

# Bright vortex solitons in Bose Condensates

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**Abstract.** We suggest the possibility of observing and studying bright vortex solitons in attractive Bose-Einstein condensates in three dimensions with a radial trap. Such systems lie on the verge of critical stability and we discuss the conditions of their stability. We study the interaction between two such solitons. Unlike the text-book solitons in one dimension, the interaction between two radially trapped and axially free three-dimensional solitons is inelastic in nature and involves exchange of particles and deformation in shape. The interaction remains repulsive for all phase  $\delta$  between them except for  $\delta \approx 0$ .

## 1 Introduction

Solitary waves or solitons are a consequence of nonlinear dynamics. A classic text-book example of soliton appears in the following one-dimensional nonlinear free Schrödinger equation in dimensionless units

$$\left[ -i \frac{\partial}{\partial t} - \frac{\partial^2}{\partial y^2} - |\Psi(y, t)|^2 \right] \Psi(y, t) = 0. \quad (1)$$

The solitons of this equation are localized solution due to the attractive nonlinear interaction  $-|\Psi(y, t)|^2$  with wave function at time  $t$  and position  $y$ :  $\Psi(y, t) = \sqrt{2|\Omega|} \exp(-i\Omega t) \operatorname{sech}(y\sqrt{|\Omega|})$ , with  $\Omega$  the energy [1].

Solitons have been noted in optics [2], high-energy physics and water waves [3], and more recently in Bose-Einstein condensates (BEC) [4, 5]. The Schrödinger equation with a nonlinear interaction  $-|\Psi|^2$  does not sustain a localized solitonic solution in three dimensions. However, a radially trapped and axially free version of this equation in three dimensions does sustain such a bright solitonic solution [6] which has been observed experimentally [4, 5]. Here we study the dynamics of these bright solitons. We also suggest that such solitons can be generated in an axially rotating nonzero angular momentum state, which are called bright vortex solitons and can be observed in BEC.

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A number of bright solitons constituting a soliton train was observed in an experiment by Strecker *et al.* [4], where they turned a repulsive BEC of  ${}^7\text{Li}$  atoms attractive by manipulating the background magnetic field near a Feshbach resonance [7]. It was found [4] that solitons in such a train usually stay apart. Also, often a soliton was found to be missing from a train [4]. There have been theoretical attempts [8, 9, 10] to simulate essentials of these experiments [4, 5].

We use the explicit numerical solution of the axially-symmetric mean-field Gross-Pitaevskii (GP) equation [11] to study the dynamics of bright solitons in a soliton train [4]. Attractive BEC's may not form vortices in a thermodynamically stable state. However, due to the conservation of angular momentum, a vortex soliton train could be generated by suddenly changing the inter-atomic interaction in an axially-symmetric rotating vortex condensate from repulsive to attractive near a Feshbach resonance [7] in the same fashion as in the experiment by Strecker *et al.* [4] for a non-rotating BEC. Alternatively, a single vortex soliton could be prepared and studied in the laboratory by forming a vortex in a small repulsive condensate and then making the interaction attractive via a Feshbach resonance and subsequently reducing the axial trap slowly.

## 2 Mean-field Model and Results

Mean-field Model: In a quantized vortex state [12], with each atom having angular momentum  $L\hbar$  along the axial  $y$  axis, the axially-symmetric wave function can be written as  $\Psi(\mathbf{r}, \tau) = \varphi(r, y, \tau) \exp(iL\theta)$  where  $\theta$  is the azimuthal angle and  $r$  the radial direction. The dynamics of the BEC in an axially-symmetric trap can be described by the following GP equation [11, 12]

$$\left[ -i\frac{\partial}{\partial t} - \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{\partial^2}{\partial y^2} + \frac{1}{4}(r^2 + \lambda^2 y^2) + \frac{L^2 - 1}{r^2} + 8\sqrt{2}\pi n \left| \frac{\varphi(r, y; t)}{r} \right|^2 \right] \varphi(r, y; t) = 0, \quad (2)$$

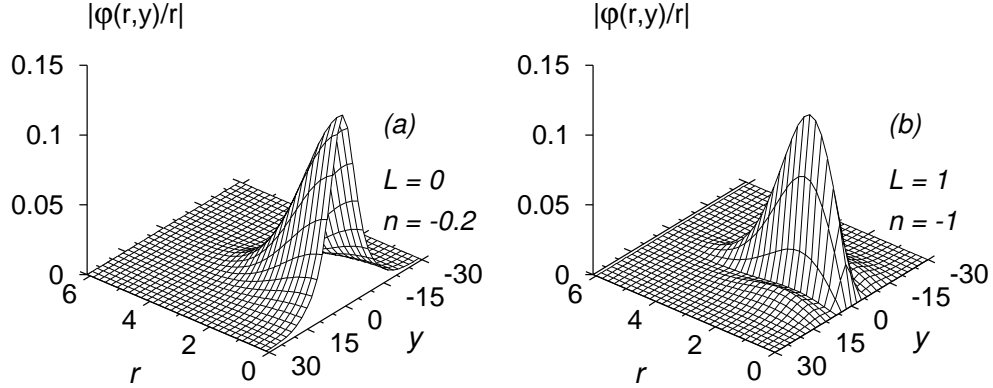
where the length, time, and wave function are expressed in units of  $\sqrt{\hbar/(2m\omega)}$ ,  $\omega^{-1}$ , and  $[r\sqrt{l^3/\sqrt{8}}]^{-1}$ , respectively. Here radial and axial trap frequencies are  $\omega$  and  $\lambda\omega$ , respectively,  $m$  is the atomic mass,  $l \equiv \sqrt{\hbar/(m\omega)}$  is the harmonic oscillator length, and  $n = Na/l$  the nonlinearity with  $a$  the interatomic scattering length. For solitonic states  $n$  is negative. In terms of the one-dimensional probability  $P(y, t)$  defined by

$$P(y, t) = 2\pi \int_0^\infty dr |\varphi(r, y, t)|^2 / r, \quad (3)$$

the normalization of the wave function is given by  $\int_{-\infty}^\infty dy P(y, t) = 1$

We solve the GP equation (2) numerically using split-step time-iteration method using the Crank-Nicholson discretization scheme described recently [13]. The details of the numerical scheme for this problem can be found in [14].

Results: For bright solitons the nonlinearity  $n$  is negative. Under the conditions  $n < 0$  and  $\lambda = 0$ , a soliton-type BEC state can be generated only for



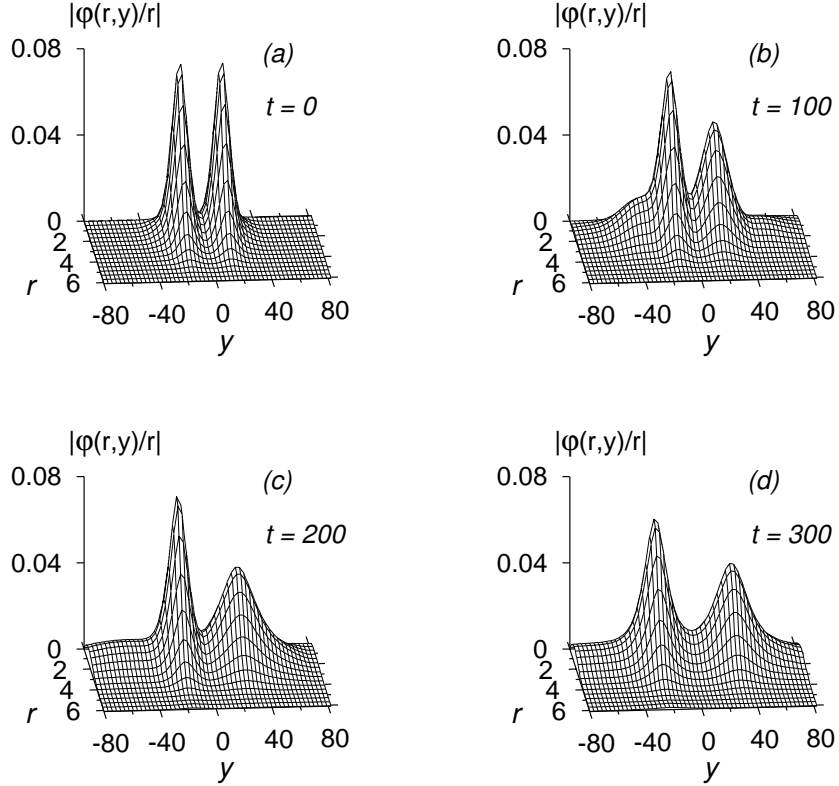
**Figure 1.** Three-dimensional wave function  $|\varphi(r, y)/r|$  vs.  $r$  and  $y$  for a single soliton with  $\lambda = 0$ , and (a)  $L = 0$ ,  $n = -0.2$ , and (b)  $L = 1$ ,  $n = -1$ .

$n$  greater than a critical value ( $n_{\text{cr}}$ ):  $n_{\text{cr}} < n < 0$ . For  $n < n_{\text{cr}}$ , the system becomes too attractive and collapses and no stable soliton could be generated. The actual value of  $n_{\text{cr}}$  is a function of the trap parameter  $\lambda$ . For the spherically symmetric case  $\lambda = 1$ , and  $n_{\text{cr}} = -0.575$  [11, 12].

Numerically solving the GP equation (2) for  $\lambda = 0$ , we find that the critical  $n$  for collapse of a single soliton is  $n_{\text{cr}} = -0.67$  for  $L = 0$  and  $n_{\text{cr}} = -2.10$  for  $L = 1$ . For  $L = 0$   $n_{\text{cr}} = -0.67$  is in close agreement with  $n_{\text{cr}} = -0.676$  obtained by other workers [6, 15]. For  $L = 1$ ,  $n_{\text{cr}} = -2.10$  should be contrasted with  $n_{\text{cr}} = -2.20$  obtained by Salasnich [16]. For  $L = 0$ ,  $\omega = 2\pi \times 800$  Hz and final scattering length  $-3a_0$  as in the experiment of Strecker *et al.* [4],  $n_{\text{cr}} = -0.67$  corresponds to about 6000  $^7\text{Li}$  atoms. One can have proportionately about three times more atoms in the  $L = 1$  state.

A  $L = 0$  soliton with  $n = -0.2$  is illustrated in Fig. 1 (a) where we plot the three-dimensional wave function  $|\varphi(r, y)/r|$  vs.  $r$  and  $y$ . For  $L = 1$  we calculated the soliton for  $n = -1$  and plot  $|\varphi(r, y)/r|$  in Fig. 1 (b). Because of the radial trap the soliton remains confined in the radial direction  $r$ , although free to move in the axial  $y$  direction. The nature of the two wave functions are different. For  $L = 0$ , the condensate has maximum density for  $r = 0$ . For  $L = 1$ , because of rotation a vortex has been generated along the axial direction corresponding to a zero density for  $r = 0$ .

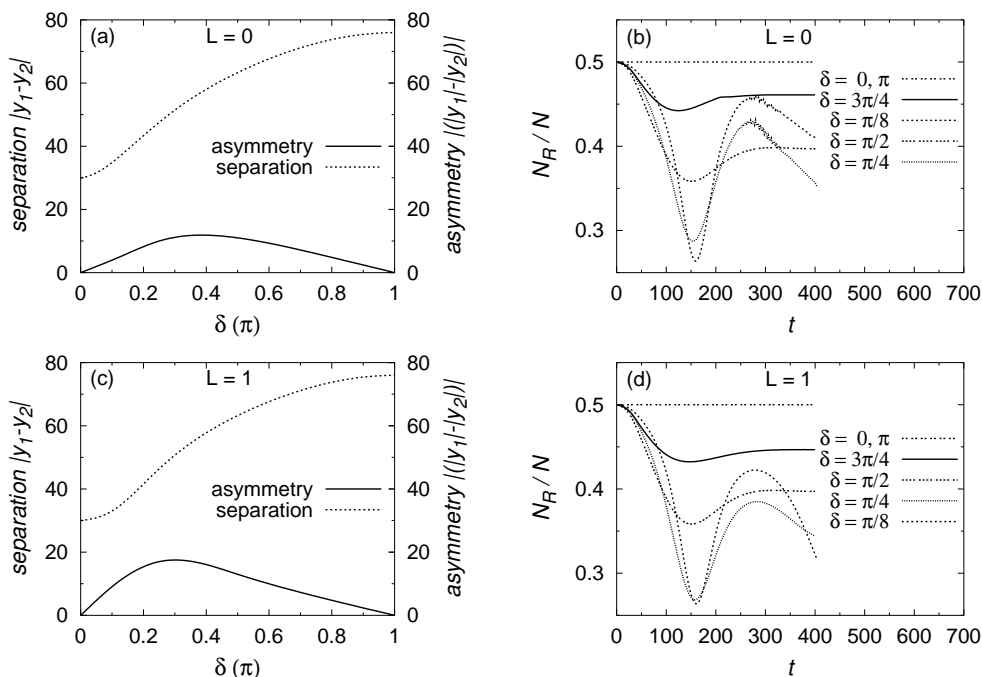
Next we consider the interaction between two solitons with a phase difference  $\delta$  given by the following superposition of two solitons  $\bar{\varphi}$  at  $\pm y_0$  at time  $t = 0$   $\varphi(r, y) = |\bar{\varphi}(r, y + y_0)| + e^{i\delta} |\bar{\varphi}(r, y - y_0)|$ . The time evolution of these two solitons is found using the solution of (2) for different  $\delta$ . In the present simulation we consider two  $L = 1$  equal vortex solitons each of  $n = -0.4$  for  $\lambda = 0$  initially at positions  $y_0 = \pm 15$  and observe them for an interval of time  $t = 400$ . We also



**Figure 2.** The wave function  $|\varphi(r, y)/r|$  vs.  $r$  and  $y$  of two  $L = 0$  solitons with  $n = -0.2$  each and  $\delta = \pi/2$  at times (a)  $t = 0$ , (b) 100, (c) 200 and (d) 300.

consider the evolution of two  $L = 0$  solitons each of  $n = -0.2$  for  $y_0 = \pm 15$ . The solitons interact by exchanging particles and after an interval of time two unequal solitons are generated from two equal solitons. The evolution of the two solitons in the  $L = 0$  case for  $\delta = \pi/2$  is shown in Fig. 2. Similar evolution for the  $L = 1$  case is reported in [14].

From Fig. 2 we find that at  $t = 0$  the two solitons are equal and symmetrically located. However, this symmetry is broken for  $t > 0$ . The asymmetry and separation in the final position of the solitons  $y_1$  and  $y_2$  are best studied via  $||y_1| - |y_2||$  and  $|y_1 - y_2|$ , respectively, at time  $t = 400$  for different phase  $\delta$  between the solitons and in Fig. 3 (a) we plot the same for different  $\delta$  for the  $L = 0$  case. We find that the asymmetry is zero for  $\delta = \pi$  and 0 and is largest for a  $\delta$  in between. However the separation increases monotonically as  $\delta$  increases from 0 to  $\pi$ . Hence the interaction is repulsive for almost all  $\delta$  except for  $\delta \approx 0$ . Closely associated with the asymmetry and separation in the final position of the solitons is the number of exchanged atoms between the two solitons, which demonstrates the change in the sizes of the solitons. Actually, the smaller soliton travels faster and the larger one travels slower. This results in the asymmetry in



**Figure 3.** (a) Final asymmetry and separation in the position of the two solitons of  $L = 0$  and  $n = -0.2$  each at  $t = 400$  vs. phase  $\delta$ . (b) Ratio  $N_R/N$  vs.  $t$  for different phase  $\delta$  for two  $L = 0$  and  $n = -0.2$  solitons initially at  $\pm 15$ .  $N_R$  is the number of atoms in the right soliton and  $N$  the total number of atoms. (c) Same as (a) for two solitons of  $L = 1$  and  $n = -0.4$ . (d) Same as (b) for two solitons of  $L = 1$  and  $n = -0.4$ .

the final positions. The change in the sizes of the solitons for  $L = 0$  is demonstrated in the plot of  $N_R/N$  vs.  $t$  for different  $\delta$  in Fig. 3 (b), where  $N_R$  is the number of atoms in the right soliton and  $N$  the total number of atoms in the two solitons. In Figs. 3 (c) and (d) we plot the same for  $L = 1$ . The variation of  $N_R/N$  is qualitatively similar for  $L = 0$  and 1 in Figs. 3 and also to that found in [10] for  $L = 0$ . However, there are quantitative differences, specially at large times. In the present simulation we find that, for the change  $\delta \rightarrow -\delta$ ,  $N_R/N \rightarrow N_L/N$ , where  $N_L \equiv (N - N_R)$  is the number of atoms in the left soliton.

If the phase difference  $\delta$  between two neighboring solitons is not close to zero, they experience overall repulsion and stay apart. However, for  $\delta$  close to zero they interact attractively and often a soliton could be lost as observed in the experiment of Strecker *et al.* [4]. Throughout this investigation in the interaction of two equal solitons we assumed that the nonlinearity  $|n|$  for each is less than  $|n_{\text{cr}}|/2$ , so that a stable solitonic condensate with total  $|n| < |n_{\text{cr}}|$  exists when the two coalesce. However, if two solitons each with  $|n| > |n_{\text{cr}}|/2$  encounter for  $\delta = 0$ , the system is expected to coalesce, collapse and emit atoms via three-body recombination. It is possible that in this case only a smaller single soliton survives. This might also explain some missing soliton(s) in experiment.

### 3 Conclusion

We emphasize the possibility of creating and studying bright vortex solitons of an attractive BEC in laboratory under radial trapping. We determine the condition of critical stability of bright solitons. Employing a numerical solution of the GP equation with axial symmetry, we have performed a realistic mean-field study of the interaction among two bright solitons in a train and find the overall interaction to be repulsive except for phase  $\delta$  between neighbors close to 0. There is an inelastic exchange of atoms between two solitons resulting in a change of size and shape. Except in the  $\delta \approx 0$  case, the solitons in a train stay apart and never cross each other as observed in the experiment by Strecker *et al.* [4]. For  $\delta \approx 0$  a single soliton can often disappear as a result of the attractive interaction among solitons, as observed experimentally by Strecker *et al.*. The  $L = 1$  vortex solitons can accommodate a larger number of atoms and the present study may motivate future experiments with them.

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