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Campos de vetores suaves por partes: Aspectos teóricos e aplicações

São José do Rio Preto

2020

Luiz Fernando Gonçalves

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Tese apresentada como parte dos requisitos para obtenção do título de Doutor em Matemática, junto ao Programa de Pós-Graduação em Matemática, do Instituto de Biociências, Letras e Ciências Exatas da Universidade Estadual Paulista "Júlio de Mesquita Filho", Câmpus de São José do Rio Preto.

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A meus amados pais José e Ilma;

A meus grandes amigos;

A meus mestres;

Dedico.

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"In the broad light of day mathematicians check their equations and their proofs, leaving no stone unturned in their search for rigour. But, at night, under the full moon, they dream, they float among the stars and wonder at the mystery of the heavens: they are inspired. Without dreams there is no art, no mathematics, no life."

(Michael Atiyah, 2010, p.1)

Resumo

Nesta tese abordaremos aspectos qualitativos e dinâmicos de problemas envolvendo campos de vetores suaves por partes, também conhecidos como campos descontínuos. Primeiramente, apresentamos aplicações da teoria de campos de vetores descontínuos em modelos de tratamento intermitente de Cancer e Vírus da Imunodeficiência Humana onde exibimos a existência de singularidades típicas e órbitas periódicas. Ainda no contexto de aplicações, revisitamos um modelo predador-presa descontínuo de modo a concluir que o mesmo tem um comportamento caótico através da existência de uma órbita de Shilnikov. Posteriormente, respondemos questões sobre existência de conjuntos minimais e caóticos para campos de vetores descontínuos na esfera bidimensional. Em seguida, partimos ao estudo de bifurcação de ciclos limites em campos de vetores descontínuos tri e bidimensionais. No primeiro caso, perturbamos um campo descontínuo tangente a uma folheação por toros de modo a gerar uma quantidade finita ou infinita de ciclos limites. No segundo caso, estudamos uma família de campos descontínuos apresentando uma dobra-dobra invisível de costura, sua ciclicidade e a relação entre os coeficientes de Lyapunov desta família e sua regularização. Além disso, estudamos campos vetoriais suaves por partes Hamiltonianos contendo uma dobra-dobra invisível de costura donde apresentamos uma fórmula explícita para o cálculo dos cinco primeiros coeficientes de Lyapunov, além de explorar os diagramas de bifurcação gerados pelo desdobramento desta singularidade.

Palavras-chave: Campo de vetores suaves por partes. Singularidades típicas. Conjuntos minimais e caóticos. Ciclos limites. Coeficientes de Lyapunov. Órbita de Shilnikov.

Abstract

In this work we discuss qualitative and dynamic features of problems involving piecewise smooth vector fields, also known as discontinuous vector fields. Firstly, we present applications of discontinuous vector field theory in Human Immunodeficiency Virus and Cancer intermittent treatment models where we exhibit typical singularities and periodic orbits. Moreover, we revisit a discontinuous predator-prey model in order to conclude that it has a chaotic behavior through the existence of a Shilnikov orbit. Next, we answer questions about the existence of minimal and chaotic sets in the bidimensional sphere for discontinuous vector fields. Subsequently, we investigate the creation of limit cycles in three and two-dimensional discontinuous vector fields. In the first case, we perturb a discontinuous vector field tangent to a foliation composed by topological nested tori to generate a finite or infinite number of limit cycles. In the second case, we analyze a family of discontinuous vector fields containing a crossing invisible fold-fold, their cyclicity and the relation between the Lyapunov coefficients of this family and their regularization. Also, we study general piecewise Hamiltonian vector fields presenting a crossing invisible fold-fold where we give an explicit formula for the computation of the five first Lyapunov coefficients in addition to the investigation of the bifurcation diagrams.

Keywords: Piecewise smooth vector fields. Typical singularities. Minimal and chaotic sets. Limit cycles. Lyapunov coefficients. Shilnikov orbit.

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Introduction

Ordinary Differential Equations (ODE's) are important mathematical tools for investigating natural phenomena. However, we know that many of them do not admit explicit solutions and this motivated great mathematicians to look for different alternatives to understand such solutions. Consequently, the way in which Ordinary Differential Equations were studied changed dramatically in the late $19th$ century. This fact is due to Henri Poincaré after the publication of his work *Mémoire sur les courbes definies par une équation différentielle* (see [100]) in which he introduces a new technique for the study of ODE's that was the basis of which today we call the Qualitative Theory of Ordinary Differential Equations. This theory gives us important and significant results and tools for studying the behavior of the orbits of the differential equation and the analysis of its phase portrait, without knowing its explicit solutions, through geometric, topological, analytical aspects, among others.

Several and important models used in applied problems, related to engineering and biology, are described by systems that are not completely differentiable, but in different parts, where a law of evolution is suddenly interrupted by another law of evolution that will begin to govern such system. The modeling of such systems consists of different vector fields defined in distinct regions separated by a switching manifold and are known as piecewise smooth vector fields, discontinuous systems or Filippov systems.

Pioneering studies initiated by Andronov [5] and Filippov [47] led to a theoretical foundation for this kind of problem and developed certain conventions for the transition of orbits between different regions, in order to define the basic objects of Qualitative Theory of Differential Equations and investigation of their dynamics. Currently, there is a wide literature on discontinuous vector fields. See, for example, the book [37] for main theory and applications, [16, 40, 82] for applications in mechanical models, [28, 32, 76] on circuits electrical, [39, 71] in relay systems, among others.

The study of minimal sets and limit sets is of great importance in the context of qualitative theory, as they help us to understand the behavior of a dynamic system. In the context of vector field flows in compact sets of the plane, and even in the sphere, the Poincaré-Bendixson Theorem establishes that the limit set of a trajectory must be an equilibrium point, a graph or a periodic orbit. Denjoy in [36] proves that if we consider in the torus (or a connected two-dimensional compact manifold) a flow of class C^2 then the minimal sets are trivial, that is, fixed points, closed orbits or all the torus, in which case the flow is topologically equivalent to an irrational flow. However, a C^1 flow can present non-trivial minimal sets, a fact that was also shown by Denjoy in [36] (see also [22, 63, 102]).

Minimal trivial sets significant in the theory of dynamical systems are isolated periodic orbits, that is, limit cycles. Studies involving bifurcations of limit cycles and the maximum number of limit cycles that can bifurcate in a family of vector fields, that is, their ciclicity, are some of the topics most covered in the qualitative theory, both for their mathematical relevance and for their applications in real models. Such an object is the subject of Hilbert's famous $16th$ problem (see [65]) which seeks for an upper bound for the number of limit cycles in planar polynomial vector fields. The appearance of non-trivial minimal sets requires a lower regularity than that required in the Denjoy-Schwartz Theorem, that is, vector fields of class C^1 or even discontinuous as can be seen in [18], [19], and [36].

The occurrence of non-trivial minimal sets in certain systems is linked to the presence of complex behaviors, such as chaotic dynamics. Chaos can be understood through the existence of an invariant compact set in which trajectories by their initial conditions exhibit transitivity and sensitive dependence (see [67, 89]) and can be traced by studying the existence of previously known objects being chaotic as this is the case of Shilnikov's homoclinic orbit, a trajectory connecting a hyperbolic equilibrium of saddle-focus type to himself (see [67, 103, 104]). Chaotic behaviors in discontinuous vector fields (see Definition 12) have recently been investigated and the first evidence of chaoticity in such systems comes from the work of Jeffrey [31] where new and rich chaotic models appear.

The theory of discontinuous vector fields has been developed taking, at first, parallel to the classical theory. It is known that the Existence and Uniqueness Theorem is not valid in this context since we have no uniqueness of solutions in the orbits defined by means of the sliding field (see Definition 1). The Poincaré-Bendixson Theorem has a version for discontinuous systems (see [19]) as well as the Peixoto Theorem (see [107]). Even a technique that allows us to relate discontinuous to regular systems has been developed. The regularization method, introduced by Sotomayor and Teixeira in [109] consists of the approximation of a discontinuous system by a one parameter family of smooth vector fields, from where we can apply the classical theory to try to obtain information about the piecewise smooth vector field.

As in the regular case, the study of limit cycles is an important problem in discontinuous systems theory. Traditionally, the study of limit cycles in a family of planar vector fields X_λ close to a continuum of periodic orbits or a center-type equilibrium can be replaced by the study of the fixed points of a first return map P_λ or, equivalently, by studying the zeros from the so-called displacement function $\delta_\lambda = P_\lambda - Id$. Such technique is still very useful in the study of crossing limit cycles in discontinuous vector fields and great effort has been devoted to determining the maximum number of limit cycles that can occur in a piecewise smooth vector field, in which even the linear case is already shown intriguing. In the planar case, if the separation curve is not a straight line, for each positive integer n, we can find a piecewise linear system with exactly n limit cycles (see [15, 92]). However, if the separation curve is a straight line, continuous piecewise linear systems can present at most one limit cycle, a fact that was conjectured by Lum and Chua in [86] and proved later in [48]. This result also proved to be valid for sewing planar piecewise linear systems as can be seen in [88]. Discontinuous vector fields, on the other hand, can have three limit cycles (see [84]) and there is a strong belief that three is the maximum number of limit cycles that can appear in crossing piecewise linear systems. There is also the appearance of crossing limit cycles for linear vector fields in which the separation line involves sliding regions as can be seen in [14].

In this way, many authors have contributed to the development of discontinuous vector field theory. Several models related to engineering and biology governed by discontinuous systems occur in families with n parameters that undergo some kind of bifurcation, making it essential to establish the notion of structural stability and the classification of bifurcations for such systems. One of the starting points in this context was the work of Teixeira [116] in which he studies vector fields in two-dimensional manifolds with boundaries. The structural stability of Filippov's vector fields had as precursors the works of Kozlova [77] and Teixeira [118] and, in the study of generic singularities, Teixeira's work in [117]. The classification of one parameter local and global bifurcations of a Filippov planar system was established later by Kuznetsov et al. em [80] and complemented by Guardia et al. in [62] which also begins the study of unfoldings of two parameters generic singularities.

We now present a brief description of the chapters of this work.

In the first chapter, we introduce the terminology and basic concepts of the theory of piecewise smooth vector fields in order to define the fundamental objects of the theory that will be essential in the development of this work.

In chapter two, we present applications of discontinuous vector field theory. We explored models of intermittent treatment of Cancer and Human Immunodeficiency Virus (HIV) and, after, we studied the chaoticity of a discontinuous predator-prey model. In the first case, we studied a system of differential equations in $\mathbb{R}^3$ that models the treatment of leukemia. We explored the transition between the model with or without the treatment given by chemotherapy and/or immunotherapy and we exhibit an interesting behavior next to the switching manifold due to the existence of typical singularities. Some of the singularities found in this work are more complicated than those already shown in the literature (see, for example, [24, 26]) as they are not yet represented by normal forms. This study resulted in the papers I and II from the list of published or submitted papers derived from this thesis available at the end of the introduction. In the second case, we studied the qualitative dynamics of a piecewise smooth system that models the intermittent treatment of HIV. We exhibit the existence of typical singularities and periodic orbits and explore the dynamics around these objects. In addition, we conclude that such a protocol is successful, since the trajectory that passes through any initial condition converges to one of these distinct orbits. Our results corroborate the observation of the real world, where the virus is not eliminated, but is controlled around a specific value. The results derived from this study are the contents of preprints III and IV from the list available at the end of this section.

Also in chapter two, we revisit a piecewise smooth vector field in $\mathbb{R}^3$ that was obtained recently by modeling the interaction between a predator and two prey in which the predator adjusts its feeding preference between a preferred prey and an alternative. Such a model proposed by Piltz et al. in [99] is given by

$$(\dot{x}, \dot{y}, \dot{z})^T = \begin{cases} X(x,y,z) = \begin{pmatrix} (r_1 - z)x \\ r_2 y \\ (eq_1 x - m)z \end{pmatrix} & h(x,y,z) \geq 0; \\ Y(x,y,z) = \begin{pmatrix} r_1 x \\ \left(r_2 - \dfrac{\beta_2}{\beta_1} z\right) y \\ \left(\dfrac{eq_2}{a_q} y - m\right) z \end{pmatrix} & h(x,y,z) \leq 0, \end{cases}$$

where $(x, y, z) \in \mathbb{R}^3_{\geq 0}$ and $h(x, y, z) = x - y$. In this model, the authors observed numerical evidences of chaotic behavior. We proved that in a codimension one submanifold of the parameter space such a system exhibits a Shilnikov connection (see Definition 15) and through this fact we conclude that the system is chaotic (see Theorem 9). This work resulted in preprint V of the list of papers derived from this thesis available at the end of this section.

The third chapter is dedicated to the study of piecewise smooth vector fields tangent to the bidimensional sphere. We show the existence of discontinuous vector fields in the two-dimensional sphere that do not present singularities, but presenting chaotic non-trivial minimal sets, following Filippov's conventions. In addition, we show the existence of a discontinuous vector field that has the entire sphere as a minimal set. As far as we know, the literature does not yet contemplate the study of piecewise smooth vector fields in the sphere via Filippov's conventions. Such studies resulted in paper VI and preprint VII of the list of papers.

In the fourth chapter, we perturb a discontinuous vector field tangent to a foliation composed by topological nested tori. Such vector field in cylindrical coordinates is given by

$$Z_0(r,\theta,z)=\begin{cases} X(r,\theta,z)=\left(r,\,0\,,\dfrac{4+17r^2-12r^4}{4r}\right) & z\geq 0,\\[2ex] Y(r,\theta,z)=\left(-r,\,0\,,\dfrac{4+17r^2-12r^4}{4r}\right) & z\leq 0.\end{cases}$$

In each half-plane $\Pi_{\theta_0}=\{(x,y,z)\in\mathbb{R}^3\,|\,x=r\cos(\theta_0),y=r\sin(\theta_0)\text{ where }r\geq 0\}$ the vector field Z_0 is invariant and behaves like a center. Through two uncoupled perturbations we consider a perturbation of Z_0 of the form

$$Z_\varepsilon(r,\theta,z)=\begin{cases} X_\varepsilon(r,\theta,z)=\left(r,\,0\,,\dfrac{4+17r^2-12r^4}{4r}+r\dfrac{\partial F_\varepsilon}{\partial r}(r)\right) & z\geq 0,\\[2ex] Y^\nu(r,\theta,z)=\left(-r,\,\nu(\theta)\,,\dfrac{4+17r^2-12r^4}{4r}\right) & z\leq 0,\end{cases}$$

in order to generate a finite, or infinite, number of invariant half-planes, with each of these half-planes having a finite, or infinite, number of limit cycles. The stability of such limit cycles is related to the derivatives of the perturbation functions involved, and we can also have multiple limit cycles. This study resulted in preprint VIII of the list of papers derived from this thesis.

Finally, in chapter five, we study a k parameters unfolding of the family of discontinuous vector fields containing an invisible fold-fold at the origin given by

$$\mathfrak{Z}_{b,k,\lambda}(x,y)=\begin{cases}\mathfrak{X}_{b,k}(x,y)=\left(1,-x+bx^{2k}\right) & y\geq 0\\ Y_\lambda(x,y)=(-1,-x+\lambda) & y\leq 0,\end{cases}$$

where λ is a small parameter, b a non-zero real number and $k=1,2,3,\ldots$. We show that, for each fixed k, there exists $\lambda\in\mathbb{R}$ sufficiently small and $a=(a_2,a_4,\ldots,a_{2k-2})$ such that the unfolding $Z_{a,k,\lambda}$ given by

$$Z_{a,k,\lambda}(x,y)=\begin{cases}X_{a,k}(x,y)=\left(1,-x+a_2x^2+\cdots+a_{2k-2}x^{2k-2}+x^{2k}\right) & y\geq 0;\\ Y_\lambda(x,y)=(-1,-x+\lambda) & y\leq 0,\end{cases}$$

has exactly k crossing limit cycles in a neighborhood of the origin, a number that is also maximum. We generalize the results found in [80] and [62] in which such a family was studied with one and two parameters, respectively, where one and two limit cycles were found and the bifurcation diagrams displayed. Given the similarity of such diagrams with classic Hopf bifurcations, we wondered if there could be any relationship between the Lyapunov coefficients of the family of discontinuous vector field and its regularization. The answer was positive and we showed the following result: For $\varepsilon > 0$ small enough, the Lyapunov coefficients of the discontinuous vector field $Z_{a,k,0}$ and the Lyapunov coefficients of its regularization Z_ε has the same sign. This work is the content of preprint IX of the list of papers derived from this thesis available at the end of this introduction.

Also in chapter five, we consider a piecewise smooth vector field of the form

$$\mathcal{Z}(x,y,\mu,\nu) = \begin{cases} \mathcal{X}(x,y,\mu,\nu), & y \geq 0; \\ \mathcal{Y}(x,y,\mu,\nu), & y \leq 0, \end{cases}$$

containing an invisible fold-fold at the origin where $\mathcal{X}$ and $\mathcal{Y}$ are Hamiltonian systems. We obtained explicit formulas, written in terms of the derivatives of the Hamiltonian functions associated with the vector fields $\mathcal{X}$ and $\mathcal{Y}$, for the calculation of the first five Lyapunov coefficients associated with a Hamiltonian piecewise smooth vector field with an invisible fold-fold at the origin (see Theorem 15 and Theorem 18). In addition, we study the bifurcation diagrams associated with such a discontinuous vector field when the second Lyapunov coefficient is nonzero or when the second Lyapunov coefficient is zero and the fourth is nonzero (see Theorems 16 e 19). In the last case, we give a general expression for the curve of non-hyperbolic periodic orbits that occurs in this diagram (see Proposition 42 and Remark 22). This study resulted in the paper X of the list below of published and submitted papers derived from this thesis.

Published and submitted papers derived from this Thesis

I. L. F. Gonçalves, D. S. Rodrigues, P. F. A. Mancera, and T. Carvalho. Sliding mode control in a mathematical model to chemoimmunotherapy: the occurrence of typical singularities. Applied Mathematics and Computation, 2019. doi: 10.1016/j.amc.2019.124782.

II. L. F. Gonçalves, D. S. Rodrigues, P. F. A. Mancera, and T. Carvalho. A mathematical model for chemoimmunotherapy of chronic lymphocytic leukemia. Applied Mathematics and Computation, 349: 118-133, 2019. doi: 10.1016/j.amc.2018.12.008

III. L. F. Gonçalves, D. C. Vicentin, P. F. A. Mancera, and T. Carvalho. Mathematical model of an antiretroviral therapy to HIV via Filippov theory. Preprint, 2019.

IV. L. F. Gonçalves, T. Carvalho, R. Cristiano, and D. J. Tonon. Global analysis of the dynamics of a mathematical model to intermittent HIV treatment using sliding mode control. Preprint, 2019.

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