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Letter Section

A Chebyshev-type quadrature rule with some interesting properties ☆

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Abstract

We give here an n-point Chebyshev-type rule of algebraic degree of precision $n-1$, but having nodes that can be given explicitly. This quadrature rule also turns out to be one with an "almost" highest algebraic degree of precision.

Key words: Chebyshev-type quadrature rules; Ullman weight function

1. Introduction

Let $w(t)$ be an integrable nonnegative weight function on $[a, b]$. Then the n-point quadrature rule

$$\int_a^b f(t)w(t)\,\mathrm{d}t = \sum_{r=1}^{n} w_r^{(n)} f\left(x_r^{(n)}\right) + \mathbb{E}_n(f),$$

where the nodes $x_r^{(n)}$ are all distinct and belong to (a, b), and $\mathbb{E}_n(f) = 0$ for $f(t) \in \mathbb{P}_N$, is said to have an *algebraic degree of precision equal to N.*

When N takes the maximum value $2n-1$, the quadrature rule is known as a Gaussian rule. Following [2], one may also refer to a Gaussian rule as one of highest algebraic degree of precision.

If the nodes $w_r^{(n)}$, $r = 1, 2, \ldots, n$, are all equal, then the quadrature rule is known either as a Chebyshev rule when $N \geqslant n$ or as a Chebyshev-type rule when $N < n$.

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The only Chebyshev rule, which also turns out to be a Gaussian rule, is that with $w(t)=(1-t^2)^{-1/2}$ in $(-1, 1)$. This quadrature rule, apart from being one of the very few Gaussian rules which have explicit nodes and weights, is the only Chebyshev rule with explicit nodes.

Let us refer to any n-point quadrature rule with a precision given by $\mathbb{E}_n(f)=0$ whenever $t^k f(t) \in \mathbb{P}_{2n-1}$ (where k is any integer, positive or negative) also as a quadrature rule of highest algebraic degree of precision. Our principal result is the following.

Theorem 1. *If* $0<a<b<\infty$ *and* $\tilde{w}(t)=\frac{1}{2}(1+\sqrt{ab}/t)/(\sqrt{b-t}\sqrt{t-a})$, *then*

$$\int_a^b f(t)\tilde{w}(t)\,\mathrm{d}t=\frac{\pi}{n}\sum_{r=1}^{n} f(x_r^{(n)}), \quad \textit{for } t^{n-1}f(t) \in \mathbb{P}_{2n-2},$$

provided that

$$x_{n+1-m}^{(n)}=\beta+\alpha\vartheta_m^{(n)}+\sqrt{\left(\beta+\alpha\vartheta_m^{(n)}\right)^2-\beta^2},$$

$$x_m^{(n)}=\frac{\beta^2}{x_{n+1-m}^{(n)}}, \quad \textit{for } m=1, 2, \ldots, \left[\tfrac{1}{2}(n+1)\right],$$

where $\vartheta_m^{(n)}=1+\cos((2m-1)\pi/n)$, $\beta=\sqrt{ab}$ *and* $\alpha=\frac{1}{4}(\sqrt{b}-\sqrt{a})^2$.

This n-point quadrature rule, which lacks one degree from being one of highest algebraic degree of precision, is a Chebyshev-type rule with precision $n-1$.

2. Proof of Theorem 1 and some related results

Proof of Theorem 1. For $x_r^{(n)}$ as given in Theorem 1, it has been shown [3] that when $F(t)$ satisfies $t^n F(t) \in \mathbb{P}_{2n-1}$,

$$\int_a^b F(t)\frac{1}{\sqrt{b-t}\sqrt{t-a}}\,\mathrm{d}t=\frac{\pi}{n}\sum_{r=1}^{n}\frac{2x_r^{(n)}}{x_r^{(n)}+\beta}F\left(x_r^{(n)}\right).$$

If we take $F(t)=\frac{1}{2}t^{-1}(t+\beta)f(t)$, then whenever $f(t)$ satisfies $t^{n-1}f(t)\in\mathbb{P}_{2n-2}$, it follows that $t^nF(t)\in\mathbb{P}_{2n-1}$, and therefore we obtain the required result of the theorem. □

One can observe that, since the Chebyshev-type rule given in Theorem 1 has degree of precision $n-1$, it is also an interpolatory quadrature rule. This is not the case for Chebyshev-type rules of lesser degree.

In Theorem 1, we now consider the linear transformations

$$t=\tfrac{1}{2}(b-z)u+\tfrac{1}{2}(b+a) \text{ and } t=-\tfrac{1}{2}(b-z)u+\tfrac{1}{2}(b+a).$$

If we take in the first case $\lambda=(\sqrt{b}-\sqrt{a})/(\sqrt{b}+\sqrt{a})$ and in the other $\lambda=-(\sqrt{b}-\sqrt{a})/(\sqrt{b}+\sqrt{a})$, the following result is obtained.

Theorem 2. *If* $0 < |\lambda| < 1$ *and*

$$\alpha_r^{(n)} = \frac{1-\lambda^2}{2\lambda} y_r^{(n)} - \frac{1+\lambda^2}{2\lambda}, \quad r = 1, 2, \ldots, n,$$

where $y_{n+1-m}^{(n)} = z_m^{(n)} + \sqrt{\left(z_m^{(n)}\right)^2 - 1}$, $y_m^{(n)} = 1/y_{n+1-m}^{(n)}$ *and*

$$z_m^{(n)} = \frac{1+\lambda^2\cos((2m-1)\pi/n)}{1-\lambda^2}, \quad m = 1, 2, \ldots, \left[\tfrac{1}{2}(n+1)\right],$$

then

$$\int_{-1}^{1} f(t)(1-t^2)^{-1/2} \frac{1+\lambda t}{1+\lambda^2+2\lambda t}\,\mathrm{d}t = \frac{\pi}{n} \sum_{r=1}^{n} f(\alpha_r^{(n)}), \quad \textit{for } f(t) \in \mathbb{P}_{n-1}.$$

This result is perhaps already known, as one can also derive it in the following manner. We note that the weight function is the same as the one considered in [4]. Taking $|\lambda| < 1$, Ullman [4] shows that for the quadrature rule to be a Chebyshev rule, the $\alpha_r^{(n)}$ must be the zeros of the polynomial

$$P_n(z) = \left\{\tfrac{1}{2}\left(z + \sqrt{z^2-1} + \lambda\right)\right\}^n + \left\{\tfrac{1}{2}\left(z - \sqrt{z^2-1} + \lambda\right)\right\}^n - \left(\tfrac{1}{2}\lambda\right)^n.$$

He proceeds to show that only when $|\lambda| \leqslant \frac{1}{2}$, the zeros of $P_n(z)$ are distinct and belong to the interval $(-1, 1)$.

Using a result of [1], one can say that the zeros of the polynomial $P_n(z) + c$, where c is any real parameter, lead to a Chebyshev-type quadrature rule with precision $n-1$. Hence, to obtain the result of the theorem, we consider the polynomial $P_n(z) + (\frac{1}{2}\lambda)^n$. When $0 < |\lambda| < 1$, it can be verified that the zeros $\alpha_r^{(n)}$ of this polynomial can be given as in the theorem.

References

[1] D.K. Kahaner, Chebyshev type quadrature formulas, *Math. Comp.* **24** (1970) 571–574.

[2] V.I. Krylov, *Approximate Calculation of Integrals* (Macmillan, New York, 1962).

[3] A. Sri Ranga, Another quadrature rule of highest algebraic degree of precision, *Numer. Math.*, to appear.

[4] J.L. Ullman, A class of weight functions that admit Tchebycheff quadrature, *Michigan Math. J.* **13** (1966) 417–423.