# Scaling Properties of Universal Tetramers

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We evidence the existence of a universal correlation between the binding energies of successive four-boson bound states (tetramers), for large two-body scattering lengths ($a$), related to an additional scale not constrained by three-body Efimov physics. Relevant to ultracold atom experiments, the atom-trimer relaxation peaks for $|a| \to \infty$ when the ratio between the tetramer and trimer energies is $\simeq 4.6$ and a new tetramer is formed. The new scale is also revealed for $a < 0$ by the prediction of a correlation between the positions of two successive peaks in the four-atom recombination process.

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*Introduction.*—Dilute ultracold atomic gases, which are being produced in laboratories, are quite convenient to observe few-body universal phenomena predicted in the past in the context of nuclear physics, such as the infinite number of three-body levels appearing for the zero two-body binding (Efimov effect [1]), or the Tjon correlation between three- and four-nucleon binding energies [2]. In this regard, we already have experimental evidences [3–7] of the appearance of the Efimov tower of three-boson excited states in the vicinity of a Feshbach resonance (when the two-body scattering length $a$ is close to $\pm\infty$). The interest and relevance of identifying simple universal correlations between few-body quantities have been illustrated in recent perspective articles by Modugno [8], Efimov [9], and Ferlaino and Grimm [10].

The Efimov phenomena appearing when $|a| \to \infty$ (unitary limit), can be related by a scale transformation to the three-body ground-state collapse when the two-body interaction range $r_0$ goes to zero [11], as in both cases $|a/r_0| \to \infty$ [12]. In view of that, the need of a three-body scale to parameterize the short-range physics is widely recognized in the actual understanding of low-energy three-body phenomena [13,14]. The possible existence of a similar effect in four-body systems, related to a proper four-body scale, (i.e., expressed as a universal correlation between four-body quantities, not constrained by two- and three-body low-energy properties), has a long history in the nuclear physics context [15], without a clear and unquestionable conclusion.

Motivated by the possibilities in cold-atom experiments, in which $a$ can be tuned near a Feshbach resonance, the discussion on the existence of an independent four-body scale, which parameterizes the four-body short-range physics, is having a renewed interest [16–22]. The collapse of the tetramer ground state $B_4^{(0)}$, found numerically within a renormalized zero-range approach [17], provided the first indication on the existence of a four-body scale, against previous conclusions presented in [16]. This debate motivated the investigation shown in [19,20] within a finite-range potential model. In the light of their analysis, the conclusions of [16] were supported.

Our aim in this Letter is to report further precise numerical results supporting the analysis of Ref. [17], by also considering tetramer excited states, and present predictions that will allow one to verify the effect of a four-body scale in cold-atom experiments near a Feshbach resonance. Here, by using a zero-range interaction model [17], we found a universal pattern in the correlation between successive tetramer binding energies, $B_4^{(N+1)}$ and $B_4^{(N)}$ ($B_4^{(0)} =$ ground state), due to the four-body scale, in the case of fixed trimer properties and scattering length. In the unitary limit, each excited tetramer emerges from the atom-trimer continuum [i.e., $B_4^{(N+1)} = B_3$] for a universal ratio $B_4^{(N)}/B_3 \simeq 4.6$, leading to a resonant atom-trimer relaxation. Furthermore, for negative scattering lengths (no two-body binding), the positions of two successive four-atom recombination peaks (where the tetramers reach the four-atom continuum) are shown to be universally correlated. The corresponding scaling behavior expresses the dependence to a four-body scale beyond the three-body one.

The existence of a proper four-body scale is obviously challenging the analysis presented in [19,20] without raising objections on the finite-range model results reported in these references, as well as in Refs. [16,18,21,23,24]. What we are specifically pointing out is the consistency of those results with the new universal scaling function or limit cycle, which gives the correlation between the energies of two successive tetramers. We should note that, within our approach one can easily introduce the few-body scales in the four-body problem, whereas in a more involved model calculation one faces difficulties to distinguish and control the effects of the independent scales. One way to avoid the sensitivity to the four-body scale and stabilize the three-boson state against collapse in the zero-range limit is by using repulsive interactions at short distances, which leads to the well-known Tjon correlation [2]. It is worthwhile to observe that a dependence on a four-body scale implies in a family of Tjon lines, with slope moving with the new parameter, in the unitary limit.

*Trimer and tetramer scalings.*—To make clear our arguments on the emergence of a proper four-body scale, we start by discussing briefly the corresponding three-body case, in which the Thomas-Efimov effect is manifested due to the sensitivity of the low-energy physics to short-range effects parameterized by a three-body scale. It was shown in [25] that, by changing the three-body scale in respect to the two-body one, all the energies of the Efimov states (near zero two-body binding) can be represented in a single limit-cycle plot, which defines an appropriate scaling function, $\mathcal{F}_3^{(N_3)}$, as

$$\sqrt{[B_3^{(N_3+1)} - \bar{B}_2]/B_3^{(N_3)}} \equiv \mathcal{F}_3^{(N_3)}\left(\pm\sqrt{B_2/B_3^{(N_3)}}\right), \qquad (1)$$

where $\bar{B}_2$ is set to the two-body binding $B_2$ in case of plus sign, and set to zero for virtual states (minus sign). This function, obtained in the zero-range limit in [25], depends on $a = \pm 1/\sqrt{B_2}$ (+ for bound and − for virtual dimers), in units of the natural length scale of a trimer (with atom mass $m = 1$ and $\hbar = 1$). The tower of weakly bound $s-$wave states is given by $\mathcal{F}_3^{(\infty)}(0) = \exp(-\pi/1.006\,24)$. The plot also shows that, for $B_2 \neq 0$, each three-body excited state emerges from the two-body threshold for a universal ratio $B_3^{(N_3)}/B_2 = B_3^{(N_3)}/B_3^{(N_3+1)} \simeq 6.925$.

The essential physics of the Efimov effect for small but non vanishing dimer energies, with three-identical particles is captured in Eq. (1). This scaling function turns out to be universal, after few cycles becoming independent on $N_3$, evidencing a renormalization-group invariant limit-cycle in three-body physics [26,27].

*How this curious picture changes by adding to a weakly bound three-boson system another identical boson?* Does it exist a scaling function, for the ratio between two successive four-body binding energies $B_4^{(N+1)}/B_4^{(N)}$, in (or near) the limit $a \to \pm\infty$, having a similar role as the one found for three-boson systems? Our answer is yes.

For convenience, by considering a given three-body energy $B_3$ and $a \to \pm\infty$, we write a tetramer scaling function $\mathcal{F}_4$ in analogy with the three-body one, being zero defined for each case that $B_3$ coincides with one of the excited four-body energy states, such that

$$\sqrt{(B_4^{(N+1)} - B_3)/B_4^{(N)}} = \mathcal{F}_4^{(N)}\left(\sqrt{B_3/B_4^{(N)}}\right). \qquad (2)$$

A way to move the tetramer energy while keeping the three-body one unchanged with contact interactions was given in [17]. The essential idea is to recognize that the interacting three-body subsystem within the four-body system carries the three-body scale. The four-body scale parameterizes the ultraviolet physics carried by the four-boson system when it propagates between different fully interacting three-body clusters or disjoint two-body clusters in the Faddeev-Yakubovsky (FY) equations. This is the core of the subtractive renormalization procedure [28,29] applied to the four-body problem [17].

The tetramer scaling function (2) is built by solving numerically the FY equations in momentum space [17,30] with the methods detailed in [31]. In order to

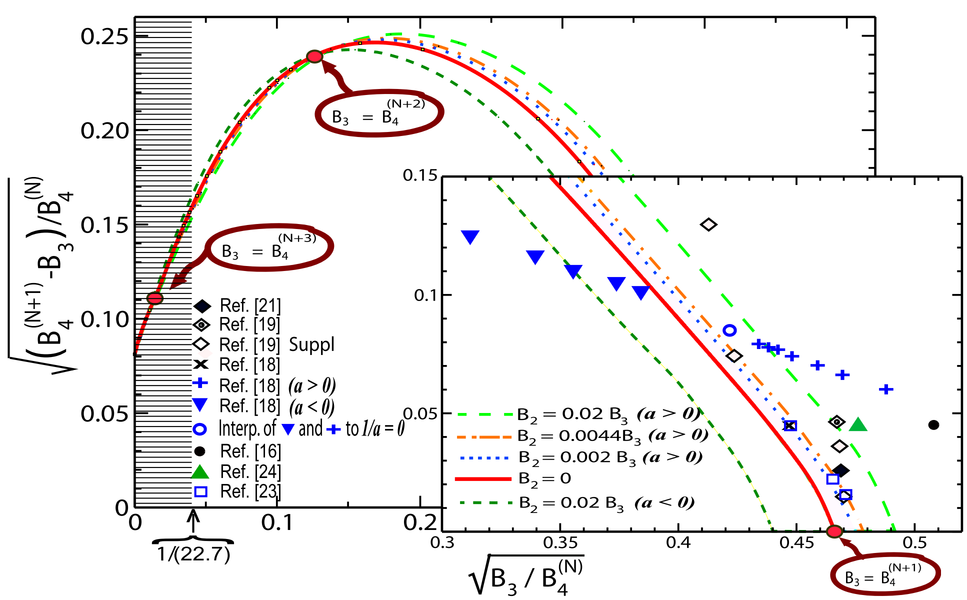


FIG. 1 (color online). Universal scaling function for successive tetramer binding energies. Our model results, with corresponding two-body energies, are indicated inside the inset, where the red-solid line is for the unitary limit. The other model calculations are indicated inside the main frame, where [18,19,21,23] report results in the unitary limit; with the others near this limit. By decreasing the ratio $B_3/B_4^{(N)}$, successive excited tetramers bind at the threshold. The shadowed region excludes tetramers for excited trimers.

achieve accurate results, a convenient choice of mesh distribution points is demanded in the regions where the FY components are more relevant. For a discussion on different approaches to the four-body problem, see Ref. [32].

At the unitary limit, an excited $N+1$ four-body state is found to emerge from the atom-trimer scattering threshold as the four-body parameter is driven to short distances or to the ultraviolet momentum region, for a universal ratio $B_4^{(N)}/B_3 \simeq 4.6$ (independent of $N$). Figure 1 illustrates this phenomenon, which corresponds to an atom-trimer resonant relaxation. We see a close analogy to the identical three-boson case near the Efimov limit, represented in the limit cycle defined in [25].

The binding energies of two successive tetramers between two trimers are represented into a single curve depicted in Fig. 1 for $|a| = \infty$. We observe that the convergence with $N$ is fast towards the limit cycle, namely, the new four-boson scaling function. The universal property of this function can be appreciated by comparing it to different model results [16,18,19,21,23,24] as shown in Fig. 1. Irrespectively to which trimer the tetramers are nearest below, the energies are scaling according to the plot shown in the figure. Results for different values of $a > 0$ and $a < 0$ are also shown. Under the condition of a resonant atom-trimer relaxation, near the unitary limit one has $B_4^{(N)} \approx 4.6B_3[1 - 0.8(a\sqrt{B_3})^{-1}]$.

For the unitary limit, we show a shadowed region in Fig. 1, $\sqrt{B_3/B_4^{(N)}} < 1/22.7$, which is physically allowed only for tetramers connected with the trimer ground-state. By increasing the tetramer scale in relation to the trimer one, as an excited tetramer emerges from the threshold defined by $B_3$, the less excited one dives even more into a nonphysical sheet of the complex energy plane, due to the decay into the next tightly-bound trimer and an atom. Therefore, in the unitary limit the superposed spectrum of trimers and tetramers presents at most three tetramers between two successive trimers. The complex analytical structure underlying this physical picture merits further investigations, which should also consider systems beyond the unitary limit.

*Effects of a four-body scale in cold atoms and how to observe them.*—Universal few-body properties have in fact been observed with trapped cold-atoms near a Feshbach resonance, by dialing $a$ over several orders of magnitude. From their resonant contribution to inelastic collisions and the corresponding trap losses, the experiments have in fact confirmed both, the presence of geometrically separated Efimov trimers (see, e.g., [3]), and two associated tetramers [4–6]. The four-body recombination experiments with tunable interaction, which are in agreement with some theoretical predictions [19], hopefully can also explore regions, for large $|a|$, where $B_3 \ll B_4^{(N)}$, in order to verify the scaling shown in Fig. 1.

The four-boson recombination rate, atom-trimer or dimer-dimer scattering lengths, can exhibit correlations not constrained by one low-energy $s$-wave three-boson observable and $a$. In the unitary limit, only three and four-body scales are relevant, and thus the correlation between the positions of the four-atom resonant recombination corresponding to successive tetramers crossing the continuum threshold is expected to be

$$a^T_{N_3,N+1} = a^-_{N_3}\mathcal{A}(a^T_{N_3,N}/a^-_{N_3}), \tag{3}$$

where $a^-_{N_3}$ is the position of the three-atom resonant recombination for $a < 0$. In Fig. 2, we show the scaling function depicted by Eq. (3). We compare our results with the corresponding positions found experimentally in Refs. [5,6], and also with the theoretical results from [19]. The reported experiments and calculations are limited to a very narrow region of Fig. 2, showing that the new universal four-body limit cycle are still unrevealed.

The coupled channel nature of the Feshbach resonance induces, by the reduction to the open channel, three and four-atom potentials, which can drive independently the corresponding physical scales. It is expected that, the magnitude of the induced forces will increase as the Feshbach resonance is approached, due to vanishing energy denominators of the virtual intermediate propagations of the subsystems in the closed channel [17]. Therefore, one cannot exclude the relevance of these interactions in actual experiments. However, the positions of the four-atom resonances seem to agree with the universal theory (see Fig. 2), with no need of a four-body parameter [6,10]. This suggests that, under the conditions of actual experiments, the interaction is dominated mainly by two-body forces with a much smaller or slowly varying three-body and four-body interactions. Thus, to observe a wider region of the plot in Fig. 2, one should perform experiments much closer to the Feshbach resonance.

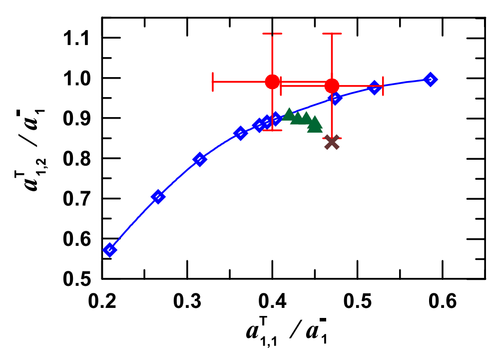


FIG. 2 (color online). Positions of four-atom recombination peaks ($a < 0$) where two successive tetramers become unbound (blue solid line with boxes). For comparison, we show results from calculations given in [19] (green triangles) and from experiments reported in [6] (red bullets with error bars) and [5] (brown ×). (Our first point from the left corresponds to $B_4 \simeq 64B_3$ at the unitary limit.)

The recombination rates measured by Zaccanti *et al.* [4], with $^{39}$K, and by Pollack *et al.* [6], with $^{7}$Li, are indicating the required change of the three-body parameter when crossing the Feshbach resonance. The same effect was also seen in an experiment of atom-dimer loss in an ultracold trapped gas of a mixture with three hyperfine states of $^{7}$Li performed by Nakajima *et al.* [33]. In this case, even contributions from two-body nonuniversal properties were excluded; moreover, it was shown that different two-body models lead to a model independent interpretation of the nonuniversal physics of the Efimov trimers, e.g., as given by the change of a short-range three-body scale. In this respect, we observe that, the correlation between the observables survives the change in the three-body parameter (in a more general universal theory where this parameter is a moving scale), supporting the conclusion on *model independence* of [33].

Once a short-range three-body force appears the very same mechanism produces effective four-body forces acting in the open channel. In the presence of three- or even four-body forces the three and four-body short-range scales can move and detach the physics of trimers from tetramers, and the position of the four-boson resonance for large and negative $a$'s will evolve as the correlation shown in Fig. 2. In addition, the position of the resonant atom-trimer relaxation is not only a function of the atom-atom scattering and the three-body scale, but it will also depend on the new four-body scale. Therefore, in this case the Efimov ratio of 22.7 between the values of the scattering length, corresponding to the position of successive resonances, is not assured.

*Conclusions.*—Our calculations of excited tetramer states support the existence of a proper four-body scale for large two-body scattering lengths, leading to new universal scaling laws for the four-boson properties. Our findings are particularly relevant in the analysis of current experiments with ultracold atoms close to a Feshbach resonance. The binding energies of the tetramer states can move in respect to the trimer one and a universal scaling in the form of a correlation between the energies of two successive tetramers expresses the sensitivity of these quantities to the short-range physics parameterized by the four-body scale (see Fig. 1). This suggests the possibility of a resonance in the atom-trimer relaxation, when a new tetramer is formed at the scattering threshold for $|a| \rightarrow \infty$. The description of the interwoven tetramer and trimer limit cycles, which builds the full tetramer spectrum, is a challenge left for a future work. Our study also predicts a universal behavior of the positions of the resonant four-atom recombination peaks when $a < 0$ (see Fig. 2), to be experimentally verified.

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