_1(x)) + \frac{\partial H}{\partial y}\left(Q_2(x) + yP_1(x)\right) + \frac{\partial H}{\partial z}(-\lambda z + zR_1(x)) = 0.$$

The coefficients of the monomials z^l in the left-hand side of the above equation are given by $-\lambda l p_{0,0,l}$. Hence, $p_{0,0,l} = 0$ for $l \in \mathbb{N}$.

Now, we study the coefficients of monomials $x^j y^k z^l$ with $l \neq 0$ in the left-hand side of the above equation. We conclude that these coefficients satisfy the following relation:

$$\begin{aligned}(j+1)p_{j+1,k-1,l} \quad &+ \quad \sum_{m=2}^{j}(j-m+1)p_{j-m+1,k,l}a_{m-1} + \sum_{m=2}^{j}(k+1)p_{j-m,k+1,l}b_m \\ &+ \quad \sum_{m=1}^{j} kp_{j-m,k,l}a_m - \lambda l p_{j,k,l} + \sum_{m=1}^{j+k-1} lp_{j-m,k,l}c_m = 0, \end{aligned} \tag{2.21}$$

where a_m, b_m, c_m are the coefficients of x^m in the expression of P_1, Q_2, R_1 respectively. Note that the summations above are expressed in terms of the coefficients $p_{j_1,j_2,l}$ with $j_1 + j_2 < j + k$. We now prove that $p_{jkl} = 0$ if $l \neq 0$ by induction over $j + k$:

Suppose $j + k = 1$. Since $p_{j_1,j_2,l} = 0$ for $j_1 + j_2 < 1$, equation (2.21) becomes

$$(j+1)p_{j+1,k-1,l} - \lambda l p_{j,k,l} = 0. \tag{2.22}$$

We conclude that $p_{0,1,l} = p_{1,0,l} = 0$. Now suppose that for $j + k < n$, $p_{j,k,l} = 0$. Then, for $j + k = n$, equation (2.21) becomes (2.22). For $(j,k,l) = (n,0,l)$, we have by (2.22) that $p_{n,0,l} = 0$. Hence, equation (2.22) sets the following chain of implications:

$$p_{n,0,l} = 0 \Rightarrow p_{n-1,1,l} = 0 \Rightarrow p_{n-2,2,l} = 0 \Rightarrow \cdots \Rightarrow p_{0,n,l} = 0.$$

Therefore $p_{j,k,l} = 0$ for $l \neq 0$ and the result follows. □

The previous lemma lets us conclude that formal integrability of the normal form (2.18) is essentially formal integrability of the planar normal form (2.19). Hence, we proceed to study the formal integrability of the formal system (2.19).

Remark 2.39. One of the consequences of Lemma 2.38 is that if system (2.2) is formally integrable, then it has a formal first integral H such that $j^2H(0) = y^2$. In fact, since (2.2) is transformed into (2.18) via near-identity changes of variables and the formal integrability of system (2.2) is equivalent to the formal integrability of system (2.18), it is enough to verify this statement for system (2.19). But this follows from the proof of Theorem 1 in paper [41].

From now on, we will establish a link between the quantities ω_n and the formal integrability of system (2.2). For a vector field X associated to system (2.2), consider a formal series $H(x,y,z) = y^2 + \sum_{n\geqslant 3} H_n(x,y,z)$, which we will also write as

$$H(x,y,z) = y^2 + \sum_{j+k+l\geqslant 3} p_{jkl}x^j y^k z^l,$$

such that $XH = \sum_{n\geqslant 4} \omega_n x^n$. Following the steps of the proof of Theorem 2.37, we have that $XH = \sum_{n\geqslant 3} T_n(H_n) + F_n$. For each n, by Lemma 2.24, there exists a homogeneous polynomial H_n such that $T_n(H_n) + F_n = \omega_n x^n$. Since $\mathrm{Im}T_n$ is isomorphic to $H^{(n)}_{(3,1)}/\ker T_n$, this polynomial is unique up to a monomial $p_{0,n,0}\, y^n$. Thus, each quantity ω_n depends on the coefficients chosen to obtain the previous ω_k, i.e., the coefficients $p_{0,k,0}$ with $k < n$ of the formal series H. These coefficients can be seen as free parameters, which we represent as $C_n^H = (p_{0,3,0}, \ldots, p_{0,n-1,0})$, for $n \geqslant 4$, and then define the functions

$$\begin{aligned} \omega_n \colon \mathbb{R}^{n-3} &\to \quad \mathbb{R} \\ C_n^H \ &\mapsto \omega_n(C_n^H). \end{aligned}$$

Thus, when searching for a formal series such that $XH = \sum_{n \geqslant 4} \omega_n x^n$, different choices of C_n^H may yield different values of ω_n. However, they are a useful tool to study the formal integrability of system (2.2) as the next result holds.

Theorem 2.40. *Let X be the vector field associated to a system* (2.2) *having a nilpotent singular point and H be a formal series as in Theorem* (2.37)*. If there exists $m \geqslant 4$ such that ω_m is a non-vanishing function, then system* (2.2) *is not formally integrable.*

Proof: Suppose that X admits a formal first integral. Therefore, by Remark 2.39, it admits a formal first integral $\tilde{H}(x, y, z)$ with $j^2\tilde{H}(0) = y^2$. Thus, we have that $X\tilde{H} \equiv 0$, and so, $\omega_n(C_n^{\tilde{H}}) = 0$ for all $n \geqslant 4$. □

The quantities ω_n provide a helpful tool to determine necessary conditions for a three-dimensional system with a nilpotent singular point to be analytically integrable. In fact, for system (2.2), if all the ω_n vanish, then the formal series $H(x, y, z)$ in Theorem 2.37 is a formal first integral with $j^2H(0) = y^2$. Under these conditions, it is possible (see the forthcoming Theorem 2.46) to obtain an analytic first integral on a neighborhood of the origin. Thus, the restriction of system (2.2) to a center manifold also admits a first integral. Additionally, if the nilpotent singular point is monodromic, then it must be a nilpotent center. Therefore, one can use the quantities ω_n to detect nilpotent centers on center manifolds.

However, not all nilpotent centers are formally integrable. For instance, consider the following system:

$$\begin{aligned} \dot{x} &= y + x^2, \\ \dot{y} &= -x^3, \\ \dot{z} &= -\lambda z. \end{aligned} \tag{2.23}$$

which has $z = 0$ as a center manifold and its restriction is a time-reversible system which implies that the origin is a center on the center manifold. If we attempt to construct a formal first integral $H(x, y, z)$ for system we obtatin $\omega_5 \equiv 2$ and by Theorem 2.40, the system does not admit a formal first integral.

By Proposition 2.8, the example (2.23) has a monodromic singular point with Andreev number $n = 2$ and satisfies monodromy condition $\beta = n - 1$ in Theorem 2.6. The lack of formal integrability for this system is not an exception. More precisely, the monodromy condition $\beta = n - 1$ is an obstruction for formal integrability as we will see in the next results.

Lemma 2.41. *Consider system* (2.19) *having a monodromic singular point satisfying monodromy condition $\beta = n - 1$ in Theorem 2.6. Then it is not formally integrable.*

Proof: For system (2.19), the functions $F(x), f(x)$ and $\Phi(x)$ in Theorem 2.6 are given by

$$F(x) = -xP_1(x),\ f(x) = Q_2(x) - xP_1^2(x),\ \Phi(x) = 2P_1(x) + x\frac{\partial P_1(x)}{\partial x}.$$

Writing $P_1(x) = \sum_{l\geqslant 1} a_l x^l$ and $Q_2(x) = \sum_{l\geqslant 2} b_l x^l$, we have:

$$f(x) = b_2x^2 + \sum_{l\geqslant 3}(b_l - A_{l-1})x^l, \quad \text{and} \quad \Phi(x) = \sum_{l\geqslant 1}(2+l)a_l x^l,$$

where $A_l = \sum_{i=1}^{l-1} a_i a_{l-i}$. Since monodromy condition $\beta = n-1$ in Theorem 2.6 is satisfied, there exist a positive integer $n \geqslant 2$ such that $(b_{2n-1} - A_{2n-2}) < 0$ is the first non-zero coefficient in the power series expansion of $f(x)$ and a_{n-1} is the first non-zero coefficient in the power series expansion of $P_1(x)$. Note that $A_k = 0$ for $k < 2(n-1)$, $b_k = 0$ for $k < 2n-1$ and $A_{2n-2} = a_{n-1}^2$ which implies $b_{2n-1} - a_{n-1}^2 < 0$.

Suppose there exists $H(x,y)$ a formal first integral for system (2.19). Thus, we can write $H(x,y) = y^2 + \sum_{j+k\geqslant 3} p_{jk}x^j y^k$. Then, the following equation holds:

$$y\frac{\partial H}{\partial x} + Q_2(x)\frac{\partial H}{\partial y} + P_1(x)\left(x\frac{\partial H}{\partial x} + y\frac{\partial H}{\partial y}\right) \equiv 0. \tag{2.24}$$

Studying the above equation by expanding its left-hand side in a power series, we conclude that the coefficients of the monomials $x^j y^k$ are given by the left-hand side of the following equation:

$$(j+1)p_{j+1,k-1} + \sum_{l=0}^{j-2}\left((k+1)b_{j-l}p_{l,k+1} + a_{j-l}(l+k)p_{l,k}\right) = 0. \tag{2.25}$$

The above equation implies that for $0 < j \leqslant n-1$, $p_{j,k} = 0$. Now, substituting $j = n-1$ in (2.25) yields $np_{n,k-1} + ka_{n-1}p_{0,k} = 0$, and so

$$p_{n,k-1} = -\frac{ka_{n-1}p_{0,k}}{n}.$$

Substituting $k = 1$ in (2.25) yields:

$$(j+1)p_{j+1,0} + \sum_{l=0}^{j-2}\left(2b_{j-l}p_{l,2} + a_{j-l}(l+1)p_{l,1}\right) = 0, \tag{2.26}$$

and for $j \leqslant n$, it follows that $p_{j,0} = 0$. Studying the above equation for $n \leqslant j \leqslant 2n-1$, we conclude that $p_{j,0} = 0$, for $j < 2n$, and $2np_{2n,0} + 2b_{2n-1} + a_{n-1}(n+1)p_{n,1} = 0$, i.e.

$$p_{2n,0} = -\frac{(2b_{2n-1} + a_{n-1}(n+1)p_{n,1})}{2n}.$$

Now, we can explicitly compute the coefficient of x^{3n-1} in the left-hand side of equation (2.24) denoted by ω_{3n-1}:

$$\omega_{3n-1} = \sum_{j=2}^{3n-3}(b_{3n-1-j}p_{j,1} + ja_{3n-1-j}p_{j,0}) = \sum_{j=n}^{2n}(b_{3n-1-j}p_{j,1} + ja_{3n-1-j}p_{j,0}) =$$

$$= b_{2n-1}p_{n,1} + 2na_{n-1}p_{2n,0} = -\frac{2a_{n-1}(n+1)(a_{n-1}^2 - b_{2n-1})}{n}.$$

Formal integrability implies that $\omega_{3n-1} = 0$, but this is not possible since $a_{n-1} \neq 0$ and $b_{2n-1} - a_{n-1}^2 < 0$. Therefore, the result holds. □

Remark 2.42. Using the notation from Theorem 2.6, the quantity ω_{3n-1} given in the previous proof is given by:

$$\omega_{3n-1} = -\frac{2ab}{n}.$$

Proposition 2.43. *Consider system* (2.1) *having a monodromic singular point satisfying monodromy condition $\beta = n-1$ in Theorem 2.6. Then it cannot admit a formal first integral.*

Proof: By Proposition 2.21, the monodromy condition $\beta = n-1$ is invariant under formal near-identity changes of variables. Moreover, the formal integrability of system (2.1) would imply the formal integrability of its normal form (2.19). Hence, by Lemma 2.41, system (2.1) cannot be formally integrable. □

Theorem 2.44. *Consider system* (2.2) *having a monodromic singular point such that its restriction to a center manifold satisfies monodromy condition $\beta = n-1$ in Theorem 2.6. Then it cannot admit a formal first integral.*

Proof: The result follows from the fact that the formal integrability of system (2.2) implies the formal integrability of the associated normal form (2.18) and, by Lemma 2.38, the formal integrability of the bidimensional normal form (2.19). By Lemma 2.41, system (2.2) cannot be formally integrable. □

The monodromy condition $\beta > n-1$ also interferes with the formal integrability of the normal forms (2.18) and (2.19). More precisely, the next results classify the normal form for the formal integrable nilpotent singular points in $\mathbb{R}^3$.

Theorem 2.45. *Consider system* (2.18) *having a monodromic singular point. If it admits formal first integral $H(x,y)$, then either $P_1(x) \equiv 0$ or $m = 2sn-1$ for some $s \in \mathbb{N}$.*

Proof: For system (2.19), the functions $F(x), f(x)$ and $\Phi(x)$ in Theorem 2.6 are given by

$$F(x) = -xP_1(x),\ f(x) = Q_2(x) - xP_1^2(x),\ \Phi(x) = 2P_1(x) + x\frac{\partial P_1(x)}{\partial x}.$$

Writing $P_1(x) = \sum_{l\geqslant 1} a_l x^l$ and $Q_2(x) = \sum_{l\geqslant 2} b_l x^l$, we have:

$$f(x) = b_2x^2 + \sum_{l\geqslant 3}(b_l - A_{l-1})x^l, \ \text{ and } \ \Phi(x) = \sum_{l\geqslant 1}(2+l)a_l x^l,$$

where $A_l = \sum_{i=1}^{l-1} a_i a_{l-i}$. The monodromy conditions for system (2.19) imply that $b_2 = 0$ and the first nonzero coefficient in the power series expansion of $f(x)$ is $(b_{2n-1} - A_{2n-2}) < 0$. Furthermore, either $P_1(x) \equiv 0$, $m > n-1$, or $m = n-1$ where m is the index of the first nonzero coefficient a_m of $P_1(x)$. By Lemma 2.41, formal integrability can only occur in the cases $m > n-1$ or $P_1(x) \equiv 0$.

For $P_1(x) \equiv 0$, $H(x,y) = y^2 - \sum_{l\geqslant 2}\frac{2b_l}{l+1}x^{l+1}$ is a formal first integral for system (2.19). Now, we suppose that $m > n-1$. Then $A_j = 0$ for $j < 2m$, $b_j = 0$ for $j < 2n-1$ and $b_{2n-1} < 0$. If there exists $H(x,y)$ a formal first integral for system (2.19), we can write $H(x,y) = y^2 + \sum_{j+k\geqslant 3} p_{jk}x^j y^k$ and the equation (2.24) holds. Moreover the coefficients $p_{j,k}$ satisfy equations (2.25) and (2.26).

Suppose $n \leqslant m < 2n-1$. Studying equations (2.25) and (2.26), we conclude that $p_{j,k} = 0$ for $0 < j \leqslant m$ and $p_{j,0} = 0$ for $j \leqslant m+1$. Substituting $j = m$ in (2.25) yields $p_{m+1,k-1} = -\dfrac{ka_m p_{0,k}}{m+1}$. By (2.26) for $m+1 \leqslant j \leqslant 2n-1$, we have that:

$$p_{j,0,} = 0,\ \text{for } j < 2n,\ p_{m+2,0} = -\frac{2b_{m+1}}{m+2},\ \text{and } p_{2n,0} = -\frac{2b_{2n-1}}{2n}.$$

Note that $p_{m+2,0} \neq 0$ only if $m = 2n-2$. With this information, we compute the coefficient ω_{2n+m} of x^{2n+m} in XH, which is given by:

$$\omega_{2n+m} = \sum_{j=2}^{2n+m-2} (b_{2n+m-j}p_{j,1} + ja_{2n+m-j}p_{j,0}) = \sum_{j=m+1}^{2n} (b_{2n+m-j}p_{j,1} + ja_{2n+m-j}p_{j,0}),$$

which simplifies to:

$$\omega_{2n+m} = 2na_m p_{2n,0} + b_{2n-1}p_{m+1,1} = -\frac{2a_m b_{2n-1}(m+2)}{m+1}.$$

Since $a_m \neq 0$ and $b_{2n-1} < 0$, it follows that $\omega_{2m+n} \neq 0$ and therefore, system (2.19) cannot be formally integrable.

Now, suppose $m > 2n-1$ and $m \neq 2sn-1$ for $s \in \mathbb{N}$. Thus, there exists $r \in \mathbb{N}$ such that $2rn - 1 < m < 2(r+1)n - 1$. By (2.25) we have that $p_{j,k} = 0$ for $0 < j \leqslant 2n-1$. For $2n-1 \leqslant j < m$, equation (2.25) becomes:

$$(j+1)p_{j+1,k-1} + \sum_{l=0}^{j-2n+1} (k+1)b_{j-l}p_{l,k+1} = 0. \tag{2.27}$$

Moreover, the quantities ω_{2sn-1} for $s \leqslant r+1$ are given by:

$$\omega_{2sn-1} = \sum_{j=2n}^{2(s-1)n} (b_{2sn-1-j}p_{j,1}). \tag{2.28}$$

Substituting $k = 2$ in (2.27), yields:

$$(j+1)p_{j+1,1} + \sum_{l=0}^{j-2n+1} (3b_{j-l}p_{l,3}) = 0. \tag{2.29}$$

For $j = 2n-1$ we have $p_{2n,1} = -\dfrac{3b_{2n-1}p_{0,3}}{2n}$ and substituting this expression in (2.28) for $s = 2$, we conclude that $\omega_{4n-1} = -\dfrac{3b_{2n-1}^2 p_{0,3}}{2n}$. Formal integrability implies $p_{0,3} = 0$. As a consequence, $p_{j,1} = 0$ for $j \leqslant 2n$.

We turn again to equation (2.26) with $j \leqslant 2n-1, p_{j,0} = 0$ and $p_{2n,0} = -\dfrac{b_{2n-1}}{n}$. The quantity ω_{2n+m} is given by:

$$\omega_{2n+m} = \sum_{j=2}^{2n+m-2} (b_{2n+m-j}p_{j,1} + ja_{2n+m-j}p_{j,0}) = -2a_m b_{2n-1} + \sum_{j=2n+1}^{m+1} (b_{2n+m-j}p_{j,1}).$$

Our goal is to prove that ω_{2n+m} cannot be zero. Recalling that $p_{j,1} = 0$ for $j \leqslant 2n$ and $p_{j,k} = 0$ for $0 < j < 2n$. By (2.29), for $j \leqslant 4n-2$, we have:

$$(j+1)p_{j+1,1} + \sum_{l=0}^{j-2n+1} 3b_{j-l}p_{l,3} = (j+1)p_{j+1,1} + 3b_j p_{0,3} = 0,$$

and thus $p_{j,1} = 0$ for $j \leqslant 4n-1$. Moreover, substituting $j = 4n-1$ in (2.29), yields

$$4np_{4n,1} + \sum_{l=0}^{2n} 3b_{4n-1-l}p_{l,3} = 4np_{4n,1} + 3b_{2n-1}p_{2n,3} = 0,$$

and from (2.27),

$$2np_{2n,3} + 5b_{2n-1}p_{0,5} = 0.$$

Hence, we obtain $\omega_{6n-1} = b_{2n-1}p_{4n,1}$ and the formal integrability implies that $p_{4n,1} = 0$ and $p_{0,5} = 0$.

Suppose, by induction, that for $j \leqslant 2sn$, $p_{j,1} = 0$ and $p_{0,3} = p_{0,5} = \cdots = p_{0,2s+1} = 0$, for $s < r$. Using equations (2.29) and (2.27) repeatedly, we obtain for $j \leqslant 2(s+1)n - 2$,

$$(j+1)p_{j+1,1} + \sum_{l=0}^{j-2n+1} 3b_{j-l}p_{l,3} = (j+1)p_{j+1,1} + \sum_{l=0}^{j-2n+1} 3b_{j-l}\left(-\frac{1}{l}\sum_{t=0}^{l-2n} 5b_{l-1-t}p_{t,5}\right).$$

The expressions of the coefficients $p_{j,k}$, by (2.27), are given in terms of nested summations whose indexes vary in intervals of natural numbers that are each instance $2n$ units less than the previous one until eventually fit in the interval $0 \leqslant j \leqslant 2n-1$. Then, all coefficients, $p_{j,k}$ are null for $j > 0$. Thus, the coefficient $p_{j+1,1}$ is a linear combination of the coefficients $p_{0,3}, p_{0,5}, \cdots, p_{0,2s+1}$. By the induction hypothesis, we conclude that $p_{j,1} = 0$ for $j \leqslant 2(s+1)n - 1$.

Computing $p_{2(s+1)n,1}$ and $\omega_{2(s+1)n-1}$, we conclude that the system (2.18) can only be formally integrable if $p_{0,2(s+1)+1} = 0$ and consequentially $p_{2(s+1)n,1} = 0$.

Thus, using (2.29), we compute $p_{m+1,1}$:

$$(m+1)p_{m+1,1} + 2a_m + \sum_{l=1}^{m-2}(3b_{m-l}p_{l,3}) = 0, \text{ which implies that } p_{m+1,1} = -\frac{2a_m}{m+1}.$$

Then ω_{2n+m} becomes:

$$\omega_{2n+m} = -2a_m b_{2n-1} - \frac{2a_m b_{2n-1}}{m+1},$$

which is not zero. Hence, system (2.18) with monodromic singular point can only be formally integrable if $P_1(x) \equiv 0$ or $m = 2sn - 1$. □

Theorem (2.45) gives us a formal normal form for formally integrable systems (2.2). The same result holds for planar systems and it was proven by a different method in [23, Theorem 4]. Although it is clear that the case $P_1(x) \equiv 0$ is always formally integrable, there are few examples of the case $m = 2sn - 1$. One of those is the following system:

$$\begin{aligned}
\dot{x} &= y + \frac{x^4}{5}, \\
\dot{y} &= -x^3 + \frac{x^7}{25} + \frac{x^3 y}{5}, \\
\dot{z} &= -\lambda z.
\end{aligned}$$

Here $n = 2$ and $m = 3 = 2n - 1$. We have that $H(x,y) = \left(y + \frac{x^4}{5} - 1\right)\exp\left(y - \frac{x^4}{20}\right)$ is a first integral for the above system. Note that $j^2H(0) = -1 + \frac{y^2}{2}$.

Even though we have considered formal integrability, the following result, proven by Mattei and Moussu in [42], tells us that formal integrability and analytic integrability are the same.

Theorem 2.46 (Theorem A in [42]). *Theorem Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be an analytic vector field, 0 an isolated singular point of F and H a formal first integral of F such that H is not a power of a formal series. Then, there exists a formal function $f : \mathbb{R} \to \mathbb{R}$, with $f(0) = 0, f'(0) \neq 0$ such that $f \circ H$ is an analytic first integral of F.*

Remark 2.47. The condition that H is not a power of a formal series, means that, any decomposition $H = h_1^{k_1} h_2^{k_2} \dots h_l^{k_l}$ in irreducible factors is such that the greatest common divisor of $k_1, \dots, k_l$ is 1.

Theorems 2.40 and 2.45 are our main results in this subsection and appear in the paper [9].

2.4.2 Inverse Jacobi Multipliers

In [43], the authors explored the relationship between the center problem for Hopf singular points and the existence of an inverse Jacobi multiplier. Inspired by the ideas from that paper, in this section we study the properties of inverse Jacobi multipliers applied to system (2.2), i.e. for the nilpotent case.

Definition 2.48. Let X be a vector field defined on an open set $U \subset \mathbb{R}^k$ and $V : U \subset \mathbb{R}^k \to \mathbb{R}$ be a C^1-function not locally null such that

$$XV - V\text{div}X \equiv 0.$$

We say that V is an *inverse Jacobi multiplier* of X.

Remark 2.49. When $k = 2$ in the above definition, the function V is usually called an *inverse integrating factor* of X.

Proposition 2.50. *Let X be the vector field associated to system* (2.2) *having a nilpotent singular point in $\mathbb{R}^3$. Any non-flat C^∞ inverse Jacobi multiplier $V(x, y, z)$ for vector field X has the form $V(x, y, z) = y^m z + O(|(x, y, z)|^{m+2})$, i.e. there exists a value $m \geqslant 0$ such that $j^{m+1}V(0) = y^m z$, up to multiplication by constant.*

Proof: Assuming that there exists an inverse Jacobi multiplier $V(x, y, z)$ for system (2.2), it must satisfy $XV = V\text{div}X$. Writting $V(x, y, z) = \sum_{k=0}^{\infty} V_k$, where V_k are homogeneous polynomials of degree k in x, y, z, the following equation holds

$$\left\langle X, \sum_{k=0}^{\infty} \nabla V_k \right\rangle - \text{div}X \left(\sum_{k=0}^{\infty} V_k \right) = 0.$$

Comparing the lower order terms in both sides of the above equation yields

$$y\frac{\partial V_m}{\partial x} - \lambda z \frac{\partial V_m}{\partial z} + \lambda V_m = 0,$$

where $m \geqslant 0$ is the smallest index for which $V_m \neq 0$. However the above equation is equivalent to the condition that V_m is in the kernel of linear operator L_m from Section 2.2. By Lemma 2.25, we must have $V_m(x, y, z) = \alpha y^{m-1} z$. □

The next result is proven in [43] and relates inverse Jacobi multipliers and invariant manifolds for generic three-dimensional systems.

Theorem 2.51 (Theorem 6 in [43])**.** *Let X be a smooth vector field defined in an open set $U \subset \mathbb{R}^3$. Assume that V is a C^∞ inverse Jacobi multiplier of the form*

$$V(x, y, z) = (z - h(x, y))\, W(x, y, z).$$

Then $M = \{z = h(x, y)\}$ is an invariant manifold of X and

$$v(x, y) = W(x, y, h(x, y))$$

is an inverse integrating factor for system $X|_M$.

The following theorem is an adaptation of the result presented in [43, Theorem 8] to system (2.2).

Theorem 2.52. *Let $V(x,y,z)$ be an inverse Jacobi multiplier for vector field X associated to system (2.2) having a nilpotent singular point and a local C^∞ center manifold $W^c = \{z = h(x,y)\}$. Consider $V|_{W^c} = V(x,y,h(x,y))$. The following holds:*

i) $V|_{W^c}$ is flat at the origin;

ii) If $W^c \subset V^{-1}(0)$, then there is a C^∞-function $W(x,y,z)$ such that $W(x,y,h(x,y)) \not\equiv 0$ and $V(x,y,z) = (z-h(x,y))W(x,y,z)$. Moreover $W(x,y,h(x,y))$ is an inverse integrating factor for system $X|_{W^c}$.

Proof: First, we prove item (i). Since $W^c = \{z = h(x,y)\}$ is a C^∞-parametrization for a center manifold, the following equation is satisfied:

$$\frac{\partial h}{\partial x}(y+P) + \frac{\partial h}{\partial y}Q = -\lambda h + R.$$

Let $u(x,y) = V(x,y,h(x,y))$. The above equation together with V being an inverse Jacobi multiplier of X yields:

$$\begin{aligned}
\left.\frac{\partial u}{\partial x}(y+P) + \frac{\partial u}{\partial y}Q\right|_{z=h(x,y)} &= \left.\frac{\partial V}{\partial x}(y+P) + \frac{\partial V}{\partial y}Q + \frac{\partial V}{\partial z}\left(\frac{\partial h}{\partial x}(y+P) + \frac{\partial h}{\partial y}Q\right)\right|_{z=h(x,y)} \\
&= \left.\frac{\partial V}{\partial x}(y+P) + \frac{\partial V}{\partial y}Q + \frac{\partial V}{\partial z}(-\lambda h + R)\right|_{z=h(x,y)} \\
&= u(-\lambda + \operatorname{div}(P,Q,R))|_{z=h(x,y)},
\end{aligned}$$

on W^c. Hence, $u(0,0) = 0$ since $\lambda \neq 0$. Suppose $j^\infty u(0) \neq 0$. There exists $u_m(x,y)$ a homogeneous polynomial of degree m such that $j^m u(0) = u_m(x,y)$. By substituting in the above equation and comparing the lowest degree terms in both sides, we obtain $y\frac{\partial u_m}{\partial x} = -\lambda u_m(x,y)$. Since u_m is homogeneous, it satisfies $x\frac{\partial u_m}{\partial x} + y\frac{\partial u_m}{\partial y} = mu_m$. Thus

$$\frac{\partial u_m}{\partial x}(\lambda x + my) + \lambda y \frac{\partial u_m}{\partial y} = 0.$$

Therefore $u_m(x,y)$ is a first integral for linear system $\dot{x} = \lambda x + my,\ \dot{y} = \lambda y$, which is not possible. Therefore $j^\infty u(0) = 0$.

Now we prove (ii). Suppose $W^c = \{z = 0\}$. By hypothesis $V(x,y,0) = 0$, and by Hadamard's lemma (see [44]), $V(x,y,z) = z^k W(x,y,z)$ where $W(x,y,0) \neq 0$, $k > 0$. By Proposition 2.50, $j^{m+1}V(0) = zy^m$ and hence $k = 1$. The result follows.

Now if $W^c = \{z = h(x,y)\}$ with $h(x,y) \neq 0$, it is enough to consider the coordinate change $x \to x,\ y \to y,\ z \to Z = z - h(x,y)$. The new system has $Z = 0$ as a center manifold and admits $\tilde{V}(x,y,Z)$ as an inverse Jacobi multiplier such that $\tilde{V}(x,y,z - h(x,y)) = V(x,y,z)$.

The fact that $V(x,y,h(x,y))$ is an inverse integrating factor for $X|_{W^c}$ follows directly from Theorem 2.51. □

The search of an inverse Jacobi multiplier has its importance for the Nilpotent center problem which will be explained in Section 2.5. The following theorem provides a direction for this search.

Theorem 2.53. *Consider a vector field X associated to system (2.2) having a nilpotent singular point. Then there exists a formal series $V(x,y,z)$ such that $XV - V\mathrm{div}X = \sum_{n\geqslant 1}\Lambda_n x^{n-1}z$.*

Proof: Consider $V(x,y,z)=\sum_{n=1}^{\infty}V_n(x,y,z)$. Then

$$\begin{aligned} XV - V\mathrm{div}X &= (y+P_2+P_3+\dots)\left(\frac{\partial V_1}{\partial x}+\frac{\partial V_2}{\partial x}+\dots\right)\\ &+(-\lambda z+R_2+R_3+\dots)\left(\frac{\partial V_1}{\partial z}+\frac{\partial V_2}{\partial z}+\dots\right)+\\ &+(Q_2+Q_3+\dots)\left(\frac{\partial V_1}{\partial y}+\frac{\partial V_2}{\partial y}+\dots\right)\\ &-(V_1+V_2+\dots)\left(-\lambda+\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}\right).\end{aligned}$$

Rewriting the above expression by organizing the homogeneous terms, we have:

$$XV - V\mathrm{div}X = \sum_{n\geqslant 1}\left(y\frac{\partial V_n}{\partial x}-\lambda z\frac{\partial V_n}{\partial z}+\lambda V_n+F_n\right)=\sum_{n\geqslant 1}L_n(V_n)+F_n,$$

where $F_n\in H^{(n)}_{(3,1)}$ are obtained by the homogeneous parts of P,Q,R and V of degree less than n. By Lemma 2.26, we can choose V_n such that $L_n(V_n)+F_n=\Lambda_n x^{n-1}z$, $\Lambda_n\in\mathbb{R}$. Making the suitable choices, $XV - V\mathrm{div}X=\sum_{n\geqslant 1}\Lambda_n x^{n-1}z$. □

The next result can be found in [45] for the C^1 case and present how the existence of an inverse Jacobi multiplier is invariant by changes of variables.

Theorem 2.54. *Let X be a formal vector field and V is a formal inverse Jacobi multiplier of X. If there is a formal change of variables ϕ that transforms X into a formal vector field Y then $W=(\det d\phi\cdot V)\circ\phi^{-1}$ is a formal inverse Jacobi multiplier of Y.*

We give a proof for this result in the Appendix A.3. An alternative proof can be found in [46, Lemma 2.3].

2.5 Nilpotent Center Problem

In this section, we will extend to the three-dimensional case, some techniques that are used to solve the nilpotent center problem for planar systems. The bulk of our results rely on the hypothesis that the restricted system (2.3) is analytic. For this reason, in what follows, we say that a singular point is an *analytic nilpotent center* for system (2.2) when it is a nilpotent center for restricted system (2.3) and this restriction is analytic.

2.5.1 The Inverse Jacobi Multiplier Method

Consider planar system (2.1) having a nilpotent monodromic singular point at the origin. The nilpotent center problem can be studied by investigating the existence of a formal inverse integrating factor, i.e. the *inverse integrating factor method* which is based in Theorem 2.9. However the existence of a formal inverse integrating factor by itself is not sufficient to prove that the singular point is a nilpotent center. The best result in this direction, also presented in [2], is the following.

Theorem 2.55 (Theorem 4 in [2])**.** *Consider analytic system* (2.1) *with a monodromic nilpotent singular point at the origin having Andreev number n and satisfying the condition $\beta > n-1$ given by the statement (i) in Theorem 2.2. If there is a formal inverse integrating factor $V(x,y) = \sum_{j\geqslant 2n} V_j(x,y)$ where V_j are $(1,n)$-quasi-homogeneous polynomials of weighted degree j (see Definition 2.10) and $V_{2n} \not\equiv 0$, then the origin is a center. The converse is not true.*

We now connect the results presented in Section 2.4.2 to the study of the center problem for the singular point at the origin of system (2.2) in $\mathbb{R}^3$ without restricting it to a center manifold, extending the inverse integrating factor method to the three-dimensional case.

Consider vector field X associated to system (2.2) having a nilpotent singular point at the origin. By Theorem 2.29, there is a formal near-identity change of variables ϕ such that Y is given by:

$$\begin{aligned}\dot{x} &= y + xP_1(x) = Y_1,\\ \dot{y} &= Q_2(x) + yP_1(x) = Y_2,\\ \dot{z} &= -\lambda z + zR_1(x) = Y_3.\end{aligned}$$

Since Y is decoupled, if $\tilde{v}(x,y)$ is an inverse integrating factor for $(Y_1,Y_2)^T$, then $z\tilde{v}(x,y)$ is an inverse Jacobi multiplier for Y. In fact:

$$Y(z\tilde{v}) - z\tilde{v}\mathrm{div}Y = zY_1\frac{\partial \tilde{v}}{\partial x} + zY_2\frac{\partial \tilde{v}}{\partial y} + Y_3\tilde{v} - z\tilde{v}\left(\mathrm{div}(Y_1,Y_2) + \frac{\partial Y_3}{\partial z}\right) = \tilde{v}\left(Y_3 - z\frac{\partial Y_3}{\partial z}\right) = 0.$$

Therefore, by Theorem 2.54 the original vector field X would also admit an inverse Jacobi multiplier. Using this argument, we have the following result:

Theorem 2.56. *Consider system* (2.2) *having a monodromic singular point with Andreev number* 2 *at the origin. If the origin is an analytic nilpotent center on a center manifold, then there exists a formal inverse Jacobi multiplier $V(x,y,z)$ for system* (2.2) *such that $j^{m+1}V(0) = y^m z$ for $m \geqslant 0$.*

Proof: Suppose the origin is an analytic nilpotent center on a center manifold $z = h(x,y)$ of vector field $X = (X_1,X_2,X_3)^T$ associated to system (2.2). Denote by $\bar{X}$ the analytic restricted vector field $X|_{z=h(x,y)}$ to the center manifold. We have that

$$\frac{\partial h}{\partial x}(x,y)X_1(x,y,h(x,y)) + \frac{\partial h}{\partial y}(x,y)X_2(x,y,h(x,y)) = X_3(x,y,h(x,y)).$$

Let $\phi = (\phi_1,\phi_2,\phi_3)$ the formal near-identity change of variables that puts X into its formal normal form (2.18), denoted by $Y = (Y_1,Y_2,Y_3)^T$, i.e. $d\phi \cdot X = Y \circ \phi$. Thus, setting $\bar{\phi}(x,y) = (\phi_1(x,y,h(x,y)), \phi_2(x,y,h(x,y)))$, we have that

$$\begin{aligned}
d\bar{\phi}\cdot\bar{X} &= \left.\begin{pmatrix} \dfrac{\partial\phi_1}{\partial x} + \dfrac{\partial\phi_1}{\partial z}\dfrac{\partial h}{\partial x} & \dfrac{\partial\phi_1}{\partial y} + \dfrac{\partial\phi_1}{\partial z}\dfrac{\partial h}{\partial y} \\ \dfrac{\partial\phi_2}{\partial x} + \dfrac{\partial\phi_2}{\partial z}\dfrac{\partial h}{\partial x} & \dfrac{\partial\phi_2}{\partial y} + \dfrac{\partial\phi_2}{\partial z}\dfrac{\partial h}{\partial y} \end{pmatrix}\right|_{z=h(x,y)} \begin{pmatrix} X_1(x,y,h(x,y)) \\ X_2(x,y,h(x,y)) \end{pmatrix} = \\
&= \left.\begin{pmatrix} \dfrac{\partial\phi_1}{\partial x}X_1 + \dfrac{\partial\phi_1}{\partial y}X_2 + \dfrac{\partial\phi_1}{\partial z}\left(\dfrac{\partial h}{\partial x}X_1 + \dfrac{\partial h}{\partial y}X_2\right) \\ \dfrac{\partial\phi_2}{\partial x}X_1 + \dfrac{\partial\phi_2}{\partial y}X_2 + \dfrac{\partial\phi_2}{\partial z}\left(\dfrac{\partial h}{\partial x}X_1 + \dfrac{\partial h}{\partial y}X_2\right) \end{pmatrix}\right|_{z=h(x,y)} = \\
&= \left.\begin{pmatrix} \dfrac{\partial\phi_1}{\partial x}X_1 + \dfrac{\partial\phi_1}{\partial y}X_2 + \dfrac{\partial\phi_1}{\partial z}X_3 \\ \dfrac{\partial\phi_2}{\partial x}X_1 + \dfrac{\partial\phi_2}{\partial y}X_2 + \dfrac{\partial\phi_2}{\partial z}X_3 \end{pmatrix}\right|_{z=h(x,y)} = \left.\begin{pmatrix} Y_1(\phi_1,\phi_2) \\ Y_2(\phi_1,\phi_2) \end{pmatrix}\right|_{z=h(x,y)} = \\
&= (Y_1, Y_2)^T \circ \bar{\phi}.
\end{aligned}$$

Therefore, $\bar{\phi}$ is a near-identity formal change of variables that transforms the restricted system into the two first coordinates of the formal normal form (2.18).

Since the Andreev number for the origin of the restricted system is 2, by Theorem 2.9 there exists a formal series $v(x,y)$ satisfying $\bar{X}v - v\,\mathrm{div}\bar{X} = 0$, i.e. $v(x,y)$ is a formal inverse integrating factor for $\bar{X}$. Therefore, there exists an inverse integrating factor $\tilde{v}(x,y)$ for the vector field associated to the first two components of the normal form (2.18). Thus, normal form (2.18) admits the inverse Jacobi multiplier $\tilde{V}(x,y,z) = z\tilde{v}(x,y)$ which implies, by Theorem 2.54, that the original system (2.2) also has a formal inverse Jacobi multiplier $V(x,y,z)$. By Proposition 2.50, we must have $j^{m+1}V(0) = zy^m$, for some $m \geqslant 0$. The result follows. □

Remark 2.57. Since the formal change of variables that transforms system (2.2) into normal form (2.18) is a near-identity one, the first non-zero jet of any formal inverse Jacobi multiplier is preserved.

The idea is to use the above results to devise an algorithm to solve the nilpotent center problem for systems (2.2) with Andreev number 2. We know, by Proposition 2.50, that any inverse Jacobi multiplier of system (2.2) must have $j^{m+1}V(0) = zy^m$. However, without reducing the possible values for m, such algorithm would not be practical for computational purposes. The next results, proven in [2], determine the possible values for m.

Proposition 2.58. *Consider system* (2.1) *having monodromic singular point at the origin with Andreev number n, satisfying monodromy condition $\beta = n-1$. If there is a formal inverse integrating factor $V(x,y)$ for* (2.1)*, then it has the form $j^2V(0) = y^2$.*

Proposition 2.59. *Consider system* (2.1) *having monodromic singular point at the origin with Andreev number n, satisfying monodromy condition $\beta > n-1$ in Theorem 2.2. If the system admits formal inverse integrating factor $V(x,y)$, then either $V(0,0) \neq 0$ or $j^2V(0) = y^2$.*

Thus, for the three-dimensional case, any inverse Jacobi multiplier have either the form $j^1V(0) = z$ or $j^3V(0) = zy^2$.

Remark 2.60. Propositions 2.58 and 2.59 are direct consequences of Propositions 7 and 8 in [2], respectively, since the monodromy conditions are invariant under changes of variables (see Proposition 2.21).

Now, we can use the above results to apply the following procedure to find necessary conditions for the origin of system (2.2) with Andreev number 2 to be an analytic nilpotent center. More precisely, we use Proposition 2.8 to determine which of the monodromy conditions in Theorem 2.6 are satisfied. Then, we consider a formal series $V(x,y,z) = \sum_{n\geqslant 1} V_n(x,y,z)$, where $V_n \in H_{3,1}^{(n)}$. By Theorem 2.53, it is always possible to find $V(x,y,z)$ such that

$$XV - V\mathrm{div}X = \sum_{n\geqslant 1} \Lambda_n x^{n-1} z.$$

The quantities Λ_n are polynomials in the parameters of system (2.2). Any non-zero Λ_n is an obstruction for $V(x,y,z)$ to be an inverse Jacobi multiplier. If there are obstructions for both possibilities $j^1V(0) = z$ and $j^3V(0) = zy^2$ then the origin cannot be an analytic nilpotent center on the center manifold. We apply this algorithm in the section 2.6.

Theorem 2.56 is one of our main results of this chapter and it is published in [12].

2.5.2 Analytic Nilpotent Centers as limits of Non-degenerated Centers in $\mathbb{R}^3$

A planar vector field X has a non-hyperbolic non-degenerate singular point p if the eigenvalues of DX_p are purely imaginary and conjugated, say $\pm\beta i$ for $\beta \neq 0$. One canonical form for such systems is

$$\begin{aligned} \dot{x} &= y + P(x,y), \\ \dot{y} &= -x + Q(x,y), \end{aligned} \tag{2.30}$$

where $j^1P(0) = j^1Q(0) = 0$. This singular point is monodromic. In regards to the Center Problem, the following result, due to Poincaré and Lyapunov, holds true:

Theorem 2.61 (Poincaré-Lyapunov Center Theorem)**.** *The origin of system* (2.30) *is a center if and only if there is an analytical first integral $H(x,y)$ such that $j^2H(0) = x^2+y^2$.*

A proof for this result can be found in [1, 40]. The Poincaré-Lypunov Center Theorem provides an algorith to solve the center problem which consists in the search for a first integral. Such algoritm, refered to as *Poincaré-Lyapunov algorithm*, is well-known and extensively studied in the literature (see, for instance [31, 15, 40, 32]).

For the nilpotent case, the authors of [3] (and its previous versions) were able to use the Poincaré-Lyapunov algorithm to investigate the nilpotent center problem for planar systems (2.1). The key of their approach is the following theorem.

Theorem 2.62 (Theorem 1 in [3])**.** *Suppose the origin of the following system*

$$\begin{aligned} \dot{x} &= y + P(x,y), \\ \dot{y} &= Q(x,y), \end{aligned}$$

is a nilpotent center. Then there are two functions $F_1(x,y)$, $F_2(x,y)$ analytical at the origin with $j^1F_1(0) = j^1F_2(0) = 0$ such that the 1-parameter family

$$\begin{aligned} \dot{x} &= y + P(x,y) + \varepsilon F_1(x,y), \\ \dot{y} &= -\varepsilon x + Q(x,y) + \varepsilon F_2(x,y). \end{aligned}$$

has a non-degenerate center at the origin for any $\varepsilon > 0$*. Also there is an analytic function* $f(x, y)$ *at the origin with* $j^1 f(0) = 0$ *such that* $(x - F_1(x, y))\dfrac{\partial f}{\partial y} = F_2(x, y)(1 + \dfrac{\partial f}{\partial x})$.

The above result tells us that for the analytical planar case, any nilpotent center is limit of non-degenerate centers. We now proceed to study a possible extension of this result to $\mathbb{R}^3$.

A three-dimensional system has a Hopf singular point if it can be put, via changes of variables and time rescalings, in the form:

$$\begin{aligned} \dot{x} &= y + P(x, y, z), \\ \dot{y} &= -x + Q(x, y, z), \\ \dot{z} &= -\lambda z + R(x, y, z), \end{aligned} \tag{2.31}$$

for which $j^1 P(0) = j^1 Q(0) = j^1 R(0) = 0$ and $\lambda \neq 0$. That is, the eigenvalues of its correspondent Jacobian matrix are two pure imaginary numbers and one non-zero real number. The Center Manifold Theorem 2.1 also holds for system (2.31) and thus it is also possible to study the Center Problem for Hopf singular points. For this matter, Theorem 2.61 has its three-dimensional counterpart (a proof is provided in [39, 40]):

Theorem 2.63 (Lyapunov Center Theorem)**.** *The origin of system* (2.31) *is a center on a center manifold if and only if there is an analytical first integral* $H(x, y, z)$ *such that* $j^2 H(0) = x^2 + y^2$*. Moreover, if the origin is a center on a center manifold, the center manifold is unique and analytical.*

There are several obstacles for the extension of Theorem 2.62 to three-dimensional system. Among them, the lack of analiticity of the center manifold is one of the most noticeable. Therefore we make some assumptions to obtain a first result. A more detailed study on the center manifolds can be found in [21, 22].

Theorem 2.64. *Suppose the origin of the following system*

$$\begin{aligned} \dot{x} &= y + P(x, y, z), \\ \dot{y} &= Q(x, y, z), \\ \dot{z} &= -\lambda z + R(x, y, z), \end{aligned} \tag{2.32}$$

is a nilpotent center on an analytical center manifold. Then there are three functions $F_1(x, y)$*,* $F_2(x, y)$*,* $F_3(x, y)$ *analytical at the origin with* $j^1 F_1(0) = j^1 F_2(0) = j^1 F_3(0) = 0$ *such that the 1-parameter family*

$$\begin{aligned} \dot{x} &= y + P(x, y, z) + \varepsilon F_1(x, y), \\ \dot{y} &= -\varepsilon x + Q(x, y, z) + \varepsilon F_2(x, y), \\ \dot{z} &= -\lambda z + R(x, y, z) + \varepsilon F_3(x, y). \end{aligned} \tag{2.33}$$

has a non-degenerate center at the origin for any $\varepsilon > 0$*. Moreover, there are analytic functions* $f(x, y), h(x, y)$ *at the origin with* $j^1 f(0) = j^1 h(0) = 0$ *such that* $(x - F_1(x, y))\dfrac{\partial f}{\partial y} = F_2(x, y)(1 + \dfrac{\partial f}{\partial x})$ *and* $F_3(x, y) = F_1(x, y)\dfrac{\partial h}{\partial x} + (F_2(x, y) - x)\dfrac{\partial h}{\partial y}$.

Proof: By hypothesis, (2.32) admits an analytical center manifold $z = h(x, y)$, $j^1 h(0) = 0$ on which the origin is a nilpotent center. Thus, the restricted system is analytical and has a nilpotent center at the origin. Consider the following change of variables

$$x = x, \; y = y, \; Z = z - h(x, y).$$

Under these new coordinates, system (2.32) becomes

$$\begin{aligned}\dot{x} &= y + P(x, y, Z + h(x, y)),\\ \dot{y} &= Q(x, y, Z + h(x, y)),\\ \dot{Z} &= -\lambda Z + Z\tilde{R}(x, y, Z).\end{aligned}$$

We have that $Z = 0$ is an analytical center manifold. Since the above system is analytical, by Theorem 2.62 there exist two functions $F_1(x, y), F_2(x, y)$ analytical at the origin with $j^1F_1(0) = j^1F_2(0) = 0$ such that

$$\begin{aligned}\dot{x} &= y + P(x, y, h(x, y)) + \varepsilon F_1(x, y),\\ \dot{y} &= -\varepsilon x + Q(x, y, h(x, y)) + \varepsilon F_2(x, y)\end{aligned}$$

has a non-degenerate center at the origin for every $\varepsilon > 0$. Hence, the three-dimensional system

$$\begin{aligned}\dot{x} &= y + P(x, y, Z + h(x, y)) + \varepsilon F_1(x, y),\\ \dot{y} &= -\varepsilon x + Q(x, y, Z + h(x, y)) + \varepsilon F_2(x, y),\\ \dot{Z} &= -\lambda Z + Z\tilde{R}(x, y, Z).\end{aligned}$$

has a non-degenerate center on $Z = 0$. Returning to the original coordinates, we conclude that system

$$\begin{aligned}\dot{x} &= y + P(x, y, z) + \varepsilon F_1(x, y),\\ \dot{y} &= -\varepsilon x + Q(x, y, z) + \varepsilon F_2(x, y),\\ \dot{z} &= -\lambda z + R(x, y, z) + \varepsilon F_3(x, y),\end{aligned}$$

with $F_3(x, y) = F_1(x, y)\dfrac{\partial h}{\partial x} + (F_2(x, y) - x)\dfrac{\partial h}{\partial y}$ has a non-degenerate center at the origin on center manifold $z = h(x, y)$. $\square$

Remark 2.65. The equation

$$(x - F_1(x, y))\frac{\partial f}{\partial y} = F_2(x, y)(1 + \frac{\partial f}{\partial x}),$$

implies that x must factor out the lowest degree homogeneous polynomial of the power series expansion of F_1. Thus, the coefficient of y^2 in its expansion must be zero.

Remark 2.66. The arguments presented in the previous proof are also valid even when the center manifold $z = h(x, y)$ is not analytical as long as the restricted system is. For instance, the decoupled systems given by

$$\begin{aligned}\dot{x} &= y + P(x, y),\\ \dot{y} &= Q(x, y),\\ \dot{z} &= -\lambda z + R(x, y, z).\end{aligned}$$

have this property since its restriction to any center manifold is completely determined by the first two components.

Now, we use the Poincaré-Lyapunov method to detect analytic nilpotent centers. First, we consider three formal series:

$$F_i = \sum_{j+k\geqslant 2} g^i_{jk} x^j y^k,\ i = 1, 2, 3,$$

with $g_{02}^1 = 0$. We then perturb system (2.32) as in (2.33) and search for obstructions for the formal series $H(x,y,z) = \varepsilon x^2 + y^2 + \sum_{n\geqslant 3} H_n(x,y,z)$ to be a first integral. Those obstructions will be the *Lyapunov quantities*. More precisely, let X_ε be the vector field associated to (2.33). We can always find $H(x,y,z)$ such that

$$X_\varepsilon H = \sum_{l\geqslant 1} \eta_l (x^2+y^2)^l,$$

where η_l is a rational function on the parameters of system (2.33), i.e. the coefficients of functions P, Q, R, the coefficients g_{jk}^i as well as ε and λ. The numerator of the Lyapunov quantities will be a polynomial on ε. For the system (2.33) to have a center on the center manifold for every $\varepsilon > 0$, we must have all the Lyapunov quantities to be null. We illustrate this method in the examples of the next section.

Theorem 2.64 is our second main result of this chapter and it is published in [12].

2.6 Applications and Examples

2.6.1 Elsonbaty–El-Sayed system

In [47], the authors study the following system,

$$\begin{aligned}\dot{x} &= y,\\ \dot{y} &= az,\\ \dot{z} &= -z + bx - cy - x^3,\end{aligned} \tag{2.34}$$

which has peculiar dynamics. For $a > 0$, and $b = c = 0$, this system has a nilpotent singular point at the origin. Via the change of variables $x = \bar{x} + \bar{z}$, $y = \bar{y} - \bar{z}$, $z = \dfrac{\bar{z}}{a}$, and dropping the bars, system (2.34) is transformed into the canonical form

$$\begin{aligned}\dot{x} &= y + a(x+z)^3,\\ \dot{y} &= -a(x+z)^3,\\ \dot{z} &= -z - a(x+z)^3.\end{aligned} \tag{2.35}$$

Note that, by Proposition 2.8, the origin is a monodromic nilpotent singular point. Computing the quantities ω_n, we obtain that the first non-zero is $\omega_6 = 2a^2$ and by Theorem 2.40, system (2.35) cannot admit a formal or analytic first integral. Note that if $a = 0$, the system admits $H(x,y,z) = y^2$ as a first integral but loses monodromy at the origin.

2.6.2 The Generalized Lorenz system

This example shows that, in some cases, the process of vanishing the ω_n can be used to find families of nilpotent centers.

The Generalized Lorenz system is one of the most studied three-dimensional systems in the literature since its dynamics are very rich. Among the particular cases of the Generalized Lorenz systems are the Lü and Chen systems. Its expression is given by

$$\begin{aligned}\dot{x} &= a(y-x),\\ \dot{y} &= bx + cy - xz,\\ \dot{z} &= dz + xy.\end{aligned} \tag{2.36}$$

We consider $ad \neq 0$ for the singular points of system (2.36) to be isolated. The origin is always a singular point whose dynamics is determined by the parameters a, b, c, d. The nondegenerate case is studied in [48, 40] and the references therein. We investigate conditions on the parameters a, b, c, d for which system (2.36) has a nilpotent singular point at the origin.

The singular points of the above system are the origin, and

$$Q_{\pm} = (\pm\sqrt{-d(b+c)}, \pm\sqrt{-d(b+c)}, b+c),$$

for $d(b+c) < 0$. The determinant of the Jacobian matrix of (2.36) at point $Q_{\pm}$ is given by $2ad(b+c)$. Thus, $Q_{\pm}$ are nilpotent singular points only when $b+c=0$ which means they coincide with the origin.

Now, the Jacobian matrix of (2.36) at the origin is

$$\begin{pmatrix} -a & a & 0 \\ b & c & 0 \\ 0 & 0 & d \end{pmatrix},$$

and for the origin to be a nilpotent singular point, we must have $b+c=0$ and $c=a$. By means of the coordinate change $\bar{x} = y$, $\bar{y} = a(y-x)$, $\bar{z} = z$, dropping the bars, system (2.36) becomes

$$\begin{aligned} \dot{x} &= y - xz + \tfrac{1}{a}yz, \\ \dot{y} &= -axz + yz, \\ \dot{z} &= dz + x^2 - \tfrac{1}{a}xy. \end{aligned} \tag{2.37}$$

We compute the quantities ω_n (Theorem 2.37) for system (2.37) and obtain the first non-zero one:

$$\omega_6 = -\frac{2\,a\,(2\,a+d)}{3d^3}.$$

Thus, by Theorem 2.40, system (2.37) can only be formally or analytically integrable if $d = -2a$. Under this condition, by Proposition 2.8, the origin is monodromic with Andreev number 2. Moreover, it satisfies monodromy condition $\beta > n-1$ from Theorem 2.6.

For $d = -2a$, the function $V(x,y,z) = x^2 - \frac{2xy}{a} + \frac{y^2}{a^2} - 2az$ defines an invariant surface $V \equiv 0$ for (2.37) which is tangent to the xy-plane, thus, it is a center manifold for system (2.37). Moreover, the restriction of the system to this center manifold is given by

$$\dot{x} = y - xz + \frac{y}{2a^2}\left(x - \frac{y}{a}\right)^2,$$

$$\dot{y} = -axz + \frac{y}{2a}\left(x - \frac{y}{a}\right)^2,$$

which is a Hamiltonian system with Hamiltonian function $H(x,y) = y^2 + \frac{x^4}{4} - \frac{x^3y}{a} + \frac{3x^2y^2}{2a^2} - \frac{xy^3}{a^3} + \frac{y^4}{4a^4}$. Hence, the origin is a nilpotent center on a center manifold.

2.6.3 Hide–Skeldon–Acheson Dynamo system

In paper [49] the authors proposed a model for self-exciting dynamo action in which a Faraday disk and coil are arranged in series with either a capacitor or a motor. The

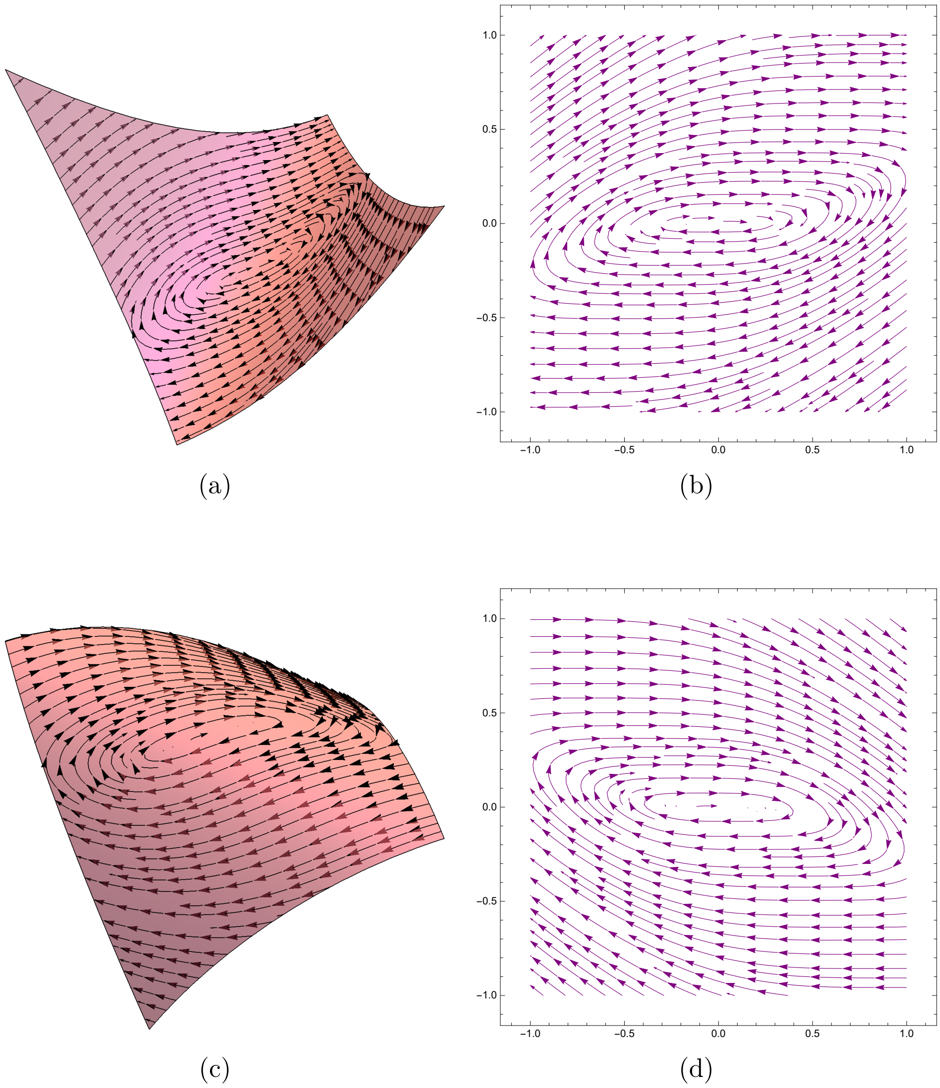


Figura 2.2: Phase portraits of systems (2.37) on the center manifold $x^2 - \frac{2xy}{a} + \frac{y^2}{a^2} - 2az = 0$ for $a = 1$ in figures (a),(b) and $a = -1$ in figures (c),(d).

proposed system for this model is the following

$$\begin{aligned} \dot{x} &= -x - \beta z + xy, \\ \dot{y} &= \alpha - \kappa y - \alpha x^2, \\ \dot{y} &= x - \lambda z. \end{aligned} \tag{2.38}$$

The physical quantities that the parameters $\alpha, \beta, \lambda, \kappa$ represent are detailed in [49, 50]. The above system has a singular point at $Q = (0, \frac{\alpha}{\kappa}, 0)$. The determinant of the Jacobian matrix of (2.38) at Q is $(\alpha - \kappa)\,\lambda - \beta\,\kappa$, and its characteristic polynomial has

$$\frac{(\lambda+1)\,\kappa^2 + (\beta - \alpha + \lambda)\,\kappa - \lambda\,\alpha}{\kappa}$$

as the coefficient of its linear term. Both these expressions must be zero for Q to be a nilpotent singular point. This occurs for $\alpha = \kappa(\lambda+1)$ and $\beta = \lambda^2$. Now, under these conditions, we translate Q to the origin and via the change of coordinates $x = \lambda\bar{x} + \bar{y},\ y = \bar{z},\ z = \bar{x}$, dropping the bars, we obtain the following system

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= \lambda xz + yz, \\ \dot{z} &= -\kappa z - \kappa(\lambda+1)(\lambda x + y)^2. \end{aligned} \tag{2.39}$$

We now investigate whether the origin is monodromic or not. The inequality (2.9) in Proposition 2.8 for system (2.39) is given by:

$$-\lambda^3(\lambda+1) < 0,$$

which is only satisfied for $\lambda < -1$ or $\lambda > 0$. For $-1 < \lambda < 0$, by Theorem 2.6 the origin of system (2.39) is not monodromic (see the proof of Proposition 2.8). Moreover, for $\lambda = -1$ or $\lambda = 0$, (2.39) does not have an isolated singular point since the x-axis is a line of equilibria in this case. Note that, by Proposition 2.8 we also know that when the origin of system (2.39) is monodromic, it satisfies the condition $\beta > n - 1$ in Theorem 2.6.

We compute the quantities ω_n described in Theorem 2.37. The first not identically zero is

$$\omega_6 = -\frac{2\lambda^5 (\lambda+1)^2 (-2\lambda + 3\kappa)}{3\kappa}.$$

By Theorem 2.40, system (2.39) can only be formally or analytically integrable when $\lambda = -1$, $\lambda = 0$ or $\lambda = \frac{3\kappa}{2}$. Assuming the latter, the next non-vanishing quantity is ω_8 which is given by:

$$\omega_8 = -\frac{6561\,\kappa^6 (3\kappa + 2)^3}{512}.$$

We obtain that $\kappa = -\frac{2}{3}$ is the only value for which system (2.38) could admit a formal or analytic first integral. However, that would imply that $\lambda = -1$ and thus the origin would not be an isolated singular point. In fact, for $\lambda = -1$, $z = 0$ is a center manifold and the flow of (2.39) is governed by the linear vector field $y\partial_x$ which does not have any periodic orbits. We conclude that monodromy and formal integrability cannot coexist for system (2.39). This implies that the singular point $Q = (0, \frac{\alpha}{\kappa}, 0)$ is never an analytically integrable nilpotent center on a center manifold for system (2.39).

2.6.4 The Song-Wang-Feng system

The following system was first considered in [24].

$$\begin{aligned} \dot{x} &= y - 2xy + axz, \\ \dot{y} &= -2x^3 + y^2 + byz, \\ \dot{z} &= -z + dxy. \end{aligned} \tag{2.40}$$

In [24] the authors solved the center problem by considering a polynomial approximation of the center manifold. Here, we solve the center problem without restricting the considered system to a center manifold.

By Proposition 2.8, the origin of (2.40) is monodromic with Andreev number $n = 2$ and satisfies the monodromy condition $\beta > n - 1$.

Using the algorithm described in Section 2.5.2, we compute the Lyapunov quantities of the perturbation (2.33) for system (2.40). We have:

$$\eta_2 = -\frac{3bd^2(a-b)}{20(4\varepsilon+1)}.$$

By Theorem 2.64, $d\,b(a-b)=0$ is a necessary condition for the origin to be an analytic nilpotent center.

Under $d=0$, system (2.40) becomes

$$\begin{aligned}\dot{x} &= y - 2xy + axz,\\ \dot{y} &= -2x^3 + y^2 + byz,\\ \dot{z} &= -z.\end{aligned}$$

which has $z=0$ as an analytical center manifold. The restricted system is the Hamiltonian system given by

$$\begin{aligned}\dot{x} &= y - 2xy,\\ \dot{y} &= -2x^3 + y^2.\end{aligned} \tag{2.41}$$

Thus, $d=0$ implies that the origin is a nilpotent center on a center manifold. Now, if $d \neq 0$, we search for obstructions for system (2.40) to have a formal inverse Jacobi multiplier. We consider both possibilities of $V(x,y,z)$. First, if $j^1V(0)=z$. We have $\Lambda_1=\cdots=\Lambda_4=0$ and:

$$\Lambda_5 = -4d\,(2a-b),\ \Lambda_6 = -a\,d.$$

Now for $j^3V(0)=zy^2$, we obtain $\Lambda_1=\cdots=\Lambda_8=0$ and:

$$\Lambda_9 = -\frac{12\,d\,(2\,a-b)}{5},\ \Lambda_{10} = -\frac{2\,d\,(9\,a-2\,b)}{15}.$$

Thus, by Theorem 2.56, the conditions $a=b=0$ or $d=0$ are necessary for the origin to be an analytic nilpotent center on a center manifold.

Now, $a=b=0$ is a center condition, since, under this condition, the first two components of (2.40) do not depend on z. Thus, the restricted system is also given by (2.41) regardless the center manifold. Therefore $a=b=0$ or $d=0$ are sufficient conditions for system (2.40) to have an analytic nilpotent center on a center manifold. We have proven the following result:

Theorem 2.67. *The origin of system* (2.40) *is an analytic nilpotent center on a center manifold if and only if $a=b=0$ or $d=0$.*

2.6.5 Jerk systems

Consider the three-dimensional system of differential equations:

$$\begin{aligned}\dot{x} &= y,\\ \dot{y} &= z,\\ \dot{z} &= f(x,y,z).\end{aligned} \tag{2.42}$$

This system is called *Jerk system* since it is equivalent to the third-order differential equation $\dddot{x} = f(x,\dot{x},\ddot{x})$ which is called *Jerk equation*. In mechanical models where $x(t)$ denotes the position of given particle at instant t, $\dddot{x}(t)$ denotes the rate of change of the accelaration of such particle, that is, its jerk. Jerk equations are broadly studied in the literature (see, for instance, [5, 51, 52]) for its use in science and engineering.

We search for nilpotent singular points for system (2.42) with $f(x, y, z)$ being a polynomial. The singular points, when they exist, are given by $Q_i = (\alpha_i, 0, 0)$ where α_i are the roots of $f(x, 0, 0)$ for $i = 1, \ldots, \deg(f)$. The Jacobian matrix J of (2.42) is always given by:

$$J = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} & \dfrac{\partial f}{\partial z} \end{pmatrix}.$$

The determinant and the trace of the above matrix are, respectively, $\det J = \frac{\partial f}{\partial x}$ and $\mathrm{tr}J = \frac{\partial f}{\partial z}$. Its characteristical polynomial $p(\eta)$ is given by:

$$p(\eta) = -\eta^3 + \eta^2 \frac{\partial f}{\partial z} + \eta \frac{\partial f}{\partial y} + \frac{\partial f}{\partial x}.$$

Note that the rank of J is greater or equal to 2. Thus, for Q_i to be a nilpotent singular point, it is sufficient that $\frac{\partial f}{\partial x}(Q_i) = \frac{\partial f}{\partial y}(Q_i) = 0 \neq \frac{\partial f}{\partial z}(Q_i)$.

Translating Q_i to the origin transforms system (2.42) into:

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= z, \\ \dot{z} &= f(x + \alpha_i, y, z). \end{aligned}$$

Let $\tau = \frac{\partial f}{\partial z}(Q_i)$. By making the change of variables $x = \bar{x} + \frac{\bar{z}}{\tau^2}$, $y = \bar{y} + \frac{\bar{z}}{\tau}$, $z = \bar{z}$ and dropping the bars, we obtain the canonical form:

$$\begin{aligned} \dot{x} &= y - \tfrac{1}{\tau^2} F_2(x, y, z), \\ \dot{y} &= -\tfrac{1}{\tau} F_2(x, y, z), \\ \dot{z} &= \tau z + F_2(x, y, z), \end{aligned} \tag{2.43}$$

where $F_2(x, y, z)$ is the nonlinear part of $f(x + \alpha_i, y, z)$ in the new variables. We further investigate the origin of system (2.43) since it is the canonical form of any jerk system having a nilpotent singular point. For now on, we denote $F_2(x, y, z) = \sum_{j+k+l \geqslant 2} g_{jkl} x^j y^k z^l$.

By Proposition 2.8, the origin is monodromic with Andreev number 2 if and only if $g_{200} = 0$ and inequality (2.9) is satisfied, which for system (2.43) becomes:

$$g_{110}^2 - 8 g_{300} \tau < 0.$$

Note that the above inequality implies that there are no quadratic jerk systems (2.43) with nilpotent singularity having Andreev number 2.

Proposition 2.68. *There is no quadratic Jerk system with an isolated monodromic nilpotent singular point.*

Proof: If $g_{200} \neq 0$, by Theorem 2.6 the origin cannot be monodromic. For $g_{200} = 0$, we prove that any parametrization $h(x, y)$ for a central manifold must be such that $h(x, 0) \equiv 0$ in a neighborhood of $x = 0$. In fact, for $g_{200} = 0$, any parametrization $h(x, y)$ must satisfy:

$$-F_2(x, 0, h(x, 0)) \left(\frac{1}{\tau^2} \frac{\partial h}{\partial x}(x, 0) + \frac{1}{\tau} \frac{\partial h}{\partial y}(x, 0) + 1 \right) - \tau h(x, 0) \equiv 0.$$

For $g_{200} = 0$, we have $F_2(x,0,h(x,0)) = g_{101}xh(x,0) + g_{002}h(x,0)^2$. Then, the above equation becomes $h(x,0)G(x) \equiv 0$, where $G(0) = -\tau$. Thus, there is a neighborhood of $x = 0$ such that $G(x) \neq 0$ and therefore $h(x,0) \equiv 0$. This implies that, for this neighborhood of $x = 0$, $F_2(x,0,h(x,0)) \equiv 0$.

Now, the restricted system is

$$\begin{aligned}\dot{x} &= y - \tfrac{1}{\tau^2}F_2(x,y,h(x,y)),\\ \dot{y} &= -\tfrac{1}{\tau}F_2(x,y,h(x,y)),\end{aligned}$$

and thus, there is an interval of singular points contained in the x-axis for the above system and there is no isolated nilpotent singular point. □

The above result tells us that, in order to study the nilpotent center problem for polynomial jerk systems, we must start investigating the cubic ones. However, due to computation difficulties, we chose a simpler form of cubic jerk system to apply our previous results in order to solve the nilpotent center problem.

Consider the following jerk system

$$\begin{aligned}\dot{x} &= y - F(x,y,z),\\ \dot{y} &= F(x,y,z)\\ \dot{z} &= -z + F(x,y,z),\end{aligned} \tag{2.44}$$

where $F(x,y,z) = g_{300}x^3 + g_{011}yz + g_{210}x^2y + g_{120}xy^2 + g_{030}y^3$. Suppose that the origin has Andreev number $n = 2$, which by Proposition 2.8 implies that $g_{300} > 0$. Note that the monodromy condition $\beta > n - 1$ is satisfied. Applying Theorem 2.53 with the formal series $V(x,y,z) = \sum_{j+k+l\geqslant 1} v_{jkl}x^jy^kz^l$, we compute the obstructions Λ_n for the existence of an Inverse Jacobi Multiplier. For $j^1V(0) = z$, we have $\Lambda_1 = \Lambda_2 = 0$ and:

$$\begin{aligned}\Lambda_3 &= 3g_{300} - g_{210},\\ \Lambda_4 &= -g_{300}\left(g_{011} - v_{011}\right),\\ \Lambda_5 &= 3\,g_{300}\left(g_{030} - g_{120} + 2\,g_{300}\right) \quad \mathrm{mod}\ \langle\Lambda_3,\Lambda_4\rangle,\\ \Lambda_6 &= g_{300}g_{011}\left(g_{120} + 3\,g_{300}\right) \quad \mathrm{mod}\ \langle\Lambda_3,\Lambda_4,\Lambda_5\rangle,\\ \Lambda_7 &= -4\,{g_{300}}^2{g_{011}}^2 \quad \mathrm{mod}\ \langle\Lambda_3,\Lambda_4,\Lambda_5\rangle.\end{aligned}$$

And for $j^3V(0) = zy^2$, we obtain $\Lambda_1 = \cdots = \Lambda_6 = 0$ and:

$$\begin{aligned}\Lambda_7 &= \tfrac{1}{6}\,g_{300}\left(g_{210} - 3\,g_{300}\right),\\ \Lambda_8 &= \tfrac{1}{4}{g_{300}}^2\left(g_{011} - 3\,v_{031}\right),\\ \Lambda_9 &= -\frac{9\,{g_{300}}^2\left(g_{030} + 2\,g_{300} - g_{120}\right)}{10} \quad \mathrm{mod}\ \langle\Lambda_7,\Lambda_8\rangle\\ \Lambda_{10} &= -\tfrac{1}{3}g_{011}{g_{300}}^2\left(3\,g_{300} + g_{120}\right) \quad \mathrm{mod}\ \langle\Lambda_7,\Lambda_8,\Lambda_9\rangle,\\ \Lambda_{11} &= \frac{10\,{g_{011}}^2{g_{300}}^3}{7} \quad \mathrm{mod}\ \langle\Lambda_7,\Lambda_8,\Lambda_9\rangle.\end{aligned}$$

Both the possibilities above yield the same conditions on the parameters for the existence of an inverse Jacobi multiplier: $g_{030} = g_{120} - 2g_{300}$, $g_{011} = 0$ and $g_{210} = 3g_{300}$. Thus, by Theorem 2.56, the origin can only be an analytic center on a center manifold if those conditions are satisfied. These are in fact center conditions, since under those, the restricted system is given by the first two equations of (2.44) and has

$$\begin{aligned}v(x,y) = 1 &+ \left(g_{120} - 3\,g_{300}\right)x^2 + 2\left(g_{120} - 3\,g_{300}\right)xy\\ &+ \frac{\left(g_{120} - 3\,g_{300}\right)\left(g_{120} - 2\,g_{300}\right)y^2}{g_{300}},\end{aligned}$$

as an inverse integrating factor. By Theorem 2.9, the origin is a center. We conclude:

Theorem 2.69. *The origin of system* (2.44) *is a nilpotent center on a center manifold if and only if* $g_{030} = g_{120} - 2g_{300}$, $g_{011} = 0$ *and* $g_{210} = 3g_{300}$.

2.6.6 A system with fixed Andreev number

If a planar system has a monodromic nilpotent singular point with Andreev number n, it can be put in the following form:

$$\begin{aligned} \dot{x} &= y + \mu x^n + \textstyle\sum_{k\geqslant n+1} P_k(x,y), \\ \dot{y} &= -nx^{2n-1} + n\mu x^{n-1}y + \textstyle\sum_{k\geqslant 2n} Q_k(x,y), \end{aligned}$$

where $P_k, Q_k \in \mathcal{P}_k^{(1,n)}$ (see [23], for more details). Thus, an interesting family of three-dimensional systems to consider are the following

$$\begin{aligned} \dot{x} &= y + \mu x^n + \textstyle\sum_{k\geqslant n} P_k(x,y,z), \\ \dot{y} &= -nx^{2n-1} + n\mu x^{n-1}y + \textstyle\sum_{k\geqslant 2n-1} Q_k(x,y,z), \\ \dot{z} &= -z + R(x,y,z), \end{aligned} \tag{2.45}$$

where $j^1R(0) = 0$ and P_k, Q_k are $(1,n,1)$-quasi-homogeneous polynomials of weighted degree k. For any parametrization $z = h(x,y)$ of a center manifold, we have that the functions $F(x), f(x)$ and $\Phi(x)$ in Theorem 2.6 are given by:

$$F(x) = -\mu x^n + O(x^{n+1}),$$

$$f(x) = -n(1+\mu^2)x^{2n-1} + O(x^{2n}),$$

$$\Phi(x) = 2n\mu x^{n-1} + O(x^n).$$

Thus, we have that the origin is monodromic with Andreev number n. Moreover, the parameter μ defines whether we have monodromy conditions $\beta = n-1$ or $\beta > n-1$.

We tackle the nilpotent center problem for a particular case of system (2.45) with $n = 2$:

$$\begin{aligned} \dot{x} &= y + \mu x^2 + a_{101}xz + a_{002}z^2, \\ \dot{y} &= -2x^3 + 2\mu xy, \\ \dot{z} &= -z + c_{200}x^2 + c_{110}xy + c_{020}y^2. \end{aligned} \tag{2.46}$$

Using Theorem 2.53, we compute the obstructions for the existence of an Inverse Jacobi Multiplier $V(x,y,z)$. For the case $j^1V(0) = z$, we compute the obstructions $\Lambda_1, \ldots, \Lambda_{10}$. Due to the size of these quantities, we exhibit only $\Lambda_1, \ldots, \Lambda_5$ and omit $\Lambda_6, \ldots, \Lambda_{10}$.

$$\begin{aligned} \Lambda_1 &= 0,\ \Lambda_2 = -4\mu, \\ \Lambda_3 &= -3\,a_{101}c_{200}, \\ \Lambda_4 &= -2\,\mu^2 v_{011} - 2\,\mu\, a_{101}c_{110} + 4\,\mu\, a_{101}c_{200} - 4\,a_{002}{c_{200}}^2 - 2\,v_{011}, \\ \Lambda_5 &= -\tfrac{1}{3}\,c_{020}\mu^2 a_{101} + 2\,c_{110}\mu^2 a_{101} - 24\,c_{200}\mu^2 a_{101} - 2\,c_{110}a_{002}\mu\, c_{200} \\ &\quad -2\,c_{200}\mu\, a_{101}v_{011} - c_{110}c_{200}{a_{101}}^2 - 8\,a_{101}c_{110} + 16\,a_{101}c_{200} \\ &\quad +3\,{c_{200}}^2{a_{101}}^2 - 2\,c_{020}a_{101}. \end{aligned}$$

Setting $\Lambda_2 = \cdots = \Lambda_{10} = 0$ yields the conditions on the parameters: $\mu = 0$ and $a_{101} = a_{002} = 0$ or $c_{200} = c_{110} = c_{020}$.

For the case $j^3V(0) = zy^2$, we compute the obstructions $\Lambda_1, \ldots, \Lambda_{15}$. Due to the size of these quantities, we exhibit only $\Lambda_1, \ldots, \Lambda_9$ and omit $\Lambda_{10}, \ldots, \Lambda_{15}$. We have $\Lambda_1 = \cdots = \Lambda_6 = 0$ and

$$\begin{aligned}
\Lambda_7 &= -\frac{c_{200}a_{101}\left(\mu^2+5\right)}{5}, \\
\Lambda_8 &= \tfrac{\mu}{3}\left(\mu^2+5\right)a_{101}c_{110} + \tfrac{2\mu}{3}\left(\mu^2+1\right)c_{200}a_{101} \\
&\quad -\tfrac{2}{3}\left(\mu^2+3\right)a_{002}{c_{200}}^2 - 2/5\,\mu\,{a_{101}}^2{c_{200}}^2 - \tfrac{1}{3}\left(\mu^4+10\,\mu^2+9\right)v_{031}, \\
\Lambda_9 &= -\frac{\left(54\,\mu^4+222\,\mu^2+168\right)a_{101}c_{110}}{35} + \frac{\left(48\,\mu^2+48\right)c_{200}a_{101}}{5} \\
&\quad -\frac{\left(27\,\mu^4+129\,\mu^2+42\right)a_{101}c_{020}}{35} + \tfrac{\mu}{35}\left(11\,\mu^2-29\right)c_{200}a_{101}v_{031} \\
&\quad +\frac{72\,\mu\,\left(\mu^2+1\right)a_{002}{c_{200}}^2}{35} + \frac{6\,\mu\,\left(9\,\mu^2+29\right)c_{200}a_{002}c_{110}}{35} \\
&\quad +\tfrac{1}{35}\left(34\,\mu^2+35\right){a_{101}}^2c_{110}c_{200} + \frac{\left(57\,\mu^2-105\right){a_{101}}^2{c_{200}}^2}{35} \\
&\quad -\tfrac{6\mu}{5}\,a_{101}\,a_{002}{c_{200}}^3.
\end{aligned}$$

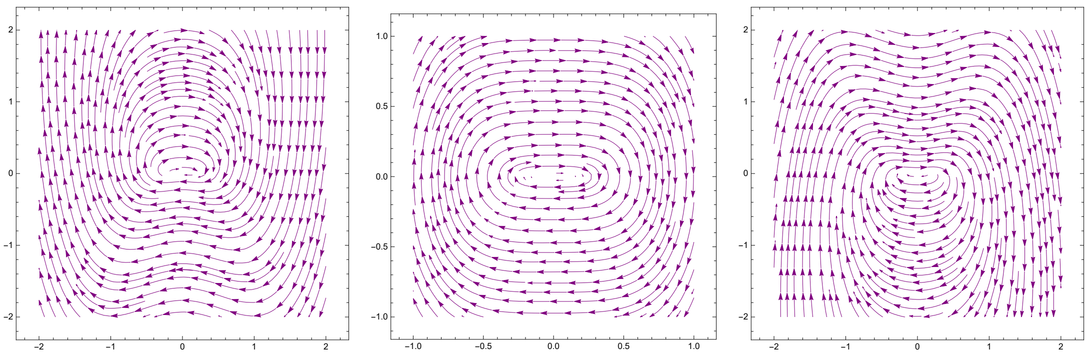
Setting $\Lambda_7 = \cdots = \Lambda_{15} = 0$ yields the following conditions on the parameters: $a_{101} = a_{002} = 0$ or $c_{200} = c_{110} = c_{020} = 0$.

The conditions for (2.46) to have a formal inverse Jacobi multiplier are $a_{101} = a_{002} = 0$ or $c_{200} = c_{110} = c_{020} = 0$. By Theorem 2.56, those are necessary conditions for the origin to be an analytic nilpotent center.

These conditions are also sufficient since under either one of those, the restricted system is

$$\begin{aligned}
\dot{x} &= y + \mu x^2, \\
\dot{y} &= -2x^3 + 2\mu xy.
\end{aligned}$$

which is invariant by the transformations $x \mapsto -x$, $y \mapsto y$, $t \mapsto -t$.



(a) $\mu < 0$. (b) $\mu = 0$ (Hamiltonian). (c) $\mu > 0$.

Figura 2.3: Phase portraits of the restriction systems (2.46) under $a_{101} = a_{002} = 0$ or $c_{200} = c_{110} = c_{020} = 0$.

Therefore we conclude:

Theorem 2.70. *The origin of system* (2.46) *is an analytic nilpotent center on a center manifold if and only if* $a_{101} = a_{002} = 0$ *or* $c_{200} = c_{110} = c_{020} = 0$.

3 Hopf singular points in $\mathbb{R}^3$ and the cyclicity problem

In this chapter, we divert from the nilpotent singular points in $\mathbb{R}^3$ to their non-degenerate counterpart: *Hopf singular points*. In this section, we consider analytical vector fields in $\mathbb{R}^3$ having a singular point for which the Jacobian matrix has a pair of purely imaginary eigenvalues and a non-zero real eigenvalue. The differential system associated with such vector fields can be put into the following canonical form, by means of a linear change of variables and time rescaling:

$$\begin{aligned} \dot{x} &= -y + P(x,y,z), \\ \dot{y} &= x + Q(x,y,z), \\ \dot{z} &= -\lambda z + R(x,y,z), \end{aligned} \tag{3.1}$$

where P, Q, R are polynomials with no linear or constant terms and $\lambda \neq 0$. For vector fields associated to (3.1), the Center Manifold Theorem 2.1 also holds. The restriction of (3.1) to a center manifold W^c is a two-dimensional differential system which has an elementary center or a focus at the origin. We recall that a singular point of a planar polynomial vector field is an *elementary center or a focus* if the eigenvalues of its Jacobian matrix, when evaluated at the singular point, are purely imaginary.

The problem of determining and locating the limit cycles for a given planar polynomial differential system is known as *Hilbert's sixteenth problem* and dates back to the year 1900. In the last century, several researchers have studied this problem and made significant advancements, which are described in detail in the surveys [53, 54] by Ilyashenko and Jibin Li, respectively. However, Hilbert's sixteenth problem has not been solved yet. There are several weaker versions of this problem, for instance, determine the number of small amplitude limit cycles that bifurcate from an elementary center or a focus of a polynomial vector field (see, for instance, [55]). In simple terms, this number is the *cyclicity* of the singular point.

One of the main techniques used to study the cyclicity of an elementary center or a focus is the computation of the *Lyapunov coefficients*, which are also related to the *center problem* [15, 32]. Christopher in [56] proved that, by performing an analysis on the linear part of the Lyapunov coefficients in its power series expansion with respect to the perturbation parameters, it is possible to estimate the cyclicity of a center. Using this idea, Torregrosa and Liang [57] proposed the *parallelization method*, which allows computations for the linear parts to be made separately, reducing computational time. This method was further explored in [58], where the authors generalized the method of parallelization for any order of the Lyapunov coefficients. Moreover, the authors improved the previous lower bounds of the cyclicity for centers of some polynomial vector fields using higher-order developments of the Lyapunov coefficients with respect to the perturbation parameters.

This problem can be naturally extended to Hopf singular points of three-dimensional systems. However, since the center manifold is not necessarily unique or analytic, computing a parametrization for W^c and then applying the planar techniques is not optimal. Fortunately, the computation of the Lyapunov coefficients can be made even without knowing a parametrization for any center manifold [4]. Furthermore, in [59] the authors proved that it is possible to study cyclicity for the Hopf singular point on center manifolds via the Lyapunov coefficients in the same way as proposed by Christopher. The authors of [60] used this approach to give new lower bounds for quadratic, cubic, quartic, and quintic three-dimensional systems. The respective lower bounds are 11, 31, 54, and 92 limit cycles.

In this chapter, we make a chart of the cyclicity of centers on center manifolds of the known quadratic systems (3.1) in the literature. In order to attain this goal, we use the higher order developments of the Lyapunov coefficients and full quadratic perturbations. In this study, we obtained an example of a quadratic three-dimensional system for which it is possible to bifurcate 13 limit cycles from the center on the center manifold which is so far the best known lower bound in the literature. This result is published in [13].

3.1 Lyapunov coefficients and Cyclicity

One of the most useful tools to study monodromic singular points is the *Poincaré map* or the *first return map*. For planar systems, it is widely known how to define those maps and their properties. In [59], the authors show how to extend those concepts to the system (3.1). We state the main results here and encourage the reader to refer to [40, 43, 59] for more details.

Introducing the change of variables $x = \rho\cos\theta$, $y = \rho\sin\theta$ and $z = \rho\omega$, we can describe the solution curves of (3.1) by the following equations:

$$\begin{aligned} \frac{d\rho}{d\theta} &= \frac{P\cos\theta + Q\sin\theta}{1+\rho[\tilde{Q}\cos\theta - \tilde{P}\sin\theta]}, \\ \frac{d\omega}{d\theta} &= \frac{-\mu\omega + \rho[\tilde{R} - \omega(\tilde{P}\cos\theta + \tilde{Q}\sin\theta)]}{1+\rho[\tilde{Q}\cos\theta - \tilde{P}\sin\theta]}. \end{aligned} \tag{3.2}$$

For each (ρ_0, ω_0) with sufficiently small $\|(\rho_0, \omega_0)\|$, let

$$\varphi(\theta, \rho_0, \omega_0) = (\rho(\theta, \rho_0, \omega_0), \omega(\theta, \rho_0, \omega_0)),$$

be the solution of system (3.2) with initial conditions $(\rho(0, \rho_0, \omega_0), \omega(0, \rho_0, \omega_0)) = (\rho_0, \omega_0)$.

Definition 3.1. For system (3.1), the map

$$d(\rho_0, \omega_0) = \varphi(2\pi, \rho_0, \omega_0) - (\rho_0, \omega_0)$$

is called *displacement map.*

Although the displacement map $d(\rho_0, \omega_0) = (d_1(\rho_0, \omega_0), d_2(\rho_0, \omega_0))$ is a two-dimensional map, there exists a unique analytical function $\tilde{\omega}(\rho_0)$ defined in a neighborhood V of $\rho_0 = 0$ such that $d_2(\rho_0, \tilde{\omega}(\rho_0)) \equiv 0$ (see [43] for a proof). The function $\mathbf{d}(\rho_0) = d_1(\rho_0, \tilde{\omega}(\rho_0))$ is analytical in V and is called *reduced displacement map*. Expanding it in a power series, we have

$$\mathbf{d}(\rho_0) = v_1\rho_0 + v_2\rho_0^2 + v_3\rho_0^3 + v_4\rho_0^4 + \cdots .$$

The coefficients v_k are called *focal values* and the coefficients $l_k = v_{2k+1}$ are called *Lyapunov coefficients*. The emphasis on the odd indexed focal values is due to the fact that the first nonzero focal value is the coefficient of an odd power of ρ_0 (see [43]).

The isolated zeros of the reduced displacement map correspond to limit cycles of system (3.1). Moreover, the system has a center on a center manifold W^c if and only if all l_k are null. Thus, the reduced displacement map is a powerful tool to study the cyclicity. However, the determination of the focal values is a difficult task.

The study of the cyclicity of Hopf singular points of three-dimensional systems has a strong connection to the problem of distinguishing whether the singular point is either a center or a focus on a center manifold, i.e. the *center problem*. The following result provides one of the most useful tools to solve the center problem for system (3.1). Its proof can be found in [4, 39].

Theorem 3.2. *Consider system* (3.1) *and let W^c be a center manifold. The following statements are equivalent:*

(i) The origin of the system restricted to W^c is a center;

(ii) System (3.1) *admits a local analytical first integral $H(x, y, z)$ such that $j^2H(0) = x^2 + y^2$;*

(iii) System (3.1) *admits a formal first integral $H(x, y, z)$ such that $j^2H(0) = x^2 + y^2$.*

Moreover, if any of the above statements holds, W^c is unique and analytic.

Following the steps of [40, 4, 59], the standard algorithm to study the center problem consists in the construction of a formal series

$$H(x, y, z) = x^2 + y^2 + \sum_{j+k+l\geqslant 3} p_{jkl}x^j y^k z^l$$

with unknown real coefficients p_{jkl} yet to be determined. Let X be the vector field associated with system (3.1). It is always possible to choose p_{jkl} such that

$$XH = \sum_{i\geqslant 2} L_{i-1}(x^2 + y^2)^i. \tag{3.3}$$

This fact was proved in [4]. Furthermore, the coefficients L_i are rational functions on the parameters of system (3.1) whose denominators depend only on λ. Any non-zero L_i is an obstruction for the origin of (3.1) to be a center on the center manifold. In practical terms, to obtain the quantity L_N, we consider a truncated expression up to order $2N + 2$ of the formal series $H(x, y, z)$ given by

$$\bar{H}(x, y, z) = x^2 + y^2 + \sum_{j+k+l=3}^{2N+2} p_{jkl}x^j y^k z^l.$$

Then, for each $m = 3, \ldots, 2N + 2$, we seek to vanish the coefficients of the homogeneous terms of degree m in the power series expansion of the Lie derivative $X\bar{H}$. Starting at $m = 3$, we aim to solve in a recurrent way the linear system of equations in the variables p_{jkl} with $j+k+l=m$ obtained from the condition that all the coefficients of the homogeneous term of degree m vanish. For odd values of m, we can always find a solution for the system. The same is true for even values of $m=2q$, with the exception of the coefficient of the homogeneous binomial $(x^2+y^2)^q$, which does not depend on p_{jkl}. Hence, solving the possible equations, we obtain the following expression.

$$X\bar{H} = \sum_{i\geqslant 2}^{N+1} L_{i-1}(x^2+y^2)^i.$$

The set of parameters of system (3.1) for which all L_i are null, i.e., the origin is a center, is called the *Bautin variety*. The coefficients L_i are the *focal coefficients* for system (3.1). The focal coefficients and the Lyapunov coefficients are related in the following way: given a positive integer $k>2$, we have

$$l_1 = \pi L_1 \quad \text{and} \quad l_i = \pi L_i \bmod \langle L_1, L_2, \dots, L_{i-1}\rangle, \tag{3.4}$$

that is, $l_i = \pi L_i$ when $L_1 = L_2 = \cdots = L_{i-1} = 0$. The proof of this result can be found in [59]. This relationship allows us to obtain the information about the reduced displacement map by computing the focal coefficients. Since the computations involving the expressions given in (3.3) are algebraic, with the help of symbolic mathematics softwares such as Maple and Mathematica, it is possible to compute a large amount of focal coefficients with less computational time.

Remark 3.3. The concepts of *focal coefficients*, *focal values*, and the *Bautin variety* exist for planar systems having an elementary center or a focus and their properties and roles are the same as in the three-dimensional case.

Having the previous tools to obtain the focal coefficients, we state the next two results, which can be found in [58, 56]. The same ideas were first presented in [61], where they were applied in a similar setting.

Theorem 3.4. *Suppose that s is a point on the Bautin variety and that the first k Lyapunov coefficients, $L_1, \dots, L_k$, have independent linear parts (with respect to the expansion of L_i about s). Then s lies on a component of the Bautin variety of codimension at least k and there are bifurcations which produce k limit cycles locally from the center corresponding to the parameter value s. If, furthermore, we know that s lies on a component of the center variety of codimension k, then s is a smooth point of the variety, and the cyclicity of the center for the parameter value s is exactly k. In the latter case, k is also the cyclicity of a generic point on this component of the Bautin variety.*

Theorem 3.5. *Suppose that s is a point in the Bautin variety such that the k first Lyapunov coefficients have independent linear parts (with respect to the expansion of L_i about s) and the next l ones have their linear parts as linear combinations of the linear parts of the k first ones. After a change of variables in the parameter space, if necessary, we can write $L_i = u_i$, for $i=1,\dots,k$ and, assuming $L_0 = L_1 = \cdots = L_k = 0$, the next Lyapunov coefficients are $L_i = h_i(u) + O(|u|^{m+1})$, for $i=k+1,\dots,k+l$, where h_i are homogeneous polynomials of degree $m \geqslant 2$ and $u=(u_{k+1},\dots,u_{k+l})$. If there exists a line η in the parameter space such that for $k<i<k+l$, $h_i(\eta)=0$, the hypersurfaces $h_i=0$ intersect transversally along η , and $h_{k+l}(\eta)\neq 0$, then there are perturbations of the center which produce $k+l$ limit cycles.*

Even though Theorems 3.4 and 3.5 were first stated to deal with planar systems, they also apply to the three-dimensional case since they are results regarding the parameter space and the Lyapunov coefficients. Furthermore, in the statement of the above theorems, one can change *Lyapunov coefficients* for *focal coefficients*, in light of the relationship (3.4) (see also [48, Proposition 34] and [40, Proposition 4.1]).

Theorem 3.5 was first stated in [56] but was revisited in [58] with a new proof and a proposed scheme for its application to the study of the cyclicity problem. Essentially, their proof rely on the study of the algebraic variety $S_L = \{L_i = 0,\ i = 1, \ldots N\}$ which is equivalent to the local surjectivity of the map $L : \mathbb{R}^N \to \mathbb{R}^N$ given by $L(u) = (L_1(u), \ldots, L_N(u))$ near $u = 0$, for $u = (u_1, \ldots, u_N)$. The intuition behind this idea is that we can use the parameters u to manipulate the coefficients $L_i(u)$ such that

$$|L_1| \ll |L_2| \ll \cdots \ll |L_N|, \quad L_k L_{k-1} < 0,$$

and consequentially the displacement function has N positive zeros.

At first, it seems that obtaining the polynomials h_j in the statement of Theorem 3.5 are computationally challenging. However, using the following result, we can obtain the variety S_L using a simplification of the h_j.

Proposition 3.6. *Let $f_i, g_i : \mathbb{R}^N \to \mathbb{R}$, $i = 1, \ldots, k$ and consider $S_f = \{f_i = 0,\ i = 1, \ldots, k\}$, $S_g = \{g_i = 0,\ i = 1, \ldots, k\}$ and the ideals $\langle f_1, \ldots, f_k\rangle$, $\langle g_1, \ldots, g_k\rangle$ in the ring of real functions. If $\langle f_1, \ldots, f_k\rangle = \langle g_1, \ldots, g_k\rangle$, then $S_f = S_g$.*

Proof: Suppose that $\langle f_1, \ldots, f_k\rangle \subset \langle g_1, \ldots, g_k\rangle$. Then, for each $i = 1, \ldots, k$, we have that $f_i = \sum_{m=1}^k A_m g_m$ with $A_m : \mathbb{R}^N \to \mathbb{R}$. Thus, for each $p \in S_g$, we have that $f_i(p) = 0$ which implies that $S_g \subset S_f$. The reverse inclusion is analogous. □

Corollary 3.7. *Given $f_i : \mathbb{R}^N \to \mathbb{R}$, $i = 1, \ldots, k$, if $g_i = f_i + \sum_{j<i} A_j^i f_j$, for $A_j^i : \mathbb{R}^N \to \mathbb{R}$, then $S_g = S_f$.*

Proof: This follows immediatly from the fact that

$$\langle f_1, \ldots, f_k\rangle = \langle f_1, f_2 + A_1^2 f_1, \ldots, f_k + \sum_{m=1}^{k-1} A_m^k f_m\rangle.$$

□

These results allow us to perform a simplification of the polynomials h_j in Theorem 3.5. Suppose that s is a point in the Bautin variety such that Theorem 3.4 holds and let k and l as in the statement of Theorem 3.5. Thus, we can perform a linear change of variables so that we can write

$$L_i(u) = u_1 + O(|u|^2),$$

for $i = 1, \ldots k$, and $u = (u_1, \ldots, u_k, u_{k+1}, \ldots u_{k+l})$ the new parameter variables. The next l Lyapunov constants write as

$$L_i(u) = \sum_{j=1}^{k} a_j^i u_j + O(|u|^2),$$

for $i = k+1, \ldots, k+l$. Now, if we consider $\bar{L}_i(u) = L_i - \sum_{j=1}^k a_j^i L_j$, $i = k+1, \ldots, k+l$, by Corollary 3.7, we have that $S_L = \{L_1(u) = \cdots = L_k(u) = \bar{L}_{k+1}(u) = \cdots = \bar{L}_{k+l}(u) = 0\}$. Now, we can focus our study on the new simplified quantities $\bar{L}_i$. We have

$$\bar{L}_i(u) = h_i(u) + O(|u|^{m+1}),$$

for $i = k+1, \ldots, k+l$, where h_i are homogeneous polynomials of degree $m \geqslant 2$. Now, we can use a similar procedure to simplify our variety once more, by taking

$$\tilde{L}_i(u) = \bar{L}_i(u) - \left(h_i(L_1, \ldots, L_k, u_{k+1}, \ldots, u_{k+l}) - h_i(0, \ldots, 0, u_{k+1}, \ldots, u_{k+l})\right),$$

for $i = k+1, \ldots, k+l$. Again, by Corollary 3.7, we have $S_L = \{L_1(u) = \cdots = L_k(u) = \tilde{L}_{k+1}(u) = \cdots = \tilde{L}_{k+l}(u) = 0\}$. Note that the lower homogeneous terms of the $\tilde{L}_i(u)$ are the polynomials $h_i(0, \ldots, 0, u_{k+1}, \ldots, u_{k+l})$, which means they do not depend on the parameters u_i, $i = 1, \ldots, k$.

Now, by the Implicit Function Theorem, there exists a change of variables

$$(u_1, \ldots, u_k, u_{k+1}, \ldots, u_{k+l}) \mapsto (L_1, \ldots, L_k, u_{k+1}, \ldots, u_{k+l}).$$

Note that this change do not modify the lower homogeneous terms of the simplified quantities $\tilde{L}_i(u)$, $i = k+1, \ldots, k+l$, i.e. the polynomials h_i depending only on $(u_{k+1}, \ldots, u_{k+l})$. These are obtainable computationally and the ones that we will consider to apply Theorem 3.5. For the sake of simplicity, in what follows, we will drop the tildes when referring to those simplified quantities.

The approach we use to study the cyclicity for systems of the form (3.1) in the present chapter is as follows: First, we consider those systems for which the origin is a center on the center manifold. Then, we make the following perturbation of such a system:

$$\begin{aligned}
\dot{x} &= -y + P(x, y, z) + \sum_{j+k+l=2} a_{jkl} x^j y^k z^l, \\
\dot{y} &= x + Q(x, y, z) + \sum_{j+k+l=2} b_{jkl} x^j y^k z^l, \\
\dot{z} &= -\lambda z + R(x, y, z) + \sum_{j+k+l=2} c_{jkl} x^j y^k z^l,
\end{aligned} \tag{3.5}$$

where $\Lambda = (a_{jkl}, b_{jkl}, c_{jkl},\ j+k+l = 2) \in \mathbb{R}^{18}$ is the tuple of perturbation parameters. Thus, $a_{jkl} = b_{jkl} = c_{jkl} = 0$ is a point on the Bautin variety for system (3.5). Let L_i^m be the degree m Taylor polynomial of the focal coefficient L_i about $\Lambda = 0$. We proceed to compute the focal coefficients L_i for the perturbed system. Once a sufficient amount N is computed, we evaluate the matrix $J(L_1, \ldots, L_N) = \left.\frac{\partial(L_1, \ldots, L_N)}{\partial(a_{jkl}, b_{jkl}, c_{jkl})}\right|_{\Lambda=0}$ and its rank r. We use Theorem 3.4 to conclude that at least r limit cycles can bifurcate from $\Lambda = 0$, retrieving a first lower bound for the cyclicity for the center of the considered system by the study of the linear parts of the focal coefficients.

After this first procedure, to obtain more limit cycles using Theorem 3.5, we have to do a suitable change of variables that allows us to write the first r focal coefficients as $L_i^1 = u_i$, $i = 1, \ldots, r$. The next step consists in simplifying the next focal coefficients L_i, $i \geqslant r+1$, vanishing their linear terms, using L_i^1, $i = 1, \ldots, r$, Corollary 3.7 and the previously discussed process. Then, we obtain the higher order terms of L_i and verify if the hypothesis from Theorem 3.5 are satisfied and apply it to obtain more limit cycles. This verification often consists in performing a blow-up change in the parameter space and finding a parameter value Λ^* for which $h_i(\Lambda^*) = 0$ for $i = r+1, \ldots, r+l-1$, $h_{r+l}(\Lambda^*) \neq 0$ and the Jacobian of $h_{r+1}, \ldots, h_{r+l-1}$ with respect to u_i at Λ^* is non zero. This procedure is the same performed in the papers [62, 58] to study the cyclicity.

If there is a noticeable increase in the number of limit cycles, we repeat this process for the next degree of expansion until no new limit cycles can be obtained.

3.2 The cyclicity study for quadratic systems with a Hopf singular point

In this section we will present lower bounds for the cyclicity of most of the known Hopf centers on center manifolds of quadratic three-dimensional systems in the literature.

3.2.1 Rössler system.

In 1976, Rössler proposed the following system:

$$\begin{aligned}\dot{x} &= -y - z,\\ \dot{y} &= x + az,\\ \dot{z} &= b - cz + xz;\end{aligned}$$

which has a chaotic behavior for some values of a, b, c [63]. For parameter values $a = b = 0$, the origin is a Hopf singular point. The change of variables $x = \bar{x} + \frac{c\bar{z}}{c^2+1}$, $y = \bar{y} - \frac{\bar{z}}{c^2+1}$, $z = \bar{z}$ transforms the above system into

$$\begin{aligned}\dot{x} &= -y - \tfrac{cxz}{c^2+1} - \tfrac{c^2z^2}{(c^2+1)^2},\\ \dot{y} &= x + \tfrac{xz}{c^2+1} + \tfrac{cz^2}{(c^2+1)^2},\\ \dot{z} &= -cz + xz + \tfrac{cz^2}{c^2+1},\end{aligned} \tag{3.6}$$

which has a center at the origin on the center manifold $W^{\mathrm{c}} = \{z = 0\}$ for all real values of c. For $c = -1$, we have the following system:

$$\begin{aligned}\dot{x} &= -y + \tfrac{xz}{2} - \tfrac{z^2}{4},\\ \dot{y} &= x + \tfrac{xz}{2} + \tfrac{z^2}{4},\\ \dot{z} &= z + xz - \tfrac{z^2}{2}.\end{aligned}$$

We consider perturbation (3.5) for the above system and then compute the first 11 focal coefficients. The linear terms of the first 3 focal coefficients are given by

$$\begin{aligned}L_1^1 &= -\frac{11c_{020}}{15} - \frac{c_{110}}{5} - \frac{3c_{200}}{5},\\ L_2^1 &= -\frac{4c_{020}}{25} - \frac{c_{110}}{25} - \frac{2c_{200}}{25},\\ L_3^1 &= -\frac{101c_{020}}{5950} - \frac{3c_{110}}{850} - \frac{c_{200}}{170}.\end{aligned}$$

We have that $\operatorname{rank} J(L_1, \ldots, L_{11}) = \operatorname{rank} J(L_1, L_2, L_3) = 3$ and thus, by Theorem 3.4, it is possible to obtain 3 limit cycles. After a suitable coordinate change and simplification process, we can write $L_i = u_i + O(|u|^2)$, for $i = 1, 2, 3$ and $L_j = h_j(u) + O(|u|^3)$, wherein h_j is a quadratic polynomial for $j = 4, 5$. More precisely,

$$\begin{aligned}h_4 &= \frac{16}{109395}\left(b_{110}b_{200} + b_{110}b_{020} - a_{110}a_{200}\right.\\ &\quad \left. + 2a_{200}b_{200} - a_{020}a_{110} - 2a_{020}b_{020}\right),\\ h_5 &= \frac{1053}{13024}h_4.\end{aligned}$$

Applying Theorem 3.5 for $k = 3$ and $l = 1$, we can retrieve one additional limit cycle. Since h_5 is a multiple of h_4, there is no solution Λ^* such that $h_4(\Lambda^*) = 0 \neq h_5(\Lambda^*)$ and no more conclusions can be drawn. Thus, the cyclicity of the Rössler system (3.6) is at least 4.

3.2.2 Lorenz system.

The celebrated Lorenz system, first proposed in 1963 [64], is one of the widely studied three-dimensional systems due to its rich dynamics. We consider the following generalization of the Lorenz system (see [48, Subsection 9.3]):

$$\begin{aligned}\dot{x} &= a(y-x),\\ \dot{y} &= bx + cy - xz,\\ \dot{z} &= dz + xz.\end{aligned} \tag{3.7}$$

For the Lorenz system to have an isolated Hopf singular point at the origin, we must have $a = c$, $ad \neq 0$ and $a(a+b) < 0$ and its Bautin variety is given by the condition $d = -2a$ (see [48, Theorem 37]). After the suitable coordinate changes are applied, system (3.7) becomes

$$\begin{aligned}\dot{x} &= -y - \tfrac{a}{b\sigma}xz + \tfrac{a^2}{b\sigma^2}yz,\\ \dot{y} &= x - \tfrac{1}{b}xz + \tfrac{a}{b\sigma}yz,\\ \dot{z} &= \tfrac{d}{\sigma}z + \tfrac{1}{b}xy - \tfrac{a}{b\sigma}y^2,\end{aligned} \tag{3.8}$$

where $\sigma = \sqrt{-a(a+b)}$. Under the condition $d = -2a$, considering the perturbation (3.5), we compute the focal coefficients for system (3.8) with parameter values $a = -1, b = 5, d = 2$, for which system (3.8) takes the form

$$\begin{aligned}\dot{x} &= -y + \tfrac{1}{10}xz + \tfrac{1}{20}yz,\\ \dot{y} &= x - \tfrac{1}{5}xz - \tfrac{1}{10}yz,\\ \dot{z} &= z + \tfrac{1}{5}xy + \tfrac{1}{10}y^2.\end{aligned}$$

The linear parts of the first two focal coefficients are

$$\begin{aligned}L_1^1 &= -\tfrac{4a_{011}}{75} - \tfrac{13a_{101}}{75} - \tfrac{7b_{011}}{75} - \tfrac{4a_{11}}{75} + \tfrac{c_{0,2,0}}{15} - \tfrac{c_{110}}{30} - \tfrac{c_{200}}{15},\\ L_2^1 &= -\tfrac{47a_{011}}{93750} - \tfrac{461a_{101}}{375000} - \tfrac{479b_{011}}{375000} - \tfrac{19b_{101}}{187500} - \tfrac{c_{002}}{1250} + \tfrac{47c_{0,2,0}}{75000} - \tfrac{47c_{110}}{150000} + \tfrac{7c_{200}}{18750}.\end{aligned}$$

We have that $\text{rank}J(L_1, \ldots, L_{11}) = \text{rank}J(L_1, L_2) = 2$ and by Theorem 3.4 it is possible to obtain 2 limit cycles from the origin. Making the appropriate coordinate change, and the simplication process, we can write $L_1 = u_1 + O(|u|^2)$, $L_2 = u_2 + O(|u|^2)$, $L_3 = h_3 + O(|u|^3)$ and $L_4 = h_4 + O(|u|^3)$, where h_3 and h_4 are quadratic polynomials given by

$$\begin{aligned}h_3 &= -\tfrac{10207}{65625000}a_{002}a_{020} - \tfrac{9193}{892500000}a_{020}^2 + \tfrac{89}{218750}a_{002}^2,\\ h_4 &= \tfrac{1941688577}{261056250000000}a_{002}a_{020} + \tfrac{6012750503}{3550365000000000}a_{020}^2 - \tfrac{309851}{25593750000}a_{002}^2.\end{aligned}$$

The line η given by $a_{002} = \left(\frac{10207}{53400} + \frac{\sqrt{50972356861}}{907800}\right)a_{020}$ is such that $h_3(\eta) = 0$ and

$$h_4(\eta) = \left(\frac{8116604827567907+13056042403\sqrt{50972356861}}{421836617475000000000}\right)a_{020}^2.$$

Using Theorem 3.5, it is possible to obtain two additional limit cycles from the center at the origin. We conclude that the cyclicity of the Lorenz system is at least 4.

3.2.3 Moon–Rand system:

The Moon–Rand [65] system is given by

$$\begin{aligned}\dot{x} &= y,\\ \dot{y} &= -x - xz,\\ \dot{z} &= -\mu z + cx^2 + bxy + ay^2,\end{aligned}$$

and its Bautin variety is $V^c = \{a = 2c - \mu b = 0\}$, see [40, 59]. Assuming the parameter values $\mu = 1$, $b = 2$, and $c = 1$, we obtain the next system.

$$\begin{aligned}\dot{x} &= y,\\ \dot{y} &= -x - xz,\\ \dot{z} &= -z + x^2 + 2xy.\end{aligned}$$

The linear parts of the first two focal coefficients of the perturbation (3.5) for the above system are given by the following expressions.

$$\begin{aligned}L_1^1 &= -2a_{101} - \frac{2}{3}b_{011} + \frac{4}{15}c_{020} + \frac{2}{15}c_{110} - \frac{4}{15}c_{200},\\ L_2^1 &= -\frac{8}{25}c_{002} + \frac{64}{375}c_{020} + \frac{2}{375}c_{110} - \frac{4}{375}c_{200} - \frac{4}{25}a_{011} - \frac{22}{25}a_{101} - \frac{38}{75}b_{011}.\end{aligned}$$

Since $\operatorname{rank} J(L_1, \ldots, L_{11}) = \operatorname{rank} J(L_1, L_2) = 2$, by Theorem 3.4 it is possible to obtain 2 limit cycles from the origin. By applying the appropriate coordinate change to the parameter space, and the simplification process, we can write $L_1 = u_1 + O(|u|^2)$, $L_2 = u_2 + O(|u|^2)$ and $L_3 = h_3 + O(|u|^3)$ and $L_4 = h_4 + O(|u|^3)$, where h_3, h_4 are quadratic polynomials. Their expressions are too cumbersome, so we will set $a_{0,2,0} = a_{002} = a_{110} = a_{200} = b_{002} = b_{0,2,0} = b_{101} = b_{110} = b_{200} = c_{002} = c_{011} = c_{101} = 0$, since these parameters do not play an important role in this particular case. Hence,

$$\begin{aligned}h_3 &= \tfrac{5016}{74375}c_{110}c_{200} - \tfrac{852}{14875}a_{011}c_{110} + \tfrac{2556}{14875}a_{011}c_{020} + \tfrac{1704}{14875}a_{011}c_{200} + \tfrac{7524}{74375}c_{020}c_{110}\\ &\quad - \tfrac{1254}{74375}c_{110}^2 - \tfrac{5016}{74375}c_{200}^2 - \tfrac{11286}{74375}c_{020}^2 - \tfrac{15048}{74375}c_{200}c_{020} + \tfrac{402}{2975}a_{011}^2,\\ h_4 &= \tfrac{616}{541875}c_{110}c_{200} - \tfrac{268}{252875}a_{011}c_{110} + \tfrac{804}{252875}a_{011}c_{020} + \tfrac{536}{252875}a_{011}c_{200} + \tfrac{308}{180625}c_{020}c_{110}\\ &\quad - \tfrac{154}{541875}c_{110}^2 - \tfrac{616}{541875}c_{200}^2 - \tfrac{462}{180625}c_{020}^2 - \tfrac{616}{180625}c_{200}c_{020} + \tfrac{274}{151725}a_{011}^2.\end{aligned}$$

There exists a line η, given by $\{c_{200} = c_{020} = 0,\ c_{110} = \frac{335}{209}a_{011}\}$, such that $L_3(\eta) = 0$ and $L_4(\eta) = -\frac{1472}{2362745}a_{011}^2$. By Theorem 3.5, it is possible to obtain two additional limit cycles from the center at the origin. We conclude that the cyclicity of the Moon–Rand system is at least 4.

Our investigation concerning the above systems is condensed into Table 3.1. In this, and in all of the following tables in this chapter, **"# Lyap."** represents the number of focal coefficients computed for each system; **"H.O."** stands for the highest order of the Taylor development of the focal coefficients; **"Cyc. ord. 1"** denotes the lower bound for the cyclicity obtained by studying the linear parts of the focal coefficients; **"Cyc. ord. 2"** represents how much this lower bound was improved by analyzing the second-order terms; and **"Cyc. H.O."** is the lower bound we obtained at the end of the study for each considered system.

	# Lyap.	H.O.	Cyc. ord. 1	Cyc. ord. 2	Cyc. H.O.
Rössler	11	2	3	+1	4
Lorenz	11	2	2	+2	4
Moon–Rand	11	2	2	+2	4

Tabela 3.1: Local cyclicity for the Rössler, Lorenz and Moon–Rand systems.

3.2.4 Jerk systems

Here, we consider again the Jerk system (2.42):

$$\begin{aligned}\dot{x} &= y,\\ \dot{y} &= z,\\ \dot{z} &= f(x,y,z).\end{aligned}$$

We now search for Hopf singular points for system (2.42) with $f(x,y,z)$ being a polynomial. We know that the singular points for this system are $Q_i = (\alpha_i, 0, 0)$ where α_i are the roots of $f(x,0,0)$ for $i = 1,\ldots,\deg(f)$. We recall the characteristic polynomial of its Jacobian matrix:

$$p(\eta) = -\eta^3 + \eta^2\frac{\partial f}{\partial z} + \eta\frac{\partial f}{\partial y} + \frac{\partial f}{\partial x}.$$

For Q_i to be a Hopf singular point, we must have $\frac{\partial f}{\partial z}(Q_i) = \tau$, $\frac{\partial f}{\partial y}(Q_i) = -\beta^2$ and $\frac{\partial f}{\partial x}(Q_i) = \beta^2\tau$ for some $\beta, \tau \neq 0$.

Translating Q_i to the origin transforms system (2.42) into

$$\begin{aligned}\dot{x} &= y,\\ \dot{y} &= z,\\ \dot{z} &= f(x+\alpha_i,y,z).\end{aligned}$$

Performing the change of variables $x = -\frac{\bar{x}}{\beta^2} + \frac{\bar{z}}{\tau^2}$, $y = \frac{\bar{y}}{\beta} + \frac{\bar{z}}{\tau}$, $z = \bar{x} + \bar{z}$, dropping the bars, and rescaling time, we obtain the canonical form:

$$\begin{aligned}\dot{x} &= -y + \beta F_2(x,y,z),\\ \dot{y} &= x - \tau F_2(x,y,z),\\ \dot{z} &= -\lambda z - \tau\lambda F_2(x,y,z);\end{aligned} \tag{3.9}$$

where the Taylor expansion of F_2 at the origin with respect to x, y, and z has neither linear nor constant terms. We further investigate the origin of system (3.9) since it is the canonical form of any jerk system having a Hopf singular point. An extensive study of the center problem for some families of jerk systems can be found in [5, 6]. We study the cyclicity of the centers given by each of the center conditions found in these papers.

We consider F_2 a quadratic polynomial, that is: $F_2(x,y,z) = a_1x^2 + a_2y^2 + a_3z^2 + a_4xy + a_5xz + a_6yz$ and $\lambda = \beta = -\tau = 1$. The center conditions for those systems, proven by the authors of [6], are:

a) $a_1 = a_2 = a_4 = 0$;

b) $a_1 - a_2 = a_3 = a_5 = a_6 = 0$;

c) $a_1 + a_2 = a_3 = a_5 = a_6 = 0$;

d) $a_1 + a_2 = 2a_2 - a_3 + a_6 = a_3 - a_4 - 2a_5 = 2a_4 + 3a_5 + a_6 = 0;$

e) $2a_1 - a_6 = 2a_2 + a_5 = 2a_3 - a_5 + a_6 = a_4 + a_5 + a_6 = 0;$

f) $a_1 - a_2 = 2a_2 + a_6 = a_4 = a_5 + a_6 = 0;$

g) $2a_1 + a_2 = 2a_2 + a_6 = 4a_3 + 5a_6 = a_4 = 2a_5 - a_6 = 0.$

For case **a**, we take the parameter values $a_3 = 1, a_5 = 2, a_6 = 3$. Then, by perturbing the system with 18 perturbation parameters, we obtain 3 limit cycles using the linear terms of the focal coefficients. For case **b**, we assume $a_2 = 2, a_4 = 4$ and obtain 5 limit cycles with the linear terms of the focal coefficients.

Analyzing case **c**, and assuming again $a_2 = 2, a_4 = 4$, we obtain 6 limit cycles using linear terms. As for case **d**, assuming $a_4 = 1$, and $a_5 = -1/2$, we also obtain 5 limit cycles using only the linear terms of the focal coefficients.

For case **e**, assuming $a_2 = 5, a_3 = -1/2$, and computing the linear terms of the first 11 focal coefficients, we obtain 6 limit cycles.

For case **f**, taking $a_2 = 3/5, a_3 = 1$, we obtain the system

$$\begin{aligned}
\dot{x} &= -y + \tfrac{3}{5}x^2 + \tfrac{6}{5}xz + \tfrac{3}{5}y^2 - \tfrac{6}{5}yz + z^2,\\
\dot{y} &= x + \tfrac{3}{5}x^2 + \tfrac{6}{5}xz + \tfrac{3}{5}y^2 - \tfrac{6}{5}yz + z^2,\\
\dot{z} &= -z + \tfrac{3}{5}x^2 + \tfrac{6}{5}xz + \tfrac{3}{5}y^2 - \tfrac{6}{5}yz + z^2.
\end{aligned}$$

Computing the first eleven focal coefficients up to order 2 for this system, we obtain 7 limit cycles using the linear terms with the perturbation parameters $a_{0,0,2}$, $a_{0,1,1}$, $a_{0,2,0}$, $a_{1,0,1}$, $a_{1,1,0}$, $a_{2,0,0}$, and $b_{0,1,1}$. Making an appropriate change of coordinates, and the simplification process for the focal coefficients L_8 through L_{11}, we apply Theorem 3.5 to obtain one more limit cycle. Therefore, it is possible to obtain at least 8 limit cycles with the focal coefficients up to order 2.

Condition	# Lyap.	H.O.	Cyc. ord. 1	Cyc. ord. 2	Cyc. H.O.
a)	11	1	3	—	3
b)	11	1	5	—	5
c)	11	1	6	—	6
d)	11	1	5	—	5
e)	11	1	6	—	6
f)	11	2	7	+1	8
g)	12	2	8	+4	12

Tabela 3.2: Local cyclicity for each center condition of the quadratic jerk systems.

Finally, among the considered jerk systems, the one with the most cyclicity was the following system:

$$\begin{aligned}
\dot{x} &= -y + 2x^2 + 4xz - 4y^2 + 8yz - 10z^2,\\
\dot{y} &= x + 2x^2 + 4xz - 4y^2 + 8yz - 10z^2,\\
\dot{z} &= -z + 2x^2 + 4xz - 4y^2 + 8yz - 10z^2.
\end{aligned}$$

The origin is a center on the center manifold since the system satisfies the center condition **g** above. To simplify the calculations, we assume $b_{020} = b_{101} = b_{110} = b_{200} = c_{002} = c_{011} = 0$ in the quadratic perturbation (3.5). We have $\operatorname{rank} J(L_1, \ldots, L_{12}) = J(L_1, \ldots, L_8) = 8$ with respect to the perturbation parameters a_{002}, a_{011}, a_{020}, a_{101}, a_{110}, a_{200}, b_{002}, and b_{011}.

We make the following change of variables: $a_{002} = \gamma^2 u_1$, $a_{011} = \gamma^2 u_2$, $a_{020} = \gamma^2 u_3$, $a_{101} = \gamma^2 u_4$, $a_{110} = \gamma^2 u_5$, $a_{200} = \gamma^2 u_6$, $b_{002} = \gamma^2 u_7$, $b_{011} = \gamma^2 u_8$, $c_{020} = u_9\gamma$, $c_{101} = u_{10}\gamma$, $c_{110} = u_{11}\gamma$, and $c_{200} = \gamma$. Considering the second-order terms in the Taylor expansion of the focal coefficients in γ, we obtain eleven equations in eleven variables. We present only the first three of them:

$$\begin{aligned}
\mathcal{L}_1 = {} & \frac{107773}{25992}u_{10} - \frac{37355}{12996}u_{11}^2 - \frac{3869}{4332}u_{10}^2 - \frac{3283}{25992}u_9^2 + \frac{30829}{2888}u_{11} + \frac{7377}{1444}u_9 \\
& + u_1 + \frac{853}{25992} - \frac{110623}{25992}u_{10}u_{11} - \frac{11047}{25992}u_9u_{10} + \frac{385}{25992}u_9u_{11}, \\
\mathcal{L}_2 = {} & \frac{10472438}{81225}u_{11}^2 - \frac{1405238}{81225}u_{10}^2 - \frac{203501}{16245}u_9^2 + \frac{2717813}{27075}u_{10}u_{11} + u_2 \\
& - \frac{72276202}{81225}u_9 + \frac{13572899}{81225}u_{11} + \frac{5841047}{16245}u_{10} + \frac{3562733}{27075}u_9u_{10} + \frac{728603}{16245}u_9u_{11} \\
& - \frac{10443633}{9025}, \\
\mathcal{L}_3 = {} & -\frac{64427829146}{9665775}u_{10}u_{11} + u_3 + \frac{35151434924}{386631}u_9 - \frac{1119816144458}{48328875}u_{11} \\
& - \frac{218875710314}{5369875}u_{10} - \frac{167865339164}{16109625}u_{11}^2 + \frac{27891420988}{9665775}u_{10}^2 + \frac{6274007402}{3221925}u_9^2 \\
& - \frac{693744306994}{48328875}u_9u_{10} - \frac{37253310546}{5369875}u_9u_{11} + \frac{1168498101998}{9665775}.
\end{aligned}$$

Solving the equations $\mathcal{L}_1 = \cdots = \mathcal{L}_{11} = 0$, with respect to the variables u_i, $i = 1, \ldots, 11$, we obtain two solutions. The first solution corresponds to a center. The second solution, denoted by $\hat{u}$, is given by

$$\begin{aligned}
u_1 &\approx -8.623904469 \times 10^1, & u_2 &\approx 1.389124570 \times 10^1, \\
u_3 &\approx -8.330814109, & u_4 &\approx 2.203360998 \times 10^3, \\
u_5 &\approx -2.361040988 \times 10^5, & u_6 &\approx 3.868219078 \times 10^7, \\
u_7 &\approx -9.106418481 \times 10^9, & u_8 &\approx 2.862902520 \times 10^{12}, \\
u_9 &\approx -1.159927815 \times 10^{15}, & u_{10} &\approx 5.898376936 \times 10^{17}, \\
u_{11} &\approx 2.550457940.
\end{aligned}$$

Evaluating $\mathcal{L}_{12}$ at this solution yields

$$\mathcal{L}_{12}(\hat{u}) \approx 1.780215068 \times 10^9.$$

Moreover, the Jacobian determinant of $\mathcal{L}_1, \ldots, \mathcal{L}_{11}$ with respect to u_i, $i = 1, \ldots, 11$, at $\hat{u}$ is approximately equal to $2.870123963 \times 10^{27}$. Thus, the line $\eta = \hat{u}\gamma$ satisfy the hypothesis of Theorem 3.5, from which we conclude that the cyclicity for this system is at least 12.

The exact values of the solution $\hat{u}$, as well as $\mathcal{L}_{12}(\hat{u})$ and the Jacobian determinant of $\mathcal{L}_1, \ldots, \mathcal{L}_{11}$ with respect to u_i, $i = 1, \ldots, 11$, are rational numbers whose fractional representations have denominators and numerators with more than 400 digits each. Thus, we have presented a numerical approximation of these numbers. The results of our investigation on the jerk systems are summed up in Table 3.2.

3.2.5 Giné–Valls systems

In [66], the authors studied the quadratic system

$$\begin{aligned}\dot{x} &= y,\\ \dot{y} &= -x + a_1x^2 + a_2xy + a_3xz + a_4y^2 + a_5yz + a_6z^2,\\ \dot{z} &= -z + c_1x^2 + c_2xy + c_3y^2\end{aligned} \tag{3.10}$$

and found conditions on the parameters $(a_1, a_2, a_3, a_4, a_5, a_6, c_1, c_2, c_3)$ for which the origin is a center on its center manifold. There are 68 conditions overall. However, many of those yield the same system. We have included only the 20 distinct ones, keeping the same notation as in [66]. These conditions are

a.1) $a_1 = a_2 = c_1 = c_2 = c_3 = 0$;

a.2) $a_1 = a_2 = a_5 = c_3 = 2c_1 - c_2 = 0$;

a.3) $a_1 = a_2 = a_3 = a_5 = a_6 = 0$;

b.1) $a_1 = a_3 = a_4 = c_1 = c_2 = c_3 = 0$;

b.3) $a_1 = a_3 = 2a_2a_4 + a_5c_2 = a_4a_5^2 - a_2^2a_6 = 2a_4^2a_5 + a_2a_6c_2 = 4a_4^3 - a_6c_2^2 = 2a_2^3a_6 + a_5^3c_2 = 2c_1 - c2 = c_3 = 0$;

b.5) $a_1 = a_3 = a_4 = a_5 = a_6 = 0$;

c.1) $a_1 = a_4 = c_1 = c_2 = c_3 = 0$;

c.2) $a_1 = a_4 = a_5 = a_6 = 2c_1 - c_2 = c_3 = 0$;

f.1) $a_2 = a_3 = a_5 = a_6 = 0$;

f.2) $a_2 = a_3 = a_5 = 2c_1 - c_2 = c_3 = 0$;

f.3) $a_2 = a_3 = c_1 = c_2 = c_3 = 0$;

g.1) $a_2 = a_4 = a_5 = 2c_1 - c_2 = c_3 = 0$;

g.2) $a_2 = a_4 = c_1 = c_2 = c_3 = 0$;

h.3) $a_2 = a_5 = 2c_1 - c_2 = c_3 = 0$;

i.1) $a_2 = a_6 = c_1 = c_2 = c_3 = 0$;

j.3) $a_3 = a_4 = a_6 = 2a_1a_2 + a_5c_2 = 2c_1 - c_2 = c_3 = 0$;

k.3) $a_3 = a_5 = a_1 + a_4 = a_6 = 0$;

k.4) $a_3 = a_5 = a_1 + a_4 = c_1 = c_2 = c_3 = 0$;

l.4) $a_3 = a_6 = a_1 + a_4 = c_1 = c_2 = c_3 = 0$;

o.3) $a_5 = a_6 = a_1 + a_4 = c_1 = c_2 = c_3 = 0$.

Condition	# Lyap.	H.O.	Cyc. ord. 1	Cyc. ord. 2	Cyc. H.O.
a.1	11	1	4	—	4
a.2	11	1	4	—	4
a.3	11	1	7	—	7
b.1	11	1	3	—	3
b.3	11	3	9	+1	10
b.5	11	1	6	—	6
c.1	11	1	3	—	3
c.2	11	1	5	—	5
f.1	11	3	8	+1	9
f.2	11	3	8	+1	9
f.3	11	1	4	—	4
g.1	11	1	7	—	7
g.2	11	1	3	—	3
h.3	11	3	8	+1	9
i.1	11	1	4	—	4
j.3	11	3	8	+1	10
k.3	11	1	7	—	7
k.4	11	1	2	—	2
l.4	11	1	4	—	4
o.3	11	1	4	—	4

Tabela 3.3: Local cyclicity for each known center condition of the Giné–Valls systems (3.10).

We go through each condition and study their cyclicity, computing the focal coefficients for a perturbation (3.5) of the system (3.10). The study of the cyclicity of the quadratic center at the origin for system (3.10) is comprised into Table 3.3.

Among the known center conditions of system (3.10), the one with the most cyclicity is condition **j.3**. Setting the parameter values at $a_2 = -2$, $a_5 = 7/8$, and $c_1 = 2/3$, we obtain the system

$$\begin{aligned}\dot{x} &= y,\\ \dot{y} &= \tfrac{7}{24}x^2 - 2xy + \tfrac{7}{8}yz - x,\\ \dot{z} &= \tfrac{2}{3}x^2 + \tfrac{4}{3}xy - z.\end{aligned}$$

Computing the focal coefficients up to order 3 shows that $\mathrm{rank} J(L_1, \ldots, L_{11}) = \mathrm{rank} J(L_1, \ldots, L_8) = 8$ with respect to the perturbation parameters a_{002}, a_{011}, a_{020}, a_{101}, a_{110}, a_{200}, b_{011}, and c_{200} is 8. Thus, by Theorem 3.4, we have eight limit cycles.

After an appropriate change of variable and simplification process, we find the second-order terms of the focal coefficients L_9, L_{10}, and L_{11}:

$$\begin{aligned}L_9 &= u_9u_{10} + O(|u|^3),\\ L_{10} &= u_9u_{10} + O(|u|^3),\\ L_{11} &= u_9u_{10} + O(|u|^3),\end{aligned}$$

where $u_9 = 992b_{002} - 147b_{020} + 336b_{101} - 504c_{002} - 147c_{101}$ and $u_{10} = 992b_{002} - 147b_{020} + 336b_{101} - 434c_{002} - 147c_{101}$. Consequently, we can obtain one more limit cycle, adding up to 9 limit cycles. Applying once more an appropriate change of coordinates and simplification process, we compute the terms of order 3 of the focal coefficients. In particular,

$$
\begin{aligned}
L_{10} &= u_{10}^2 u_{11} + O(|u|^4),\\
L_{11} &= u_{10}^2 u_{11} + O(|u|^4),
\end{aligned}
$$

where $u_{11} = 16905b_{020} - 3920b_{101} + 7595c_{011} + 22785c_{020} - 5880c_{101} - 288u_{10}$. Hence, we can obtain one more limit cycle. Therefore, there are perturbations of this system that yield ten limit cycles.

3.2.6 Edneral–Mahdi–Romanovski–Shafer quadratic systems

In paper [4], the Lyapunov method was used for solving the center problem for the following family of quadratic three-dimensional systems having a Hopf singular point:

$$
\begin{aligned}
\dot{x} &= -y + ax^2 + ay^2 + cxz + dyz,\\
\dot{y} &= x + bx^2 + by^2 + exz + fyz,\\
\dot{z} &= -z + Sx^2 + Sy^2 + Txz + Uyz.
\end{aligned}
\tag{3.11}
$$

We go through every component of the Bautin variety of system (3.11) and compute the focal coefficients of the perturbation (3.5).

The five components of the Bautin variety of system (3.11) obtained in [4] are

a) $a = b = c + f = 0$, $S = 1$ with $c = d + e = 0$;

b) $a = b = c+f = 0$, $S = 1$ with $8c+T^2-U^2 = 4(e-d)-T^2-U^2 = 2(e+d)+TU = 0$;

c) $d + e = c = f = 0$, $S = 1$ with $T - 2a = U - 2b = 0$;

d) $d + e = c = f = 0$, $S = 1$ with $d = e = 0$;

e) $S = 0$.

As in the previous cases, we sum up our investigation for system (3.11) in Table 3.4.

Condition	# Lyap.	H.O.	Cyc. ord. 1	Cyc. ord. 2	Cyc. H.O.
a)	11	4	9	+0	9
b)	11	4	9	+0	9
c)	11	4	9	+0	9
d)	11	4	8	+0	8
e)	11	3	5	+2	7

Tabela 3.4: Local cyclicity for each component of the Bautin variety of the Edneral–Mahdi–Romanovski–Shafer systems systems (3.11).

3.2.7 Rigid Systems in $\mathbb{R}^3$

We exhibit a quick overview of rigid systems in $\mathbb{R}^3$, which are studied with more detail in [67].

Definition 3.8 (Rigid systems in $\mathbb{R}^3$)**.** We say that the three-dimensional system (3.1) is rigid when its restriction to a center manifold is rigid, i.e., the restricted system has constant angular speed. Moreover, if the origin is a center on the center manifold, we say that the origin is a rigid center.

In order to avoid restricting system (3.1) to a center manifold, which is not effective computationally, the authors in [67] introduced a subclass of rigid systems.

Definition 3.9. We say that system (3.1) is rigid by cylindrical coordinates when, in cylindrical coordinates $(x, y, z) \to (\rho\cos\theta, \rho\sin\theta, z)$, the transformed system satisfies $\dot{\theta} = 1$.

The orbits of rigid systems by cylindrical coordinates rotate around the z-axis with constant angular speed. The following result classify rigid systems (3.1).

Proposition 3.10 (Corollary 2.5 in [67])**.** *Consider system* (3.1)*. Then,*

(a) it is a rigid system if and only if

$$(xQ(x, y, z) - yP(x, y, z))\,|_{z=h(x,y)} \equiv 0,$$

where $h(x, y)$ is the local expression of a local center manifold;

(b) it is a rigid system by cylindrical coordinates if and only if $xQ(x, y, z) - yP(x, y, z) \equiv 0$.

Consider the following system:

$$\begin{aligned} \dot{x} &= -y + xF_n(x, y, z), \\ \dot{y} &= x + yF_n(x, y, z), \\ \dot{z} &= -\lambda z + R_m(x, y, z), \end{aligned} \tag{3.12}$$

where F_n is a homogeneous polynomial of degree n and R_m is a homogeneous polynomial of degree m. It is easy to see by Proposition 3.10, that it is a rigid system. In [67, Theorems 4.3 and 4.4] the authors obtained center conditions for particular values of m and n. For the quadratic case, i.e. $n = 1$ and $m = 2$, the center conditions are

a) $\{a_{001} = 0\}$;

b) $R_2(x, y, 0) \equiv 0$;

c) $\{a_{100} = a_{010} = 0,\ b_{101} = b_{011} = 0,\ b_{002} = 2a_{001},\ b_{020} = -b_{200}\}$;

d) $\{b_{101} = 2a_{100} + a_{010},\ b_{020} = -b_{200},\ b_{011} = -a_{100} + 2a_{010},\ b_{002} = 2a_{001}\}$;

e) $\{b_{101} = 2a_{100},\ b_{020} = -b_{200},\ b_{011} = 2a_{010},\ b_{002} = 2a_{001},\ (a_{100}^2 - a_{010}^2)(b_{200} - b_{110}) + a_{100}a_{010}(b_{110} + 4b_{200}) = 0\}$;

f) $\Big\{ b_{200} = \dfrac{a_{100}a_{010}(b_{002} - a_{001})}{a_{001}^2}, b_{110} = \dfrac{(a_{100}^2 - a_{010}^2)(a_{001} - b_{002})}{a_{001}^2},$
$b_{101} = \dfrac{b_{002}(a_{100} + a_{010}) - a_{010}a_{001}}{a_{001}}, b_{020} = \dfrac{a_{100}a_{010}(a_{001} - b_{002})}{a_{001}^2},$
$b_{011} = \dfrac{b_{002}(a_{010} - a_{100}) + a_{100}a_{001}}{a_{001}} \Big\}.$

As in the previous sections, we studied the cyclicity for system (3.12) which is condensed into Table 3.5.

Condition	# Lyap.	H.O.	Cyc. ord. 1	Cyc. ord. 2	Cyc. H.O.
a)	12	2	8	+0	8
b)	12	2	5	+0	5
c)	12	2	3	+0	3
d)	13	2	9	+4	13
e)	12	2	10	+0	10
f)	12	2	11	+0	11

Tabela 3.5: Local cyclicity for each center condition of the rigid system (3.12).

The following result gives us the highest lower bound of the cyclicity of Hopf centers for quadratic systems (3.1), as far as we know.

Theorem 3.11. *Consider the rigid system*

$$\begin{aligned}\dot{x} &= -y + xF_1(x,y,z),\\ \dot{y} &= x + yF_1(x,y,z),\\ \dot{z} &= -z + R_2(x,y,z),\end{aligned} \tag{3.13}$$

with $F_1(x,y,z) = x + 2y + 3z$ *and* $R_2(x,y,z) = \frac{2}{3}x^2 + xy + 4xz - \frac{2}{3}y^2 + 3yz + 6z^2$. *The origin is a rigid center on the center manifold and at least 13 limit cycles bifurcate from it under quadratic perturbations.*

Proof: Note that system (3.13) satisfies the center condition (d) and therefore the origin is a center on the center manifold. Considering a quadratic perturbation (3.5) for system (3.13), we compute the first 13 focal coefficients. The rank of $J(L_1,\ldots,L_9)$ is 9 and by Theorem 3.4, it is possible to obtain 9 limit cycles bifurcating from the origin using the perturbation parameters $(a_{jkl}, b_{jkl}, c_{jkl})$ as well as a perturbation of the trace. To simplify the computations, we assign $c_{011} = c_{020} = c_{101} = c_{110} = c_{200} = 0$, since these parameters do not play a role when we apply Theorems 3.4 and 3.5 to obtain a higher estimate for the cyclicity. Due to the size of the expressions, we only exhibit the linear parts of L_1, L_2, L_3:

$$\begin{aligned}
L_1^1 &= \tfrac{22}{45}a_{011} + 4a_{020} - \tfrac{4}{45}a_{101} + \tfrac{2}{3}a_{110} + \tfrac{4}{3}a_{200} + \tfrac{4}{45}b_{011} - \tfrac{2}{3}b_{020}\\
&\quad + \tfrac{22}{45}b_{101} - \tfrac{4}{3}b_{110} - 2b_{200},\\
L_2^1 &= \tfrac{3713}{25}b_{200} + \tfrac{4}{3}c_{002} - \tfrac{2}{9}a_{002} - \tfrac{39937}{1125}a_{011} - \tfrac{7297}{25}a_{020} + \tfrac{6409}{1125}a_{101}\\
&\quad - \tfrac{3389}{75}a_{110} - \tfrac{1463}{15}a_{200} - \tfrac{9409}{1125}b_{011} + \tfrac{3839}{75}b_{020} - \tfrac{37687}{1125}b_{101} + \tfrac{1433}{15}b_{110},\\
L_3^1 &= \tfrac{44084}{1785}a_{002} + \tfrac{28460414}{7875}a_{011} + \tfrac{442099341}{14875}a_{020} - \tfrac{76527541}{133875}a_{101} - \tfrac{5168}{35}c_{002}\\
&\quad + \tfrac{203415874}{44625}a_{110} + \tfrac{444002837}{44625}a_{200} + \tfrac{5707}{1785}b_{002} + \tfrac{116062741}{133875}b_{011}\\
&\quad - \tfrac{233857774}{44625}b_{020} + \tfrac{26669714}{7875}b_{101} - \tfrac{433328537}{44625}b_{110} - \tfrac{225465802}{14875}b_{200}.
\end{aligned}$$

We now turn to Theorem 3.5. After a suitable change of variables in the perturbation parameters $(a_{002}, a_{011}, a_{020}, a_{101}, a_{110}, a_{200}, b_{002}, b_{011}, b_{020})$, we can write

$$L_i^1 = u_i, \quad i = 1, \ldots, 9.$$

Considering the new variables $u = (u_1, \ldots, u_9, b_{101}, b_{110}, b_{200}, c_{002})$, after the simplification process, we find a line η such that $L_{10}^2(\eta) = L_{11}^2(\eta) = L_{12}^2(\eta) = 0$ and $L_{13}^2(\eta) \neq 0$. Since

$$\begin{aligned}
L_{10}^2 &= -\tfrac{42463...00071}{62328...85264} b_{101}b_{200} - \tfrac{40320...92721}{31164...42632} b_{110}b_{101} - \tfrac{65360...49715}{93493...27896} b_{101}c_{002} \\
&\quad -\tfrac{49690...18261}{12119...05468} b_{110}b_{200} - \tfrac{91271...35547}{81806...36909} b_{101}^2 - \tfrac{46335...13063}{36358...16404} b_{110}^2 \\
&\quad +\tfrac{58144...10977}{14543...65616} b_{200}^2 - \tfrac{27934...30905}{34627...15848} c_{002}b_{200} - \tfrac{20443...06675}{13754...68284} c_{002}b_{110}, \\
L_{11}^2 &= \tfrac{20939...26951}{30054...46160} b_{101}b_{200} + \tfrac{19880...86461}{15027...23080} b_{110}b_{101} + \tfrac{64454...32783}{90164...33848} b_{101}c_{002} \\
&\quad +\tfrac{40002...58387}{95411...04640} b_{110}b_{200} + \tfrac{25716...92939}{22541...84620} {b_{101}}^2 + \tfrac{65259...07273}{50091...74360} {b_{110}}^2 \\
&\quad -\tfrac{16384...09829}{40072...94880} {b_{200}}^2 + \tfrac{33058...21677}{40072...59488} c_{002}b_{200} + \tfrac{57083...74365}{37899...60712} c_{002}b_{110}, \\
L_{12}^2 &= -\tfrac{81825...04719}{11696...02500} b_{101}b_{200} - \tfrac{14372...72644}{10818...73125} b_{110}b_{101} - \tfrac{93191...38049}{12982...67750} b_{101}c_{002} \\
&\quad -\tfrac{16194...76093}{38467...60000} b_{110}b_{200} - \tfrac{40196...79299}{35088...07500} {b_{101}}^2 - \tfrac{37739...55031}{28850...95000} {b_{110}}^2 \\
&\quad +\tfrac{94762...70553}{23080...60000} {b_{200}}^2 - \tfrac{19118...35489}{23080...76000} c_{002}b_{200} - \tfrac{65905...19663}{43657...69800} c_{002}b_{110},
\end{aligned}$$

such a line η is given by:

$$\begin{aligned}
b_{101} &= \varepsilon, \\
b_{200} &= \frac{\alpha}{3}\varepsilon, \\
b_{110} &= \tfrac{99478...56893\alpha^2 - 15595...75578\alpha - 61109...90912}{74402...67872\alpha^2 + 13857...14244\alpha - 17563...30264}\varepsilon, \\
c_{002} &= -\tfrac{14326...21024\alpha^4 + 41672...01633\alpha^3 - 54197...40400\alpha^2 - 12687...07756\alpha + 19077...08656}{65412...11040\alpha^3 + 29236...40360\alpha^2 + 16155...29280\alpha - 40172...14560}\varepsilon,
\end{aligned}$$

or, in numerical approximation,

$$\begin{aligned}
b_{101} &= \varepsilon, \\
b_{200} &= 0.8198666763\varepsilon, \\
b_{110} &= 0.2553429989\varepsilon, \\
c_{002} &= -1.912377388\varepsilon,
\end{aligned}$$

and

$$L_{13}^2(\eta) = \tfrac{60274...32675\alpha^2 - 33282...47954\alpha - 87785...71976}{19996...30400}\varepsilon^2 \approx 97.51353566\varepsilon^2,$$

where α is a real root of the cubic polynomial

$$p(x) = 37499\ldots25384x^3 + 97562\ldots80503x^2 - 13970\ldots27210x - 27339\ldots94856.$$

Note that $L_{13}^2(\eta) = \varepsilon^2 g(\alpha)$ com $g(\alpha) \neq 0$. Moreover, since the Jacobian determinant

$$\det \left.\frac{\partial(L_{10}^2, L_{11}^2, L_{12}^2)}{\partial(b_{110}, b_{200}, c_{002})}\right|_{u=\eta} = \tfrac{15316...73503\alpha^2 - 11623...71962\alpha - 78417...92584}{20165...60960}\varepsilon^3 \approx -7115998.546\varepsilon^3,$$

is non-zero, we have that the hypersurfaces $L_i^2 = 0, i = 10, 11, 12$ intersect transversally along η. Therefore, by Theorem 3.5, there are perturbations of the center at the origin which yield 13 limit cycles. □

The tables presented in this chapter establish a chart of the cyclicity of Hopf singular points of quadratic systems. The performed computations suggest that it is very difficult to obtain examples of centers bifurcating more than 10 limit cycles. In the literature, besides the systems we have worked on, the paper [68] presents one more example for which 10 limit cycles can unfold. As in the case of 11 limit cycles, the only known example is presented in [60]. The authors of [60] also conjectured that the maximum number of limit cycles bifurcating from centers of systems (3.1) is 12. However, Theorem 3.11 shows us that this lower bound must be at least 13.

4 The period of limit cycles bifurcating from a persistent polycycle

In the two previous chapters we dealt with problems related to periodic orbits in the neighborhood of monodromic singular points. In this chapter, we change our attention to the independent variable (also referred as time). More precisely, we study the period of limit cycles arising in bifurcations for families of smooth planar vector fields $\{X_\mu\}_{\mu\in\Lambda}$, where $\Lambda \subset \mathbb{R}^N$. From the classification of first-order structurally unstable vector fields (see, for instance [8, 17, 69]), the generic compact isolated bifurcations for one-parameter (i.e., $N = 1$) families of smooth planar vector fields which give rise to periodic orbits for $\mu \to \mu_0$ are: the Andronov-Hopf bifurcation, the bifurcation of a semi-stable periodic orbit, the saddle-node loop and the saddle loop bifurcations. These are referred to as the *elementary bifurcations*. In [7] the authors determined the behavior of the period $\mathscr{T}(\mu)$ of the limit cycle of X_μ arising from an elementary bifurcation as $\mu \to \mu_0$. More precisely, they obtained the principal term of the expression of $\mathscr{T}(\mu)$ which is comprised in the following list:

(i) $\mathscr{T}(\mu) \sim T_0 + T_1(\mu - \mu_0)$ for the Andronov-Hopf bifurcation;

(ii) $\mathscr{T}(\mu) \sim T_0 + T_1\sqrt{|\mu - \mu_0|}$ for the bifurcation from a semi-stable periodic orbit;

(iii) $\mathscr{T}(\mu) \sim T_0/\sqrt{|\mu - \mu_0|}$ for the saddle-node loop bifurcation;

(iv) $\mathscr{T}(\mu) \sim T_0 \log|\mu - \mu_0|$ for the saddle loop bifurcation.

Here T_0 and T_1 are constants with $T_0 \neq 0$ and we use the notation $\mathscr{T}(\mu) \sim a+f(\mu-\mu_0)$ as μ tends to μ_0 meaning that $\lim_{\mu\to\mu_0}(\mathscr{T}(\mu) - a)/f(\mu - \mu_0) = 1$. Accordingly the principal term of the period of the periodic orbit arising from generic elementary bifurcations characterizes the bifurcation. In what concerns to applications, the information concerning $\mathscr{T}(\mu)$ can be useful to estimate parameters associated to a system, for instance, when studying neuron activities in the brain with the aim of determining the synaptic conductance that it receives (see [70]).

When studying the case (iv) above, the authors in [7] (see also [71]) assume the breaking of the saddle loop connection, which is "the first order condition" for the bifurcation of a limit cycle to occur. The present work started as a natural continuation of this previous study by considering the case in which the homoclinic connection remains unbroken and the parameter space is not necessarily one-dimensional. That being said, the tools and techniques applied in our approach enabled us to extend the results obtained for the saddle loop case to more general situations, namely to any hyperbolic polycycle that is persistent (i.e. such that none of the separatrix connections is broken along the perturbation). For a persistent polycycle Γ, we have an associated return map defined from a fixed transversal section Σ, parametrized smoothly by $\sigma(s;\mu)$, which writes as

$$\mathscr{R}(s;\mu) = s^{r(\mu)}\left(A(\mu) + R(s;\mu)\right), \tag{4.1}$$

where $r(\mu)$ is the product of the hyperbolicity ratios of all saddles in Γ (also called the *graphic number* [72]) and $R(s;\mu)$ is a well-behaved remainder. The expression in (4.1) can be seen as a very particular case of the general results by Mourtada (see [73, 74, 75]). However we shall work with a remainder with better flat properties. It is well known (see [76]) that the stability of Γ is determined by the graphic number in case that $r(\mu_0) \neq 1$, which is the case considered in [7]. There are however other situations that may occur. The list, from the most to the least generic case, is the following:

(a) A saddle connection breaks and $r(\mu_0) \neq 1$;

(b) A saddle connection breaks and $r(\mu_0) = 1$;

(c) No saddle connection breaks and $r(\mu_0) \neq 1$;

(d) No saddle connection breaks and $r(\mu_0) = 1$.

As we already said, case (a) is treated in [7], whereas there is no bifurcation of limit cycle in case (c), see Remark 4.15. In the next sections, we will address case (d).

4.1 Definitions and preliminary results

We now define precisely the notions of persistent polycycle and the cyclicity of a polycycle.

Definition 4.1. Let X be a two-dimensional vector field. A *graphic* Γ for X is a compact non-empty invariant subset which is a continuous image of $\mathbb{S}^1$ and consists of a finite number of (not necessarily distinct) isolated singular points $\{p_1,\ldots,p_m,p_{m+1}=p_1\}$ and compatibly oriented separatrices $\{s_1,\ldots,s_m\}$ connecting them (meaning that s_i has $\{p_i\}$ as the α-limit set and $\{p_{i+1}\}$ as the ω-limit set). A graphic for which all its singular points are hyperbolic saddles is said to be *hyperbolic*. A *polycycle* is a graphic with a return map defined on one of its sides.

Definition 4.2 (Persistent polycycle)**.** Let $\{X_\mu\}_{\mu\in\Lambda}$ be a smooth family of planar vector fields such that Γ is a hyperbolic polycycle of X_{μ_0}. We say that Γ is a *persistent polycycle* when all of its separatrix connections remain unbroken inside the family $\{X_\mu\}_{\mu\in\Lambda}$.

Definition 4.3. Let $\{X_\mu\}_{\mu\in\Lambda}$ be a family of vector fields on $\mathbb{S}^2$ and suppose that Γ is a polycycle for X_{μ_0}. We say that Γ has finite cyclicity in the family $\{X_\mu\}_{\mu\in\Lambda}$ if there exist $\kappa\in\mathbb{N}$, $\varepsilon>0$ and $\delta>0$ such that any X_μ with $|\mu-\mu_0|<\delta$ has at most κ limit cycles γ_i with $\mathrm{dist_H}(\Gamma,\gamma_\mathrm{i})<\varepsilon$ for $i=1,\ldots,\kappa$. The minimum of such κ when δ and ε go to zero is called *cyclicity* of Γ in $\{X_\mu\}_{\mu\in\Lambda}$ at $\mu=\mu_0$ and denoted by $\mathrm{Cycl}(\Gamma,\mu_0)$.

We now introduce the notion of finitely flat functions that play a substantial role in the results of this chapter.

Definition 4.4. Consider $K \in \mathbb{N} \cup \{\infty\}$ and an open set $U \subset \mathbb{R}^N$. We say that a function $\psi(s;\mu)$ belongs to class $C^K_{s>0}(U)$, if there exists an open neighborhood Ω of $0 \times U$ in $\mathbb{R}^{N+1}$ such that $(s;\mu) \mapsto \psi(s,\mu)$ is C^K on $\Omega \cap ((0,+\infty) \times U)$.

Definition 4.5 (Finitely flat functions)**.** Consider $K \in \mathbb{N} \cup \{\infty\}$ and an open set $U \subset \mathbb{R}^N$. Given $L \in \mathbb{R}$ and $\mu_0 \in U$, we say that $\psi(s;\mu) \in C^K_{s>0}(U)$ is (L,K)-flat with respect to s at μ_0, and we write $\psi \in \mathcal{F}^K_L(\mu_0)$, if for each $\nu = (\nu_0, \ldots, \nu_N) \in \mathbb{N}^{N+1}$ with $|\nu| \leqslant K$, there exist a neighborhood V of μ_0 and C, $s_0 > 0$ such that

$$\partial_\nu \psi(s;\mu) := \left| \frac{\partial^{|\nu|} \psi(s;\mu)}{\partial s^{\nu_0} \partial \mu_1^{\nu_1} \cdots \partial \mu_N^{\nu_N}} \right| \leqslant C s^{L-\nu_0} \text{ for all } s \in (0, s_0) \text{ and } \mu \in V.$$

If W is a (not necessarily open) subset of U, then $\mathcal{F}^\infty_L(W) = \bigcap_{\mu_0 \in W} \mathcal{F}^\infty_L(\mu_0)$.

The next two results were proven in [77] and will be very useful to work with finitely flat functions.

Lemma 4.6. *Let U and U' be open sets of $\mathbb{R}^N$ and $\mathbb{R}^{N'}$ respectively and consider $W \subset U$ and $W' \subset U'$. Then, the following holds:*

(a) $\mathcal{F}^K_L(W) \subset \mathcal{F}^K_L(\hat{W})$ *for any* $\hat{W} \subset W$*;*

(b) $\mathcal{F}^K_L(W) \subset \mathcal{F}^K_L(W \times W')$*;*

(c) $C^K(U) \subset \mathcal{F}^K_0(W)$*;*

(d) If $K \geqslant K'$ *and* $L \geqslant L'$ *then* $\mathcal{F}^K_L(W) \subset \mathcal{F}^{K'}_{L'}(W)$*;*

(e) $\mathcal{F}^K_L(W)$ *is closed under addition;*

(f) If $f \in \mathcal{F}^K_L(W)$ *and* $\nu \in \mathbb{N}^{N+1}$ *with* $|\nu| \leqslant K$ *then* $\partial_\nu f \in \mathcal{F}^{K-|\nu|}_{L-\nu_0}(W)$*;*

(g) $\mathcal{F}^K_L(W) \cdot \mathcal{F}^{K'}_{L'}(W) \subset \mathcal{F}^K_{L+L'}(W)$*;*

(h) Assume that $\phi : U' \to U$ *is a* C^K *function with* $\phi(W') \subset W$ *and let us take* $g \in \mathcal{F}^K_{L'}(W')$ *with* $L' > 0$ *and verifying* $g(s;\eta) > 0$ *for all* $\eta \in W'$ *and* $s > 0$ *small enough. Consider also any* $f \in \mathcal{F}^K_L(W)$*. Then* $h(s;\eta) := f(g(s;\eta);\phi(\eta))$ *is a well-defined function that belongs to* $\mathcal{F}^K_{LL'}(W')$*.*

Lemma 4.7. *Let U be an open set of $\mathbb{R}^N$, $K \in \mathbb{N}$ and $g(s;\mu) \in \mathscr{C}^K_{s>0}(U)$ such that, for some $W \subset U$ and $L \in \mathbb{R}$, $g(s;\mu) \in \mathcal{F}^K_L(W)$. If $L > K$, then g extends to a $\mathscr{C}^K$ function $\tilde{g}$, defined in some open neighborhood of $\{0\} \times W \in \mathbb{R}^{N+1}$, satisfying $\partial_\nu \tilde{g}(0;\mu) = 0$ for all $\mu \in W$ and $\nu \in \mathbb{N}^{N+1}$ with $|\nu| \leqslant K$.*

4.2 Dulac map and time

Since we will deal with persistent hyperbolic polycycles, we will need to work with the Dulac map and Dulac time associated to hyperbolic saddles. In this section we defined these concepts in a particular setting where the expression of these maps is known. For more details, we refer the reader to [77, 78] where the specifics are carried out extensively.

We consider an open set $\Lambda \subset \mathbb{R}^N$ and the family $\{X_\mu\}_{\mu \in \Lambda}$ of vector fields given by:

$$X_\mu := \frac{1}{x^{n_1} y^{n_2}} \left(xP(x,y;\mu)\partial_x + yQ(x,y;\mu)\partial_y \right). \tag{4.2}$$

Here,

- $\mathtt{n} := (n_1, n_2) \in \mathbb{N}^2$;
- $P, Q \in C^\infty(V \times \Lambda)$, for some open set $V \subset \mathbb{R}^2$ containing the origin;
- $P(x, 0, \mu) > 0$ and $Q(0, y, \mu) < 0$, for all $(x, 0), (0, y) \in V$ and $\mu \in \Lambda$. This means that the origin is a hyperbolic saddle of $x^{n_1} y^{n_2} X_\mu$ with the y-axis being the stable manifold and x-axis the unstable manifold;
- $\lambda(\mu) = -\dfrac{Q(0,0;\mu)}{P(0,0;\mu)}$ is the hyperbolic ratio of the saddle.

For $i = 1, 2$, let $\sigma_i : (-\epsilon, \epsilon) \times \Lambda \to \Sigma_i$ be transverse sections of X_μ to the axis such that

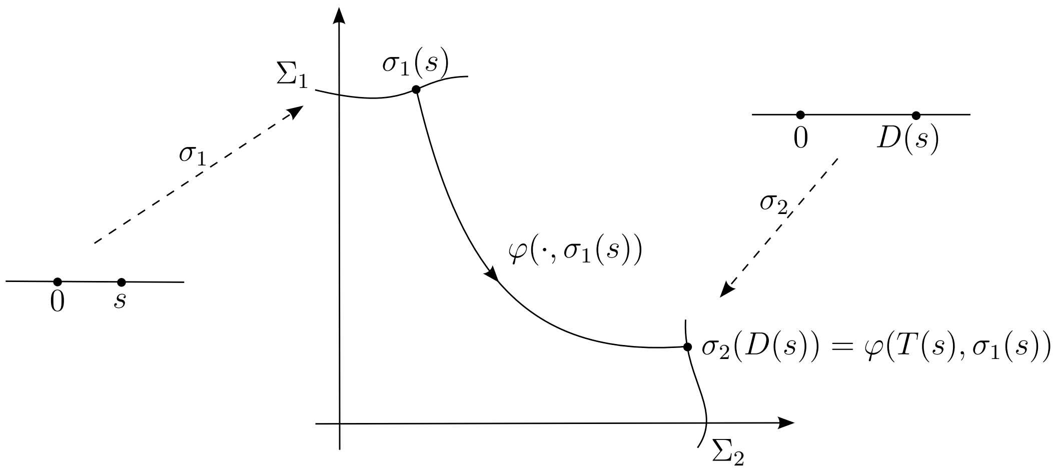


Figura 4.1: The Dulac map and time.

$\sigma_1(0; \mu) \in \{(0, y) : y > 0\}$ and $\sigma_2(0, \mu) \in \{(x, 0) : x > 0\}$ for all $\mu \in \Lambda$. The Dulac map $D(\cdot; \mu)$ and the Dulac time $T(\cdot; \mu)$ are defined by the following relationship:

$$\varphi(T(s;\mu), \sigma_1(s;\mu); \mu) = \sigma_2(D(s;\mu), \mu), \ \forall s \in (0, \epsilon),$$

where $\varphi(t, p_0; \mu)$ is the orbit of X_μ passing through $p_0 \in V$ at $t = 0$ (see Figure 4.1).

The following result gathers Theorems A and B in [78] to give an asymptotic development of the Dulac map and time.

Theorem 4.8. *Let $D(s;\mu)$ and $T(s;\mu)$ be respectively the Dulac map and time of the hyperbolic saddle* (4.2) *from Σ_1 to Σ_2. Then, there exists $\Delta_{00} \in C^\infty(\Lambda)$ such that for any $\lambda_0 > 0$ and $\epsilon > 0$ small enough so $\epsilon < \min\{\lambda_0, 1\}$, for which*

$$D(s;\mu) = s^\lambda (\Delta_{00}(\mu) + \mathcal{F}_\epsilon^\infty(\mu_0)).$$

And for $\mathtt{n} = (n_1, 0)$ or $\mathtt{n} = (0, n_2)$, there exists $T_{00} \in C^\infty(\Lambda)$ such that for any $\lambda_0 > 0$ and $\epsilon > 0$ small enough so $\epsilon < \min\{\lambda_0, 1\}$, we have

$$T(s;\mu) = T_0(\mu) \log s + T_{00}(\mu) + \mathcal{F}_\epsilon^\infty(\mu_0),$$

where $T_0(\mu) = \begin{cases} 0, & \text{if } \mathtt{n} \neq (0,0), \\ \frac{-1}{P(0,0;\mu)}, & \text{if } \mathtt{n} = (0,0). \end{cases}$

Note that the above result applies to hyperbolic saddles for which the separatrices are contained in the orthogonal axis. However, this is not a restrictive assumption since we can rectify the separatrices of hyperbolic saddles via a family C^∞ diffeomorphism [78, Lemma 4.3]. For the sake of the reader, we present the result below.

Lemma 4.9 (Lemma 4.3 in [78]). *Consider a C^∞ family $\{X_\mu\}_{\mu\in\mathbb{R}^N}$ of planar vector fields defined in an open set of $\mathbb{R}^2$. For a fixed $\mu_0 \in \mathbb{R}^N$ and assume that for all μ in a neighborhood of μ_0, X_μ has a hyperbolic saddle p_μ with stable and unstable separatrices S^+_μ and S^-_μ, respectively. Consider two closed connected arcs $l^\pm \subset S^\pm_{\mu_0}$, having both an endpoint at p_{μ_0}. In the case of a homoclinic connection ($S^+_{\mu_0} = S^-_{\mu_0}$), we require additionally that $l^+ \cap l^- = \{p_{\mu_0}\}$. Then, there exists a neighborhood V of $(l^+ \cup l^-) \times \mu_0$ in $\mathbb{R}^2 \times \mathbb{R}^N$ and a C^∞ diffeomorphism $\Phi : V \to \Phi(V) \subset \mathbb{R}^2 \times \mathbb{R}^N$ with $\Phi(x, y, \mu) = (\phi_\mu(x, y), \mu)$ such that $(\phi_\mu)_*(X_\mu) = xP(x, y; \mu)\partial_x + yQ(x, y; \mu)\partial_y$, with $P, Q \in C^\infty(\Phi(V))$.*

4.3 The period of the limit cycle bifurcating from a persistent polycycle

We now present the main results obtained concerning the period of the limit cycle bifurcating from a persistent polycycle. We work with smooth families of planar polynomial vector fields $\{X_\mu\}_{\mu\in\Lambda}$, where Λ is an open subset of $\mathbb{R}^N$, for which there is a hyperbolic polycycle Γ that is persistent.

For some families $\{X_\mu\}_{\mu\in\Lambda}$ of polynomial vector fields, we may find an unbounded polycycle Γ. Thus, to investigate the behavior of the trajectories for a given vector field X_μ near infinity, we can consider its Poincaré compactification $p(X_\mu)$ (see [15, §5], for details), which is an analytically equivalent vector field defined on $\mathbb{S}^2$. The points at infinity of $\mathbb{R}^2$ are in bijective correspondence with the points at the equator of $\mathbb{S}^2$, denoted by ℓ_∞. Furthermore the trajectories of $p(X_\mu)$ in $\mathbb{S}^2$ are symmetric with respect to the origin and so it suffices to draw only its flow in the closed northern hemisphere, the so-called Poincaré disc.

For us to have access to the entirety of the previous results, we need to work with a family $\{X_\mu\}_{\mu\in\Lambda}$ of polynomial vector fields such that for $\mu = \mu_0$, Γ is a persistent polycycle for $\{X_\mu\}_{\mu\in\Lambda}$. If $\Gamma \cap \ell_\infty \neq \emptyset$, we assume additionally that the infinite line ℓ_∞ is invariant in $p(X_\mu)$ for all $\mu \in \Lambda$ (see Figure 4.2).

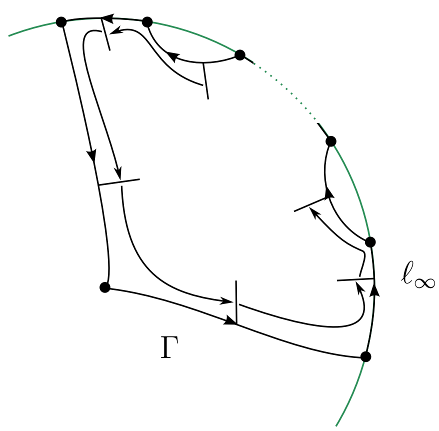


Figura 4.2: A representation of a polycycle that suits our setting.

The return map $\mathscr{R}(\cdot;\mu)$ associated to a hyperbolic polycycle is broadly studied in the literature. Its general expression is $\mathscr{R}(s;\mu)=A(\mu)s^{r(\mu)}+o(s^{r(\mu)})$ (see [72, 76, 79], for instance) where $r(\mu)$ is the product of the hyperbolicity ratios of all the saddles in the polycycle, also referred to as *graphic number*. It is known that if $r(\mu_0)\neq 1$ then the cyclicity of the polycycle at $\mu=\mu_0$ is zero and if $r(\mu_0)=1$ and $A(\mu_0)\neq 1$ then the cyclicity is at most 1. Since we are interested in the case which the cyclicity of the persistent polycycle is exactly 1, we work with the additional assumption that $A(\mu)$ changes sign at $\mu=\mu_0$. In this setting, we have determined the behavior of the period function in a neighborhood of μ_0 when the graphic number $r(\mu_0)=1$. The generic statement of this result is as follows:

Theorem 4.10. *Let $\{X_\mu\}_{\mu\in\Lambda}$ be a smooth family of planar polynomial vector fields with uniformly bounded degree. Let us fix $\mu_0\in\Lambda$ such that X_{μ_0} has a persistent hyperbolic polycycle Γ. If $\Gamma\cap\ell_\infty\neq\emptyset$ we assume additionally that the infinite line ℓ_∞ is invariant for all $\mu\in\Lambda$. Fix a transversal section Σ to Γ parametrized smoothly by $s\mapsto\sigma(s;\mu)$ with $\sigma(0;\mu)\in\Gamma$ for all $\mu\in\Lambda$. Then the following holds:*

(*a*) *The associated return map writes as*

$$\mathscr{R}(s;\mu)=s^{r(\mu)}\left(A(\mu)+\mathcal{F}_\epsilon^\infty(\mu_0)\right),$$

for any $\epsilon>0$ small enough, where $r(\mu)$ is the product of all the hyperbolicity ratios of the saddles in Γ and $A\in C^\infty(\Lambda)$ with $A(\mu_0)>0$;

(*b*) *The associated return time is given by*

$$\mathcal{T}(s;\mu)=\bar{T}_0(\mu)\log s+\mathcal{F}_0^\infty(\mu_0),$$

where $\bar{T}_0\in\mathscr{C}^\infty(\Lambda)$. If at least one of the saddles in Γ is finite then $\bar{T}_0(\mu)<0$.

Moreover, if $r(\mu)-1$ changes sign at μ_0 and $A(\mu_0)\neq 1$ then the following also holds:

(*c*) $\mathrm{Cycl}(\Gamma,\mu_0)=1$ *and the limit cycle γ_μ that bifurcates from the polycycle approaches it at least exponentially fast, i.e., there exists $K>0$ such that*

$$\lim_{\mu\to\mu_0}\frac{s(\mu)}{e^{\frac{-K}{|1-r(\mu)|}}}=0,$$

where $s(\mu)$ is the location of γ_μ at Σ, i.e., the unique positive solution of $\mathscr{R}(s;\mu)=s$;

(*d*) *If at least one of the saddles at Γ is finite then the period $\mathscr{T}(\mu)$ of γ_μ goes to infinity as $\frac{1}{|1-r(\mu)|}$ i.e. $\lim_{\mu\to\mu_0}|1-r(\mu)|\mathscr{T}(\mu)$ is a real positive number.*

In the following subsections, we will prove this result and provide some particular cases to help the reader better understand the generic statement. Theorem 4.10 is our main result of this chapter and it appears in [10].

4.3.1 The machinery

In what follows, we present several technical results which will be applied in the study of limit cycles bifurcating from hyperbolic persistent polycycles. These are essentially the machinery that we will use to obtain our main results.

Lemma 4.11. *Let $f_1(\mu), \ldots, f_m(\mu)$ be C^{k+1} functions on an open subset $\Lambda \subset \mathbb{R}^N$ and $k \in \mathbb{N}$. For each $j = 1, \ldots, m$, let $\partial_\mu^{\leqslant k} f_j(\mu)$ denote the vector whose $\binom{N+k}{k}$ entries are all the partial derivatives of order less or equal to k of $f_j(\mu)$. Then for every $P \in \mathbb{R}[x_1, \ldots, x_{m\binom{N+k}{k}}]$ and each $i \in \{1, \ldots, N+1\}$ there exists $Q_i \in \mathbb{R}[x_1, \ldots, x_{m\binom{N+k+1}{k+1}}]$ such that if $G(\mu) = P(\partial_\mu^{\leqslant k} f_1, \ldots, \partial_\mu^{\leqslant k} f_m)$ then $\partial_{\mu_i} G(\mu) = Q_i(\partial_\mu^{\leqslant k+1} f_1, \ldots, \partial_\mu^{\leqslant k+1} f_m)$.*

The above lemma is a direct consequence of the chain rule and the fact that the ring of polynomials is closed under differentiation.

Lemma 4.12. *For any $L > 0$ and $K \in \mathbb{N}$, the function $s(\alpha; \mu, z) = c(\mu)^{-1/\alpha}(1 + z)$, with $c(\mu)$ being a C^∞ function on an open subset $U \subset \mathbb{R}^N$ and $c(\mu_0) > 1$ belongs to $\mathcal{F}_L^K(\mu_0, 0)$.*

Proof: First we compute $\lim\limits_{\alpha \to 0^+} \alpha^{-L} c(\mu)^{-1/\alpha}$. Since $c(\mu_0) > 1$, there is an open neighborhood of U' containing μ_0 such that $c(\mu) > \delta > 1$ for all $\mu \in U'$. Moreover,

$$\log\left(\frac{c(\mu)^{-1/\alpha}}{\alpha^L}\right) = -\frac{1}{\alpha}\log c(\mu) - L\log\alpha = -\frac{1}{\alpha}\left(\log c(\mu) - L\alpha\log\alpha\right),$$

and then, we have that $\lim\limits_{\alpha \to 0^+} \log\left(\alpha^{-L} c(\mu)^{-1/\alpha}\right) = -\infty$ and therefore $\lim\limits_{\alpha \to 0^+} \alpha^{-L} c(\mu)^{-1/\alpha} = 0$. Since $c(\mu) > \delta > 1$, these limits are uniform and hence, there exist $C > 0$ and a neighborhood V of $(\mu_0, 0)$ such that $|s(\alpha; \mu, z)| \leqslant C\alpha^L$ for α small enough and $(\mu, z) \in V$. Thus, $s(\alpha; \mu, z) \in \mathcal{F}_L^0(\mu_0, 0)$.

Now, to see that $s(\alpha; \mu, z) \in \mathcal{F}_L^1(\mu_0, 0)$, it is sufficient to verify that the first derivatives of $s(\alpha; \mu, z)$ are bounded in a neighborhood of $(\mu_0, 0)$. This is a direct consequence of their expressions, given as follows:

$$\begin{aligned}
\frac{\partial s(\alpha; \mu, z)}{\partial z} &= c(\mu)^{-1/\alpha}, \\
\frac{\partial s(\alpha; \mu, z)}{\partial \alpha} &= c(\mu)^{-1/\alpha}\alpha^{-2}(1 + z)\log c(\mu), \\
\frac{\partial s(\alpha; \mu, z)}{\partial \mu_i} &= -\frac{1}{\alpha}c(\mu)^{-1/\alpha-1}(1 + z)\frac{\partial c}{\partial \mu_i}(\mu) = \\
&= -\frac{1}{\alpha}c(\mu)^{-1/\alpha}(1 + z)\frac{\partial c}{\partial \mu_i}(\mu)c(\mu)^{-1}.
\end{aligned} \tag{4.3}$$

Thus, we have

$$\lim_{\alpha \to 0^+} \alpha^{-L}\left|\frac{\partial s(\alpha; \mu, z)}{\partial\alpha^{\nu_0}\partial\mu_1^{\nu_1}\cdots\partial\mu_N^{\nu_N}\partial z^{\nu_{N+1}}}\right| = 0,$$

uniformly, for $\nu_0 + \cdots + \nu_{N+1} = 1$. Therefore $s(\alpha; \mu, z) \in \mathcal{F}_L^1(\mu_0, 0)$.

We then proceed by induction to prove the following claim:
Claim. For any $\nu \in \mathbb{N}^{N+2}$ with $|\nu| = k \leqslant K$, we have

$$\begin{aligned}
\partial_\nu s(\alpha; \mu, z) = \frac{\partial^{|\nu|} s(\alpha; \mu, z)}{\partial\alpha^{\nu_0}\partial\mu_1^{\nu_1}\cdots\partial\mu_N^{\nu_N}\partial z^{\nu_{N+1}}} &= \\
= \alpha^{-2k}c(\mu)^{-1/\alpha}P_\nu\left(\log c(\mu), \alpha, z, c(\mu)^{-1}, \partial_\mu^{\leqslant k}c(\mu)\right)&,
\end{aligned}$$

where P_ν is a polynomial with real coefficients and $\partial_\mu^{\leqslant k}c(\mu)$ denotes a vector with all the partial derivatives of $c(\mu)$ of order less or equal to k on μ.

For $k=1$, by equation (4.3), this is true. Now, assume it holds for $k=\tilde{k}\geqslant 1$. For $\nu\in\mathbb{N}^{N+2}$ with $|\nu|=\tilde{k}+1$, we must consider one of the following cases:

1. $\nu_{N+1}\geqslant 2$. In this case, by Schwarz Theorem, $\partial_\nu s(\alpha;\mu,z)=0$ and the claim holds.

2. $\nu_0\geqslant 1$. In this case, $\partial_\nu s(\alpha;\mu,z)=\partial_\alpha\partial_{\nu-e_0}s(\alpha;\mu,z)$, where $e_0=(1,0,\dots,0)\in\mathbb{N}^{N+2}$. Thus, since $|\nu-e_0|=\tilde{k}$, we have

$$\begin{aligned}\partial_\alpha\partial_{\nu-e_0}s(\alpha;\mu,z)=&\partial_\alpha\left(\alpha^{-2\tilde{k}}c(\mu)^{-1/\alpha}P_{\nu-e_0}\left(\log c(\mu),\alpha,z,c^{-1}(\mu),\partial_\mu^{\leqslant\tilde{k}}c(\mu)\right)\right)=\\ =&-2\tilde{k}\alpha^{-2\tilde{k}-1}c(\mu)^{-1/\alpha}P_{\nu-e_0}+\alpha^{-2\tilde{k}-2}c(\mu)^{-1/\alpha}\log c(\mu)P_{\nu-e_0}\\ &+\alpha^{-2\tilde{k}}c(\mu)^{-1/\alpha}\partial_{x_2}P_{\nu-e_0}=\\ =&\alpha^{-2\tilde{k}-2}c(\mu)^{-1/\alpha}\left(-2\tilde{k}\alpha P_{\nu-e_0}+\log c(\mu)P_{\nu-e_0}+\alpha^2\partial_{x_2}P_{\nu-e_0}\right).\end{aligned}$$

Then, the claim holds.

3. $\nu_i\geqslant 1$ for some $0<i<N+1$. In this case, $\partial_\nu s(\alpha;\mu,z)=\partial_{\mu_i}\partial_{\nu-e_i}s(\alpha;\mu,z)$, where $e_i\in\mathbb{N}^{N+2}$ has 1 at the $(i+1)$th position and zeros elsewhere. As in the previous case, since $|\nu-e_i|=\tilde{k}$, we have

$$\begin{aligned}\partial_{\mu_i}\partial_{\nu-e_i}s(\alpha;\mu,z)&=\partial_{\mu_i}\left(\alpha^{-2\tilde{k}}c(\mu)^{-1/\alpha}P_{\nu-e_i}\left(\log c(\mu),\alpha,z,c^{-1}(\mu),\partial_\mu^{\leqslant\tilde{k}}c(\mu)\right)\right)=\\ =\alpha^{-2\tilde{k}}&\left(-\frac{1}{\alpha}c(\mu)^{-1/\alpha-1}\frac{\partial c}{\partial\mu_i}(\mu)P_{\nu-e_i}+c(\mu)^{-1/\alpha}\partial_{\mu_i}P_{\nu-e_i}\left(\log c(\mu),\alpha,z,c^{-1}(\mu),\partial_\mu^{\leqslant\tilde{k}}c(\mu)\right)\right)=\\ =\alpha^{-2\tilde{k}-2}&c(\mu)^{-1/\alpha}\left(-\alpha c(\mu)^{-1}\frac{\partial c}{\partial\mu_i}(\mu)P_{\nu-e_i}+\alpha^2\partial_{\mu_i}P_{\nu-e_i}\left(\log c(\mu),\alpha,z,c^{-1}(\mu),\partial_\mu^{\leqslant\tilde{k}}c(\mu)\right)\right).\end{aligned}$$

By Lemma 4.11, the claims hold true.

Now, since for any $\nu\in\mathbb{N}^{N+2}$ with $|\nu|=k\leqslant K$, we have

$$\partial_\nu s(\alpha;\mu,z)=\alpha^{-2k}c(\mu)^{-1/\alpha}P_\nu\left(\log c(\mu),\alpha,z,c^{-1}(\mu),\partial_\mu^{\leqslant k}c(\mu)\right).$$

Since $\lim_{\alpha\to0^+}\alpha^{-L}c(\mu)^{-1/\alpha}=0$ for any $L>0$ and P_ν is a real polynomial and thus is limited for $(\alpha;\mu,z)$ in a neighborhood of $(0;\mu_0,0)$ then $\lim_{\alpha\to0^+}\alpha^{-L}|\partial_\nu s(\alpha;\mu,z)|=0$ uniformly and thus all the derivatives of order less or equal to K are bounded, which implies that $s(\alpha;\mu,z)\in\mathcal{F}_L^K(\mu_0,0)$. □

Lemma 4.13. *Let $h\in\mathscr{C}^1(0,\delta)$, for $\delta>0$, such that $h(\alpha)>0$ for $\alpha>0$,*

$$\lim_{\alpha\to0^+}h(\alpha)=\lim_{\alpha\to0^+}\alpha h'(\alpha)=\lim_{\alpha\to0^+}\frac{\alpha\log\alpha}{h(\alpha)}=0.$$

Then, for any $L>0$, $s(\alpha;z)=(1+h(\alpha))^{-1/\alpha}(1+z)\in\mathcal{F}_L^1(0)$.

Proof: We have that

$$\begin{aligned}\log\left(\frac{s(\alpha;z)}{\alpha^L}\right)&=\log\left(\frac{(1+h(\alpha))^{-1/\alpha}(1+z)}{\alpha^L}\right)=\\ &=-\frac{1}{\alpha}\log(1+h(\alpha))-L\log\alpha+\log(1+z).\end{aligned}$$

The hypothesis $\lim_{\alpha\to0^+}\frac{\alpha\log\alpha}{h(\alpha)}=0$ implies that $\lim_{\alpha\to0^+}\frac{h(\alpha)}{\alpha}=+\infty$. Thus,

$$\lim_{\alpha\to0^+}\log(\alpha^{-L}s(\alpha;z))=\lim_{\alpha\to0^+}-\frac{h(\alpha)}{\alpha}\left(1-L\frac{\alpha\log\alpha}{h(\alpha)}\right)+\log(1+z)=-\infty,$$

which implies that $\lim_{\alpha\to 0^+} \alpha^{-L}s(\alpha;z) = 0$ and therefore $s(\alpha;z) \in \mathcal{F}^0_L(\mathbb{R}) \subset \mathcal{F}^0_L(0)$. Now taking the first derivatives of $s(\alpha;z)$ we obtain:

$$\begin{aligned}
\partial_z s(\alpha;z) &= (1+h(\alpha))^{-1/\alpha} = \frac{1}{1+z}s(\alpha;z)\\
\partial_\alpha s(\alpha;z) &= \frac{s(\alpha;z)}{\alpha^2}\left(\log(1+h(\alpha)) - \alpha\frac{h'(\alpha)}{1+h(\alpha)}\right),
\end{aligned}$$

since $\lim_{\alpha\to 0^+} \alpha^{-L}s(\alpha;z) = 0$ and, by hypothesis, $\lim_{\alpha\to 0^+} \alpha h'(\alpha) = 0$, we have that

$$\lim_{\alpha\to 0^+} \alpha^{-L}\partial_\alpha s(\alpha;z) = \lim_{\alpha\to 0^+} \alpha^{-L}\partial_z s(\alpha;z) = 0.$$

Hence, $s(\alpha;z) \in \mathcal{F}^1_L(\mathbb{R}) \subset \mathcal{F}^1_L(0)$. □

Lemma 4.14. *Let μ_0 be a point in an open subset $\Lambda \subset \mathbb{R}^N$. For a given $k \geqslant 1$, consider the equation*

$$s^{-\alpha} - c(\mu) + f(s;\mu) = 0, \tag{4.4}$$

where $\alpha \in \mathbb{R}\setminus\{0\}$, $f(s;\mu) \in \mathcal{F}^k_\epsilon(\mu_0)$, for some $0 < \epsilon$ and $c(\mu)$ is a C^k function at $\mu = \mu_0$ such that $c(\mu_0) \neq 1$ and $c(\mu_0) > 0$. Then, there exists an open neighborhood $\tilde{U}$ of μ_0, an open real interval I with 0 as an endpoint and a C^k function $z(\alpha;\mu)$ defined in a neighborhood of $(0,\mu_0)$ such that $z(0;\mu_0) = 0$ and

$$s = c(\mu)^{-1/\alpha}(1 + z(\alpha;\mu)),$$

is a solution to equation (4.4) for $(\alpha;\mu) \in I \times \tilde{U}$. More precisely, for $c(\mu_0) > 1$, we have $I = (0,\delta)$ and if $c(\mu_0) < 1$, then $I = (-\delta, 0)$, for a sufficiently small $\delta > 0$.

Proof: We have that $s = c(\mu)^{-1/\alpha}$ is a solution of equation (4.4) disregarding the $\mathcal{F}^\infty_\epsilon(\mu_0)$ terms. It is a well defined function, since $c(\mu_0) > 0$. We set the following ansatz to estimate the solution of (4.4).

$$s(\alpha;\mu,z) = c(\mu)^{-1/\alpha}(1+z).$$

Substituting into (4.4) we obtain:

$$c(\mu)\left((1+z)^{-\alpha} - 1\right) + h(\alpha;\mu,z) = 0,$$

where $h(\alpha;\mu,z) := f(c(\mu)^{-1/\alpha}(1+z);\mu)$. Note that $((1+z)^{-\alpha} - 1) = -\alpha z g(\alpha,z)$ with g being an analytic function at $(0,0)$ with $g(0,0) = 1$. The above equation is equivalent, for $\alpha \neq 0$, to

$$-c(\mu)zg(\alpha,z) + \tfrac{1}{\alpha}h(\alpha;\mu,z) = 0,$$

Define the functions

$$\begin{aligned}
H(\alpha;\mu,z) &= -c(\mu)zg(\alpha,z) + \tfrac{1}{\alpha}h(\alpha;\mu,z),\\
G(\alpha;\mu,z) &= \tfrac{1}{\alpha}h(\alpha;\mu,z).
\end{aligned}$$

Now, since $c(\mu_0) \neq 1$, there is a neighborhood of μ_0 such that $c(\mu) - 1$ maintains its sign. Notice that $s(\alpha;\mu,z)$ has different behaviors depending on the sign of $c(\mu) - 1$. However, the behaviors are similar regarding the flatness of the function as α goes to zero. More precisely, for $c(\mu) > 1$ (and $c(\mu) < 1$), $s(\alpha;\mu,z) \to 0$ exponentially flat as $\alpha \to 0^+$ (respectively $\alpha \to 0^-$).

Suppose that $c(\mu_0) > 1$. The case $c(\mu_0) < 1$ is similar with the only difference is that the interval I in the statement of the theorem will lie in a different semi-axis. By Lemma 4.12, $s(\alpha; \mu, z) \in \mathcal{F}_L^\infty(\mu_0, 0)$ for any $L > 0$. Now, since $f(s;\mu) \in \mathcal{F}_\epsilon^\infty(\mu_0)$, using item (h) in Lemma 4.6, with $W = (\mu_0)$, $W' = (\mu_0, 0)$, $\eta = (\mu, z)$ $\phi(\mu, z) = \mu$, we have that $f(s(\alpha;\mu,z);\mu) \in \mathcal{F}_{\epsilon L}^\infty(\mu_0, 0)$. Now, item (g) in Lemma 4.6 implies that $G(\alpha;\mu,z) \in \mathcal{F}_{\epsilon L-1}^\infty(\mu_0, 0)$.

Given a non negative integer $k \geqslant 1$, choosing $L > (k+1)/\epsilon$, by Lemma 4.7, there exists an extension $\tilde{G}(\alpha;\mu,z)$ of $G(\alpha;\mu,z)$, which is C^k on an open neighborhood of $(0;\mu_0,0)$ such that $\tilde{G}(0;\mu_0,0) = 0$ and $\partial_z \tilde{G}(0;\mu_0,0) = 0$. Consequentially, $H(\alpha;\mu,z)$ admits an extension $\tilde{H}(\alpha;\mu,z)$ which is C^k on the same open neighborhood of $(0;\mu_0,0)$ such that

$$\tilde{H}(0;\mu_0,0) = 0, \quad \partial_z \tilde{H}(0;\mu_0,0) = c(\mu_0) \neq 0.$$

By the Implicit Function Theorem, there exists a unique C^k function $z(\alpha;\mu)$ defined in an open set $(-\delta,\delta) \times \tilde{U}$ containing $(0;\mu_0)$ for which

$$\tilde{H}(\alpha;\mu,z(\alpha,\mu)) \equiv 0, \quad \text{and } z(0;\mu_0) = 0.$$

Since H is the restriction of $\tilde{H}$, $s = s(\alpha;\mu,z(\alpha,\mu))$ is a solution for (4.4) for $(\alpha;\mu)$ in the open set $I \times \tilde{U}$ for $I = (0,\delta)$ where $\delta > 0$ is sufficiently small.

For the case $c(\mu_0) < 1$, it is enough to write $s = \hat{c}(\mu)^{-1/\hat{\alpha}}(1 + z(\hat{\alpha},\mu))$ with $\hat{c} = 1/c$ and $\hat{\alpha} = -\alpha$ and the proof will follow with the same arguments. □

Our goal is to study the zeros of the function

$$\mathscr{D}(s;\mu) = A(\mu) - B(\mu)s^{-\alpha(\mu)} + \mathcal{F}_\epsilon^k(\mu_0),$$

for $\epsilon > 0$, $k \geqslant 1$, with $A(\mu), B(\mu) > 0$, $\alpha(\mu)$ being C^k functions on Λ. For any parameter $\mu_0 \in \Lambda$, we define $Z(\mathscr{D},\mu_0) := \inf_{\delta,\rho>0} N(\delta,\rho)$, where

$$N(\delta,\rho) := \sup_{\mu \in B_\rho(\mu_0)} \#\{\text{isolated zeros of } \mathscr{D}(\cdot;\mu) \text{ in } (0,\delta)\}.$$

These definitions are motivated by the notion of cyclicity. In fact, the isolated zeros of $\mathscr{D}(s;\mu)$ in (4.5) correspond to limit cycles of a vector field near a persistent polycycle and, in that case, $Z(\mathscr{D},\mu_0)$ is precisely the cyclicity of the polycycle at parameter value $\mu = \mu_0$.

Remark 4.15. For $\alpha(\mu_0) \neq 0$, we must have $Z(\mathscr{D},\mu_0) = 0$. In fact, for $\alpha(\mu_0) < 0$, we have:

$$\lim_{(s,\mu)\to(0,\mu_0)} \mathscr{D}(s;\mu) = A(\mu_0) > 0.$$

And for $\alpha(\mu_0) > 0$,

$$\lim_{(s,\mu)\to(0,\mu_0)} s^\alpha \mathscr{D}(s;\mu) = -B(\mu_0) < 0.$$

In both cases, there exist an open neighborhood U of μ_0 and $\varepsilon > 0$ small enough such that $\mathscr{D}(s;\mu) \neq 0$ for $(s;\mu) \in (0,\varepsilon) \times U$. This implies that no limit cycle bifurcate from a persistent polycycle with graphic number different from one.

Thus, for zeros of $\mathscr{D}(\cdot;\mu)$ to exist near $s = 0$ for a small neighborhood of μ_0, we must have $\alpha(\mu_0) = 0$. For the proof of the next theorem, we need to define the *Écalle–Roussarie compensator.*

Definition 4.16. The function defined for $s > 0$ and $\alpha \in \mathbb{R}$ by means of

$$\omega(s;\alpha) = \begin{cases} \frac{s^{-\alpha}-1}{\alpha} & \text{if } \alpha \neq 0, \\ -\log s & \text{if } \alpha = 0, \end{cases}$$

is called *Écalle–Roussarie compensator.*

Remark 4.17. It is known from [78, Lemma A.3] that $\omega(s;\alpha), \frac{1}{\omega(s;\alpha)} \in \mathcal{F}^{\infty}_{-\delta}(0)$ for every $\delta > 0$.

Theorem 4.18. *For a fixed value $\mu_0 \in \Lambda$, consider the function*

$$\mathscr{D}(s;\mu) = A(\mu) - B(\mu)s^{-\alpha(\mu)} + \mathcal{F}^k_\epsilon(\mu_0), \tag{4.5}$$

where $\epsilon > 0$ and $A, B, \alpha \in C^k(\Lambda)$ with $k \in \mathbb{N} \cup \{\infty\}$. Assume that

(i) $\alpha(\mu)$ changes sign at μ_0, in particular $\alpha(\mu_0) = 0$;

(ii) $A(\mu_0)$ and $B(\mu_0)$ are positive and different.

Then $Z(\mathscr{D}, \mu_0) = 1$. Moreover there exist a neighborhood V of μ_0 and $\delta_0 > 0$ such that $W := \{\mu \in V : \alpha(\mu)(A - B)(\mu) > 0\} \neq \emptyset$ and for every $\mu \in W$ there is an unique $s(\mu) \in (0, \delta_0)$ for which $\mathscr{D}(s(\mu);\mu) = 0$. Furthermore, there exists $\hat{z} \in C^k(V)$, with $\hat{z}(\mu_0) = 0$ such that $s(\mu) = \left(\frac{A}{B}(\mu)\right)^{-1/\alpha(\mu)}(1 + \hat{z}(\mu))$ for all $\mu \in W$.

Proof: We first show that $Z(\mathscr{D}, \mu_0) \leqslant 1$, we express the function $\mathscr{D}(s;\mu)$ as follows

$$\mathscr{D}(s;\mu) = (A - B)(\mu) - \alpha(\mu)B(\mu)\omega(s;\alpha(\mu)) + \mathcal{F}^k_\epsilon(\mu_0),$$

and apply the derivation-division algorithm. Let

$$D_1(s;\mu) := \frac{\mathscr{D}(s;\mu)}{\omega(s;\alpha(\mu))} = -\alpha(\mu)B(\mu) + \frac{(A - B)(\mu)}{\omega(s;\alpha(\mu))} + \mathcal{F}^k_{\epsilon-\delta}(\mu_0).$$

Here, we used the fact that $1/\omega(s;\alpha(\mu)) \in \mathcal{F}^{\infty}_{-\delta}(\mu_0)$ for any $\delta > 0$ (Remark 4.17). Then,

$$\partial_s D_1(s;\mu) = \frac{(A - B)(\mu)s^{-\alpha-1}}{\omega^2(s;\alpha(\mu))} + \mathcal{F}^k_{\epsilon-\delta-1}(\mu_0).$$

Since $\omega(s;\alpha), s^\alpha \in \mathcal{F}^k_{-\delta}(\alpha = 0)$, taking $\delta \in (0, \epsilon/4)$, we obtain

$$\lim_{(s,\mu)\to(0,\mu_0)} s^{\alpha+1}\omega^2(s;\alpha(\mu))\partial_s D_1(s;\mu) = \lim_{(s,\mu)\to(0,\mu_0)} (A - B)(\mu) + \mathcal{F}^k_{\epsilon-4\delta}(\mu_0) = (A - B)(\mu_0) \neq 0,$$

which implies, using Rolle's Theorem, that there are small enough neighborhood U of μ_0 and $\varepsilon > 0$ such that $D_1(\cdot;\mu)$ and so $\mathscr{D}(\cdot;\mu)$ have at most one zero for $0 < s < \varepsilon$, i.e. $Z(\mathscr{D}, \mu_0) \leqslant 1$.

Now, to show that $Z(\mathscr{D}, \mu_0) = 1$, we use the fact that these zeros are solutions of the following equation:

$$s^{-\alpha(\mu)} - \frac{A(\mu)}{B(\mu)} + \mathcal{F}^k_\epsilon(\mu_0) = 0. \tag{4.6}$$

Since $(A - B)(\mu_0) \neq 0$, by Lemma 4.14, there is a solution of the form

$$s = \left(\frac{A}{B}(\mu)\right)^{-1/\alpha}(1 + z(\alpha;\mu)),$$

for $(\alpha;\mu)$ in an open set $I \times \tilde{U} \subset \mathbb{R} \times \mathbb{R}^N$ for equation (4.6) if we consider α as a free variable. However, since $\alpha(\mu)$ changes signs at μ_0, there exists a neighborhood V of μ_0 such that $W := \{\mu \in V : \alpha(\mu)(A - B)(\mu) > 0\} \subset \tilde{U} \cap \alpha^{-1}(I)$ is not empty. Hence, for every $\mu \in W$, $s(\mu) = \left(\frac{A}{B}(\mu)\right)^{-1/\alpha(\mu)}(1 + z(\alpha(\mu);\mu))$ is such that $\mathscr{D}(s(\mu);\mu) = 0$. Therefore, $Z(\mathscr{D}, \mu_0) = 1$. $\square$

4.3.2 Period of the limit cycle unfolded from an unbroken saddle loop

Consider a family $\{X_\mu\}_{\mu\in\Lambda}$ of vector fields depending on a bifurcation parameter μ on an open subset Λ of $\mathbb{R}^N$ such that for $\mu_0 \in \Lambda$, X_{μ_0} has a saddle loop Γ for a hyperbolic saddle point p_1. In [7] the authors studied the asymptotic development of the period of the limit cycle unfolding from Γ when this saddle connection is broken. Here, we will have an important distinction: We assume that for $\mu \in \Lambda$ the loop remains unbroken. Our goal is to determine the behavior of the period of the only limit cycle that bifurcates from the loop Γ when its cyclicity at μ_0 is exactly 1.

We now prove Theorem 4.10 for the particular case in which Γ is a (finite) persistent saddle loop.

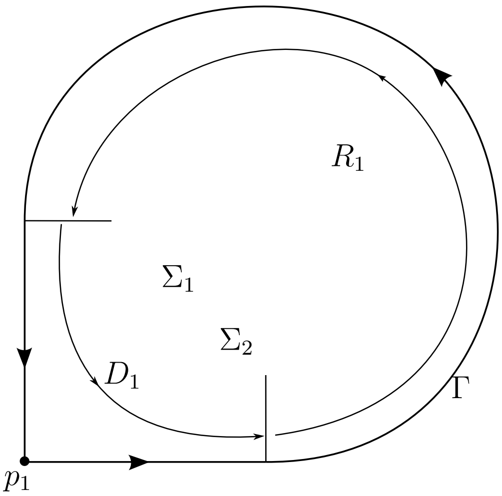


Figura 4.3: The transversal sections and the maps defined around a homoclinic loop Γ of a hyperbolic saddle p_1.

We place two transversal sections Σ_1 and Σ_2 along the homoclinic connection labeled according to the direction of the flow. Now, consider D_1 and T_1 the Dulac map and time from Σ_1 to Σ_2 and R_1 and T_2 the corresponding maps associated to the passage from Σ_1 to Σ_2 using the flow of $-X_\mu$, as depicted in Figure 4.3. More precisely, $\sigma_2(R_1(s;\mu)) = \varphi(-T_2(s;\mu), \sigma_1(s);\mu)$, where σ_i is a parametrization of the transversal section Σ_i, $\varphi(t,q;\mu)$ is the orbit of X_μ passing through $q \in \mathbb{R}^2$ at $t = 0$.

The return map is given by

$$\mathscr{R}(s;\mu) = R_1 \circ D_1(s;\mu).$$

Note that the function $R_1(s;\mu)$ is a diffeomorphism. Since, the saddle loop is unbroken for $\mu \in \Lambda$, we have that $R_1(0;\mu) = 0$. Thus, we obtain that $R_1(s;\mu) = a_1(\mu)s + \mathcal{F}_2^\infty(\mu_0)$, with $a_1(\mu)$ being a strictly positive function.

Now, in order to study the Dulac map and time, we compactify X_μ by a means of projective changes of variables instead of the Poincaré compactification $p(X_\mu)$. Note that there exists a straight line l not intersecting the flow from Σ_1 to Σ_2 for μ near μ_0 and such that $p_1 \notin l$. Up to rotation and translation, we can assume that $l = \{y = 0\}$. Then, the projective change of variables $\{u = \frac{x}{y}, v = \frac{1}{y}\}$ takes the vector field $X_\mu(x,y)$ into the form

$$\hat{X}_\mu(u,v) = \frac{1}{v^{d-1}}\left(\hat{P}(u,v;\mu)\partial_u + v\hat{Q}(u,v;\mu)\partial_v\right),$$

where $d \geqslant 2$ is the degree of X_μ, and $\hat{P}, \hat{Q}$ are polynomials with the coefficients depending smoothly on μ. It is important to remark that X_μ and $\hat{X}_\mu$ are conjugated, which is essential in the study of the Dulac time. The above transformation brings the infinite line to $\{v = 0\}$. Now we can apply Lemma 4.9 to conclude that there exists a neighborhood V of (p_1, μ) and a C^∞ diffeomorphism $\Phi : V \to \Phi(V)$ with $\Phi(x, y, \mu) = (\phi_\mu(x, y), \mu)$ such that $(\phi_\mu)_*(\hat{X}_\mu|_V) = \tilde{X}_\mu$, where

$$\tilde{X}_\mu(u, v) = u\tilde{P}(u, v; \mu)\partial_u + v\tilde{Q}(u, v; \mu)\partial_v \tag{4.7}$$

where $\tilde{P}, \tilde{Q} \in C^\infty(\Phi(V))$. Now, we can apply Theorem 4.8 to obtain that the Dulac map is given by

$$D_1(s; \mu) = s^{\lambda_1} \left(\Delta_{10}(\mu) + \mathcal{F}_\epsilon^\infty(\mu_0)\right) = \Delta_{10}(\mu) s^{\lambda_1} \left(1 + \mathcal{F}_\epsilon^\infty(\mu_0)\right),$$

where λ_1 is the hyperbolicity ratio of p_1, $\Delta_{10} \in \mathscr{C}^\infty(\Lambda)$ and we choose $\epsilon > 0$ small enough so $0 < \epsilon < \min\{\lambda_1(\mu_0), 1\}$. Therefore, the return map is given by

$$\mathscr{R}(s; \mu) = \bar{A}(\mu) s^{\lambda_1} \left(1 + \mathcal{F}_\epsilon^\infty(\mu_0)\right), \tag{4.8}$$

with $\bar{A}(\mu) = a_1(\mu)\Delta_{10}(\mu)$, which proves item (a) in Theorem 4.10 for this particular case. The return time $\mathcal{T}(s; \mu)$ of a point in position s at Σ_1 is:

$$\mathcal{T}(s; \mu) = T_1(s; \mu) + T_2(D_1(s; \mu); \mu).$$

Now $T_2(s; \mu)$ is the regular time from Σ_2 to Σ_1, which is a $\mathscr{C}^\infty$ function in a neighborhood of $(s; \mu) = (0; \mu_0)$. Thus $T_2 \in \mathcal{F}_0^\infty(\mu_0)$. As for the Dulac time $T_1(s; \mu)$, by (4.7), the vector field can be locally expressed as (4.2) with $\mathtt{n} = (0, n_2)$. Therefore, by Theorem 4.8, the associated time map T_i is given by

$$T_1(s; \mu) = T_0^1(\mu) \log s + T_{00}^1(\mu) + \mathcal{F}_\epsilon^\infty(\mu_0) = T_0^1(\mu) \log s + \mathcal{F}_0^\infty(\mu_0),$$

with $T_0^1(\mu) < 0$. Therefore,

$$\mathcal{T}(s; \mu) = T_0^1(\mu) \log s + \mathcal{F}_0^\infty(\mu_0). \tag{4.9}$$

This proves item (b) in Theorem 4.10 for the saddle loop. The limit cycles that bifurcate from the saddle loop Γ correspond to small positive zeros of the proportional displacement map, which by (4.8), is given by:

$$\mathscr{D}(s; \mu) = s^{-\lambda_1} \left(\mathscr{R}(s; \mu) - s\right) = \bar{A}(\mu) - s^{1-\lambda_1} + \mathcal{F}_\epsilon^\infty(\mu_0).$$

By Theorem 4.18, the sufficient conditions for Γ to have cyclicity 1 at μ_0 are $r(\mu) - 1 = \lambda_1(\mu) - 1$ changes sign at μ_0 and $\hat{A}(\mu_0) \neq 1$ and under these conditions, the location of the lone limit cycle is given by

$$s(\mu) = \bar{A}(\mu)^{-1/(r(\mu)-1)}(1 + z(\mu)),$$

for $\mu \in W := \{\mu \in V : (r(\mu) - 1)(\bar{A}(\mu) - 1) > 0\}$, V being a small enough neighborhood of μ_0. Then, choosing $K_0 \in (0, |\log(\bar{A}(\mu_0))|)$ we have

$$\lim_{\substack{\mu \to \mu_0 \\ \mu \in W}} \frac{s(\mu)}{\exp\left(\frac{-K_0}{|1-r(\mu)|}\right)} = \lim_{\substack{\mu \to \mu_0 \\ \mu \in W}} \exp(|\log(\bar{A}(\mu))| - K_0)^{\frac{-1}{|1-r(\mu)|}}(1 + z(\mu)) = 0,$$

which concludes the proof of item (c).

The period $\mathscr{T}(\mu)$ of a limit cycle γ_μ unfolded from Γ at the parameter value μ is given by $\mathscr{T}(\mu) = \mathcal{T}(s(\mu);\mu)$. Thus, substituting $s(\mu)$ into (4.9), we conclude that the asymptotic behavior of the period of the limit cycle γ_μ is

$$\lim_{\substack{\mu\to\mu_0\\ \mu\in W}} (\lambda_1(\mu)-1)\mathscr{T}(\mu) = -T_0^1(\mu_0)|\log(\bar{A}(\mu_0))|,$$

which proves item (d).

Remark 4.19. If $\bar{A}(\mu_0) > 1$, the limit must be taken as $r(\mu)\to 1^+$ and for $\bar{A}(\mu_0) < 1$, $r(\mu)\to 1^-$. This assures we are always at a well defined position $s(\mu)$ for the limit cycle γ_μ as we approach the polycycle Γ.

Remark 4.20. In [7] the authors proved that the period of the limit cycle unfolding from a saddle loop when the connection breaks is given by $\mathscr{T}(\mu) = c\log|\mu| + O(1)$. The above arguments show that the behavior when the loop is persistent is significantly different.

4.3.3 Period of the limit cycle unfolded from a persistent polycycle (Proof of Theorem 4.10)

We now consider the most general case. For our considered persistent hyperbolic polycycle Γ, let $p_1,\ldots,p_m$ be the m hyperbolic saddles that we conveniently label according to the direction of the flow. We denote by $\lambda_i(\mu)$ the hyperbolicity ratio of p_i. We consider the case $m\geqslant 2$ since the case $m=1$ is the saddle loop case which is already explored in Section 4.3.2.

Since the infinite line ℓ_∞ is invariant for the flow of $p(X_\mu)$, for each saddle located at infinity we have that exactly one of its separatrices is contained in ℓ_∞. This also implies that Γ has an even number of singularities at infinity assembled in pairs. For $i=1,\ldots,m$, let Σ_i be the transversal section to the connection from p_{i-1} to p_i (set $p_0=p_m$), and $D_i = D_i(\cdot;\mu)$ and $T_i = T_i(\cdot;\mu)$ the corresponding Dulac maps and times from Σ_i to Σ_{i+1} (see Figure 4.4 for a schematic). It is clear that we can assume that Σ_1 is the transversal section given in the statement.

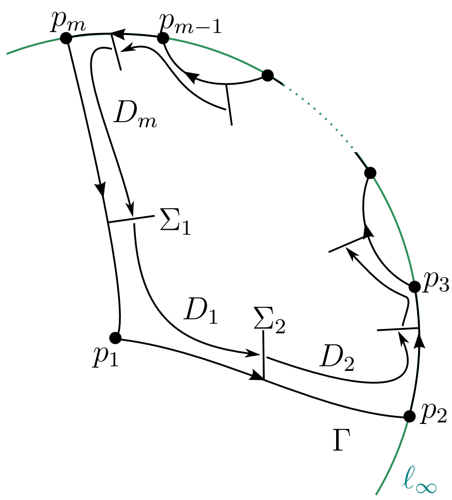


Figura 4.4: A representation of the considered maps and saddles on the polycycle Γ.

By following a point in position s at Σ_1 until its return to the transversal section, we determine the return map given by

$$\mathscr{R}(s;\mu) = D_m \circ D_{m-1} \circ \cdots \circ D_1(s;\mu).$$

Again, in order to study each Dulac map and time, we compactify X_μ by a means of projective changes of variables instead of the Poincaré compactification $p(X_\mu)$. Note that, for each saddle p_i at infinity, there exists a straight line l_i not intersecting the flow from Σ_i to Σ_{i+1} for μ near μ_0 and such that $p_i \notin l_i \cap l_\infty$. For a fixed i, by means of rotation and translation, we can assume that $l_i = \{y = 0\}$. Then, we perform the projective change of variables $\{u = \frac{x}{y}, v = \frac{1}{y}\}$ which transforms $X_\mu(x,y)$ into

$$\hat{X}^i_\mu(u,v) = \frac{1}{v^{d-1}}\left(\hat{P}(u,v;\mu)\partial_u + v\hat{Q}(u,v;\mu)\partial_v\right),$$

where $d \geqslant 2$ is the degree of X_μ, and $\hat{P}, \hat{Q}$ are polynomials with the coefficients depending smoothly on μ. This transformation brings the infinite line to $\{v = 0\}$. This process can be repeated for each saddle at infinity. Now we are ready to work with all saddles in Γ (the finite ones, and the infinite ones after the above procedure). For each of them we can apply Lemma 4.9 to conclude that for each $i = 1, \ldots, m$ there exists a neighborhood V_i of (p_i, μ) and a C^∞ diffeomorphism $\Phi_i : V_i \to \Phi_i(V_i)$ with $\Phi_i(x,y,\mu) = (\phi^i_\mu(x,y), \mu)$ such that $(\phi^i_\mu)_*(\hat{X}^i_\mu|_{V_i}) = \tilde{X}^i_\mu$, where

$$\tilde{X}^i_\mu(u,v) = \frac{1}{v^{\kappa_i}}\left(uP_i(u,v;\mu)\partial_u + vQ_i(u,v;\mu)\partial_v\right) \text{ with } \kappa_i = \begin{cases} d-1, \text{if } p_i \in l_\infty, \\ 0, \text{if } p_i \notin l_\infty, \end{cases} \tag{4.10}$$

where $P_i, Q_i \in C^\infty(\Phi(V_i))$. Now, we can apply Theorem 4.8 to obtain that each Dulac map is given by

$$D_i(s;\mu) = s^{\lambda_i}\left(\Delta_{i0}(\mu) + \mathcal{F}^\infty_\epsilon(\mu_0)\right) = \Delta_{i0}(\mu)s^{\lambda_i}\left(1 + \mathcal{F}^\infty_\epsilon(\mu_0)\right),$$

where $\Delta_{i0} \in \mathscr{C}^\infty(\Lambda)$ and we choose $\epsilon > 0$, small enough so $0 < 2\epsilon < \min\{1, \lambda_1(\mu_0), \ldots, \lambda_m(\mu_0)\}$. This also implies that $D_i \in \mathcal{F}^\infty_{\lambda_i(\mu_0)-\epsilon}(\mu_0) \subset \mathcal{F}^\infty_\epsilon(\mu_0)$. Using the properties listed in Lemma 4.6, we can see that:

$$\begin{aligned} D_2 \circ D_1(s;\mu) &= \Delta_{20}(\mu)\left(\Delta_{10}(\mu)s^{\lambda_1}\left(1 + \mathcal{F}^\infty_\epsilon(\mu_0)\right)\right)^{\lambda_2}\left(1 + \mathcal{F}^\infty_{\epsilon^2}(\mu_0)\right) = \\ &= \Delta_{20}(\mu)\Delta_{10}^{\lambda_2}(\mu)s^{\lambda_1\lambda_2}\left(1 + \mathcal{F}^\infty_\epsilon(\mu_0)\right)^{\lambda_2}\left(1 + \mathcal{F}^\infty_{\epsilon^2}(\mu_0)\right) = \\ &= \Delta_{20}(\mu)\Delta_{10}^{\lambda_2}(\mu)s^{\lambda_1\lambda_2}\left(1 + \mathcal{F}^\infty_\epsilon(\mu_0)\right)\left(1 + \mathcal{F}^\infty_{\epsilon^2}(\mu_0)\right) = \\ &= \Delta_{20}(\mu)\Delta_{10}^{\lambda_2}(\mu)s^{\lambda_1\lambda_2}\left(1 + \mathcal{F}^\infty_{\epsilon^2}(\mu_0)\right), \end{aligned}$$

where in the third equality we use the fact that $(s;\mu) \to (1+s)^{\lambda_i(\mu)} - 1$ belongs to $\mathcal{F}^\infty_1(\mu_0)$. Notice that the composition of two Dulac maps has a Dulac map like expression. Proceeding analogously, we obtain the composition of k consecutive Dulac maps:

$$D_k \circ D_{k-1} \circ \cdots \circ D_1(s;\mu) = A_k(\mu)s^{\lambda_1 \ldots \lambda_k}\left(1 + \mathcal{F}^\infty_{\epsilon^k}(\mu_0)\right), \tag{4.11}$$

where

$$A_k(\mu) = \prod_{i=1}^k \Delta_{i0}^{\Lambda_{ik}}(\mu), \quad \Lambda_{ik} = \prod_{i<j\leqslant k} \lambda_j \text{ and } \Lambda_{kk} = 1.$$

Therefore, the return map is given by

$$\mathscr{R}(s;\mu)=A_m(\mu)s^{\lambda_1\dots\lambda_m}\left(1+\mathcal{F}^{\infty}_{\epsilon^m}(\mu_0)\right), \tag{4.12}$$

which proves item (a) in Theorem 4.10.

The return time of a point in position s at Σ_1 is determined by the following expression:

$$\mathcal{T}(s;\mu)=T_1(s;\mu)+\sum_{i=2}^{m}T_i(D_{i-1}\circ\dots\circ D_1(s;\mu)). \tag{4.13}$$

For each saddle p_i in Γ, by (4.10), the vector field can be locally expressed as (4.2) with $\mathtt{n}=(0,n_2)$. Therefore, by Theorem 4.8, the associated time map T_i is given by

$$T_i(s;\mu)=T_0^i(\mu)\log s+T_{00}^i(\mu)+\mathcal{F}^{\infty}_{\epsilon}(\mu_0)=T_0^i(\mu)\log s+\mathcal{F}^{\infty}_{0}(\mu_0).$$

By (4.11) and Lemma 4.6 for $i\geqslant 2$, we have

$$\begin{aligned}T_i(D_{i-1}\circ\dots\circ D_1(s;\mu)) &= T_0^i(\mu)\Lambda_{0(i-1)}\log s+T_0^i(\mu)\log A_{i-1}(\mu)+\mathcal{F}^{\infty}_{0}(\mu_0)=\\ &= T_0^i(\mu)\Lambda_{0(i-1)}\log s+\mathcal{F}^{\infty}_{0}(\mu_0).\end{aligned}$$

Thus, substituting into (4.13), we obtain

$$\mathcal{T}(s;\mu)=\bar{T}_0(\mu)\log s+\mathcal{F}^{\infty}_{0}(\mu_0), \tag{4.14}$$

for $\bar{T}_0(\mu)=T_0^1(\mu)+\sum_{i=2}^{m}T_0^i(\mu)\Lambda_{0(i-1)}$. Note that, by Theorem 4.8, $T_0^i\leqslant 0$ and the equality holds if and only if p_i is an infinite saddle. If there is at least one finite saddle, then $\bar{T}_0(\mu)$ is a strictly negative function. This proves item (b).

The limit cycles bifurcating from Γ correspond to small positive zeros of the proportional displacement map $\mathscr{D}(s;\mu)=s^{-\lambda_1\dots\lambda_n}(\mathscr{R}(s;\mu)-s)$, which by (4.12) writes as

$$\mathscr{D}(s;\mu)=A_m(\mu)-s^{1-\lambda_1\dots\lambda_m}+\mathcal{F}^{\infty}_{\epsilon^m}(\mu_0). \tag{4.15}$$

Recall that, by assumption, $r(\mu)-1=(\lambda_1\dots\lambda_m)(\mu)-1$ changes sign at μ_0 and $A_m(\mu_0)\neq 1$. Hence, by Theorem 4.18, Γ has cyclicity 1 at μ_0. Furthermore, there exist a neighbourhood V of μ_0 and a smooth function $z\in C^{\infty}(V)$ with $z(\mu_0)=0$ such that the location of the bifurcating limit cycle is given by $s=s(\mu)$ with

$$s(\mu)=A_m(\mu)^{-1/(r(\mu)-1)}\Big(1+z(\mu)\Big), \tag{4.16}$$

for $\mu\in W:=\{\mu\in V:(r(\mu)-1)(A_m(\mu)-1)>0\}$. Then, choosing $K_0\in(0,|\log(A_m(\mu_0))|)$ we have

$$\lim_{\substack{\mu\to\mu_0\\ \mu\in W}}\frac{s(\mu)}{\exp\left(\frac{-K_0}{|1-r(\mu)|}\right)}=\lim_{\substack{\mu\to\mu_0\\ \mu\in W}}\exp(|\log(A_m(\mu))|-K_0)^{\frac{-1}{|1-r(\mu)|}}(1+z(\mu))=0, \tag{4.17}$$

which concludes the proof of (c). It only remains to be proved the assertion (d) about the period of the bifurcating limit cycle. To this end we assume that at least one of the saddles in Γ is finite. The period $\mathscr{T}(\mu)$ of the limit cycle γ_μ bifurcating from Γ is obtained inserting the location of the limit cycle given by (4.16) into the return time (4.14). Hence,

$$\begin{aligned}&\lim_{\substack{\mu\to\mu_0\\ \mu\in W}}|1-r(\mu)|\mathscr{T}(\mu)=\lim_{\substack{\mu\to\mu_0\\ \mu\in W}}|1-r(\mu)|\mathcal{T}(s(\mu);\mu)=\\ &=\lim_{\substack{\mu\to\mu_0\\ \mu\in W}}-\bar{T}_0(\mu)|\log A_m(\mu)|=-\bar{T}_0(\mu_0)|\log A_m(\mu_0)|>0.\end{aligned}$$

□

Theorem 4.10 provides sufficient conditions for the cyclicity of the persistent polycycle to be exactly 1 and also describes how fast the unfolded limit cycle approaches the polycycle. Moreover, the asymptotic behavior of the period of the limit cycle is the same as the function $\frac{1}{|1-r(\mu)|}$ where $r(\mu)$ is the graphic number. This result encompasses a wide range of polycycles, among them the Kolmogorov polycycles studied in [79].

Under the hypothesis of Theorem 4.10, if the parameter space is one-dimensional (i.e., $N = 1$) then the graphic number writes as $r(\mu) = 1 + \frac{1}{T_0}(\mu - \mu_0)^k + O(\mu^{k+1})$ for an odd positive integer k. Thus, by applying Theorem 4.10, the behavior of the period of the limit cycle bifurcating from a persistent polycycle is $\mathscr{T}(\mu) \sim T_0/|\mu - \mu_0|^k$, which is an entirely different behavior from any of the elementary bifurcations considered in [7]. However, it coincides with the behavior of the generalized Andronov-Hopf bifurcation considered in [7, Theorem 7]. Thus, we conclude that one cannot classify bifurcations of limit cycles based uniquely on the behavior of their periods.

In the case which the above conditions on $r(\mu)$ and $A(\mu)$ are not satisfied, although we may have cyclicity exactly 1, the behavior of the period of the bifurcating limit cycle is not determined. In fact, we solve a sort of inverse problem when $r(\mu_0) = A(\mu_0) = 1$ and the gradients of A and r are linearly independent at μ_0. We will detail this discussion in the next section.

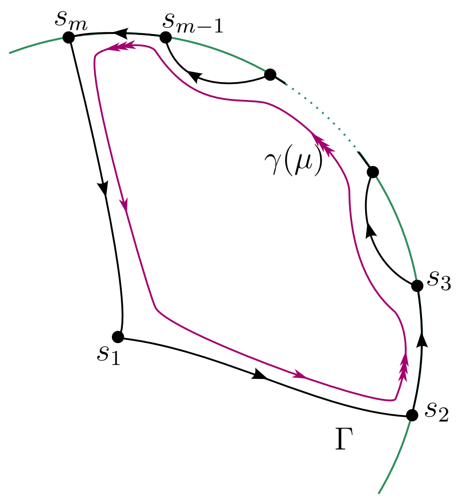


Figura 4.5: A representation of the time spent in the limit cycle $\gamma(\mu)$ near the finite and infinite saddles of a polycycle Γ.

4.4 The undetermined period of the limit cycle unfolded from a center-type polycycle

Up to this point, we studied families of polynomial systems with a persistent polycycle whose cyclicity is 1 at a parameter μ_0 but the return map near the polycycle is not the identity. In those families, by Theorem 4.10, the period of the limit cycle unfolded from the polycycle has a determined behavior. However, that may not be the case when the return map near the polycycle is the identity at μ_0. In this case, we say that the polycycle is of the *center-type*.

In [79, Theorem 3.1, item (c)], the authors exhibit an example of a center-type polycycle. It is present in the following family

$$X_\mu = x(1 + x + x^2 + axy + py^2)\partial_x + y(-1 - y + qy^2 + axy - y^2)\partial_y,$$

where $\mu = (a, p, q) \in \mathbb{R}^3$ with $p < -1$, $q > 1$. The displacement function associated to the polycycle of this Kolmogorov system is, after a change of variables in the parameter space, proportional to the following expression

$$\mathscr{D}(s; \mu) = s^{-\alpha(\mu)} - (1 + \varepsilon(\mu)) + f(s; \mu).$$

Moreover, if $\mu_0 = (0, p_0, -p_0)$ then $\mathscr{D}(s; \mu_0) \equiv 0$, $f(s; \mu) \in \mathcal{F}_\epsilon^\infty(\mu_0)$, $\epsilon > 0$ and $\alpha(\mu), \varepsilon(\mu)$ are such that $\alpha(\mu_0) = \varepsilon(\mu_0) = 0$ and $\operatorname{rank}\{\nabla\alpha, \nabla\varepsilon\}|_{\mu=\mu_0} = 2$.

Note that in this case, we cannot apply Theorem 4.18. However, the cyclicity of the polycycle at μ_0 is 1.

This example motivates the next result, which will focus in families of polynomial vector fields $\{X_\mu\}_{\mu\in\Lambda}$ such that X_{μ_0} has a persistent hyperbolic polycycle Γ with the infinite line ℓ_∞ being invariant for all $\mu \in \Lambda$ and with at least one finite saddle, with the additional assumption that for a fixed transversal section we can write the proportional displacement function near the the polycycle as

$$\mathscr{D}(s; \mu) = s^{-\alpha(\mu)} - (1 + \varepsilon(\mu)) + f(s; \mu),$$

with $f(s; \mu) \in \mathcal{F}_\epsilon^\infty(\mu_0)$, $\epsilon > 0$, with $\alpha(\mu_0) = \varepsilon(\mu_0) = 0$ and $\nabla\alpha, \nabla\varepsilon$ are linearly independent at μ_0. These assumptions encompass the cases for which the polycycle Γ is of the center-type or the cyclicity of the polycycle is greater than 1. The next result is our main result of this section and it appears in [10].

Theorem 4.21. *Let $\{X_\mu\}_{\mu\in\Lambda}$ be a smooth family of planar polynomial vector fields with uniformly bounded degree. Let us fix $\mu_0 \in \Lambda$ such that X_{μ_0} has a persistent hyperbolic polycycle Γ with at least a finite saddle. If $\Gamma \cap \ell_\infty \neq \emptyset$ we assume additionally that the infinite line ℓ_∞ is invariant for all $\mu \in \Lambda$. Fix a transversal section Σ to Γ parametrized smoothly by $s \mapsto \sigma(s; \mu)$ with $\sigma(0; \mu) \in \Gamma$ for all $\mu \in \Lambda$. Then the associated return map writes as*

$$\mathscr{R}(s; \mu) = s^{r(\mu)}\left(A(\mu) + \mathcal{F}_\epsilon^\infty(\mu_0)\right),$$

for any $\epsilon > 0$ small enough, where $r(\mu)$ is the product of all the hyperbolicity ratios of the saddles in Γ and $A \in C^\infty(\Lambda)$ with $A(\mu_0) > 0$. Suppose that $A(\mu_0) = r(\mu_0) = 1$ and $\operatorname{rank}\{\nabla A, \nabla r\}|_{\mu=\mu_0} = 2$. *Then for every function $\tau \in \mathscr{C}^1(0, \delta)$ satisfying $\tau(\alpha) > 0$ for $\alpha > 0$ and*

$$\lim_{\alpha\to 0^+} \alpha\tau(\alpha) = \lim_{\alpha\to 0^+} \alpha^2\tau'(\alpha) = \lim_{\alpha\to 0^+} \frac{\log\alpha}{\tau(\alpha)} = 0,$$

there exists a differentiable arc at μ_0 in the parameter space Λ, i.e., a C^1 map $m : (0, \delta_0) \to \Lambda$ with $\lim_{\alpha\to 0^+} m(\alpha) = \mu_0$, such that $X_{m(\alpha)}$ has a limit cycle that approaches the polycycle Γ as $\alpha \to 0^+$ with period $\mathscr{T}(\alpha)$ verifying that $\lim\limits_{\alpha\to 0^+} \mathscr{T}(\alpha)/\tau(\alpha)$ is a positive real number.

Proof: By item (a) in Theorem 4.10, the return map associated to Γ writes as

$$\mathscr{R}(s; \mu) = s^{r(\mu)}\left(A(\mu) + f(s; \mu)\right),$$

with $f(s; \mu) \in \mathcal{F}_\epsilon^\infty(\mu_0)$ for $\epsilon > 0$ small enough. Let $\alpha(\mu) = r(\mu) - 1$ and $\varepsilon(\mu) = A(\mu) - 1$. Since $\operatorname{rank}\{\nabla A, \nabla r\}|_{\mu=\mu_0} = 2$, there exists a local diffeomorphism $\varphi(\mu) =$

$(\alpha(\mu), \varepsilon(\mu), \bar{\mu}(\mu))$ defined on a small neighborhood of μ_0 such that $\varphi(\mu_0) = (0, 0, \bar{\mu}_0)$. Thus, we can work locally with the parameter set $(\alpha, \varepsilon, \bar{\mu})$. The parameter $\bar{\mu}$ plays a secondary role in our proof and therefore, we will omit it in the rest of the argument to simplify the notation. In our new parameters, the proportional displacement map $\mathscr{D}(s; \alpha, \varepsilon) = -s^{-\alpha-1}\left(\mathscr{R}(s; \varphi^{-1}(\alpha, \varepsilon)) - s\right)$ is given by

$$\mathscr{D}(s; \alpha, \varepsilon) = s^{-\alpha} - (1+\varepsilon) + \bar{f}(s; \alpha, \varepsilon),$$

where $\bar{f}(s; \alpha, \varepsilon) = f(s; \varphi^{-1}(\alpha, \varepsilon))$. We have $\bar{f}(s; \alpha, \varepsilon) \in \mathcal{F}_\epsilon^\infty(0,0)$. For the given function τ, consider the function $h(\alpha) = \alpha\tau(\alpha)$ and the $\mathscr{C}^1$ curve $m(\alpha) = \varphi^{-1}(\alpha, h(\alpha))$ in the parameter space defined for small $\alpha > 0$ (here, we consider $\bar{\mu} = \bar{\mu}_0$ fixed). It is clear that $\lim_{\alpha\to 0^+} m(\alpha) = \mu_0$. For parameters on this curve, the proportional displacement map becomes

$$M(s; \alpha) := \mathscr{D}(s; \alpha, h(\alpha)) = s^{-\alpha} - (1 + h(\alpha)) + \bar{f}(s; \alpha, h(\alpha)). \tag{4.18}$$

The zeros of $M(\cdot; \alpha)$ correspond to limit cycles of the corresponding vector field $X_{m(\alpha)}$. To search for these zeros, we establish the following ansatz

$$s(\alpha; z) = (1 + h(\alpha))^{-1/\alpha}(1 + z).$$

Substituting into (4.18), we obtain

$$\begin{aligned} H(\alpha, z) &:= \tfrac{1}{\alpha} M(s(\alpha; z), \alpha) = \tfrac{1}{\alpha}\Big((1 + h(\alpha))(1+z)^{-\alpha} - (1 + h(\alpha)) + \bar{f}(s(\alpha; z); \alpha, h(\alpha))\Big) = \\ &= (1 + h(\alpha))\omega(1 + z; \alpha) + \tfrac{1}{\alpha}\bar{f}((1 + h(\alpha))^{-1/\alpha}(1+z); \alpha, h(\alpha)). \end{aligned}$$

Since $\omega(1 + z; \alpha) = F(\alpha\log(1 + z))\log(1 + z)$ where $F(x) = \frac{e^{-x}-1}{x}$, the function $(1 + h(\alpha))\omega(1+z; \alpha)$ admits a continuous extension $g_1(\alpha, z)$ in a neighborhood of $(\alpha, z) = (0, 0)$ such that $g_1(0, 0) = 0$ and $\partial_z g_1$ is continuous with

$$\partial_z g_1(\alpha, z) = \tfrac{1+h(\alpha)}{1+z}\left(F'(\alpha\log(1 + z))\alpha\log(1 + z) + F(\alpha\log(1 + z))\right),$$

and $\partial_z g_1(0, 0) = -1$. Moreover, the assumptions on τ imply that h is a $\mathscr{C}^1$ function for small $\alpha > 0$, such that $h(\alpha) > 0$ for $\alpha > 0$, and $\lim_{\alpha\to 0^+} h(\alpha) = \lim_{\alpha\to 0^+} \alpha h'(\alpha) = \lim_{\alpha\to 0^+} \frac{\alpha\log\alpha}{h(\alpha)} = 0$. Thus, by Lemma 4.13, for any $L > 0$, $s(\alpha; z) \in \mathcal{F}_L^1(0)$. Now, since $\bar{f}(s; \alpha, \varepsilon) \in \mathcal{F}_\epsilon^\infty(0, 0)$, by items (g) and (h) in Lemma 4.6, we have that $g(\alpha; \varepsilon, z) = \alpha^{-1}\bar{f}(s(\alpha; z); \alpha, \varepsilon) \in \mathcal{F}_{\epsilon L-1}^1(0, 0)$. Taking $L > 2/\varepsilon$, by Lemma 4.7, there is an extension $\tilde{g}(\alpha; \varepsilon, z)$ of $g(\alpha; \varepsilon, z)$ which is C^1 on an open neighborhood of $(0; 0, 0)$ such that $\tilde{g}(0; 0, 0) = \partial_z\tilde{g}(0; 0, 0) = 0$. Thus, we can use this extension to conclude that the function $g(\alpha; h(\alpha), z)$ admits a continuous extension $g_2(\alpha, z)$ with continuous partial derivative $\partial_z g_2$ such that $g_2(0, 0) = \partial_z g_2(0, 0) = 0$.

Now, we have that $H(\alpha, z)$ admits $\tilde{H}(\alpha, z) := g_1(\alpha, z) + g_2(\alpha, z)$ as a continuous extension with continuous partial derivative $\partial_z\tilde{H}$ such that $\tilde{H}(0, 0) = 0$ and $\partial_z(0, 0) = -1$. By the Continuous Implicit Function Theorem [80, Theorem 2], there exists a unique function $\hat{z}(\alpha)$, continuous, defined in an open set $(-\delta, \delta)$, for which

$$\tilde{H}(\alpha; \hat{z}(\alpha)) \equiv 0, \text{ and } \hat{z}(0) = 0.$$

Therefore $s = s(\alpha; \hat{z}(\alpha))$ is a zero for (4.18) for $\alpha > 0$ small enough. We recall that $s(\alpha; \hat{z}(\alpha))$ corresponds to the location of the intersection of a limit cycle for vector field

$X_{m(\alpha)}$ and the transversal section Σ. Since $\lim_{\alpha\to 0^+} s(\alpha;\hat{z}(\alpha)) = 0$, this limit cycle approaches the polycycle as $\alpha \to 0^+$. We now proceed to compute its period $\mathscr{T}(\alpha)$. By item (b) in Theorem 4.10 the return time associated to Γ is given by

$$\mathcal{T}(s;\mu) = \bar{T}_0(\mu)\log s + \mathcal{F}_0^\infty(\mu_0),$$

where $\bar{T}_0 \in \mathscr{C}^\infty(\Lambda)$ with $\bar{T}_0(\mu) < 0$. Then, we have that $\mathscr{T}(\alpha) = \mathcal{T}(s(\alpha;\hat{z}(\alpha)); m(\alpha))$ and

$$\begin{aligned}\lim_{\alpha\to 0+}\frac{\mathscr{T}(\alpha)}{\tau(\alpha)} &= \lim_{\alpha\to 0+}\frac{1}{\tau(\alpha)}\bar{T}_0(m(\alpha))\log\left(1+h(\alpha)\right)^{-1/\alpha} = \\ &= \lim_{\alpha\to 0+} -\frac{\bar{T}_0(m(\alpha))}{\alpha\tau(\alpha)}\log\left(1+\alpha\tau(\alpha)\right) = -\bar{T}_0(\mu_0) > 0.\end{aligned}$$

This concludes the proof of the result. $\square$

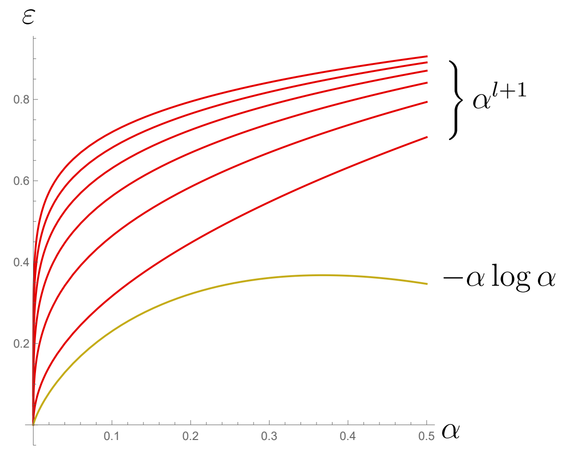


Figura 4.6: The functions $h(\alpha) = \alpha^{l+1}$, for $l \in (-1,0)$ in relation to the threshold $-\alpha\log\alpha$ for which we can apply Lemma 4.13.

Theorem 4.21 tells us that there is not a single determinate behavior for the period of a limit cycle unfolding from the polycycle for the considered family of vector fields X_μ. In particular, if we consider the functions $\tau(\alpha) = \alpha^l$, for $l \in (-1,0)$, we can always find a path in the parameter space approaching $\alpha = 0$ for which the limit cycle has period $\mathscr{T}(\alpha) \sim \alpha^l$. This is substantially different from the cases encompassed by Theorem 4.10, since the behavior of the period of the limit cycle does not depend on the way we approach the parameter value μ_0. In Figure 4.6 we represent graphically the behavior of the functions $h(\alpha) = \alpha\tau(\alpha)$ compared to the function $-\alpha\log\alpha$ showing some functions which are suitable to be the period of a limit cycle bifurcating from the polycycle in the case encompassed by Theorem 4.21.

5 Final Considerations

This thesis presents advancements in the qualitative theory of differential equations, focusing on three relevant problems.

Firstly, we address the nilpotent center problem for three-dimensional systems associated with vector fields with linear part $y\partial_x - \lambda z\partial_z$, with $\lambda \neq 0$. In our approach, we obtained a formal normal form for such systems, which allowed us to characterize the formally integrable systems. Furthermore, we proved that nilpotent systems having an analytical nilpotent center on a center manifold, with Andreev number 2, admit a formal inverse Jacobi multiplier. We also proved that nilpotent centers of three-dimensional systems, on analytic center manifolds, are limits of Hopf-type centers.

Secondly, we studied the cyclicity problem for Hopf singular points of quadratic three-dimensional systems. In this endeavor, we used the analysis of the higher-order terms of the Lyapunov coefficients to obtain new lower bounds for the cyclicity of the considered systems, including an example with 13 limit cycles bifurcating from the center, which, as far as we know, is the highest known lower bound on the cyclicity of Hopf singular points in quadratic systems.

Lastly, we investigated the period of the limit cycle bifurcating from a persistent polycycle Γ in families of polynomial planar vector fields depending on a perturbation parameter μ. We proved that, when some generic conditions on the return map depending on μ are satisfied, excatly one limit cycle bifurcates from Γ whose period function behaves as $\dfrac{1}{|1-r(\mu)|}$ where $r(\mu)$ is the graphic number associated to the polycycle Γ.

Overall, our work enhances our understanding of nilpotent centers, cyclicity and bifurcation phenomena. The results of this thesis are distributed in papers [9, 10, 11, 12, 13, 14].

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A Appendix

A.1 Computing the focal values for system (2.1) with odd Andreev number

We consider analytic system (2.1) having an isolated monodromic nilpotent singular point at the origin. We will work in the case which the Andreev number n is odd. By Theorem 2.18, the computation of the focal values is only necessary when the monodromy condition $\beta > n-1$ is satisfied.

By Proposition 2.11, (2.1) can be transformed into (2.10) with $\mu = 0$. From this canonical form, by performing the change of variables $x \to \sqrt[2n-2]{n}\, x$, $y \to -\sqrt[2n-2]{n}\, y$, we obtain the following system

$$\begin{aligned} \dot{x} &= -y + \textstyle\sum_{k \geqslant n+1} P_k(x,y), \\ \dot{y} &= x^{2n-1} + \textstyle\sum_{k \geqslant 2n} Q_k(x,y), \end{aligned} \tag{A.1}$$

where $P_k, Q_k \in \mathcal{P}_k^{(1,n)}$. The generalized polar coordinates transform (A.1) into the following system

$$\begin{aligned} \dot{\rho} &= \rho^{n+1} \textstyle\sum_{k \geqslant 0} \rho^k R_{3n+k}(\theta), \\ \dot{\theta} &= \rho^{n-1} \left(1 + \textstyle\sum_{k \geqslant 1} \rho^k \Theta_{2n+k}(\theta)\right), \end{aligned} \tag{A.2}$$

where $R_k(\theta)$ and $\Theta_k(\theta)$ are $(1,n)$-quasi-homogeneous polynomials of weighted degree k in the variables $\mathrm{Cs}\,\theta$, $\mathrm{Sn}\,\theta$. We have that, for $k \geqslant 0$,

$$\begin{aligned} R_{3n+k}(\theta) &= \mathrm{Cs}^{2n-1}\theta\, P_{n+1+k}(\mathrm{Cs}\,\theta, \mathrm{Sn}\,\theta) + \mathrm{Sn}\,\theta\, Q_{2n+k}(\mathrm{Cs}\,\theta, \mathrm{Sn}\,\theta), \\ \Theta_{2n+1+k}(\theta) &= -n\mathrm{Sn}\,\theta\, P_{n+1+k}(\mathrm{Cs}\,\theta, \mathrm{Sn}\,\theta) + \mathrm{Cs}\,\theta\, Q_{2n+k}(\mathrm{Cs}\,\theta, \mathrm{Sn}\,\theta). \end{aligned} \tag{A.3}$$

We say that the focal value $v_k(T)$ is the *k-th generalized Lyapunov coefficient* when it is the first non-zero focal value. The following result, [27, Theorem 1] provides a method to compute the generalized Lyapunov coefficient for system (A.2).

Theorem A.1. *Consider system* (A.2). *It can be written in the form $dH + \omega_1 + \omega_2 + \cdots = 0$, where $H = \frac{\rho^{2n}}{2n}$ and $\omega_k(\rho,\theta)$ are the 1-forms*

$$\omega_k(\rho,\theta) = \rho^{2n+k-1}\Theta_{2n+k}(\theta) d\rho - \rho^{2n+k} R_{3n+k-1}(\theta) d\theta.$$

Then, the k-th Lyapunov coefficient can be computed as

$$v_k(T) = -\frac{1}{(\sqrt[2n]{2n\tilde{\rho}})^{k+1}} \int_{H=\tilde{\rho}} \sum_{l=1}^{k-1} h_{k-1-l}\omega_l,$$

where $h_0 = 1$, and for $m = 1, \ldots, k-1$, h_m are defined by the recurrent expression $d\left(\sum_{l=1}^{m} h_{m-l}\omega_l\right) = -d(h_m dH)$.

The following lemma, [27, Lemma 4], simplifies the application of the above theorem.

Lemma A.2. *Let ω be the 1-form $\omega = \alpha(\rho,\theta)d\rho + \beta(\rho,\theta)d\theta$ where α, β are T-periodic functions in the variable θ and $H = \frac{\rho^{2n}}{2n}$. Then, the following hold*

i) $\int_{H=\tilde{\rho}} \omega = \int_0^T \beta(\sqrt[2n]{2n\tilde{\rho}}, \theta)d\theta$;

ii) The function $h = -\dfrac{1}{\rho^{2n-1}} \int_0^\theta \frac{\partial \alpha}{\partial \theta}(\rho,\varphi) - \frac{\partial \beta}{\partial \rho}(\rho,\varphi)d\varphi$ *satisfies* $d\omega = -d(hdH)$. *Moreover, it turns out that if* $\int_{H=\tilde{\rho}} \omega = 0$, *then h is also T-periodic.*

Now, we have the tools to compute the third Lyapunov coefficient. By Lemma 2.16 and Proposition 2.14, we have that $v_1(T) = v_2(T) = 0$. Thus, we compute $v_3(T)$. We show the explicit computations below to better illustrate the method. Let $P_k(x,y) = \sum_{i+nj=k} a_{ij}x^iy^j$ and $Q_k(x,y) = \sum_{i+nj=k} b_{ij}x^iy^j$. By Theorem A.1, we have:

$$v_3(T) = -\frac{1}{(\sqrt[2n]{2n\tilde{\rho}})^4} \int_{H=\tilde{\rho}} h_1\omega_1 + \omega_2, \tag{A.4}$$

for which h_1 must satisfy $d\omega_1 = -d(h_1 dH)$. Since $\omega_1 = \rho^{2n}\Theta_{2n+1}(\theta)d\rho - \rho^{2n+1}R_{3n}(\theta)d\theta$, using Lemma A.2, we have:

$$h_1 = -\rho \int_0^\theta \Theta'_{2n+1}(\varphi) + (2n+1)R_{3n}(\varphi)d\varphi. \tag{A.5}$$

By (A.3), we have:

$$\begin{aligned}
R_{3n}(\theta) &= a_{n+1,0}\mathrm{Cs}^{3n}\theta + (a_{11} + b_{2n,0})\mathrm{Cs}^{2n}\theta\,\mathrm{Sn}\,\theta \\
&\quad + b_{n,1}\mathrm{Cs}^{n}\theta\,\mathrm{Sn}^2\theta + b_{02}\mathrm{Sn}^3\theta, \\
R_{3n+1}(\theta) &= a_{n+2,0}\mathrm{Cs}^{3n+1}\theta + (a_{21} + b_{2n+1,0})\mathrm{Cs}^{2n+1}\theta\mathrm{Sn}\,\theta \\
&\quad + b_{n+1,1}\mathrm{Cs}^{n+1}\theta\mathrm{Sn}^2\theta + b_{12}\mathrm{Cs}\,\theta\mathrm{Sn}^3\theta, \\
\Theta_{2n+1}(\theta) &= b_{2n,0}\mathrm{Cs}^{2n+1}\theta + (b_{n,1} - na_{n+1,0})\mathrm{Cs}^{n+1}\theta\,\mathrm{Sn}\,\theta \\
&\quad + (b_{02} - na_{11})\mathrm{Cs}\,\theta\,\mathrm{Sn}^2\theta,
\end{aligned} \tag{A.6}$$

and so

$$\Theta'_{2n+1}(\varphi) + (2n+1)R_{3n}(\varphi) = (a_{11} + 2b_{02})\mathrm{Sn}\,\theta + (b_{n,1} + (n+1)a_{n+1,0})\mathrm{Cs}^n\theta.$$

Thus, by (A.5), we have:

$$h_1 = \left((a_{11} + 2b_{02})\mathrm{Cs}\,\theta - (b_{n,1} + (n+1)a_{n+1,0})\Psi_n(\theta) + (a_{11} + 2b_{02})\right)\rho, \tag{A.7}$$

where $\Psi_n(\theta) = \int_0^\theta \mathrm{Cs}^n\varphi d\varphi$. Note that, since n is odd, $\int_0^T \mathrm{Cs}^n\theta d\theta = 0$. Also, h_1 is T-periodic in the variable θ, which implies that the coefficients of $d\rho$ and $d\theta$ in $h_1\omega_1 + \omega_2$ are T-periodic functions. Using Lemma A.2 once more yields

$$\int_{H=\tilde{\rho}} h_1\omega_1 + \omega_2 = \int_0^T -\rho^{2n+1}h_1R_{3n}(\theta) - \rho^{2n+2}R_{3n+1}(\theta)d\theta\Bigg|_{\rho=\sqrt[2n]{2n\tilde{\rho}}}.$$

Thus, by (A.4),

$$v_3(T) = (\sqrt[2n]{2n\tilde{\rho}})^{2n-2} \int_0^T \frac{h_1}{\rho}R_{3n}(\theta) + R_{3n+1}(\theta)d\theta. \tag{A.8}$$

We set $\varepsilon_3 = (\sqrt[2n]{2n\tilde{\rho}})^{2n-2}$ for the sake of simplicity. Studying the expressions (A.6) and (A.7), we conclude that the integrand of (A.8) is a linear combination of the functions

- $\mathrm{Cs}^{3n+1}\theta$;
- $\mathrm{Cs}^{3n}\theta$;
- $\mathrm{Cs}^{3n}\theta\Psi_n(\theta)$;
- $\mathrm{Cs}^{n+1}\theta\,\mathrm{Sn}^2\theta$;
- $\mathrm{Cs}^{n}\theta\,\mathrm{Sn}^2\theta$;
- $\mathrm{Cs}^{n}\theta\,\mathrm{Sn}^2\theta\Psi_n(\theta)$;
- $\mathrm{Cs}^{2n+1}\theta\,\mathrm{Sn}\,\theta$;
- $\mathrm{Cs}^{2n}\theta\,\mathrm{Sn}\,\theta$;
- $\mathrm{Cs}^{2n}\theta\,\mathrm{Sn}\,\theta\Psi_n(\theta)$;
- $\mathrm{Cs}\,\theta\,\mathrm{Sn}^3\theta$;
- $\mathrm{Sn}^3\theta$;
- $\mathrm{Sn}^3\theta\Psi_n(\theta)$.

Since n is odd, only the first two are not zero when integrated from 0 to T. Thus, (A.8) simplifies to

$$\begin{aligned} v_3(T) &= \varepsilon_3\int_0^T (a_{n+2,0}+a_{n+1,0}(a_{11}+2b_{02}))\mathrm{Cs}^{3n+1}\theta\,d\theta \\ &\quad +\varepsilon_3\int_0^T (b_{n+1,1}+b_{n,1}(a_{11}+2b_{02}))\mathrm{Cs}^{n+1}\theta\,\mathrm{Sn}^2\theta\,d\theta. \end{aligned}$$

By a simple integration by parts, it is easy to see that

$$\int_0^T \mathrm{Cs}^{3n+1}\theta\,d\theta = (n+2)\int_0^T \mathrm{Cs}^{n+1}\theta\,\mathrm{Sn}^2\theta\,d\theta.$$

Since n is odd,

$$\int_0^T \mathrm{Cs}^{n+1}\theta\,\mathrm{Sn}^2\theta\,d\theta = \frac{2\Gamma(\frac{3}{2})\Gamma(\frac{n+2}{2n})}{n^{\frac{3}{2}}\Gamma(\frac{2n+1}{n})},$$

hence

$$\begin{aligned} v_3(T) &= \epsilon_3\left((n+2)(a_{n+2,0}+a_{n+1,0}(a_{11}+2b_{02}))\right. \\ &\quad \left. +b_{n+1,1}+b_{n,1}(a_{11}+2b_{02})\right), \end{aligned}$$

for a non-zero constant ϵ_3.

A.2 Proof of Proposition 2.14

We work with the canonical form (2.10) for C^r systems. The change of variables $x\to \sqrt[2n-2]{n}\,x,\ y\to -\sqrt[2n-2]{n}\,y$ transforms the canonical form into

$$\begin{aligned} \dot{x} &= y+\mu x^n+P(x,y), \\ \dot{y} &= -x^{2n-1}+n\mu x^{n-1}y+Q(x,y), \end{aligned} \tag{A.9}$$

where $P(x,y),Q(x,y)$ are C^r functions for $2n-1<r$. The generalized polar coordinates transform (A.9) into

$$\begin{aligned} \dot{\rho} &= \mu\rho^n\mathrm{Cs}^{n-1}\theta+\mathrm{Cs}^{2n-1}\theta\,P(\rho\mathrm{Cs}\,\theta,\rho^n\mathrm{Sn}\,\theta)+\rho^{-(n+1)}\mathrm{Sn}\,\theta\,Q(\rho\mathrm{Cs}\,\theta,\rho^n\mathrm{Sn}\,\theta), \\ \dot{\theta} &= \rho^{-(n+1)}\left(\rho^{2n}-n\rho^n\mathrm{Sn}\,\theta\,P(\rho\mathrm{Cs}\,\theta,\rho^n\mathrm{Sn}\,\theta)+\rho\mathrm{Cs}\,\theta\,Q(\rho\mathrm{Cs}\,\theta,\rho^n\mathrm{Sn}\,\theta)\right), \end{aligned} \tag{A.10}$$

from which we obtain

$$\frac{d\rho}{d\theta}=\frac{\dot{\rho}}{\dot{\theta}}=\frac{R(\rho,\theta)}{1+\Theta(\rho,\theta)}. \tag{A.11}$$

Using Proposition 2.5, and (A.10), we conclude that (A.11) is invariant by the transformations

$$\rho\mapsto-\rho,\ \theta\mapsto\theta+\frac{T}{2},\ \text{for } n \text{ odd},$$

$$\rho \mapsto -\rho,\ \theta \mapsto -\left(\theta + \frac{T}{2}\right),\ \text{for } n \text{ even.}$$

From here, we will follow the steps described in [31, Section 24 of Chapter IX]. We will assume n even, since for n odd, the results are exactly the same as those written in [31]. We also adopt the same notation as [31] for the convenience of the reader.

Let $f(\theta;\theta_0,\rho_0)$ be the solution of (A.11) with initial condition $\rho = \rho_0$ for $\theta = \theta_0$. The displacement function is given by $d(\rho_0) = f(T;0,\rho_0) - \rho_0$. We assume that $|\rho_0| < \delta$ for δ sufficiently small so the denominator of (A.11) is not null and the function f is well-defined. Since (A.11) is invariant by $\rho^* = -\rho$, $\theta^* = -(\theta + \frac{T}{2})$, any solution of (A.11) satisfies

$$\frac{d\rho^*}{d\theta^*} = \frac{d(-\rho)}{d(-(\theta+\frac{T}{2}))} = \frac{R(\rho,\theta)}{1+\Theta(\rho,\theta)} = \frac{R(\rho^*,\theta^*)}{1+\Theta(\rho^*,\theta^*)}. \tag{A.12}$$

Lemma A.3. *For equation* (A.11)*, we have* $d(-\rho_0) = -(f(-\frac{3T}{2};-\frac{T}{2},\rho_0) - \rho_0)$.

Proof: Consider the solution of (A.12) with initial condition $\rho^* = -\rho = \rho_0$ for $\theta^* = -(\theta + \frac{T}{2}) = -\frac{T}{2}$. It is given by:

$$\rho^* = -\rho = f\left(\theta^*; -\tfrac{T}{2}, \rho_0\right) = f(-(\theta + \tfrac{T}{2}); -\tfrac{T}{2}, \rho_0).$$

The initial condition is equivalent to the initial condition $\rho = -\rho_0$ for $\theta = 0$. Thus, by the uniqueness of solutions, we have that

$$f(\theta;0,-\rho_0) = -f(-(\theta+\tfrac{T}{2}); -\tfrac{T}{2}, \rho_0).$$

Therefore, we have

$$d(-\rho_0) = f(T;0,-\rho_0) + \rho_0 = -f(-\tfrac{3T}{2}; -\tfrac{T}{2}, \rho_0) + \rho_0 = -(f(-\tfrac{3T}{2}; -\tfrac{T}{2}, \rho_0) - \rho_0).$$

□

Lemma A.4. *If there exists* $r_1 > 0$ *such that for all* $0 < \rho_0 < r_1$, $d(\rho_0) < 0$ *(or* $d(\rho_0) > 0$*), then there exists* $r_2 > 0$ *such that* $d(-\rho_0) < 0$ *(respectively* $d(-\rho_0) > 0$*). Thus, for* $\rho_0 < \min\{r_1, r_2\}$, $d(\rho_0)d(-\rho_0) > 0$.

Proof: We consider the case where $d(\rho_0) < 0$ for $0 < \rho_0 < r_1$. The case $d(\rho_0) > 0$ is analogous. Under this assumption, the origin is a focus, and the solution curves parametrized by $(f(\theta;0,\rho_0),\theta)$ spiral towards the $(0,0)$ as θ increases. Note that these curves spiral away from the focus as θ decreases. Thus, the curves cross the rays $\theta \equiv c$ for c constant and in particular $\theta = -\frac{T}{2}$. Then, the function $f(\theta; -\frac{T}{2}, \rho_0)$ can be defined for $0 < \rho_0 < r_2$ where r_2 is a positive number. Since the curve $(f(\theta;0,\rho_0),\theta)$ spirals away from the focus as θ decreases, we have:

$$f(-\tfrac{3T}{2}; -\tfrac{T}{2}, \rho_0) > f(-\tfrac{T}{2}; -\tfrac{T}{2}, \rho_0) = \rho_0.$$

Thus, by Lemma A.3, $d(-\rho_0) < 0$ for $0 < \rho_0 < r_2$. The result holds. □

Proof of Proposition 2.14, for n even: By Lemma 2.16, we know that $d(\rho_0) \in O(\rho_0^2)$. Suppose that the first non-zero focal value is $u_k(T)$. Then $d(\rho_0) = v_k(T)\rho_0^k + O(\rho_0^{k+1})$ and there exists $r_1 > 0$ such that for all $0 < \rho_0 < r_1$, $d(\rho_0)$ and $v_k(T)$ have the same sign. By Lemma A.4 $d(-\rho_0)$ must have the same sign as $v_k(T)$ for sufficiently small ρ_0. However, this is only possible when k is an even number. □

A.3 Proof of Theorem 2.54

First we prove Theorem 2.54 for the case where V is a C^1 function. Essentially, we organize the proof presented in [45]. We recall the well-known definition:

Definition A.5. Let X be a vector field defined on an open set $U \subset \mathbb{R}^k$ and $M : U \subset \mathbb{R}^k \to \mathbb{R}$ be a C^1-function not locally null such that

$$\mathrm{div}(MX) = XM + M\mathrm{div}X \equiv 0.$$

We say that M is a *Jacobi multiplier* of X.

Let $x \in \mathbb{R}^k$ and $\varphi(t, \cdot)$ the flow of vector field X at time t. We now proceed to compute the following quantity:

$$\left(\frac{d}{dt}\int_{V_t} M(x)dx\right)\bigg|_{t=0},$$

where $V_t = \varphi(t, V)$ is the volume resulting from the temporary evolution of a volume V through $\varphi(t, \cdot)$. Since $\varphi(t, \cdot)$ is a diffeomorphism, we have:

$$\begin{aligned}\int_{V_t} M(x)dx &= \int_{\varphi(t,V)} M(x)dx = \int_V M \circ \varphi^{-1}(t, x) \det(d\varphi_x^{-1}(t, x))dx = \\ &= \int_V M \circ \varphi(-t, x) \det(d\varphi_x(-t, x))dx\end{aligned}$$

Taking the derivative with respect to time in the above equation yields:

$$\begin{aligned}\frac{d}{dt}\int_{V_t} M(x)dx &= \frac{d}{dt}\int_V M \circ \varphi(-t, x) \det(d\varphi_x(-t, x))dx = \\ &= \int_V \Big(dM(\varphi(-t, x))(-\dot{\varphi}(-t, x)) \det(d\varphi_x(-t, x)) \\ &\quad + M \circ \varphi(-t, x)\frac{d}{dt}(\det(d\varphi_x(-t, x)))\Big)\, dx, \end{aligned} \tag{A.13}$$

Now, $d\varphi_x(t, x)$ is the fundamental matrix solution to the matrix system

$$\dot{A} = dX(\varphi(x; t))A$$

with initial conditions $A(0) = I$. Then, we may apply the Liouville formula, which gives us:

$$\det(d\varphi_x(t, x)) = \exp\left(\int_0^t \mathrm{div}X \circ \varphi(\tau, x)d\tau\right),$$

which implies that

$$\left(\frac{d}{dt}\det(d\varphi_x(t, x))\right)\bigg|_{t=0} = \left(\exp\left(\int_0^t \mathrm{div}X \circ \varphi(\tau, x)d\tau\right)\mathrm{div}X \circ \varphi(t, x)\right)\bigg|_{t=0} = \mathrm{div}X.$$

Coming back to equation (A.13), and evaluating at $t = 0$ we have

$$\left(\frac{d}{dt}\int_{V_t} M(x)dx\right)\bigg|_{t=0} = -\int_V (dM(x)X + M(x)\mathrm{div}X)\, dx = -\int_V \mathrm{div}(MX)dx.$$

Thus, M is a Jacobi multiplier of X if and only if $\left.\left(\frac{d}{dt}\int_{V_t} M(x)dx\right)\right|_{t=0} = 0$. Now, introducing the change of variables $y = \phi(x)$, we have that

$$\int_{V_t} M(x)dx = \int_{\phi(V_t)} M \circ \phi^{-1}(y)\det(d\phi^{-1}(y))dy.$$

Hence, if M is a Jacobi multiplier of X then $N = \det(d\phi^{-1})M \circ \phi^{-1}$ is a Jacobi multiplier of the vector field $Y = d\phi \cdot X \circ \phi^{-1}$ in the new variables. Now to complete the proof of Theorem 2.54 it is enough to note that V is an inverse Jacobi multiplier of X if and only if $M = 1/V$ is a Jacobi multiplier of X. Then, N is a Jacobi multiplier of Y which implies that $W = 1/N = (\det(d\phi)V) \circ \phi^{-1}$ is an inverse Jacobi multiplier of Y.

Now, to address the formal case. Let ϕ and V be formal series. We then turn to Borel Lemma, which states

Lemma A.6 (Borel Lemma)**.** *For each formal mapping F, there exists a C^∞ mapping $\hat{F} : \mathbb{R}^n \to \mathbb{R}^m$ such that its Taylor series at 0 is $j^\infty \hat{F}(0) = F$.*

Thus, there exists C^∞ mappings $\hat{\phi}$ and $\hat{V}$ for which the respective power series expansion are ϕ and V. Once the algebraic relationships hold for the C^1 case, that is

$$XV = V\mathrm{div}X \Rightarrow YW = W\mathrm{div}Y,$$

in particular, they hold for $\hat{V}$ and $\hat{\phi}$. Hence, these equations hold term-by-term in the power series expansions. This completes the proof.

Remark A.7. One can also verify that the equation $YW = W\mathrm{div}Y$ hold by computing each side and comparing both of them term-by-term. However the computations get longer and longer as the dimension of the given system X increases. The algebraic verification of the simpler case, i.e. the one-dimensional is presented below:

In the one-dimensional case, $Y = \phi' X \circ \phi^{-1}$ and $W = (\det \phi' V) \circ \phi^{-1} = (\phi' V) \circ \phi^{-1}$. If V is an inverse Jacobi multiplier of X, then $V'X = VX'$. Then,

$$W'Y = (\phi'' V + \phi' V')(\phi^{-1})' \phi' X \circ \phi^{-1} = (\phi'' V + \phi' V')X \circ \phi^{-1} = (\phi'' VX + \phi' V'X) \circ \phi^{-1},$$

and

$$WY' = ((\phi' V) \circ \phi^{-1})(\phi'' X + \phi' X')(\phi^{-1})' = (\phi'' VX + \phi' VX') \circ \phi^{-1}.$$