IFT - UNESP
INSTITUTO DE FÍSICA TEÓRICA

MASTER'S DISSERTATION IFT–D.007/2025

A Study on the Gravitational Polarizability of Schwarzschild Black Holes

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September 2025

D192s

Dantas, Gabriel Macedo

A study on the gravitacional polarizability of Schwarzschild black holes / Gabriel Macedo Dantas. -- São Paulo, 2025

135 f: il. color.

Dissertação (mestrado) – Universidade Estadual Paulista (Unesp), Instituto de Física Teórica (IFT), São Paulo

Orientador: Riccardo Sturani

1. Gravitação. 2. Teoria de campos (Física). 3. Buracos negros (Astronomia). I. Título

Sistema de geração automática de fichas catalográficas da Unesp. Biblioteca do Instituto de Física Teórica (IFT), São Paulo. Dados fornecidos pelo autor(a).

A STUDY ON THE GRAVITATIONAL POLARIZABILITY OF SCHWARZSCHILD BLACK HOLES

Dissertação de Mestrado apresentada ao Instituto de Física Teórica do Câmpus de São Paulo, da Universidade Estadual Paulista "Júlio de Mesquita Filho", como parte dos requisitos para obtenção do título Mestre em Ciências, área: Física.

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Conceito: Aprovado

São Paulo, 02 de julho de 2025.

À minha mãe e à minha avó

Agradecimentos

Frases que o vento vem às vezes me lembrar
— Lô Borges, *O trem azul*

É impossível mensurar a gratidão que carrego comigo para com todos que me ajudaram a sair da cidade, pegar o trem azul e ouvir o vento. Aqui vai minha singela tentativa de expressá-la.

Agradeço ao Prof. Arsen pelo imenso apoio em minha trajetória na UnB, por todos os conselhos e ensinamentos. Eu certamente não teria chegado ao IFT sem a sua ajuda.

Agradeço ao meu orientador, Riccardo, por me aceitar como aluno e me guiar durante o mestrado. Agradeço ao Prof. Gabriel Menezes pela colaboração durante os últimos dois anos e pelas valiosas lições (de Física e de Vida).

Agradeço aos meus queridos amigos que transformaram a cidade de São Paulo no palco onde colhi algumas das minhas memórias mais especiais: João, Rodrigo, Eric, Matheus, Jorge, Juliana, Lu, Isa, Marcelo, Washington e Eggon. Em especial, agradeço aos meus colegas de orientação, Vidal, Alan e Almeida (*the first*), pela parceria, conversas animadas, desabafos, risadas e por me ensinarem muito do que aprendi nos últimos anos. Ao Vidal, agradeço pela colaboração que culminou nesse projeto.

Agradeço aos meus amigos de longa data de Brasília – com imensa alegria digo que vocês são numerosos demais para serem listados – por manterem a chama da amizade viva e por me apoiarem nessa jornada.

Agradeço ao IFT, seus professores e servidores, pelo ambiente de ensino, pesquisa e extensão da mais alta qualidade. Agradeço o apoio da Fundação de Amparo à Pesquisa do Estado de São Paulo (FAPESP) pelo suporte financeiro através do processo de número 2023/00295-2.

Por fim e no mais alto grau de importância, agradeço à minha família, e em especial, à minha mãe e à minha avó pelo amor incondicional e irrestrito. Devo tudo a vocês. Essa vitória é de todos nós. Obrigado (um milhão de vezes).

Passamos metade da vida a estudar os restos mortais dos outros,
e a outra metade a deixar os nossos.

— Domenico Starnone, *Línguas*

Resumo

Dentre as inúmeras frentes de investigação surgidas no contexto da astronomia de ondas gravitacionais, um dos tópicos mais fascinantes – em parte devido a sua natureza interdisciplinar – é o estudo das deformações de maré e as propriedades absorptivas de objetos compactos como as estrelas de nêutrons e buracos negros. Notavelmente, estes últimos exibem zero deformações de maré estáticas em quatro dimensões espaço-temporais, isto é, seus números de Love estáticos são nulos.

A função de resposta – também chamada de polarizabilidade gravitacional – de buracos negros de Schwarzschild é um tópico vigoroso de estudo em abordagens de teorias efetivas de campos aplicadas à dinâmica gravitacional. No presente trabalho, nós consideramos a validade das relações de dispersão de Kramers-Kronig para a função de resposta e discutimos sua compatibilidade com o desaparecimento dos números de Love estáticos. Para tal, realizamos o *matching* de funções de Wightman gravitacionais de um campo teste no background de Schwarzschild com um modelo efetivo do tipo linha-mundo (*worldline*) construído para analisar deformações de maré. Os estados de vácuo de Boulware, Unruh e Hartle-Hawking são considerados. Seguindo a aplicação das relações de dispersão à função de resposta, propomos uma regra de soma para a função de Wightman definida na teoria efetiva. Por fim, discutimos a possibilidade de modificar tal regra de soma tendo em vista efeitos de cauda (*tail effects*) na resposta do buraco negro.

A primeira parte dessa dissertação consiste em uma revisão bibliográfica do formalismo de teoria efetiva de campos conhecido como Relatividade Geral Não-Relativística. Escrito com uma audiência de leitores inexperientes em mente, foi realizado um esforço para tornar o texto o mais autocontido e pedagógico possível. O último capítulo lida com o problema exposto acima.

Palavras Chaves: Polarizabilidade; Buracos Negros; Ondas Gravitacionais; Teoria de Campos; Teoria Efetiva.

Áreas do conhecimento: Física; Relatividade Geral; Teorias Efetivas de Campos.

Abstract

Amongst the wealth of research prospects brought forth by gravitational wave astronomy, one of the most fascinating topics – partly due to its multidisciplinary nature – is the investigation of tidal deformations and absorption properties of compact objects such as neutron stars and black holes. Remarkably, the latter exhibit zero static tidal deformability in four spacetime dimensions, i.e., their static Love numbers vanish.

To date, the response function – also called the polarizability – of Schwarzschild black holes has been vigorously studied with the framework of effective field theories (EFTs) applied to gravitational dynamics. In this work we consider the validity of Kramers-Kronig dispersion relations for such response function and discuss their compatibility with the vanishing of the static Love numbers. To do so, we match gravitational Wightman functions of a long-wavelenght test field in a Schwarzschild background to a worldline EFT model for tidal deformations. The Boulware, Unruh, and Hartle-Hawking vacua states are considered. Following the application of dispersion relations to the EFT response, a sum rule for the EFT Wightman function is proposed. We discuss its validity in light of late-time tails effects in the black hole response.

The first part of this dissertation reviews the EFT framework of Non-Relativistic General Relativity (NRGR). Written with unexperienced readers in mind, an effort was made to present a text as self-contained and pedagogic as possible. The last chapter deals with the aforementioned puzzle.

Keywords: Polarizability; Black Holes; Gravitational Waves; Field Theory; Effective Theory.

Fields of knowledge: Physics; General Relativity; Effective Field Theories.

Contents

Chapter 1

The NRGR Framework

1.1 Introduction

Since its advent in late 1915, the pillar which underpines our understanding of gravity is Einstein's theory of general relativity (GR). By recasting the laws of gravitation as a consequence of spacetime's curvature, one is able to not only recover the celebrated predicitons of Newtonian gravity, but also systematically investigate phenomena that lie beyond the scope of Newton's formalism. This list of remarkable effects include: the theory of stellar collapse and formation of black holes (BHs), the expansion of our Universe, and the propagation of gravitational waves (GWs).

The first observational confirmation of gravitational-wave theory came with the observations of the Hulse-Taylor pulsar system in 1974, in which the loss of energy of the binary matched with great precision the predictions of general relativity regarding GW emission [75] [74]. The direct observation of GWs, however, had to wait the turn of the century. In 2015, September 14, the LIGO/Virgo collaboration detected the first signal of gravitational waves emitted by the inspiral and merger of a binary (BH) system [1].

So far, the LIGO/Virgo/KAGRA collaboration has successfully detected more than one hundred events due to coalescence of compact binaries [2] [32]. Moving beyond laser interferometry experiments, in 2023, the NANOGrav collaboration presented compelling evidence for a stochastic background of low-frequency GWs [3]. Their experiment unveils the correlation between the angular separation in the sky of a pulsar array. Such a correlation is expected in a stochastic background of GWs, such as those produced by supermassive BH mergers or the inflationary epoch in the early Universe. Although the source of the signal is yet to be confirmed, this partial result boosts our confidence in the theoretical and experimental status of gravitational-wave theory. In the next decade, both next generation ground-based interferometers [111] [109] and space-based interferometers [10] [84] are expected to operate alongside current generation detectors.

Gravity is the weakest fundamental interaction, and the detection of such waves is an enormous feat of engineering and data science. Crucially, the observation relies on a tremendous theoretical effort spanning decades in the modelling of compact binary systems and their associated gravitational radiation. To detect such events one must know the shape of the predicted signal in great detail.[1] Such *numerical templates* form the basis of the *matched filtering* method [86], which itself is a central technique in the data analysis. In the case of GW experiments, one needs to provide the experimentalist with an accurate model for the gravitational *waveform*, the radiated gravitational field at null infinity. Moreover,

[1] As discussed in [38], a reliable data analysis is unfeasible unless a 3PN/3.5PN accuracy in the waveform is attained.

as shown in equation (1.19), the observed GW phase $\Delta\Phi(t)$ is sensitive to the radiated power and the equations of motion of the binary system, so the source dynamics must also be determined precisely. The coalescence of a compact binary system comprises three distinct stages:

(i) **Inspiral**: The initial stage in which the objects are widely separated and their motion can be regarded as non-relativistic. For typical masses M of order $1-10\ M_\odot$, this means an orbital separation around 100 Km to a few $(M/M_\odot)$ Km and typical velocities in the range $0.1 < v/c < 0.4$. The inspiral stage is thus amenable to perturbative treatments, as we shall discuss in more detail below.

(ii) **Merger**: The objects collide and the non-linear strong-field regime of GR dominates the dynamics, rendering any perturbative method unfeasible. The standard technique used to analyze the merger is *numerical relativity* [82].

(iii) **Ringdown**: The compact objects form a single perturbed BH which oscillates towards an equilibrium configuration. This stage is well-described by the methods of *black hole perturbation theory* (BHPT) [108].

The search for an analytic solution in general relativity that describes all three coalescence stages is a hopeless endeavor; one must resort to approximation schemes within *perturbation theory* or numerical modelling. In this former approach, deviations from the Newtonian equations of motion, for example, are organized in a series of small corrections controlled by a reasonable expansion parameter. The appropriate choice of parameter depends on the approximation scheme, the most important being the *post-Newtonian* (PN) and *post-Minkowskian* (PM) expansions.[2] See [23] for an authoritative account of PN theory.

The PN approximation is applied whenever one is dealing with slowly moving and weakly self-interacting sources (e.g., the inspiral stage of coalescence). The expansion parameters are $\epsilon_1 \equiv v/c$ and $\epsilon_2 \equiv r_s/d$, where v is the typical orbital velocity, d is the typical orbital separation, and r_s is the characteristic size of the binary constituents (the Schwarzschild radius). These two ratios are related to one another by the virial theorem, which implies that $v^2/c^2 \sim r_s/d$ in a gravitationally bound system. Corrections of order $\mathcal{O}(\epsilon_1^n)$ with respect to the Newtonian dynamics are called "nPN" corrections. The validity of the PN approximation is restricted to the so-called *near-zone* of the source. Let $\lambda\!\!\!^{-} = c/\omega$ denote the typical reduced wavelength of the gravitational radiation and ω be its associated frequency. Then, we separate the region outside the source $(r \geq r_s)$ as:

(i) **Near-zone** ($r \ll \lambda\!\!\!^{-}$): The retardation effects associated to the emission of radiation are negligible and one works with "quasi-instantaneous" gravitational potentials. The PN approximation is well-suited here, as long as we are dealing with weak-field sources.

[2]It is worth mentioning a third approximation scheme naturally adapted to describe extreme mass-ratio inspirals: the *self-force* approach [98], where one solves Einstein's field equations as a formal series in the ratio $m_1/m_2 \ll 1$. So far, this technique has produced a number of important cross-checks of results obtained in the PN and PM approximations. See [12] for an example.

(ii) **Intermediate-zone** ($r \sim \lambdabar$): Also called the "induction-zone" in electrodynamics. Traditional GR techniques often require "matching conditions" in this region to connect the near-zone dynamics to the far-zone boundary conditions associated to radiation emission and retardation.

(iii) **Far-zone** ($r \gg \lambdabar$): The region in which the propagation of GWs takes place. Retardation effects are crucial and the PN expansion by itself is no longer appropriate.[3] The emission of GWs alters the near-zone dynamics via *radiation-reaction* effects.

In figure 1.1 we display the parameter space $(\epsilon_1^2, \epsilon_2)$ and the range of validity of the PN approximation. It is worth mentioning that each of the regions introduced above may be further segmented depending on the regime one wishes to investigate and the techniques used. For example, when describing strong-field sources such as neutron stars and black holes, it is often useful to split the near-zone into a *strong-field* near-zone (where r is roughly a few r_s) and a *weak-field* near-zone.

In light of the near/far description given above, one cannot rely on the PN scheme in the near-zone. Instead, a *post-Minkowskian* approximation is traditionally used. The expansion parameter is G, Newton's constant, and we are no longer restricted to slowly-moving sources.[4] As one can easily infer from the virial theorem, an observable known at some order in G can be further expanded in powers of v, allowing cross-checks between quantities computed with different approximations. Since the PM expansion is able to probe sources with relativistic speeds, this approximation is also ideal for studying scattering events and their associated gravitational radiation [79]. Both PN and PM approximations are supplemented by a multipole expansion controlled by the ratio r_s/L, where L is the typical distance from the source.

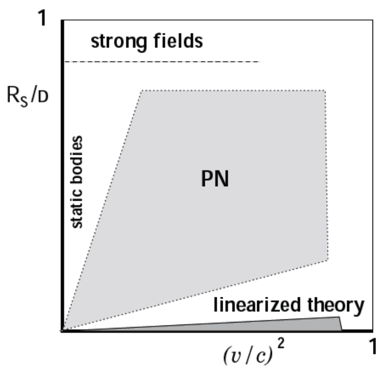


Figure 1.1: Part of the parameter space of a compact binary system and the approximation regime encompassed by post-Newtonian theory. Adapted from [86].

Beyond the traditional methods used in GR, the two approximation schemes just described are amenable to *field-theoretic* treatments, where one exploits the framework of Lagrangian field theory to extract GR observables from Feynman diagrams describing the

[3]One cannot impose boundary conditions at infinity ($r \ll \lambdabar$) within the PN approximation. See footnote 12 in chapter 5 of [86].

[4]A correction of order G^n with respect to the Newtonian dynamics is said to be a nPM correction.

interaction of gravitons and matter fields (or classical worldlines). The literature on this subject is huge, and we make no effort to survey it completely in this introduction.

In the context of QFT/amplitude techniques applied to the PM approximation, the first milestone was reached by Z. Bern and collaborators [16] in 2019, with the computation of the 3PM correction to the conservative two-body Hamiltonian for spinless black holes, pushing forward the state of the art at the time. Their methods involve a mixture of double copy, effective field theory, and generalized unitarity. Further developments (up to 2022) are surveyed in [22, 78]. Since then, many other frameworks have been developed for extracting classical physics from amplitudes, including: the KMOC formalism [77], the eikonal approach [44], heavy-mass EFT [27], and the worldline QFT [91].

This dissertation deals with *Non-Relativistic General Relativity (NRGR)*; an effective field theory approach to post-Newtonian gravitational dynamics [59, 50, 104, 83, 120]. In this framework, gravitons are coupled to classical worldline actions which describe the compact objects. By exploiting the natural *separation of scales* in the two-body problem, one constructs a tower of effective theories by integrating out "short-distance" degrees of freedom in the path integral. The small expansion parameter is the orbital velocity $v \ll 1$.

At a first level, the *internal scale* of the binary is integrated out, leaving behind an effective action consisting of the bulk gravitational field and point-particle worldlines supplemented by higher-dimensional operators enconding finite-size effects. The graviton field is then split into *potential, off-shell* modes and *radiative, on-shell* modes. This separation allows for a clear PN scaling of each term in the effective action, and thus, for each Feynman diagram as well. While potential modes are associated with quasi-instantaneous propagators, the on-shell modes require the complete relativistic propagator. Feynman rules for vertices and graviton-worldline interactions follow from usual path-integral techniques [97].

The starting point for a near-zone description of the dynamics is the integration of potential modes, which results in an effective potential for two-body interactions. The PN corrections to the equations of motion are extracted from the real part of this effective action. By computing graphs including couplings to external radiative graviton legs, one takes into account far-zone interactions. The effective action then picks up an imaginary part from which the instantaneous radiated power is extracted via optical theorem. In the parlance of effective field theory, this is a *top-bottom* approach to carry out computations.

Radiation-reaction forces are systematically captured by implementing the *in-in formulation* of NRGR, an extension of the textbook *in-out* approach which allows one to analyze dissipative systems at the level of the Lagrangian [55]. So far, the conservative sector of NRGR has been completely determined up to 4PN order [51, 49] and 5PN order [26].

The to GW emission can also be investigated by constructing the most general effective action compatible with the symmetries of the two-body of the far-zone action. This is a *bottom-up* approach to NRGR computations. The action thus constructed is naturally organized in multipole moments of the system, whose expression in terms of orbital variables is *a priori* unknown to us. The Wilson coefficients of this action have a clear PN scaling and a fixed dimension; they encode all information regarding the near-zone physics. These coefficients are determined via a *matching* calculation, i.e., a direct comparison between observables computed in the EFT and a given model for the short-scale physics (be it phenomenological or not). Once these constants are fixed, the EFT machinery can be put to use with full predictive power within its realm of validity.

Even though this text does not discuss the structure of ultraviolet (UV) and infrared (IR) divergences in NRGR, the ability to organize and systematize such divergences is one of the most attractive features of the EFT approach [107]. We also focus on *spinless* bodies – the basic formalism of NRGR was expended to incorporate spin degrees of freedom by Porto in the early days of the subject [102, 103]. More recently the formalism was modified to systematically compute observables in the PM approximation [80].

Structure. The work of this dissertation is structured as follows. Our **literature review** spans chapters 1 to 3. The goal of these chapters is to discuss the foundations of NRGR (chapter 1), including the *in-in* formulation (chapter 2), and absorption effects (chapter 3). This first part of the text contains a fair amount of detailed calculations of basic NRGR results. Students and newcomers to the field may benefit from an expanded discussion of key results presented in early NRGR papers, and our exposition was written with special attention to this audience.

The experienced reader may wish to head straight to chapter 4, which contains the bulk work of this dissertation. This chapter is based on the preprint [124], a work done in collaboration with G. Vidal, R. Sturani, and G. Menezes. The grand theme of 4 is the use of linear response theory applied to the study of long wavelenght perturbations of a Schwarzschild BH. In particular, inspired by previous work on the subject [61, 57], we devise a worldline EFT inspired in NRGR to attack this problem. The main problem we investigate is the conciliation of Kramers-Kronig dispersion relations – derived under causality requirements – and the vanishing of the static response of four-dimensional Schwarzschild BHs. We find that the classical part of the two-point correlation function of the BH quadrupole (of electric parity) depends on the semiclassical vacuum boundary conditions of the spacetime. We also explore the consequence of this fact to the validity of the Kramers-Kronig dispersion relations. In particular, we propose a sum rule for the Wightman function of the electric quadrupole. We conclude that the inclusion of late-time tail[5] effects may contaminate the BH response, which requires a modification of the aforementioned sum rule in order to take into account the overlap of tidal responses and tail effects.

[5] By tail effect we mean the scattering of GW waves by static gravitational fields. See chapter 4 for more context.

List of conventions

- We adopt the mostly-minus signature $(+---)$. Greek indices $\mu, \nu, \cdots$ are spacetime indices ranging from 0 to 3, while Latin indices $i, j, \cdots$ are purely spatial indices ranging from 1 to 3. Einstein's summation convention is implied whenever a pair of repeated indices appears contracted.

- Partial and covariant derivatives are denoted by $\partial_\mu = \frac{\partial}{\partial x^\mu}$ and ∇_μ, respectively.

- The Riemann tensor is defined as $R^\rho{}_{\mu\sigma\nu} = \partial_\sigma \Gamma^\rho_{\mu\nu} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\lambda\sigma}\Gamma^\lambda_{\mu\nu} - \Gamma^\rho_{\lambda\nu}\Gamma^\lambda_{\mu\sigma}$, where $\Gamma^\alpha_{\mu\nu}$ is the Levi-Civita connection, and the Ricci tensor is $R_{\mu\nu} = R^\rho{}_{\mu\rho\nu}$.

- Newton's constant (in three spatial dimensions) is $G = (32\pi M_{\text{Pl}}^2)^{-1}$, where M_{Pl} denotes Planck's mass. We work in natural units such that $c = \hbar = 1$.

- The proper time is denoted by τ and we use either x^0 or t to denote coordinate time (unless otherwise mentioned). Time derivatives with respect to coordinate time are often written with the dot notation: $\dot{f} \equiv \frac{\mathrm{d}f}{\mathrm{d}t}$. For $n \geq 3$, we also write $\frac{\mathrm{d}^n f}{\mathrm{d}t^n} = f^{(n)}(t)$. All three-dimensional vectors $\mathbf{x}, \mathbf{y}, \cdots$ are boldface and unit vectors carry a hat: $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \cdots$. When a spacetime function f depends on $x^\mu, y^\mu, \cdots$ we omit the Greek indices in the argument and simply write $f(x, y, \cdots)$.

- The Fourier transform is defined as $f(x) = \int_k e^{-ikx} \tilde{f}(k) \equiv \int \frac{\mathrm{d}^4 k}{(2\pi)^4} e^{-ikx} \tilde{f}(k)$. We also denote $\int_{\mathbf{k},\mathbf{p},\cdots} \equiv \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} \cdots$ and $\int_{k^0,p^0,\cdots} \equiv \int \frac{\mathrm{d}k^0}{2\pi} \frac{\mathrm{d}p^0}{2\pi} \cdots$. Whenever the integration limits of a one-dimensional integral are suppressed, it is implied that they range from $-\infty$ to $+\infty$.

- Depending on which choice minimizes notational clutter in a given expression, complex conjugation is either denoted by a $*$ symbol (z^*) or a bar $(\bar{z})$.

- In certain equations we employ the multi-index notation of Blanchet and Damour: $I^L = I^{i_1, i_2, \cdots, i_\ell}$. Symmetric and trace-free (STF) tensors are written as $[T^{ij}]_{\text{STF}} = \frac{1}{2}(T^{ij} + T^{ji}) - \frac{1}{3}\delta^{ij} T$, where $T = \delta_{kl} T^{kl}$. The operation of removing traces is similarly written as $[T^{ij}]_{\text{TF}} = T^{ij} - \frac{1}{3}\delta^{ij} T$.

- Green's functions play a key role in our discussion. In our **literature review** (chapters 1 to 3), we use the conventions of the seminal reference [55], which coincide with those used in [59, 60, 68]. In this way, the reader that wishes to read our review to become familiar with the basics of NRGR may directly compare intermediate steps of key calculations with the exposition of the well-stablished aforementioned papers. Our chapter 4 is an expanded version of the **original work** presented in the preprint [124]. As such, in chapter 4 – *and chapter* 4 *only* – we follow [124]. Further details and a comprehensive dictionary for translating both conventions back and forth can be found in section A.2 of the appendix.

1.1.1 The linearized theory of gravity

The theory of gravitational waves begins with the linearization of Einstein's field equations:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu}. \tag{1.1}$$

We consider slowly-moving sources in weak gravitational fields, so that it makes sense to treat v and G as independent expansion parameters. In particular, we shall treat the background spacetime near the sources as flat and discuss the generation of GWs in a small v approximation.

This simplified approach does not accurately describe the realistic case of a compact binary system held together by gravitational forces, where the background spacetime curvature is important. Since our goal is to simply review fundamental aspects of GWs, we will not pursue the systematic implementation of the post-Newtonian formalism in general relativity, which is the appropriate framework for dealing with realistic sources (See chapter 5 of [86]).

With these remarks in mind, we write the spacetime metric $g_{\mu\nu}$ as

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad |h| \ll 1. \tag{1.2}$$

We assume that there is a coordinate system where the separation above holds on a sufficiently large spacetime region. Losing general covariance is inevitable if we wish to isolate the degrees of freedom that describe the propagation of waves. The linearized theory as a whole is invariant under finite Poincaré transformations and local infinitesimal local transformations. The metric perturbation $h_{\mu\nu}$ introduced above is a Lorentz tensor. Keep only terms linear in h, the Riemann tensor becomes

$$R_{\rho\sigma\mu\nu} = \frac{1}{2}(\partial_\sigma\partial_\nu h_{\mu\rho} + \partial_\rho\partial_\mu h_{\nu\sigma} - \partial_\sigma\partial_\rho h_{\mu\nu} - \partial_\mu\partial_\nu h_{\rho\sigma}). \tag{1.3}$$

Under $x^\mu \to x'^\mu + \xi^\mu$, the metric perturbation transforms as $h_{\mu\nu} \to h'_{\mu\nu} = h_{\mu\nu} - 2\partial_{(\mu}\xi_{\nu)}$. The linearized Riemann tensor above is invariant under this transformation of $h_{\mu\nu}$. Defining the trace-reversed metric[6]

$$\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h, \quad h = \eta^{\alpha\beta}h_{\alpha\beta}, \tag{1.4}$$

one can easily show that it is always possible to choose ξ^μ such that

$$\partial_\mu \bar{h}^{\mu\nu} = 0, \quad (\text{de Donder gauge}). \tag{1.5}$$

There is a residual gauge freedom defined by the set of transformations that satisfy $\Box\xi^\mu = 0$. The linearized equation (1.1) takes the following form in de Donder gauge:

$$\Box\bar{h}_{\mu\nu} = -16\pi G T_{\mu\nu}, \quad \Box \equiv \eta^{\mu\nu}\partial_\mu\partial_\nu. \tag{1.6}$$

[6] Not to be confused with the radiative graviton modes introduced later in the text.

The vacuum field equations $\Box\bar{h}_{\mu\nu} = 0$ and the condition (1.5) *do not* completely fix the gauge freedom, as mentioned above. Assume a vacuum spacetime $T_{\mu\nu} = 0$ (we are considering the propagation of waves outside the source) and that $h_{\mu\nu} \to 0$ sufficiently far away (asymptotic flatness). Then, it can be shown that one can further specialize the gauge choice to the *transverse-traceless (TT) gauge*:

$$h^{0\mu} = 0, \quad h = 0, \quad \partial^i h_{ij} = 0. \tag{1.7}$$

This gauge completely fixes the gauge freedom. Given a plane GW propagating in the $\hat{\mathbf{n}}$ direction in the de Donder gauge, one can project it onto the TT gauge with the help of the symmetric and traceless "lambda projector" $\Lambda_{ijkl}(\hat{\mathbf{n}})$, defined as

$$\Lambda_{ijkl}(\hat{\mathbf{n}}) \equiv P_{ik}P_{jl} - \frac{1}{2}P_{ij}P_{kl}, \tag{1.8}$$

where $P_{ij} \equiv \delta_{ij} - n_i n_j$. We are, of course, interested in the case where $T_{\mu\nu}$ is non-zero due to the presence of some source. In this situation, the full metric perturbation $h_{\mu\nu}$ can no longer be put in the TT gauge. The previous procedure in which pure gauge and longitudinal degrees of freedom were removed from h, leaving only a $h^{\mathrm{TT}}_{\mu\nu}$ component, can no longer be applied. The perturbation now contains three classes of degrees of freedom:

(i) Non-physical, pure gauge degrees of freedom;

(ii) Physical, longitudinal degrees of freedom which are tied to the matter sources;

(iii) Physical, radiative degrees of freedom which describe the propagation of gravitational waves.

By carefully performing a scalar-vector-tensor decomposition of (1.6), one can show that only the TT component of $h_{\mu\nu}$ satisfies a wave equation [48]. To quote A. Eddington, the non-TT components propagate at the "speed of thought." The two independent components of h^{TT}_{ij} thus encode the two polarization modes of a GW. Detectors based on laser interferometry are particularly sensitive to the combination

$$R_{0i0j} = -\frac{1}{2}\ddot{h}^{\mathrm{TT}}_{ij}. \tag{1.9}$$

Far away from the source, (1.6) can be solved for $\bar{h}^{\mathrm{TT}}_{ij}$ as a *multipolar expansion* of the fundamental solution obtained by convolving T^{ij} with the retarded Green function of the $\Box$ operator. Exploiting the conservation equation $\partial_\mu T^{\mu\nu}$, one finds the following far-zone metric in the quadrupolar approximation:[7]

$$h^{\mathrm{TT}}_{ij}(t, \mathbf{x}) = \frac{2G}{r}\Lambda_{ijkl}(\hat{\mathbf{n}})\ddot{I}_{ij}(t - r), \tag{1.10}$$

[7]Notice that, in the TT gauge, there is no difference between h_{ij} and its trace reverse.

where $r \equiv |\mathbf{x}|$ and I_{ij} denotes the tracefree quadrupole moment of the source:

$$I_{ij}(t) \equiv \int \mathrm{d}^3\mathbf{x}'\, T^{00}(t,\mathbf{x}')\Big(x'_i x'_j - \frac{1}{3}\delta_{ij}|\mathbf{x}'|^2\Big). \tag{1.11}$$

Since one can only make sense of the energy carried by a wave in a averaged sense (over a few wavelengths), the energy flux transported by GWs is given by[8]

$$\mathcal{P}_{\text{gw}} = \frac{r^2}{32\pi G}\int \mathrm{d}\Omega\, \langle \dot{h}^{\text{TT}}_{ij}\dot{h}^{\text{TT}}_{ij}\rangle, \tag{1.12}$$

This is the rate of emitted energy per unit time through a unit sphere integrated over the angles (θ,ϕ). Using (1.10), we obtain the celebrated *quadrupole formula*:

$$\mathcal{P}_{\text{quad}} = \frac{G}{5}\big\langle I^{(3)}_{ij} I^{(3)}_{ij}\big\rangle. \tag{1.13}$$

We shall derive this result using the formalism of NRGR towards the end of section 1.3.4.

The two-body problem in GR. As mentioned earlier, a systematic description of a binary system held together by gravitational forces is given by the post-Newtonian approximation scheme. Let us now say a few words on the necessity of high-precision calculations in perturbation theory.

Since we are dealing with a gravitationally bound state, we cannot have arbitrary orbital velocities v for any given separation r. Rather, these parameters are related to each other by the virial theorem:

$$v^2 = \frac{Gm}{r}, \tag{1.14}$$

where $m = m_1 + m_2$ is the total mass of the binary. This is nothing but Kepler's third law. The orbital frequency of the source is related to the orbital velocity by

$$\omega_{\text{s}} = \frac{v^3}{Gm}. \tag{1.15}$$

The laser interferometer laboratories such as LIGO and Virgo are particularly sensitive to the *gravitational wave phase* $\Delta\Phi(t)$. To define it, let us consider without loss of generality the case of a circular orbit. To be more precise, this derivation – which culminates in (1.19) – assumes an "adiabatic inspiral," where the binary is in quasi-circular motion at each time t. This requires $\dot{\omega}_{\text{s}}/\omega_{\text{s}}^2 \ll 1$, which guarantees that the derivative of the orbital radius $|\dot{R}|$ is small [86]. With these remarks in mind, we write

$$\Delta\Phi(t) = \int^t \mathrm{d}t'\, \omega_{\text{gw}}(t') = 2\int^t \mathrm{d}t'\, \omega_{\text{s}}(t') = \frac{2}{Gm}\int^t \mathrm{d}t'\, v^3. \tag{1.16}$$

Conservation of energy requires that the loss of orbital energy matches the power radiated

[8] See [86] and [48] for more technical details on the definition of the energy carried by GWs.

as GWs. Therefore,

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -\mathcal{P}_{\mathrm{gw}}. \tag{1.17}$$

In general, when considering radiation reaction onto the source, the instantaneous flux computed from the far-zone metric does not match the flux derived from the near-zone radiation reaction forces; they rather differ by total time derivatives called *Schott terms* (See [88] for a modern textbook discussion in electrodynamics and §3.2.2 of [23] for comments in the context of the PN approximation).

For example, the quadrupolar flux $\mathcal{P}_{\mathrm{quad}}$ derived from Einstein' formula differs from the Burke-Thorne flux $\mathcal{P}_{\mathrm{BT}}$ derived from the leading radiation reaction force (2.56):

$$\mathcal{P}_{\mathrm{BT}} = \mathcal{P}_{\mathrm{quad}} + \frac{G}{5}\frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{I}_{ij}I^{(4)}_{ij} - \ddot{I}_{ij}I^{(3)}_{ij}\right). \tag{1.18}$$

The quantity on the right-hand side does not affect the equation of motion directly, and it vanishes when time averaged. In the absence of time averages the Schott terms consistently redefine the conserved charges of the near-zone at each PN order [54] [106]. The Burke-Thorne radiation reaction force will be derived in chapter 2 using the *in-in* formulation of NRGR.

Moving on, from (1.17) we obtain

$$\Delta\Phi(t) = -\frac{2}{Gm}\int_{v(t_0)}^{v(t)} \mathrm{d}v\, v^3 \frac{1}{\mathcal{P}_{\mathrm{gw}}}\frac{\mathrm{d}E}{\mathrm{d}v}. \tag{1.19}$$

Using the machinery of NRGR we can compute $\mathcal{P}_{\mathrm{gw}}$ and $\mathrm{d}E/\mathrm{d}v$ in a post-Newtonian expansion, i.e., as power series in v. The radiated power is extracted from the imaginary part of the effective NRGR action[9] $S^{\mathrm{NRGR}}_{\mathrm{eff}}$, while $\mathrm{d}E/\mathrm{d}v$ is determined from the associated two-body Lagrangian extracted from the real part of $S^{\mathrm{NRGR}}_{\mathrm{eff}}$.

1.2 Generalities on effective field theories

Before developing the basic formalism of NRGR, we now make some general remarks on effective field theories. It is not our goal to provide a comprehensive description of EFT techniques (See the text [28] instead). We will simply sketch the general construction of EFTs and introduce the uninitiated reader to the "effective" way of thinking. Our discussion is inspired by [101].

Common sense tells us that we do not need to understand quantum chromodynamics to formulate the equation of state of an ideal gas. This miracle is only possible because physical phenomena are organized by *scales*. One does not need to understand the full dynamics of high-frequency/UV degrees of freedom to understand low-frequency/IR physics and make meaningful predictions. This is the organizing principle of effective field theories, which are nothing but models tailored to describe a certain subset of IR degrees

[9] As explained in section 2, to accurately capture radiation reaction effects one must implementat the *in-in* formulation of NRGR.

of freedom while systematically incorporating UV data in the form of unknown parameters fixed *a posteriori*. It is not a stretch to claim that all physical theories are effective.

Using the EFT machinery, the way that the UV affects the IR is entirely described by (i) the IR degrees of freedom, (ii) interactions local in spacetime, (iii) a set of unknown coefficients (the Wilson coefficients) which must be determined by either comparison with experiments or confrontation with a more fundamental theory which governs the UV degrees of freedom. Regardless of the method used, this process of fixing the unknown parameters is called "matching." In NRGR, the traditional EFT approach is supplemented by the method of regions, see section 1.3.3.

Let us sketch the general EFT procedure. Consider a quantum field theory of a scalar field ϕ with characteristic energy scale E_0. Suppose we wish to describe physics at energies E such that $E \ll E_0$. Most systems have many different scales but we can focus on one at a time. First, pick a cutoff Λ at E_0 (or slightly below it). Then, split the fields into high-frequency and low-frequency parts:

$$\phi = \phi_{\mathrm{L}} + \phi_{\mathrm{H}}, \tag{1.20}$$

where ϕ_{L} and ϕ_{H} describe physics at $\omega < \Lambda$ and $\omega > \Lambda$, respectively. It does not matter how the separation of the field is done; it may be sharp or smooth, it may break some symmetries or preserve them. Assuming that the separation has been done, we integrate out high-frequency degrees of freedom from the path integral:

$$\int \mathcal{D}\phi_{\mathrm{L}}\mathcal{D}\phi_{\mathrm{H}}\, e^{iS[\phi_{\mathrm{L}},\phi_{\mathrm{H}}]} = \int \mathcal{D}\phi_{\mathrm{L}}\, e^{iS_\Lambda[\phi_{\mathrm{L}}]}, \tag{1.21}$$

where

$$e^{iS_\Lambda[\phi_{\mathrm{L}}]} \equiv \int \mathcal{D}\phi_{\mathrm{H}}\, e^{iS[\phi_{\mathrm{L}},\phi_{\mathrm{H}}]} \tag{1.22}$$

is the *Wilsonian effective action*, which we will simply refer to as the "effective action" from now on. We can expand $S_\Lambda[\phi_{\mathrm{L}}]$ in terms of local operators:

$$S_\Lambda[\phi_{\mathrm{L}}] = \int \mathrm{d}^D x \sum_i g_i \mathcal{O}_i \tag{1.23}$$

The infinite sum above runs over all local operators allowed by the symmetries of the theory. Because S_Λ is generated by integrating out modes with frequencies $\omega > \Lambda$, the effective action is non-local in time on the scale $1/\Lambda$.

The effective action S_Λ is computed by standard perturbative techniques in terms of Feynman diagrams. This leads us to another key feature of EFTs: the power counting scheme that controls the perturbative order at which each operator enters the effective action.

Let us indulge ourselves in some dimensional analysis. Set $\hbar = c = 1$. If an operator $\mathcal{O}_i$ has units E^{δ_i}, then δ_i is known as its dimension. Therefore, g_i has units $E^{D-\delta_i}$ to ensure a dimensionless $S_\Lambda[\phi_L]$. Use the free action to assign units to the fields:[10]

$$S_0 = \frac{1}{2}\int \mathrm{d}^D x\, \partial_\mu \phi\, \partial^\mu \phi. \tag{1.24}$$

This implies that ϕ has units of $E^{-1+D/2}$. A general operator $\mathcal{O}_i$ constructed from M powers of ϕ and N derivatives has dimension

$$\delta_i = M\left(\frac{D}{2} - 1\right) + N. \tag{1.25}$$

Define dimensionless couplings $\lambda_i = \Lambda^{\delta_i - D} g_i$. Given that Λ is basically the characteristic scale of the system, the constants λ_i are presumably of order 1. Now consider a process at energy E. The i-th term of $S_\Lambda[\phi_L]$ scales as

$$\int \mathrm{d}^D x\, g_i \mathcal{O}_i \sim \lambda_i \left(\frac{E}{\Lambda}\right)^{\delta_i - D}. \tag{1.26}$$

This leads to the following classification of operators:

- ($\delta_i < D$) The operator's contribution grows with E. We thus call it a *relevant* term or *superrenormalizable*;
- ($\delta_i = D$) The operator stays constant as E changes. It is called a *marginal* term or *strictly renormalizable*;
- ($\delta_i > D$) The operator's contribution falls with E. We thus call it *irrelevant* or *non-renormalizable*.

In most cases, there is a finite number of *relevant* and *marginal* terms, which implies that the low energy physics only depends on a finite number of parameters.

Why insist on using the *free* action to set the dimensions of the fields? This is so because we are assuming a *weakly-coupled theory*, so that S_0 really determines the size of typical fluctuations (or matrix elements) of the fields. It is necessary that the coefficient of the dominant term in the action is made dimensionless, as shown in S^0 above. This can be achieved by rescaling the fields. This was used in the estimate (1.26) because we implicitly assumed that the only dimensionful quantity is the energy scale.

The approach illustrated above is often called a "*top-down*" approach to EFTs. One begins with an action describing all degrees of freedom and integrates out high-frequency modes to obtain an effective theory for the low-frequency modes. An alternative "*bottom-up*" approach is to first write down the most general effective action compatible with the symmetries of the problem. The Wilson coefficients of this general EFT action are *a priori* unknown. We must perform a matching computation where a certain observable is computed using the EFT and the "complete theory" which governs all degrees of freedom.[11]

[10]The free action controls the typical size of the fluctuations of the fields, assuming a weakly coupled theory. We will emphasize this point later.

[11]Experimental input and/or phenomenological models can also be used for matching computations. Indeed, this is common practice in particle physics [117].

Then, one can match results and fix the Wilson coefficients, ensuring the predictability of the effective theory (as long as we remain within its application bounds).

Taken *as a whole*, one can think of non-relativistic general relativity (NRGR), and other similar worldline methods, as "top-down" approaches to gravitational dynamics since we know in advance the full theory: general relativity. We reason we bother developing worldline EFTs is because of their convenience. The separation of scales found in many gravitational systems is exploited to produce EFTs which are easier to handle than GR, calculation-wise.[12] As we shall see in detail, one can *implement* NRGR in either a "top-down" (integrating out degrees of freedom from the path integral) or "bottom-up" (parametrizing general actions and matching) manner. Both are incredibly useful and necessary. Matter of fact, NRGR is a *tower* of effective theories constructed for each scale of the gravitational two-body problem, and both methods described above are used.

1.3 NRGR: An effective field theory for gravitational dynamics

Among the vastitude of field-theoretic formalisms developed in recent years to analyze the gravitational two-body problem, the work presented in this dissertation makes use of *NRGR*, an effective field theory of gravitationally bound objects which is well tailored to describe the binary dynamics in the *inspiral* phase.

This initial phase of the coalescence of compact binaries is kinematically favorable to analytical methods, as mentioned in 1.1. The EFT approach, in particular, fundamentally relies on the *separation of scales* of the two-body problem. We have

(i) A *radiation scale* whose characteristic length scale is $\lambda_{\rm gw}$, the wavelenght of the GWs emitted by the binary;

(ii) An *orbital scale* chatacterized by r, the orbital separation between the binary constituents;

(iii) The *internal scale* ruled by finite size effects and set by the Schwarzschild radius $r_{\rm s} \sim Gm$.

Under the assumption that the leading-order dynamics is purely Newtonian – so that the virial theorem holds – the parameters above are related by

$$\lambda_{\rm gw} \gg r \gg r_{\rm s}. \tag{1.27}$$

The scale suppresion is given by powers of $v \ll 1$, the typical velocity of the binary in the

[12] They also offer an avenue for harnessing the powers of modern amplitude methods. It is worth mentioning that there is a law of "conservation of difficulty" when comparing formalisms, but that does not stop us from developing new and productive forms describing "old" physics.

inspiral phase. Denoting by ω_s the orbital frequency of the source, we have

$$\lambda_\mathrm{gw} \sim \frac{1}{\omega_\mathrm{s}} = \frac{Gm}{v^3} \sim \frac{r}{v}, \tag{1.28a}$$

$$v^2 \sim \frac{r_\mathrm{s}}{r} \ll 1. \tag{1.28b}$$

In a specific field theory setup – soon to be introduced – the hierarchy (1.27) will be exploited to construct effective theories adapted for describing the dynamics at each scale of the problem. The simplicity of the EFT approach lies in the fact that each theory in the "tower of theories" is easier to handle than general relativity in its full glory. Moreover, the manifest power counting in v will streamline the computation of observables, ensuring that only a finite number of Feynman diagrams contributes to each PN order.

Before separating the scales at the level of the degrees of freedom themselves, let us first setup the stage. The bulk gravitational field is assumed to be governed by the usual Einstein-Hilbert action (with a gauge-fixing term)[13]

$$S_\mathrm{bulk} \equiv S_\mathrm{EH} + S_\mathrm{gf} \equiv 2M_\mathrm{Pl}^2 \int \mathrm{d}^4x \, \sqrt{-g} \left(R[g] - \frac{1}{2}\Gamma^\mu \Gamma_\mu \right), \tag{1.29}$$

where $\Gamma^\mu = g^{\alpha\beta}\Gamma^\mu_{\alpha\beta}$ is the contracted Christoffel symbol with respect to the full metric $g_{\mu\nu}$. We work in the harmonic/de Donder gauge, defined by $\Gamma^\mu = 0$. The compact objects which constitute the binary are described by a classical point-particle action

$$S_\mathrm{pp}[g_{\mu\nu}, x^\mu_a] \equiv -\sum_{a=1}^{2} m_a \int \mathrm{d}\tau_a = -\sum_{a=1}^{2} m_a \int \mathrm{d}t \left(g_{\mu\nu}[x_a(t)] \frac{\mathrm{d}x^\mu_a}{\mathrm{d}t} \frac{\mathrm{d}x^\nu_a}{\mathrm{d}t} \right)^{1/2}. \tag{1.30}$$

The full action is simply the sum of (1.29) and (1.30). The second expression above makes manifest the coupling of the worldlines $x^\mu_a(t)$ with the metric tensor. If we restrict ourselves to (1.30), we are assuming that the sources have no finite extension. Notice that no extra fields are introduced to describe the binary, and consequently, no propagators associated with the classical compact objects appear in NRGR.

The action (1.30) only describes geodesic motion. In principle, one must also include an infinite set of operators (as dictated by diffeomorphism invariance) which encode finite-size effects, i.e., deviations from pure geodesic motion. This is independent of the inclusion (or exclusion) of spin degrees of freedom, which we will not discuss in this dissertation. When (1.30) is supplemented with such higer-dimensional operators which capture finite-size effects, the corresponding action can be thought as the result of completely "integrating out" the internal scale r_s.

According to the power counting scheme that we will later introduce, the first two finite

[13] In this work we restrict our attention to general relativity. The investigation of modified gravity theories in the context of NRGR is a well-established line of research, of course. See, for example, references [8] and [45].

size operators are

$$S_{\text{pp}} \supset \sum_a c_a \int \mathrm{d}\tau_a\, R[x_a] + \sum_a \tilde{c}_a \int \mathrm{d}\tau_a\, R_{\mu\nu}[x_a]\dot{x}_a^\mu \dot{x}_a^\nu + \cdots\,. \tag{1.31}$$

The Wilson coefficients c_a and $\tilde{c}_a$ are determined by matching a given observable with the predictions of a dynamical model for the finite size effects. It happens that the two operators in (1.31) are spurious; their effects can be removed by field redefinitions [60]. In terms of the Weyl tensor $C^{\mu\nu\alpha\beta}$, the simplest *non-trivial* finite-size operators are $C_{\mu\nu\alpha\beta}C^{\mu\nu\alpha\beta}$, $\dot{x}^\rho \dot{x}_\sigma C_{\mu\nu\alpha\rho}C^{\mu\nu\alpha\sigma}$, and $\dot{x}^\alpha \dot{x}^\beta \dot{x}_\rho \dot{x}_\sigma C_{\mu\alpha\nu\beta}C^{\mu\rho\nu\sigma}$.

At last, since we are interested in investigating the GWs emitted by the system, we write the full metric as a flat spacetime Minkowski metric plus a linear perturbation:

$$g_{\mu\nu} = \eta_{\mu\nu} + \frac{1}{M_{\text{Pl}}} h_{\mu\nu}. \tag{1.32}$$

With this setup we expect to recover the results of the linearized theory presented in 1.1.1.

1.3.1 Separation of scales

Let us now discuss how the EFT approach organizes the relevant degrees of freedom. Instead of working with the fully relativistic propagator for $h_{\mu\nu}$ coming from (1.29), we decompose the metric perturbation as[14]

$$h_{\mu\nu} = \bar{h}_{\mu\nu} + H_{\mu\nu}. \tag{1.33}$$

The field $\bar{h}_{\mu\nu}$ represents the long-wavelength *radiation modes* which are always on-shell, whereas $H_{\mu\nu}$ are the off-shell *potential modes* describing the longitudinal degrees of freedom of the gravitational field. The momenta of the potential and radiation modes, respectively, scale as

$$k^\mu_{\text{pot}} \sim (v/r, 1/r), \tag{1.34a}$$

$$k^\mu_{\text{rad}} \sim (v/r, v/r). \tag{1.34b}$$

Therefore, the scaling rules for the derivatives of $H_{\mu\nu}$ and $\bar{h}_{\mu\nu}$ are

$$\partial_0 H_{\mu\nu} \sim \frac{v}{r} H_{\mu\nu}, \tag{1.35a}$$

$$\partial_i H_{\mu\nu} \sim \frac{1}{r} H_{\mu\nu}. \tag{1.35b}$$

and also

$$\partial_\mu \bar{h}_{\alpha\beta} \sim \frac{v}{r} \bar{h}_\alpha. \tag{1.36}$$

[14] The field $\bar{h}_{\mu\nu}$ mode should not be confused the trace-reversed metric perturbation which appears exclusively in section 1.1.1.

It is advantageous to work with the Fourier transform of $H_{\mu\nu}$ with respect to the spatial coordinates:

$$H_{\mu\nu}(x) = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\, e^{i\mathbf{k}\cdot\mathbf{x}} H_{\mathbf{k}\mu\nu}(x^0). \tag{1.37}$$

This guarantees that all derivatives of both graviton modes scale the same:

$$\partial_0 H_{\mathbf{k}\alpha\beta}(x^0) \sim \frac{v}{r} H_{\mathbf{k}\alpha\beta}(x^0), \tag{1.38a}$$

$$\partial_\mu \bar{h}_{\alpha\beta}(x) \sim \frac{v}{r} \bar{h}_{\alpha\beta}. \tag{1.38b}$$

The scaling of the modes themselves will be determined once we derive their associated propagators.

The radiation modes are the only ones that can appear as asymptotic states of the gravitational field in NRGR. An EFT which aims to describe the coupled dynamics of the worldlines and radiation modes must "integrate out" the potential modes. Thus,

$$\exp\left(iS_{\text{eff}}^{\text{WR}}[\bar{h}_{\mu\nu}, x_a^\mu]\right) \equiv \int \mathcal{D}H_{\mu\nu} \exp\left(iS_{\text{EH}}[\bar{h} + H] + iS_{\text{gf}} + iS_{\text{pp}}\right), \tag{1.39}$$

where the wordline-radiation effective action $iS_{\text{eff}}^{\text{WR}}$ is introduced.[15] The point-particle enters the path integral above as a background, non-dynamical source. We also emphasize that, since no graviton loops are requires to describe the *classical* dynamics of the binary (see **Lemma 2** of this chapter), there is no need to introduce ghost fields.

At this stage one must be careful with the gauge-fixing term. We need to ensure that the effective action is invariant under diffeomorphisms of $\bar{g}_{\mu\nu} \equiv \eta_{\mu\nu} + \bar{h}_{\mu\nu}/M_{\text{Pl}}$, which is the background metric for the radiation gravitons in the EFT action above. The correct harmonic gauge-fixing term is thus

$$S_{\text{gf}}[H_{\mu\nu}, \bar{g}_{\mu\nu}] \equiv -M_{\text{Pl}}^2 \int \mathrm{d}^4x\, \sqrt{-\bar{g}}\, \bar{g}^{\mu\nu} \Gamma_\mu^{(H)} \Gamma_\nu^{(H)}, \tag{1.40}$$

where the Christoffel symbol $\Gamma_\mu^{(H)}$ is defined with respect to the covariant derivative $\bar{\nabla}_\mu$ compatible with $\bar{g}$, i.e.,

$$\Gamma_\mu^{(H)} \equiv \frac{1}{M_{\text{Pl}}} \bar{g}^{\rho\sigma} \left(\bar{\nabla}_\sigma H_{\rho\mu} - \frac{1}{2} \bar{\nabla}_\mu H_{\rho\sigma} \right). \tag{1.41}$$

The final NRGR effective action is obtained by integrating out the remaining external radiation modes:

$$\exp\left(iS_{\text{eff}}^{\text{NRGR}}[x_a^\mu]\right) \equiv \int \mathcal{D}\bar{h}_{\mu\nu} \exp\left(iS_{\text{eff}}^{\text{WR}}[x_a^\mu, \bar{h}_{\mu\nu}] + iS_{\text{gf}}[\bar{h}]\right), \tag{1.42}$$

[15]This nomenclature is not standard in the NRGR literature; we use it in this dissertation to emphasize the difference between the two effective actions $iS_{\text{eff}}^{\text{WR}}[\bar{h}_{\mu\nu}, x_a^\mu]$ and $iS_{\text{eff}}^{\text{NRGR}}[x_a^\mu]$.

where $S_{\rm gf}[\bar h_{\mu\nu}]$ is the familiar

$$S_{\rm gf}[\bar h_{\mu\nu}] \equiv -M_{\rm Pl}^2 \int {\rm d}^4x\, \bar\Gamma_{\mu\nu}\bar\Gamma^{\mu\nu}, \quad \bar\Gamma_\mu \equiv \frac{1}{M_{\rm Pl}}\left(\partial^\nu \bar h_{\nu\mu} - \frac{1}{2}\partial_\mu \bar h\right). \tag{1.43}$$

The gauge-fixed WR effective action that enters the path integral (1.42) has the following form:

$$S_{\rm eff}^{\rm WR}[x_a^\mu, \bar h_{\mu\nu}] + S_{\rm gf} = S_{\rm cons}[x_a^\mu] + S^{(2)}[\bar h_{\mu\nu}] + S_{\rm int}[x_a^\mu, \bar h_{\mu\nu}] + \cdots\,, \tag{1.44}$$

where $S^{(2)}[\bar h_{\mu\nu}]$ is the quadratic action from which the propagator of the radiative modes is derived, $S_{\rm int}[x_a^\mu, \bar h_{\mu\nu}]$ is an interaction term describing the coupling of the worldlines to the radiation, and $S_{\rm cons}[x_a^\mu]$ is the PN-expanded action that yields the conservative equations of motion, which is, of course, unaffected by the integration of $\bar h_{\mu\nu}$ in (1.42). Finally, the ellipsis in (1.44) contains all pure-graviton vertices.

The diagramatic evaluation of the path integrals (1.39) and (1.42) will be discussed in the following sections.

1.3.2 Near-zone dynamics

Propagators. Before computing the near-zone diagrams that determine the conservative two-body Lagrangian, we need to derive a basic set of Feynman rules. First, we consider the graviton propagator.

Expanding the gauge-fixed Einstein-Hilbert action to quadratic order in the metric perturbation, one finds the usual Feynman propagator:

$$\begin{aligned}\left\langle T\{h_{\mu\nu}(x)h_{\alpha\beta}(y)\}\right\rangle &= \mathbb{P}_{\mu\nu\alpha\beta}\, D_{\rm f}(x-y) \\ &= \mathbb{P}_{\mu\nu\alpha\beta}\int \frac{{\rm d}^4k}{(2\pi)^4}\frac{i}{k^2+i\mathsf{a}}\, e^{-ik(x-y)},\end{aligned} \tag{1.45}$$

where the projector

$$\mathbb{P}_{\mu\nu\alpha\beta} \equiv \frac{1}{2}(\eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\mu\beta}\eta_{\nu\alpha} - \eta_{\mu\nu}\eta_{\alpha\beta}) \tag{1.46}$$

satisfies

$$\mathbb{P}_{\alpha\beta\mu\nu}\mathbb{P}^{\alpha\beta}{}_{\rho\sigma} = \frac{1}{2}(\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho}) \equiv \mathbb{1}_{\mu\nu|\rho\sigma}, \tag{1.47}$$

and $\mathbb{1}_{\mu\nu|\rho\sigma}$ is the identity in the space of symmetric rank-2 tensors. While the fully-relativistic propagator (1.45) is the correct one for both radiation and potential modes, to keep a manifest power-counting in v we perform a "quasi-instantaneous" expansion for the potential mode propagator. Since these modes never go on-shell, one can write

$$\frac{1}{\omega^2 - |\mathbf{k}|^2 + i\mathsf{a}} = -\frac{1}{|\mathbf{k}|^2}\sum_{n=0}^{\infty}\frac{\omega^{2n}}{\mathbf{k}^{2n}}. \tag{1.48}$$

The rigorous justification for the small $\omega/|\mathbf{k}|$ expansion under the integral sign, however, requires the *method of regions*, in which one carefully separates the different kinematical regions of the Feynman integral – See section 1.3.3 and reference [52]. One may wonder if the omission of the Feynman prescription in (1.48) leads to undesirable consequences in the long run. As long as we limit ourselves to the computation of the conservative two-body Lagrangian, nothing bad happens.[16] Off-shell modes are innocuous to the choice of Green's functions. Radiation reaction contributions coming from far-zone interactions (first treated in § 1.3.4) naturally require both potential and radiative modes, so that a careful analysis with the *in-in* formalism must be done. The basic elements of the *in-in* approach are presented in section 2.

Notice that the n-th term (for $n \neq 0$) of (1.48) is a v^n correction of the leading term. When writing the Feynman rules for the potential modes we will treat all $n \geq 0$ terms as operator insertions of the instantaneous $1/|\mathbf{k}|^2$ propagator. It is not difficult to see that the v^0 propagator in the mixed Fourier representation is

$$\left\langle H_{\mathbf{k}\mu\nu}(t_1) H_{\mathbf{q}\alpha\beta}(t_2) \right\rangle_{v^0} = -\mathbb{P}_{\mu\nu\alpha\beta} \frac{i}{\mathbf{k}^2} (2\pi)^3 \delta^{(3)}(\mathbf{k}+\mathbf{q}) \delta(t_1 - t_2). \tag{1.49}$$

The remaining contributions are also easy to compute. For example, at order v^2 one simply trades the factor of ω^2 in the numerator by derivates with respect to time. This gives

$$\left\langle H_{\mu\nu}(x_1) H_{\alpha\beta}(x_2) \right\rangle_{v^2} = -i\mathbb{P}_{\mu\nu\alpha\beta} \frac{\partial^2}{\partial t_1 \partial t_2} \delta(t_1 - t_2) \int_{\mathbf{k}} \frac{e^{i\mathbf{k}\cdot(\mathbf{x}_1 - \mathbf{x}_2)}}{\mathbf{k}^4}. \tag{1.50}$$

Thus,

$$\left\langle H_{\mathbf{k}\mu\nu}(t_1) H_{\mathbf{q}\alpha\beta}(t_2) \right\rangle_{v^2} = -\mathbb{P}_{\mu\nu\alpha\beta} \frac{i}{\mathbf{k}^4} (2\pi)^3 \delta^{(3)}(\mathbf{k}+\mathbf{q}) \frac{\partial^2}{\partial t_1 \partial t_2} \delta(t_1 - t_2). \tag{1.51}$$

At last, we introduce the diagrammatic rules. All instantaneous graviton propagators are depicted by a solid blue line, whereas velocity insertions are represented by a cross in the middle of the propagator line:

$$= -\mathbb{P}_{\mu\nu\alpha\beta} \frac{i}{\mathbf{k}^2} (2\pi)^3 \delta^{(3)}(\mathbf{k}+\mathbf{q}) \delta(t_1 - t_2). \tag{1.52a}$$

$$= -\mathbb{P}_{\mu\nu\alpha\beta} \frac{i}{\mathbf{k}^4} (2\pi)^3 \delta^{(3)}(\mathbf{k}+\mathbf{q}) \frac{\partial^2}{\partial t_1 \partial t_2} \delta(t_1 - t_2). \tag{1.52b}$$

Finally, all radiation gravitons are depited as wavy blue lines. For example, when computing the NRGR effective action $S_{\text{eff}}^{\text{NRGR}}[x_a^\mu]$ the following rule will be used:

$$= \mathbb{P}_{\mu\nu\alpha\beta} \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{i}{k^2 + i\mathsf{a}} e^{-ik(x-y)}. \tag{1.53}$$

When drawing the Feynman diagrams that describe the coupling of radiation gravitons with the wordlines, the external radiation modes will also be depicted by the wavy line above even though they do not have any propagator attached.

[16] We are glossing over some important subleties here, to be honest. High powers of inverse momenta coming from such propagators lead to IR divergencies [104] that can nevertheless be controlled.

Power counting scheme. From (1.51) it is easy do deduce that potential modes scale as $r^2 v^{1/2}$. Since the propagator of the radiative modes scales as $1/p^2 \sim r^2/v^2$, we have $\bar{h}_{\mu\nu} \sim v/r$. The basic set of power counting rules is summarized in the table 1.1 below.

Quantity	m/M_{Pl}	$H_{\mathbf{k}\mu\nu}$	$\bar{h}_{\mu\nu}$	$\partial_0 H_{\mathbf{k}\mu\nu}$	$\partial_\alpha \bar{h}_{\mu\nu}$
Scaling	$(Lv)^{1/2}$	$r^2 v^{1/2}$	v/r	$(v/r) H_{\mathbf{k}\mu\nu}$	$(v/r)\bar{h}_{\mu\nu}$

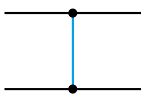
Table 1.1: Basic power counting rules for NRGR.

From these rules one can determine the precise PN order of any NRGR diagram involved in the computation of $S_{\text{eff}}^{\text{NRGR}}[x_a^\mu]$. For example, the Newtonian term in the point particle action scales as

$$S_{\text{pp}} \supset \frac{1}{2} \int \mathrm{d}t \sum_a m_a v_a^2 \sim mrv \sim L, \tag{1.54}$$

where L denotes the orbital angular momentum. For typical binaries of compact objects, this quantity is huge! Nevertheless, as we shall see in forthcoming examples, all diagrams contributing to $S_{\text{eff}}^{\text{NRGR}}[x_a^\mu]$ scale as L times some power of v, which is a small parameter in the inspiral phase. We thus have a well-organized perturbative expansion.

Besides the leading kinetic term, the single exchange-diagram that yields the Newtonian potential also scales as L (see (1.62) for more details on this simple computation). The result is, of course



$$= i \int \mathrm{d}t \, \frac{G m_1 m_2}{|\mathbf{x}_1(t) - \mathbf{x}_2(t)|} \sim \left(\mathrm{d}t \, \mathrm{d}^3\mathbf{k} \, \frac{m}{M_{\text{Pl}}} H_{\mathbf{k}} \right)^2 \sim L. \tag{1.55}$$

Regarding generic NRGR diagrams, the following lemmata can be established [60].

Lemma 1. *All contributions to the effective action $S_{\text{eff}}^{\text{NRGR}}[x_a^\mu]$ are down by powers of v with respect to the Newtonian kinetic and potential energies.*

Lemma 2. *After all particle worldlines are stripped off the NRGR diagrams, graviton loops are supressed by powers of $1/L$ with respect to trees.*

The lesson we extract from **Lemma 1** is that all effective operators scaling with Lv^n, for some positive integer n, will be treated perturbatively with respect to the leading Lv^0 operators. Moreover, **Lemma 2** teaches us that graviton loops can be neglected in the classical limit.

Proof (Lemma 2). Consider a general NRGR diagram with N_g graviton self-interaction vertices, N_m insertions of the point-particle worldline, and P graviton propagators. Since the Einstein-Hilbert action (1.29) is proportional to M_{Pl}^2, each graviton self-interaction vertex brings a factor of M_{Pl}^2 to the diagram. Each worldline coupling brings a factor of m, and the only other scale that can enter the power counting is the orbital distance r (besides powers of v, which is obviously dimensionless). Recall that we are using natural units such that $c = \hbar = 1$, so mass has units of inverse length. Since $S_{\text{eff}}^{\text{NRGR}}$ is dimensionless, a general connected NRGR diagram must scale as

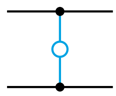
$$m^{N_m}(M_{\text{Pl}}^2)^{N_g}\left(\frac{1}{M_{\text{Pl}}^2}\right)^P r^{N_m+2N_g-2P}v^{\#}, \tag{1.56}$$

for some power $\#$ of v, which we do not specify. Using the virial theorem $v^2 \sim G_{\text{N}}m/r$, we can replace $M_{\text{Pl}}^{-2} \sim v^2 r/m$. Thus, our generic diagram scales as

$$(mr)^{N_m+N_g-P}v^{\#} \sim L^{V-P}v^{\#}, \tag{1.57}$$

where $V \equiv N_m + N_g$ is the total number of vertices in the graph and $L \sim mrv$ is the angular momentum. The simplest diagram one can think of is the trivial free wordline graph with $V = 1$, $P = 0$. Building up from this diagram, we can add propagators and vertices as we like to form more complicated diagrams. To guarantee that the resulting graph is *connected*, for each new vertex we must include at least one graviton line, so that $V - P \leq 1$.

Now, recall that the number of graviton loops ℓ in a graph is equal to the number of *undetermined momenta*, i.e., unconstrained by momentum conservation delta functions. This leads to $\ell = P - (V - 1)$. The reason we subtract $V - 1$ (and not simply V) in this relation is because the last vertex added does not constrain internal momenta; it simply enforces overall momentum conservation. We conclude that a generic NRGR diagram scales as $L^{1-\ell}v^{\#}$, for integer $\ell \geq 0$. For example, a potential exchange diagram with one graviton loop scales as



$$\sim\ L^0 v^{\#}. \tag{1.58}$$

A similar graph with two loops scales as $L^{-1}v^{\#}$, and so on. If we restore units of $\hbar$ for a moment, the diagrammatic series for $S_{\text{eff}}^{\text{NRGR}}$ has the following overall structure:

$$\exp\left(\frac{i}{\hbar}S_{\text{eff}}^{\text{NRGR}}[x_a]\right) = \exp\left[\frac{iL}{\hbar}\left(v^{\#} + \frac{\hbar}{L}v^{\#} + \frac{\hbar^2}{L^2}v^{\#} + \cdots\right)\right]. \tag{1.59}$$

Since $\hbar/L \ll 1$ for typical compact binary sytems, we can safely neglect graviton loops. This finishes the proof that graviton loops are supressed by powers of $1/L$ with respect to trees. ■

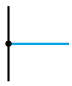
As a final remark, we mention that the relation between classical physics and loop diagram in quantum field theory is more subtle than what is usually expressed in textbook references. As Holstein and Donoghue show in their seminal paper [64], when at least two massless propagators are present in a loop diagram, its expression may contain contributions to classical observables (e.g., exchange potentials between charged particles), even though most standard textbook discussions seem to suggest the naive picture that “the loop expansion is purely quantum because it is an $\hbar$ expansion.” Crucially, the argument put forth in [64] requires that the loop diagrams in question involves both massless and massive propagators. This is not the case in NRGR; we have massless particles (gravitons) interacting with classical sources which do not have any massive propagator attached. If we were to replace the NRGR worldlines by massive propagators, then one can show that the classical contribution of loop diagrams is isolated by performing a large-mass limit [120].

In recent years, a wealth of formalisms have been developed for systematically extracting classical physics from "genuine" quantum field theoretic calculations (by "genuine" we mean that no wordline methods are employed, just quantum fields). Partial reviews of such developments can be found in [22] [78].

Graviton-worldline vertices. The final set of rules we need to derive before turning to the explicit computation of more complicated near-zone diagrams consists of graviton-wordline vertices. Expanding the point particle action, the H_{00} term is found to be

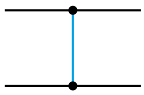
$$iS_{\rm pp}^{H_{00}} = -\frac{im}{2M_{\rm Pl}}\int {\rm d}x^0 \int_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}(x^0)}\eta_{\mu 0}\eta_{\nu 0}H_{\mathbf{k}\mu\nu}(x^0). \tag{1.60}$$

Therefore,



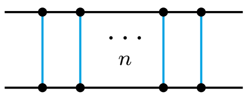
$$= -\frac{im}{2M_{\rm Pl}}\int {\rm d}t \int_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}(t)}\eta^{\mu 0}\eta^{\nu 0} \sim L^{1/2}. \tag{1.61}$$

With this rule in hands one can easily derive the Newtonian potential. We have

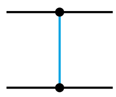


$$\begin{aligned} &= \frac{im_1}{2M_{\rm Pl}}\int {\rm d}t_1 \int_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}_1(t_1)}\eta^{\mu 0}\eta^{\nu 0}\langle H_{\mathbf{k}\mu\nu}(t_1)H_{\mathbf{q}\alpha\beta}(t_2)\rangle_{v^0} \\ &\quad\times \frac{im_2}{2M_{\rm Pl}}\int {\rm d}t_2 \int_{\mathbf{q}} e^{i\mathbf{q}\cdot\mathbf{x}_2(t_2)}\eta^{\alpha 0}\eta^{\beta 0}. \end{aligned} \tag{1.62}$$

Using $\mathbb{P}_{0000} = 1/2$, the formula (1.49) and the fundamental integral (A.2), one obtains (1.55), as expected. By definition of the effective action, the single-exchange diagram exponentiates to yield the correct path integral as defined in (1.39). As a result, there is no need to compute diagrams with two potential mode exchanges, for example, in order to recover the Newtonian potential. It is worth emphasizing that such exponentiation relies on the fact that wordlines have no propagators associated to them, so that diagrams with multiple potential exchanges are simply products of the fundamental graph (1.62). Therefore,



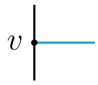
$$= \frac{1}{n!}\left(\quad\right)^n \tag{1.63}$$



In general, as we shall see in the computation of the complete 1PN Lagrangian, the action $iS_{\rm eff}^{\rm NRGR}[x_a^\mu]$ is the sum of all diagrams that remain connected when all particle worldlines are removed. Moving on with the derivation of the vertices, we consider the $H_{0i}v^i$ term of the point-particle action:

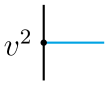
$$\begin{aligned} iS_{\text{pp}}^{H_{0i}v^i} &= -\frac{im}{M_{\text{Pl}}}\int \mathrm{d}x^0\, H_{0i}(x)v^i \\ &= -\frac{im}{M_{\text{Pl}}}\int \mathrm{d}x^0 \int_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}(x^0)} H_{\mathbf{k}\nu\mu}\eta_{i\mu}\eta_{\nu 0}v^i. \end{aligned} \tag{1.64}$$

The inferred diagrammatic rule is thus



$$= -\frac{im}{M_{\text{Pl}}}\int \mathrm{d}x^0 \int_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}(x^0)}\eta^{\nu 0}\eta^{i\mu}v^i(x^0) \sim L^{1/2}v. \tag{1.65}$$

The $H_{00}v^2$ rule follows similarly. One finds

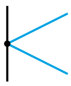


$$= -\frac{im}{2M_{\text{Pl}}}\int \mathrm{d}x^0 \int_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}(x^0)} \left(\eta_{i\mu}\eta_{j\nu}v^iv^j + \frac{1}{2}\eta_{\mu 0}\eta_{\nu 0}v^2\right) \sim L^{1/2}v^2. \tag{1.66}$$

Vertices with multiple gravitons attached to the wordline are also straightforward to derive. For instance, consider the H_{00}^2 vertex that is needed in the computation of the 1PN Lagrangian. The relevant term of the point-particle action is

$$iS_{\text{pp}}^{H_{00}^2} = \frac{im}{8M_{\text{Pl}}^2}\int \mathrm{d}x^0 \int_{\mathbf{k},\mathbf{q}} e^{i(\mathbf{k}+\mathbf{q})\cdot\mathbf{x}(x^0)} H_{\mathbf{k}00}(x^0)H_{\mathbf{q}00}(x^0). \tag{1.67}$$

Since there are two possible ways to assign two pairs of indices to the potential modes above, we have



$$= \frac{im}{4M_{\text{Pl}}^2}\int \mathrm{d}x^0 \int_{\mathbf{k},\mathbf{q}} e^{i(\mathbf{k}+\mathbf{q})\cdot\mathbf{x}(x^0)}\, \eta_{\mu 0}\,\eta_{\nu 0}\,\eta_{\alpha 0}\,\eta_{\beta 0} \sim v^2. \tag{1.68}$$

The Einstein-Infeld-Hoffmann Lagrangian. As an illustrate application of the formalism developed so far, we compute the complete 1PN conservative Lagrangian of the two-body problem. The diagrams we need to evaluate at this order are presented in the figures 1.2 and 1.3.

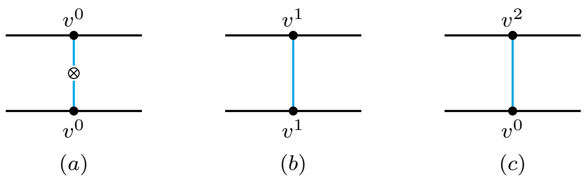


Figure 1.2: Single-exchange near-zone diagrams required in the computation of the EIH Lagrangian.

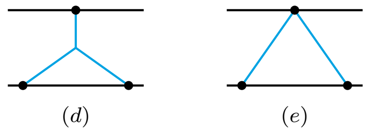
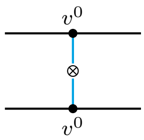


Figure 1.3: The three-graviton vertex graph (d) and the seagull (e) diagram entering at 1PN order. All graviton-worldline couplings are of order v^0 here.

Diagram (a)**.** We start with the computation of the single-exchange diagrams of figure 1.2. For the first diagram, the application of the NRGR rules leads to the expression



$$= -\frac{m_1 m_2}{4M_{\rm Pl}^2}\int \mathrm{d}t_1\,\mathrm{d}t_2 \int_{\mathbf{k},\mathbf{q}} e^{i\mathbf{k}\cdot\mathbf{x}_1(t_1)} e^{i\mathbf{q}\cdot\mathbf{x}_2(t_2)} \big\langle H_{\mathbf{k}00}(t_1) H_{\mathbf{q}00}(t_2)\big\rangle_{v^2}. \tag{1.69}$$

Substituting (1.50) and simplifying, we find

$$(a) = \frac{i m_1 m_2}{8M_{\rm Pl}^2}\int \mathrm{d}t_1\,\mathrm{d}t_2 \int_{\mathbf{q}} \frac{1}{\mathbf{q}^4}\, e^{i\mathbf{q}\cdot[\mathbf{x}_2(t_2)-\mathbf{x}_1(t_1)]}\,\frac{\partial^2}{\partial t_1 \partial t_2}\,\delta(t_1 - t_2). \tag{1.70}$$

To deal with the derivatives of Dirac deltas we exploit the following distributional identity, which holds for well-behaved functions $f(x)$ [116]:

$$\int \mathrm{d}x\, f(x)\,\frac{\mathrm{d}^n}{\mathrm{d}x^n}\,\delta(x - x') = (-1)^n \frac{\mathrm{d}^n f(x)}{\mathrm{d}x^n}\bigg|_{x=x'}. \tag{1.71}$$

Therefore,

$$\begin{aligned}(a) &= \frac{i m_1 m_2}{8M_{\rm Pl}^2}\int \mathrm{d}t \int_{\mathbf{q}} \frac{1}{\mathbf{q}^4}\,[i\mathbf{q}\cdot\mathbf{v}_2(t)][-i\mathbf{q}\cdot\mathbf{v}_1(t)]\, e^{i\mathbf{q}\cdot[\mathbf{x}_2(t)-\mathbf{x}_1(t)]} \\ &= \frac{i m_1 m_2}{8M_{\rm Pl}^2}\int \mathrm{d}t\, v_1^i(t) v_2^j(t) \int_{\mathbf{q}} \frac{q_i q_j}{\mathbf{q}^4}\, e^{i\mathbf{q}\cdot[\mathbf{x}_2(t)-\mathbf{x}_1(t)]}.\end{aligned} \tag{1.72}$$

Evaluating the remaining three-dimensional integral with (A.1) and simplifying, one obtains

$$(a) = i\int \mathrm{d}t\, \frac{G m_1 m_2}{2r}\left[\mathbf{v}_1\cdot\mathbf{v}_2 - \frac{(\mathbf{r}\cdot\mathbf{v}_1)(\mathbf{r}\cdot\mathbf{v}_2)}{r^2}\right], \tag{1.73}$$

where $r \equiv |\mathbf{x}_1(t) - \mathbf{x}_2(t)|$.

Diagram (b)**.** Using the vertex (1.65) and substituting (1.49), we have

$$\begin{aligned}(b) &= \frac{i m_1 m_2}{M_{\rm Pl}^2}\int \mathrm{d}t \int_{\mathbf{k},\mathbf{q}} e^{i\mathbf{k}\cdot\mathbf{x}_1(t)} e^{i\mathbf{q}\cdot\mathbf{x}_2(t)} v_1^i(t) v_2^j(t)\, \mathbb{P}_{i0j0}\, \frac{1}{\mathbf{k}^2}\,(2\pi)^3 \delta^{(3)}(\mathbf{k}+\mathbf{q}) \\ &= -\frac{i m_1 m_2}{2M_{\rm Pl}^2}\int \mathrm{d}t\,(\mathbf{v}_1\cdot\mathbf{v}_2)\int_{\mathbf{q}} \frac{1}{\mathbf{q}^2}\, e^{i\mathbf{q}\cdot[\mathbf{x}_2(t)-\mathbf{x}_1(t)]}.\end{aligned} \tag{1.74}$$

Once again, using (A.2) we find

$$(b) = -4i \int \mathrm{d}t\, \frac{Gm_1 m_2}{r}\, (\mathbf{v}_1 \cdot \mathbf{v}_2). \tag{1.75}$$

Diagram (c)**.** Without loss of generality, let us take particle 1 to be represented by the upmost worldline in the (c) diagram of figure 1.2. Using (1.66), we find

$$(c) = \frac{i m_1 m_2}{4M_{\mathrm{Pl}}^2} \int \mathrm{d}t_1\, \mathrm{d}t_2 \int_{\mathbf{k},\mathbf{q}} e^{i\mathbf{k}\cdot\mathbf{x}_1(t_1)} e^{i\mathbf{q}\cdot\mathbf{x}_2(t_2)} \eta_{\alpha 0}\eta_{\beta 0} \left(\eta_{\mu i}\eta_{\nu j} v_1^i v_1^j + \frac{1}{2}\eta_{\mu 0}\eta_{\nu 0}\mathbf{v}_1^2 \right) \\ \times \mathbb{P}_{\mu\nu\alpha\beta}\, \frac{1}{\mathbf{k}^2}\, \delta(t_1 - t_2)\, (2\pi)^3 \delta^{(3)}(\mathbf{k}+\mathbf{q}). \tag{1.76}$$

Simplifying the integrals with the Dirac deltas,

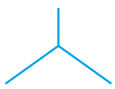
$$(b) = \frac{i m_1 m_2}{4M_{\mathrm{Pl}}^2} \int \mathrm{d}t \int_{\mathbf{q}} \frac{1}{\mathbf{q}^2}\, e^{i\mathbf{q}\cdot[\mathbf{x}_2(t) - \mathbf{x}_1(t)]} \left(\mathbb{P}_{ij00} v_1^i v_1^j + \frac{1}{2}\mathbb{P}_{0000}\mathbf{v}_1^2 \right). \tag{1.77}$$

Finally, using $\mathbb{P}_{ij00} = \frac{1}{2}\delta_{ij}$ and $\mathbb{P}_{0000} = \frac{1}{2}$, one gets (after adding the contribution from the mirror-image diagram under $1 \leftrightarrow 2$ exchange)

$$(c) = i \int \mathrm{d}t\, \frac{3Gm_1 m_2}{2r}\, (\mathbf{v}_1^2 + \mathbf{v}_2^2). \tag{1.78}$$

Diagram (d)**.** Before computing this diagram, it is instructive to pause and ponder on why we must consider this topology at 1PN order in the first place [56]. Let us begin our brief dicussion by first considering *static* couplings to the worldlines, so that only the h_{00} polarization is taken into account. To leading-order, and in the absence of propagator insertions, the only possible source of powers of v is the three-graviton vertex. *Naively*, since this vertex receives contributions from terms like $h_{00}(\partial_0 h_{00})^2$, one expects contributions of order v^2 due to the time derivatives. How come this topology contribute to the 1PN Lagrangian?

One must remember that the graviton propagator $\langle T\{h_{\mu\nu}h_{\alpha\beta}\}\rangle$ is proportional to the projector $\mathbb{P}_{\mu\nu\alpha\beta}$. Consequently, the h_{00} polarization can mix with the trace of h_{ij} given that $\langle T\{h_{00}h_{ij}\}\rangle \sim \delta_{ij}$. As a result, the h_{00} gravitons coupled to the worldlines may end up as either h_{00} or h_{ii} when the three-graviton vertex is inserted. In conclusion, the correct vertex rule to be used in diagram (d) is *not* the complete three-$h_{\mu\nu}$ vertex, but rather the three-point function $\langle T\{H_{00}H_{00}H_{00}\}\rangle$. To derive its expression, one contracts the complete three-$h_{\mu\nu}$ vertex with three propagators and sets all remaining free indices to zero. The terms without time derivatives conspire to produce [60]



$$= -\frac{i}{4M_{\mathrm{Pl}}}\delta(t_1 - t_2)\delta(t_1 - t_3)(2\pi)^3\delta^{(3)}(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3)\, i^3\, \frac{(\mathbf{k}_1^2 + \mathbf{k}_2^2 + \mathbf{k}_3^2)}{\mathbf{k}_1^2\,\mathbf{k}_2^2\,\mathbf{k}_3^2}. \tag{1.79}$$

It is not difficult to see that this vertex scales as $L^{-1/2}v^2$. For the sake of clarity, the explicit power counting of diagram (d) is

$$(d) \sim (L^{1/2})^3 \left(\mathrm{d}x^0 \, (\mathrm{d}^3\mathbf{k})^3 \delta^{(3)}(\mathbf{k}) \, \frac{\mathbf{k}^2 H_{00}^3}{M_{\rm Pl}} \right) \sim L^{3/2} \, \frac{r}{v} \, \frac{1}{(r^3)^3} \, r^3 \, \frac{(r^2 v^{1/2})^3}{r^2 M_{\rm Pl}} \sim Lv^2. \tag{1.80}$$

Proceeding to the actual computation, we have

$$\begin{aligned}(d) = \frac{i m_1 m_2}{16 M_{\rm Pl}^3} \int \mathrm{d}t_1 \, \mathrm{d}t_2 \, \mathrm{d}t_2' \int_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3} e^{i\mathbf{k}_1\cdot\mathbf{x}_1(t_1)} e^{i\mathbf{k}_2\cdot\mathbf{x}_2(t_2)} e^{i\mathbf{k}_3\cdot\mathbf{x}_2(t_2')} \, \frac{(-i)^2}{4M_{\rm Pl}} \\ \times \, \delta(t_1 - t_2)\delta(t_1 - t_2')(2\pi)^3 \delta^{(3)}(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3) \, \frac{(\mathbf{k}_1^2 + \mathbf{k}_2^2 + \mathbf{k}_3^2)}{\mathbf{k}_1^2 \, \mathbf{k}_2^2 \, \mathbf{k}_3^2}\end{aligned} \tag{1.81}$$

As done previously when computing diagram (c), we take the upmost wordline to be particle 1. We have three terms contributing to (1.81), which we write as

$$(d)_1 = \int_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3} \frac{1}{\mathbf{k}_2^2 \, \mathbf{k}_3^2} \, e^{i\mathbf{k}_1\cdot\mathbf{x}_1(t)} e^{i(\mathbf{k}_2+\mathbf{k}_3)\cdot\mathbf{x}_2(t)} (2\pi)^3 \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3), \tag{1.82a}$$

$$(d)_2 = \int_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3} \frac{1}{\mathbf{k}_1^2 \, \mathbf{k}_3^2} \, e^{i\mathbf{k}_1\cdot\mathbf{x}_1(t)} e^{i(\mathbf{k}_2+\mathbf{k}_3)\cdot\mathbf{x}_2(t)} (2\pi)^3 \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3), \tag{1.82b}$$

$$(d)_3 = \int_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3} \frac{1}{\mathbf{k}_1^2 \, \mathbf{k}_2^2} \, e^{i\mathbf{k}_1\cdot\mathbf{x}_1(t)} e^{i(\mathbf{k}_2+\mathbf{k}_3)\cdot\mathbf{x}_2(t)} (2\pi)^3 \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3), \tag{1.82c}$$

so that

$$(d) = -\frac{i m_1 m_2^2}{64 M_{\rm Pl}^4} \int \mathrm{d}t \, \big[(d)_1 + (d)_2 + (d)_3\big]. \tag{1.83}$$

Consider the term $(d)_2$ for example. The momentum-conserving delta can be used to eliminate the integral over $\mathbf{k}_2$ by setting $\mathbf{k}_2 = -\mathbf{k}_1 - \mathbf{k}_3$. Then,

$$(d)_2 = \left(\int_{\mathbf{k}_3} \frac{1}{\mathbf{k}_3^2} \right) \int_{\mathbf{k}_1} \frac{1}{\mathbf{k}_1^2} \, e^{i\mathbf{k}_1\cdot(\mathbf{x}_1 - \mathbf{x}_2)}. \tag{1.84}$$

The integral in parenthesis is scaleless, so it vanishes in dimensional regularization (see also the discussion on §6.3.1 of [104]). This term and $(d)_3$ amount to contact interactions of the form depicted in figure 1.4. At last, $(d)_1$ yields the actual value of the diagram.

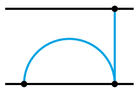


Figure 1.4: Singular "self-energy" diagram.

Eliminating $\mathbf{k}_1$, one finds

$$(d) = -\frac{i m_1 m_2^2}{64 M_{\rm Pl}^4} \int \mathrm{d}t \int_{\mathbf{k}_2,\mathbf{k}_3} \frac{1}{\mathbf{k}_2^2 \, \mathbf{k}_3^2} \, e^{i\mathbf{k}_2\cdot(\mathbf{x}_2 - \mathbf{x}_1) + i\mathbf{k}_3\cdot(\mathbf{x}_2 - \mathbf{x}_1)}. \tag{1.85}$$

Using (A.2) and including the mirror image graph, the answer takes the form

$$(d) = -i \int \mathrm{d}t \, \frac{G^2 m_1 m_2 (m_1 + m_2)}{r^2}. \tag{1.86}$$

Diagram (e)**.** Finally, we consider the seagull diagram. Including the symmetry factor, we have

$$\begin{aligned}(e) = &\frac{1}{2} \frac{i m_1}{4 M_{\mathrm{Pl}}^2} \int \mathrm{d}t_1 \int_{\mathbf{k},\mathbf{q}} e^{i(\mathbf{k}+\mathbf{q})\cdot \mathbf{x}_1(t_1)} \eta_{\sigma 0}\eta_{\rho 0}\eta_{\xi 0}\eta_{\lambda 0} \\ &\times \frac{i m_2}{2 M_{\mathrm{Pl}}} \int \mathrm{d}t_2 \int_{\mathbf{k}'} e^{i\mathbf{k}'\cdot \mathbf{x}_2(t_2)} \eta_{\mu 0}\eta_{\nu 0} \frac{i m_2}{2 M_{\mathrm{Pl}}} \int \mathrm{d}t_2' \int_{\mathbf{q}'} e^{i\mathbf{q}'\cdot \mathbf{x}_2(t_2')} \eta_{\alpha 0}\eta_{\beta 0} \\ &\times \big\langle H_{\mathbf{k}'\mu\nu}(t_2) H_{\mathbf{k}\sigma\rho}(t_1) \big\rangle \big\langle H_{\mathbf{q}'\alpha\beta}(t_2') H_{\mathbf{q}\xi\lambda}(t_1) \big\rangle.\end{aligned} \tag{1.87}$$

Inserting the propagators and simplifying, one obtains

$$(e) = \frac{i m_1 m_2^2}{2^7 M_{\mathrm{Pl}}^4} \int \mathrm{d}t \int_{\mathbf{k},\mathbf{q}} e^{i\mathbf{k}\cdot(\mathbf{x}_2 - \mathbf{x}_1)} e^{i\mathbf{q}\cdot(\mathbf{x}_2-\mathbf{x}_1)} \frac{1}{\mathbf{k}^2 \mathbf{q}^2}. \tag{1.88}$$

Thus,

$$(e) = i \int \mathrm{d}t \, \frac{G^2 m_1 m_2 (m_1 + m_2)}{2 r^2}. \tag{1.89}$$

where the mirror image diagram was added.

Collecting all terms. We can finally write down the complete 1PN Lagrangian. But first, we need the $\mathcal{O}(v^2)$ correction to the kinetic term. Simply expanding the relativistic point-particle action one easily finds that

$$K_{\mathrm{1PN}} = \frac{1}{8} \sum_a m_a \mathbf{v}_a^4. \tag{1.90}$$

Combining (1.73), (1.75), (1.78), (1.86), (1.89), and (1.90), we obtain the famous Einstein-Infeld-Hoffmann Lagrangian:

$$\begin{aligned}L_{\mathrm{EIH}} = &\frac{1}{8} \sum_a m_a \mathbf{v}_a^4 + \frac{G m_1 m_2}{2r} \bigg[3(\mathbf{v}_1^2 + \mathbf{v}_2^2) - 7(\mathbf{v}_1 \cdot \mathbf{v}_2) \\ &- \frac{(\mathbf{r}\cdot\mathbf{v}_1)(\mathbf{r}\cdot\mathbf{v}_2)}{r^2} \bigg] - \frac{G^2 m_1 m_2 (m_1+m_2)}{2r^2}.\end{aligned} \tag{1.91}$$

The derivation of the 1PN Lagrangian finishes our discussion on the near-zone dynamics within the NRGR framework. The computations sketched here have been vigorously developed in recent years. Notably, EFT methods were successfull in the derivation of the complete (spinless) 4PN conservative Lagrangian [51] [49]. To do so, one must take into account the *tail effect*, a hereditary process taking place in the far-zone. These results perfectly agree with traditional GR methods [69] [19] [39] [70] [40] [17] [18].

1.3.3 Interplay between near and far zones

Before showing how one can construct a far-zone effective action starting from near-zone computation, let us make some general comments regarding the near-far separation in NRGR. The following discussion is adapted from [52].

As a concrete example, consider a "tree-level" NRGR process[17] involving arbitrary graviton-worldline vertices V but only one graviton exchange (i.e., only one propagator). Instead of performing the quasi-instantaneous expansion from the outset, let us retain the full expression of the retarded propagator. For later use of dimensional regularization, we write the contribution of this process to the effective action in $d+1$ dimensions. Therefore, in the remainder of this section the $(d+1)$-dimensional Newton's is given by $\mathcal{G} \equiv (32\pi\Lambda^2)^{-1}$ and momentum integrals are written as

$$\int_k \equiv \int \frac{\mathrm{d}^{d+1}k}{(2\pi)^{d+1}}. \tag{1.92}$$

Moreover, write $x_a = (t_a, \mathbf{x}_a(t_a))$ and $x_b = (t_b, \mathbf{x}_b(t_b))$. Then, for the process described before, the effective action receives a contribution of the form

$$S_{\text{eff}} = \frac{1}{\Lambda^2}\sum_{a,b=1}^{2}\int \mathrm{d}t_a\,\mathrm{d}t_b \int_{k,p} V_a(t_a)\,e^{-ikx_a} V_b(t_b)\,e^{-ipx_b}\frac{1}{R(k)}\,\delta(k+p), \tag{1.93}$$

where $V(t)$ denotes the graviton-worldline vertex and

$$R(k) \equiv (\omega + i\mathsf{a})^2 - \mathbf{k}^2, \tag{1.94a}$$

$$A(k) \equiv (\omega - i\mathsf{a})^2 - \mathbf{k}^2. \tag{1.94b}$$

are the denominators of the retarded and advanced Green functions, respectively. The delta function $\delta(k+p)$ is assumed to be $(d+1)$-dimensional as well. Next, we define the off-shell instantaneous expansion of the momentum-space propagator by

$$\frac{1}{\omega^2 - |\mathbf{k}|^2 + i\mathsf{a}} \longrightarrow N(k) \equiv -\frac{1}{|\mathbf{k}|^2}\sum_{n=0}^{\infty}\left(\frac{\omega}{|\mathbf{k}|}\right)^{2n}. \tag{1.95}$$

The expression (1.93) includes all $1 \leftrightarrow 2$ interactions and all "self-interactions." Notice that using either $R(k)$ and $A(k)$ in S_{eff} leads to the same expression, since $R(-\omega,\mathbf{k}) = A(\omega,\mathbf{k})$ and we integrate over all momenta and time coordinates. With this setup, one can prove the following statement.

Claim. "The integral over the momentum k of the exact Green's function can be recast as

[17] By "tree-level" we mean leading order in G_{N}, as opposed to contributions which carry post-Minkowskian corrections such as the tail effect [58].

the sum of instantaneous propagators $N(k)$ and exact propagators multiplying multipole-expanded sources terms

$$e^{-ik_a x_a} = e^{-i\omega t_a + i\mathbf{k}\cdot\mathbf{x}_a} = e^{-i\omega t}\sum_{n=0}^{\infty}\frac{i^n}{n!}[\mathbf{k}\cdot\mathbf{x}_a(t_a)]^n, \quad |\mathbf{k}|r_a < 1. \tag{1.96}$$

where $r_a \equiv |\mathbf{x}_a|$."

The proof of this result amounts to write (1.93) as

$$\begin{aligned} S_{\text{eff}} = \sum_{a,b=1}^{2}\int \mathrm{d}t_a\,\mathrm{d}t_b\int_k V_a(t_a)V_b(t_b)\bigg[e^{-i\omega t_{ab}+i\mathbf{k}\cdot\mathbf{x}_{ab}}N(k) \\ + \sum_{n,m\geq 0}\frac{(i\mathbf{k}\cdot\mathbf{x}_a)^n}{n!}\frac{(-i\mathbf{k}\cdot\mathbf{x}_b)^m}{m!}\frac{1}{2}\left(\frac{1}{R(k)}+\frac{1}{A(k)}\right)+\mathcal{S}_{ab}(k)\bigg], \end{aligned} \tag{1.97}$$

where $t_{ab} \equiv t_a - t_b$, $x_{ab} \equiv |\mathbf{x}_a - \mathbf{x}_b|$, and

$$\begin{aligned} \mathcal{S}_{ab}(k) \equiv e^{-i\omega t_{ab}-i\mathbf{k}\cdot\mathbf{x}_{ab}}\left[\frac{1}{2}\left(\frac{1}{R(k)}-\frac{1}{A(k)}\right)-N(k)\right] \\ - \sum_{n,m\geq 0}\frac{(i\mathbf{k}\cdot\mathbf{x}_a)^n}{n!}\frac{(-i\mathbf{k}\cdot\mathbf{x}_b)^m}{m!}\frac{1}{2}\left(\frac{1}{R(k)}+\frac{1}{A(k)}\right). \end{aligned} \tag{1.98}$$

It is a simple matter to substitute $\mathcal{S}_{ab}(k)$ back into (1.97) and recover the original expression for S_{eff} given by (1.93).

Then, one shows that the integral of $\mathcal{S}_{ab}(k)$ over the four-momentum k vanishes after appropriately separating the kinematical regions and using dimensional regularization to make a scaleless term disappear. Let us perform these steps in detail now.

Proof. Introduce a momentum scale κ such that $\frac{1}{r} < \kappa < \Omega$, where Ω is the characteristic far-zone frequency scale. We split spatial momenta $|\mathbf{k}|$ into two regions: a *hard* region where $|\mathbf{k}| > \kappa$ and a *soft* region where $|\mathbf{k}| < \frac{1}{r}$. In the former region we are allowed to expand the Green functions as $N(k)$, whereas in the latter region the exponentials are expanded as (1.96). We write

$$\mathcal{I} \equiv \int_k \mathcal{S}_{ab}(k) = \int_k \big[\theta(|\mathbf{k}|-\kappa)\,\mathcal{S}_{ab}(k) + \theta(\kappa-|\mathbf{k}|)\,\mathcal{S}_{ab}(k)\big]. \tag{1.99}$$

Notice that $\mathcal{S}_{ab}(k)$ is a sum of two terms. We split the four terms in (1.99) as

$$
\begin{aligned}
\mathcal{I} = & \int_k \theta(|\mathbf{k}| - \kappa)\, e^{-i\omega t_{ab} + i\mathbf{k}\cdot\mathbf{x}_{ab}} \left[\frac{1}{2}\left(\frac{1}{R} + \frac{1}{A}\right) - N\right] \\
& - \int_k \theta(|\mathbf{k}| - \kappa) \sum_{n,m\geq 0} \frac{(i\mathbf{k}\cdot\mathbf{x}_a)^n}{n!} \frac{(-i\mathbf{k}\cdot\mathbf{x}_b)^m}{m!} N \\
& + \int_k \theta(\kappa - |\mathbf{k}|) \left[e^{-i\omega t_{ab} + i\mathbf{k}\cdot\mathbf{x}_{ab}} - \sum_{n,m\geq 0} \frac{(i\mathbf{k}\cdot\mathbf{x}_a)^n}{n!} \frac{(-i\mathbf{k}\cdot\mathbf{x}_b)^m}{m!}\right] \frac{1}{2}\left(\frac{1}{R} + \frac{1}{A}\right) \\
& - \int_k \theta(\kappa - |\mathbf{k}|)\, e^{-i\omega t_{ab} + i\mathbf{k}\cdot\mathbf{x}_{ab}} N.
\end{aligned}
\tag{1.100}
$$

The exponentials in the last line above can be expanded inside the soft-momenta integral. Then, we combine this term with the one in the second line using the identity $\theta(|\mathbf{k}| - \kappa) + \theta(\kappa - |\mathbf{k}|) = 1$. The result is

$$
\begin{aligned}
\mathcal{I} = & \int_k \theta(|\mathbf{k}| - \kappa)\, e^{-i\omega t_{ab} + i\mathbf{k}\cdot\mathbf{x}_{ab}} \left[\frac{1}{2}\left(\frac{1}{R} + \frac{1}{A}\right) - N\right] \\
& + \int_k \theta(\kappa - |\mathbf{k}|) \left[e^{-i\omega t_{ab} + i\mathbf{k}\cdot\mathbf{x}_{ab}} - \sum_{n,m\geq 0} \frac{(i\mathbf{k}\cdot\mathbf{x}_a)^n}{n!} \frac{(-i\mathbf{k}\cdot\mathbf{x}_b)^m}{m!}\right] \frac{1}{2}\left(\frac{1}{R} + \frac{1}{A}\right) \\
& - \int_k \sum_{n,m\geq 0} \frac{(i\mathbf{k}\cdot\mathbf{x}_a)^n}{n!} \frac{(-i\mathbf{k}\cdot\mathbf{x}_b)^m}{m!} N.
\end{aligned}
\tag{1.101}
$$

The last integral is scaleless because it is a product of momentum monomials; it vanishes in dimensional regularization. The first and second terms vanish identically because of the expansion of Green functions and exponentials in the appropriate momentum regions. This finishes the proof that $\mathcal{I}$ is zero. ■

This analysis can be extended to more complicated diagrams involving hereditary processes as well as diagrams within the *in-in* formulation of NRGR. The takeaway message remains the same: both near and far-zone contributions can be neatly traced back to Feynman diagrams with complete propagators. When studying more general graphs with the Schwinger-Keldysh doubling of degrees of freedom, the appropriate propagator substitution can become much more subtle; see [52] for the complete picture.

1.3.4 Top-down: from the near-zone to the far-zone

Setup. So far we have neglected the coupling of the radiation gravitons in the NRGR effective action. This is fine as long as we are working below the PN order in which radiation reaction starts playing a role; otherwise the conservative dynamics of the near-zone is modified as a result of the energy loss due to GW emission. Moreover, and most importantly, the discussion so far does not say anything about the radiated power loss by

GWs, a key observable in this discipline.

Starting from the near-zone NRGR rules, we now discuss the **top-down** construction of the far-zone effective action to lowest non-trivial PN order. Since both types of graviton modes will appear in the diagrams from now on, we must perform a *multipole expansion* of the radiation modes $\bar{h}_{\mu\nu}$ in order to preserve the *manifest power counting in* v.

The reason for doing so is easy to understand by considering a generic diagram in which a potential and radiative mode interact with each other. For example, consider the figure 1.5 below. This diagram – when included in a complete NRGR graph – contains an

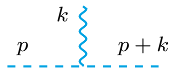


Figure 1.5: An incoming (with respect to the vertex node) radiation mode with four-momentum k meets a potential mode with momentum p. The outgoing potential graviton has momentum $p + k$.

instantaneous propagator proportional to $1/(\mathbf{p}+\mathbf{k})^2$. Since $\mathbf{p} \sim \mathcal{O}(1/r)$ and $\mathbf{k} \sim \mathcal{O}(v/r)$, this term does not scale homogeneously with the power counting parameter v. The solution is to expand it as a power series of $|\mathbf{k}|/|\mathbf{p}| \sim v$. This is achieved by performing a multipole[18] expansion (at the level of the action) of the radiation modes, so that

$$\bar{h}_{\mu\nu}(x^0, \mathbf{x}) = \bar{h}_{\mu\nu}(x^0, \mathbf{X}) + \delta x^i \partial_i \bar{h}_{\mu\nu}(x^0, \mathbf{X}) + \frac{1}{2}\delta x^i \delta x^j \partial_i \partial_j \bar{h}_{\mu\nu}(x^0, \mathbf{X}) + \cdots , \quad (1.102)$$

where $\delta X^i \equiv x^i - X^i$ and $\mathbf{X}$ is some reference point. We choose X^i to be the center of mass of the system, which is taken to coincide with the origin of the reference frame. The introduction of $H_{\mathbf{k}\mu\nu}(x^0)$ and the expansion above are simply field redefinitions.

With this consideration in mind, the construction of the effective action in (1.39) will proceed as follows. The goal is to compute the diagrams with one external radiation mode and sum over all possible combinations of potential-worldline and potential-radiation couplings consistent to a given PN order.[19] To $(Lv^5)^{1/2}$ order, we have the three graphs of figure 1.6. Notice that already at the lowest non-trivial PN order it is necessary to include

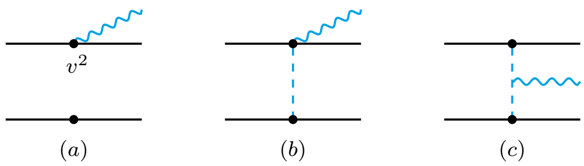


Figure 1.6: Near-zone diagrams with one external radiation graviton needed for the computation of the effective action to order $(Lv^5)^{1/2}$.

graphs in which potential and radiation modes interact. The omission of the diagrams (b)

[18] Objects such as neutron stars can exhibit both intrinsic and induced multipoles, the latter arising from interactions with external tidal fields. Four-dimensional black holes in general relativity do not deform under such circumstances. See chapter 3 for a more thorough discussion on multipoles.

[19] In general, a term of $S_{\text{eff}}^{\text{WR}}[\bar{h}_{\mu\nu}, x_a^\mu]$ describing the coupling of n radiation modes is given by the *sum* of all diagrams with n external radiative graviton legs and any number of internal potential lines.

and (c) of figure 1.6 would lead – as we shall see – to a problematic effective action. Before computing these diagrams with potential exchanges, let us derive the relevant worldline couplings. As a byproduct, this procedure will naturally yield the correct result for the diagram (a).

Lowest-order couplings: $(Lv)^{1/2}$ and $(Lv^3)^{1/2}$. Consider the following expansion

$$\frac{\mathrm{d}\tau}{\mathrm{d}t} = 1 - \frac{v^2}{2} + \frac{1}{2M_{\rm Pl}}(\bar{h}_{00} + 2\bar{h}_{0i}v^i + \bar{h}_{ij}v^iv^j) - \frac{1}{8}\left[v^4 - \frac{2v^2}{M_{\rm Pl}}(\bar{h}_{00} + 2\bar{h}_{0i}v^i + \bar{h}_{ij}v^iv^j) + \frac{1}{M_{\rm Pl}^2}(\bar{h}_{00}^2 + 4\bar{h}_{0i}\bar{h}_{0j}v^iv^j + \cdots)\right]. \tag{1.103}$$

The associated point-particle Lagrangian is identified by substituting the expansion above back into (1.30). Using the power counting rules of table 1.1, it is easy to see that the $\mathcal{O}\big(L^{1/2}v^{1/2}\big)$ term comes from the orange h_{00} piece:

$$L_{v^{1/2}} = -\frac{1}{2M_{\rm Pl}}\sum_a m_a\bar{h}_{00}. \tag{1.104}$$

The blue term $2h_{0i}v^i$ yields $L_{v^{3/2}}$:

$$L_{v^{3/2}} = -\frac{1}{M_{\rm Pl}}\sum_a m_a v_a^i\bar{h}_{0i} = -\frac{1}{M_{\rm Pl}}\left(P_{\rm cm}\right)^i\bar{h}_{0i}, \tag{1.105}$$

where the center of mass momentum $\mathbf{P}_{\rm cm}$ is defined as

$$\mathbf{P}_{\rm cm} \equiv m\frac{\mathrm{d}\mathbf{X}_{\rm cm}}{\mathrm{d}t} = \sum_a m_a\mathbf{v}_a, \tag{1.106}$$

and $m = \sum_a m_a$. This, however, is not the end of the story. As mentioned before, to determine the complete coupling at order $(Lv^3)^{1/2}$ one must Taylor expand the radiation modes:

$$\bar{h}_{\mu\nu}(x^0,\mathbf{x}) = \sum_{n=0}^{\infty}\frac{1}{n!}\,\delta x^N\partial_N\bar{h}_{\mu\nu}(x^0,\mathbf{X}_{\rm cm}), \tag{1.107}$$

where $\delta x^i = x^i - X^i_{\rm cm}$. This expansion – when applied to the point-particle Lagrangian determined so far – will naturally yield a number of terms which enter at order $(Lv^5)^{1/2}$ too, as we shall now discuss. In the following we omit the argument of $\bar{h}_{\mu\nu}$ in the multipole expansion.

Next order in v: $(Lv^5)^{1/2}$ term. Collecting $L_{v^{1/2}}$ and $L_{v^{3/2}}$, we get

$$L_{v^{1/2}+v^{3/2}} = -\frac{1}{2M_{\rm Pl}}\sum_a m_a\bar{h}_{00} - \frac{1}{M_{\rm Pl}}\sum_a m_a v_a^i\bar{h}_{0i} \tag{1.108}$$

Part of the terms contributing at $\mathcal{O}\left(L^{1/2}v^{5/2}\right)$ come from multipole expanding the radiation gravitons in the formula above. The remaining ones come directly from (1.103). For the sake of clarity, we reproduce this formula below and color in green the terms which enter at the next order in v:

$$\begin{aligned}\frac{\mathrm{d}\tau}{\mathrm{d}t} &= 1 - \frac{v^2}{2} + \frac{1}{2M_{\mathrm{Pl}}}(\bar{h}_{00} + 2\bar{h}_{0i}v^i + \bar{h}_{ij}v^iv^j) \\ &- \frac{1}{8}\left[v^4 - \frac{2v^2}{M_{\mathrm{Pl}}}(\bar{h}_{00} + 2\bar{h}_{0i}v^i + \bar{h}_{ij}v^iv^j) + \frac{1}{M_{\mathrm{Pl}}^2}(\bar{h}_{00}^2 + 4\bar{h}_{0i}\bar{h}_{0j}v^iv^j + \cdots)\right].\end{aligned} \tag{1.109}$$

These terms contribute to $L_{\text{multipolar}}$ as

$$L_{\text{multipolar}} \supset -\frac{1}{4}\sum_a \frac{m_a}{M_{\mathrm{Pl}}}\, v_a^2\, \bar{h}_{00} - \frac{1}{2}\sum_a \frac{m_a}{M_{\mathrm{Pl}}}\, \bar{h}_{ij}v_a^iv_a^j, \tag{1.110}$$

where by $L_{\text{multipolar}}$ we denote the complete multipole-expanded point-particle action to order $(Lv^5)^{1/2}$. At last, consider the contributions coming from the multipole expansion of (1.108). In the center of mass frame, we write:

$$\begin{aligned}L_{v^{1/2}+v^{3/2}} = &-\frac{1}{2M_{\mathrm{Pl}}}\sum_a m_a\left(\bar{h}_{00} + x^i\partial_i\bar{h}_{00} + \frac{1}{2}x^ix^j\partial_i\partial_j\bar{h}_{00} + \cdots\right) \\ &- \frac{1}{M_{\mathrm{Pl}}}\sum_a m_a v_a^i\left(\bar{h}_{0i} + x^j\partial_j\bar{h}_{0i} + \cdots\right).\end{aligned} \tag{1.111}$$

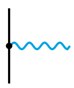
The orange and blue terms above are the complete $(Lv)^{1/2}$ and $(Lv^3)^{1/2}$ terms, respectively. It follows that

$$L_{\text{multipolar}} = -\sum_a \frac{m_a}{2M_{\mathrm{Pl}}}\left(\frac{1}{2}v_a^2\,\bar{h}_{00} + \frac{1}{2}x_a^ix_a^j\partial_i\partial_j\bar{h}_{00} + 2x_a^iv_a^j\partial_i\bar{h}_{0j} + \bar{h}_{ij}v_a^iv_a^j\right). \tag{1.112}$$

This Lagrangian cannot possibly be the complete effective action. It has two problems which are not independent from one another:

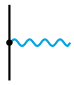
- It is not gauge invariant under infinitesimal coordinate transformations;
- It predicts that the unphysical $\bar{h}_{00}$ and $\bar{h}_{0i}$ polarizations can be emitted by the binary system.

The full result will only come after adding the results of diagrams (b) and (c) in fig. 1.6. Before doing so, let's write the Feynman rules for the radiation-wordline couplings we have just derived. The $\mathcal{O}\left(L^{1/2}v^{1/2}\right)$ rule is simply

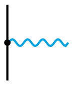


$$= -\frac{im}{2M_{\mathrm{Pl}}}\int \mathrm{d}t\,\bar{h}_{00} \sim (Lv)^{1/2}. \tag{1.113}$$

The $\mathcal{O}\left(L^{1/2}v^{3/2}\right)$ rule is

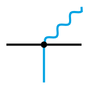
v 
$$= -\frac{im}{2M_{\rm Pl}}\int \mathrm{d}t\,(x^i\partial_i\bar{h}_{00} + 2\bar{h}_{0i}v^i), \tag{1.114}$$

while the $\mathcal{O}\left(L^{1/2}v^{5/2}\right)$ rule is

v^2 
$$= -\frac{im}{2M_{\rm Pl}}\int \mathrm{d}t\,\bigg(\bar{h}_{ij}v^iv^j + \frac{1}{2}\bar{h}_{00}v^2 + 2x^iv^j\partial_i\bar{h}_{0j} + \frac{1}{2}x^ix^j\partial_i\partial_j\bar{h}_{00}\bigg). \tag{1.115}$$

From (1.115), it is easy to write down the expression of diagram (a). We will do this after evaluating diagrams (b) and (c).

Diagram (b)**.** To compute this graph we need the following worldline coupling involving both radiation and potential modes:


$$= \frac{im}{4M_{\rm Pl}}\int \mathrm{d}t\int_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}(t)}\eta^{\mu0}\eta^{\nu0}\bar{h}_{00}. \tag{1.116}$$

The derivation of this rule is straightforward and follows the steps outlined in previous sections. Using it one finds

$$(b) = \frac{m_1m_2}{8M_{\rm Pl}^3}\int \mathrm{d}t_1\,\mathrm{d}t_2\int_{\mathbf{k},\mathbf{q}} e^{i\mathbf{k}\cdot\mathbf{x}_1(t_1)}e^{i\mathbf{q}\cdot\mathbf{x}_2(t_2)}\langle H_{\mathbf{k}00}(t_1)H_{\mathbf{q}00}(t_2)\rangle\bar{h}_{00}. \tag{1.117}$$

Substituting (1.49), we obtain

$$(b) = -\frac{i}{2M_{\rm Pl}}\int \mathrm{d}t\,\frac{Gm_1m_2}{|\mathbf{x}_1(t)-\mathbf{x}_2(t)|}\bar{h}_{00}(t,\mathbf{0}). \tag{1.118}$$

To derive the corresponding contribution to the effective action one must add up the mirror image diagram under $1 \leftrightarrow 2$ exchange. In the case of diagram (b), this amounts to multiplying (1.118) by a simple factor of 2.

Diagram (c)**.** The computation of (c) is a bit more involved because it requires a particular case of the three-graviton vertex. We quote without proof the relevant term from the gauge-fixed Einstein-Hilbert action:

$$\begin{aligned}\mathcal{L}_{\bar{h}H^2} = \mathbf{k}^2\bar{h}_{00}\bigg[&-\frac{1}{4}H^{\mu\nu}_{\mathbf{k}}H_{-\mathbf{k}\mu\nu}+\frac{1}{8}H_{\mathbf{k}}H_{-\mathbf{k}}+H^{\mu 0}_{\mathbf{k}}H_{-\mathbf{k}\mu 0}-\frac{1}{2}H_{\mathbf{k}00}H_{-\mathbf{k}}\bigg]\\
&+\mathbf{k}^2\bar{h}_{0i}\left[2H^{00}_{\mathbf{k}}H_{-\mathbf{k}0i}-H_{\mathbf{k}0i}H_{-\mathbf{k}}\right]+\mathbf{k}^2\delta^{ij}\frac{\bar{h}_{ij}}{4}\left[H^{\mu\nu}_{\mathbf{k}}H_{-\mathbf{k}\mu\nu}-\frac{1}{2}H_{\mathbf{k}}H_{-\mathbf{k}}\right]\\
&+\frac{\bar{h}_{ij}}{2}\left[2\mathbf{k}^2H^{i\mu}_{\mathbf{k}}H_{-\mathbf{k}}{}^{j}{}_{\mu}-k^ik^jH^{\mu\nu}_{\mathbf{k}}H_{-\mathbf{k}\mu\nu}-\mathbf{k}^2H_{\mathbf{k}ij}H_{-\mathbf{k}}+\frac{1}{2}k^ik^jH_{\mathbf{k}}H_{-\mathbf{k}}\right].\end{aligned} \tag{1.119}$$

First, consider the contribution coming from the $\bar{h}_{00}$ term of $\mathcal{L}_{\bar{h}H^2}$. There are two possible Wick contractions of the potential graviton fields, but since they both yield the same result we will just carry out the computation with one of the possible contractions and multiply the resulting expression by an overall factor of two. Doing so leads to

$$\begin{aligned}(c)_{\bar{h}_{00}} = 2\times&\frac{im_1}{2M_{\text{Pl}}}\frac{im_2}{2M_{\text{Pl}}}\int \mathrm{d}t_1\,\mathrm{d}t_2\int_{\mathbf{q},\mathbf{q}'}e^{i\mathbf{q}\cdot\mathbf{x}_1(t_1)}e^{i\mathbf{q}'\cdot\mathbf{x}_2(t_2)}\bar{h}_{00}\\
&\times\frac{i}{M_{\text{Pl}}}\int \mathrm{d}\bar{t}\int_{\mathbf{k}}\mathbf{k}^2\left(-\frac{1}{4}\eta^{\mu\alpha}\eta^{\nu\beta}+\frac{1}{8}\eta^{\mu\nu}\eta^{\alpha\beta}+\eta^{\mu\alpha}\eta^{\nu 0}\delta^{\beta}{}_{0}-\frac{1}{2}\eta^{\alpha\beta}\delta^{\mu}{}_{0}\delta^{\nu}{}_{0}\right)\\
&\times\langle H_{\mathbf{q}00}(t_1)H_{\mathbf{k}\mu\nu}(\bar{t})\rangle\langle H_{-\mathbf{k}\alpha\beta}(\bar{t})H_{\mathbf{q}'00}(t_2)\rangle\end{aligned} \tag{1.120}$$

Inserting (1.49) and simplifying the tensor structure, we obtain

$$(c)_{\bar{h}_{00}} = \frac{im_1m_2}{8M^3_{\text{Pl}}}\int \mathrm{d}t\int_{\mathbf{k}}\frac{e^{-i\mathbf{k}\cdot[\mathbf{x}_1(t)-\mathbf{x}_2(t)]}}{\mathbf{k}^2}\,\frac{3}{2}\,\bar{h}_{00}. \tag{1.121}$$

The $\bar{h}_{0i}$ contribution simply vanishes due to its tensor structure. Moving on, the evaluation of the trace $\delta^{ij}\bar{h}_{ij}$ and the $k^ik^j\bar{h}_{ij}$ term is similar to the $\bar{h}_{00}$ case. Repeating the usual calculations, one eventually gets

$$(c) = \frac{im_1m_2}{8M^3_{\text{Pl}}}\int \mathrm{d}t\int_{\mathbf{k}}\frac{e^{-i\mathbf{k}\cdot(\mathbf{x}_1-\mathbf{x}_2)}}{\mathbf{k}^4}\left(\frac{3}{2}\,\mathbf{k}^2\bar{h}_{00}+\frac{1}{2}\,\mathbf{k}^2\delta^{ij}\bar{h}_{ij}-k^ik^j\bar{h}_{ij}\right). \tag{1.122}$$

The remaining integrals over $\mathrm{d}^3\mathbf{k}$ are standard. Using (A.1), the result is

$$\begin{aligned}(c) = \frac{im_1m_2}{8M^3_{\text{Pl}}}\int \mathrm{d}t\,\frac{1}{8\pi|\mathbf{x}_1-\mathbf{x}_2|}\bigg\{&\left(3\bar{h}_{00}+\delta^{ij}\bar{h}_{ij}\right)\\
&-\bar{h}_{ij}\left[\delta_{ij}-\frac{(x_1-x_2)^i(x_1-x_2)^j}{|\mathbf{x}_1-\mathbf{x}_2|^2}\right]\bigg\}.\end{aligned} \tag{1.123}$$

To further simplify this expression we must use the $\mathcal{O}(v^0)$ equations of motion:

$$\ddot{x}_1 = -\frac{Gm_2}{|\mathbf{x}_1 - \mathbf{x}_2|^3}(x_1 - x_2)^j, \tag{1.124a}$$

$$\ddot{x}_2 = \frac{Gm_1}{|\mathbf{x}_1 - \mathbf{x}_2|^3}(x_1 - x_2)^j. \tag{1.124b}$$

At last, we find

$$(c) = \frac{i}{M_{\text{Pl}}}\int \mathrm{d}t \left[\frac{3}{2}\frac{Gm_1m_2}{|\mathbf{x}_1(t) - \mathbf{x}_2(t)|}\bar{h}_{00}(t,\mathbf{0}) - \frac{1}{2}\sum_a m_a x_a^i \ddot{x}_a^j \bar{h}_{ij}(t,\mathbf{0})\right]. \tag{1.125}$$

Summing up all diagrams. Collecting our results, the complete effective action becomes

$$L^{\text{NRGR}}_{v^{5/2}} = -\frac{\bar{h}_{00}}{2M_{\text{Pl}}}E_{\text{N}} - \frac{1}{2M_{\text{Pl}}}\sum_a\left(\frac{1}{2}m_a x_a^i x_a^j \partial_i\partial_j\bar{h}_{00} + 2m_a x_a^i v_a^j \partial_i \bar{h}_{0j} + m_a v_a^i v_a^j \bar{h}_{ij} + m_a x_a^i \ddot{x}_a^j \bar{h}_{ij}\right), \tag{1.126}$$

where E_{N} is the Newtonian orbital energy:

$$E_{\text{N}} \equiv \sum_a \frac{1}{2}m_a v_a^2 - \frac{Gm_1m_2}{|\mathbf{x}_1 - \mathbf{x}_2|}. \tag{1.127}$$

Omitting the a label and introducing the linearized Riemann tensor

$$\bar{R}_{0i0j} = -\frac{1}{2}(\ddot{\bar{h}}_{ij} - \partial_i\dot{\bar{h}}_{0j} - \partial_j\dot{\bar{h}}_{0i} + \partial_i\partial_j\bar{h}_{00}), \tag{1.128}$$

the term inside parenthesis on (1.126), momentarily denoted by $\#$, becomes

$$\# = -x^i x^j \bar{R}_{0i0j} - \frac{1}{2}x^i x^j \ddot{\bar{h}}_{ij} + v^i v^j \bar{h}_{ij} + x^{(i}\ddot{x}^{j)}\bar{h}_{ij} + 2x^i v^j \partial_i \bar{h}_{0j} + \frac{1}{2}x^i x^j(\partial_i\dot{\bar{h}}_{0j} + \partial_j\dot{\bar{h}}_{0i}). \tag{1.129}$$

Dropping total derivatives, the blue terms cancel upon integration by parts. By the same procedure, the remaining green terms can be written as

$$2x^i v^j \partial_i\bar{h}_{0j} + \frac{1}{2}x^i x^j(\partial_i\dot{\bar{h}}_{0j} + \partial_j\dot{\bar{h}}_{0i}) = (x^j v^i - x^i v^j)\partial_j\bar{h}_{0i}. \tag{1.130}$$

Finally, introducing the Newtonian orbital angular momentum

$$L_k \equiv \sum_a \epsilon^{klm} m_a x_a^l v_a^m, \tag{1.131}$$

it is not difficult to see that

$$L_{v^{5/2}}^{\text{NRGR}} = -\frac{\bar{h}_{00}}{2M_{\text{Pl}}}E_{\text{N}} + \frac{1}{2M_{\text{Pl}}}Q^{ij}\bar{R}_{0i0j} - \frac{1}{2M_{\text{Pl}}}\epsilon_{jik}L_k\partial_j\bar{h}_{0i}, \tag{1.132}$$

where $Q^{ij} = \sum_a m_a x_a^i x_a^j$ is the quadrupole moment of the binary. A few remarks on this result are needed.

(i) After integrating out all the potential modes by summing up the diagrams in fig. 1.6, we have obtained an effective action in which all the dependence on the orbital scale r appears in the *multipole moments* of the binary. This is a reflection of the separation of scales. Thus,

> *The near-zone non-relativistic physics decouples from the far-zone radiation at the level of the NRGR Lagrangian.*

This feature is similar in spirit to the "near-decoupling" of the external dynamics from the internal dynamics of self-gravitating Newtonian bodies [100].

(ii) For on-shell radiation modes, the linearized Riemann tensor $\bar{R}_{0i0j}$ is traceless:

$$\bar{R}_{0i0i} = \partial_0\left(\partial_\mu\bar{h}^\mu{}_0 - \frac{1}{2}\partial_0\bar{h}\right) - \frac{1}{2}\Box\bar{h}_{00} = 0. \tag{1.133}$$

Therefore, $\bar{R}_{0i0j}$ couples to the *traceless* quadrupole moment in the effective Lagrangian:

$$I_{ij} \equiv Q_{ij} - \frac{1}{3}\delta_{ij}Q, \tag{1.134}$$

where $Q = \delta^{kl}Q_{kl}$.

(iii) The orbital energy and angular momentum are conserved quantities at 2.5PN order, and thus only the quadrupole moment can radiate. This is intimately connected with the fact that E_{N} and L_k couple to the non-radiative graviton polarizations $\bar{h}_{00}$ and $\bar{h}_{0i}$, respectively.

(iv) The Lagrangian (1.132) is manifestly invariant under infinitesimal coordinate transformations $x \mapsto x + \xi$. This is not difficult to check. First, the $\mathcal{O}(\bar{h})$ Riemann tensor is invariant, by construction. Next, the $\bar{h}_{00}$ term changes by a total time derivative

$$E_{\text{N}}\bar{h}_{00} \to E_{\text{N}}(\bar{h}_{00} + 2\partial_0\xi_0), \tag{1.135}$$

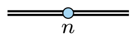
and the same happens to the $\bar{h}_{0i}$ term:

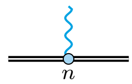
$$\epsilon_{ijk}L_k\partial_j\bar{h}_{0i} \to \epsilon_{ijk}L_k\partial_j\bar{h}_{0i} + \epsilon_{ijk}\partial_0(L_k\partial_j\xi_i). \tag{1.136}$$

Radiated power. With the result (1.132) in hands, we can finally derive the famous Einstein quadrupole formula within the framework of NRGR. But first, we need to understand diagramatically the effect of integrating out the remaining external radiation gravitons. Consider once again the effective action

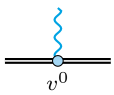
$$S_{\text{eff}}[x_a^\mu, \bar{h}_{\mu\nu}] = iS_{\bar{h}}[\bar{h}] + iS_{\text{cons}}[x_a^\mu] + \big(iS_{\text{diss}}[x_a^\mu, \bar{h}^{\mu\nu}] + \cdots\big). \tag{1.137}$$

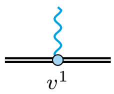
where $iS_{\text{cons}}[x_a^\mu]$ and $iS_{\text{diss}}[x_a^\mu, \bar{h}^{\mu\nu}]$ are schematically given by

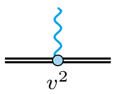
$$iS_{\text{cons}}[x_a^\mu] = \sum_n \Big($$  $$\Big), \tag{1.138a}$$

$$iS_{\text{diss}}[x_a^\mu, \bar{h}^{\mu\nu}] = \sum_n \Big($$  $$\Big) \sim L^{1/2} v^{n+1/2}, \tag{1.138b}$$

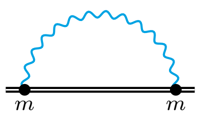
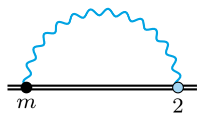
and $iS_{\bar{h}}$ is the sum of all pure-graviton vertices. The ellipsis in (1.137) contains n-graviton emission diagrams. For simplicity we'll focus on the single emission diagrams of (1.138b). Notice that (1.132) is contained in iS_{diss}, whereas the EIH Lagrangian (1.91) contributes to iS_{cons}. The results of our previous discussion imply that

 $$= -\frac{im}{2M_{\text{Pl}}} \int \mathrm{d}t\, \bar{h}_{00}, \tag{1.139a}$$

 $$= 0 \quad \text{(in the center-of-mass frame)}, \tag{1.139b}$$

 $$= -\frac{i}{2M_{\text{Pl}}} \int \mathrm{d}t\, \big[E_{\text{N}} \bar{h}_{00} - \epsilon^{ijk} L_k \partial_j \bar{h}_{0i} - I^{ij} \bar{R}_{0i0j}\big]. \tag{1.139c}$$

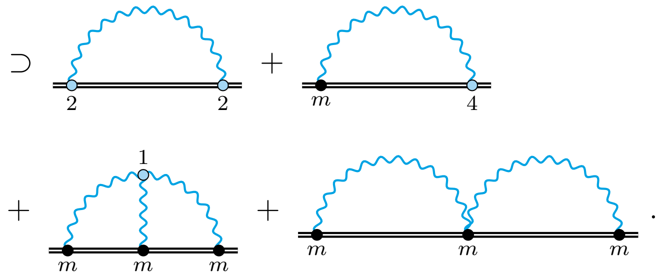
From the single-emission diagrams of $iS_{\text{diss}}[x_a^\mu, \bar{h}^{\mu\nu}]$, the NRGR effective action $iS_{\text{eff}}^{\text{NRGR}}[x_a^\mu]$ is constructed by summing up all trees[20] in which the only external "legs" are the world-lines of the binary constituents. The conservative term iS_{cons} survives the integration of radiation gravitons untouched. For example, the first few terms of $iS_{\text{eff}}^{\text{NRGR}}[x_a^\mu]$ are

$$iS_{\text{eff}}^{\text{NRGR}}[x_a^\mu] \supset iS_{\text{cons}}[x_a^\mu] +$$  $$+$$  $$. \tag{1.140}$$

[20]By trees we mean diagrams with no pure graviton loops.

From left to right, the "self-energy" diagrams above scale as Lv and Lv^3. Moving on to Lv^5 (or 2.5PN) order, we have the following graphs:

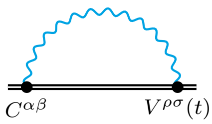
$iS_{\text{eff}}^{\text{NRGR}}[x_a^\mu] \supset$



(1.141)

Luckily, as a consequence of a general result proved in [55], we do not need to compute most of these diagrams.

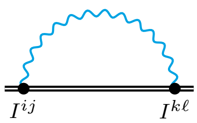
Lemma 3 (Galley, Tiglio)**.** *Let* $C^{\alpha\beta}$ *be a conserved quantity and* $V^{\rho\sigma}(t)$ *an arbitrary vertex. Then, the following diagram in NRGR vanishes:*



$$= 0. \tag{1.142}$$

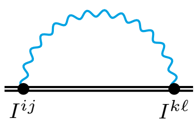
From now on this result will be refereed to as the "GT lemma." Its proof is presented in appendix A. Since the mass m is conserved at 2.5PN order, it follows that the only diagram we need to compute in $iS_{\text{eff}}^{\text{NRGR}}$ at this order is the first one in (1.141). Moreover, only the quadrupole coupling in the Feynman rule (1.139c) survives. Thus,

$$iS_{\text{eff}}^{\text{NRGR}}[x_a^\mu] = iS_{\text{cons}}[x_a^\mu] +$$



$$+ \mathcal{O}\big(Lv^6\big). \tag{1.143}$$

We now proceed to compute the diagram above. Using (1.139c), we find



$$= \frac{1}{2}\left(\frac{i}{2M_{\text{Pl}}}\right)^2 \int \mathrm{d}t\,\mathrm{d}t'\, I^{ij}(t) I^{k\ell}(t') \times \langle T\{\bar{R}_{0i0j}(t,\mathbf{0})\bar{R}_{0k0\ell}(t',\mathbf{0})\}\rangle. \tag{1.144}$$

The overall $1/2$ is the symmetry factor of the graph. Also, notice that the Riemann tensor is evaluated at the spatial origin due to the multipole expansion. The computation of the two-point function above is straightforward albeit a little tedious. Written in terms of the usual Feynman propagator, the result is

$$\begin{aligned}\langle T\{\bar{R}_{0i0j}(t,\mathbf{0})\bar{R}_{0k0\ell}(t',\mathbf{0})\}\rangle &= \frac{i}{8}\int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{e^{-ik^0(t-t')}}{k^2+i\mathsf{a}} \\ &\times \Big[k_ik_jk_kk_\ell + (k^0)^4(\delta_{ik}\delta_{j\ell}+\delta_{i\ell}\delta_{jk}-\delta_{ij}\delta_{k\ell}) \\ &+ (k^0)^2(\delta_{ij}k_kk_\ell+\delta_{k\ell}k_ik_j-\delta_{j\ell}k_ik_k-\delta_{jk}k_ik_\ell-\delta_{ik}k_jk_\ell-\delta_{i\ell}k_jk_k)\Big].\end{aligned} \tag{1.145}$$

Exploiting the rotational invariance of the integral, we make the replacements:

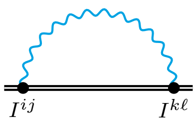
$$k_ik_j \to \frac{1}{3}|\mathbf{k}|^2\delta_{ij}, \tag{1.146a}$$

$$k_ik_jk_kk_\ell \to \frac{1}{15}|\mathbf{k}|^4(\delta_{ij}\delta_{k\ell}+\delta_{ik}\delta_{j\ell}+\delta_{i\ell}\delta_{jk}). \tag{1.146b}$$

Since k is an on-shell momentum we can set $|\mathbf{k}| = k^0$. Therefore,

$$\begin{aligned}\langle T\{\bar{R}_{0i0j}(t,\mathbf{0})\bar{R}_{0k0\ell}(t',\mathbf{0})\}\rangle &= \frac{i}{20}\int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{(k^0)^4}{k^2+i\mathsf{a}}\,e^{-ik^0(t-t')} \\ &\times\left(\delta_{ik}\delta_{j\ell}+\delta_{i\ell}\delta_{jk}-\frac{2}{3}\delta_{ij}\delta_{k\ell}\right).\end{aligned} \tag{1.147}$$

Substituting (1.147) back into (1.143) and using the fact that I_{ij} is traceless, we find



$$\begin{aligned}&= -\frac{i}{80M_{\mathrm{Pl}}^2}\int \mathrm{d}t\,\mathrm{d}t'\, I_{ij}(t)I^{ij}(t')\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} \\ &\times\left(\mathsf{PV}\int_{k^0}\frac{(k^0)^4}{(k^0)^2-|\mathbf{k}|^2}\,e^{-ik^0(t-t')} - i\pi\int_{k^0}(k^0)^4\delta[(k^0)^2-|\mathbf{k}|^2]\,e^{-ik^0(t-t')}\right)\end{aligned} \tag{1.148}$$

The derivation of this expression makes use of the Sokhostki-Plemelj formula:

$$\lim_{\mathsf{a}\to 0}\frac{1}{x\pm i\mathsf{a}} = \mathsf{PV}\,\frac{1}{x}\mp i\pi\delta(x). \tag{1.149}$$

Notice that (1.148) is a sum of two terms. The principal-value piece will yield an imaginary contribution to the diagram whereas the second integral over the delta function produces a real term. Let us evaluate the latter first. Using the fact the $I_{ij}(t)$ is a real quantity – so that its Fourier transform satisfies $\tilde{I}_{ij}(-\omega) = \tilde{I}^*_{ij}(\omega)$ – we obtain

$$\begin{aligned}
\mathcal{I}_{\rm D} &\equiv -\frac{i}{2}\int_{\mathbf{k}}\int \mathrm{d}k^0 \int \mathrm{d}t\,\mathrm{d}t'\, e^{-ik^0(t-t')} I_{ij}(t) I^{ij}(t')(k^0)^4 \delta[(k^0)^2 - |\mathbf{k}|^2] \\
&= -\frac{i}{2}\int_{\mathbf{k}}\int \mathrm{d}k^0\, \tilde{I}_{ij}(k^0)\tilde{I}^{ij}(-k^0)\,\frac{(k^0)^4}{2|\mathbf{k}|}\left[\delta(k^0+|\mathbf{k}|)+\delta(k^0-|\mathbf{k}|)\right] \\
&= -\frac{i}{2}\int_{\mathbf{k}} |\mathbf{k}|^3 |\tilde{I}_{ij}(|\mathbf{k}|)|^2 .
\end{aligned} \tag{1.150}$$

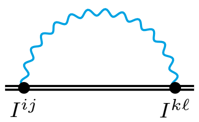
The principal-value integral in (1.148) can be evaluated using (A.7). Setting $t' = t + s$ and Taylor-expanding, we find

$$\begin{aligned}
\mathcal{I}_{\rm R} &\equiv \int \mathrm{d}t\,\mathrm{d}s\, I_{ij}(t) I^{ij}(t+s) \int_{\mathbf{k}} \mathsf{PV}\int_{k^0} \frac{(k^0)^4}{(k^0)^2 - |\mathbf{k}|^2}\, e^{-ik^0 s} \\
&= \int \mathrm{d}t\, I_{ij}(t) \sum_{n=0}^{\infty} \frac{1}{n!}\frac{\mathrm{d}^n I^{ij}(t)}{\mathrm{d}t^n} \int \mathrm{d}s \int_{\mathbf{k}} \mathsf{PV}\int_{k^0} \frac{s^n (k^0)^4}{(k^0)^2-|\mathbf{k}|^2}\, e^{ik^0 s} \\
&= \int \mathrm{d}t\, I_{ij}(t) \sum_{n=0}^{\infty} \frac{1}{n!}\frac{\mathrm{d}^n I^{ij}(t)}{\mathrm{d}t^n}\, \mathbb{M}_2^{(4)}(n,4,0).
\end{aligned} \tag{1.151}$$

In the second line above we also set $k^0 \mapsto -k^0$. Since only the $n = 5$ master integral is non-zero, we find

$$\mathcal{I}_{\rm R} = \frac{1}{4\pi}\int \mathrm{d}t\, I_{ij}(t)\, \frac{\mathrm{d}^5 I^{ij}(t)}{\mathrm{d}t^5}. \tag{1.152}$$

Substituting (1.150) and (1.152) back into (1.148), one gets



$$\begin{aligned}
&= -\frac{i}{80 M_{\rm Pl}^2}(\mathcal{I}_{\rm R} + \mathcal{I}_{\rm D}) \\
&= -\frac{iG}{10}\int \mathrm{d}t\, I^{ij} I^{(5)}_{ij} - \frac{\pi G}{5}\int_{\mathbf{k}} |\mathbf{k}|^3 |\tilde{I}_{ij}(|\mathbf{k}|)|^2,
\end{aligned} \tag{1.153}$$

which is the first non-trivial contribution to $iS_{\rm diss}[x_a^\mu]$ to 2.5 PN order. To extract the radiated power we must identify the *graviton emission rate* $\frac{\mathrm{d}^2\Gamma}{\mathrm{d}\omega\mathrm{d}\Omega}$ by means of the formula

$$\frac{1}{T}\, 2\,\mathrm{Im}\, S^{\rm NRGR}_{\rm eff}[x_a^\mu] \equiv \int_0^\infty \mathrm{d}\omega\,\mathrm{d}\Omega\, \frac{\mathrm{d}^2\Gamma}{\mathrm{d}\omega\mathrm{d}\Omega}, \tag{1.154}$$

where T is some characteristic time such as the binary's period. Let's take a step back and try to understand the intuition behind this definition. In non-relativistic quantum mechanics, the probability $p(t)$ that a normalized state $|\psi(t)\rangle = e^{-iEt}|\psi(0)\rangle$, with $E \in \mathbb{C}$, will not decay is given by $p(t) = \langle\psi(t)|\psi(t)\rangle = e^{2(\mathrm{Im}\,E)t}$. The quantity $2(\mathrm{Im}\,E)t$, which plays the role of the effective action in this toy model, can be used to extract a decay width τ defined by $p(t) \equiv e^{t/\tau}$. As we shall discuss in §1.3.5, the definition (1.154) can also be understood in light of the optical theorem.

The preceding digression suggests that the integrand of (1.154), the graviton emission rate, should be identified with the average number of emitted gravitons per frequency per solid angle unit. The radiated power $\mathcal{P}$ is simply given by

$$\mathcal{P} = \int_0^\infty \mathrm{d}\omega\, \mathrm{d}\Omega\, \omega\, \frac{\mathrm{d}^2\Gamma}{\mathrm{d}\omega \mathrm{d}\Omega}. \tag{1.155}$$

From (1.153) we deduce that

$$S_{\text{eff}}^{\text{NRGR}}[x_a^\mu] = S_{\text{cons}}[x_a^\mu] - \frac{G}{10}\int \mathrm{d}t\, I^{ij} I_{ij}^{(5)} + \frac{i\pi G}{5}\int_{\mathbf{k}} |\mathbf{k}|^3 |\tilde{I}_{ij}(|\mathbf{k}|)|^2 + \mathcal{O}\big(Lv^6\big), \tag{1.156}$$

and the graviton emission rate is identified by writing

$$\begin{aligned} 2\,\mathrm{Im}\, S_{\text{eff}}^{\text{NRGR}} &= \frac{2\pi G}{5}\int \mathrm{d}\Omega \int_0^\infty \frac{\mathrm{d}|\mathbf{k}|}{(2\pi)^3}\, |\mathbf{k}|^5 |\tilde{I}_{ij}(|\mathbf{k}|)|^2 \\ &= \int_0^\infty \mathrm{d}\omega \int \mathrm{d}\Omega\, \frac{G}{20\pi^2}\, \omega^5 |\tilde{I}_{ij}(\omega)|^2 \Big|_{|\mathbf{k}|=\omega}. \end{aligned} \tag{1.157}$$

Having identified the emission rate, the leading-order radiated power follows:

$$\begin{aligned} \mathcal{P}_{\text{LO}} &= \frac{1}{T}\frac{G}{20\pi^2}\int \mathrm{d}\Omega \int_0^\infty \mathrm{d}\omega\, \omega^6\, |\tilde{I}_{ij}(\omega)|^2 \\ &= \frac{1}{2T}\frac{G}{5\pi}\int_{-\infty}^\infty \mathrm{d}\omega\, \omega^6 \int \mathrm{d}t\, \mathrm{d}t'\, e^{i\omega(t-t')} I_{ij}(t) I^{ij}(t'). \end{aligned} \tag{1.158}$$

The powers of ω in the integrant can be traded by time derivatives acting on $e^{i\omega(t-t')}$. Integrating by parts, one recovers Einstein's quadrupole formula:

$$\begin{aligned} \mathcal{P}_{\text{LO}} = \frac{1}{T}\frac{G}{10\pi}\int \mathrm{d}t\, \mathrm{d}t'\; 2\pi\delta(t-t') I_{ij}^{(3)}(t) I_{ij}^{(3)}(t') &= \frac{G}{5}\frac{1}{T}\int \mathrm{d}t\, I_{ij}^{(3)}(t) I_{ij}^{(3)}(t) \\ &= \frac{G}{5}\langle I_{ij}^{(3)} I_{ij}^{(3)} \rangle_T. \end{aligned} \tag{1.159}$$

If the imaginary part of (1.156) is responsible for the instantaneous power loss, what is the role of the real term? In principle, it produces a contribution to the equations of motion, much like the conservative Lagrangian iS_{cons} does. However, since this term is obtained by integrating out the radiation gravitons, it is not part of the *conservative* potential per se, but rather a *dissipative* potential. Of course, such radiation reaction contributions are expected when describing the dynamics of a radiating system in a consistent manner.

Unfortunately, as we shall discuss in detail in section 2, the real piece of (1.156) does *not* yield the correct 2.5PN radiation reaction force as given by the well-known Burke-Thorne formula; it produces a boundary term in the effective action. This problem can be fixed by implementing the so-called *in-in* formalism.

The state-of-the-art result (derived from traditional GR methods) for the GW phasing in quasi-circular orbits lies at 4.5PN order [25].

1.3.5 Bottom-up: amplitudes and the matching procedure

We now turn to the bottom-up approach to NRGR, first devised by Goldberger and Ross in [58]. This particular setup to the far-zone computations has lead to a number of important results in PN theory; see [54] [5] [6] [7] for an incomplete list in the spinless sector. Moreover, as mentioned in §1.1, this approach feels closer in spirit to the way that EFTs are used in particle physics.

In the latter scenario, one typically does not have access to a more fundamental theory from which the low-energy dynamics under investigation can be derived, rendering the "handcrafted" computation of the effective action illustrated in §1.3.4 impossible.

Instead, we now exploit the symmetries of our system to write the most possible effective worldline action, which takes the form of a multipole expansion. We are *a priori* agnostic about the multipole moments; their precise expressions in terms of the orbital variavels are inferred *a posteriori* via a matching computation. This matching computation is done via a top-down calculation in the style of §1.3.2. Furthermore, in contrast with the near-zone computation, we now work with the full propagators. The quasi-instantaneous approximation of the near-zone is ill-suited to describe far-zone process in which retardation effects are guaranteed to be important.

As before, the separation of scales of the two-body problem will be exploited to determine the multipole moments order by order in the PN expansion.

An action for linearized gravity. Consider a self-gravitating skeletonized[21] body in GR which perturbs the background Minkowski spacetime. Write

$$g_{\mu\nu} = \eta_{\mu\nu} + \frac{1}{M_{\text{Pl}}} h_{\mu\nu}, \tag{1.160}$$

where $h_{\mu\nu}$ represents a linear perturbation. At this level, one can describe the coupling of the gravitational field to the matter by the action[22]

$$S_{\text{int}} = -\frac{1}{2M_{\text{Pl}}} \int \mathrm{d}^4x\, \mathcal{T}^{\mu\nu}(x) h_{\mu\nu}(x). \tag{1.161}$$

In the equation above, $\mathcal{T}^{\mu\nu}(x)$ denotes the pseudo stress-energy tensor of the system. In the case of the two-body problem, this tensor receives contributions both from the gravitational binding energy (as a by product of integrating out potential modes) and the matter point-particles. Since we are interested in a long-wavelength description of a radiating system,

[21] By skeletonized body we mean a point particle supplemented by multipole moments which capture finite-size structure effects.

[22] This is a straightforward generalization of the interaction Lagrangian $J^\mu(x)A_\mu(x)$ extensively used in Maxwell's theory, where J^μ and A_μ are the current and gauge potential four-vectors, respectively [88].

we multipole expand the $h_{\mu\nu}(x)$ field:

$$h_{\mu\nu}(t,\mathbf{x}) = \sum_{n=0}^{\infty} \frac{1}{n!} x^N \partial_N h_{\mu\nu}(t,\mathbf{0}). \tag{1.162}$$

We take the origin of the coordinate system to coincide with the location of the self-gravitating body. Furthermore, consider a frame such that the four-velocity can be written as $u^\mu_{\text{body}} = (1,\mathbf{0})$. From now on, we work in a *locally-flat comoving frame* $\{\mathbf{e}^\mu_\alpha\}$ such that

$$e^\mu_0 = u^\mu, \tag{1.163a}$$

$$g^{\mu\nu} = e^\mu_0 e^\nu_0 - \delta^{ij} e^\mu_i e^\nu_j. \tag{1.163b}$$

Notice that the frame metric is $\eta^{\alpha\beta} = \text{diag}(1,-1,-1,-1)$. After substituting (1.162) into (1.161), one makes extensive use of Young tableaux and the conservation of the pseudo stress-energy tensor $\partial_\mu \mathcal{T}^{\mu\nu} = 0$ to express the action in terms of symmetric tracefree (STF) moments of $\mathcal{T}^{\mu\nu}$. This procedure is described in detail in reference [114]. The result is

$$\begin{aligned} S_{\text{int}} = &- \frac{1}{2M_{\text{Pl}}} \int \mathrm{d}t \left(M h_{00} + 2P^i h_{0i} + M X^i \partial_i h_{00} + \epsilon_{ijk} L^i \partial^j h^{0k}\right) \\ &+ \int \mathrm{d}t \sum_{\ell=2}^{\infty} \left(\frac{1}{\ell!} I^L \partial_{L-2} E_{i_{\ell-1} i_\ell} - \frac{2\ell}{(\ell+1)!} J^L \partial_{L-2} B_{i_{\ell-1} i_\ell} \right), \end{aligned} \tag{1.164}$$

where E, P^i, X^i, and L^i are defined by

$$M \equiv \int \mathrm{d}^3\mathbf{x}\, \mathcal{T}^{00}, \tag{1.165a}$$

$$P^i \equiv \int \mathrm{d}^3\mathbf{x}\, \mathcal{T}^{0i}, \tag{1.165b}$$

$$M X^i \equiv \int \mathrm{d}^3\mathbf{x}\, \mathcal{T}^{00} x^i, \tag{1.165c}$$

$$L^i \equiv - \int \mathrm{d}^3\mathbf{x}\, \epsilon^{ijk} \mathcal{T}_{0j} x_k. \tag{1.165d}$$

The STF tensors I^L and J^L are the rank-ℓ electric and magnetic *source* multipole moments, respectively (also called mass and current moments). For example, I^{ij} is the usual STF mass quadrupole and J^{ijk} is the STF current octupole. This identification is only possible due to the conservation of the pseudo stress-energy tensor, which is one of the ingredientes used by the authors of [114] to derive the general formulae for I^L and J^L (the other ingredient being the representation theory of the rotation group). For completeness, the general formulae for these moments are

$$\begin{aligned} I^L = &\sum_{p=0}^{\infty} \frac{(2\ell+1)!!}{(2p)!!(2\ell+2p+1)!!}\left[1+\frac{8p(\ell+p+1)}{(\ell+1)(\ell+2)}\right]\left[\int d^3\mathbf{x}\,\partial_t^{2p}\mathcal{T}^{00}r^{2p}x^L\right]_{\text{STF}} \\ &+\sum_{p=0}^{\infty} \frac{(2\ell+1)!!}{(2p)!!(2\ell+2p+1)!!}\left[1+\frac{4p}{(\ell+1)(\ell+2)}\right]\left[\int d^3\mathbf{x}\,\partial_t^{2p}\mathcal{T}^{aa}r^{2p}x^L\right]_{\text{STF}} \\ &-\sum_{p=0}^{\infty} \frac{(2\ell+1)!!}{(2p)!!(2\ell+2p+1)!!}\frac{4}{\ell+1}\left(1+\frac{2p}{\ell+2}\right)\left[\int d^3x\,\partial_t^{2p+1}\mathcal{T}^{0a}r^{2p}x^{aL}\right]_{\text{STF}} \\ &+\sum_{p=0}^{\infty} \frac{(2\ell+1)!!}{(2p)!!(2\ell+2p+1)!!}\frac{2}{(\ell+1)(\ell+2)}\left[\int d^3\mathbf{x}\,\partial_t^{2p+2}\mathcal{T}^{ab}r^{2p}x^{abL}\right]_{\text{STF}}, \end{aligned} \tag{1.166}$$

and

$$\begin{aligned} J^L = &\sum_{p=0}^{\infty} \frac{(2\ell+1)!!}{(2p)!!(2\ell+2p+1)!!}\left(1+\frac{2p}{\ell+2}\right)\left[\int d^3\mathbf{x}\,\epsilon^{k_\ell ba}\partial_t^{2p}\mathcal{T}^{0a}r^{2p}x^{bL-1}\right]_{\text{STF}} \\ &-\sum_{p=0}^{\infty} \frac{(2\ell+1)!!}{(2p)!!(2\ell+2p+1)!!}\frac{1}{\ell+2}\left[\int d^3\mathbf{x}\,\epsilon^{k_\ell ba}\partial_t^{2p+1}\mathcal{T}^{ac}r^{2p}x^{bcL-1}\right]_{\text{STF}}. \end{aligned} \tag{1.167}$$

The $E_{\mu\nu}$ and $B_{\mu\nu}$ tensors in (1.164) are the “electric” and “magnetic” components of the Weyl tensor:

$$E_{\mu\nu} \equiv C_{\mu\alpha\nu\beta}u^\alpha u^\beta, \tag{1.168a}$$

$$B_{\mu\nu} \equiv \frac{1}{2}\,\epsilon_{\mu\rho\sigma\alpha}C^{\rho\sigma}{}_{\nu\beta}u^\alpha u^\beta. \tag{1.168b}$$

Both tensors are traceless with respect to the full metric and transverse to the four-velocity of the self-gravitating body: $g^{\mu\nu}E_{\mu\nu} = g^{\mu\nu}B_{\mu\nu} = 0$, and $E_{\mu\nu}u^\nu = B_{\mu\nu}u^\nu = 0$. Recall that for on-shell perturbations $h_{\mu\nu}$ in vacuum, the Ricci tensor and Ricci scalar vanish and the Weyl tensor coincides with the Riemann tensor. In the frame where $u^\mu_{\text{body}} = (1, \mathbf{0})$, the non-zero components of $E_{\mu\nu}$ and $B_{\mu\nu}$ are

$$E_{ij} = R_{0i0j} = \frac{1}{2M_{\text{Pl}}}(\partial_0\partial_j h_{0i} + \partial_0\partial_i h_{0j} - \partial_i\partial_j h_{00} - \partial_0^2 h_{ij}), \tag{1.169a}$$

$$B_{ij} = \frac{1}{2}\,\epsilon_{ikl}R_{0j}{}^{kl} = \frac{1}{2M_{\text{Pl}}}\,\epsilon_{ikl}(\partial_0\partial^l h_j{}^k + \partial_j\partial^k h_0{}^l). \tag{1.169b}$$

In particular, the moments X^i, P^i, and L^i vanish with this choice of frame, thus simplifying the action (1.164).

Separation of scales and the two-body problem. Consider once again the gravitational two-body problem and the separation between potential and radiative modes. To construct the bottom-up effective action we must first list all the symmetries we have at our disposal.

According to (1.39), after integrating out the potential modes we are left with an effective theory describing a single skeletonized body in a background metric

$$\bar{g}_{\mu\nu} \equiv \eta_{\mu\nu} + \frac{1}{M_{\rm Pl}}\,\bar{h}_{\mu\nu}. \tag{1.170}$$

In the absence of spin degrees of freedom, the symmetries of this theory are *worldline reparametrization invariance* and *diffeomorphisms with respect to* $\bar{g}_{\mu\nu}$. The latter property is ensured by promoting the partial derivatives ∂_μ in (1.164) to covariant derivatives $\bar{\nabla}_\mu$ compatible with $\bar{g}_{\mu\nu}$. Reparametrization invariance is made manifest by integrating the effective Lagrangian with respect to the proper time $(\bar{g}_{\mu\nu}\,{\rm d}x^\mu\,{\rm d}x^\nu)^{1/2}$, or ${\rm d}t\,(\bar{g}_{00})^{1/2}$, in our choice of frame. The result is the following multipolar action:

$$\begin{aligned} S_{\rm multi}[x^a_\mu, \bar{h}_{\mu\nu}] \equiv \int {\rm d}t\,(\bar{g}_{00})^{1/2}\Bigg[-M + \frac{1}{2M_{\rm Pl}}\sum_{\ell=2}^{\infty}\bigg(\frac{1}{\ell!}\,I^L(t)\bar{\nabla}_{L-2}\mathcal{E}_{i_{\ell-1}i_\ell} \\ - \frac{2\ell}{(\ell+1)!}\,J^L(t)\bar{\nabla}_{L-2}\mathcal{B}_{i_{\ell-1}i_\ell}\bigg)\Bigg], \end{aligned} \tag{1.171}$$

where $\mathcal{E}_{\mu\nu}$ and $\mathcal{B}_{\mu\nu}$ are defined as

$$\mathcal{E}_{\mu\nu} \equiv 2M_{\rm Pl}E_{\mu\nu}[\bar{g}], \tag{1.172a}$$

$$\mathcal{B}_{\mu\nu} \equiv 2M_{\rm Pl}B_{\mu\nu}[\bar{g}]. \tag{1.172b}$$

The effective action $S^{\rm WR}_{\rm eff}[x^\mu_a, \bar{h}_{\mu\nu}]$ of equation (1.44) is thus given by

$$S^{\rm WR}_{\rm eff}[x^\mu_a, \bar{h}_{\mu\nu}] + S_{\rm gf} = S_{\rm cons}[x^\mu_a] + S^{(2)}[\bar{h}_{\mu\nu}] + S_{\rm multi}[x^\mu_a, \bar{h}_{\mu\nu}]. \tag{1.173}$$

Comparing this expression with the definition given in the top-bottom approach, we see that (1.171) plays the role of $S_{\rm int}[x^\mu_a, \bar{h}_{\mu\nu}]$. In principle, $S_{\rm int}$ contains operators involving all powers of $\bar{h}$, and these should be added to the multipolar action just constructed. Such terms like $\mathcal{E}_{ij}\mathcal{E}^{ij}$ probe the finite size of the binary constituents, and their effects can be analyzed by splitting the multipole moments into "background" and "response" terms. The latter is associated to tidal effects, as we shall discuss in more detail in chapter 3. For the moment, we restrict ourselves to the single-graviton action $S_{\rm multi}[x^\mu_a, \bar{h}_{\mu\nu}]$ and neglect higher-order operators.

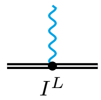
As it stands, the action (1.171) accounts not only for the emission of GWs but also their propagation on the curved spacetime geometry of the binary. To make our lives easier, we can first consider the propagation of waves in a Minkowski background and include *a posteriori* non-linear effects – such as the *tail* effect – associated to the spacetime curvature. Formally, the expressions for the radiated power and the waveform will not change; instead, the difference arises in the definition of the multipole moments. When non-linearities are "turned on" one must work with the so-called *radiative multipole moments*.

From (1.171) it is easy to write down the linearized action. We replace ${\rm d}t\,(\bar{g}_{00})^{1/2}$ by

$\mathrm{d}t$ and $\bar{\nabla}_\mu$ by ∂_μ, yielding

$$S_{\text{linear}}[x_a^\mu, \bar{h}_{\mu\nu}] = \frac{1}{2M_{\text{Pl}}} \sum_{\ell=2}^{\infty} \int \mathrm{d}t \left(\frac{1}{\ell!} I^L(t) \partial_{L-2} \tilde{E}_{i_{\ell-1}, i_\ell} - \frac{2\ell}{(\ell+1)!} J^L(t) \partial_{L-2} \tilde{B}_{i_{\ell-1} i_\ell} \right), \tag{1.174}$$

where $\tilde{E}_{i_{\ell-1}i_\ell}$ and $\tilde{B}_{i_{\ell-1}i_\ell}$ are simply (1.169a) and (1.169b) multiplied by a factor of $2M_{\text{Pl}}$, respectively. Extracting the Feynman rules out of (1.174) is straightforward and leads to an expression for the *single-graviton emission amplitude* $i\mathcal{A}_h$, which we define diagramatically by

$$i\mathcal{A}_h \equiv \sum_{\ell=2}^{\infty} \Bigg($$


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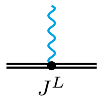
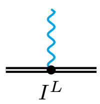


$$\Bigg)_{\text{on-shell}} . \tag{1.175}$$

Let us now derive the explicit form of the amplitudes shown above. First, consider the contribution from the electric multipoles. Working in the TT gauge and Fourier transforming (1.174), we find[23]

$$\begin{aligned}
(\mathsf{E}) &\equiv \int \mathrm{d}t \, \frac{1}{\ell!} I^L(t) \partial_{L-2} \tilde{E}_{i_{\ell-1}i_\ell} \\
&= \int \mathrm{d}t \, \frac{1}{\ell!} \int_\omega e^{-i\omega t} I^L(\omega) \partial_{L-2} \left[-\partial_t^2 \int_{\omega', \mathbf{k}} e^{-i\omega' t} e^{i\mathbf{k}\cdot\mathbf{x}} \, \bar{h}_{i_{\ell-1}i_\ell}(\omega', \mathbf{k}) \right]_{\mathbf{x}=0} \\
&= \int_{\omega, \mathbf{k}} \frac{1}{\ell!} I^L(\omega) \omega^2 (-i)^{\ell-2} k_{L-2} \, \bar{h}_{i_{\ell-1}i_\ell}(-\omega, \mathbf{k}).
\end{aligned} \tag{1.176}$$

Thus, the amplitude for the electric multipoles is



$$= \frac{i}{2M_{\text{Pl}}} \frac{(-i)^{\ell-2}}{\ell!} \omega^2 I^L(\omega) k_{L-2} \, \varepsilon^*_{i_{\ell-1}i_\ell}(\mathbf{k}, h). \tag{1.177}$$

The polarization tensor $\varepsilon_{\mu\nu}(\mathbf{k}, h)$ satisfies the following properties in the TT gauge:

$$\varepsilon_{\mu 0}(\mathbf{k}, h) = 0, \tag{1.178a}$$

$$\delta^{ij} \varepsilon_{ij}(\mathbf{k}, h) = 0, \tag{1.178b}$$

$$k^i \varepsilon_{ij}(\mathbf{k}, h) = 0. \tag{1.178c}$$

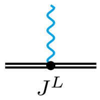
In addition, the reality of the graviton field $h_{\mu\nu}$ implies $\varepsilon^*_{\mu\nu}(\omega, \mathbf{k}; h) = \varepsilon_{\mu\nu}(-\omega, \mathbf{k}; h)$.

[23] Recall that our convention for the metric signature implies $\eta^{ij} = -\delta^{ij}$.

Moving on to the magnetic term, we have

$$\begin{aligned}(\mathsf{B}) &\equiv \int \mathrm{d}t\,\frac{2\ell}{(\ell+1)!}J^L(t)\partial_{L-2}\tilde{B}_{i_{\ell-1}i_\ell}\\ &= \int \mathrm{d}t\,\frac{2\ell}{(\ell+1)!}\int_\omega e^{-i\omega t}J^L(\omega)\partial_{L-2}\\ &\qquad\times\left[\epsilon_{i_{\ell-1}mn}\,\partial_t\,\partial^n\int_{\omega',\mathbf{k}} e^{-i\omega' t}e^{i\mathbf{k}\cdot\mathbf{x}}\,\bar{h}_{\ell_\ell}{}^m(\omega',\mathbf{k})\right]_{\mathbf{x}=0}\\ &= \int_{\omega,\mathbf{k}}\frac{2\ell}{(\ell+1)!}(-i)^{\ell-2}\omega J^L(\omega)\,\epsilon_{i_{\ell-1}mn}k^n k_{L-2}\,\bar{h}_{\ell_\ell}{}^m(-\omega,\mathbf{k}).\end{aligned} \tag{1.179}$$

Therefore,



$$= -\frac{i}{2M_{\mathrm{Pl}}}(-i)^{\ell-2}\frac{2\ell}{(\ell+1)!}\,\omega J^L(\omega)\epsilon_{i_{\ell-1}mn}k^n k_{L-2}\,\varepsilon^{*m}{}_{i_\ell}(\mathbf{k},h). \tag{1.180}$$

Combining (1.177) and (1.180), we finally obtain the single-graviton emission amplitude for all multipole moments:

$$\begin{aligned} i\mathcal{A}_h(\omega,\mathbf{k}) = \frac{i}{2M_{\mathrm{Pl}}}\sum_{\ell=2}^{\infty}\frac{(-i)^{\ell-2}}{\ell!}\Big[&\omega^2 I^L(\omega)k_{L-2}\,\varepsilon^*_{i_{\ell-1}i_\ell}(\mathbf{k},h)\\ &-\frac{2\ell}{(\ell+1)}\,\omega J^L(\omega)\epsilon_{i_{\ell-1}mn}k^n k_{L-2}\,\varepsilon^{*m}{}_{i_\ell}(\mathbf{k},h)\Big]_{\text{on-shell}}.\end{aligned} \tag{1.181}$$

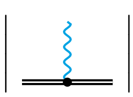
With $i\mathcal{A}_h$ in hands the computation of the radiated power follows immediately from the *optical theorem*, which says that the graviton emission rate $\mathrm{d}\Gamma$ can be obtained by squaring the on-shell amplitude:

$$\mathrm{d}\Gamma = \sum_h \frac{1}{T}\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3 2|\mathbf{k}|}\left|\mathcal{A}_h(|\mathbf{k}|,\mathbf{k})\right|^2. \tag{1.182}$$

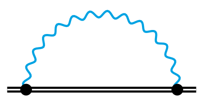
The radiated power follows as

$$\mathcal{P} = \int \mathrm{d}\Gamma\,|\mathbf{k}|. \tag{1.183}$$

Schematically, the formula (1.182) is the statement that



$$\Big|\ \cdot\ \Big|^2_{\text{on-shell}} = 2\,\mathrm{Im}\left(\ \cdot\ \right). \tag{1.184}$$



This relation was already used, albeit implicitly, in the top-bottom approach when we split the propagator in (1.153) into its real and imaginary parts. Notice that the perfect square structure of $\mathcal{I}_{\mathrm{D}}$, the term used to extract the radiated power $\mathcal{P}$, is similar to $\left|\mathcal{A}_h\right|^2$. The only difference is that in the top-bottom approach the amplitude $i\mathcal{A}_h$ was not introduced explicitly.[24] To make this connection clearer, consider the leading order formula (1.158) and (1.183). Comparing both expressions, we infer

$$\sum_h \frac{1}{T}\frac{|\mathbf{k}|^3}{(2\pi)^3 2|\mathbf{k}|}|\mathcal{A}_h(|\mathbf{k}|,\mathbf{k})|^2 = \frac{1}{T}\frac{G}{20\pi^2}|\mathbf{k}|^6\big|\tilde{I}_{ij}(|\mathbf{k}|)\big|^2. \tag{1.185}$$

The relation above can be easily checked by squaring the $\ell = 2$ amplitude

$$i\mathcal{A}_h(|\mathbf{k}|,\mathbf{k})\bigg|_{\ell=2} = \frac{i}{4M_{\text{Pl}}}|\mathbf{k}|^2 I^{ij}(\omega)\,\varepsilon^*_{ij}(\mathbf{k},h) \tag{1.186}$$

and performing the sum over graviton polarizations. Indeed, using the standard formula

$$\begin{aligned}\sum_h \varepsilon_{ij}(\mathbf{k},h)\varepsilon^*_{kl}(\mathbf{k},h) = \frac{1}{2}\bigg[&\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \delta_{ij}\delta_{kl} + \frac{1}{|\mathbf{k}|^2}(\delta_{ij}k_k k_l + \delta_{kl}k_i k_j) \\ &- \frac{1}{|\mathbf{k}|^2}(\delta_{ik}k_j k_l + \delta_{il}k_j k_k + \delta_{jk}k_i k_l + \delta_{jl}k_i k_k) + \frac{1}{|\mathbf{k}|^4}k_k k_j k_k k_l\bigg],\end{aligned} \tag{1.187}$$

one finds

$$\begin{aligned}\sum_h |\mathcal{A}_h(|\mathbf{k}|,\mathbf{k})|^2 &= \frac{1}{16M_{\text{Pl}}^2}|\mathbf{k}|^4 I^*_{ij}(\omega)I_{kl}(\omega)\sum_h \varepsilon_{ij}(\mathbf{k},h)\varepsilon^*_{kl}(\mathbf{k},h) \\ &= \frac{4\pi G}{5}|\mathbf{k}|^4|I_{ij}(\omega)|^2.\end{aligned} \tag{1.188}$$

When summing over the polarization tensors, we replaced

$$k_i k_j \to \frac{1}{3}|\mathbf{k}|^2\delta_{ij}, \tag{1.189a}$$

$$k_i k_j k_k k_l \to \frac{1}{15}|\mathbf{k}|^4(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}). \tag{1.189b}$$

This is justified because the quantity $\sum_h |\mathcal{A}_h(|\mathbf{k}|,\mathbf{k})|^2$ will be integrated with respect to the spherically symmetric measure $\mathrm{d}^3\mathbf{k}$. Since I_{ij} is traceless, we drop terms proportional to δ_{ij} and δ_{kl}. After straightforward manipulations, the simplified polarization sum rule used to derive (1.188) takes the form

$$\sum_h \varepsilon_{ij}(\mathbf{k},h)\varepsilon^*_{kl}(\mathbf{k},h) \to \frac{1}{5}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}). \tag{1.190}$$

With (1.188) in hands, the verification of (1.185) becomes a matter of elementary algebra and the derivation of $\mathcal{P}_{\text{LO}}$ follows identically. We thus see that the bottom-up approach correctly reproduces Einstein's quadrupole formula, as it should.

[24] This is simply a matter of practicality: the emission amplitude is a quantity most useful – conceptually speaking – in the bottom-up approach to the far-zone.

Since we have the all-multipoles amplitude (1.181) at our disposal, computing the flux for generic ℓ is conceptually easy, albeit technically demanding. A sketch of the intermediates steps for the case of scalar radiation – as well as the complete formulae for the electromagnetic and gravitational fluxes – can be found in reference [114]. We also point out to §3.8.2 of reference [4] as an illuminating guide to this computation.

The GW flux sourced by the magnetic quadrupole and the electric octupole moments contribute as [104]

$$\mathcal{P} = \frac{G}{T}\int_0^\infty \frac{\mathrm{d}\omega}{\pi}\left[\frac{1}{5}\,\omega^6|I^{ij}(\omega)|^2 + \frac{16}{45}\,\omega^6|J^{ij}(\omega)|^2 + \frac{1}{189}\,\omega^8|I^{ijk}(\omega)|^2 + \cdots\right]. \quad (1.191)$$

The first term is the leading-order result, clearly. As before, powers of ω can be traded by time derivatives of the multipole moments. The all-orders formula in the time-domain is

$$\begin{aligned}\mathcal{P} = G\sum_{\ell=2}^{\infty}\frac{(\ell+1)(\ell+2)}{\ell(\ell-1)\ell!(2\ell+1)!!}\left\langle\left(\frac{\mathrm{d}^{\ell+1}I^L}{\mathrm{d}t^{\ell+1}}\right)^2\right\rangle \\ + G_{\mathrm{N}}\sum_{\ell=2}^{\infty}\frac{4\ell(\ell+2)}{(\ell-1)(\ell+1)!(2\ell+1)!!}\left\langle\left(\frac{\mathrm{d}^{\ell+1}J^L}{\mathrm{d}t^{\ell+1}}\right)^2\right\rangle,\end{aligned} \quad (1.192)$$

where the $\langle\cdots\rangle$ brackets denote the time average over the period T.

Power-counting the multipole moments. So far no connection was made between (1.192) and the PN expansion. To do so, we must understand how each multipole moment scales with v, the power counting parameter of NRGR. Naturally, this investigation also yields the scaling of $\mathcal{T}^{\mu\nu}$.

Let m denote the typical mass scale of the binary. It suffices to consider the *source* multipole moments, in which case the integrals over $\mathrm{d}^3\mathbf{x}$ of $\mathcal{T}^{00}$, $\mathcal{T}^{0i}$, and $\mathcal{T}^{ij}$ scale as m, mv, and mv^2, respectively. Since $m/M_{\mathrm{Pl}} \sim (Lv)^{1/2}$, moments of $\mathcal{T}^{\mu\nu}$ scale as

$$\frac{1}{M_{\mathrm{Pl}}}\int \mathrm{d}^3\mathbf{x}\,\mathcal{T}^{00}x^L \sim (Lv)^{1/2}r^\ell, \quad (1.193a)$$

$$\frac{1}{M_{\mathrm{Pl}}}\int \mathrm{d}^3\mathbf{x}\,\mathcal{T}^{0i}x^L \sim (Lv^3)^{1/2}r^\ell, \quad (1.193b)$$

$$\frac{1}{M_{\mathrm{Pl}}}\int \mathrm{d}^3\mathbf{x}\,\mathcal{T}^{ij}x^L \sim (Lv^5)^{1/2}r^\ell. \quad (1.193c)$$

With these rules, we easily determine the scaling of the leading-order flux (1.158). Notice that

$$I^{ij}_{\mathrm{LO}} = \int \mathrm{d}^3\mathbf{x}\,\mathcal{T}^{00}x^{\langle i}x^{j\rangle} \sim mr^2, \quad (1.194)$$

and so $I^{ij}_{\mathrm{LO}}(\omega) \sim mr^3/v$. It follows that

$$\mathcal{P}_{\mathrm{LO}} \sim \frac{Lv^7}{r^2}. \quad (1.195)$$

The manner in which diagrams like (1.139) are combined to produce $iS_{\text{eff}}^{\text{NRGR}}$ suggests that the next-to-leading order (NLO) flux will be suppressed by v^2 with respect to $\mathcal{P}_{\text{LO}}$. Let us now list all multipole moments one must compute to determine the NLO flux. We write

$$I^{ij} = I_{\text{LO}}^{ij} + I_{\text{NLO}}^{ij} + \cdots\,, \tag{1.196a}$$

$$J^{ij} = J_{\text{LO}}^{ij} + J_{\text{NLO}}^{ij} + \cdots\,, \tag{1.196b}$$

and so on, for all moments. Besides the NLO electric quadrupole, both LO magnetic quadrupole and electric octupole contribute to $\mathcal{P}_{\text{NLO}}$. This is not difficult to verify. From the general formula (1.166), the NLO I^{ij} tensor is

$$I_{\text{NLO}}^{ij} = \int \mathrm{d}^3\mathbf{x}\left(\mathcal{T}^{k}_{\ k} - \frac{4}{3}\dot{\mathcal{T}}_{0k}x^k + \frac{11}{42}\ddot{\mathcal{T}}^{00}|\mathbf{x}|^2\right)\left[x^i x^j\right]_{\text{STF}} \sim mr^2v^2. \tag{1.197}$$

These are all the terms of I^L (for $\ell = 2$) that carry an additional power of v^2 with respect to I_{LO}^{ij}. It follows that $I_{\text{NLO}}^{ij}(\omega) \sim mr^3v$. Therefore, $I_{\text{LO}}^{ij}(\omega)I_{\text{NLO}}^{ij}(\omega) \sim m^2r^6$, which is indeed down by a factor of v^2 compared to $|I_{\text{LO}}^{ij}(\omega)|^2$. Moving on, the leading magnetic quadrupole moment is

$$J_{\text{LO}}^{ij} = -\int \mathrm{d}^3\mathbf{x}\left[\epsilon^i_{\ kl}\mathcal{T}^{0k}x^jx^l\right]_{\text{STF}} \sim mr^2v. \tag{1.198}$$

Then, $J_{\text{LO}}^{ij}(\omega) \sim mr^3$ and we find once again that $|J_{\text{LO}}^{ij}(\omega)|^2 \sim m^2r^6$. At last, consider the leading electric octupole

$$I_{\text{LO}}^{ijk} = \int \mathrm{d}^3\mathbf{x}\,\mathcal{T}^{00}\left[x^ix^jx^k\right]_{\text{STF}} \sim mr^3. \tag{1.199}$$

It is clear that $|I_{\text{LO}}^{ijk}(\omega)|^2 \sim m^2r^8/v^2$. Since this term appears multiplying a factor of ω^8 in the formula (1.191), it also produces a NLO contribution to the flux. Taking into account all three contributions. Taking into account the three contributions discussed so far and the general result (1.191), the NLO flux takes the form

$$\mathcal{P}_{\text{NLO}} = \frac{G}{T}\int_0^\infty \frac{\mathrm{d}\omega}{\pi}\left[\frac{2}{5}\,\omega^6 I_{\text{NLO}}^{ij}(I_{\text{LO}})_{ij} + \frac{16}{45}\,\omega^6|J_{\text{LO}}^{ij}(\omega)|^2 + \frac{1}{189}\,\omega^8|I_{\text{LO}}^{ijk}(\omega)|^2\right]. \tag{1.200}$$

It is not difficult to check that $\mathcal{P}_{\text{NLO}} \sim Lv^9/r^2$, as expected. The factor of 2 in the electric quadrupole piece appears because there are *two* terms of the form $I_{\text{NLO}}^{ij}(I_{\text{LO}})_{ij}$ when expanding the product $(I_{\text{LO}} + I_{\text{NLO}})_{ij}(I_{\text{LO}} + I_{\text{NLO}})^{ij}$.

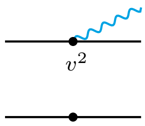
The matching calculation. So far we have not determined the precise form of the pseudo stress-energy tensor. To do so, one must perform the matching procedure. Following the near-zone calculations of §1.3.2, we now evaluate $\mathcal{T}^{\mu\nu}$ perturbatively for our binary system, taking into account both contributions from the matter particles and the gravitational binding energy.

Separating these two contributions, we write $u^\mu = (1, \mathbf{v})$ and $\mathcal{T}^{\mu\nu} = \mathcal{T}^{\mu\nu}_{\text{pp}} + \mathcal{T}_H$, where

$$\mathcal{T}^{\mu\nu}_{\text{pp}} \equiv \sum_a m_a \frac{u^\mu u^\nu}{\sqrt{1-\mathbf{v}^2}} \delta^{(3)}(\mathbf{x} - \mathbf{x}_a(t)) \tag{1.201}$$

and $\mathcal{T}^{\mu\nu}_H$ arises from integrating out the potential modes $H_{\mathbf{k}\mu\nu}$. The expansion in powers of $\mathbf{v}$ is straightforward for $\mathcal{T}^{\mu\nu}_{\text{pp}}$, as can be seen from the expression above.

The diagrams for the point-particle contribution are simply single-emission diagrams like (1.115), where no potential modes interact with the wordlines (nor with the external radiation gravitons). For example, the formula (1.110), which is *not multipole-expanded*, is precisely the v^2 rule below:



$$\begin{aligned} \sum_{a=1}^{2} \left(\text{diagram} \right) &\equiv -\frac{i}{2M_{\text{Pl}}} \int \mathrm{d}^4x \, \mathcal{T}^{\mu\nu}_{\text{pp}} \bar{h}_{\mu\nu} \Big|_{v^2} \\ &= -\frac{i}{2M_{\text{Pl}}} \sum_a m_a \int \mathrm{d}t \left(\frac{v_a^2}{2} \bar{h}_{00} + \bar{h}_{ij} v_a^i v_a^j \right). \end{aligned} \tag{1.202}$$

It is worth stressing that the derivation of the rule above does not require any vertices coming from the gauge-fixed Einstein-Hilbert action; we only expanded the point-particle stress-energy tensor to the desired PN order.

Similarly, the diagrams entering the computation of $-\frac{i}{2M_{\text{Pl}}} \int \mathrm{d}^4x \, \mathcal{T}^{\mu\nu}_H \bar{h}_{\mu\nu}$ are of the same type as diagrams (b) and (c) in figure 1.6. Such graphs require vertices with both potential and radiation modes. With these remarks in mind, we are ready to compute all the moments in the NLO flux.

Leading J^{ij} and I^{ijk} moments. The leading-order moments only involve the point-particle contribution $\mathcal{T}^{\mu\nu}_{\text{pp}}$. To order v^0, we simply have

$$\mathcal{T}^{00}_{\text{pp}}(t, \mathbf{x}) \Big|_{v^0} = \sum_a m_a \delta^{(3)}(\mathbf{x} - \mathbf{x}_a(t)). \tag{1.203a}$$

$$\mathcal{T}^{0i}_{\text{pp}}(t, \mathbf{x}) \Big|_{v^0} = \sum_a m_a v^i \delta^{(3)}(\mathbf{x} - \mathbf{x}_a(t)). \tag{1.203b}$$

Substituting (1.203a) into (1.199), one finds

$$I^{ijk}_{\text{LO}}(t) = \sum_a m_a \int \mathrm{d}^3\mathbf{x} \, \delta^{(3)}(\mathbf{x} - \mathbf{x}_a(t)) \left[x^i x^j x^k\right]_{\text{STF}} = \sum_a m_a \left[x_a^i x_a^j x_a^k\right]_{\text{STF}}. \tag{1.204}$$

Finally, substituting (1.203b) into (1.198), we obtain

$$\begin{aligned} J^{ij}_{\text{LO}}(t) &= -\sum_a m_a \int \mathrm{d}^3\mathbf{x}\, \delta^{(3)}(\mathbf{x}-\mathbf{x}_a(t)) \big[\epsilon^i{}_{kl} v^k x^l x^j\big]_{\text{STF}} \\ &= \sum_a m_a \big[(\mathbf{x}_a \times \mathbf{v}_a)^i x^j_a\big]_{\text{STF}}. \end{aligned} \tag{1.205}$$

Next-to-leading I^{ij} moment. We now compute the four terms entering the complete NLO expression for the electric quadrupole moment:

$$\begin{aligned} I^{ij}_{\text{LO}} + I^{ij}_{\text{NLO}} &= \int \mathrm{d}^3\mathbf{x} \left(\mathcal{T}^{00} + \mathcal{T}^k_{\ k} - \frac{4}{3}\dot{\mathcal{T}}_{0k} x^k + \frac{11}{42}\ddot{\mathcal{T}}^{00}|\mathbf{x}|^2 \right) \big[x^i x^j\big]_{\text{STF}} \\ &\equiv (I^{ij})_{(\text{a})} + (I^{ij})_{(\text{b})} + (I^{ij})_{(\text{c})} + (I^{ij})_{(\text{d})}. \end{aligned} \tag{1.206}$$

First, consider the last two terms $(I^{ij})_{(\text{c})}$ and $(I^{ij})_{(\text{d})}$. The time derivatives acting on $\mathcal{T}^{\mu\nu}$ bring additional powers of v in the power counting, so that one can simply use the leading-order expression of the pseudo stress-energy tensor when evalutating them. For example,

$$\begin{aligned} (I^{ij})_{(\text{d})} &= \frac{11}{42}\sum_a m_a \int \mathrm{d}^3\mathbf{x}\, \frac{\mathrm{d}^2}{\mathrm{d}t^2}\big[\delta^{(3)}(\mathbf{x}-\mathbf{x}_a(t))\big]|\mathbf{x}|^2 \big[x^i x^j\big]_{\text{STF}} \\ &= \frac{11}{42}\sum_a m_a \frac{\mathrm{d}^2}{\mathrm{d}t^2}\Big(|\mathbf{x}_a|^2 \big[x^i_a x^j_a\big]_{\text{STF}}\Big). \end{aligned} \tag{1.207}$$

Similarly,[25]

$$\begin{aligned} (I^{ij})_{(\text{c})} &= \frac{4}{3}\sum_a m_a \int \mathrm{d}^3\mathbf{x}\, \frac{\mathrm{d}}{\mathrm{d}t}\big[\delta^{(3)}(\mathbf{x}-\mathbf{x}_a(t))\big] v_k x^k \big[x^i x^j\big]_{\text{STF}} \\ &= -\frac{4}{3}\sum_a m_a \frac{\mathrm{d}}{\mathrm{d}t}\Big(\mathbf{v}_a\cdot\mathbf{x}_a \big[x^i_a x^j_a\big]_{\text{STF}}\Big). \end{aligned} \tag{1.208}$$

Next, we turn to $(I^{ij})_{(\text{a})}$ and $(I^{ij})_{(\text{b})}$. To determine these terms we must compute the combination $\mathcal{T}^{00} + \mathcal{T}^k_{\ k}$ to NLO in v. As mentioned before, this sum receives contributions both from the point-particles and the gravitational binding energy, according to the diagrams in figure 1.6.

[25] Notice that $(I^{ij})_{(\text{c})}$ has an overall minus sign because we have lowered *one* spatial index of $\mathcal{T}^{0k}$.

Consider the point particle contribution first. From (1.202), we read off

$$\mathcal{T}^{00}_{\text{pp}}(t,\mathbf{x})\bigg|_{v^2} = \frac{1}{2}\sum_a m_a v^2 \delta^{(3)}(\mathbf{x}-\mathbf{x}_a(t)), \tag{1.209a}$$

$$\mathcal{T}^{ij}_{\text{pp}}(t,\mathbf{x})\bigg|_{v^2} = \sum_a m_a v^i v^j \delta^{(3)}(\mathbf{x}-\mathbf{x}_a(t)) \tag{1.209b}$$

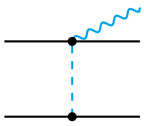
Thus,

$$\left[\mathcal{T}^{00}(t,\mathbf{x}) + \mathcal{T}^{k}_{\ k}(t,\mathbf{x})\right]_{\text{pp}} = \frac{3}{2}\sum_a m_a v^2 \delta^{(3)}(\mathbf{x}-\mathbf{x}_a(t)) + \mathcal{O}(v^4). \tag{1.210}$$

This sum contributes to the quadrupole moment as

$$\int \mathrm{d}^3\mathbf{x}\,(\mathcal{T}^{00} + \mathcal{T}^{k}_{\ k})_{\text{pp}}\left[x^i_a x^j_a\right]_{\text{STF}} = \frac{3}{2}\sum_a m_a v_a^2 \left[x^i_a x^j_a\right]_{\text{STF}} + \mathcal{O}(v^4). \tag{1.211}$$

Now we compute the contribution to I^{ij} coming from integrating out potential modes. These terms come from diagrams (b) and (c) of figure 1.6, and their stress-energy counterparts are denoted by $\mathcal{T}^{\mu\nu}_{H1}$ and $\mathcal{T}^{\mu\nu}_{H2}$, respectively. From (1.118), we write



$$\equiv -\frac{i}{2M_{\text{Pl}}}\int \mathrm{d}^4x\,\mathcal{T}^{\mu\nu}_{\text{H1}}\bar{h}_{\mu\nu}\bigg|_{v^2}$$
$$= -\frac{i}{M_{\text{Pl}}}\int \mathrm{d}t\,\frac{Gm_1m_2}{r}\,\bar{h}_{00}. \tag{1.212}$$

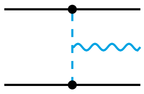
Introducing a sum $\sum_{b\neq a} \equiv \sum_{\substack{a,b=1\\ b\neq a}}^{2}$, we write this result as

$$-\frac{i}{2M_{\text{Pl}}}\int \mathrm{d}t \sum_{b\neq a}\frac{Gm_am_b}{|\mathbf{x}_{ab}|}\,\bar{h}_{00} = -\frac{i}{M_{\text{Pl}}}\int \mathrm{d}t\,\frac{Gm_1m_2}{r}\,\bar{h}_{00}. \tag{1.213}$$

In the equation above, $|\mathbf{x}_{ab}| \equiv |\mathbf{x}_a - \mathbf{x}_b| = r$. We then extract the following contribution to the 00 component of the electric quadrupole:

$$\int \mathrm{d}^3\mathbf{x}\,\mathcal{T}^{00}_{H1}(t,\mathbf{x})\left[x^i_a x^j_a\right]_{\text{STF}} = \sum_{b\neq a}\frac{Gm_am_b}{|\mathbf{x}_{ab}|}\left[x^i_a x^j_a\right]_{\text{STF}}. \tag{1.214}$$

Next, we extract $\mathcal{T}^{\mu\nu}_{H2}$ from (1.123). We write



$$\begin{aligned} &\equiv -\frac{i}{2M_{\rm Pl}}\int \mathrm{d}^4x\, \mathcal{T}^{\mu\nu}_{\rm H2}\bar{h}_{\mu\nu}\bigg|_{v^2} \\ &= \frac{i}{2M_{\rm Pl}}\int \mathrm{d}t \sum_{b\neq a}\frac{Gm_a m_b}{2|\mathbf{x}_{ab}|}\left\{\frac{3}{2}\bar{h}_{00}+\frac{1}{2}\frac{x^i_{ab}x^j_{ab}}{|\mathbf{x}_{ab}|^2}\bar{h}_{ij}\right\}. \end{aligned} \tag{1.215}$$

Repeating the same procedure, one finds

$$\int \mathrm{d}^3\mathbf{x}\,(\mathcal{T}^{00}+\mathcal{T}^{k}_{\ k})_{H2}\big[x^i_a x^j_a\big]_{\rm STF} = -2\sum_{b\neq a}\frac{Gm_a m_b}{|\mathbf{x}_{ab}|}\big[x^i_a x^j_a\big]_{\rm STF}. \tag{1.216}$$

Combining (1.207), (1.208), (1.211), (1.214) and (1.216), we finally obtain[26]

$$\begin{aligned} I^{ij} &= \sum_{a=1}^{2} m_a\Bigg(1+\frac{3}{2}v_a^2-\sum_{\substack{b=1\\ b\neq a}}^{2}\frac{Gm_b}{|\mathbf{x}_{ab}|}\Bigg)\big[x^i_a x^j_a\big]_{\rm TF} \\ &-\frac{4}{3}\sum_{a=1}^{2} m_a\frac{\mathrm{d}}{\mathrm{d}t}\Big(\mathbf{v}_a\cdot\mathbf{x}_a\big[x^i_a x^j_a\big]_{\rm TF}\Big)+\frac{11}{42}\sum_{a=1}^{2} m_a\frac{\mathrm{d}^2}{\mathrm{d}t^2}\Big(|\mathbf{x}_a|^2\big[x^i_a x^j_a\big]_{\rm TF}\Big). \end{aligned} \tag{1.217}$$

We are treating the binary system as an isolated body in the center of mass of the two-body system. Therefore, one must write (1.217) and the other moments using the appropriate center of mass coordinates.

Center-of-mass variables. First of all, denote $\mathbf{r}\equiv\mathbf{x}_1-\mathbf{x}_2$ and $\mathbf{v}=\dot{\mathbf{r}}$. Define the symmetric mass ratio η and the reduced mass μ by

$$\eta\equiv\frac{\mu}{m}\equiv\frac{m_1m_2}{m^2},\qquad m\equiv\sum_a m_a. \tag{1.218}$$

To express the electric and magnetic source multipoles in terms of the center of mass variables, we must determine M and X^i as given by (1.165a) and (1.165c), respectively. Recycling the previous matching computation, one easily finds

$$M=\sum_{a=1}^{2} m_a\Bigg(1+\frac{1}{2}v_a^2-\frac{1}{2}\sum_{\substack{b=1\\ b\neq a}}^{2}\frac{Gm_b}{|\mathbf{x}_{ab}|}\Bigg)+\mathcal{O}(v^4). \tag{1.219}$$

The first two terms above come from the point-particle contribution to the energy-momentum tensor whereas the last term comes from the integration of potential gravitons. One also

[26] Since the combination $x^i_a x^j_a$ is already symmetric, the term $\big[x^i_a x^j_a\big]_{\rm STF}$ is simply $x^i_a x^j_a$ with the trace $-\frac{1}{3}\delta^{ij}|x_a|^2$ subtracted. Hence, the "TF" label (tracefree).

finds

$$\begin{aligned}\mathbf{X} &= \frac{1}{m}\sum_{a=1}^{2} m_a\left(1+\frac{1}{2}v_a^2-\frac{1}{2}\sum_{\substack{b=1\\b\neq a}}^{2}\frac{Gm_b}{|\mathbf{x}_{ab}|}\right)\mathbf{x}_a+\mathcal{O}\big(v^4\big)\\ &= \frac{1}{m}\left[m_1\mathbf{x}_1+m_2\mathbf{x}_2+\frac{m_1}{2}v_1^2\mathbf{x}_1+\frac{m_2}{2}v_2^2\mathbf{x}_2-\frac{G\mu m}{2r}(\mathbf{x}_1+\mathbf{x}_2)\right]+\mathcal{O}\big(v^4\big)\end{aligned} \tag{1.220}$$

We take this center of mass position to coincide with the origin, so that $X^i = 0$. Using the leading-order relations

$$\mathbf{v}_1 = \frac{m_2}{m}\mathbf{v}+\mathcal{O}\big(v^3\big), \quad \mathbf{v}_2 = -\frac{m_1}{m}\mathbf{v}+\mathcal{O}\big(v^3\big), \tag{1.221}$$

the coordinates $\mathbf{x}_{1,2}$ can be determined by solving the following linear system to order v^2:

$$\mathbf{r} = \mathbf{x}_1 - \mathbf{x}_2, \tag{1.222a}$$

$$\mathbf{0} = m_1\mathbf{x}_1 + m_2\mathbf{x}_2 + \frac{\eta v^2}{2}(m_2\mathbf{x}_1+m_1\mathbf{x}_2) - \frac{G\mu m}{2r}(\mathbf{x}_1+\mathbf{x}_2). \tag{1.222b}$$

The result is

$$\mathbf{x}_1 = \left[\frac{m_2}{m}+\frac{\eta}{m}(m_1-m_2)\left(\frac{v^2}{2}-\frac{Gm}{r}\right)\right]\mathbf{r}+\mathcal{O}\big(v^4\big), \tag{1.223a}$$

$$\mathbf{x}_2 = \left[-\frac{m_1}{m}+\frac{\eta}{m}(m_1-m_2)\left(\frac{v^2}{2}-\frac{Gm}{r}\right)\right]\mathbf{r}+\mathcal{O}\big(v^4\big) \tag{1.223b}$$

Let ω_c be the orbital frequency of the circular orbit. Introduce the standard PN parameter

$$\mathsf{x} \equiv (Gm\omega_\mathrm{c})^{2/3} \sim v^2. \tag{1.224}$$

Since we are dealing with a circular orbit, the condition $\mathbf{r}\cdot\mathbf{v} = 0$ holds. Substituting (1.223a) and (1.223b) back into (1.204), (1.205), (1.217), and using the 1PN equations of motion derived from (1.91) to eliminate acceleration terms, one finds (after a lengthy calculation):

$$I^{ij}(t) = \mu\left[1-\left(\frac{1}{42}+\frac{39}{42}\eta\right)\mathsf{x}\right]\left[r^i r^j\right]_\mathrm{STF}+\frac{11}{42}\mu r^2(1-3\eta)\left[v^i v^j\right]_\mathrm{STF}, \tag{1.225a}$$

$$J^{ij}(t) = \mu\sqrt{1-4\eta}\,\left[(\mathbf{r}\times\mathbf{v})^i r^j\right]_\mathrm{STF}, \tag{1.225b}$$

$$I^{ijk}(t) = \mu\sqrt{1-4\eta}\,\left[r^i r^j r^k\right]_\mathrm{STF}. \tag{1.225c}$$

To determined the flux $\mathcal{P}_\mathrm{NLO}$ we must square the Fourier transform of these moments. Parametrizing the orbit as

$$\mathbf{r}(t) = \big(r\cos(\omega_\mathrm{c}t), r\sin(\omega_\mathrm{c}t), 0\big), \tag{1.226}$$

we find[27]

$$|I^{ij}(\omega)|^2 = \frac{\pi T \eta^2 \mathsf{x}^5}{2G^2\omega_c^6}\left[1 + \left(-\frac{107}{21} + \frac{55}{21}\eta\right)\mathsf{x}\right]\delta(\omega - 2\omega_c), \tag{1.227a}$$

$$|J^{ij}(\omega)|^2 = \frac{\pi T \eta^2 \mathsf{x}^6}{2G^2\omega_c^6}(1 - 4\eta)\delta(\omega - \omega_c), \tag{1.227b}$$

$$|I^{ijk}(\omega)|^2 = \frac{\pi T \eta^2 \mathsf{x}^6}{4G^2\omega_c^8}(1 - 4\eta)\left[\delta(\omega - 3\omega_c) + \frac{3}{5}\delta(\omega - \omega_c)\right], \tag{1.227c}$$

where $T \equiv 2\pi\delta(0)$. Identifying the leading-order flux in terms of the x parameter as

$$\mathcal{P}_{\text{LO}} = \frac{32}{5G}\eta^2 \mathsf{x}^5, \tag{1.228}$$

we finally obtain the famous result [23]

$$\frac{\mathcal{P}_{\text{NLO}}}{\mathcal{P}_{\text{LO}}} = 1 - \left(\frac{1247}{336} + \frac{35}{12}\eta\right)\mathsf{x}. \tag{1.229}$$

At the moment, the (energy) flux has been determined at 4.5PN order using the traditional post-Newtonian multipolar post-Minkowskian (PN-MPM) framework [25] [24]. The state of the art with NRGR methods is at 3PN order [9].

[27]These formulae are obtained by neglecting terms of the form $\delta(\omega)$ and $\delta(\omega + \xi\omega_c)$, for $\xi > 0$, since the frequency ω has support for positive values only.

Chapter 2

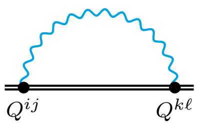
The *in-in* Formulation of NRGR

The formalism of NRGR presented thus far relies on the field-theoretic path integral approach, and as such, the usual scattering boundary conditions are inherited. This approach is dubbed "*in-out*." Consequently, we are forced to use the *time-symmetric* Feynman propagator to construct our theory. Our system of interest, however, undergoes a wealth of *time-asymmetric* dissipative processes associated to the reaction of gravitational radiation. As we shall see in the next section, these effects are not (yet) captured by the effective theory.

2.1 Shortcomings of the *in-out* approach

The loss of energy and momentum due to gravitational wave emission alters the motion of the binary itself. This is the meaning of "radiation reaction." At lowest non-trivial PN order in the effective action, one expects to find the first dissipative correction to the two-body potential in the real part of the graph (1.153). Let's define a function V_{diss} such that

$$\int \mathrm{d}t\, V_{\text{diss}} \equiv \mathrm{Re}\Bigg(i$$



$$\Bigg) \tag{2.1}$$

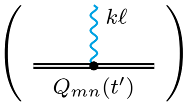
This dissipative potential also receives corrections from higher-order diagrams. For the moment let's focus our attention on the simple quadrupole-quadrupole graph. A closer look at (1.153) reveals that this *in-out* diagram does not contribute to the equations of motion of the binary; it produces a boundary term. After two integration by parts, we find

$$\int \mathrm{d}t\, V_{\text{diss}}\Big|_{\text{2.5 PN}} = \frac{G}{10}\int \mathrm{d}t\, Q^{(5)}_{ij} Q_{ij} = \frac{G}{20}\int \mathrm{d}t\, \frac{\mathrm{d}}{\mathrm{d}t}\,(\ddot{Q}_{ij}\ddot{Q}^{ij}). \tag{2.2}$$

Rather than a failure of NRGR, this feature signals a limitation of the *in-out* Lagrangian formalism. To describe real-time radiation-reaction effects one must abandon the Feynman propagator!

Besides the radiation reaction problem, it is straightforward to see that the gravitational waveform evaluated with the Feynman propagator is non-causal. To see this we must compute the NRGR one-point function. Introducing the TT gauge projector $\Lambda_{ijk\ell}(\hat{\mathbf{n}})$, we then compute the 2.5PN graph for the one-graviton emission. Schematically, this is

$$\langle \bar{h}^{\rm TT}_{ij}(x)\rangle = \Lambda_{ijk\ell} \tag{2.3}$$



At this order only the quadrupole coupling in (1.139c) survives the projection onto the TT gauge. Translating the formula above using the NRGR rules, we write[1]

$$\langle \bar{h}^{\rm TT}_{ij}(x)\rangle = \frac{i}{2M_{\rm Pl}}\Lambda_{ijk\ell}\int \mathrm{d}\tau\,\langle T\{\bar{h}^{k\ell}(x)\bar{R}_{0m0n}(\tau,\mathbf{0})\}\rangle Q^{mn}(\tau) \tag{2.4}$$

Since we are isolating the TT degrees of freedom, we can substitute $\bar{R}_{0m0n}$ by $\frac{1}{2}\partial_\tau^2\bar{h}_{mn}$ inside the correlator. The ∂_τ^2 derivatives then act on $\mathbb{P}^{k\ell}{}_{mn}D_{\rm f}(t-\tau,\mathbf{x})$. After integrating by parts, one finds

$$\langle \bar{h}^{\rm TT}_{ij}(x)\rangle = \frac{i}{4M_{\rm Pl}}\Lambda_{ijk\ell}\int \mathrm{d}\tau\, D_{\rm f}(t-\tau,\mathbf{x})\ddot{Q}^{k\ell}(\tau). \tag{2.5}$$

To extract the real part of this function we employ the identity (A.36), which leads to

$$\begin{aligned}\mathrm{Re}\left[iD_{\rm f}(t-\tau,\mathbf{x})\right] &= \frac{1}{2}\left[G_{\rm ret}(t-\tau,\mathbf{x}) + G_{\rm adv}(t-\tau,\mathbf{x})\right]\\ &= \frac{1}{4\pi}\,\delta\left[(t-\tau)^2 - |\mathbf{x}|^2\right].\end{aligned} \tag{2.6}$$

Therefore,

$$\begin{aligned}\frac{1}{M_{\rm Pl}}\mathrm{Re}\,\langle \bar{h}^{\rm TT}_{ij}(x)\rangle &= \frac{1}{16\pi M_{\rm Pl}^2}\,\Lambda_{ijk\ell}\int \mathrm{d}\tau\,\delta\left[(t-\tau)^2 - |\mathbf{x}|^2\right]\ddot{Q}^{k\ell}(\tau).\\ &= 2G\Lambda_{ijk\ell}\int \mathrm{d}\tau\,\frac{1}{2|\mathbf{x}|}\left[\delta(\tau - t_{\rm adv}) + \delta(\tau - t_{\rm ret})\right]\ddot{Q}^{k\ell}(\tau)\\ &= \frac{G}{|\mathbf{x}|}\Lambda_{ijk\ell}\left[\ddot{Q}^{k\ell}(t_{\rm ret}) + \ddot{Q}^{k\ell}(t_{\rm adv})\right],\end{aligned} \tag{2.7}$$

where $t_{\rm ret} = t - |\mathbf{x}|$ and $t_{\rm adv} = t + |\mathbf{x}|$. The expression above for the waveform is manifestly non-causal due to its dependence on the *advanced* time.

At the heart of this problem lies the Feynman propagator, which is a natural outcome of the *in-out* Lagrangian formalism. The correct way to implement retarded boundary conditions at the level of the action relies on the *in-in* formalism. But before we get to its implementation in NRGR, let's review some fundamental aspects of the *in-out* approach.

[1] The time τ is simply a dummy integration variable. Unless explicitly mentioned, it should not be confused with the proper time.

2.2 Review: *in-out* one-point functions

Let's take one step back and ask ourselves: "How does one compute one-point functions in the *in-out* path integral formalism?" The starting point is $W[\mathbf{j}_1, \mathbf{j}_2, J^{\mu\nu}]$, the generator of connected correlation functions:

$$e^{iW} = \int \prod_{a=1}^{2} \mathcal{D}\mathbf{x}_a \mathcal{D}\bar{h}_{\mu\nu} \exp\left(iS_{\text{WR}}[\mathbf{x}_a, \bar{h}] + i \int \mathrm{d}t\, \mathbf{j}_a \cdot \mathbf{x}_a + i \int \mathrm{d}^4x\, J^{\mu\nu} \bar{h}_{\mu\nu} \right). \tag{2.8}$$

In this path integral, S_{WR} is the action introduced in section 1.3.1 which is obtained by integrating out the potential modes of the gravitational field. Also, there is an implicit sum over the label a in the argument of the exponential above. Varying W with respect to $J^{\mu\nu}$ produces the one-point function

$$\langle \bar{h}_{\mu\nu}(x) \rangle_{\text{in-out}} = \left. \frac{\delta W}{\delta J^{\mu\nu}(x)} \right|_0 . \tag{2.9}$$

As before, the vertical bar above means: "take the sources to zero after computing all variations." Similarly, one can vary W with respect to $\mathbf{j}_a$ to derive $\langle \hat{\mathbf{x}}_a(t) \rangle_{\text{in-out}}$. The so-called *effective action* Γ is constructed by Legendre transforming the W functional:

$$\Gamma[\langle \hat{\mathbf{x}}_a(t) \rangle, J^{\mu\nu}] = W[\mathbf{j}_a, J^{\mu\nu}] - \sum_{a=1}^{2} \int \mathrm{d}t\, \mathbf{j}_a \cdot \langle \hat{\mathbf{x}}_a \rangle_{\text{in-out}}. \tag{2.10}$$

Previously, when writing (1.30) we had already integrated out the short scale modes by including point-particle actions (and possible finite-size corrections) in the full action. Now we made manifest the integration over the modes $\mathbf{x}_a$ governing the short-scale dynamics of the compact objects.

After obtaining the effective action Γ, one can derive the conservative equations of motion of the binary from

$$0 = \left. \frac{\delta(\operatorname{Re}\Gamma)}{\delta \langle \hat{\mathbf{x}}_a(t) \rangle_{\text{in-out}}} \right|_0 \tag{2.11}$$

This is yet another way to derive the equations of motion, which is slightly different from the near-zone computations of section 1.3.2. From the point of view of the radiation gravitons, the fields $\langle \hat{\mathbf{x}}_a(t) \rangle_{\text{in-out}}$ play the role of external sources (as far as the classical limit is concerned).

We now argue that this *in-out* one-point function is not the true expectation value of the position operator, and as such, it fails to evolve causally in time and it is not a real-valued quantity.

Scalar toy-model. Consider a real massive scalar field coupled to a physical source Q. The action of this system is

$$S[\phi] \equiv S_0[\phi] + S_Q[\phi] = \frac{1}{2} \int \mathrm{d}^4x\, (\partial_\mu\phi\partial^\mu\phi + m^2\phi^2) + \int \mathrm{d}^4x\, Q\phi. \tag{2.12}$$

The partition function of the theory is

$$Z[J] = e^{iW[J]} = \int \mathcal{D}\phi \exp\left[i \int \mathrm{d}^4x \left(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}m^2\phi^2 + \mathbb{J}\phi\right)\right], \tag{2.13}$$

where $\mathbb{J} \equiv J + Q$ denotes the sum of the physical and the auxiliary sources. The time-evolution operator in the interaction picture is given by the standard formula [97]:

$$\hat{U}_{\mathbb{J}}(t,t') = T \exp\left[i \int_{t'}^{t} \mathrm{d}t\, \mathbb{J}(x)\phi_{\mathrm{I}}(x)\right], \tag{2.14}$$

where T is the time-ordering symbol. This is the usual Dyson's formula, where the term $\mathbb{J}\phi_{\mathrm{I}}$ plays the role of the interaction Hamiltonian. The partition function is simply the expectation value of the vacuum-to-vacuum amplitude:

$$e^{iW[J]} = \int \mathcal{D}\phi\, \hat{U}_{\mathbb{J}}(\infty, -\infty)\, e^{iS_0[\phi]} = {}_{\mathrm{out}}\langle 0|\hat{U}_{\mathbb{J}}(+\infty, -\infty)|0\rangle_{\mathrm{in}} \tag{2.15}$$

In the distant past and future we adiabatically turn off the sources so that the *in-out* vacuum states correspond to the usual Minkowski vacuum which is singled out by Poincaré symmetry. Hence the name of the formalism itself.

In more general situations (e.g., non-flat background spacetimes), the *in-out* vacua need not to coincide. This is also the case in the *in-in* formalism, as we shall see. From (2.15), the one-point function is found by

$$\langle\phi(x)\rangle_{\text{in-out}} = \left.\frac{\delta W[J]}{\delta J(x)}\right|_{J=0}. \tag{2.16}$$

Contrary to what our notation suggests, this object is not the true expectation value of the field with respect to the $|0\rangle$ vacuum. To see this, we must use the formula

$$\frac{\delta}{\delta J(x)}\hat{U}_{\mathbb{J}}(\infty, -\infty) = i\hat{U}_{\mathbb{J}}(+\infty, t)\phi_{\mathrm{I}}(x)\hat{U}_{\mathbb{J}}(t, -\infty), \tag{2.17}$$

whose derivation follows directly from (2.14). The computation of $\langle\phi(x)\rangle_{\text{in-out}}$ is easy now. First, compute the functional derivative of the partition function:

$$e^{iW[J]}\frac{\delta W[J]}{\delta J(x)} = {}_{\mathrm{out}}\langle 0|\hat{U}_{\mathbb{J}}(\infty, t)\phi_{\mathrm{I}}\hat{U}_{\mathbb{J}}(t, -\infty)|0\rangle_{\mathrm{in}}. \tag{2.18}$$

Writing the interaction picture field in terms of its Heisenberg picture representation

$$\phi_{\mathrm{I}}(x) = \hat{U}_{\mathbb{J}}(t, -\infty)\phi_{\mathrm{H}}(x)\hat{U}_{\mathbb{J}}(-\infty, t) \tag{2.19}$$

and setting $J = 0$, we find

$$\langle\phi(x)\rangle_{\text{in-out}} = {}_{\mathrm{out}}\langle 0|\hat{U}_Q(\infty, -\infty)\phi_{\mathrm{H}}(x)|0\rangle_{\mathrm{in}} \equiv \langle Q|\phi_{\mathrm{H}}(x)|0\rangle_{\mathrm{in}}. \tag{2.20}$$

Therefore, since $\langle Q| \neq \langle 0|$ in general, the *in-out* one-point function is *not* the expectation value of the field.

Let us now show that $\langle\phi(x)\rangle_{\text{in-out}}$ does not evolve causally in time. Given that the action (2.12) is quadratic, we can compute the Gaussian path integral exactly. A direct computation of the one-point function then yields[2]

$$\langle\phi(x)\rangle_{\text{in-out}} = \frac{\delta}{\delta J(x)}\left(\frac{i}{2}\mathbb{J}\cdot\Delta_{\text{f}}\cdot\mathbb{J}\right)\Bigg|_{J=0} = \int \mathrm{d}^4y\, i\Delta_{\text{f}}(x-y)Q(y). \tag{2.21}$$

The acausal nature of $\langle\phi(x)\rangle_{\text{in-out}}$ then follows from the fact that $\Delta_{\text{f}}(x-y)$ is a non-causal combination of the retarded and advanced Green functions, as indicated by the identity (A.36).

2.3 The *in-in* approach

Having discussed in detail the problems of using *in-out* one-point functions to describe classical radiation fields, we now systematically introduce the *in-in formalism* for the scalar toy model introduced in the previous section. The fundamental object from which the formalism is constructed is the vacuum expectation value (VEV) of the field:

$$\begin{aligned}\langle\phi(x)\rangle_{\text{in-in}} &= {}_{\text{in}}\langle 0|\phi_{\text{H}}(x)|0\rangle_{\text{in}} \\ &= {}_{\text{in}}\langle 0|\hat{U}_Q(-\infty,t)\phi_{\text{I}}\hat{U}_Q(t,-\infty)|0\rangle_{\text{in}}.\end{aligned} \tag{2.22}$$

Since $\hat{U}$ is unitary and ϕ is Hermitian, it follows trivially that this VEV is real. As we will later prove, this function evolves causally in time. First, let's determine the generating functional that produces this VEV. The correct functional is

$$e^{iW[J_1,J_2]} \equiv \langle 0|\hat{U}_{J_2+Q}(-\infty,\infty)\hat{U}_{J_1+Q}(\infty,-\infty)|0\rangle, \tag{2.23}$$

where *two* auxiliary sources are introduced: J_1 for the forward evolution in time and J_2 for the evolution backwards in time. This way we break the time-symmetric nature of the in-out functions, paving our way towards a description which does not need to specify *ab initio* the final state of the system. After doing all computations with the duplicated variables, one must take the physical limit in which $\phi_1 = \phi_2 \equiv \phi$ and so on, for all degrees of freedom.

Crucially, when we rewrite Dyson's formula, the time integration must be performed along the closed time path (CTP) contour depicted in the figure 2.1 below.[3]

[2] This compact "inner product" notation is defined by

$$\mathbb{J}\cdot\Delta_{\text{f}}\cdot\mathbb{J} \equiv \int \mathrm{d}^4x\,\mathrm{d}^4y\,\mathbb{J}(x)\Delta_{\text{f}}(x-y)\mathbb{J}(y).$$

[3] Recall that in the *in-out* approach one already integrates over a deformed contour from $-\infty(1-i\epsilon)$ to t in Dyson's formula. This is necessary to guarantee that the interacting vacuum $|\Omega\rangle$ is projected onto the free

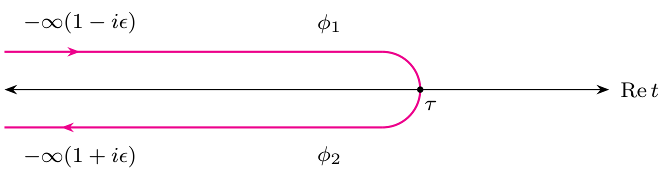


Figure 2.1: The "closed time path" contour used in the evaluation of the *in-in* path integral. Correlations between two points located in the upmost segment are always time-ordered, similarly to the usual *in-out* approach. Correlations in the lowermost segment are *anti*-time-ordered, and correlations mixing points in both branches give rise to the non-diagonal elements in the propagator matrix G^{AB}.

A simple computation with (2.23) and (2.17) shows that

$$\langle\phi_{\mathrm{H}}^{(1)}(x)\rangle_{\mathrm{in}} = +\frac{\delta W}{\delta J_1(x)}\bigg|_{J_A=0} = \langle 0|\hat{U}_Q(-\infty,\infty)\hat{U}_Q(\infty,t)\phi_{\mathrm{I}}(x)\hat{U}_Q(t,-\infty)|0\rangle, \quad (2.24a)$$

$$\langle\phi_{\mathrm{H}}^{(2)}(x)\rangle_{\mathrm{in}} = -\frac{\delta W}{\delta J_2(x)}\bigg|_{J_A=0} = \langle 0|\hat{U}_Q(-\infty,t)\phi_{\mathrm{I}}(x)\hat{U}_Q(t,\infty)\hat{U}_Q(\infty,-\infty)|0\rangle \quad (2.24b)$$

As expected, the one-point function derived from both fields coincide:

$$\langle\phi_{\mathrm{H}}^{(1)}(x)\rangle_{\mathrm{in}} = \langle\phi_{\mathrm{H}}^{(2)}(x)\rangle_{\mathrm{in}} = \langle 0|\phi_{\mathrm{H}}|0\rangle. \quad (2.25)$$

Theorists working on cosmological correlators [13] use a Hamiltonian formulation of the definition (2.22). Taking into account the doubling of degrees of freedom displayed in (2.23), they write the following "*in-in* master formula":

$$\langle\mathbb{O}(t)\rangle_{\text{in-in}} = \langle 0|\bar{T}e^{i\int_{-\infty(1+i\epsilon)}^{t}\mathrm{d}\tau\hat{H}_{\mathrm{I}}(\tau)}\mathbb{O}(t)Te^{-i\int_{-\infty(1-i\epsilon)}^{t}\mathrm{d}\tau\hat{H}_{\mathrm{I}}(\tau)}|0\rangle, \quad (2.26)$$

valid for arbitrary operators $\mathbb{O}(t)$. Since NRGR favors path integrals, we consider the following partition function along the CTP contour:

$$e^{iW[J_1,J_2]} = \int_{\mathrm{CTP}} \mathcal{D}\phi_1\mathcal{D}\phi_2\,\exp\big(iS_1[\phi_1] - iS_2[\phi_2] + iJ_1\cdot\phi_1 - iJ_2\cdot\phi_2\big). \quad (2.27)$$

Since the actions S_1 and S_2 are the same quadratic actions, we can compute the Gaussian partition function exactly once again. Now that we have twice as many degrees of freedom, the most general form of the W functional is

$$\begin{aligned} W[J_1,J_2] &= -i\log Z[J_1,J_2] \\ &= \frac{1}{2}\int \mathrm{d}^4x\,\mathrm{d}^4y\,[J_A(x)+Q_A(x)]iG^{AB}(x,y)[J_B(y)+Q_B(y)], \end{aligned} \quad (2.28)$$

where the *in-in* indices A and B run from 1 to 2. These indices must be summed over, as in the usual summation convention. The physical source is taken to satisfy $Q_1 = Q_2$, but we

vacuum $|0\rangle$ in the asymptotic past [97].

still write the formula above with different indices to be consistent with the notation. An important property of $G^{AB}(x,y)$ is the following symmetry relation under simultaneous exchange of the *in-in* indices and its arguments:

$$G^{AB}(x,y) = G^{BA}(y,x). \tag{2.29}$$

Notice that $G^{AB}(x,y)$ replaces the Feynman propagator now. But what are the components of this propagator matrix? To answer this question we must use the CTP contour of figure 2.1 and the representations (2.27) and (2.28) given above. Computing the two-point function using these different starting points, we find:

$$\frac{\delta}{i\delta J_1(x)}\frac{\delta}{i\delta J_1(x')}Z^{\text{in}}\bigg|_{J_A=0} = G^{11}(x,x') = G^{11}(x',x) = \langle T\{\phi_1(x)\phi_1(x')\}\rangle, \tag{2.30a}$$

$$\frac{\delta}{i\delta J_2(x)}\frac{\delta}{i\delta J_2(x')}Z^{\text{in}}\bigg|_{J_A=0} = G^{22}(x,x') = G^{22}(x',x) = \langle \bar{T}\{\phi_2(x)\phi_2(x')\}\rangle. \tag{2.30b}$$

We see that the diagonal terms are simply the Feynman and the Dyson propagators. The derivation of the off-diagonal terms requires a bit more care. We compute

$$\frac{\delta}{i\delta J_1(x)}\frac{\delta}{i\delta J_2(x')}Z^{\text{in}}\bigg|_{J_A=0} = \frac{1}{2}\left[G^{12}(x,x') + G^{21}(x',x)\right] = -\langle\phi_1(x)\phi_2(x')\rangle, \tag{2.31a}$$

$$\frac{\delta}{i\delta J_2(x)}\frac{\delta}{i\delta J_1(x')}Z^{\text{in}}\bigg|_{J_A=0} = \frac{1}{2}\left[G^{12}(x',x) + G^{21}(x,x')\right] = -\langle\phi_2(x)\phi_1(x')\rangle. \tag{2.31b}$$

Using (2.29), the two equations above become

$$G^{12}(x,x') = -\langle\phi_1(x)\phi_2(x'), \tag{2.32a}$$

$$G^{21}(x,x') = -\langle\phi_2(x)\phi_1(x')\rangle. \tag{2.32b}$$

To correctly identify these two-point functions consider the case $x'_0 < x_0$. By definition, the Feynman propagator $G^{11}(x,x')$ becomes

$$G^{11}(x,x')\bigg|_{x'_0<x_0} = \Delta_+(x-x') = \langle\phi_1(x)\phi_2(x')\rangle. \tag{2.33}$$

Now consider (2.32a) once again with the choice $x'_0 < x_0$. Since ϕ_2 always comes after ϕ_1 in the CTP path, this two-point function has a "backwards" time-ordering of its arguments (remember that earlier times in time-ordered n-point functions are always read from right to left).

Thus, $G^{12}(x,x')$ is simply (minus) the *negative frequency Wightman function*. A similar argument applies to $G^{21}(x,x')$, leading to the identification

$$G^{12}(x,x') = -\Delta_-(x-x'), \tag{2.34a}$$

$$G^{21}(x,x') = -\Delta_+(x-x'). \tag{2.34b}$$

We conclude that the components of the matrix $G^{AB}(x,x')$ are

$$G^{AB}(x,x') = \begin{pmatrix} \Delta_{\rm f}(x-y) & -\Delta_-(x-y) \\ -\Delta_+(x-y) & \Delta_{\rm d}(x-y) \end{pmatrix}. \tag{2.35}$$

From this expression we can derive the analog of (2.21) in the *in-in* formalism:

$$\begin{aligned} \langle \phi(x) \rangle_{\rm in} = \left. \frac{\delta W}{\delta J_1(x)} \right|_{J_A=0} = \frac{1}{2} \int {\rm d}^4 y \, \big[& 2i\Delta_{\rm F}(x-y) Q_1(y) - i\Delta_-(x-y)Q_2(y) \\ & - i\Delta_+(x-y)Q_2(y) \big]. \end{aligned} \tag{2.36}$$

Remember that Q_1 and Q_2 coincide. Using the property $\Delta_+(y-x) = \Delta_-(x-y)$ and the identity (A.27), we obtain

$$\langle \phi(x) \rangle_{\rm in} = \int {\rm d}^4 y \, iG_{\rm ret}(x-y)Q(y). \tag{2.37}$$

This formula is manifestly causal since it only depends on the retarded Green function.

The Keldysh representation. The identity (A.36) shows that not all Green functions are independent. We thus have some freedom when choosing a basis for writing the matrix $G^{AB}(x,y)$. A convenient choice is the *Keldysh representation*, where

$$J_- = J_1 - J_2, \tag{2.38a}$$
$$J_+ = \frac{1}{2}(J_1 + J_2), \tag{2.38b}$$

and similarly for all other quantities. This includes, in particular, the time x^0. The choice in this case is obvious though: one sets $(x_a^0)_- = 0$ and $(x_a^0)_+ = x_a^0 = t_a$, where t_a is the usual non-relativistic coordinate time. A simple calculation shows that

$$G^{AB}(x-y) = \begin{pmatrix} 0 & -iG_{\rm adv}(x-y) \\ -iG_{\rm ret}(x-y) & \frac{1}{2}\Delta_{\rm H}(x-y) \end{pmatrix}, \tag{2.39}$$

where the *in-in* indices now refer to the $\{+,-\}$ parametrization. These indices are raised and lowered with the matrix:

$$c_{AB} = c^{AB} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{2.40}$$

For example, it is not difficult to show that $J_\pm = J^\mp$. In this parametrization, the *in-in* one-point function is given by

$$\langle \phi_{\rm H}(x) \rangle_{\rm in} = \langle \phi_{\rm H}^+(x) \rangle_{\rm in} \Big|_{J_\pm=0} = \left. \frac{\delta W}{\delta J^+} \right|_{J_\pm=0} = \left. \frac{\delta W}{\delta J^-} \right|_{J_\pm=0}. \tag{2.41}$$

In the Keldysh representation, the physical limit consists in setting to zero all the "minus" variables and identifying the "plus" variables with the physical degrees of freedom. This prescription will become clear once we discuss the examples of section 2.4.

Applying the *in-in* formalism to NRGR. We now have all the relevant ingredients to implement retarded boundary conditions at the level of the NRGR action. After integrating out the potential gravitons, the *in-in* partition function takes the form

$$e^{iW[x^\mu_{aA},J^{\mu\nu}_A]} = \int_{\text{CTP}} \mathcal{D}\bar{h}^1_{\mu\nu}\mathcal{D}\bar{h}^2_{\mu\nu} \exp\left(iS^{\text{WR}}_{\text{eff}}[x^\mu_{a1},\bar{h}^1_{\mu\nu}] - iS^{\text{WR}}_{\text{eff}}[x^\mu_{a2},\bar{h}^2_{\mu\nu}]\right) \\ \times \exp\left(iS_{\text{gf}}[\bar{h}^1_{\mu\nu}] - iS_{\text{gf}}[\bar{h}^2_{\mu\nu}] + iJ^{\mu\nu}_1\cdot\bar{h}^1_{\mu\nu} - iJ^{\mu\nu}_2\cdot\bar{h}^2_{\mu\nu}\right). \tag{2.42}$$

In this expression, S_{gf} is the usual de Donder gauge-fixing action defined in (1.29). All the forward (labeled "1") and backward (labeled "2") quantities in this formula are set to coincide at infinity. Also, remember that $S^{\text{WR}}_{\text{eff}}$ is defined as the following sum:

$$S^{\text{WR}}_{\text{eff}}[x^\mu_a,\bar{h}_{\mu\nu}] + S_{\text{gf}}[\bar{h}_{\mu\nu}] = S_{\text{cons}}[x^\mu_a] + S^{(2)}[\bar{h}_{\mu\nu}] + S_{\text{int}}[x^\mu_a,\bar{h}_{\mu\nu}], \tag{2.43}$$

where S_{cons} produces all conservative near-zone graphs with any number of potential exhanges and $S^{(2)}$ is the gauge-fixed quadratic term of the EH action. The term S_{int} is the sum of two components: the pure graviton vertices and S_{diss}, which produces graphs with any number of potential exchanges and external radiatives modes attached to the worldlines.

The perturbative computation of the W functional follows the usual approach: we substitute the $\bar{h}^{1,2}_{\mu\nu}$ fields in the interaction term by functional derivatives and evaluate

$$e^{iW} = \exp\left(iS_{\text{cons}}[x^\mu_{a1}] - iS_{\text{cons}}[x^\mu_{a2}]\right) \\ \times \exp\left(iS_{\text{int}}\left[x^\mu_{a1},x^\mu_{a2},\frac{\delta}{i\delta J^{\mu\nu}_1},\frac{\delta}{i\delta J^{\mu\nu}_2}\right]\right) Z^{\text{in}}_0[J^{\mu\nu}_A], \tag{2.44}$$

where Z^{in}_0 denotes the CTP Gaussian path integral obtained from $S^{(2)}[\bar{h}^{\mu\nu}_A]$. In the Keldysh representation, this free partition function is

$$Z^{\text{in}}_0[J^{\mu\nu}_\pm] = \exp\left(\frac{i}{2}\int \mathrm{d}^4x\,\mathrm{d}^4y\, J^A_{\mu\nu}(x) iG^{\mu\nu\alpha\beta}_{AB}(x,y) J^B_{\alpha\beta}(y)\right), \tag{2.45}$$

where $G^{\mu\nu\alpha\beta}_{AB}(x,y) \equiv \mathbb{P}^{\mu\nu\alpha\beta}G_{AB}(x,y)$. The corresponding modifications in the NRGR Feynman rules are modest. They are summarized below:

(i) Include a factor of $G^{\mu\nu\alpha\beta}_{AB}(x,y)$ connecting particle-field vertices labeled with *in-in* indices A, B at spacetime points x, y;

(ii) A vertex coupled to n gravitons is labeled with n *in-in* indices (one for each graviton);

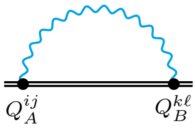
(iii) In the Keldysh representation, include $G^{-A}_{\mu\nu\alpha\beta}(x,y)$ for each radiation graviton that connects a vertex at y with index A to the field point x;

(iv) Sum over all *in-in* indices.

2.4 Examples

To illustrate the formalism and the *in-in* Feynman rules, we derive two important results: the 2.5PN radiation reaction force and the leading waveform.

2.5PN radiation reaction force. Let's derive the lowest-order contribution to the dissipative potential defined in (2.1). For such, we recalculate (1.153) with the *in-in* rules. The resulting formula is formally identical to the *in-out* computation:



$$= \frac{1}{2}\left(\frac{i}{2M_{\rm Pl}}\right)^2 \int \mathrm{d}t\,\mathrm{d}t'\, Q_A^{ij}(t)\langle \bar{R}^A_{0i0j}(t,\mathbf{0})\bar{R}^B_{0k0\ell}(t',\mathbf{0})\rangle Q_B^{k\ell}(t') \equiv iS_{\rm diss}^{Q^2} \tag{2.46}$$

Previously we found the following expression for the two-point function of the linearized Riemann tensor (with the Feynman prescription):

$$\langle T\{\bar{R}_{0i0j}(t,\mathbf{0})\bar{R}_{0k0\ell}(t',\mathbf{0})\}\rangle = \frac{1}{20}\,\partial_t^2\partial_{t'}^2\,\Delta_{\rm f}(t-t',\mathbf{0}) \times\left(\delta_{ik}\delta_{j\ell}+\delta_{i\ell}\delta_{kj}-\frac{2}{3}\delta_{ij}\delta_{k\ell}\right). \tag{2.47}$$

The corresponding two-point function in the *in-in* formalism is essentially the same; simply replace the Feynman propagator by the matrix G^{AB}. Then, after summing over the *in-in* indices and using $G_{\rm adv}(t-t',\mathbf{0}) = G_{\rm ret}(t'-t,\mathbf{0})$, one finds

$$\begin{aligned} iS_{\rm diss}^{Q^2} &= -\frac{1}{80M_{\rm Pl}^2}\int \mathrm{d}t\,\mathrm{d}t'\, Q_A^{ij}(t)\left[\partial_t^2\partial_{t'}^2\, G^{AB}(t-t',\mathbf{0})\right]Q_{Bij}(t') \\ &= +\frac{1}{80M_{\rm Pl}^2}\int \mathrm{d}t\,\mathrm{d}t'\left\{2iQ_-^{ij}(t)\left[\partial_t^2\partial_{t'}^2 G_{\rm ret}(t-t',\mathbf{0})\right]Q_{+ij}(t')\right. \\ &\qquad \left. -\frac{1}{2}Q_-^{ij}(t)\left[\partial_t^2\partial_{t'}^2\Delta_{\rm H}(t-t',\mathbf{0})\right]Q_{-ij}(t')\right\}. \end{aligned} \tag{2.48}$$

Explicitly, the quadrupole moments are

$$Q_+^{ij} = \frac{1}{2}(Q_1^{ij}+Q_2^{ij}) = \sum_a m_a\left(x^i_{a+}x^j_{a+} - \frac{1}{3}|\mathbf{x}_{a+}|^2\right) + \mathcal{O}\left(x_{a-}^3\right), \tag{2.49a}$$

$$Q_-^{ij} = Q_1^{ij}-Q_2^{ij} = \sum_a m_a\left(x^i_{a-}x^j_{a+} + x^i_{a+}x^j_{a-} - \frac{2}{3}\,\delta^{ij}\mathbf{x}_{a+}\cdot\mathbf{x}_{a-}\right). \tag{2.49b}$$

In [55] it is argued that $\mathcal{O}\big(x_{a-}^2\big)$ terms are associated to quantum corrections to the classical motion of the binary. In practice, this means that we'll drop the term proportional to $\Delta_{\rm H}$ in (2.48). Since Q_+^{ij} is symmetric and tracefree, the result simplifies a bit and we obtain

$$iS_{\rm diss}^{Q^2} = \frac{i}{20M_{\rm Pl}^2}\sum_a m_a \int {\rm d}t\,{\rm d}t'\,\big[\partial_t^2\partial_{t'}^2 G_{\rm ret}(t-t',\mathbf{0})\big]x_{a-}^i(t)x_{a+}^j(t)Q_{+ij}(t'). \tag{2.50}$$

The integral over ${\rm d}t'$ is given by (2.51):

$$\int {\rm d}\tau\, f(\tau)\,\frac{\partial^2}{\partial\tau^2}\frac{\partial^2}{\partial t^2}G_{\rm ret}(t-\tau,\mathbf{0}) = -\frac{1}{4\pi}f^{(5)}(t). \tag{2.51}$$

Proof. Going to Fourier space and integrating over τ, we find

$$\int {\rm d}\tau\, f(\tau)\,\frac{\partial^2}{\partial\tau^2}\frac{\partial^2}{\partial t^2}G_{\rm ret}(t-\tau,\mathbf{0}) = \int_\omega f(\omega)(i\omega)^2(-i\omega)^2 e^{-i\omega t}\int_{\mathbf{k}} G_{\rm ret}(\omega,\mathbf{k}). \tag{2.52}$$

Using the Fourier representation of the retarded Green function as written in (A.35c), the remaining integral in d dimensions becomes $\mathbb{M}_1^{(d)}(1,\omega^2)$, defined in (A.3). When $d\to 3$, we obtain

$$\begin{aligned}\int {\rm d}\tau\, f(\tau)\,\frac{\partial^2}{\partial\tau^2}\frac{\partial^2}{\partial t^2}G_{\rm ret}(t-\tau,\mathbf{0}) &= -\int_\omega f(\omega)(-i\omega)^2(-i\omega)^2 e^{-i\omega t}\,\mathbb{M}_1^{(3)}(1,\omega^2)\\ &= -\frac{1}{4\pi}\int_\omega f(\omega)(-i\omega)^5 e^{-i\omega t},\end{aligned} \tag{2.53}$$

which is the Fourier space expression of $-\frac{1}{4\pi}f^{(5)}(t)$. ■

Back to our calculation, the final result reads

$$iS_{\rm diss}^{Q^2} = -\frac{i}{80\pi M_{\rm Pl}^2}\sum_a m_a\int {\rm d}t\, x_{a-}^i(t)x_{a+}^j(t)Q_{+ij}^{(5)}(t). \tag{2.54}$$

According to the definition (2.1), this diagram contributes to the dissipative potential as

$$\int {\rm d}t\, V_{\rm diss}(t) = \frac{2G}{5}\int {\rm d}t\left(\sum_a m_a x_{a-}^i x_{a+}^j\right)Q_{+ij}^{(5)}(t). \tag{2.55}$$

The radiation-reaction force is thus given by

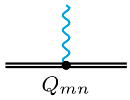
$$\frac{\delta}{\delta x_{a-}^i}\left(\int {\rm d}t\, V_{\rm diss}\right)\bigg|_{\sf PL} = \frac{2m_aG}{5}\,x_a^j\,Q_{ij}^{(5)} \equiv \mathcal{F}_i^{\sf BT}, \tag{2.56}$$

where the "physical limit" $\sf PL$ is obtained by setting $x_{a-}^i = 0$ and $x_{a+}^i = x_a^i$. The formula above is precisely the Burke-Thorne force [122, 29].

Leading waveform. Using the *in-in* effective action of NRGR (2.44), the one-point function (2.41) from which the waveform is extracted is determined by

$$\langle \bar{h}^{\text{TT}}_{ij}(x)\rangle = \Lambda_{ij}{}^{k\ell} \frac{\delta W}{\delta J^{k\ell}_{-}(x)}\bigg|_0 , \tag{2.57}$$

where the "0" subscript means that the sources are taken to zero after all derivatives are computed. Since we're interested in the TT projection of the one-point-function, the only coupling we need comes from



$$\supset \frac{i}{4M_{\text{Pl}}} \int \mathrm{d}t'\, Q^{mn}_A(t')\, \frac{\partial^2}{\partial t'^2}\, \bar{h}^A_{mn}(t', \mathbf{0}). \tag{2.58}$$

Therefore, the only contribution to the iS_{int} action in (2.44) that we must consider is (after integrating by parts twice and substituting the external graviton by a functional derivative):

$$iS^{\text{leading}}_{\text{int}} = \frac{i}{4M_{\text{Pl}}} \int \mathrm{d}t'\, \ddot{Q}^{mn}_A(t')\, \frac{\delta}{i\delta J^{mn}_A(t', \mathbf{0})}. \tag{2.59}$$

Expanding $e^{iS_{\text{int}}}$ and keeping the linear term, we find

$$\langle \bar{h}_{k\ell}(x)\rangle = \frac{i}{4M_{\text{Pl}}} \int \mathrm{d}t'\, \ddot{Q}^{mn}_A(t')\, \frac{\delta}{\delta J^{k\ell}_{-}(t, \mathbf{x})} \frac{\delta}{i\delta J^{mn}_A(t', \mathbf{0})}\, Z^{\text{in}}_0[J_\pm]\bigg|_0 . \tag{2.60}$$

The computation of the functional derivatives is straightforward with the rule

$$\frac{\delta J^A_{\mu\nu}(x^0, \mathbf{x})}{\delta J^{\alpha\beta}_B(y^0, \mathbf{y})} = \delta^{AB} \delta_{\mu\alpha}\delta_{\nu\beta}\, \delta(x^0 - y^0)\delta^{(3)}(\mathbf{x} - \mathbf{y}). \tag{2.61}$$

Using the symmetry relation (2.29), one finds

$$\langle \bar{h}_{k\ell}(x)\rangle = \frac{i}{4M_{\text{Pl}}} \int \mathrm{d}t'\, G^{-A}(t - t', \mathbf{x}) \mathbb{P}_{k\ell mn} \ddot{Q}^{mn}_A(t')\bigg|_{\text{PL}} . \tag{2.62}$$

The physical limit imposes $\ddot{Q}^{mn}_A \to \ddot{Q}^{mn}_+$. At last, invoking the direct space representation $G_{\text{ret}}(t, \mathbf{x}) = \frac{1}{4\pi r}\, \delta(t - r)$, the one-point function becomes

$$\frac{1}{M_{\text{Pl}}} \langle \bar{h}_{k\ell}(x)\rangle = \frac{2G}{|\mathbf{x}|}\, \ddot{Q}^{k\ell}(t_{\text{ret}}). \tag{2.63}$$

Therefore,

$$\frac{1}{M_{\text{Pl}}} \langle \bar{h}^{\text{TT}}_{ij}(x)\rangle = \frac{2G}{|\mathbf{x}|} \Lambda_{ijk\ell}\, \frac{\mathrm{d}^2}{\mathrm{d}t^2}\, Q^{k\ell}(t - |\mathbf{x}|), \tag{2.64}$$

which is, of course, the leading-order waveform that yields Einstein's quadrupole formula.

Chapter 3

Absorption Effects in NRGR

3.1 Introducing dissipation

One of the most attractive features of detecting GWs from coalescing binaries is the prospect of investigating the internal structure of the compact objects. It is widely known that, for example, the equation of state of a neutron star leaves an imprint in the observed waveform [47]. These absorption effects are closely tied to the deformability properties of compact objects, i.e., how their shape changes under the perturbation of external fields, which include the emitted GWs themselves.

In the case of BHs, the event horizon is responsible for the absorption of gravitational energy, which in turn changes the mass of the BH, as dictated by the no-hair theorem. This information can be extracted by studying the *response* function of a compact object (essentially the retarded Green's function). In Fourier space, the complex response is parametrized in terms of *tidal Love numbers*, which appear in the real, even-frequency terms, and *dissipation numbers* appearing in the imaginary, odd-frequency terms.

The actual values of the deformability parameters are interesting in their own right. For instance, the tidal Love numbers of a neutron star satisfy a set of (quasi)-universal relations called "I-Love-Q" relations [126, 95]. Remarkably, the static Love numbers of four-dimensional Schwarzschild [20] and Kerr BHs [81] vanish identically, a feature that was shown to originate in a symmetry enhancement called "Love symmetry" [35, 66]. In the Schwarzschild case, the non-linear static Love number vanishes too [112, 37, 72].

In this chapter, we use worldline EFT methods to investigate absorption effects between BH horizons and external long-wavelength test fields. We closely follow [60].

3.1.1 An electrodynamical toy-model

As a warm-up for the gravitational case, in this section we study the absorption of photons in the long wavelength limit by the horizon of a BH. The dynamical fields at our disposal are $A^\mu(x)$, the gauge potential, and $x^\mu_a(\tau)$, the worldlines. The symmetries from which we'll construct an effective action for the photon-BH interactions are:

1. Diffeomorphism invariance;
2. Electromagnetic, i.e., $U(1)$ gauge invariance;
3. Worldline reparametrization invariance.

If the typical frequency of the external fields in our problem is ω, then the aforementioned long wavelength limit is simply $r_s\omega \ll 1$. In this regime, the *non-dissipative* interactions of the BH with the photons can be systematically described by an effective action $S_{\text{eff}}^{(\text{ND})}$ constructed with fields and symmetries just mentioned.

By including all possible operators which consistent with the allowed symmetries, we are guaranteed to capture all finite-size effects of BH electrodynamics order by order in the power-counting parameter $r_s\omega$. The lowest-order non-trivial terms of the action $S_{\text{eff}}^{(\text{ND})}$ are

$$S_{\text{eff}}^{(\text{ND})}[x_a^\mu, A_\mu] = eQ \int \mathrm{d}x^\mu\, A_\mu(x) + \frac{c_{\text{E}}}{2} \int \mathrm{d}\tau\, E^\mu E_\mu + \frac{c_{\text{B}}}{2} \int \mathrm{d}\tau\, B^\mu B_\mu + (\cdots). \quad (3.1)$$

The ellipsis contains all higher-dimensional operators. By "non-trivial terms" we mean terms that cannot be absorbed into local counterterms by field redefinitions. For the purposes of this chapter, it suffices to consider the terms explicitly written in the action above.

The electric and magnetic fields, respectively, are defined by

$$E_\mu \equiv v^\nu F_{\nu\mu}, \quad (3.2a)$$

$$B_\mu \equiv \frac{1}{2}\epsilon_{\mu\alpha\beta\sigma} v^\sigma F^{\alpha\beta}. \quad (3.2b)$$

The vector v^μ is simply the four-velocity associated to the worldline: $v^\mu = \frac{\mathrm{d}}{\mathrm{d}\tau}x^\mu(\tau)$. Since we'll only consider neutral BHs, we set $Q = 0$. The action (3.1) can reproduce, for example, the elastic scattering amplitudes for photons to arbitrary order in $r_s\omega$. Unfortunately, it cannot be used to reproduce BH absorption effects.

A deeper discussion on the finite-size terms in (3.1) (and their analogues in the gravitational case) is presented at the end of this chapter, in section 3.3. We shall see that the effect of integrating out short-distance degrees results into worldline effective operators that can be matched to linear response theory operators.

In principle, one is agnostic with respect to the dynamics of the gapless degrees of freedom ξ responsible for dissipation. The EFT paradigm, however, allows one the use the symmetries of the BH solution to determine the *spectrum of possible composite operators* of any effective theory that incorporates ξ. Since we're considering the Schwarzschild solution and parity-conserving fields, the background symmetries dictate that the spectrum will be organized in terms of $SO(3)$ representations and parity eigenvalues.

The first terms of the effective action describing the interactions of the horizon with the photons can thus be written as

$$S_{\text{int}} = eQ \int \mathrm{d}x^\mu\, A_\mu(x) - \int \mathrm{d}\tau\, p_a(\tau) E^a(\tau) - \int \mathrm{d}\tau\, m_a(\tau) B^a(\tau), \quad (3.3)$$

where $E^a = \mathrm{e}^a{}_\mu E^\mu$ and $B^a = \mathrm{e}^a{}_\mu B^\mu$ are the electric and magnetic fields in the BH frame. The composite operators $p_a(\tau)$ and $m_a(\tau)$ correspond to the electric and magnetic dipole moments of the BH. Notice that the isometry group of the background, $SO(3)$, acts as a global symmetry in the worldline theory. The operators p_a, m_a are then split into *background* and *response* contributions. For example,

$$p_a = \langle p_a \rangle_{\mathcal{S}} + (p_a)_{\mathcal{R}}. \tag{3.4}$$

The labels $\mathcal{S}$ and $\mathcal{R}$ stand for "source" and "response", respectively. The first term is a background expectation value over short-distance degrees of freedom (the ξ's). Since an isolated Schwarzschild BH cannot have permanent dipole moments, we only consider the "response" term, which arises due to interactions with long-distance probe fields such as E_a and B_a.

There's no first principle computation which determines the correlators of the worldline operators such as the Feynman two-point function $\langle T\{p_a(\tau)p_b(\tau')\}\rangle$, for example. In the EFT, they are rather found by *matching* the results which come from the Feynman rules to some known observables in the "full theory" up to a given order in the power counting parameter. In our case, we must compare the EFT to the theory of electromagnetic scattering in a Schwarzschild background.

To determine the effects of *horizon dissipation* it is sufficient to compare the low-energy absorption cross section for polarized photons

$$\sigma_{\text{abs}}(\omega, h) = \frac{4\pi}{3}(r_s^2\omega)^2 + \mathcal{O}\big(\omega^3\big) \tag{3.5}$$

to the predictions of (3.3). The label h is the helicity of the incoming state. This result, first obtained by Starobinsky [119] and Page [93], can be derived in the full theory by solving Maxwell's equations in a Schwarzschild background.

Lowest-order Amplitudes. To carry out the predictions of the EFT to the photon absorption cross section – or any other observable – we must first derive the Feynman rules for the amplitudes. We begin with the electric term of the interaction action:

$$S_{\text{int}}^{\mathsf{E}} \equiv -\int \mathrm{d}t\, p_a(t)\, \mathbf{e}^a{}_\mu F^{\nu\mu} v_\nu + \text{non-linearities}. \tag{3.6}$$

In the BH frame, the four-velocity is $v^\nu = (1,0,0,0)$. Using $\mathbf{e}^a{}_j = \delta^a{}_j$, we then find

$$S_{\text{int}}^{\mathsf{E}} = -\int \mathrm{d}t\, p_j(t) F^{0j} v_0 = -\int \mathrm{d}t\, p_j(t)(\partial^0 A^j - \partial^j A^0). \tag{3.7}$$

In Fourier space, this expression becomes[1]

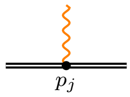
$$S_{\text{int}}^{\mathsf{E}} = -\int_{\omega,\mathbf{k}} p_j(\omega)[i\omega A^j(-\omega,\mathbf{k}) + ik^j A^0(-\omega,\mathbf{k})]. \tag{3.8}$$

As previously explained, all external fields are subject to a multipole expansion when deriving worldline couplings or amplitudes. If we choose the center of mass to be the reference point for the expansion and further choose it as the origin of the frame, then no

[1]Remember that we're using the mostly-minus convention for the metric. Therefore, raising or lowering an odd number of spatial indices costs a minus sign: $\partial^j(k^i x_i) = -k^j$.

terms of the type $e^{i\mathbf{k}\cdot\mathbf{x}}$ appear in the Feynman rules.

Dropping the A^0 term (it does not corresponde to a propagating radiative degree of freedom), the associated Feynman rule is inferred to be[2]



$$= \omega p_j(\omega)\varepsilon_j^*(\mathbf{k}, h). \tag{3.9}$$

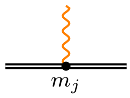
Next, consider the magnetic term

$$S_{\text{int}}^{\mathsf{B}} \equiv -\frac{1}{2}\int \mathrm{d}t\, m_a(t)\, \mathbf{e}^a{}_\mu \epsilon^{\mu\alpha\beta\sigma} v_\sigma F_{\alpha\beta} + \text{non-linearities}. \tag{3.10}$$

Using $\epsilon^{0jkl} = \epsilon^{jkl}$ and specializing to the BH frame, we find

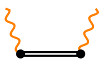
$$S_{\text{int}}^{\mathsf{B}} = \int \mathrm{d}t\, m_j(t)\epsilon^{jkl}\partial_k A_l(t, \mathbf{0}) = -\int_{\omega,\mathbf{k}} m_j(\omega)\epsilon^{jkl} i k_k A_l(-\omega, \mathbf{k}). \tag{3.11}$$

The Feynman rule for the magnetic term is thus



$$= m_j(\omega)(\mathbf{k}\times\boldsymbol{\varepsilon}^*)_j. \tag{3.12}$$

Computing σ_{abs}. To derive the leading-order expression for the absorption cross section, we consider the following diagram with leading multipole operator insertions:

$i\mathcal{M} =$ 

$$= -\int \mathrm{d}x^0\, e^{i\omega x^0}\big[\omega^2 \varepsilon_a^*\varepsilon_b \langle p_a(x^0)p_b(0)\rangle + (\mathbf{k}\times\boldsymbol{\varepsilon}^*)_a(\mathbf{k}\times\boldsymbol{\varepsilon})_b\langle m_a(x^0)m_b(0)\rangle\big]. \tag{3.13}$$

Notice that $\mathcal{M}$ has no real component. We now use the optical theorem to extract the absorption cross section. Considering a $2\to 2$ scattering event, we write [97]

$$2\,\mathrm{Im}\,\mathcal{M}(a\to a) = 4\big[(p_1\cdot p_2)^2 - m_1^2 m_2^2\big]^{1/2}\sigma_{\text{t}}(a\to \text{anything}), \tag{3.14}$$

where $\mathcal{M}(a\to a)$ denotes the forward scattering amplitude just derived, and σ_{t} is the total cross section.

Remember that we are working in the BH frame. Since the photon is massless, kinematics dictates that

$$\sigma_{\text{abs}} = \frac{1}{2M\omega}\,\mathrm{Im}\,\mathcal{M}, \tag{3.15}$$

where M is the mass of the BH. The two-point functions in (3.13) are supposed to be computed in the "ground state" $|\Omega\rangle$ of the worldline theory. Since the states on which the operators act correspond to a classical object – the black hole – they are subject to a non-relativistic normalization [85].

[2] The extra factor of i comes from the iS that appears in the path integral.

As written in (3.13), our amplitude implicitly assumes the usual relativistic normalization of states. The recipe for translating one normalization into the other is straightforward: simply replace $|\cdots\rangle \mapsto M^{1/2}|\cdots\rangle_{\mathsf{NR}}$. Thus,

$$\sigma_{\text{abs}}(\omega, h) = \frac{1}{2\omega}\int \mathrm{d}x^0\, e^{i\omega x^0}\big[\omega^2 \varepsilon_a^* \varepsilon_b \langle p_a(x^0) p_b(0)\rangle \\ + (\mathbf{k}\times\boldsymbol{\varepsilon}^*)_a(\mathbf{k}\times\boldsymbol{\varepsilon})_b\langle m_a(x^0) m_b(0)\rangle\big]. \tag{3.16}$$

We now see how the EFT determines the absorption cross section in terms of two-point functions of worldline operators such as p_a and m_a.

Fluctuation-dissipation relation. It is more convenient to work with time-ordered correlation functions rather than the usual two-point Wightman functions $\langle p_a(x^0)p_b(0)\rangle$ and $\langle m_a(x^0)m_b(0)\rangle$. The relation between the two types of correlators is given by the *fluctuation-dissipation theorem*:

$$\mathbb{W}_+^{LL'}(\omega) = 2\,\mathrm{Im}\left(i\int \mathrm{d}\tau\, e^{i\omega\tau}\langle T\{\mathbb{O}^L(\tau)\mathbb{O}^{L'}(0)\}\rangle\right), \quad \omega > 0, \tag{3.17}$$

where $\mathbb{O}^L$ is an arbitrary worldline operator with multi-index L and $\mathbb{W}_+^{LL'}(\omega)$ is the positive-frequency Wightman function that we define below. Before continuing our discussion, let's derive this formula following [68].

Proof. Define the positive and negative frequency Wightman functions as

$$\mathbb{W}_+^{LL'}(\tau-\tau') \equiv \langle \mathbb{O}^L(\tau)\mathbb{O}^{L'}(\tau')\rangle, \tag{3.18a}$$
$$\mathbb{W}_-^{LL'}(\tau-\tau') \equiv \langle \mathbb{O}^{L'}(\tau')\mathbb{O}^{L}(\tau)\rangle. \tag{3.18b}$$

These functions satisfy the obvious symmetry

$$\mathbb{W}_\pm^{LL'}(\tau-\tau') = \mathbb{W}_\mp^{L'L}(\tau'-\tau). \tag{3.19}$$

The Feynman and retarded Green functions are defined[3]

$$\mathbb{G}_{\text{f}}^{LL'}(\tau-\tau') = \theta(\tau-\tau')\mathbb{W}_+^{LL'}(\tau-\tau') + \theta(\tau'-\tau)\mathbb{W}_-^{LL'}(\tau-\tau'), \tag{3.20a}$$
$$\mathbb{G}_{\text{ret}}^{LL'}(\tau-\tau') = i\theta(\tau-\tau')\big[\mathbb{W}_+^{LL'}(\tau-\tau') - \mathbb{W}_-^{LL'}(\tau-\tau')\big]. \tag{3.20b}$$

If we assume that $\mathbb{O}^L$ is a Hermitian operator, then the positive and negative frequency Wightman functions are mapped into each other via complex conjugation:

$$\big[\mathbb{W}_+^{LL'}(\tau-\tau')\big]^* = \mathbb{W}_-^{LL'}(\tau-\tau'). \tag{3.21}$$

Equivalently, one can understand the condition above as a corollary of the fact that

[3] Recall our conventions for the *literature review*.

$\mathbb{G}_{\text{ret}}^{LL'}(\tau-\tau')$ is a real function. From (3.21) it follows that

$$\left[\mathbb{G}_{\text{f}}^{LL'}(\tau-\tau')\right]^* = -\mathbb{G}_{\text{d}}^{LL'}(\tau-\tau'), \tag{3.22}$$

where $\mathbb{G}_{\text{d}}^{LL'}(\tau-\tau')$ is the *Dyson function*, defined as the anti-time ordered combination of Wightman functions. These are all the building blocks we need for establishing (3.17). Next, we need the *dispersive representation* of the retarded Green function:

$$\mathbb{G}_{\text{ret}}^{LL'}(\omega) = -\int \frac{\mathrm{d}\omega'}{2\pi}\frac{[\mathbb{W}_+^{LL'}(\omega') - \mathbb{W}_-^{LL'}(\omega')]}{\omega-\omega'+i\mathsf{a}}. \tag{3.23}$$

Next, we split the real and imaginary parts of $\mathbb{G}_{\text{ret}}^{LL'}(\omega)$ using the Sokhotski-Plemelj formula:

$$\lim_{\mathsf{a}\to 0^+}\frac{1}{x\pm i\mathsf{a}} = \mathsf{PV}\,\frac{1}{x} \mp i\pi\delta(x). \tag{3.24}$$

In what follows, we introduce the function

$$\mathcal{W}^{LL'}(\omega) \equiv \mathbb{W}_+^{LL'}(\omega) - \mathbb{W}_-^{LL'}(\omega). \tag{3.25}$$

From (3.23), we get

$$\begin{aligned}\operatorname{Re}\mathbb{G}_{\text{ret}}^{LL'}(\omega) + i\operatorname{Im}\mathbb{G}_{\text{ret}}^{LL'}(\omega) &= \frac{i}{2}\operatorname{Re}\mathcal{W}^{LL'}(\omega) - \frac{1}{2}\operatorname{Im}\mathcal{W}^{LL'}(\omega)\\ &\quad - \mathsf{PV}\int_{\omega'}\frac{\operatorname{Re}\mathcal{W}^{LL'}(\omega')}{\omega-\omega'} - i\,\mathsf{PV}\int_{\omega'}\frac{\operatorname{Im}\mathcal{W}^{LL'}(\omega')}{\omega-\omega'}.\end{aligned} \tag{3.26}$$

Equating real and imaginary parts, one obtains

$$\operatorname{Re}\mathbb{G}_{\text{ret}}^{LL'}(\omega) = -\frac{1}{2}\operatorname{Im}\mathcal{W}^{LL'}(\omega) - \mathsf{PV}\int_{\omega'}\frac{\operatorname{Re}\mathcal{W}^{LL'}(\omega')}{\omega-\omega'}, \tag{3.27a}$$

$$\operatorname{Im}\mathbb{G}_{\text{ret}}^{LL'}(\omega) = \frac{1}{2}\operatorname{Re}\mathcal{W}^{LL'}(\omega) - \mathsf{PV}\int_{\omega'}\frac{\operatorname{Im}\mathcal{W}^{LL'}(\omega')}{\omega-\omega'}. \tag{3.27b}$$

The pair of equations above can be further simplified by exploiting the property that $\mathbb{G}_{\text{ret}}^{LL'}(\tau-\tau')$ is real, which implies $[\mathbb{G}_{\text{ret}}^{LL'}(\omega)]^* = \mathbb{G}_{\text{ret}}^{LL'}(-\omega)$. Therefore, on the real ω-axis, the functions $\operatorname{Re}\mathbb{G}_{\text{ret}}^{LL'}(\omega)$ and $\operatorname{Im}\mathbb{G}_{\text{ret}}^{LL'}(\omega)$ are *even* and *odd* in ω, respectively. The symmetry properties of (3.25) are summarized in table 3.1. It follows that

]	$\omega \mapsto -\omega$	$L \leftrightarrow L'$
$\operatorname{Re}\mathcal{W}^{LL'}(\omega)$	odd	even
$\operatorname{Im}\mathcal{W}^{LL'}(\omega)$	even	odd

Table 3.1: Parity properties of the real and imaginary components of $\mathcal{W}^{LL'}(\omega)$.

$$\operatorname{Re}\mathbb{G}^{LL'}_{\text{ret}}(\omega) = -\frac{1}{2}\operatorname{Im}\mathcal{W}^{LL'}(\omega) - \mathsf{PV}\int_0^\infty \frac{\mathrm{d}\omega'}{\pi}\frac{\omega'\operatorname{Re}\mathcal{W}^{LL'}(\omega')}{\omega^2-(\omega')^2}, \tag{3.28a}$$

$$\operatorname{Im}\mathbb{G}^{LL'}_{\text{ret}}(\omega) = \frac{1}{2}\operatorname{Re}\mathcal{W}^{LL'}(\omega) - \omega\,\mathsf{PV}\int_0^\infty \frac{\mathrm{d}\omega'}{\pi}\frac{\operatorname{Im}\mathcal{W}^{LL'}(\omega')}{\omega^2-(\omega')^2}. \tag{3.28b}$$

For Wightman functions defined in a Schwarzschild black hole, the only tensor structure compatible with spherical symmetry is

$$\mathbb{W}^{LL'}_{+}(\omega) \equiv w_+(\omega)\,\delta^{\langle L\rangle}_{\langle L'\rangle}. \tag{3.29}$$

With this additional assumption, one can easily prove that $\mathbb{W}^{LL'}_{+}(\omega)$ is real when $\omega \in \mathbb{R}$. To establish the fluctuation-dissipation theorem, consider

$$\begin{aligned}\mathbb{W}^{LL'}_{+}(\omega) &= \int_{-\infty}^{\infty}\mathrm{d}\tau\, e^{i\omega\tau}\,\mathbb{W}^{LL'}_{+}(\tau)\\ &= \int_{-\infty}^{\infty}\mathrm{d}\tau\, e^{i\omega\tau}\big[\delta(\tau)\mathbb{G}^{LL'}_{\text{f}}(\tau) + \theta(-\tau)\overline{\mathbb{G}}^{LL'}_{\text{f}}(\tau)\big].\end{aligned} \tag{3.30}$$

By definition, the Feynman Green function is translation invariant and symmetric under exchange of its arguments:

$$\langle T\{Q^L(-\tau)Q^{L'}(0)\}\rangle = \langle T\{Q^L(0)Q^{L'}(\tau)\}\rangle, \tag{3.31a}$$

$$\langle T\{Q^L(\tau)Q^{L'}(0)\}\rangle = \langle T\{Q^L(0)Q^{L'}(\tau)\}\rangle. \tag{3.31b}$$

Setting $\tau \mapsto -\tau$ in the second term of (3.30) and exploiting (3.31), we find

$$\begin{aligned}\mathbb{W}^{LL'}_{+} &= \int_0^\infty \mathrm{d}\tau\, e^{i\omega\tau}\langle T\{Q^L(\tau)Q^{L'}(0)\}\rangle + \int_0^\infty \mathrm{d}\tau\, e^{-i\omega\tau}\langle T\{Q^L(0)Q^{L'}(\tau)\}\rangle^*\\ &= 2\operatorname{Re}\left(\int_0^\infty \mathrm{d}\tau\, e^{i\omega\tau}\langle T\{Q^L(\tau)Q^{L'}(0)\}\rangle\right).\end{aligned} \tag{3.32}$$

Alternatively, one can write

$$\mathbb{W}^{LL'}_{+}(\omega) = 2\operatorname{Im}\left(i\int_0^\infty \mathrm{d}\tau\, e^{i\omega\tau}\langle T\{Q^L(\tau)Q^{L'}(0)\}\rangle\right). \tag{3.33}$$

This is the most general form of the fluctuation-dissipation theorem. To derive (3.17), however, one must further specialize to the case of classical Schwarzschild black holes. The appropriate vaccum state in this scenario is the *Boulware state*, which is compatible with the hypothesis of a purely absorptive Wightman function [68]

$$w_+(\omega < 0) = 0. \tag{3.34}$$

From this condition and (3.33), we can once again set $\tau \mapsto -\tau$ and use the properties

(3.31) to derive

$$
\begin{aligned}
0 = \mathbb{W}_{+}^{LL'}(-\omega) &= 2\,\mathrm{Im}\left(i\int_0^{\infty} \mathrm{d}\tau\, e^{-i\omega\tau}\langle T\{Q^L(\tau)Q^{L'}(0)\}\rangle\right) \\
&= 2\,\mathrm{Im}\left(i\int_{-\infty}^{0} \mathrm{d}\tau\, e^{i\omega\tau}\langle T\{Q^L(\tau)Q^{L'}(0)\}\rangle\right).
\end{aligned}
\tag{3.35}
$$

Therefore, it is possible to extended the integration range of (3.33) to the entire real ω-axis, thus establishing (3.17). ■

Having proved the fluctuation-dissipation theorem, let us return to the evaluation of the absorption cross section (3.16). First, we specialize the general worldline operator Q^L to the electric and magnetic dipole operators p_a and m_a. Then, owning to the tensor decomposition in (3.29), we define

$$
i\int \mathrm{d}x^0\, e^{i\omega x^0}\langle T\{p_a(0)p_b(x^0)\}\rangle \equiv \delta_{ab}\mathcal{F}(\omega), \tag{3.36}
$$

where $\mathcal{F}(\omega)$ is a function yet to be determined. By definition, notice that this function must be even: $\mathcal{F}(\omega) = \mathcal{F}(-\omega)$. Since Maxwell's equations in a Schwarzschild background are invariant under the duality transformation $E^a \to -B^a$, $B^a \to E^a$, our effective action (3.3) must exhibit this symmetry. This requires, of course, that $p_a \to -m_a$ under the duality operation.

As a result, the two-point functions $\langle T\{p_a(0)p_b(x^0)\}\rangle$ and $\langle T\{m_a(0)m_b(x^0)\}\rangle$ contribute equally to σ_{abs}. Without loss of generality, we focus our attention on the electric dipole operator contribution to (3.16). The inclusion of the magnetic term amounts to an overall factor of 2 in σ_{abs}. With these considerations in mind, we find[4]

$$
\begin{aligned}
\sum_h \sigma_{\text{abs}}(\omega,h) &= 2\times\frac{1}{2\omega}\sum_h\int \mathrm{d}x^0\, e^{i\omega x^0}\omega^2\,\bar{\varepsilon}_a\varepsilon_b\langle p_a(x^0)p_b(0)\rangle \\
&= 2\omega\,\mathrm{Im}\,\mathcal{F}(\omega)\sum_h \delta_{ab}\,\bar{\varepsilon}_a\varepsilon_b.
\end{aligned}
\tag{3.37}
$$

For each polarization state $p = \pm 1$ of the photon, the absorption cross section is

$$
\sigma_{\text{abs},p}(\omega) = 2\omega\,\mathrm{Im}\,\mathcal{F}(\omega). \tag{3.38}
$$

Matching this result to the full theory expression [119], [93]

$$
\sigma_{\text{abs},p}(\omega) = \frac{4\pi}{3}r_s^4\omega^2 + \mathcal{O}(\omega^3), \tag{3.39}
$$

[4]The overall factor of 2 in the first line of (3.37) takes care of the electromagnetic duality discussed above.

one extracts

$$\mathrm{Im}\,\mathcal{F}(\omega) = \frac{2\pi}{3} r_s^4 |\omega|. \tag{3.40}$$

The scaling of p_a with the power-counting parameter can be immediatly determined from this expression. Since $\langle p_a(x^0) p_b(0) \rangle \sim G_N^4 M^4 \omega^2$, the rules of table 1.1 imply

$$p_a \sim \frac{1}{M_{\text{Pl}}} (L v^7)^{1/2}. \tag{3.41}$$

At last, we make a few comments on the meaning of (3.40).

(i) First, notice that the imaginary part of $\mathcal{F}(\omega)$, which describes absorption, has an odd power of ω. The *elastic* scattering of photons, however, can only produce a cross section which is even in ω.

(ii) When expanding the *gravitational* response function in momentum space, the coefficient multiplying $|\omega|$ is called the *first dissipation number*. The coefficient $2\pi r_s^4/3$ we just derived is the electrodynamical analog of such number.

We shall return to a thorough discussion on dissipation numbers – as well as *Love numbers* – once we discuss absorption in a gravitational context. The curious reader is encouraged to read section 3.3.

Application: Absorption of photons by a BH horizon. Having determined $\mathrm{Im}\,\mathcal{F}(\omega)$, it is now possible to recycle this information and investigate more complex phenomena within the scope of our EFT. Working under the usual point-particle approximation, we can compute a number of observables regarding the interaction of a BH horizon and external electrodynamical effects.

As an example, consider the setup of a charged point-particle with mass m and charge e orbiting a non-rotating BH [60]. It is assumed that the orbital radius is much bigger than r_s and the motion of the test-particle is non-relativistic, so that $v^2 \ll 1$. Following the footsteps of NRGR, we define an effective action $S_{\text{eff}}[x_a]$ that depends on the worldlines x_a via the path integral

$$\exp(iS_{\text{eff}}[x_a]) \equiv \int \mathcal{D}h_{\mu\nu} \mathcal{D}A_\mu \mathcal{D}\xi_a \, \exp(iS[x_a, \xi_a, A_\mu, h_{\mu\nu}]), \tag{3.42}$$

where ξ_a denotes the unknown degrees of freedom responsible for dissipation on the BH worldline. These degrees of freedom are formally enconded in operators such as p_a and m_a discussed before.

The action S contains a number of terms. The *bulk* components consist of the Einstein-Hilbert and Maxwell's actions, together with appropriate gauge-fixing terms (we need propagators, after all). In addition, one has point-particles actions for the test-particle and the black hole, which may be supplemented by finite-size operators. At last, S contains interaction terms encoding ξ_a, either in the form of an explicit Lagrangian $L[\xi_a]$ dependent on the horizon microphysics, or parametrized as worldline operators such as p_a, m_a to be later determined by matching to a UV theory.

Let $v^\mu \equiv \frac{\mathrm{d}}{\mathrm{d}\tau}x^\mu$ be the four-velocity of the charged particle. Also, specialize to the BH rest frame so that the vierbein $\mathbf{e}^a{}_\mu$ coincides with a globally flat Cartesian coordinate system. The interaction action (3.3), together with the point-particle actions for the BH and the particle, is given by

$$\begin{aligned} S_{\text{int}} \supset & -m\int \mathrm{d}\tau \left(1+\frac{1}{M_{\text{Pl}}}h_{\mu\nu}v^\mu v^\nu\right)^{1/2} - M\int \mathrm{d}t\left(1+\frac{1}{M_{\text{Pl}}}h_{00}\right)^{1/2} \\ & + e\int \mathrm{d}\tau\, v^\mu A_\mu(x) - \int \mathrm{d}t\, p_a(t)E^a(t) - \int \mathrm{d}t\, m_a(t)B^a(t). \end{aligned} \tag{3.43}$$

The leading-order contribution to S_{eff} is extracted from the following terms:

$$S_0 \equiv -\frac{m}{2M_{\text{Pl}}}\int \mathrm{d}\tau \int \mathrm{d}^3\mathbf{x}\,\delta^{(3)}(\mathbf{x}-\mathbf{x}(\tau))h_{\mu\nu}v^\mu v^\nu - \frac{M}{2M_{\text{Pl}}}\int \mathrm{d}t\int \mathrm{d}^3\mathbf{x}\,\delta^{(3)}(\mathbf{x})h_{00}. \tag{3.44}$$

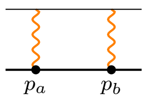
Both interactions terms above are linear in the graviton field, and consequentely, their contribution to the path integral (3.42) is straightforward to evaluate. Recall that the Gaussion partition function of any free field theory has the heuristic form

$$Z_0[J] = \exp\left(\frac{i}{2}\,J\cdot i\Delta_{\text{F}}\cdot J\right), \tag{3.45}$$

where J are auxiliary sources and Δ_{F} is a placeholder for the bulk propagators. To leading-order in v, the leading term of iS_{eff} is thus

$$iS_{\text{eff}} = -\frac{mM}{4M_{\text{Pl}}^2}\int \mathrm{d}\tau\,\mathrm{d}t\, v^\mu(\tau)v^\nu(\tau)\langle T\{h_{\mu\nu}(x(\tau))h_{00}(0,t)\}\rangle \overset{\text{LO}}{=} i\int \mathrm{d}t\,\frac{GMm}{r}, \tag{3.46}$$

which is nothing but the Newtonian potential. The lowest-order term of the effective action that captures dissipative effects of the horizon is



$$= \frac{e^2}{2}\int \mathrm{d}x^\mu\,\mathrm{d}\overline{x}^\nu\,\mathrm{d}t\,\mathrm{d}\overline{t}\,\langle T\{p_a(t)p_b(\overline{t})\}\rangle\langle T\{A_\mu(x)E^a(t)\}\rangle \times \langle T\{A_\nu(\overline{x})E^b(\overline{t})\}\rangle, \tag{3.47}$$

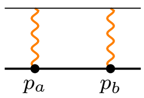
where the top horizontal line of the graph represents the point-charge worldline. The overall $1/2$ is the symmetry factor of the graph. The same diagram with insertions of magnetic operators m_a is suppresed by an additional power of v with respect to (3.47).

We have already determined the correlator $\langle T\{p_a(t)p_b(\overline{t})\}\rangle$ from the matching calculation. Focusing on the lowest-order terms of (3.47), the remaining two-point function we needs to compute is[5]

$$\langle T\{A_0(x^0,\mathbf{x})E_i(0)\}\rangle = -\partial_i\int_p e^{-ip^0x^0}e^{i\mathbf{p}\cdot\mathbf{x}}\frac{-i\eta^{00}}{p^2+i\mathsf{a}} = -\frac{i}{4\pi}\delta(x^0)\frac{x_i}{|\mathbf{x}|^3}. \tag{3.48}$$

[5]Notice that we use the photon propagator in the Feynman gauge [97].

As usual, we have performed an instantaneous approximation of the fully relativistic propagator. The result would be the same as if we had split A_μ into radiative and potential modes and considered the instantaneous propagator of the potential modes in the diagram. Obviously, such splitting is necessary to maintain the manifest power counting in v in the long run. Using (3.36), one easily finds



$$= \frac{ie^2}{32\pi^2}\int \frac{\mathrm{d}\omega}{2\pi}\,|d_i(\omega)|^2\mathcal{F}(\omega) = iS_{\text{eff}}, \tag{3.49}$$

where the dipole vector is defined as $d_i(t) \equiv x_i(t)/|\mathbf{x}(t)|^3$. Notice that this vector is real, so that $d_i^*(-\omega) = d_i(\omega)$. Having determined the relevant term of the effective action, one extracts the emission rate Γ by computing

$$\frac{2}{T}\,\mathrm{Im}\,S_{\text{eff}} \equiv \int \mathrm{d}\omega\,\frac{\mathrm{d}\Gamma}{\mathrm{d}\omega}. \tag{3.50}$$

As before, T denotes the typical timescale of the system and the absorbed power must be understood in a time-averaged sense. Using (3.40), one gets

$$\frac{\mathrm{d}\mathcal{P}_{\text{abs}}}{\mathrm{d}\omega} \equiv \omega\,\frac{\mathrm{d}\Gamma}{\mathrm{d}\omega} = \frac{1}{T}\frac{1}{24\pi^2}\,r_{\mathrm{s}}^4\omega^2|d_i(\omega)|^2. \tag{3.51}$$

Integrating this expression over the range $\omega > 0$, one finally obtains the following time-domain expression:

$$\mathcal{P}_{\text{abs}} = \frac{e^2}{24\pi}\,r_{\mathrm{s}}^4\langle\dot{\mathbf{d}}\cdot\dot{\mathbf{d}}\rangle_T, \tag{3.52}$$

where $\langle\cdots\rangle_T$ denotes a time average. Comparing this result with Larmor's formula, $\mathcal{P} = \frac{2}{3}\langle|\ddot{\mathbf{d}}|^2\rangle_T$, we see that absorption is suppresed by a factor of v^6 with respect to the emission of dipolar radiation in the non-relativistic limit.

3.2 Dissipation in NRGR

Following the example of our electrodynamical detour, we investigate dissipative effects and the absorption of gravitational energy within the framework of NRGR. The first non-trivial terms of the action which encode finite-size effects are the non-minimal couplings

$$S_{\text{fs}} \equiv \frac{\alpha}{2}\int \mathrm{d}\tau\,E^{\mu\nu}E_{\mu\nu} + \frac{\beta}{2}\int \mathrm{d}\tau\,B^{\mu\nu}B_{\mu\nu}, \tag{3.53}$$

where $E_{\mu\nu}$ and $B_{\mu\nu}$ are the electric and magnetic components of the Weyl tensor, respectively:

$$E_{\mu\nu} \equiv C_{\mu\alpha\nu\beta}v^\alpha v^\beta, \tag{3.54a}$$

$$B_{\mu\nu} \equiv \frac{1}{2}\epsilon_{\mu\alpha\beta\rho}C^{\alpha\beta}{}_{\nu\sigma}v^\rho v^\sigma. \tag{3.54b}$$

Since our compact object of interest is a Schwarzschild BH – a vacuum solution of Einstein's field equations – we can write any EFT operator in terms of $E_{\mu\nu}$ and $B_{\mu\nu}$. Notice that α and β – our Wilson coefficients[6] – must scale as r_s^3, since $E_{\mu\nu}$ and $B_{\mu\nu}$ are quadratic derivative operators and the action must be dimensionless.

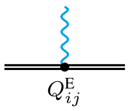
As before, we introduce an action to describe the effective dynamics of the horizon degrees of freedom responsible for dissipation and their coupling to the gravitational field. The allowed terms are chosen based on consistency with the symmetries of our problem, namely, worldline reparametrization invariance, general covariance, and parity invariance. Limiting ourselves to the simplest operators involving two derivatives, we write

$$S_{\text{int}} = -\int \mathrm{d}\tau\, Q^{\mathrm{E}}_{ab} E^{ab} - \int \mathrm{d}\tau\, Q^{\mathrm{B}}_{ab} B^{ab}, \tag{3.55}$$

where $E^{ab} = \mathbf{e}^a{}_\mu \mathbf{e}^b{}_\nu E^{\mu\nu}$ and $B^{ab} = \mathbf{e}^a{}_\mu \mathbf{e}^b{}_\nu B^{\mu\nu}$. We see that the electric and magnetic components of the Weyl tensor couple, respectively, to the electric and magnetic quadrupolar operators Q^{E}_{ab} and Q^{B}_{ab}. The former are responsible for encoding the unknown dynamics of the horizon degrees of freedom.

By matching the predictions of this theory for a given observable to the results of black hole perturbation theory (BHPT), one is able to infer the two-point functions of $Q^{\mathrm{E/B}}_{ab}$. Then, exploiting the universality of such correlators, one proceeds without loss of predictability to compute non-trivial observables which can be probed by the EFT.

The first step of the matching computation is to infer the Feynman rules of (3.55). The procedure is identical to the derivation of (1.181) and (3.9). For example, from the electric term of S_{int} we infer the amplitude



$$= -\frac{i}{2M_{\text{Pl}}}\,\omega^2 Q^{\mathrm{E}}_{ij}(\omega)\,\bar{\varepsilon}^{\,ij}(\mathbf{k},h), \tag{3.56}$$

where $\varepsilon^{ij}(k,h)$ is the STF spin-2 polarization tensor. Similarly, one derives the magnetic amplitude sourced by Q^{B}_{ij}.

The observable we choose to compute in the matching calculation is the (unpolarized) absorption cross section for gravitons in the low-energy limit, i.e., to lowest non-trivial order in $r_s\omega$. Following the same steps that lead to (3.16), we obtain

$$\begin{aligned}\sigma_{\text{abs}}(\omega) = \frac{\omega}{8M_{\text{Pl}}^2}\int \mathrm{d}x^0\, e^{i\omega x^0}\big[&\omega^2\,\bar{\varepsilon}_{ab}\varepsilon_{cd}\langle Q^{\mathrm{E}}_{ab}(x^0)Q^{\mathrm{E}}_{cd}(0)\rangle \\ &+(\mathbf{k}\times\bar{\boldsymbol{\varepsilon}})_{ab}(\mathbf{k}\times\boldsymbol{\varepsilon})_{cd}\langle Q^{\mathrm{B}}_{ab}(x^0)Q^{\mathrm{B}}_{cd}(0)\rangle\big],\end{aligned} \tag{3.57}$$

where $(\mathbf{k}\times\boldsymbol{\varepsilon})_{ab} \equiv \epsilon_{acd}k_c\varepsilon_{db}$. Once again, we shall use the fluctuation-dissipation theorem (3.17) to express the Wightman functions in terms of Feynman two-points functions. For the latter, the spherical symmetry of the Schwarzschild solution dictates the following

[6] The physical meaning of these terms will be explained in the end of this chapter.

tensor structure:

$$i \int \mathrm{d}x^0 \, e^{i\omega x^0} \langle T\{Q^{\mathrm{E}}_{ab}(x^0) Q^{\mathrm{E}}_{cd}(0)\}\rangle \equiv \frac{1}{2}\left(\delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc} - \frac{2}{3}\delta_{ab}\delta_{cd}\right)\mathcal{F}(\omega). \tag{3.58}$$

Similarly to the electromagnetic case, it suffices to *explictly* consider the electric term of $\sigma_{\text{abs}}(\omega)$. This is due to the invariance of Teukolsky equation[7] under the duality operation $E_{ab} \mapsto -B_{ab}$, $B_{ab} \mapsto E_{ab}$, which implies that both even and odd parity correlators contribute equally to the absorption cross section. This property is somewhat non-trivial when compared with the analogous duality of Maxwell's equations. An argument which justifies this symmetry in the regime we are interested here can be found in [103].

Combining (3.17), (3.57), and (3.58), we obtain the following cross section for polarized gravitons (including the contribution of both parities)

$$\sigma_{\text{abs},p}(\omega) = \frac{\omega^3}{2M_{\text{Pl}}^2}\,\text{Im}\,\mathcal{F}(\omega). \tag{3.59}$$

Matching this result to Page's formula [93]

$$\sigma_{\text{abs},p}(\omega) = \frac{4\pi}{45} r_{\text{s}}^6 \omega^4, \tag{3.60}$$

we obtain, for positive ω,

$$\text{Im}\,\mathcal{F}(\omega) = \frac{16}{45} G^5 M^6 |\omega|. \tag{3.61}$$

Having determined $\text{Im}\,\mathcal{F}(\omega)$, we now proceed to tackle two non-trivial problems regarding horizon dissipation.

Application I: Absorption of background gravitational energy by a SBH. Consider the motion of an isolated SBH in a background vacuum spacetime whose typical curvature scale is much larger than r_{s}. In the interaction action (3.55) one must split the quadrupole operator into "background" and "response" contributions, as in (3.4). Since we are interested in a SBH, which exhibits spherical symmetry, the background expectation value of $Q^{\mathrm{E,B}}_{ab}$ is zero.[8] It is thus implied that the quadrupole operators coincide with their "response" contributions from now on.

[7] The Teukolsky equation is the master equation governing linearized perturbations over a Kerr background solution [108]. It plays the same role of Maxwell's equations in our electrodynamical toy model.

[8] If spin is included, one parametrizes the background expectation value of Q^{E}_{ij} as

$$\langle Q^{\mathrm{E}}_{ij}\rangle_{\mathcal{S}} \equiv \frac{C_{\text{ES}^2}}{2M} S^{ik} S_k{}^j,$$

where S^{ij} is the spin tensor and C_{ES^2} is a Wilson coefficient determined by matching. For example, $C_{\text{ES}^2} = 1$ for an isolated rotating BH [104].

From (3.4), the leading term of the effective action (3.42) is

$$iS_{\text{eff}} \supset -\frac{1}{2}\int \mathrm{d}\tau\,\mathrm{d}\overline{\tau}\,\langle T\{Q^{\text{E}}_{ab}(\tau)Q^{\text{E}}_{cd}(\overline{\tau})\}\rangle\big[E^{ab}(\tau)E^{cd}(\overline{\tau}) + B^{ab}(\tau)B^{cd}(\overline{\tau})\big]. \tag{3.62}$$

At this order we are safe to ignore gravitational fluctuations of the sources E_{ab}, B_{ab}, so they are taken as their "background" values. The power absorbed by the BH will be extracted from the imaginary part of (3.62), and by construction, this should match the change in mass of the BH as measured by a local observer.

Obviously, this change in mass will impact the motion of the BH itself. However, such secondary effects shall not concern us, at least not at this order in perturbation theory. We focus on simply determining the rate of absorbed energy to first non-trivial PN order. With the help of (3.61) and some Fourier transforms, we find

$$S_{\text{eff}} = \frac{1}{2}\int_{-\infty}^{\infty}\frac{\mathrm{d}\omega}{2\pi}\,\mathcal{F}(\omega)E_{ab}(\omega)E^{ab}(-\omega), \tag{3.63}$$

and thus

$$\operatorname{Im} S_{\text{eff}} = \frac{16}{45}G^5M^6\int_0^{\infty}\frac{\mathrm{d}\omega}{2\pi}\,\omega\big[E_{ab}(\omega)E^{ab}(-\omega) + B_{ab}(\omega)B^{ab}(-\omega)\big]. \tag{3.64}$$

The extraction of the absorbed power $\mathcal{P}_{\text{abs}}$ is straightforward. We have

$$\begin{aligned}\mathcal{P}_{\text{abs}} = \int_0^{\infty}\mathrm{d}\omega\,\omega\,\frac{\mathrm{d}\Gamma}{\mathrm{d}\omega} &= \frac{1}{T}\frac{16}{45\pi}G^5M^6\int_0^{\infty}\mathrm{d}\omega\,\omega^2\left[E_{ab}(\omega)E^{ab}(-\omega) + (E\leftrightarrow B)\right]\\ &= \frac{1}{T}\frac{16}{45}G^5M^6\int\mathrm{d}t\,\big[\dot{E}_{ab}(t)\dot{E}^{ab}(t) + (E\leftrightarrow B)\big],\end{aligned} \tag{3.65}$$

where $\dot{E}_{ab} \equiv \frac{\mathrm{d}}{\mathrm{d}t}E_{ab}$. The result is

$$\mathcal{P}_{\text{abs}} = \frac{16}{45}G^5M^6\big\langle\dot{E}_{ab}\dot{E}^{ab} + \dot{B}_{ab}\dot{B}^{ab}\big\rangle_T. \tag{3.66}$$

Notice that, since we are dealing with non-spinning BHs, we have $\dot{E}^{ab} = \mathbf{e}^a{}_\mu\mathbf{e}^b{}_\nu v^\alpha\nabla_\alpha E^{\mu\nu}$, and similarly for the $\dot{B}^{ab}$ tensor. Thus, the expression given above for the observable $\mathcal{P}_{\text{abs}}$ is perfectly covariant, as it should be.

Application II: Binding energy loss due to absorption in an inspiraling binary. At last, consider the description of gravitational energy absorption by a binary system. It is assumed that the bound motion is non-relativistic and the constituents are slowly inspiralling as they transverse the orbit. As before, we work in the point-particle limit.

To organize the relevant interaction terms to lowest non-trivial PN order, we must discuss the scaling of various quantities. First, we need the scaling of Q_{E}^{ab}. The argument is identical to the one leading to (3.41). Since $\omega \sim v/r$ and $\langle Q_{\mathrm{E}}^{ab} Q_{\mathrm{E}}^{cd} \rangle \sim G^5 M^6 \omega^2$, we find

$$Q_{\mathrm{E}}^{ab} \sim \frac{1}{M_{\mathrm{Pl}}} L v^4. \tag{3.67}$$

Therefore, by considering only the leading potential gravitons in $E_{ij} = -\frac{1}{2M_{\mathrm{Pl}}} \partial_i \partial_j H_{00}$, we find that the electric term of the action (3.55) scales as

$$\int \mathrm{d}\tau\, Q^{\mathrm{E}}_{ab} E^{ab} \sim v^{13/2}. \tag{3.68}$$

In the derivation of this result we have used $GM/r \sim v^2$, which is nothing but the virial theorem for a gravitationally bound system. A similar computation reveals that the magnetic term of (3.55) also scales as $v^{13/2}$. What happens if we consider radiative modes in E_{ij}? A straightforward application of the power counting rules shows that this leads to an interaction term suppressed by $v^{5/2}$ with respect to (3.68).

Therefore, to leading non-trivial PN order it suffices to consider the coupling to potential modes. What diagrams shall we compute, then? The constituents of the binary system are assumed to be non-rotating BHs. Therefore, after separating the *background* and *response* contributions to E^{E}_{ab} we must take

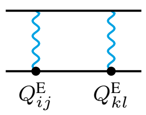
$$\langle Q^{\mathrm{E}}_{ab} \rangle_{\mathcal{S}} = 0, \tag{3.69}$$

where, as in (3.4), the angled brackets with the $\mathcal{S}$ label denote the background, "source," expectation value (our Schwarzschild BHs). We are of course interested in the *response* contribution to the quadrupole, since this is the term that yields horizon dissipation in our setup. It must be mentioned that spinning compact objects admit a non-zero source quadrupole. We briefly discuss this matter in section 3.3.

Having ruled out diagrams proportional to quadrupolar one-point functions with a little help from spherical symmetry, we now consider graphs depending on the two-point functions. In this case there are further constraints set by the parity invariance of the effective action. For instance, a "box" diagram with one insertion of both Q^{E}_{ab} and Q^{B}_{cd} must necessarily be zero since it violates such parity invariance.

We are thus led to consider box diagrams with two insertions of either the eletric or magnetic quadrupole. To leading-order, B_{ij} is linear in the $0i$ polarizations of $H_{\mu\nu}$, which in turn, *do not couple* to leading-order to the 00 polarization; one must couple them to terms of the form $v_i H_{0i}$, see (1.64), for example. This naturally introduces additional powers of v and the "magnetic box" diagram only contributes at a higher PN order.

The leading term $E_{ij} = -\frac{1}{2M_{\mathrm{Pl}}} \partial_i \partial_j H_{00}$ however, does couple to the 00 polarizations. As a result, two insertions of the simple (1.60) vertex are required in the "electric box" diagram. We write



$$= \frac{m_2^2}{8M_{\mathrm{Pl}}^2} \int \mathrm{d}t_1\, \mathrm{d}\bar{t}_1\, \mathrm{d}t_2\, \mathrm{d}\bar{t}_2\, \langle T\{Q^{\mathrm{E}}_{ij}(t_1) Q^{\mathrm{E}}_{kl}(\bar{t}_1)\}\rangle$$
$$\times \langle T\{H_{00}(t_2) E_{ij}(t_1)\}\rangle \langle T\{H_{00}(\bar{t}_2) E_{kl}(\bar{t}_1)\}\rangle + (1 \leftrightarrow 2). \tag{3.70}$$

To simplify this expression we have suppresed the spatial dependence of the H_{00} gravitons and the E_{ij} tensors. The $1/8$ above is a result of the $(1/2)^2$ factor coming from the two insertions of (1.60) and the $1/2$ symmetry factor of the diagram. Before proceeding we compute the following two-point function in the instantaneous approximation:

$$\begin{aligned}\langle T\{H_{00}(x^0,\mathbf{x})E_{ij}(0)\}\rangle &= -\frac{1}{2M_{\rm Pl}}\partial_i\partial_j\int\frac{{\rm d}^4p}{(2\pi)^4}\,e^{-ip^0x^0+i\mathbf{p}\cdot\mathbf{x}}\left(-\frac{i}{\mathbf{p}^2}\right)\mathbb{P}_{0000}\\ &= \frac{i}{16\pi M_{\rm Pl}}\delta(x^0)\partial_i\partial_j\frac{1}{|\mathbf{x}|}.\end{aligned}\tag{3.71}$$

To simplify the remaining formulae we define

$$q_{ij}^{(a)}(t)\equiv\partial_i^a\partial_j^a\frac{1}{|\mathbf{x}_{12}(t)|},\qquad a=1,2,\tag{3.72}$$

where $\mathbf{x}_{12}\equiv\mathbf{x}_1-\mathbf{x}_2$. Using (3.58), a routine calculation yields

$$\begin{aligned}{\rm Im}\,S_{\rm eff}^{\text{E-box}} &= \frac{G^2}{2}\int\frac{{\rm d}\omega}{2\pi}\,m_2^2\,|q_{ij}^{(1)}(\omega)|^2\,{\rm Im}\,\mathcal{F}^{(1)}(\omega)+(1\leftrightarrow 2)\\ &= \frac{G^2}{2}\sum_{a\neq b}\int\frac{{\rm d}\omega}{2\pi}\,m_b^2\,|q_{ij}^{(a)}(\omega)|^2\,{\rm Im}\,\mathcal{F}^{(a)}(\omega).\end{aligned}\tag{3.73}$$

Substituting (3.61), we can easily extract the graviton emission rate:

$$\frac{2}{T}\,{\rm Im}\,S_{\rm eff}^{\text{E-box}}=\int_0^\infty{\rm d}\omega\,\frac{{\rm d}\Gamma}{{\rm d}\omega}=\frac{1}{T}\int_0^\infty{\rm d}\omega\left(\frac{32}{45}\sum_{a\neq b}G^7m_a^6m_b^2\,\frac{\omega}{2\pi}\left|q_{ij}^{(a)}(\omega)\right|^2\right).\tag{3.74}$$

The flux of gravitons which is absorbed by the horizon is thus

$$\mathcal{P}_{\rm abs}=\int_0^\infty{\rm d}\omega\,\omega\,\frac{{\rm d}\Gamma}{{\rm d}\omega}=\frac{32}{45}\sum_{a\neq b}G^7m_a^6m_b^2\left\langle\frac{1}{2}\,\dot{q}_{ij}^{(a)}\,\dot{q}_{ij}^{(a)}\right\rangle_T.\tag{3.75}$$

A more illuminating expression is obtained by recasting this formula in terms of the orbital variables. Since $\dot{q}_{ij}^{(1)}\dot{q}_{ij}^{(1)}$ is the same function as $\dot{q}_{ij}^{(2)}\dot{q}_{ij}^{(2)}$, one can safely drop the (1) and (2) labels. In what follows, we denote $r=|\mathbf{x}|=|\mathbf{x}_1-\mathbf{x}_2|$ and $\mathbf{v}=\frac{{\rm d}\mathbf{x}}{{\rm d}t}$. A straightforward calculation gives

$$\frac{1}{2}\,\dot{q}_{ij}\dot{q}_{ij}=9\left[\frac{v^2}{r^8}+2\frac{(\mathbf{x}\cdot\mathbf{v})^2}{r^{10}}\right].\tag{3.76}$$

Therefore, (3.75) becomes

$$\mathcal{P}_{\rm abs}=\frac{32}{5}G^7(m_1^6m_2^2+m_2^6m_1^2)\left[\frac{v^2}{r^8}+2\frac{(\mathbf{x}\cdot\mathbf{v})^2}{r^{10}}\right].\tag{3.77}$$

In the small $\eta=\mu/M$ regime, comparison with Einstein's quadrupole formula applied to a circular binary,

$$\mathcal{P}_{\text{quad}} = \frac{32}{5G}\eta^2 v^{10}, \quad \eta = \frac{\mu}{M} = \frac{m_1 m_2}{(m_1+m_2)^2}, \tag{3.78}$$

gives[9]

$$\frac{\mathcal{P}_{\text{abs}}}{\mathcal{P}_{\text{quad}}} = v^8, \quad \eta \ll 1. \tag{3.79}$$

This result agrees with the seminal black hole perturbation theory calculations of Poisson and Sasaki [99]. As the authors point out in their paper, black hole absorption is indeed an extremely small effect.

The computations presented in this section within the framework of NRGR were promptly extended to spinning black holes by Porto [103] after the publication of [60].

3.3 Love and dissipations numbers

The purpose of this section is twofold. First, we shall elaborate on the construction of the finite-size terms in (3.1) and (3.53). We will do so by reconsidering the construction of the effective action through the lenses of linear response theory. This will naturally lead us to our second goal: interpret the Wilson coefficients as tidal Love numbers, which describe how a massive self-gravitating object deforms under the interaction of long-wavelength perturbations.

As we discussed, one has to introduce gapless degrees of freedom on the worldline to describe absorption effects. Their influence on an isolated self-gravitating compact object was parametrized in the form of multipole moments $Q_L(X)$ which naturally split into “background" and “response" terms. For spin-s long-wavelength probe fields (the perturbing fields) and general multipolar order ℓ, one writes the following interaction action [68]

$$S^{(s)}_{\text{int}} = -\sum_{\ell} \int \mathrm{d}\tau\, Q^{(s)}_L(X,\tau) E^{(s)L}(x(\tau)), \tag{3.80}$$

where $E^{(s)_L}$ is the natural generalization of (3.2a) and (3.54a). Nothing new so far. Now, we parametrize the induced multipole moments by a *linear response theory* (LRT) Ansatz:

$$\langle Q_L(\tau)\rangle_{\text{in-out}} = \int \mathrm{d}\tau'\, iG^{LL'}_{\text{f}}(\tau-\tau') E_{L'}(x(\tau)), \tag{3.81}$$

where $G^{LL'}_{\text{f}}$ is the Feynman two-point function. The *in-in* expectation value, as we have seen in section 2, is rather given by convolving the retarded Green function. By plugging

[9]When specializing (3.77) to the $\eta \ll 1$ limit, it suffices to consider the v^2/r^8 term to recover the result of Poisson and Sasaki.

(3.81) back into the *in-out* action (3.80), we find[10]

$$(S_{\text{int}})\Big|_{\text{in-out}} = \frac{i}{2}\int \mathrm{d}\tau\,\mathrm{d}\tau'\,\langle T\{Q_L(\tau)Q_{L'}(\tau')\}\rangle E^L(\tau)E^{L'}(\tau'). \tag{3.82}$$

The matching procedure used in our EFT so far was judiciously chosen to determine $G_{\text{f}}^{LL'}$ in a small $(r_s\omega)$ expansion, which is usually written as[11]

$$\begin{aligned} i\int \mathrm{d}x^0\, e^{i\omega x^0}\langle T\{Q_{L_1}(x^0)Q_{L_2}(0)\}\rangle &= \delta_{L_1L_2}\mathcal{F}_\ell(\omega) \\ &= \delta_{L_1L_2}\big(C_\ell + iC_{\ell,\omega}|\omega| + C_{\ell,\omega^2}\omega^2 + \cdots\big), \end{aligned} \tag{3.83}$$

where the constant C are free parameters. They are, essentially, the Wilson coefficients appearing in (3.53). The connection will soon be made exact in an explicit example. Our calculation in the previous sections show that, indeed, dissipative effects are captured by odd terms in the frequency-space expansion.

Let us make some general remarks on such terms. Given the $iC_{\ell,\omega}|\omega|$ term in (3.83), we infer that the function $G_{\text{f}}^{LL'}$ is not analytic around $\omega = 0$. In contrast, the retarded Green function is analytic around $\omega = 0$, as can be seen from its dispersive representation (3.23). In the case of a purely absorptive Schwarzschild BH, all odd terms of $G_{\text{ret}}^{LL'}(\omega)$ are fixed in terms of the coefficients in (3.83). This follows from

$$\operatorname{Im} G_{\text{ret}}^{LL'}(\omega) = \operatorname{Re} G_{\text{f}}^{LL'}(\omega), \tag{3.84}$$

which is a consequence of (3.32) and the condition (3.34) applied to the representation (3.28b). Similarly, all even coefficients of the retarded function are also uniquely fixed in terms of the coefficients in (3.83); this follows from the equality between retarded and Feynman propagators for off-shell modes, i.e., conservative effects [104] [68]. All this leads to

$$G_{\text{ret}}^{LL'}(\omega) = \delta^{LL'}\big(C_\ell + iC_{\ell,\omega}\omega + C_{\ell,\omega^2}\omega^2 + \cdots\big), \tag{3.85}$$

where the response function $G_{\text{ret}}^{LL'}(\omega)$ is related to the *in-in* expectation value by

$$\langle Q_L(\tau)\rangle_{\text{in-in}} = \int \mathrm{d}\tau'\, G_{\text{ret}}^{LL'}(\tau-\tau')E_{L'}(x(\tau)). \tag{3.86}$$

The even terms (the *tidal Love numbers*) in (3.85) can be absorbed into local counterterms. They simply renormalize the values of the Wilson coefficients in (3.53). The odd terms (the *dissipation numbers*) cannot be absorbed into a local worldline action.

[10] This procedure has the same effect as "integrating out" the unknown short-distance degrees of freedom ξ from the path integral (3.42).

[11] Assuming a Schwarzschild BH to simplify the tensor structure.

Example. To make the connection between the C coefficients and the Wilson coefficients more precise, consider the simple case of a state $\mathcal{O}(r_s\omega)$ response. In this case, we have

$$\langle T\{Q_L(\tau)Q_{L'}(\tau')\}\rangle = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, e^{-i\omega(\tau-\tau')} G_f^{LL'}(\omega) = -iC_\ell \delta(\tau-\tau')\delta^{LL'}. \tag{3.87}$$

Plugging this back into the (3.82), we find

$$(S_{\text{int}})\Big|_{\text{in-out}} = \frac{1}{2} C_\ell \int d\tau\, E_L(\tau) E^L(\tau) \equiv \frac{\lambda_\ell}{2\ell!} \int d\tau\, E_L(\tau) E^L(\tau), \tag{3.88}$$

where the coefficients λ_ℓ define the *static Love numbers*, whose dynamical generalizations are straightforward to define in the EFT setting [96]. It is the quantity λ_ℓ that reduces to the Newtonian tidal Love numbers in the appropriate limit [34].

Even though the Wilson coefficients are universal, it must be emphasized that the matching computation is not unique. For example, one could have obtained the coefficients α and β in (3.53) by perturbatively computing the one-point function of the metric far outside an isolated self-gravitating body. The matching of this result is obtained by comparison with the metric perturbation computed in black-hole perturbation theory. An illustration of such calculation can be found in references [68] [65] [34].

Chapter 4

Dispersion Relations and Horizon Dynamics

4.1 Overview

It is a well-known fact that four-dimensional BHs in GR have zero *static*[1] tidal Love numbers [41, 20, 76, 62]. These compact objects, however, can absorb external perturbations, as investigated by Page in a seminal paper [94]. From the effective field theory point of view, the vanishing of Love numbers is puzzling because it implies a fine-tuning of the Wilson coefficients in the effective action [105]. Recently, this fascinanting property has been shown to stem from a symmetry enhancement dubbed "Love symmetry." See, for example, the papers [65, 66, 35].

Tidal deformations and absorption effects due to linear responses to external fields can be related to one another via dispersion relations. For example, in the worldline EFT setup, the BH dynamics is effectively encoded in a point particle action supplemented by higher derivative operators, whose Wilson coefficients, or rather, their real and imaginary parts, are mapped to tidal deformability and absorption parameters, respectively. This procedure requires the use of the well-known Schwinger-Keldysh (or *in-in*) formalism [118, 73, 61, 71].

Given this framework, it is tantalizing to try to relate the real and imaginary parts of the BH response to external fields with dispersion relations à la Kramers-Kronig. Naively, however, the vanishing of the static tidal deformation *and* the presence of absorption are incompatible with dispersion relations – or more broadly – with the fluctuation-dissipation theorem, which, loosely speaking, forbids the existence of fluctuations without dissipation and vice-versa.

In this chapter we explain how this issue arises and we frame it in light of the current understanding of the BH linear response function (also called the *gravitational polarizability*), complementing previous EFT-based results [59, 60, 61].

Chapter structure. First, in section 4.2 we review the derivation of the Kramers-Kronig relations within the scope of linear response theory. Then, in the beginning of section 4.3 we introduce the EFT setup and the standard Regge-Wheeler approach to gravitational scattering. This will pave the way for the EFT computation of the BH polarizability, which is performed by matching the numerical solution of the Regge-Wheeler equation to an EFT computation as proposed in [60].[2] The detail of the EFT calculation are presented in

[1] See [72] for a derivation of the vanishing of non-linear static Love numbers.

[2] See also [71], where the *absorptive* cross section is related to the *quadrupole* two-point function.

section 4.3.2. At last, in section 4.4 we interpret our results in terms of dispersion relations for the BH polarizability and propose a sum rule.

4.2 Kramers-Kronig relations

We now review how the dispersion relations of Kramers and Kronig arise from linear response theory and certain causality requirements. By "dispersion relation" we mean an integral relation between two observable quantities where the integration range is restricted to values of the argument which are physically meaningful. In our setup, this means positive frequencies ($\omega > 0$) in Fourier space.

After the standard theory is covered, we turn to applications in a worldline EFT inspired by NRGR. This section is largely based on the references [30, 90, 88].

The Hilbert transform pair. Let $f(z)$ be a complex function which is analytic in the upper half-plane (UHP) and satisfies $|f(z)| \to 0$ as $|z| \to \infty$ in this region. Let $\alpha \in \mathbb{R}$ and consider the integral

$$I \equiv \oint_C \frac{f(z)}{z-\alpha}\,\mathrm{d}z \tag{4.1}$$

along the contour C depicted in the figure 4.1 below. Since the integrand of I is analytic

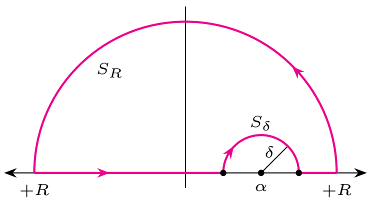


Figure 4.1: The contour C used in the derivation of the Hilbert transform pair. The large semicircle with radius R is denoted S_R whereas the small semicircle with radius δ is denoted S_δ.

within and on the contour, Cauchy's theorem says that

$$0 = \mathsf{PV}\int_{-R}^{R} \frac{f(x)}{x-\alpha}\,\mathrm{d}x + \int_{S_\delta} \frac{f(z)}{z-\alpha}\,\mathrm{d}z + \int_{S_R} \frac{f(z)}{z-\alpha}\,\mathrm{d}z\,. \tag{4.2}$$

The principal value PV appears because δ is taken to be infinitesimally small. For the integral along the big semicircle C_R we set $z = Re^{i\theta}$ and write

$$\int_{S_R} \frac{f(z)}{z-\alpha}\,\mathrm{d}z = i\int_0^{\pi} \frac{f(Re^{i\theta})}{Re^{i\theta}-\alpha} Re^{i\theta}\,\mathrm{d}\theta\,. \tag{4.3}$$

Then, using the inequality

$$|Re^{i\theta} - \alpha| = (R^2 + \alpha^2 - 2R\alpha\cos\theta)^{1/2} \geq R - \alpha, \tag{4.4}$$

we find

$$\left|\int_{S_R} \frac{f(z)}{z-\alpha}\,\mathrm{d}z\right| \leq \frac{R}{R-\alpha}\int_0^\pi |f(Re^{i\theta})|\,\mathrm{d}\theta\,. \tag{4.5}$$

As a result, for R suitably big, the ratio $R/(R-\alpha)$ tends to 1 and the remaining integral becomes arbitrarily small. Therefore,

$$\begin{aligned}\lim_{R\to\infty} \mathsf{PV}\int_{-R}^{R} \frac{f(x)}{x-\alpha}\,\mathrm{d}x &= -\int_{S_\delta}\frac{f(z)}{z-\alpha}\,\mathrm{d}z \\ &= -f(\alpha)\int_{S_\delta}\frac{\mathrm{d}z}{z-\alpha} - \int_{S_\delta}\frac{[f(z)-f(\alpha)]}{z-\alpha}\,\mathrm{d}z\,.\end{aligned} \tag{4.6}$$

Now consider the second integral in the last line above. Since $|z-\alpha| = \delta$ on the infinitesimally small semicircle C_δ, the continuity of $f(z)$ at $z = \alpha$ guarantees that $|f(z) - f(\alpha)| < \epsilon$, for all $\epsilon > 0$. This term can thus be made arbitrarily small:

$$\left|\oint_{S_\delta} \frac{f(z)-f(\alpha)}{z-\alpha}\,\mathrm{d}z\right| \leq \oint_{C_\delta}\frac{|f(z)-f(\alpha)|}{|z-\alpha|}|\mathrm{d}z| < \frac{\epsilon}{\delta}(2\pi\delta) = 2\pi\epsilon. \tag{4.7}$$

On the other hand, setting $z-\alpha = \rho e^{i\phi}$ in the first term yields

$$f(\alpha)\int_{S_\delta}\frac{\mathrm{d}z}{z-\alpha} = if(\alpha)\int_0^\pi \mathrm{d}\phi = -i\pi f(\alpha). \tag{4.8}$$

Collecting our intermediate results, the formula (4.6) becomes

$$\mathsf{PV}\int_{-\infty}^{\infty}\frac{f(x)}{x-\alpha}\,\mathrm{d}x = i\pi f(\alpha). \tag{4.9}$$

Writing $f(x) = \operatorname{Re} f(x) + i\operatorname{Im} f(x)$, we finally obtain

$$\operatorname{Re} f(\alpha) = \frac{1}{\pi}\,\mathsf{PV}\int_{-\infty}^{\infty}\frac{\operatorname{Im} f(x)}{x-\alpha}\,\mathrm{d}x\,, \tag{4.10a}$$

$$\operatorname{Im} f(\alpha) = -\frac{1}{\pi}\,\mathsf{PV}\int_{-\infty}^{\infty}\frac{\operatorname{Re} f(x)}{x-\alpha}\,\mathrm{d}x\,. \tag{4.10b}$$

Any pair of functions (such as $\operatorname{Re} f(x)$ and $\operatorname{Im} f(x)$ shown above) satisfying (4.10) is called a *Hilbert transform pair* [30].

Linear response theory. Consider an input $I(t)$ which generates a response $R(t)$ such that

$$R(t) = \int_{-\infty}^{\infty} G(t,t')I(t')\,\mathrm{d}t'\,, \tag{4.11}$$

i.e., the response function is a time-domain convolution of the input with a certain kernel $G(t)$. Let us now discuss the mathematical properties of a physically well-motivated class of kernels.

First, if we shift the input by an amount τ, we expect the response to be also shifted by τ. To make the discussion concrete, take as an input the sharply-peaked function $I(t') = I_0\delta(t' - t_0)$. Now consider

$$I_1(t') = I_0\delta(t' - t_0), \tag{4.12a}$$
$$I_2(t') = I_0\delta(t' - t_0 - \tau). \tag{4.12b}$$

Plugging these functions back into the integral (4.11), we find $R_1(t) = I_0 G(t, t_0)$ and also $R_2(t) = I_0 G(t, t_0 + \tau)$. Then, the equality $R_2(t + \tau) = R_1(t)$ can only be satisfied if the kernel is a function of $t - t'$. What more can we say about $G(\tau)$ on physical grounds?

(i) **Causality**: "*An input at t should not give rise to a response at times prior to t.*" Therefore, the response only depends on the past history of the input:

$$G(\tau) = 0, \quad \tau < 0. \tag{4.13}$$

This is sometimes called the "primitive causality" requirement [90].

(ii) **Finiteness**: "*$G(\tau)$ cannot be singular for any finite τ when considering a sharp input.*" This means that $G(t-t_0)$ must be finite for all positive values of its argument:

$$R_{\text{sharp}}(t) = I_0 G(t - t_0) < \infty, \quad \forall\, t > t_0. \tag{4.14}$$

(iii) **Bad Memory**: "*The remote past does not appreciably influence the present.*" In other words, real physical systems have dissipation mechanisms which eventually "dilute" the response as time passes:

$$\lim_{\tau\to\infty} G(\tau) = 0. \tag{4.15}$$

Notice that the definition of the response function as a convolution integral suggests that computations will be easier in Fourier space:

$$R(t) = (G * I)(t) \equiv \int_{-\infty}^{\infty} G(t - t')I(t')\,\mathrm{d}t'\,. \tag{4.16}$$

The convolution theorem then says

$$\tilde{R}(\omega) = \tilde{G}(\omega)\tilde{I}(\omega). \tag{4.17}$$

Before enunciating the precise conditions to be imposed on $G(\tau)$, let us explore the consequences of the condition

$$\int_0^{\infty} |G(\tau)|\,\mathrm{d}\tau < \infty. \tag{4.18}$$

If the equation above holds, then a bounded Fourier transform $\tilde{G}(\omega)$ exists for all ω. Then, the physically motivated assumptions (1), (2), and (3) can be summarized as

(i) $G(\tau) = 0$ *for* $\tau < 0$;

(ii) $G(\tau)$ *is bounded for all* τ;

(iii) $|G(\tau)|$ *is integrable, so* $G \to 0$ *faster than* $1/\tau$ *as* $\tau \to \infty$.

Notice that (ii) and (iii) together imply that $G(\tau)$ *is square integrable* and hence $\tilde{G}(\omega)$ is square integrable as well. Our goal now is to extend the function $\tilde{G}(\omega)$ into the complex z-plane in such a way that it satisfies the conditions under which the Hilbert transform holds. The condition (i) above implies

$$\tilde{G}(\omega) = \int_0^\infty G(t)e^{i\omega t}\,\mathrm{d}t = \int_{-\infty}^\infty \theta(t)G(t)e^{i\omega t}\,\mathrm{d}t\,. \tag{4.19}$$

Notice that we are only integrating over half the real line now. We extend this relation to the complex plane by defining z as

$$z = \mathrm{Re}\,z + i\,\mathrm{Im}\,z = \omega + i\omega', \tag{4.20}$$

so that

$$\tilde{G}(z) \equiv \tilde{G}(\omega, \omega') = \int_0^\infty G(t)e^{i(\omega-\omega')t}\,\mathrm{d}t\,. \tag{4.21}$$

Causality requires $t > 0$, so we restrict ourselved to the upper half-plane ($\omega' > 0$) to ensure that $e^{-\omega' t}$ is a decaying exponential. This guarantees uniform convergence. Now define $M_G = \max_{t>0}|G(t)|$. The condition (ii) implies that, for $\theta \in (0, \pi)$, the following estimate holds:

$$|\tilde{G}(\omega)| \leq M_G \int_0^\infty |e^{it(\rho\cos\theta + i\rho\sin\theta)}|\,\mathrm{d}t\,, \tag{4.22}$$

where ρ is the absolute value of z. It follows that

$$|\tilde{G}(\omega)| \leq M_G \int_0^\infty e^{-|z|\sin\theta t}\,\mathrm{d}t = \frac{M_G}{|z|\sin\theta}. \tag{4.23}$$

As a result, we find

$$\lim_{|z|\to\infty} |\tilde{G}(z)| = 0, \quad \theta \in (0, \pi). \tag{4.24}$$

For $\theta = 0$ or $\theta = \pi$, the function $\tilde{G}(\omega)$ of equation (4.19) is recovered:

$$\tilde{G}(\omega) = \tilde{G}(\omega, 0) = \int_0^\infty G(t)e^{i\omega t}\,\mathrm{d}t\,. \tag{4.25}$$

As argued before, since $G(t)$ is square integrable, so is its Fourier transform $\tilde{G}(\omega, 0)$. Thus,

$$\lim_{\omega\to\infty} |\tilde{G}(\omega, 0)| = 0. \tag{4.26}$$

Strictly speaking, the limit above only holds if $\tilde{G}(\omega, 0)$ belongs to the Sobolev space (see Lemma 2.30 of [121]), but since this space is dense in $L^2(\mathbb{R})$ and we're not interested in mathematical subtleties, the conclusion remains the same.

From these results we see that $|\tilde{G}(z)| \to 0$ as $|z| \to \infty$ in any direction in the UHP. Let us now argue that $\tilde{G}(z)$ is *analytic* in this region. If $\omega' > 0$, then

$$\frac{\mathrm{d}^n}{\mathrm{d}z^n}\tilde{G}(z) = \int_0^\infty G(t)\,\frac{\mathrm{d}^n}{\mathrm{d}z^n}\,e^{izt}\,\mathrm{d}t = i^n \int_0^\infty t^n G(t)\,e^{i\omega t}e^{-\omega' t}\,\mathrm{d}t\,. \tag{4.27}$$

Once again, the decaying exponential $e^{-\omega' t}$ guarantees uniform convergence. It is clear that the argument does not work if $\omega' = 0$. How can we be sure that $\tilde{G}(z)$ is analytic on the real axis? Let us make an extra assumption about $\tilde{G}(z)$ and require this function to be *bounded* on the real axis. So the only singularities that may appear when $\omega' = 0$ are of the *branch point variety*. In particular, notice that the branch singularities of the type $1/\sqrt{z}$ or $\log z$ are excluded by the boundedness requirement.

The derivation of the Hilbert transform pair can be modified to account for these branch point singularities on the real axis. It suffices to take a small semicircular detour around each of them [30]. As a result, the equations (4.10a) and (4.10b) remain unaltered (recall that branches can always be chosen to avoid the upper half-plane). To eliminate the existence of such branch singularities altogether one must assume a exponential type fallout of $G(\tau)$ as $\tau \to \infty$.

With all the previous considerations in mind, we can write the Hilbert transform pair for the frequency-space kernel:

$$\mathrm{Re}\,\tilde{G}(\omega) = \frac{1}{\pi}\,\mathrm{PV}\int_{-\infty}^{\infty} \frac{\mathrm{Im}\,\tilde{G}(\bar{\omega})}{\bar{\omega}-\omega}\,\mathrm{d}\bar{\omega}\,, \tag{4.28a}$$

$$\mathrm{Im}\,\tilde{G}(\omega) = -\frac{1}{\pi}\,\mathrm{PV}\int_{-\infty}^{\infty} \frac{\mathrm{Re}\,\tilde{G}(\bar{\omega})}{\bar{\omega}-\omega}\,\mathrm{d}\bar{\omega}\,. \tag{4.28b}$$

We thus see that the key assumption of causality (together with reasonable mathematical assumptions) allowed us to show that the real and imaginary parts of $\tilde{G}(\omega)$ are related to each other for real values of the argument. To put the Hilbert transform pair relations into the *dispersive representation* we exploit the fact that $G(t)$ is real, so its Fourier transform satisfies $\tilde{G}^*(\omega) = \tilde{G}(-\omega)$. In terms of the real and imaginary pieces of $\tilde{G}(\omega)$, this reality condition implies

$$\mathrm{Re}\,\tilde{G}(\omega) - i\,\mathrm{Im}\,\tilde{G}(\omega) = \mathrm{Re}\,\tilde{G}(-\omega) + i\,\mathrm{Im}\,\tilde{G}(-\omega). \tag{4.29}$$

Therefore,

$$\mathrm{Re}\,\tilde{G}(\omega) = \mathrm{Re}\,\tilde{G}(-\omega), \tag{4.30a}$$

$$\mathrm{Im}\,\tilde{G}(\omega) = -\,\mathrm{Im}\,\tilde{G}(-\omega), \tag{4.30b}$$

meaning that $\mathrm{Re}\,\tilde{G}$ and $\mathrm{Im}\,\tilde{G}$ are even and odd functions of ω, respectively. We now return to (4.10a) and split the integration range into two intervals:

$$\operatorname{Re}\tilde{G}(\omega) = \frac{1}{\pi}\,\mathsf{PV}\int_{-\infty}^{0} \frac{\operatorname{Im}\tilde{G}(\bar{\omega})}{\bar{\omega}-\omega}\,\mathrm{d}\bar{\omega} + \frac{1}{\pi}\,\mathsf{PV}\int_{0}^{\infty} \frac{\operatorname{Im}\tilde{G}(\bar{\omega})}{\bar{\omega}-\omega}\,\mathrm{d}\bar{\omega}\,. \tag{4.31}$$

Let $\bar{\omega} \mapsto -\bar{\omega}$ in the first integral:

$$\operatorname{Re}\tilde{G}(\omega) = \frac{1}{\pi}\,\mathsf{PV}\int_{\infty}^{0} \frac{\operatorname{Im}\tilde{G}(-\bar{\omega})}{\bar{\omega}+\omega}\,\mathrm{d}\bar{\omega} + \frac{1}{\pi}\,\mathsf{PV}\int_{0}^{\infty} \frac{\operatorname{Im}\tilde{G}(\bar{\omega})}{\bar{\omega}-\omega}\,\mathrm{d}\bar{\omega}\,. \tag{4.32}$$

Using $\operatorname{Im}\tilde{G}(-\bar{\omega}) = -\operatorname{Im}\tilde{G}(\bar{\omega})$, we find

$$\begin{aligned} \operatorname{Re}\tilde{G}(\omega) &= \frac{1}{\pi}\,\mathsf{PV}\int_{0}^{\infty} \left[\frac{\operatorname{Im}\tilde{G}(\bar{\omega})}{\bar{\omega}-\omega} + \frac{\operatorname{Im}\tilde{G}(\bar{\omega})}{\bar{\omega}+\omega}\right]\mathrm{d}\bar{\omega} \\ &= \frac{2}{\pi}\,\mathsf{PV}\int_{0}^{\infty} \frac{\bar{\omega}\operatorname{Im}\tilde{G}(\bar{\omega})}{\bar{\omega}^2-\omega^2}\,\mathrm{d}\bar{\omega}\,. \end{aligned} \tag{4.33}$$

The formula above involve only positive, experimentally accessible frequencies. The same manipulations can be done with (4.10b). The resulting formulae are the famous *Kramers-Kronig dispersion relations*:

$$\operatorname{Re}\tilde{G}(\omega) = \frac{2}{\pi}\,\mathsf{PV}\int_{0}^{\infty} \frac{\bar{\omega}\operatorname{Im}\tilde{G}(\bar{\omega})}{\bar{\omega}^2-\omega^2}\,\mathrm{d}\bar{\omega}\,, \tag{4.34a}$$

$$\operatorname{Im}\tilde{G}(\omega) = -\frac{2\omega}{\pi}\,\mathsf{PV}\int_{0}^{\infty} \frac{\operatorname{Re}\tilde{G}(\bar{\omega})}{\bar{\omega}^2-\omega^2}\,\mathrm{d}\bar{\omega}\,. \tag{4.34b}$$

Once-subtracted dispersion relations. It often happens that $f(z)$ in (4.1) does not tend to zero as $|z| \to \infty$. Realistically speaking, we may know nothing about the precise behavior at infinity, except that the quantity remains *bounded* for large $|z|$. If this is the case, one proceeds as follows. Let $f(z)$ be analytic at $\alpha_0 \in \mathbb{R}$. Define

$$D(z) \equiv \frac{f(z)-f(\alpha_0)}{z-\alpha_0} \tag{4.35}$$

is not singular at $z = \alpha_0$ and $|D(z)| \to 0$ as $|z| \to \infty$. If $f(z)$ is analytic in the UHP, so is $D(z)$ and we may write a dispersion relation for $D(z)$:

$$i\pi\left[\frac{f(\alpha)-f(\alpha_0)}{\alpha-\alpha_0}\right] = \mathsf{PV}\int_{-\infty}^{\infty} \frac{[f(x)-f(\alpha_0)]}{(x-\alpha)(x-\alpha_0)}\,\mathrm{d}x\,. \tag{4.36}$$

But notice that

$$\frac{1}{(x-\alpha)(x-\alpha_0)} = \frac{1}{\alpha-\alpha_0}\left[\frac{1}{x-\alpha} - \frac{1}{x-\alpha_0}\right]. \tag{4.37}$$

Consequently, the dispersion relation takes the form

$$\begin{aligned} i\pi f(\alpha) = i\pi f(\alpha_0) + (\alpha-\alpha_0)\,\mathsf{PV}\int_{-\infty}^{\infty}\frac{f(x)}{(x-\alpha)(x-\alpha_0)}\,\mathrm{d}x \\ - f(\alpha_0)\,\mathsf{PV}\int_{-\infty}^{\infty}\frac{\mathrm{d}x}{x-\alpha} + f(\alpha_0)\,\mathsf{PV}\int_{-\infty}^{\infty}\frac{\mathrm{d}x}{x-\alpha_0}. \end{aligned} \tag{4.38}$$

The last two integrals vanish and we are left with

$$i\pi f(\alpha) = i\pi f(\alpha_0) + (\alpha-\alpha_0)\,\mathsf{PV}\int_{-\infty}^{\infty}\frac{f(x)}{(x-\alpha)(x-\alpha_0)}\,\mathrm{d}x\,, \tag{4.39}$$

which is a modified version of (4.9). Separating f into its real and imaginary components, we find

$$\operatorname{Re} f(\alpha) = \operatorname{Re} f(\alpha_0) + \frac{(\alpha-\alpha_0)}{\pi}\,\mathsf{PV}\int_{-\infty}^{\infty}\frac{\operatorname{Im} f(x)}{(x-\alpha)(x-\alpha_0)}\,\mathrm{d}x\,, \tag{4.40a}$$

$$\operatorname{Im} f(\alpha) = \operatorname{Im} f(\alpha_0) - \frac{(\alpha-\alpha_0)}{\pi}\,\mathsf{PV}\int_{-\infty}^{\infty}\frac{\operatorname{Re} f(x)}{(x-\alpha)(x-\alpha_0)}\,\mathrm{d}x\,. \tag{4.40b}$$

These are the *once-subtracted dispersion relations*. How do we use them in a particular physical problem? We must know $\operatorname{Re} f(\alpha_0)$ for some α_0 and assume that $|f(z)|$ is bounded for large $|z|$, in addition to analiticity in the UHP, as we required when deriving the Hilbert transform pair (4.10a) and (4.10b).

If the properties of $f(z)$ for large $|z|$ are not optimal (e.g., the ratio $|f(z)/z|$ tends to a non-zero constant), then one can introduce more subtraction points α_1, α_2, and so on. In this situation, instead of integrating $\tilde{G}(\omega')/(\omega-\omega')$ along the contour 4.1, one considers the function

$$\frac{1}{2\pi i}\frac{\tilde{G}(\omega')}{(\omega'-\omega)\prod_{s=1}^{k}(\omega'-\omega_s)}. \tag{4.41}$$

The same analysis then leads to a pair of k*-subtracted dispersion relations.*

Reconstruction of the response function. Given the Hilbert transform relations (4.34), one can also reconstruct the complete response function $\tilde{G}(\omega)$ as an integral over its imaginary component. To accomplish this, we use the Sokhotski-Plemelj formula:

$$\lim_{\epsilon\to 0}\frac{1}{x\pm i\epsilon} = \mathsf{PV}\,\frac{1}{x} \mp i\pi\delta(x). \tag{4.42}$$

The identity above holds in the sense of distributions. Choosing the $-i\epsilon$ sign, we write

$$\operatorname{Re}\tilde{G}(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\operatorname{Im}\tilde{G}(\bar{\omega})}{\bar{\omega}-\omega-i\epsilon} - \int_{-\infty}^{\infty} i\operatorname{Im}\tilde{G}(\bar{\omega})\delta(\bar{\omega}-\omega)\,\mathrm{d}\bar{\omega}\,. \tag{4.43}$$

Thus,

$$\tilde{G}(\omega) = \mathrm{Re}\,\tilde{G}(\omega) + i\,\mathrm{Im}\,\tilde{G}(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\mathrm{Im}\,\tilde{G}(\bar{\omega})}{\bar{\omega}-\omega-i\epsilon}\,\mathrm{d}\bar{\omega}\,. \tag{4.44}$$

A curious feature of this formula is that, in order to investigate $\tilde{G}(\omega)$ for small ω (the typical scenario in EFTs tailored to linear response calculations), one must know $\mathrm{Im}\,\tilde{G}(\omega)$ for large values of ω. Besides, one typically cannot say much about the precise behaviour of $|\tilde{G}(z)|$ as $|z| \to \infty$, except that it remains bounded.

This is when the *subtracted dispersion relations* become most useful. These modified relations give more weight to the response function evaluated at a "subtracted" point $\omega_0 \in \mathbb{R}$, while also requiring more mild conditions on the behavior of the response at infinity. Once again, using (4.42), we can write

$$\frac{\mathrm{Re}\,\tilde{G}(\omega) - \mathrm{Re}\,\tilde{G}(\omega_0)}{\omega-\omega_0} + i\,\mathrm{Im}\,\tilde{G}(\omega_0) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\mathrm{Im}\,\tilde{G}(\bar{\omega})}{(\bar{\omega}-\omega-i\epsilon)(\bar{\omega}-\omega_0)}\,\mathrm{d}\bar{\omega}\,. \tag{4.45}$$

If $\mathrm{Im}\,\tilde{G}(\omega_0)$ vanishes and $\mathrm{Re}\,\tilde{G}(\omega_0)$ is known, then

$$\frac{\mathrm{Re}\,\tilde{G}(\omega) - \mathrm{Re}\,\tilde{G}(\omega_0)}{\omega-\omega_0} = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\mathrm{Im}\,\tilde{G}(\bar{\omega})}{(\bar{\omega}-\omega-i\epsilon)(\bar{\omega}-\omega_0)}\,\mathrm{d}\bar{\omega}\,. \tag{4.46}$$

It is worth mentioning once more the assumptions involved in the derivation of this expression. First, as a complex function of z, $|\tilde{G}(z)|$ must remain bounded as $|z| \to \infty$. Second, $\tilde{G}(z)$ must be analytic in the UHP and at $z = \omega_0 \in \mathbb{R}$.

4.3 An EFT for the horizon

Having discussed the basic elements of linear response theory and the derivation of the Kramers-Kronig dispersion relations, we now introduce our EFT setup for studying long-wavelength tidal deformations of BHs. In the following, we denote the rank-2 STF projector as

$$\delta^{LL'} = \delta^{i_1 i_2, j_1 j_2} \equiv \frac{1}{2}\left(\delta^{i_1 j_1}\delta^{i_2 j_2} + \delta^{i_1 j_2}\delta^{i_2 j_1} - \frac{2}{3}\delta^{i_1 i_2}\delta^{j_1 j_2}\right). \tag{4.47}$$

EFT setup. Consider an effective worldline theory for a spinless BH. We want to model the effects of a dynamically induced multipole. The starting point is a multipolar action for a Schwarzschild BH of mass M endowed with *unknown* internal degrees of freedom collectively denoted by X which are responsible, for instance, for dissipation and quasi-normal mode excitations.

We make the following minimal assumptions regarding these degrees of freedom: (i) their dynamics is ruled by a local but otherwise generic Lagrangian $\mathcal{L}[X]$, and (ii) they determine the induced multipoles. In this work we focus on the electric quadrupole $Q_{ij}[X]$, which couples linearly to the traceless, electric part of the Riemann tensor E_{ij} via the worldline action [59, 33]

$$S_{\text{worldline}} \supset \int \mathrm{d}\tau \left(-M + \mathcal{L}[X] + \frac{1}{2} Q_{ij}[X] E^{ij} + \cdots \right), \tag{4.48}$$

where the ellipsis contains higher order electric multipoles and magnetic multipoles. The action above reproduces the monopole and quadrupole part of the standard multipolar action of a compact body. Following the idea proposed in [60, 61], one can write the effective action obtained by "integrating out" the X's, which can be heuristically written as

$$S_{\text{eff}} = -M \int \mathrm{d}\tau + \int \mathrm{d}\tau \int_0^\infty \mathrm{d}t\, \chi_{ij,kl}(\tau - t) E^{ij}(t) E^{kl}(\tau), \tag{4.49}$$

where it has been assumed that the induced quadrupole response to the external field is *causal* and *linear*:

$$Q_{ij}(t) = \int_{-\infty}^{t} \mathrm{d}t'\, \chi_{ij,kl}(t - t') E^{kl}(t'), \tag{4.50}$$

and where we have introduced the time-dependent polarizability tensor $\chi_{ij,kl}(t)$, whose tensor structure $\chi_{ij,kl} = \frac{1}{2}\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \frac{2}{3}\delta_{ij}\delta_{kl}\right)\chi$ can be factorized, as required by spherical symmetry. Moving to Fourier space and Taylor expanding the electric, quadrupolar polarizability, one can write:

$$S_{\text{eff}}^{(E)} \sim \int \frac{\mathrm{d}\omega}{2\pi} \left[\tilde{\chi}_0 - i\tilde{\chi}_1 M\omega - \tilde{\chi}_2 (M\omega)^2 + \cdots\right] |\tilde{E}_{ij}(\omega)|^2, \tag{4.51}$$

where the coefficients $\tilde{\chi}_n$ are real numbers. Strictly speaking, this effective action needs to be recast in the *in-in* formalism, as it will be discussed later. The small parameter of expansion here is $M\omega$, so this is an expansion in derivatives of the external gravitational field. For example, $\tilde{\chi}_0$ in (4.51) is the only term for an instantaneous static response – defining the standard static tidal Love number – while the coefficients $\chi_{n\geq 1}$ are its dynamical generalizations.[3] We remark that dynamical Love numbers present EFT running, as discussed, for instance, in [67], and the two-loop beta function for all scalar tidal operators exhibit a universal conservative contribution arising from the post-Minkowskian expansion.

Coefficients with even ($\tilde{\chi}_{2n}$) and odd ($\tilde{\chi}_{2n+1}$) indices encode tidal and dissipative properties of the BH, respectively. As mentioned before, one must employ the *in-in* formalism [118, 73], otherwise terms involving *odd* powers of ω would be total derivative terms. For simplicity, we have written a schematic in-out action in (4.51). In turn, absorption effects can also be addressed via on-shell scattering amplitudes, see for instance reference [11].

Lightning-fast review of the GR description. The scattering of gravitational waves in a Schwarzschild background is a classical problem in GR, whose solution within the scope of the black-hole perturbation theory methods of Teukolsky and Press has been subject

[3] See also the discussion at the end of chapter 3 for more context.

to many seminal studies. See, for example, [94, 63]. We summarize a small part of these findings in this section.

By decomposing the GW perturbation

$$h_{\alpha\beta} \equiv \sum_{a=1}^{10} \sum_{\ell,m} R^a_{\omega\ell m}(r) \left[T^a_{\ell m}(\theta,\phi)\right]_{\alpha\beta} e^{-i\omega t} \tag{4.52}$$

using the *tensorial* spherical harmonics $T^a_{\ell m}$, one obtains a Schrödinger-like equation for the scalar radial function $R_{\omega\ell}(r)$:

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}r_*^2} + \omega^2 - V_\ell(r)\right] (rR_{\omega\ell m}(r)) = 0, \tag{4.53}$$

where $r_* \equiv r + 2M \log\left(\frac{r}{2M} - 1\right)$, and $V_\ell(r)$ is an ℓ-dependent potential. For axial perturbations – governed by the famous Regge-Wheeler equation – the potential[4] the explicit form [110]

$$V_\ell^{\mathrm{RW}}(r) = \left(1 - \frac{2M}{r}\right)\left[\frac{\ell(\ell+1)}{r^2} - \frac{6M}{r^3}\right]. \tag{4.54}$$

For the the polar perturbation (Zerilli) the form of the potential is quantitatively different, but in both cases cases $V(r) \sim 1/r^2$ for large r.

The scattering/absorption process can be described in GR with the Regge-Wheeler and Zerilli equations, see, for example, chapter 12 of the standard textbook [87]. We consider the scattering of an incoming *plane* GW with wave vector $\mathbf{k} = \omega\hat{\mathbf{z}}$ by the central potential generated by the Schwarzschild black hole into an outgoing spherical wave with the *same* polarization. To simplify the computation, we consider as in [36] a Cartesian component, for which one can apply the standard formula for plane-wave expansion, and obtain the elastic, absortive and total scattering cross sections $\sigma_{\mathrm{e,a,t}}$ in terms of the real inelasticity parameter η_ℓ $(0 \leq \eta_\ell \leq 1)$ and a real phase shift δ_ℓ:

$$\sigma_{\mathrm{e}}^{\ell=2} = \frac{\pi}{\omega^2} \sum_m |1 - \eta_2 e^{2i\delta_2}|^2, \tag{4.55a}$$

$$\sigma_{\mathrm{a}}^{\ell=2} = \frac{\pi}{\omega^2} \sum_m \left(1 - \eta_2^2\right), \tag{4.55b}$$

$$\sigma_{\mathrm{t}} \equiv \sigma_{\mathrm{e}} + \sigma_{\mathrm{a}}, \tag{4.55c}$$

where $\sigma_{\mathrm{e,a,t}}$ are respectively the elastic, absorption and total cross sections. The solutions to (4.53) in the free case ($M = 0$) come in two classes: pure left-moving $\overleftarrow{R}_{\omega\ell}(r)$ and right-moving $\vec{R}_{\omega\ell}(r)$ modes. Apart from an overall normalization factor, their asymptotic

[4]Under parity transformations, which in polar coordinates can be written as $r \to r$, $\theta \to \pi - \theta$, $\phi \to \phi + \pi$, seven out of the ten tensor harmonics pick a factor $(-1)^l$ and are said to be *polar*; the three others pick a factor $(-1)^{l+1}$ and are said to be *axial*, or odd.

forms can be parameterized as [43]

$$\overleftarrow{R}_{\omega\ell}(r) = \frac{1}{r}\begin{cases} B_\ell(\omega)\, e^{-i\omega r_*}, & r \to 2M \\ e^{-i\omega r_*} + \overleftarrow{A}_\ell(\omega)\, e^{i\omega r_*}, & r \to \infty \end{cases} \tag{4.56a}$$

$$\vec{R}_{\omega\ell}(r) = \frac{1}{r}\begin{cases} e^{-i\omega r_*} + \vec{A}_\ell(\omega)\, e^{i\omega r_*}, & r \to 2M \\ B_\ell(\omega)\, e^{-i\omega r_*}, & r \to \infty \end{cases} \tag{4.56b}$$

The fact that B_ℓ is the same coefficient in both the left-moving and the right-moving mode is a simple consequence of the constancy of the Wronskian:

$$\vec{R}_{\omega\ell}(r)\,\frac{\mathrm{d}\overleftarrow{R}_{\omega\ell}(r)}{\mathrm{d}r_*} - \frac{\mathrm{d}\vec{R}_{\omega\ell}(r)}{\mathrm{d}r_*}\,\overleftarrow{R}_{\omega\ell}(r) = \text{constant in } r, \tag{4.57}$$

which also requires

$$\vec{A}_\ell = -\frac{B_\ell^*}{B_\ell}\,\overleftarrow{A}_\ell. \tag{4.58}$$

Moreover, probability conservation enforces $|A_\ell|^2 + |B_\ell|^2 = 1$ (Notice that $|\vec{A}_\ell| = |\overleftarrow{A}_\ell|$). From our numerical solutions of the Regge-Wheeler equation (4.53) we obtain $\overleftarrow{R}_{\omega\ell}$, from which $\overleftarrow{A}_\ell(\omega)$ and $B_\ell(\omega)$ have been extracted. Their relation to $\sigma_{\text{e,a}}$ can be obtained by expanding the incident plane wave and then following the standard textbook procedure, which gives

$$\sigma_{\ell=2,\text{e}} = \frac{\pi}{\omega^2}\sum_m |1 + \overleftarrow{A}_2|^2, \tag{4.59a}$$

$$\sigma_{\ell=2,\text{a}} = \frac{\pi}{\omega^2}\sum_m |B_2|^2. \tag{4.59b}$$

Our numerical results for $(\omega^2/5\pi)\,\sigma^{\text{e,a}}_{\ell=2}$ as a function of $M\omega$ are reported in figure 4.2, where the limits $M\omega \to 0$ and $M\omega \gg 1$ are also shown. The $M\omega \to 0$ limit of σ_{a} can be read from [94]:

$$\sigma^{\text{a}}_{\ell=2} = \frac{256\pi}{45} M^6\omega^4. \tag{4.60}$$

Comparing this formula with (4.55b) gives

$$\eta_2 \simeq 1 - \frac{128}{225}(M\omega)^6, \quad M\omega \ll 1. \tag{4.61}$$

Also, we obtain $|B_\ell| \simeq \frac{16}{15}(M\omega)^3$ from (4.59) for $M\omega \ll 1$, which agrees with the value than can be deduced from equations (12) and (13) of [14]. The perturbative analytic result for the scattering phase in the Regge-Wheeler equation is [99, 15]

$$\delta_2^{(\text{RW})} = \left(\log(4M\omega) + \gamma_{\text{E}} - \frac{5}{3}\right) 2M\omega + \frac{107\pi}{105}(2M\omega)^2 + \mathcal{O}\big([M\omega]^3\big), \tag{4.62}$$

which together with (4.61) gives

$$\sigma^{\mathrm{e}}_{\ell=2} = 80\pi M^2 \bigg(\log(4M\omega) + \gamma_{\mathrm{E}} - \frac{5}{3} \bigg) \bigg[\log(4M\omega) + \gamma_{\mathrm{E}} - \frac{5}{3} + \frac{107\pi}{210}(2M\omega) + \mathcal{O}\big([M\omega]^2\big) \bigg], \tag{4.63}$$

where γ_{E} denotes the Euler-Mascheroni constant. For $M\omega \gg 1$ one has $\eta_\ell \to 0$, leading to $A_\ell \to 0$, $B_\ell \to 1$, and $\sigma^{\mathrm{e,a}}_{\ell=2} \to 5\pi/\omega^2$.

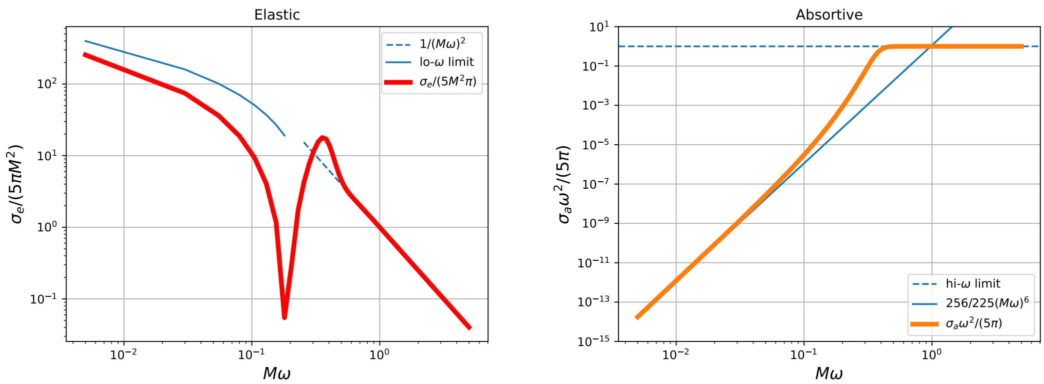


Figure 4.2: Normalized elastic (left) and absorption (right) cross sections and their asymptotic values. The $M\omega \to 0$ limit of σ_{e} is given in (4.63), whereas $\frac{\omega^2}{5\pi}\sigma_{\mathrm{a},\ell=2} \to \frac{256}{225}(M\omega)^6$. For $M\omega \gg 1$, one has $\sigma^{\ell=2}_{\mathrm{e,a}} \to \frac{5\pi}{\omega^2}$.

Introducing the EFT two-point function. While the scattering of radiation by longitudinal gravitational modes does not give rise to absorption but is responsible for the leading order elastic scattering, following [60] the leading order absorption process can be related to the *Feynman* two-point function:

$$G_{\mathrm{f}}^{LL'}(\omega) \equiv \int \frac{\mathrm{d}t}{2\pi}\, e^{-i\omega t} \langle T\{Q^L(t) Q^{L'}(0)\}\rangle, \tag{4.64}$$

where L, L' are collective indices, used here for the spin-2 representation of the rotation group. Now we consider the coupling of h_L to the source quadrupole, as discussed in detail in section 3.2. Following the standards methods explained in the previous chapter, we fix the imaginary part of the Feynman two-point function (in the low-frequency limit) as

$$\operatorname{Im} \tilde{G}_{\mathrm{f}}^{LL'}(\omega) = \delta^{LL'} \frac{16}{45} M^6 |\omega|, \tag{4.65}$$

This is the same result as (3.61). The Feynman two-point function is related to the advanced and retarded Green functions via

$$G_{\mathrm{f}} = \frac{1}{2}(G_{\mathrm{adv}} + G_{\mathrm{ret}}) - \frac{i}{2}(\Delta_+ + \Delta_-) = G_{\mathrm{ret}} - i\Delta_-, \tag{4.66}$$

where $\Delta_{\pm}^{LL'}(t)$ are the standard Wightman functions:

$$\Delta_{+}^{LL'}(t,t') \equiv \langle Q^{L}(t)Q^{L'}(t')\rangle, \tag{4.67a}$$
$$\Delta_{-}^{LL'}(t,t') \equiv \langle Q^{L}(t)Q^{L'}(t')\rangle = \Delta_{+}^{L'L}(t',t). \tag{4.67b}$$

From now on we exploit the rotational symmetry of the Schwarzschild background to impose $\Delta_{\pm}^{LL'}(t,t') = \delta^{LL'}\Delta_{\pm}(t,t')$, so that

$$\tilde{\Delta}_{+}(\omega) = \tilde{\Delta}_{-}(-\omega). \tag{4.68}$$

Further assuming that Q_L is an Hermitian operator ($Q_L = Q_L^{\dagger}$), one has

$$\tilde{\Delta}_{+}^{LL'}(\omega) = \int_{\omega} e^{i\omega(t-t')}\langle Q_L^{\dagger}(t)Q_{L'}^{\dagger}(t')\rangle = \left(\int_{\omega} e^{-i\omega(t-t')}\langle Q_{L'}(t')Q_L(t)\rangle\right)^{*}, \tag{4.69}$$

so that

$$\tilde{\Delta}_{+}^{LL'}(\omega) = \left[\tilde{\Delta}_{-}^{LL'}(-\omega)\right]^{*} = \left[\tilde{\Delta}_{+}^{L'L}(\omega)\right]^{*} = \delta^{LL'}\tilde{\Delta}_{+}^{*}(\omega), \tag{4.70}$$

in agreement with equation (3.24) of [57] and the assumption of spherical symmetry.

4.3.1 The gravitational Wightman function

Following the approach of [61], we now compute (perturbatively) the semiclassical graviton two-point function with EFT methods and compare it with the results coming from quantum field theory over a Schwarzschild background, the "full theory."

The presence of the BH modifies the Wightman function of the bulk field $G_{\rm EFT}^{LL'}(x,x')$, and these corrections can be computed perturbatively within the *in-in* formalism. The result, accurate to leading-order interactions with the BH, is[5]

$$G_{\rm EFT}^{LL'}(x,x') = \langle h^{L}(x)h^{L'}(x')\rangle_{\rm M} + \frac{\delta^{LL'}}{(4\pi r)^2}\int_{\omega} e^{-i\omega(t-t')}\left[\theta(-\omega)\mathbb{A}_{+}(\omega) + \theta(\omega)\mathbb{A}_{-}(\omega)\right], \tag{4.71}$$

where by $\langle h^{L}(x)h^{L'}(x')\rangle_{\rm M}$ we denote the two-point function in Minkowski spacetime, i.e., without the BH contribution. The object $\mathbb{A}_{+}$ above is the frequency-space Wightman function of the worldline operator $\mathbb{O}(\tau)$, as we will properly define in the following section. The second term in (4.71) is evaluated at the spatial coincidence limit $|\mathbf{x}| = |\mathbf{x}|'$.

This bulk two-point function is traditionally computed in the full theory with different boundary conditions, the most famous ones being the *Boulware* (B), the *Unruh* (U), and the *Hartle-Hawking* (HH) states. From the perspective of quantum field theory over curved spacetimes, such conditions reflect the non-uniqueness of the vacua when Poincaré symmetry is absent (See the appendix B for more context). The states mentioned above

[5] Crucially, we obtain a result that differs from the one reported in eq. (2.21) of [61]: the sign in front of their θ function should be changed to obtain our result. This small modification has important consequences for the matching procedure, as we shall discuss.

correspond, respectively, to the "no radiation," the "only outgoing radiation," and the "thermal bath" boundary conditions.

Adapting the classical result of [31], the explicit form of our bulk field two-point function can be matched. But first, we discuss the derivation of (4.71).

4.3.2 *in-in* computations

Without loss of generality, we consider in the full theory a scalar two-point function $\langle\Psi|\phi(x)\phi(x')|\Psi\rangle$, where $|\Psi\rangle$ is some known and fixed initial state. For the purposes of the *in-in* calculation on the EFT side, it suffices to consider the scalar case since the tensor structure of (4.71) is entirely contained in $\delta^{LL'}$ (assuming a non-spinning BH). The computation the follows thus assumes a linear coupling (at the level of the action) between a scalar, perturbing long-wavelength field ϕ to a monopole worldline operator $\mathbb{O}(\tau)$.

The quantity we must match in the EFT is the *in-in* correlator:

$$G_{\text{EFT}}(x,x') \equiv \langle\text{in}|\phi(x)\phi(x')|\text{in}\rangle \equiv \langle\phi^2\rangle_{\text{in}}. \tag{4.72}$$

We will focus on the case in which the full theory is either in the *Boulware* ($|\mathsf{B}\rangle$), the *Unruh* ($|\mathsf{U}\rangle$), or the *Hartle-Hawking* ($|\mathsf{H}\rangle$) states. The full theory result at $r \to \infty$ suggests that initial $|\text{in}\rangle$ state in the EFT ought to be described by a density matrix of the form:

$$\rho_{\text{In}} = |0\rangle\langle 0| \otimes \rho_{\text{BH}}, \tag{4.73}$$

where $|0\rangle$ is the usual, Poincaré invariant, free field vacuum, and ρ_{BH} is some density matrix acting on the Hilbert space of "BH states." It is assumed that the scalar field is minimally coupled to the static gravitational field of the point source at the origin (the BH). In addition, the scalar ϕ couples to the source via the worldline interation in S_{BH}. Then, to leading non-trivial order in the power counting, the *in-in* formalism produces

$$\begin{aligned} G_{\text{EFT}}(x,x') &= \Delta_+(x-x') \\ &+ \int \mathrm{d}\tau\,\mathrm{d}\tau'\, D^{2a}(x,x(\tau))\,\langle\mathbb{O}_a(\tau)\mathbb{O}_b(\tau')\rangle\, D^{1b}(x',x(\tau')). \end{aligned} \tag{4.74}$$

The second term, whose associated graph is diagram (a) of figure 4.3, can be understood as a dressing of the flat-spacetime Wightman function as a consequence of the insertion of worldline operatos $\mathbb{O}_a(\tau)$.

In the correlator expression above, $x(\tau) = (\tau, 0)$ is the worldline of the BH at rest at the origin of the coordinate system. Notice that this path has no spatial part. The object $D^{ab}(x,x')$ is the *Schwinger-Keldish propagator matrix* of the free scalar field:

$$D^{ab}(x,x') = \begin{pmatrix} -iD_{\text{f}}(x-x') & -W_0(x'-x) \\ -W_0(x-x') & -iD_{\text{d}}(x-x') \end{pmatrix}, \tag{4.75}$$

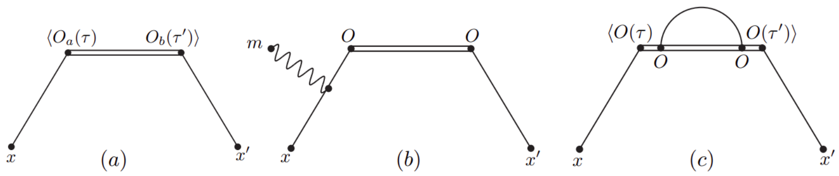


Figure 4.3: Corrections to the Wightman function in the EFT. Diagram (a) is LO, whereas (b) and (c) are NLO in the power counting parameter. The operators $O_a(\tau)$ depicted above are our $\mathbb{O}_a(\tau)$. Adapted from [61].

where

$$D_{\rm f}(x-x') = \langle 0|T\{\phi(x)\phi(x')\}|0\rangle = [D_{\rm d}(x-x')]^* \tag{4.76a}$$
$$W_0(x-x') = \langle 0|\phi(x)\phi(x')|0\rangle\,. \tag{4.76b}$$

The function $W_0(x-x')$ is the flat spacetime positive-frequency Wightman function

$$W_0(x-x') = \Delta_+(x-x') = \Delta_-(x'-x). \tag{4.77}$$

The matrix of EFT two-point functions is

$$\mathbb{G}_{ab}(\tau,\tau') \equiv \langle \mathbb{O}_a(\tau)\mathbb{O}_b(\tau')\rangle = \begin{pmatrix} -i\mathbb{D}_{\rm f}(\tau-\tau') & \mathbb{W}_-(\tau-\tau') \\ \mathbb{W}_+(\tau-\tau') & -i\mathbb{D}_{\rm d}(\tau-\tau') \end{pmatrix}, \tag{4.78}$$

where the components are denoted as

$$\mathbb{D}_{\rm f}(\tau-\tau') = \langle T\{\mathbb{O}(\tau)\mathbb{O}(\tau')\}\rangle\,, \tag{4.79a}$$
$$\mathbb{D}_{\rm d}(\tau-\tau') = \langle \tilde{T}\{\mathbb{O}(\tau)\mathbb{O}(\tau')\}\rangle. \tag{4.79b}$$

and

$$\mathbb{W}_+(\tau-\tau') = \mathbb{W}_-(\tau'-\tau) = \langle \mathbb{O}(\tau)\mathbb{O}(\tau')\rangle\,, \tag{4.80}$$

where $\tilde{T}$ is the anti-time ordering symbol. The *in-in* correlator $\langle\phi^2\rangle_{\rm In} \equiv \langle {\rm in}|\phi(x)\phi(x')|{\rm in}\rangle$ we wish to compute is

$$\left\langle\phi^2\right\rangle_{\rm in} = W_0(x-x') + \int {\rm d}\tau\,{\rm d}\tau'\,D^{2a}(x,x(\tau))\mathbb{G}_{ab}(\tau,\tau')D^{1b}(x',x(\tau')). \tag{4.81}$$

We denote the *second term* of the EFT two-point function at spatial coincidence by $\Delta G_{\rm in}$. We write it as

$$\Delta G_{\rm in} \equiv \int {\rm d}\tau\,{\rm d}\tau'\,D^{2a}(x,x(\tau))\mathbb{G}_{ab}(\tau,\tau')D^{1b}(x,x(\tau')). \tag{4.82}$$

Notice that D^{1b} is written as $D^{1b}(x,x(\tau'))$ at spatial coincidence. Since $x(\tau) = (\tau,0)$ has no spatial component, we may unambiguously write (at spatial coincidence):

$$x - x(\tau) = (t - \tau, \mathbf{x}) \tag{4.83a}$$
$$x - x(\tau') = (t' - \tau', \mathbf{x}) \tag{4.83b}$$

The quantity (4.82) is a sum of four terms:

$$\Delta G_{\text{In}} = (\mathbf{I}) + (\mathbf{II}) + (\mathbf{III}) + (\mathbf{IV}), \tag{4.84}$$

where

$$(\mathbf{I}) = +\int \mathrm{d}\tau\, \mathrm{d}\tau'\, W_0(x - x(\tau))\mathbb{D}_{\text{f}}(\tau - \tau')D_{\text{f}}(x - x(\tau')) \tag{4.85a}$$

$$(\mathbf{II}) = +\int \mathrm{d}\tau\, \mathrm{d}\tau'\, W_0(x - x(\tau))\mathbb{W}_{-}(\tau - \tau')W_0(x(\tau') - x) \tag{4.85b}$$

$$(\mathbf{III}) = -\int \mathrm{d}\tau\, \mathrm{d}\tau'\, D_{\text{d}}(x - x(\tau))\mathbb{W}_{+}(\tau - \tau')D_{\text{f}}(x - x(\tau')) \tag{4.85c}$$

$$(\mathbf{IV}) = +\int \mathrm{d}\tau\, \mathrm{d}\tau'\, D_{\text{d}}(x - x(\tau))\mathbb{D}_{\text{d}}(\tau - \tau')W_0(x(\tau') - x). \tag{4.85d}$$

Computing (II) and (III). First, we denote the frequency space Wightman function of the $\mathbb{O}$ operator by

$$\mathbb{W}_{+}(\tau - \tau') \equiv \int_\omega e^{-i\omega(\tau - \tau')}\mathbb{A}_{+}(\omega). \tag{4.86}$$

Using the formula (A.26), we calculate:

$$(\mathbf{II}) = \frac{1}{4\pi^2 r^2}\int_0^\infty \frac{\mathrm{d}\omega}{2\pi}\,\mathbb{A}_{+}(-\omega)\sin^2(\omega r)e^{-i\omega(t-t')}. \tag{4.87}$$

Expanding $\sin^2(\omega r)$ and dropping the rapidly oscillating term, one obtains

$$(\mathbf{II}) = \frac{2}{(4\pi r)^2}\int_\omega e^{-i\omega(t-t')}\theta(\omega)\mathbb{A}_{+}(-\omega). \tag{4.88}$$

We now turn our attention to $(\mathbf{III})$. First, notice that

$$D_{\text{d}} = -[D_{\text{f}}]^* = -\left[-\int_\omega e^{-i\omega t}\frac{e^{i|\omega|r}}{4\pi r}\right]^* = \int_\omega e^{i\omega t}\frac{e^{-i|\omega|r}}{4\pi r}, \tag{4.89}$$

where the representation (A.31) was used. Thus, substituting $\omega \mapsto -\omega$ in the formula above yields

$$D_{\text{d}} = -[D_{\text{f}}]^* = \int_\omega e^{-i\omega t}\frac{e^{-i|\omega|r}}{4\pi r}. \tag{4.90}$$

The computation of (**III**) then leads to

$$(\mathbf{III}) = \frac{1}{(4\pi r)^2}\int_\omega e^{-i\omega(t-t')}\mathbb{A}_+(\omega). \tag{4.91}$$

Computing (I) and (IV). We first consider (**I**). Substituting the relevant two-point functions, we find

$$\begin{aligned}(\mathbf{I}) &= -\int \mathrm{d}\tau\,\mathrm{d}\tau' \int_0^\infty \frac{\mathrm{d}\omega}{2\pi} e^{-i\omega(t-\tau)}\frac{\sin(\omega r)}{2\pi r}\mathbb{D}_\mathrm{f}(\tau-\tau')\int_{\bar\omega} e^{-i\bar\omega(t'-\tau')}\frac{e^{i|\bar\omega|r}}{4\pi r} \\ &= -\frac{2}{(4\pi r)^2}\int_\omega e^{-i\omega(t-t')}e^{i\omega r}\sin(\omega r)\theta(\omega)\mathbb{D}_\mathrm{f}(\omega)\end{aligned} \tag{4.92}$$

Expanding $e^{i\omega r}\sin(\omega r)$ and dropping rapidly oscillating terms in the $r\to\infty$ limit, we find:

$$\begin{aligned}(\mathbf{I}) &= -\frac{2}{(4\pi r)^2}\int_\omega e^{-i\omega(t-t')}\left[\frac{1}{2}\sin(2\omega r) + i\sin^2(\omega r)\right]\theta(\omega)\mathbb{D}_\mathrm{f}(\omega) \\ &= -\frac{i}{(4\pi r)^2}\int_\omega e^{-i\omega(t-t')}\theta(\omega)\mathbb{D}_\mathrm{f}(\omega).\end{aligned} \tag{4.93}$$

Therefore,

$$(\mathbf{I}) = -\frac{i}{(4\pi r)^2}\int_\omega e^{-i\omega(t-t')}\theta(\omega)\mathbb{D}_\mathrm{f}(\omega). \tag{4.94}$$

At last we compute (**IV**). Using (4.90) one finds

$$\begin{aligned}(\mathbf{IV}) &= \int \mathrm{d}\tau\,\mathrm{d}\tau' \int_\omega e^{-i\omega(t-\tau)}\frac{e^{-i|\omega|r}}{4\pi r}\mathbb{D}_\mathrm{d}(\tau-\tau')\int_0^\infty \frac{\mathrm{d}\bar\omega}{2\pi}e^{-i\bar\omega(\tau'-t')}\frac{\sin(\bar\omega r)}{2\pi r} \\ &= \frac{2}{(4\pi r)^2}\int_\omega \theta(\omega)e^{-i\omega(t-t')}\left[\frac{1}{2}\sin(2\omega r) - i\sin^2(\omega r)\right]\mathbb{D}_\mathrm{d}(\omega)\end{aligned} \tag{4.95}$$

As before, expanding $e^{-i\bar\omega r}\sin(\bar\omega r)$ and dropping rapidly oscillating terms in the $r\to\infty$ limit yields

$$(\mathbf{IV}) = -\frac{i}{(4\pi r)^2}\int_\omega e^{-i\omega(t-t')}\theta(\omega)\mathbb{D}_\mathrm{d}(\omega). \tag{4.96}$$

We now combine all the terms.

$$\begin{aligned}\Delta G_\text{in} = &\frac{1}{(4\pi r)^2}\int_\omega e^{-i\omega(t-t')}\mathbb{A}_+(\omega) + \frac{2}{(4\pi r)^2}\int_\omega e^{-i\omega(t-t')}\theta(\omega)\mathbb{A}_+(-\omega) \\ &- \frac{i}{(4\pi r)^2}\int_\omega e^{-i\omega(t-t')}\theta(\omega)\left[\mathbb{D}_\mathrm{f}(\omega)+\mathbb{D}_\mathrm{d}(\omega)\right]\end{aligned} \tag{4.97}$$

It is not difficult to see that

$$\mathbb{D}_{\mathrm{f}}(\omega) + \mathbb{D}_{\mathrm{d}}(\omega) = -i\big[\mathbb{A}_+(\omega) + \mathbb{A}_-(\omega)\big] = -i\big[\mathbb{A}_+(\omega) + \mathbb{A}_+(-\omega)\big]. \tag{4.98}$$

Therefore,

$$\begin{aligned} G_{\mathrm{EFT}}(x,x') &= W_0(x-x') \\ &+ \frac{1}{(4\pi r)^2}\int_\omega e^{-i\omega(t-t')}\big\{\mathbb{A}_+(\omega) - \theta(\omega)\left[\mathbb{A}_+(\omega) - \mathbb{A}_+(-\omega)\right]\big\}. \end{aligned} \tag{4.99}$$

We must match the formula above to the corresponding Wightman functions at spatial coincidence when the background vacuum state is either $|\mathsf{B}\rangle$, $|\mathsf{U}\rangle$ or $|\mathsf{H}\rangle$.

At last, it can be verified that the large distance limit of the equal time commutator $[h_L(t,r\hat{\mathbf{n}}), h_{L'}(t,r'\hat{\mathbf{n}}')]$ vanishes, for both the EFT formula (4.71) and the result of the full theory with the three different boundary conditions (4.103), (4.113), and (4.117).

4.3.3 Matching the EFT calculations

To ease the matching, it is convenient to choose spatial points with same angular variables: $\mathbf{x} = r\hat{\mathbf{n}}$, $\mathbf{x}' = r'\hat{\mathbf{n}}$. This allows one to use the coincidence limit of the addition theorem for tensorial spherical harmonics [92, 89],

$$\epsilon_L\epsilon^*_{L'}\sum_m [T^a_{\ell m}(\theta,\phi)]_L[T^b_{\ell m}(\theta,\phi)]_{L'} = \frac{2\ell+1}{4\pi}\delta^{ab}|\epsilon|^2, \tag{4.100}$$

where ϵ_L is the polarization tensor.

Boulware. In the Boulware state, we have

$$\begin{aligned} \epsilon_L\epsilon^*_{L'}\langle h^L(t,\mathbf{x})h^{L'}(t',\mathbf{x}')\rangle_{\mathrm{B}} = |\epsilon|^2\int_0^\infty \frac{\mathrm{d}\omega}{(4\pi)^2\omega}\, e^{-i\omega(t-t')}\sum_\ell(2\ell+1) \\ \times\left[\vec{R}_{\omega\ell}(r)\vec{R}^*_{\omega\ell}(r') + \overleftarrow{R}_{\omega\ell}(r)\overleftarrow{R}^*_{\omega\ell}(r')\right]. \end{aligned} \tag{4.101}$$

Expanding this result using the asymptotic forms (for $r\to\infty$) of the fundamental solutions to the Regge-Wheeler equation

$$\overleftarrow{R}_{\omega\ell}(r) = \frac{1}{r}\begin{cases} B_\ell(\omega)\, e^{-i\omega r_*}, & r\to 2M \\ e^{-i\omega r_*} + \overleftarrow{A}_\ell(\omega)\, e^{i\omega r_*}, & r\to\infty \end{cases} \tag{4.102a}$$

$$\vec{R}_{\omega\ell}(r) = \frac{1}{r}\begin{cases} e^{-i\omega r_*} + \vec{A}_\ell(\omega)\, e^{i\omega r_*}, & r\to 2M \\ B_\ell(\omega)\, e^{-i\omega r_*}, & r\to\infty \end{cases} \tag{4.102b}$$

we find

$$\epsilon_L\epsilon'_L\langle h^L(t,\mathbf{x})h^{L'}(t',\mathbf{x}')\rangle_{\mathrm{B}} \simeq |\epsilon|^2\int_0^\infty \frac{\mathrm{d}\omega}{16\pi^2\omega}\frac{e^{-i\omega(t-t')}}{rr'}\sum_\ell(2\ell+1) \times\left[|B_\ell|^2 e^{i\omega(r_*-r'_*)}+\left(e^{-i\omega r_*}+\overleftarrow{A}_\ell\, e^{i\omega r_*}\right)\left(e^{i\omega r'_*}+\overleftarrow{A}^*_\ell e^{-i\omega r'_*}\right)\right]. \tag{4.103}$$

To facilitate the physical interpretation of this expression, let us compare it with the corresponding two-point function in Minkowski spacetime (A.26) expanded in the same limit:

$$\epsilon_L\epsilon^*_{L'}\langle h^L(t,\mathbf{x})h^{L'}(t',\mathbf{x}')\rangle_{\mathrm{M}} \simeq \frac{|\epsilon|^2}{rr'}\int_0^\infty \frac{\mathrm{d}\omega}{16\pi^2\omega}\,e^{-i\omega(t-t')}\sum_\ell(2\ell+1) \times\left[e^{-i\omega r}-(-1)^\ell e^{i\omega r}\right]\left[e^{i\omega r'}-(-1)^\ell e^{-i\omega r'}\right]. \tag{4.104}$$

Simple algebraic manipulations take (4.103) to

$$\begin{aligned}\epsilon_L\epsilon^*_{L'}\langle h^L(t,r\hat{\mathbf{n}}),h^{L'}(t',r\hat{\mathbf{n}}')\rangle_{\mathrm{B}} &\simeq \epsilon_L\epsilon^*_{L'}\langle h^L(t,r\hat{\mathbf{n}}),h^{L'}(t',r\hat{\mathbf{n}}')\rangle_{\mathrm{M}} \\ &+\frac{|\epsilon|^2}{r^2}\int_0^\infty\frac{\mathrm{d}\omega}{16\pi^2\omega}\,e^{-i\omega(t-t')}\sum_\ell(2\ell+1)P_\ell(\cos\gamma) \\ &\times\left[|B_\ell|^2+|\overleftarrow{A}_\ell+(-1)^\ell|^2-2\left(1+(-1)^\ell\,\mathrm{Re}\,\overleftarrow{A}_\ell\right)\right],\end{aligned} \tag{4.105}$$

where fast-oscillating exponential terms like $e^{\pm i\omega(r'+r)}$ were suppresed in the large r, r' limit with $r=r'$. The last line of (4.105) vanishes as a consequence of $|\overleftarrow{A}_\ell|^2+|B_\ell|^2=1$.

Using the symmetry relation (4.68), the matching of (4.105) onto (4.71) leads to $\mathbb{A}_+(\omega)\propto\theta(\omega)$. This means that the Wightman function $\mathbb{A}_+(\omega)$ has no support over negative frequencies. Even though the same conclusion was reached by the authors of [61], our EFT result is slightly different, which leads to a different matching calculation as well.

In [61], the following expansion is used:

$$\sum_{\ell=0}^\infty(2\ell+1)|\overleftarrow{R}_\ell(\omega|r)|^2\sim 4\omega^2, \quad r\to\infty, \tag{4.106}$$

which cannot possibly take into account the EFT terms that correspond to deviations from the flat Minkowski background, i.e., $\overleftarrow{A}_\ell$ dependent terms; see (4.102). In particular, contributions involving $\mathbb{A}_+(\omega)$ (the A_+ function of [61]) are not captured.[6] As a consistency check of our results, we verify that the radial flux of radiation also disappears, as expected

[6]Using (4.106), the aforementioned authors find that the matching onto the Boulware state gives a non-vanishing contribution to the bulk two-point function with BH interactions, which implies (to leading order in $M\omega$ and with $r=r'\to\infty$):

$$G^{LL'}_{\mathrm{EFT}}(x,x')=\langle h^L(x)h^{L'}(x')\rangle_{\mathrm{M}}+\frac{1}{4\pi^2}\left(\frac{2M}{r}\right)^2\int_0^\infty\mathrm{d}\omega\,\omega\,e^{-i\omega(t-t')}.$$

for the Boulware state:

$$\begin{aligned}\langle \text{in}|T^{rt}(x)|\text{in}\rangle &= \frac{1}{2}\lim_{x\to x'}(\partial_r\partial_{t'}+\partial_t\partial_{r'})\,G_{\text{EFT}}(x,x') \\ &= \frac{1}{16\pi^2r^2}\int_\omega \omega^2 e^{-i\omega(t-t')}\big[\theta(-\omega)\mathbb{A}_+(\omega)+\theta(\omega)\mathbb{A}_-(\omega)\big] \\ &= 0.\end{aligned} \tag{4.107}$$

Unruh. Next, we consider the matching onto the Unruh state. As before, the starting point is the full theory result:

$$\begin{aligned}\epsilon_L\epsilon^*_{L'}\langle h^L(t,r\mathbf{n})h^{L'}(t',r'\mathbf{n})\rangle_{\text{U}} = \delta_{LL'}&\int_{-\infty}^{\infty}\frac{\text{d}\omega}{(4\pi)^2\omega}\,e^{-i\omega(t-t')}\sum_\ell(2\ell+1) \\ &\times\left[\frac{\vec{R}_{\omega\ell}(r)\vec{R}^*_{\omega\ell}(r')}{1-e^{-8\pi M\omega}}+\theta(\omega)\overleftarrow{R}_{\omega\ell}(r)\overleftarrow{R}^*_{\omega\ell}(r')\right],\end{aligned} \tag{4.108}$$

whose $r,r'\to\infty$ limit gives

$$\begin{aligned}\langle h_L(t,\mathbf{x})h_{L'}(t',\mathbf{x}')\rangle_{\text{U}} \simeq \delta_{LL'}&\int_{-\infty}^{\infty}\frac{\text{d}\omega}{16\pi^2\omega}\frac{e^{-i\omega(t-t')}}{rr'}\sum_\ell(2\ell+1) \\ &\times\left(\frac{|B_\ell|^2}{1-e^{-8\pi M\omega}}\,e^{i\omega(r_*-r'_*)}+\theta(\omega)\mathcal{U}_{\omega\ell}(r,r')\right),\end{aligned} \tag{4.109}$$

where the function $\mathcal{U}_{\omega\ell}(r,r')$ is defined as

$$\begin{aligned}\mathcal{U}_{\omega\ell}(r,r') \equiv{}& \big[e^{-i\omega r}-(-1)^\ell e^{i\omega r}\big]\big[e^{i\omega r'}-(-1)^\ell e^{-i\omega r'}\big] \\ &+|\overleftarrow{A}_\ell+(-1)^\ell|^2e^{i\omega(r-r')}-2\big[1+(-1)^\ell\,\text{Re}\,\overleftarrow{A}_\ell\big]e^{i\omega(r-r')} \\ &+\big[\overleftarrow{A}^*_\ell+(-1)^\ell\big]e^{-i\omega(r+r')}+\big[\overleftarrow{A}_\ell+(-1)^\ell\big]e^{i\omega(r+r')}.\end{aligned} \tag{4.110}$$

Specializing this result to the $r=r'\to\infty$ limit, we find (after suppresing fast-oscillating exponentials):

$$\begin{aligned}\epsilon_L\epsilon^*_{L'}\langle h^L(t,r\hat{\mathbf{n}})h^{L'}(t',r\hat{\mathbf{n}})\rangle_{\text{U}} ={}& \epsilon_L\epsilon^*_{L'}\langle h^L(t,r\hat{\mathbf{n}})h^{L'}(t',r\hat{\mathbf{n}})\rangle_{\text{M}} \\ &+\frac{|\epsilon|^2}{r^2}\int_{-\infty}^{\infty}\frac{\text{d}\omega}{16\pi^2\omega}\,e^{-i\omega(t-t')}\sum_{\ell=0}^{\infty}(2\ell+1)\,\times \\ &\times\frac{|B_\ell|^2}{1-e^{-8\pi M\omega}}+\theta(\omega)\big\{|\overleftarrow{A}_\ell+(-1)^\ell|^2-2\big[1+(-1)^\ell\,\text{Re}\,\overleftarrow{A}_\ell\big]\big\}.\end{aligned} \tag{4.111}$$

Notably, the $1/r^2$ piece of (4.111) can be written as

$$\frac{|B_\ell|^2}{e^{8\pi M|\omega|}-1}\big[\theta(\omega)-\theta(-\omega)\big] = \frac{|B_\ell|^2}{e^{8\pi M|\omega|}-1}\,\mathrm{sgn}(\omega), \tag{4.112}$$

where sgn is the sign function. Our final expression is thus

$$\begin{aligned}&\epsilon_L\epsilon_{L'}\langle h^L(t,r\hat{\mathbf{n}})h^{L'}(t',r\hat{\mathbf{n}})\rangle_{\mathrm{U}} = \epsilon_L\epsilon_{L'}\langle h^L(t,r\hat{\mathbf{n}})h^{L'}(t',r\hat{\mathbf{n}})\rangle_{\mathrm{M}}\\ &+\frac{|\epsilon|^2}{r^2}\int_{-\infty}^{\infty}\frac{\mathrm{d}\omega}{16\pi^2\omega}\,e^{-i\omega(t-t')}\sum_\ell(2\ell+1)\,\frac{|B_\ell|^2}{e^{8\pi M|\omega|}-1}\,\mathrm{sgn}(\omega).\end{aligned} \tag{4.113}$$

Matching this expression to the perturbative EFT calculation (4.71), one finds

$$\mathbb{A}_-(\omega>0) = \frac{5}{2\pi|\omega|}\frac{|B_2|^2}{e^{8\pi M|\omega|}-1}, \tag{4.114a}$$

$$\mathbb{A}_+(\omega<0) = \frac{5}{2\pi|\omega|}\frac{|B_2|^2}{e^{8\pi M|\omega|}-1}, \tag{4.114b}$$

which is consistent with both the spherical symmetry and Hermiticity conditions expressed in (4.68) and (4.70), respectively. For the energy-momentum tensor, we obtain

$$T^{tr} = \frac{\omega^2}{16\pi^2r^2}\big[\theta(\omega)\Delta_-(\omega)+\theta(-\omega)\Delta_+(\omega)\big] = \frac{4\omega}{e^{8\pi M|\omega|}-1}\,|B_\ell|^2, \tag{4.115}$$

which is consistent which a net positive flux of particles escaping to $r\to\infty$.

Hartle-Hawking. When the BH is in equilibirum with a thermal bath of temperature $\beta = 4\pi r_s = 8\pi GM$, the relevant two-point function is

$$\begin{aligned}\epsilon_L\epsilon_{L'}\langle h^L(t,r\mathbf{n})h^{L'}(t',r'\mathbf{n})\rangle_{\mathrm{H}} &= |\epsilon|^2\int_{-\infty}^{\infty}\frac{\mathrm{d}\omega}{(4\pi)^2\omega}\sum_\ell(2\ell+1)\\ &\times\left[e^{-i\omega(t-t')}\frac{\vec{R}_{\omega\ell m}(r)\vec{R}^*_{\omega\ell m}(r')}{1-e^{-8\pi M\omega}}+e^{i\omega(t-t')}\frac{\overleftarrow{R}^*_{\omega\ell m}(r)\overleftarrow{R}_{\omega\ell m}(r')}{e^{8\pi M\omega}-1}\right]\\ &\overset{r=r'\to\infty}{=}\delta_{LL'}\int_{-\infty}^{\infty}\frac{\mathrm{d}\omega}{16\pi^2\omega}\frac{e^{-i\omega(t-t')}}{rr'}\sum_\ell(2\ell+1)\\ &\times\frac{1}{1-e^{-8\pi M\omega}}\big[|B_\ell|^2+\mathcal{U}_{\omega\ell}(r,r')\big].\end{aligned} \tag{4.116}$$

Matching this result to the Minkowksi spacetime thermal two-point function

$$\langle h_L(t,r\hat{\mathbf{n}})h_{L'}(t',r\hat{\mathbf{n}})\rangle^\beta_{\mathrm{M}} \equiv \frac{\delta_{LL'}}{2\pi}\int_{-\infty}^{\infty}\frac{\mathrm{d}\omega}{2\pi}\frac{\omega}{1-e^{-8\pi M\omega}}, \tag{4.117}$$

one obtains

$$\theta(-\omega)\mathbb{A}_+(\omega) + \theta(\omega)\mathbb{A}_-(\omega) = 0, \tag{4.118}$$

which is the same result of the Boulware case. Analogously, the flux at infinity derived from the energy-momentum tensor disappears, leaving behind the flat spacetime thermal contribution only. This is consistent with the results presented in table I of [31].

4.4 A gravitational sum rule

We now relate the real and imaginary parts of the polarizability function, i.e., the retarded two-point function of the quadrupole EFT operator, denoted by $\mathbb{D}_{\text{ret}}^{LL'}(\omega)$ in Fourier space. The polarizability function is defined in terms of the Wightman functions $\mathbb{A}_\pm$ analogously to its flat spacetime counterpart $D_{\text{ret}}(\omega)$; see (A.22c). Since $\mathbb{D}_{\text{ret}}^{LL'}(\tau-\tau')$ is real:

$$\mathbb{D}_{\text{ret}}^{LL'}(\omega) = \left[\mathbb{D}_{\text{ret}}^{LL'}(-\omega)\right]^*, \tag{4.119}$$

we can combine the dispersive representation (A.34), the identity

$$\lim_{\epsilon\to 0}\frac{1}{x-i\epsilon} = \mathsf{PV}\,\frac{1}{x} + i\pi\delta(x), \tag{4.120}$$

the spherical symmetry relation (4.68), and the reality condition (4.70) to deduce[7]

$$\operatorname{Re}\mathbb{D}_{\text{ret}}^{LL'}(\omega) = \delta^{LL'}\mathsf{PV}\int_{-\infty}^{\infty}\frac{d\omega'}{2\pi}\frac{\mathbb{A}(\omega')-\mathbb{A}(-\omega')}{\omega-\omega'}, \tag{4.121a}$$

$$\operatorname{Im}\mathbb{D}_{\text{ret}}^{LL'}(\omega) = -\frac{\delta^{LL'}}{2}\left[\mathbb{A}(\omega)-\mathbb{A}(-\omega)\right], \tag{4.121b}$$

These formulae are in agreement with equations (3.36) and (3.37) of reference [57].

Using the analyticity property of $\mathbb{D}_{\text{ret}}^{LL'}$ and assuming that $\mathbb{D}_{\text{ret}}(\omega)\to 0$ as $\omega\to\infty$ in the UHP, the usual Kramers-Kronig relations follow:

$$\operatorname{Re}\mathbb{D}_{\text{ret}}^{LL'}(\omega) = \frac{1}{\pi}\mathsf{PV}\int\frac{d\omega'}{2\pi}\frac{\operatorname{Im}\mathbb{D}_{\text{ret}}^{LL'}(\omega')}{\omega'-\omega}, \tag{4.122a}$$

$$\operatorname{Im}\mathbb{D}_{\text{ret}}^{LL'}(\omega) = -\frac{1}{\pi}\mathsf{PV}\int\frac{d\omega'}{2\pi}\frac{\operatorname{Re}\mathbb{D}_{\text{ret}}^{LL'}(\omega')}{\omega'-\omega}. \tag{4.122b}$$

Using (4.65) and (4.66) one has

$$\operatorname{Re}\mathbb{D}_{\text{ret}}^{LL'}(\omega) = \operatorname{Re}\mathbb{D}_{\text{f}}^{LL'}(\omega), \tag{4.123a}$$

$$\operatorname{Im}\mathbb{D}_{\text{ret}}^{LL'}(\omega) = \operatorname{Im}\mathbb{D}_{\text{f}}^{LL'}(\omega) + \delta^{LL'}\mathbb{A}(-\omega) \simeq \delta^{LL'}\left[\frac{16}{45}M^6|\omega| + \mathbb{A}(-\omega)\right], \tag{4.123b}$$

where the low-$M\omega$ limit of $\mathbb{D}_{\text{f}}^{LL'}(\omega)$ was substituted. The vanishing of the static response

[7] In the equations that follow we denote $\mathbb{A}(\omega) \equiv \mathbb{A}_+(\omega)$.

$\mathbb{D}_{\rm ret}(0) = 0$ leads to a sum rule [115] that can be rewritten as

$$\int_0^\infty \mathrm{d}\omega\, \frac{\left[\mathbb{A}(\omega) - \mathbb{A}(-\omega)\right]}{\omega} = 0. \tag{4.124}$$

On one hand, in the Boulware and Hartle-Hawking vacuum states – where $\mathbb{A}(\omega) \propto \theta(\omega)$ – the sum rule above requires that $\mathbb{A}(\omega > 0)$ cannot be always positive. This is problematic because the absorption cross section $\sigma_{\rm abs}$ is such that $\sigma_{\rm abs} \propto \mathbb{A}(\omega)$. On the other hand, in the Unruh vacuum the sum rule (4.124) can be fulfilled non-trivially since $\mathbb{A}(-\omega) \neq 0$; see (4.114).

Standard lore says that retarded and advanced Green functions constructed from the commutator of Wightman functions only encode classical information, and thus, being c-numbers and not quantum operators, they cannot have vacuum-dependent expectation values. Indeed, the "quantum" part of the Feynman two-point function is proportional to a symmetric combination of Wightman functions; see (4.66). Anyway, the classical Green function depends on the choice of boundary condition, which in turn is dictated by the choice of the underlying vacuum state.[8]

Our conclusions culminating in (4.124) are only in *apparent* contradiction with the standard lore mentioned above, as we shall now explain in more detail.

Discussion. We have analyzed the apparent puzzle of reconciling the vanishing static tidal deformability of a BH and the Kramers-Kronig dispersion relations, which follow from general causality considerations within linear response theory. In particular, we find that in the Unruh vacuum state, which corresponds to a black hole emitting Hawking radiation, is compatible with a quadrupole two-point function with support on negative frequencies, as required by the outgoing net flux of radiation to infinity

It must be noticed, however, that by taking into account terms beyond the lowest order coupling to the quadrupole two point functions, one should include the tail process. As pointed out by [42], the tail effect, i.e., the scattering of radiation off the static curvature induced by the BH, alters the response function by multiplying it by a complex number, which can have the effect of mixing its real and imaginary parts, and thus changing the holomorphicity property of the retarded function as a result of the branch cut introduced by the logarithmic tail.

Matter of fact, the EFT derivation presented in this work cannot capture a possible analytic continuations, since these are oblivious to the imaginary part of retarded Green function (or the absorption cross section).

This is well-known from quantum scattering amplitudes methods; there are terms that can be reproduced by unitary cuts (which are in turn associated with discontinuities along branch cuts), but there are also *contact terms* that do not contribute to those cuts. For instance, the derivation of the Feynman propagator in quantum field theory by means of Cauchy's theorem requires a contour which contains (in principle) contributions from arcs at infinity in the UHP, resulting in possible polynomial subtraction terms [129]. This is also

[8] See [128] and [127] for similar results for the electromagnetic field in Minkowski and Schwarzschild spacetime.

known from Britto-Cachazo-Feng-Witten-type recursion relations, in which the shifted amplitude, as a function of the shift parameter z, can be written in terms of a pole term and a purely polynomial piece – the constant term from this piece is what we call the residue of the pole at $z = \infty$, the boundary term.

The discussion above implies that the real part of the retarded Green function could in principle contain an undetermined real polynomial piece $F(\omega)$ [35] that should be added to the right hand side of (4.123a), and which can be written in terms of a polynomial in ω. This polynomial piece corresponds to the Love numbers [35]. Now at $\omega \to 0$, due to the vanishing of static Love numbers for Schwarzschild BHs, the aforementioned sum rule (4.124) would gain a constant term outside the integral sign.

The physical interpretation of this sum rule may be better understood by taking into account late-time tail effects as one can rewrite the dispersion-integral representation of the retarded Green function in such a way that a connection between the near and wave zones is manifest [42]. At low frequencies the scattering is dominated by the leading instantaneous contribution associated with tidal effects. However, also at low frequencies, the overlap between potential and radiation modes produces tail contributions, so physically the modified sum rule should be interpreted in terms of an overlap between the tidal response and tail effects.

Chapter 5

Conclusion

In this work we have reviewed the fundamental aspects of NRGR and some of its applications. Equipped with this knowledge, we tackled the problem of investigating Kramers-Kronig dispersion relations to the BH response function in a worldline EFT approach motivated by NRGR.

In chapter 1 we review the linearized theory of GWs and introduced a worldline EFT to describe the dynamics of classical compact objects and gravity. As an illustrative example of the NRGR machinery, we derive the 1PN conservative Lagrangian step by step. Then, we discuss the *top-bottom* approach, where *matching* calculations and gravitational amplitudes are first introduced. In this approach, we sketch the derivation of the NLO energy flux. Next, in chapter 2 we extended the traditional *in-out* framework of NRGR to its *in-in* counterpart. This allowed us to correctly extract the 2.5PN radiation reaction force and derive a causal waveform at such order.

After discussing these foundational topics, in chapter 3 we turn to applications of the worldline EFT methods to dissipative and absorptive dynamics. We first explore an electrodynamical toy model, showing how one can compute the electromagnetic energy absorbed by a BH horizon. At last, the absorption of gravitons is considered, culminating in a calculation of the energy dissipated by the horizon in a binary BH system. The frequency-space expansion of the BH response function is introduced and Love numbers are defined.

At last, in chapter 4 we considered the central question investigated in [124]; the conciliation of Kramers-Kronig relations with the vanishing of the static tidal Love numbers. The quadrupole two-point function defined in the EFT is matched to standard formulae in quantum field theory over a Schwarzschild background. In the Unruh vacuum state, we found a EFT Wightman function with support on negative frequencies, as expected. A sum rule for the frequency-space Wightman function is proposed. On account of tail effects – which must be considered in higher-order calculations – one must be careful in distinguishing the tidal response from tails. This suggests that our sum rule ought to be modified in such scenario.

Recently, the universality of late-time power law tails was addressed in [113]. The authors' analysis include a calculation of the branch cut contribution to the BH response. It is tempting to wonder whether or not such data can be incorporated explicitly into our sum rule. This question and others are left to future work.

Appendix A

Useful integrals and identities

A.1 Master integrals

The basic three-dimensional integral which is extensively used in the near-zone computations of section 1.3.2 is

$$\int_{\mathbf{k}} \frac{k^i k^j}{\mathbf{k}^4} e^{-i\mathbf{k}\cdot\mathbf{x}} = \frac{1}{8\pi|\mathbf{x}|}\left(\delta_{ij} - \frac{x_i x_j}{|\mathbf{x}|^2}\right). \tag{A.1}$$

By contracting both side of this formula with δ_{ij}, we also obtain

$$\int_{\mathbf{k}} \frac{1}{\mathbf{k}^2} e^{-i\mathbf{k}\cdot\mathbf{x}} = \frac{1}{4\pi|\mathbf{x}|}. \tag{A.2}$$

Next, consider the d-dimensional "tadpole" integral:

$$\mathbb{M}_1^{(d)}(n,\omega^2) = \int \frac{\mathrm{d}^d\mathbf{k}}{(2\pi)^d}\frac{1}{(\mathbf{k}^2-\omega^2)^n} = \frac{\Gamma\left(n-\frac{d}{2}\right)}{(4\pi)^{\frac{d}{2}}\Gamma(n)}(-\omega^2)^{\frac{d}{2}-n}. \tag{A.3}$$

Proof. Writing the integrand as

$$\frac{1}{(\mathbf{k}^2-\omega^2)^n} = \frac{1}{\Gamma(n)}\int_0^\infty x^{n-1}e^{x\omega^2}e^{-x\mathbf{k}^2}\,\mathrm{d}x \tag{A.4}$$

and integrating over $\mathrm{d}^3\mathbf{k}$, we find

$$\mathbb{M}_1^{(d)}(n,\omega^2) = \frac{1}{(4\pi)^{d/2}\Gamma(n)}\int_0^\infty e^{n-d/2-1}e^{x\omega^2}\,\mathrm{d}x = \frac{\Gamma\left(n-\frac{d}{2}\right)}{(4\pi)^{\frac{d}{2}}\Gamma(n)}(-\omega^2)^{\frac{d}{2}-n}. \tag{A.5}$$

This establishes (A.3). ■

We now derive the master formula of Galley and Tiglio [55]. This result is used twice in chapter 1: first, to derive the 2.5PN effective action in the *in-out* formulation of NRGR;

second, in the proof of the "GT lemma" given in (1.142).

$$
\begin{aligned}
\mathbb{M}_2^{(D)}(n,p,q) &= \int_{-\infty}^{\infty} \mathrm{d}s \int \frac{\mathrm{d}^{D-1}\mathbf{k}}{(2\pi)^{D-1}} \,\mathsf{PV}\int_{k^0} \frac{s^n e^{ik^0 s}}{(k^0)^2-\mathbf{k}^2}(k^0)^p k_{i_1}\cdots k_{i_q} \\
&= \frac{i^{n-D+3}\sec\left(\frac{\pi D}{2}\right)}{2^{D-1}\pi^{\frac{(D-3)}{2}}\Gamma\left(\frac{D-1}{2}\right)} \frac{\delta_{i_1\cdots i_q}}{(q+1)!!} n! \quad (n=p+q+D-3).
\end{aligned}
\tag{A.6}
$$

When $D=4$, the expression above simplifies to

$$
\mathbb{M}_2^{(4)}(n,p,q) = \frac{i^{n-1}n!}{4\pi(q+1)!!}\delta_{i_1\cdots i_q} \quad (n=p+q+1). \tag{A.7}
$$

Proof. The master integral vanishes when q is odd. When q is even, we can write

$$
\int \mathrm{d}s\, s^n e^{ik^0 s} = \frac{1}{i^n}\frac{\mathrm{d}^n}{\mathrm{d}(k^0)^n}\int \mathrm{d}s\, e^{ik^0 s} = \frac{2\pi}{i^n}\frac{\mathrm{d}^n}{\mathrm{d}(k^0)^n}\delta(k^0). \tag{A.8}
$$

Integrating by parts n times, we then find

$$
\begin{aligned}
\mathbb{M}_2 &= (-i)^n \int \frac{\mathrm{d}^{D-1}\mathbf{k}}{(2\pi)^{D-1}} \,\mathsf{PV}\int \mathrm{d}k^0 \frac{(k^0)^p k_{i_1}\cdots k_{i_q}}{(k^0)^2-\mathbf{k}^2}\frac{\mathrm{d}^n}{\mathrm{d}(k^0)^n}\delta(k^0) \\
&= i^n \frac{\mathrm{d}^n}{\mathrm{d}(k^0)^n}\left[(k^0)^p \int \frac{\mathrm{d}^{D-1}\mathbf{k}}{(2\pi)^{D-1}} \frac{k_{i_1}\cdots k_{i_q}}{(k^0)^2-\mathbf{k}^2}\frac{\mathrm{d}^n}{\mathrm{d}(k^0)^n}\right]_{k^0=0}
\end{aligned}
\tag{A.9}
$$

Exploiting the spherical symmetry of the integrand, we can replace

$$
k_{i_1}\cdots k_{i_q} \longrightarrow \frac{|\mathbf{k}|^q}{(q+1)!!}\delta_{i_1\cdots i_q} = (\delta_{i_1 i_2}\cdots\delta_{i_{q-1}}\delta_{i_q} + \text{all pairs}). \tag{A.10}
$$

Then,

$$
\mathbb{M}_2 = \frac{i^n}{(q+1)!!}\frac{\mathrm{d}^n}{\mathrm{d}(k^0)^n}\left[\frac{(k^0)^p}{(2\pi)^{D-1}}\int \mathrm{d}\Omega_{D-1}\int_0^\infty \mathrm{d}|\mathbf{k}|\frac{|\mathbf{k}|^{D-2+q}}{(k^0)^2-|\mathbf{k}|^2}\delta_{i_1\cdots i_q}\right]_{k^0=0}. \tag{A.11}
$$

Next, consider the integral over $|\mathbf{k}|$. We first compute its "Euclidean version" before performing a Wick rotation to the expression given above. Let $\mathbb{I}^{\mathsf{E}}$ be defined as

$$
\mathbb{I}^{\mathsf{E}} \equiv \int_0^\infty \mathrm{d}|\mathbf{k}| \frac{(|\mathbf{k}|^2)^{\frac{D-2+q}{2}}}{(k^0)^2+|\mathbf{k}|^2}. \tag{A.12}
$$

A standard computation using Schwinger's parametrization and Euler's reflection formula yields

$$
\mathbb{I}^{\mathsf{E}} = \frac{\pi}{2}(k^0)^{D-3+q}\sec\left(\frac{\pi(D-2+q)}{2}\right). \tag{A.13}
$$

The integral over $\mathrm{d}|\mathbf{k}|$ that appears in (A.11), which we denote by $\mathbb{I}^{\mathsf{M}}$, is expressed in terms of $\mathbb{I}^{\mathsf{E}}$ as $\mathbb{I}^{\mathsf{M}} = -\mathbb{I}^{\mathsf{E}}(-ik^0)$. It follows that

$$\mathbb{M}_2 = \frac{\delta_{i_1\cdots i_q}}{(q+1)!!}\frac{i^n\,\pi^{\frac{D-1}{2}}(-\pi)(-i)^{D-3+q}}{(2\pi)^{D-1}\Gamma\left(\frac{D-1}{2}\right)\cos\left(\frac{\pi(D-2+q)}{2}\right)}\frac{\mathrm{d}^n}{\mathrm{d}(k^0)^n}\,(k^0)^{p+q+D-3}\Big|_{k^0=0}. \tag{A.14}$$

Since q must be an even number, we write $q=2N$, for natural N, and simplify the equation above using the identity

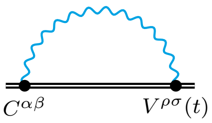
$$\cos\left(\frac{\pi D}{2}+\frac{\pi(q-2)}{2}\right) = (-1)^{N-1}\cos[\pi(N-1)] = i^{q-2}\cos[\pi(N-1)]. \tag{A.15}$$

Then, one finds

$$\mathbb{M}_2 = \frac{\delta_{i_1\cdots i_q}}{(q+1)!!}\frac{i^{n-D+3}\sec\left(\frac{\pi D}{2}\right)}{2^{D-1}\pi^{\frac{D-3}{2}}\Gamma\left(\frac{D-1}{2}\right)}\frac{(p+q+D-3)!}{(p+q+D-3-n)!}(k^0)^{p+q+D-3-n}\Big|_{k^0=0}. \tag{A.16}$$

This expression vanishes when $n \neq p+q+D-3$. Otherwise, one recovers (A.6). ■

Having established (A.6), we can finally prove the GT lemma. Using the NRGR rules, the diagram in (1.142) becomes



$$\begin{aligned}
&= \int \mathrm{d}t\,\mathrm{d}t'\, iC^{\alpha\beta}\mathbb{P}_{\alpha\beta\rho\sigma}D_{\mathrm{F}}(t-t',\mathbf{0})iV^{\rho\sigma}(t') \\
&= iC^{\alpha\beta}\mathbb{P}_{\alpha\beta\rho\sigma}\int \mathrm{d}t\,\mathrm{d}t' \int_{\mathbf{k},k^0}\frac{ie^{-ik^0(t-t')}}{(k^0)^2-|\mathbf{k}|^2+i\mathsf{a}}\,iV^{\rho\sigma}(t') \\
&\equiv -iC^{\alpha\beta}\mathbb{P}_{\alpha\beta\rho\sigma}(\mathcal{I}_1^{\rho\sigma}+\mathcal{I}_2^{\rho\sigma}),
\end{aligned} \tag{A.17}$$

where $\mathcal{I}_1^{\rho\sigma}$ and $\mathcal{I}_2^{\rho\sigma}$ are defined as

$$\mathcal{I}_1^{\rho\sigma} \equiv \int \mathrm{d}t\,\mathrm{d}t' \int_{\mathbf{k}}\mathsf{PV}\int_{k^0}\frac{e^{-ik^0(t-t')}}{(k^0)^2-|\mathbf{k}|^2}V^{\rho\sigma}(t'), \tag{A.18a}$$

$$\mathcal{I}_2^{\rho\sigma} \equiv -i\pi\int \mathrm{d}t\,\mathrm{d}t' \int_{\mathbf{k},k^0} e^{-ik^0(t-t')}\delta[(k^0)^2-|\mathbf{k}|^2]V^{\rho\sigma}(t') \tag{A.18b}$$

Setting $t' = t + s$, the integral (A.18a) becomes

$$\begin{aligned} \mathcal{I}_1^{\rho\sigma} &= \int \mathrm{d}t\, \mathrm{d}s \int_{\mathbf{k}} \mathsf{PV} \int_{k^0} \frac{e^{-ik^0 s}}{(k^0)^2 - |\mathbf{k}|^2} V^{\rho\sigma}(t+s) \\ &= \int \mathrm{d}t \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\mathrm{d}^n V^{\rho\sigma}(t)}{\mathrm{d}t^n} \int \mathrm{d}s \int_{\mathbf{k}} \mathsf{PV} \int_{k^0} \frac{s^n e^{ik^0 s}}{(k^0)^2 - |\mathbf{k}|^2} \\ &= \int \mathrm{d}t \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\mathrm{d}^n V^{\rho\sigma}(t)}{\mathrm{d}t^n} \mathbb{M}_2^{(4)}(n,0,0). \end{aligned} \tag{A.19}$$

Only the $n = 1$ term is non-zero. As in (1.151), we set $k^0 \mapsto -k^0$ in the second line above. It follows that

$$\mathcal{I}_1^{\rho\sigma} = \frac{1}{4\pi} \int \mathrm{d}t\, \dot{V}^{\rho\sigma}(t). \tag{A.20}$$

This is a surface term, so we drop it. Moving on, we compute

$$\begin{aligned} \mathcal{I}_2^{\rho\sigma} &= -i\pi \int \mathrm{d}t \int_{\mathbf{k},k^0} e^{-ik^0 t} \delta[(k^0)^2 - |\mathbf{k}|^2] \int \mathrm{d}t'\, e^{ik^0 t'} V^{\rho\sigma}(t') \\ &= -i\pi \tilde{V}^{\rho\sigma}(0) \int_{\mathbf{k}} \delta(\mathbf{k}^2) \\ &= 0. \end{aligned} \tag{A.21}$$

This finishes the proof of the GT lemma.

A.2 Green functions

In our **literature review** (chapters 1 to 3), we use the conventions of the seminal reference [55], which coincide with those used in [59, 60, 68]. In this way, the reader that wishes to read our review to become familiar with the subject may directly compare intermediate steps of key calculations with the exposition of the well-stablished aforementioned papers. Our chapter 4 is an expanded version of the **original work** presented in the preprint [124]. As such, in chapter 4 – *and chapter 4 only* – we adopt the conventions of [124].

Chapter 4 - Original work. As mentioned above, throughout chapter 4 – the bulk work of this dissertation – we adopt:

$$G_{\mathrm{f}}(x) = -i\big[\theta_+(x^0)\Delta_+(x) + \theta_-(x^0)\Delta_-(x)\big], \tag{A.22a}$$

$$G_{\mathrm{d}}(x) = -i\big[\theta_+(x^0)\Delta_-(x) + \theta_-(x^0)\Delta_+(x)\big], \tag{A.22b}$$

$$G_{\mathrm{ret}}(x) = -i\theta_+(x^0)\big[\Delta_+(x) - \Delta_-(x)\big], \tag{A.22c}$$

$$G_{\mathrm{adv}}(x) = +i\theta_-(x^0)\big[\Delta_+(x) - \Delta_-(x)\big], \tag{A.22d}$$

where G_{f} and G_{d} stand for "Feynman" and "Dyson", respectively. These two functions are related to one another by complex conjugation: $G_{\mathrm{f}}^*(x) = G_{\mathrm{d}}(x)$. Besides these, we also define the Hadamard function as the symmetric combination

$$\Delta_{\mathrm{h}}(x) = \Delta_+(x) + \Delta_-(x). \tag{A.23}$$

In the equations above, $\Delta_\pm$ denotes the positive and negative frequency flat-space Wightman functions, which are defined by

$$\begin{aligned}\Delta_+(x-x') = \Delta_-(x'-x) &= \int_{\omega,\mathbf{k}} \theta(\omega) 2\pi\delta(\omega^2 - |\mathbf{k}|^2)\, e^{-i\omega(t-t')+i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} \\ &\equiv \int_{\omega,\mathbf{k}} \Delta_+(\omega)\, e^{-i\omega(t-t')+i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}.\end{aligned} \tag{A.24}$$

The Feynman, Dyson, and Hadamard functions are symmetric under $x \mapsto -x$, whereas

$$G_{\mathrm{ret}}(x) = G_{\mathrm{adv}}(-x). \tag{A.25}$$

It is not difficult to show that Δ_+ can also be written as

$$\Delta_+(x-x') = \int_0^\infty \frac{\mathrm{d}\omega}{2\pi}\, e^{-i\omega(t-t')}\, \frac{\sin(\omega|\mathbf{x}-\mathbf{x}'|)}{2\pi|\mathbf{x}-\mathbf{x}'|}. \tag{A.26}$$

This particular representation is used in chapter 4 of this dissertation. The formulas (A.22a) – (A.22d) imply that not all functions are independent. Rather, they satisfy

$$G_{\mathrm{f}}(x) = \frac{1}{2}\big[G_{\mathrm{ret}}(x) + G_{\mathrm{adv}}(x)\big] - \frac{i}{2}\Delta_{\mathrm{h}}(x). \tag{A.27}$$

As mentioned before, these are the conventions employed in the preprint [124] that constitutes the basis for chapter 4, the core of this dissertation. We derive a number of important identities satisfied by the Wightman functions and the Green functions constructed from them. First, using the integral representation of the step function

$$\theta(\pm t) = \mp\frac{1}{i}\int_\omega \frac{e^{-i\omega t}}{\omega + i\mathsf{a}}, \tag{A.28}$$

one can show that (writing $r = |\mathbf{x}|$):

$$G_{\mathrm{ret}}(t,\mathbf{x}) = \int_{\omega,\mathbf{k}} \frac{e^{i\omega t + i\mathbf{k}\cdot\mathbf{x}}}{(\omega + i\mathsf{a})^2 - |\mathbf{k}|^2} = -\frac{1}{4\pi r}\,\delta(t-r), \tag{A.29a}$$

$$G_{\mathrm{adv}}(t,\mathbf{x}) = \int_{\omega,\mathbf{k}} \frac{e^{i\omega t + i\mathbf{k}\cdot\mathbf{x}}}{(\omega - i\mathsf{a})^2 - |\mathbf{k}|^2} = -\frac{1}{4\pi r}\,\delta(t+r) \tag{A.29b}$$

The Feynman Green function also admits an integral representation similar to (A.29). Once again, using (A.24) and (A.28), we find

$$G_{\rm f}(t,\mathbf{x}) = \int_{\omega,\mathbf{k}} \frac{e^{-i\omega t+\mathbf{k}\cdot\mathbf{x}}}{\omega^2-\mathbf{k}^2+i\mathsf{a}}. \tag{A.30}$$

Using (A.29), (A.26), and the "all-functions identity" (A.27), the following alternative representation can also be derived:

$$G_{\rm f}(t,\mathbf{x}) = -\int_{\omega} e^{-i\omega t}\,\frac{e^{i|\omega|r}}{4\pi r}. \tag{A.31}$$

Finally, we derive the so-called "dispersive representation" of the retarded Green function. Using (A.22c) and (A.28), we write

$$G_{\rm ret}(t,\mathbf{x}) = \int\frac{{\rm d}\omega'}{2\pi}\int_{\omega,\mathbf{k}}\frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{\omega'+i\epsilon}\left[\Delta_+(\omega)e^{-i(\omega'+\omega)t}-\Delta_-(\omega)e^{-i(\omega'+\omega)t}\right]. \tag{A.32}$$

Let $\omega'\mapsto\omega''-\omega$ now. After combining common terms and doing a simple relabeling of the integration variables, we obtain

$$G_{\rm ret}(t,\mathbf{x}) = \int_{\omega,\omega',\mathbf{k}}\frac{[\Delta_+(\omega')-\Delta_-(\omega')]}{\omega-\omega'+i\epsilon}\,e^{-i\omega t+i\mathbf{k}\cdot\mathbf{x}} \equiv \int_{\omega,\mathbf{k}} G_{\rm ret}(\omega)\,e^{-i\omega t+i\mathbf{k}\cdot\mathbf{x}}. \tag{A.33}$$

Therefore,

$$G_{\rm ret}(\omega) = \int_{\omega'}\frac{[\Delta_+(\omega')-\Delta_-(\omega')]}{\omega-\omega'+i\epsilon}. \tag{A.34}$$

This representation will play a key role in chapter 4, where we establish the gravitational sum rule for Schwarzschild black holes.

Chapters 1 to 3 - Literature review. In the paper [55]which introduces the *in-in* formulation of NRGR, the authors employ the following conventions:

$$G_{\rm f}(x) = \theta_+(x^0)\Delta_+(x)+\theta_-(x^0)\Delta_-(x), \tag{A.35a}$$

$$G_{\rm d}(x) = \theta_+(x^0)\Delta_-(x)+\theta_-(x^0)\Delta_+(x), \tag{A.35b}$$

$$G_{\rm ret}(x) = +i\theta_+(x^0)\big[\Delta_+(x)-\Delta_-(x)\big], \tag{A.35c}$$

$$G_{\rm adv}(x) = -i\theta_-(x^0)\big[\Delta_+(x)-\Delta_-(x)\big]. \tag{A.35d}$$

Once again, the Hadamard function is defined as (A.23). With this choice, one obtains

$$G_{\rm f}(x) = -\frac{i}{2}\big[G_{\rm ret}(x)+G_{\rm adv}(x)\big]+\frac{1}{2}\Delta_{\rm h}(x). \tag{A.36}$$

The dictionary for translating the conventions of [55] and [124] is

$$(G_{\mathrm{f}})^{\text{Our chapter 4}} = -i(G_{\mathrm{f}})^{\text{Our lit. review}}, \tag{A.37a}$$

$$(G_{\mathrm{ret}})^{\text{Our chapter 4}} = -(G_{\mathrm{ret}})^{\text{Our lit. review}}, \tag{A.37b}$$

$$(G_{\mathrm{adv}})^{\text{Our chapter 4}} = -(G_{\mathrm{adv}})^{\text{Our lit. review}}. \tag{A.37c}$$

The Wightman and Hadamard functions are unchanged. The conventions of [55] are identical to [68], which is an important reference for chapter 3. As far as the definition of the Feynman function is concerced, the choice above matches the one used in [60], a seminal reference on dissipative effects in NRGR.

Appendix B

Scalar fields in a Schwarzschild background

In this appendix we present a self-contained summary of key ideas in quantum field theory over curved spacetimes, providing broader context to the use of Candelas' [31] results in the matching calculation of chapter 4. In particular, we focus on the quantization of a free massless scalar field in a Schwarzschild background, with special attention to the issue of boundary conditions and the choice of a vacuum state.

As in the usual quantization procedure in Minkowski space, we must first investigate the classical solutions to the free equations of motion and properly define a vacuum state. Only then one can speak of a Fock space and all the subsequent field-theoretic constructions. The Poincaré symmetry of Minkowski space renders the choice of a vacuum state almost trivial. In this case, there exists a globally defined Killing vector which allows one to define a "positive frequency mode function" associated to the Fourier expansion of our field, and most importantly, all inertial observers agree with this definition. The vacuum state $|0\rangle$ is defined as the state which is annihilated by the operator multiplying the positive frequency mode. The state $|0\rangle$ obtained this way is unique.

This procedure is far less obvious in the absence of Poincaré symmetry. However, if our spacetime satisfies certain technical requirements (to be mentioned later), a choice of vacuum state is also straightforward, albeit not unique anymore; it will depend on the choice of boundary conditions we are interested in ascribing to the system. For concreteness, consider the action of a massless scalar field $\phi(x)$ minimally coupled to a background metric $g_{\mu\nu}$:

$$S = \frac{1}{2}\int \mathrm{d}^4x\,\sqrt{-g}g^{\mu\nu}(\nabla_\mu\phi)(\nabla_\nu\phi). \tag{B.1}$$

As usual, $g \equiv \det g_{\mu\nu}$. We take the metric to be the Schwarzschild solution, whose line element in the conventional (t, r, θ, φ) coordinate chart reads[1]

$$\mathrm{d}s^2 = \left(1 - \frac{2M}{r}\right)\mathrm{d}t^2 - \left(1 - \frac{2M}{r}\right)^{-1}\mathrm{d}r^2 - r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2). \tag{B.2}$$

For practical reasons, it will be useful to define the functions $A(r)$ and $B(r)$:

$$A(r) \equiv \frac{1}{B(r)} \equiv 1 - \frac{2M}{r}. \tag{B.3}$$

[1]We denote the azimuthal angle by φ to avoid any potential confusion with the scalar field ϕ.

From (B.1) one finds the following classical equation of motion:

$$\Box\phi = 0, \tag{B.4}$$

where the d'Alembertian operator is

$$\Box \equiv g^{\mu\nu}\nabla_\mu\nabla_\nu = \frac{1}{\sqrt{-g}}\,\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu). \tag{B.5}$$

After suitably defining an inner product on the space of test functions, one can define a complete set of solutions to (B.4). But before defining this inner product and investigating the mode expansion of ϕ, we must review a number of properties of the Schwarzschild solution. In particular, it will be important to review its *maximal analytical extension* (see §18.1 of [46] for the precise definition and a thorough discussion).

The construction of the Kruskal manifold which defines the maximal extension of the Schwarzschild solution begins with the coordinate patch $(\bar{u}, \bar{v}, \theta, \varphi)$, where

$$\bar{u} \equiv -4M\, e^{-u/4M}, \tag{B.6a}$$

$$\bar{v} \equiv 4M\, e^{v/4M}. \tag{B.6b}$$

In the equations above, u and v are the lightcone coordinates

$$u \equiv t - r_*, \tag{B.7a}$$

$$v \equiv t + r_*, \tag{B.7b}$$

where r_* is the tortoise coordinate defined by $r_* \equiv r + 2M \log|r/2M - 1|$. The future ($\mathcal{H}^+$) and past ($\mathcal{H}^-$) horizons are defined by the null curves $\bar{u} = 0$ and $\bar{v} = 0$, respectively. The line element takes the form

$$\mathrm{d}s^2 = \frac{2M}{r}\, e^{-r/2M}\,\mathrm{d}\bar{u}\,\mathrm{d}\bar{v} - r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2). \tag{B.8}$$

In the exterior region $r > 2M$ we can also define the coordinates (u, v, θ, φ) such that

$$4Mu \equiv \frac{1}{2}(\bar{v} - \bar{u}) = \left(\frac{r}{2M} - 1\right)^{1/2} e^{r/4M} \cosh\left(\frac{t}{4M}\right), \tag{B.9a}$$

$$4Mv \equiv \frac{1}{2}(\bar{v} + \bar{u}) = \left(\frac{r}{2M} - 1\right)^{1/2} e^{r/4M} \sinh\left(\frac{t}{4M}\right). \tag{B.9b}$$

Then, one obtains the famous *Kruskal solution*

$$\mathrm{d}s^2 = \frac{32M^3}{r}\, e^{-r/2M}(\mathrm{d}v^2 - \mathrm{d}u^2) - r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2). \tag{B.10}$$

The penrose diagram associated to the Kruskal manifold is presented in figure (B.1). Noticeably, one sees that the maximally extended Schwarzschild solution has four distinct

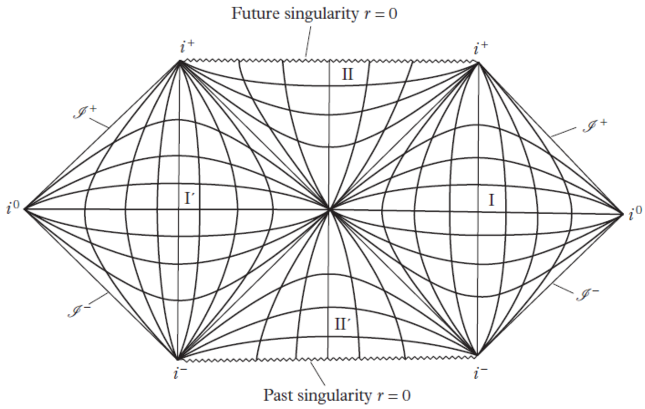


Figure B.1: The conformal Penrose diagram of the Kruskal solution. Since the coordinates (θ, φ) are suppressed, both past and future singularities $r = 0$ are depicted as lines instead of single points. The surfaces $\mathscr{I}^{\pm}$ are the future and past null infinities, whereas $i^{\pm}$ are the future and past timelike infinities. At last, i^0 is spacelike infinity. Adapted from [46].

regions, labeled from I to IV. The regions I and III are the usual exterior ($r > 2M$) and interior ($r < 2M$) regions associated to the black hole. The regions II and IV are "reversed copies" of the asymptotically flat universe formed by I and III.

Understanding this analytical extension is crucial to the quantization procedure; physically inequivalent choices for the vacuum state are closely related to the different choices of coordinate patches one can make in the Schwarzschild geometry. We shall return to this point later.

We now define an inner product on the space of solutions to (B.4). Let Σ be a spacelike hypersurface of the spacetime manifold M.[2] The volume element on Σ is $\mathrm{d}\Sigma^\mu = n^\nu \, \mathrm{d}\Sigma$, where n^μ is a future-directed unit vector orthogonal to Σ. Then, for any two solutions f and g of (B.4) with compact support, one defines [21]

$$(f, g)_{\mathsf{KG}} \equiv i \int \mathrm{d}\Sigma^\mu \sqrt{-g_\Sigma} \left[g^* (\partial_\mu f) - (\partial_\mu g^*) f \right], \tag{B.11}$$

where g_Σ denotes the induced metric on Σ, naturally. A key property of this inner product is that it is *independent of* Σ. The proof of this result is simple and follows directly from a generalization of Gauss' theorem (see §21.2.1 of [87]). It is not difficult to show that, for any test functions f, g, h and complex numbers α, β, this inner product satisfies

$$(f, g)^*_{\mathsf{KG}} = (g, f)_{\mathsf{KG}}, \tag{B.12a}$$

$$(f^*, g^*)_{\mathsf{KG}} = -(g, f)_{\mathsf{KG}}, \tag{B.12b}$$

$$(f, \alpha g + \beta h)_{\mathsf{KG}} = \alpha (f, g)_{\mathsf{KG}} + \beta (f, h)_{\mathsf{KG}}, \tag{B.12c}$$

$$(\alpha f + \beta g, h)_{\mathsf{KG}} = \alpha^* (f, h)_{\mathsf{KG}} + \beta^* (g, h)_{\mathsf{KG}}. \tag{B.12d}$$

[2]For technical reasons, it is assumed that M is a globally hyperbolic spacetime and Σ is a Cauchy surface. See chapter 4 of [125] for further mathematical details.

We now investigate the solutions to (B.4). Since the Schwarzschild solution exhibits spherical symmetry, we write the following *Ansatz*:

$$\phi(t, r, \theta, \varphi) \equiv \frac{1}{r} \sum_{m=-\ell}^{\ell} \sum_{\ell=0}^{\infty} u_{\ell m}(t, r) Y_{\ell m}(\hat{\mathbf{n}}), \tag{B.13}$$

where $Y_{\ell m}$ are the spin-0 spherical harmonics and the unit vector $\hat{\mathbf{n}}$ is a placeholder for the dependence on th angles (θ, φ). Inserting (B.13) back into (B.4), one finds a differential equation for $u_{\ell m}(t, r)$:

$$A(r)\partial_r\big[A(r)\partial_r u_{\ell m}(t, r)\big] - \partial_t^2 u_{\ell m}(t, r) - V_\ell(r) u_{\ell m}(t, r) = 0, \tag{B.14}$$

where the central potential $V_\ell(r)$ is defined as

$$V_\ell(r) \equiv A(r)\left[\frac{\ell(\ell+1)}{r^2} + \frac{2M}{r^3}\right]. \tag{B.15}$$

Switching to the tortoise coordinate r_*, a simple computation reveals that the Fourier transform $\tilde{u}_{\ell m}(\omega, r)$ satisfies a one-dimensional Schrödinger equation:

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}r_*^2} + V_\ell(r)\right] \tilde{u}_{\ell m}(\omega, r) = \omega^2 \tilde{u}_{\ell m}(\omega, r). \tag{B.16}$$

The equation (B.16) is the famous *spin-0 Regge-Wheeler (RW) equation*. One possible (non-normalized) basis for the solution space of (B.4) is the set

$$w^{\mathsf{a}}_{\omega\ell m}(x) \equiv R^{\mathsf{a}}_{\omega\ell}(r) Y_{\ell m}(\theta, \varphi)\, e^{-i\omega t}, \tag{B.17}$$

where $\mathsf{a} = (\leftarrow, \rightarrow)$ is an abstract index which distinguishes *modes* incoming from $\mathscr{I}^-$ and outgoing from $\mathcal{H}^-$, respectively. It is clear that $R^{\mathsf{a}}_{\omega\ell}(r)$ plays the same role as the function $\frac{1}{r}\,\tilde{u}_{\ell m}(\omega, r)$, i.e., up to an exponential factor, it also solves the spin-0 RW equation.

It must be stressed that the use of the word "mode" in the last paragraph is not a stylistic choice. One *defines* mode functions as those which are orthonormal with respect to the inner product (B.11), meaning that[3]

$$(u_I, u_J)_{\mathsf{KG}} = \delta_{IJ}, \tag{B.18a}$$

$$(u_I, u_J^*)_{\mathsf{KG}} = 0. \tag{B.18b}$$

In the equations above, we used capital latin indices to represent the labels a and $(\omega\ell m)$ previously used in (B.17). In the remainder of this appendix, it is assumed that the basis (B.17) satisfies (B.18).

The role of Killing vectors. The link between mode functions and Killing vectors is best understood by making a quick detour through the simplest possible scenario: a

[3] Notice that, as an immediate consequence of (B.12b), the condition (B.18a) implies $(u_I^*, u_J^*)_{\mathsf{KG}} = -\delta_{IJ}$.

scalar field $\hat{\phi}$ in Minkowski spacetime. This paragraph is based on reference [53]. Let $\varphi_{\mathbf{k}}(x) = A_k e^{-ikx} = A_k e^{i(\mathbf{k}\cdot\mathbf{x}-\omega_k t)}$ be a basis of solutions to the massless Klein-Gordon equation in Fourier space. Here, A_k is a set of constants and ω_k satisfies $\omega_k^2 = \mathbf{k}^2$. Let ω_k be positive in what follows. The quantities $\varphi_{\mathbf{k}}$ are the mode functions from which the usual field expansion of $\hat{\phi}$ is performed:

$$\hat{\phi} = \int_{\mathbf{k}} (\hat{a}_{\mathbf{k}}\varphi_{\mathbf{k}} + \hat{a}^\dagger_{\mathbf{k}}\varphi^*_{\mathbf{k}}). \tag{B.19}$$

It is not difficult to show that $(\varphi_{\mathbf{k}}, \varphi_{\mathbf{q}}) > 0$ with respect to the Klein-Gordon inner product (B.11) (specialized to the Minkowski spacetime). Thus, $\varphi_{\mathbf{k}}$ are said to be of *positive frequency modes*. Since we choose $\omega_k > 0$, the positive-frequency condition can be expressed as

$$i\partial_t \varphi_{\mathbf{k}} = \omega_k \varphi_{\mathbf{k}}, \quad \omega_k > 0. \tag{B.20}$$

Notice that $i\partial_t$ is the globally defined Killing vector of Minkowski spacetime associated to time-translation isometries. It is straightforward to show that (B.20) is a Lorentz-invariant statement. If one cannot define a Killing vector over the entire spacetime manifold, the separation of positive/negative frequency modes ceases to be unique.

Let us now return to the Schwarzschild solution. When studying the scattering of a scalar field in this background, the functions $R^{\mathsf{a}}_{\omega\ell}(r)$ admit asymptotic solutions similar to those employed in potential barrier problems in non-relativistic quantum mechanics. Since this particular setup will be relevant for the discussions on chapter 4, where the reader can find the explict form of such asymptotic solutions. We now expand the field $\hat{\phi}(x)$ as

$$\begin{aligned}\hat{\phi}(x) &= \sum_{\mathsf{a}}\sum_{\ell,m}\int_0^\infty \frac{\mathrm{d}\omega}{\sqrt{4\pi\omega}}\{w^{\mathsf{a}}_{\omega\ell m}(x)\hat{b}^{\mathsf{a}}_{\omega\ell m} + [w^{\mathsf{a}}_{\omega\ell m}(x)]^*(\hat{b}^\dagger_{\omega\ell m})^{\mathsf{a}}\} \\ &= \sum_{\mathsf{a}}\sum_{\ell,m}\int_0^\infty \frac{\mathrm{d}\omega}{\sqrt{4\pi\omega}}\{w^{\mathsf{a}}_{\omega\ell m}(x)\hat{b}^{\mathsf{a}}_{\omega\ell m} + \text{h.c}\},\end{aligned} \tag{B.21}$$

where "h.c" stands for hermitian conjugation. The (non-zero) commutation relations between the annihilation $\hat{b}^{\mathsf{a}}_{\omega\ell m}$ and creation $(\hat{b}^\dagger_{\omega\ell m})^{\mathsf{a}}$ operators are

$$\left[\hat{b}^{\mathsf{a}}_{\omega\ell m}, (\hat{b}^\dagger_{\omega'\ell'm'})^{\mathsf{a}'}\right] = \delta^{\mathsf{aa}'}\delta_{\ell\ell'}\delta_{mm'}\delta(\omega-\omega'). \tag{B.22}$$

The vacuum state $|0\rangle_{\mathrm{B}}$ defined by the condition $\hat{b}^{\mathsf{a}}_{\omega\ell m}|0\rangle_{\mathrm{B}} = 0$ is called the *Boulware* vacuum. The mode functions $u^{\mathsf{a}}_{\omega\ell m}$ are positive frequency with respect to the ∂_t Killing vector, where t is the standard Schwarzschild coordinate time. Other choices are possible.

Using the Kruskal null coordinates (U, V) on the past and future horizons, respectively, one can show that ∂_U and ∂_V are Killing vectors on such horizons. By construction, this pair of coordinates is regular on the entire Schwarzschild manifold. Basis modes can be constructed following using the standard method and the vaccum state $|0\rangle_{\mathrm{H}}$ annihilated by the resulting annihilation operator is called the Hartle-Hawking vacuum.

Alternatively, we can take incoming ($\mathsf{a} = \leftarrow$) modes to be positive frequency with respect to ∂_t and outgoing modes ($\mathsf{a} = \rightarrow$) to be negative frequency with respect to ∂_U; this set of mode functions oscillates infinitely fast on the past horizon $\mathcal{H}_-$. The vaccum states which follows from the quantization procedure is dubbed the *Unruh* vacuum.

Explicit formulae for the field expansion in both the Hartle-Hawking and Unruh vacua can be found in the seminal reference [123]. Wightman functions are determined in [31].

Bibliography

[1] B. P. ABBOTT ET AL., *Observation of gravitational waves from a binary black hole merger*, Physical Review Letters, 116 (2016).

[2] R. ABBOTT ET AL., *GWTC-3: Compact Binary Coalescences Observed by LIGO and Virgo during the Second Part of the Third Observing Run*, Phys. Rev. X, 13 (2023).

[3] G. AGAZIE ET AL., *The NANOGrav 15-year Data Set: Evidence for a Gravitational-Wave Background*, Astrophys. J. Lett., 951 (2023).

[4] G. L. ALMEIDA, *Classical Amplitudes in Gravitational-Wave Physics*, PhD thesis, Universidade Federal do Rio Grande do Norte, 2023.

[5] G. L. ALMEIDA, S. FOFFA, AND R. STURANI, *Gravitational multipole renormalization*, Phys. Rev. D, 104 (2021).

[6] ——, *Tail contributions to gravitational conservative dynamics*, Phys. Rev. D, 104 (2021).

[7] G. L. ALMEIDA, A. MÜLLER, S. FOFFA, AND R. STURANI, *Conservative binary dynamics from gravitational tail emission processes*, Phys. Rev. D, 108 (2023).

[8] G. L. ALMEIDA AND S.-Y. ZHOU, *Post-Newtonian dynamics of spinning black hole binaries in Einstein-scalar-Gauss-Bonnet gravity*, Phys. Rev. D, 110 (2024).

[9] L. AMALBERTI, Z. YANG, AND R. A. PORTO, *Gravitational radiation from inspiralling compact binaries to N^3LO in the Effective Field Theory approach*, Phys. Rev. D, 110 (2024).

[10] P. AMARO-SEOANE ET AL., *Laser Interferometer Space Antenna*, 1702.00786 [astro.ph.IM], (2017).

[11] R. AOUDE, A. CRISTOFOLI, A. ELKHIDIR, AND M. SERGOLA, *Inelastic Coupled-Channel Eikonal Scattering*, 2411.02294 [hep-th], (2024).

[12] L. BARACK ET AL., *Comparison of post-Minkowskian and self-force expansions: Scattering in a scalar charge toy model*, Phys. Rev. D, 108 (2023).

[13] D. BAUMANN, *TASI Lectures on Inflation*, in Theoretical Advanced Study Institute in Elementary Particle Physics: Physics of the Large and the Small, 2011.

[14] Y. F. BAUTISTA, Y.-T. HUANG, AND J.-W. KIM, *Absorptive effects in black hole scattering*, Phys. Rev. D, 111 (2025).

[15] Y. F. BAUTISTA, M. KHALIL, M. SERGOLA, C. KAVANAGH, AND J. VINES, *Post-Newtonian observables for aligned-spin binaries to sixth order in spin from gravitational self-force and Compton amplitudes*, 2408.01871 [gr-qc], (2024).

[16] Z. BERN, C. CHEUNG, R. ROIBAN, C.-H. SHEN, M. P. SOLON, AND M. ZENG, *Black Hole Binary Dynamics from the Double Copy and Effective Theory*, JHEP, 10 (2019).

[17] L. BERNARD, L. BLANCHET, A. BOHÉ, G. FAYE, AND S. MARSAT, *Fokker action of nonspinning compact binaries at the fourth post-Newtonian approximation*, Phys. Rev. D, 93 (2016).

[18] ——, *Energy and periastron advance of compact binaries on circular orbits at the fourth post-Newtonian order*, Phys. Rev. D, 95 (2017).

[19] D. BINI AND T. DAMOUR, *Analytical determination of the two-body gravitational interaction potential at the fourth post-Newtonian approximation*, Phys. Rev. D, 87 (2013).

[20] T. BINNINGTON AND E. POISSON, *Relativistic theory of tidal Love numbers*, Phys. Rev., D80 (2009).

[21] N. D. BIRRELL AND P. C. W. DAVIES, *Quantum Fields in Curved Space*, Cambridge Monographs on Mathematical Physics, Cambridge University Press, Cambridge, UK, 1982.

[22] N. E. J. BJERRUM-BOHR, P. H. DAMGAARD, L. PLANTE, AND P. VANHOVE, *The SAGEX Review on Scattering Amplitudes, Chapter 13: Post-Minkowskian expansion from Scattering Amplitudes*, J. Phys. A, 55 (2022).

[23] L. BLANCHET, *Post-Newtonian Theory for Gravitational Waves*, Living Rev. Rel., 17 (2014).

[24] L. BLANCHET, G. FAYE, Q. HENRY, F. LARROUTUROU, AND D. TRESTINI, *Gravitational Wave Flux and Quadrupole Modes from Quasi-Circular Non-Spinning Compact Binaries to the Fourth Post-Newtonian Order*, Phys. Rev. D, 108 (2023).

[25] ——, *Gravitational-Wave Phasing of Quasicircular Compact Binary Systems to the Fourth-and-a-Half Post-Newtonian Order*, Phys. Rev. Lett., 131 (2023).

[26] J. BLÜMLEIN, A. MAIER, P. MARQUARD, AND G. SCHÄFER, *Gravity in binary systems at the fifth and sixth post-Newtonian order*, PoS, LL2022 (2022).

[27] A. BRANDHUBER, G. R. BROWN, G. CHEN, S. DE ANGELIS, J. GOWDY, AND G. TRAVAGLINI, *One-loop Gravitational Bremsstrahlung and Waveforms from a Heavy-Mass Effective Field Theory*, JHEP, 06 (2023).

[28] C. P. BURGESS, *Introduction to Effective Field Theory*, Cambridge University Press, 12 2020.

[29] W. L. BURKE, *Gravitational Radiation Damping of Slowly Moving Systems Calculated Using Matched Asymptotic Expansions*, J. Math. Phys., 12 (1971).

[30] F. W. BYRON AND R. W. FULLER, *Mathematics of Classical and Quantum Physics*, Dover, 1992.

[31] P. CANDELAS, *Vacuum Polarization in Schwarzschild Space-Time*, Phys. Rev. D, 21 (1980).

[32] E. CAPOTE ET AL., *Advanced LIGO detector performance in the fourth observing run*, 2411.14607 [gr-qc], (2024).

[33] S. CHAKRABARTI, T. DELSATE, AND J. STEINHOFF, *Effective action and linear response of compact objects in Newtonian gravity*, Phys. Rev. D, 88 (2013).

[34] P. CHARALAMBOUS, S. DUBOVSKY, AND M. M. IVANOV, *On the Vanishing of Love Numbers for Kerr Black Holes*, JHEP, 05 (2021).

[35] ——, *Love symmetry*, JHEP, 10 (2022).

[36] P. L. CHRZANOWSKI, R. A. MATZNER, V. D. SANDBERG, AND M. P. RYAN, *Zero Mass Plane Waves in Nonzero Gravitational Backgrounds*, Phys. Rev. D, 14 (1976), pp. 317–326.

[37] O. COMBALUZIER-SZTEINSZNAIDER, L. HUI, L. SANTONI, A. R. SOLOMON, AND S. S. C. WONG, *Symmetries of Vanishing Nonlinear Love Numbers of Schwarzschild Black Holes*, 2410.10952 [gr-qc], (2024).

[38] C. CUTLER ET AL., *The Last Three Minutes: Issues in Gravitational Wave Measurements of Coalescing Compact Binaries*, Phys. Rev. Lett., 70 (1993).

[39] T. DAMOUR, P. JARANOWSKI, AND G. SCHÄFER, *Nonlocal-in-time action for the fourth post-Newtonian conservative dynamics of two-body systems*, Phys. Rev. D, 89 (2014).

[40] ——, *Conservative dynamics of two-body systems at the fourth post-Newtonian approximation of general relativity*, Phys. Rev. D, 93 (2016).

[41] T. DAMOUR AND A. NAGAR, *Relativistic tidal properties of neutron stars*, Phys. Rev., D80 (2009).

[42] V. DE LUCA, A. GAROFFOLO, J. KHOURY, AND M. TRODDEN, *Tidal Love numbers and Green's functions in black hole spacetimes*, Phys. Rev. D, 110 (2024).

[43] B. S. DEWITT, *Quantum Field Theory in Curved Space-Time*, Phys. Rept., 19 (1975), pp. 295–357.

[44] P. DI VECCHIA, C. HEISSENBERG, R. RUSSO, AND G. VENEZIANO, *The Eikonal Approach to Gravitational Scattering and Radiation at $\mathcal{O}(G^3)$*, JHEP, 07 (2021).

[45] R. F. DIEDRICHS, D. SCHMITT, AND L. SAGUNSKI, *Binary systems in massive scalar-tensor theories: Next-to-leading order gravitational wave phase from effective field theory*, Phys. Rev. D, 110 (2024).

[46] R. D'INVERNO AND J. VICKERS, *Introducing Einstein's Relativity: A deeper understanding*, Oxford University Press, 2022.

[47] E. E. FLANAGAN AND T. HINDERER, *Constraining neutron star tidal Love numbers with gravitational wave detectors*, Phys. Rev. D, 77 (2008).

[48] E. E. FLANAGAN AND S. A. HUGHES, *The Basics of Gravitational Wave Theory*, New Journal of Physics, 7 (2005).

[49] S. FOFFA, R. A. PORTO, I. ROTHSTEIN, AND R. STURANI, *Conservative dynamics of binary systems to fourth Post-Newtonian order in the EFT approach II: Renormalized Lagrangian*, Phys. Rev. D, 100 (2019).

[50] S. FOFFA AND R. STURANI, *Effective field theory methods to model compact binaries*, Class. Quant. Grav., 31 (2014).

[51] ——, *Conservative dynamics of binary systems to fourth Post-Newtonian order in the EFT approach I: Regularized Lagrangian*, Phys. Rev. D, 100 (2019).

[52] ——, *Near and far zones in two-body dynamics: An effective field theory perspective*, Phys. Rev. D, 104 (2021).

[53] E. FRODDEN AND N. VALDÉS, *UNRUH EFFECT: Introductory Notes to Quantum Effects for Accelerated Observers*, Int. J. Mod. Phys. A, 33 (2018).

[54] C. R. GALLEY, A. K. LEIBOVICH, R. A. PORTO, AND A. ROSS, *Tail effect in gravitational radiation reaction: Time nonlocality and renormalization group evolution*, Phys. Rev. D, 93 (2016).

[55] C. R. GALLEY AND M. TIGLIO, *Radiation reaction and gravitational waves in the effective field theory approach*, Phys. Rev. D, 79 (2009).

[56] J. B. GILMORE AND A. ROSS, *Effective field theory calculation of second post-Newtonian binary dynamics*, Phys. Rev. D, 78 (2008).

[57] W. D. GOLDBERGER, J. LI, AND I. Z. ROTHSTEIN, *Non-conservative effects on spinning black holes from world-line effective field theory*, JHEP, 06 (2021).

[58] W. D. GOLDBERGER AND A. ROSS, *Gravitational radiative corrections from effective field theory*, Phys. Rev. D, 81 (2010).

[59] W. D. GOLDBERGER AND I. Z. ROTHSTEIN, *An Effective field theory of gravity for extended objects*, Phys. Rev. D, 73 (2006).

[60] ——, *Dissipative effects in the worldline approach to black hole dynamics*, Phys. Rev. D, 73 (2006).

[61] ——, *An Effective Field Theory of Quantum Mechanical Black Hole Horizons*, JHEP, 04 (2020).

[62] N. GÜRLEBECK, *No-hair theorem for Black Holes in Astrophysical Environments*, Phys. Rev. Lett., 114 (2015).

[63] F. A. HANDLER AND R. A. MATZNER, *Gravitational wave scattering*, Phys. Rev. D, 22 (1980), pp. 2331–2348.

[64] B. R. HOLSTEIN AND J. F. DONOGHUE, *Classical Physics and Quantum Loops*, Phys. Rev. Lett., 93 (2004).

[65] L. HUI, A. JOYCE, R. PENCO, L. SANTONI, AND A. R. SOLOMON, *Static response and Love numbers of Schwarzschild black holes*, JCAP, 04 (2021).

[66] ——, *Ladder Symmetries of Black Holes. Implications for Love Numbers and No-Hair Theorems*, JCAP, 01 (2022).

[67] M. M. IVANOV, Y.-Z. LI, J. PARRA-MARTINEZ, AND Z. ZHOU, *Gravitational Raman Scattering in Effective Field Theory: A Scalar Tidal Matching at $\mathcal{O}(G^3)$*, Phys. Rev. Lett., 132 (2024).

[68] M. M. IVANOV AND Z. ZHOU, *Revisiting the matching of black hole tidal responses: A systematic study of relativistic and logarithmic corrections*, Phys. Rev. D, 107 (2023).

[69] P. JARANOWSKI AND G. SCHÄFER, *Dimensional regularization of local singularities in the 4th post-Newtonian two-point-mass Hamiltonian*, Phys. Rev. D, 87 (2013).

[70] ——, *Derivation of local-in-time fourth post-Newtonian ADM Hamiltonian for spinless compact binaries*, Phys. Rev. D, 92 (2015).

[71] C. R. T. JONES AND M. S. RUF, *Absorptive effects and classical black hole scattering*, JHEP, 03 (2024).

[72] A. KEHAGIAS AND A. RIOTTO, *Black Holes in a Gravitational Field: The Non-linear Static Love Number of Schwarzschild Black Holes Vanishes*, 2410.11014 [gr-qc], (2024).

[73] L. V. KELDYSH, *Diagram technique for nonequilibrium processes*, Zh. Eksp. Teor. Fiz., 47 (1964), pp. 1515–1527.

[74] D. KENNEFICK, *Traveling at the Speed of Thought: Einstein and the Quest for Gravitational Waves*, Princeton University Press, 2007.

[75] D. KENNEFICK, *Relativistic lighthouses: The role of the binary pulsar in proving the existence of gravitational waves*, 1407.2164 [physics.hist-ph], (2014).

[76] B. KOL AND M. SMOLKIN, *Black hole stereotyping: Induced gravito-static polarization*, JHEP, 02 (2012), p. 010.

[77] D. A. KOSOWER, B. MAYBEE, AND D. O'CONNELL, *Amplitudes, Observables, and Classical Scattering*, JHEP, 02 (2019), p. 137.

[78] D. A. KOSOWER, R. MONTEIRO, AND D. O'CONNELL, *The SAGEX Review on Scattering Amplitudes, Chapter 14: Classical Gravity from Scattering Amplitudes*, J. Phys. A, 55 (2022).

[79] S. J. KOVACS AND K. S. THORNE, *The Generation of Gravitational Waves. 4. Bremsstrahlung*, Astrophys. J., 224 (1978).

[80] G. KÄLIN AND R. A. PORTO, *Post-Minkowskian Effective Field Theory for Conservative Binary Dynamics*, JHEP, 11 (2020).

[81] A. LE TIEC, M. CASALS, AND E. FRANZIN, *Tidal Love Numbers of Kerr Black Holes*, Phys. Rev. D, 103 (2021).

[82] L. LEHNER AND F. PRETORIUS, *Numerical Relativity and Astrophysics*, Ann. Rev. Astron. Astrophys., 52 (2014).

[83] M. LEVI, *Effective Field Theories of Post-Newtonian Gravity: A comprehensive review*, Rept. Prog. Phys., 83 (2020).

[84] J. LUO ET AL., *Progress of the TianQin project*, 2502.11328 [gr.qc], (2025).

[85] M. MAGGIORE, *A Modern Introduction to Quantum Field Theory*, Oxford University Press, 2005.

[86] ——, *Gravitational Waves: Theory and Experiments*, vol. 1, Oxford University Press, 2008.

[87] ——, *Gravitational Waves: Astrophysics and Cosmology*, vol. 2, Oxford University Press, 2018.

[88] ——, *A Modern Introduction to Classical Electrodynamics*, Oxford University Press, 2023.

[89] V. MICHEL AND K. SEIBERT, *A Mathematical View on Spin-Weighted Spherical Harmonics and Their Applications in Geodesy*, Springer, Berlin, Heidelberg, 2019, pp. 1–113.

[90] S. MIZERA, *Physics of the Analytic S-Matrix*, Physics Reports, 1047 (2024).

[91] G. MOGULL, J. PLEFKA, AND J. STEINHOFF, *Classical black hole scattering from a worldline quantum field theory*, JHEP, 02 (2021).

[92] A. MONTEVERDI AND E. WINSTANLEY, *Some Addition Theorems for Spin-Weighted Spherical Harmonics*, Universe, 10 (2024), p. 461.

[93] D. N. Page, *Particle emission rates from a black hole. II. Massless particles from a rotating hole*, Phys. Rev. D, 14 (1976).

[94] ——, *Particle emission rates from a black hole: Massless particles from an uncharged, nonrotating hole*, Phys. Rev., D13 (1976), pp. 198–206.

[95] G. Pappas and T. A. Apostolatos, *Effectively universal behavior of rotating neutron stars in general relativity makes them even simpler than their Newtonian counterparts*, Phys. Rev. Lett., 112 (2014).

[96] M. Perry and M. J. Rodriguez, *Dynamical Love Numbers for Kerr Black Holes*, 2310.03660 [gr-qc], (2023).

[97] M. E. Peskin, *An Introduction to Quantum Field Theory*, CRC Press, 2018.

[98] E. Poisson, A. Pound, and I. Vega, *The motion of point particles in curved spacetime*, Living Rev. Rel., 14 (2011).

[99] E. Poisson and M. Sasaki, *Gravitational radiation from a particle in circular orbit around a black hole. 5: Black hole absorption and tail corrections*, Phys. Rev. D, 51 (1995).

[100] E. Poisson and C. M. Will, *Gravity: Newtonian, Post-Newtonian, Relativistic*, Cambridge University Press, 2014.

[101] J. Polchinski, *Effective Field Theory and the Fermi Surface*, in Theoretical Advanced Study Institute (TASI 92): From Black Holes and Strings to Particles, 6 1992, pp. 0235–276.

[102] R. A. Porto, *Post-Newtonian corrections to the motion of spinning bodies in NRGR*, Phys. Rev. D, 73 (2006).

[103] ——, *Absorption effects due to spin in the worldline approach to black hole dynamics*, Phys. Rev. D, 77 (2008).

[104] R. A. Porto, *The Effective Field Theorist's Approach to Gravitational Dynamics*, Physics Reports, 633 (2016).

[105] R. A. Porto, *The Tune of Love and the Nature(ness) of Spacetime*, Fortsch. Phys., 64 (2016), pp. 723–729.

[106] R. A. Porto, M. M. Riva, and Z. Yang, *Nonlinear Gravitational Radiation Reaction: Failed Tail, Memories & Squares*, 2409.05860 [gr-qc], (2024).

[107] R. A. Porto and I. Z. Rothstein, *Apparent ambiguities in the post-Newtonian expansion for binary systems*, Phys. Rev. D, 96 (2017).

[108] A. Pound and B. Wardell, *Black hole perturbation theory and gravitational self-force*, Handbook of Gravitational Wave Astronomy, (2022), pp. 1411–1529.

[109] M. Punturo et al., *The Einstein Telescope: A third-generation gravitational wave observatory*, Class. Quant. Grav., 27 (2010).

[110] T. Regge and J. A. Wheeler, *Stability of a Schwarzschild singularity*, Phys. Rev., 108 (1957), pp. 1063–1069.

[111] D. Reitze et al., *Cosmic Explorer: The U.S. Contribution to Gravitational-Wave Astronomy beyond LIGO*, Bull. Am. Astron. Soc., 51 (2019), p. 035.

[112] M. M. Riva, L. Santoni, N. Savić, and F. Vernizzi, *Vanishing of nonlinear tidal Love numbers of Schwarzschild black holes*, Phys. Lett. B, 854 (2024).

[113] R. F. Rosato and P. Pani, *On the universality of late-time ringdown tails*, 2505.08877 [gr-qc], (2025).

[114] A. Ross, *Multipole expansion at the level of the action*, Phys. Rev. D, 85 (2012).

[115] I. Z. Rothstein, *Progress in effective field theory approach to the binary inspiral problem*, Gen. Rel. Grav., 46 (2014).

[116] L. Schwartz, *Mathematics for the Physical Sciences*, Addison-Wesley, 1966.

[117] M. D. Schwartz, *Quantum Field Theory and the Standard Model*, Cambridge University Press, 3 2014.

[118] J. S. Schwinger, *Brownian motion of a quantum oscillator*, J. Math. Phys., 2 (1961), pp. 407–432.

[119] A. A. Starobinsky, *Amplification of waves reflected from a rotating "black hole."*, Sov. Phys. JETP, 37 (1973).

[120] R. Sturani, *Fundamental Gravity and Gravitational Waves*, Symmetry, 13 (2021).

[121] G. Teschl, *Mathematical Methods in Quantum Mechanics*, vol. 157 of Graduate Studies in Mathematics, AMS, 2014.

[122] K. S. Thorne, *Nonradial Pulsation of General-Relativistic Stellar Models. IV. The Weak-Field Limit*, Astrophys. J., 158 (1969).

[123] W. G. Unruh, *Notes on black-hole evaporation*, Phys. Rev. D, 14 (1976), p. 870.

[124] G. Vidal, G. M. Dantas, R. Sturani, and G. Menezes, *Gravitational Polarizability of Schwarzschild Black Holes*, 2410.23384 [gr-qc], (2024).

[125] R. M. Wald, *Quantum Field Theory in Curved Spacetime and Black Hole Thermodynamics*, Chicago Lectures in Physics, University of Chicago Press, Chicago, IL, 1994.

[126] K. Yagi and N. Yunes, *I-Love-Q Relations in Neutron Stars and their Applications to Astrophysics, Gravitational Waves and Fundamental Physics*, Phys. Rev. D, 88 (2013).

[127] W. Zhou and H. Yu, *Spontaneous excitation of a static multilevel atom coupled with electromagnetic vacuum fluctuations in Schwarzschild spacetime*, Class. Quant. Grav., 29 (2012).

[128] Z. Zhu, H. W. Yu, and S. Lu, *Spontaneous excitation of an accelerated hydrogen atom coupled with electromagnetic vacuum fluctuations*, Phys. Rev. D, 73 (2006).

[129] R. Zwicky, *A brief Introduction to Dispersion Relations and Analyticity*, in Quantum Field Theory at the Limits: from Strong Fields to Heavy Quarks, 2017, pp. 93–120.