Relativistic three-particle scattering equations

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We derive a set of relativistic three-particle scattering equations in the three-particle c.m. frame employing a relativistic three-particle propagator suggested long ago by Ahmadzadeh and Tjon in the c.m. frame of a two-particle subsystem. We make the coordinate transformation of this propagator from the c.m. frame of the two-particle subsystem to the three-particle c.m. frame. We also point out that some numerical applications of the Ahmadzadeh and Tjon propagator to the three-nucleon problem use unnecessary nonrelativistic approximations which do not simplify the computational task, but violate constraints of relativistic unitarity and/or covariance.

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There are a variety of approaches for writing three-particle relativistic scattering equations [1–7]. About 25 years ago in a classic paper Ahmadzadeh and Tjon [8] (AT) suggested a relativistic three-particle propagator in the center of momentum (c.m.) frame of a two-particle subsystem of the three particles [their Eq. (2.12)], using a dispersion relation in the two-particle invariant energy, to be used in relativistic three-particle scattering equations. The AT propagator in the two-particle c.m. frame satisfies constraints of relativistic covariance and unitarity. Here we derive a relativistic three-particle scattering equation using the transformed AT propagator in the three-particle c.m. frame.

Also, there have been several numerical calculations on studies of the relativistic effect in the three-nucleon system using the AT propagator [9–13]. The original AT propagator was derived in the c.m. frame of the two-particle subsystem. However, these numerical calculations employed relativistic three-particle scattering equations in the three-particle c.m. frame. One needs to transform the AT propagator to the three-particle c.m. frame for this purpose. The three-particle relativistic propagators used in these numerical calculations [9–13] involve unwanted and unnecessary nonrelativistic approximations, which do not facilitate the computational task but violate constraints of relativistic unitarity and/or covariance.

Relativistic three-particle scattering equations used in these studies are diagramatically represented in Fig. 1. Each single line represents propagation of a particle, and each double line represents propagation of the bound state or isobar of two particles. The labels in the figure represent the four-momentum of the propagating particle; P is the total four-momentum of the system of three particles in the three-particle c.m. frame and is given by $(\sqrt{s}, 0, 0, 0)$. Usual relativistic three-particle scattering equations employ a three-dimensional reduction of this equation in the form

$$T(\mathbf{q},\mathbf{k},s) = 2B(\mathbf{q},\mathbf{k},s) + \frac{2}{(2\pi)^3}\int \frac{d\mathbf{p}}{2\omega_{\mathbf{p}}} B(\mathbf{q},\mathbf{p},s)\tau(\sigma_p)T(\mathbf{p},\mathbf{k},s). \tag{1}$$

Here we consider three equal-mass particles of mass m, so that $\omega_{\mathbf{p}} = (|\mathbf{p}|^2 + m^2)^{1/2}$, etc. In Eq. (1) T is the t matrix for scattering of a particle from the bound state or isobar of two particles, τ is the dressed propagator for the two-particle bound state or the isobar, and g's are the vertex function for these bound states or isobars.

The AT three-particle propagator is given, in the c.m. frame of the two-particle subsystem, with two particles of momentum $\mathbf{p}$ and $-(\mathbf{p}+\mathbf{q})$ in the three-particle c.m. frame (see, Fig. 1), respectively, by [Eq. (2.12) of AT]

$$G = \frac{\hat\omega_{\mathbf{p}} + \hat\omega_{\mathbf{p}+\mathbf{q}}}{\hat\omega_{\mathbf{p}}\hat\omega_{\mathbf{p}+\mathbf{q}}} \frac{2}{(\sqrt{s}-\omega_{\mathbf{q}})^2 - |\mathbf{q}|^2 - (\hat\omega_{\mathbf{p}} + \hat\omega_{\mathbf{p}+\mathbf{q}})^2 + i0}. \tag{2}$$

Here we have changed the i, j, k labels of AT to explicit momentum labels for identifying particles. The carets over the ω's mean that the corresponding energies are to be calculated in the two-particle c.m. frame. For two equal-mass particles these $\hat\omega$'s are given by Eq. (5) below.

The relativistic relative momentum of the two particles with momentum $\mathbf{p}$ and $-(\mathbf{p}+\mathbf{q})$ (for the upper vertex in the homegeneous term of Fig. 1) is given by [13, 14]

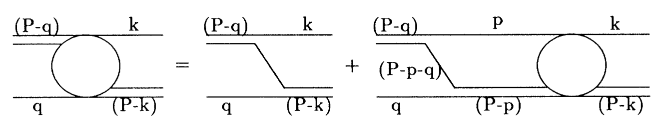

FIG. 1. The diagramatic form of the three-particle scattering equation. Each single-line represents propagation of a particle, each double line represents propagation of the bound state or isobar of two particles.

$$\boldsymbol{\mathcal{P}} = \mathbf{p} + \rho(|\mathbf{p}|, |\mathbf{q}|, \theta)\mathbf{q}. \tag{3}$$

Note that $\boldsymbol{\mathcal{P}}$ is also the momentum of a particle in the two-particle c.m. system. Here θ is the angle between $\mathbf{p}$ and $\mathbf{q}$ and the function ρ is given by [13, 14]

$$\rho(|\mathbf{p}|, |\mathbf{q}|, \theta) = \xi_q^{-1/2}\left(\omega_{\mathbf{p}} + \frac{\mathbf{p}.\mathbf{q}}{\xi_q^{1/2} + \omega_{\mathbf{p}} + \omega_{\mathbf{p+q}}}\right), \tag{4}$$

where $\xi_q = (\omega_{\mathbf{p}} + \omega_{\mathbf{p+q}})^2 - |\mathbf{q}|^2$. Note that in the nonrelativistic limit $\rho = 1/2$ and one has the usual relative momentum $\boldsymbol{\mathcal{P}} = \mathbf{p} + \mathbf{q}/2$. A completely analogous expression exists for $\mathbf{p}$ and $\mathbf{q}$ interchanged.

In the c.m. frame of the two-particle subsystem, where Eq. (2) is valid, obviously the two particles have equal and opposite momentum $\boldsymbol{\mathcal{P}}$ and $-\boldsymbol{\mathcal{P}}$, given by Eq. (3), and consequently for two equal-mass particles,

$$\hat{\omega}_{\mathbf{p}} = \hat{\omega}_{\mathbf{p+q}} = (|\boldsymbol{\mathcal{P}}|^2 + m^2)^{1/2} \equiv \omega_{\boldsymbol{\mathcal{P}}}. \tag{5}$$

As a result the AT propagator is rewritten as

$$G = \frac{2}{\omega_{\boldsymbol{\mathcal{P}}}}\frac{2}{(\sqrt{s} - \omega_{\mathbf{q}})^2 - |\mathbf{q}|^2 - 4\omega_{\boldsymbol{\mathcal{P}}}^2 + i0}. \tag{6}$$

This propagator is, however, to be evaluated in the c.m. system of the two-particle subsystem. The relevant phase space in this system is given by $d\boldsymbol{\mathcal{P}}/(2\pi)^3$. Consequently, the AT three-dimensional reduction of the homogeneous part of the equation depicted in Fig. 1, in the c.m. frame of the two-particle subsystem, is given by

$$2B\tau T = \frac{2}{(2\pi)^3}\int \frac{d\boldsymbol{\mathcal{P}}}{2\omega_{\boldsymbol{\mathcal{P}}}}\frac{2}{(\sqrt{s} - \omega_{\mathbf{q}})^2 - |\mathbf{q}|^2 - 4\omega_{\boldsymbol{\mathcal{P}}}^2 + i0} g\tau g T. \tag{7}$$

In Eq. (7) the vertex function g, the pair propagator τ, and the t matrix T are to be expressed in terms of the momentum variables in the c.m. frame of the two-particle subsystem.

Usually, the relativistic three-particle scattering equations are conveniently written in the three-particle c.m. system. For that purpose one has to make a transformation of variables $\boldsymbol{\mathcal{P}} \to \mathbf{p}$ in Eq. (7) where $\boldsymbol{\mathcal{P}}$ and $\mathbf{p}$ are related by Eq. (3). As a first step of making this transformation we note that the expression for ρ given by Eq. (4) can be conveniently rewritten in the following two useful forms:

$$\rho(|\mathbf{p}|, |\mathbf{q}|, \theta) = \frac{\sqrt{\xi_q} + 2\omega_{\mathbf{p}}}{2(\sqrt{\xi_q} + \omega_{\mathbf{p}} + \omega_{\mathbf{p+q}})} \tag{8}$$

$$= \frac{1}{2} - \frac{1}{2|\mathbf{q}|^2}(\omega_{\mathbf{p}} - \omega_{\mathbf{p+q}})(\sqrt{\xi_q} - \omega_{\mathbf{p}} - \omega_{\mathbf{p+q}}). \tag{9}$$

With ρ given by Eq. (8) the square of $\boldsymbol{\mathcal{P}}$ of Eq. (3) is given, after some straightforward algebra, by [11]

$$\boldsymbol{\mathcal{P}}^2 = \frac{1}{4}(\omega_{\mathbf{p}} + \omega_{\mathbf{p+q}})^2 - \frac{1}{4}|\mathbf{q}|^2 - m^2. \tag{10}$$

Consequently, one has

$$4\omega_{\boldsymbol{\mathcal{P}}}^2 = (\omega_{\mathbf{p}} + \omega_{\mathbf{p+q}})^2 - |\mathbf{q}|^2. \tag{11}$$

The Jacobian J of the transformation $\boldsymbol{\mathcal{P}} \to \mathbf{p}$ is given by

$$J \equiv \det\left|\frac{d\boldsymbol{\mathcal{P}}}{d\mathbf{p}}\right| = 1 + (\mathbf{q}\cdot\boldsymbol{\nabla}_{\mathbf{p}})\rho, \tag{12}$$

where $\boldsymbol{\nabla}_{\mathbf{p}}$ is the gradient with respect to $\mathbf{p}$. Using Eqs. (3) and (9) this Jacobian is evaluated after some straightforward algebra to yield

$$J = \frac{1}{2}\frac{\omega_{\mathbf{p}} + \omega_{\mathbf{p+q}}}{\omega_{\mathbf{p}}\omega_{\mathbf{p+q}}}\omega_{\boldsymbol{\mathcal{P}}}. \tag{13}$$

In arriving at Eq. (13) we have made use of the identity $\sqrt{\xi_q} = 2\omega_{\boldsymbol{\mathcal{P}}}$.

Using Eqs. (11) and (13) , Eq. (7) reduces to

$$2B\tau T = \frac{2}{(2\pi)^3}\int \frac{d\mathbf{p}}{2\omega_{\mathbf{p}}}\frac{(\omega_{\mathbf{p}} + \omega_{\mathbf{p+q}})}{\omega_{\mathbf{p+q}}[(\sqrt{s} - \omega_{\mathbf{q}})^2 - (\omega_{\mathbf{p}} + \omega_{\mathbf{p+q}})^2 + i0]} g\tau(\sigma_p) g T(\mathbf{p}, \mathbf{k}; s). \tag{14}$$

Comparing Eqs. (1) and (14) we identify the Born term of Eq. (1) as

$$B(\mathbf{q}, \mathbf{p}, s) = \frac{g(\omega_{\mathbf{p}} + \omega_{\mathbf{p+q}})g}{\omega_{\mathbf{p+q}}[(\sqrt{s} - \omega_{\mathbf{q}})^2 - (\omega_{\mathbf{p}} + \omega_{\mathbf{p+q}})^2 + i0]}. \tag{15}$$

Note that the denominator in Eq. (14) has two poles. One of them given by $\sqrt{s} = (\omega_{\mathbf{p}} + \omega_{\mathbf{q}} + \omega_{\mathbf{p+q}})$ refers to the propagation of three particles in the intermediate state and is responsible for maintaining the three-particle unitarity. The other pole, given by $\sqrt{s} = (\omega_{\mathbf{q}} - \omega_{\mathbf{p}} - \omega_{\mathbf{p+q}})$, represents the propagation of a particle and two antiparticles, and does not contribute to three-particle unitarity.

Hence the AT propagator should reduce to Eqs. (1) and (15) in the three-particle c.m. system. The numerical applications of Refs. [9–13], however, do not use Eqs. (1) and (15) in the c.m. frame of the three-particle system. For example, Jackson and Tjon [9], Hammel *et al.* [10], Rupp and Tjon [12], and Sammarruca *et al.* [13] use Eq. (7) with the Jacobian set equal to unity, instead of given by Eq. (13). Explicitly they used

$$2B\tau T = \frac{2}{(2\pi)^3}\int \frac{d\mathbf{p}}{2\omega_{\mathbf{p}}}\frac{2}{(\sqrt{s}-\omega_{\mathbf{q}})^2-|\mathbf{q}|^2-4\omega_{\mathbf{p}}^2+i0}g\tau(\sigma_p)gT(\mathbf{p},\mathbf{k};s), \tag{16}$$

in place of Eq. (14). Note that approximation $J=1$ is a nonrelativistic approximation. The three-particle propagator of formulation (16) has the correct pole, given by $\sqrt{s}=(\omega_{\mathbf{p}}+\omega_{\mathbf{q}}+\omega_{\mathbf{p}+\mathbf{q}})$, for the propagation of three particles in the intermediate state provided that one uses the relativistic expression (11) for $\omega_{\mathbf{P}}$ in Eq. (16). However, in actual calculation the nonrelativistic approximation $\omega_{\mathbf{P}} \simeq (m^2+|\mathbf{p}+\mathbf{q}/2|^2)^{1/2}$ has been used. (See, for example, Eq. (54) of Ref. [12] and Eq. (4) of Ref. [13].) This approximation in the propagator will destroy the correct pole for three-particle propagation in the intermediate state, and the resultant equation will not have the correct relativistic threshold for scattering. Even if the proper relativistic expression (11) is used, Eq. (16) will imply a wrong residue at the pole for propagation of three particles implying violation of unitarity. Equation (16) is a nonrelativistic approximation to Eqs. (1) and (15), which violates constraints of relativistic unitarity and/or covariance.

Garcilazo *et al.* [11] used the following expression:

$$2B\tau T = \frac{2}{(2\pi)^3}\int \frac{d\mathbf{p}}{2\omega_{\mathbf{p}}}\frac{[(\omega_{\mathbf{p}}+\omega_{\mathbf{p}+\mathbf{q}})^2-|\mathbf{q}|^2]^{1/2}}{\omega_{\mathbf{p}+\mathbf{q}}[(\sqrt{s}-\omega_{\mathbf{q}})^2-(\omega_{\mathbf{p}}+\omega_{\mathbf{p}+\mathbf{q}})^2+i0]}g\tau(\sigma_p)gT(\mathbf{p},\mathbf{k};s), \tag{17}$$

in place of Eq. (14). This specific application of the AT propagator is distinct from Eq. (16), used in Refs. [9, 12, 13], and uses another unnecessary approximation to the Jacobian. This approximation will have the correct pole position for the propagation of three particles in the intermediate state but will have the wrong residue. This will also imply violations of conditions of relativistic unitarity.

It should be noted that the unwanted and unnecessary approximations made in these numerical calculations do not in any way facilitate the computational task. It is equally easy to use the exact equations (1) and (15) in numerical applications.

In summary, we have derived the set of relativistic three-particle equations (1) and (15) using the AT propagator [8] in the three-particle c.m. system. Originally, the AT propagator was suggested in the c.m. frame of the two-particle subsystem. Here we provide the necessary transformation of the AT propagator to the three-particle c.m. system. We point out that the numerical applications of this equation [9–13], however, use unwanted and unnecessary nonrelativistic approximations which do not facilitate the computational task but violate constraints of relativistic unitarity and/or covariance.

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