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**On the dynamic behavior of
mono-coupled continuous structures
attached to nonlinear springs**

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On the dynamic behavior of mono-coupled continuous structures attached to nonlinear springs

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ATA DA DEFESA PÚBLICA DA TESE DE DOUTORADO DE DOUGLAS ROCA SANTO, DISCENTE DO PROGRAMA DE PÓS-GRADUAÇÃO EM ENGENHARIA MECÂNICA, DA FACULDADE DE ENGENHARIA - CÂMPUS DE BAURU.

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Prof. Dr. PAULO JOSÉ PAUPITZ GONÇALVES

In memory of my Father, Leopoldo Marcos Santo.

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If i have seen further than others,
it is by standing upon the shoulders of giants.

Sir Isaac Newton

Abstract

SANTO, D. R., ON THE DYNAMIC BEHAVIOR OF MONO-COUPLED CONTINUOUS STRUCTURES ATTACHED TO NONLINEAR SPRINGS, São Paulo State University (UNESP), School of Engineering, Bauru, Brazil, 2021, 130 p., Phd Thesis (Phd's degree).

The use of the concept of periodicity has become an interesting solution for structural vibration reduction in many engineering applications. Structures built using the repetitive assembling of identical elements are called periodic structures, which can be used to achieve frequency regions where the propagating waves are highly attenuated, called attenuation zones. The objective of this work is to investigate the harmonic response of vibrating structures attached to nonlinear springs of the cubic type. The proposed analysis aims at investigating the effect of periodic local nonlinearities on the dynamic behavior and wave propagation properties in waveguide structures. In order to solve the nonlinear dynamic stiffness matrix problem, a closed-form solution using the methodology to solve polynomial equations is proposed. This yields a scalar polynomial equation, which is well suited for accurately computing the nonlinear receptance functions at some point in the structure considering a small number of unit cells. Alternatively, a method based on a perturbation approach is proposed to calculate the nonlinear frequency responses, resulting in a cubic matrix equation that can be solved numerically. It is found that the resonance peaks shift in frequency when compared to the use of linear springs, interesting features for the passive control of these structures. The effects of nonlinear springs on continuous mono-coupled periodic structures based on the concept of transmissibility of a single cell of finite structures are analyzed. Numerical simulations are carried out showing the influence of nonlinear springs over the structure bandgaps. The investigation showed that the vibration control in periodic structures can be improved by the use of nonlinear springs.

Keywords: periodic structures, cubic stiffness, polynomial method, perturbation method, transmissibility.

Resumo

SANTO, D. R., COMPORTAMENTO DINÂMICO DE ESTRUTURAS CONTÍNUAS MONO-ACOPLADAS A MOLAS NÃO LINEARES, Universidade Estadual de São Paulo (UNESP), Faculdade de Engenharia, Bauru, Brasil, 2021, 130 p., Tese (doutorado).

O uso do conceito de periodicidade tem demonstrado ser uma solução interessante na redução de vibração estrutural em problemas de engenharia. Estruturas construídas utilizando o conceito de periodicidade são chamadas de estruturas periódicas e possuem regiões em que as ondas propagadas na estrutura são atenuadas. O objetivo deste trabalho é investigar a resposta harmônica de estruturas vibratórias acopladas a molas não lineares do tipo cúbica, com foco em analisar o efeito de não linearidades locais no comportamento dinâmico e propriedades de propagação de onda em estruturas guia. É proposta uma solução de forma fechada utilizando a metodologia de solução de polinômios, a fim de resolver a matriz de rigidez não linear. Esta metodologia resulta em uma equação escalar e mostra-se adequada para encontrar de forma precisa as funções receptância em pontos da estrutura considerando um pequeno número de células unitárias. Alternativamente, é proposta uma abordagem para calcular a resposta em frequência utilizando um método de perturbação, resultando numa equação cúbica na forma matricial que pode ser resolvida numericamente. Os resultados mostram o deslocamento na frequência dos picos de ressonância em relação ao uso de molas lineares, características interessante no controle passivo destas estruturas. Os efeitos de molas não lineares sobre estruturas periódicas mono-acopladas baseado no conceito de transmissibilidade de célula unitária de uma estrutura finita é analisado. São realizadas simulações numéricas demonstrando a influência das molas não lineares nas bandas de atenuação da estrutura. A investigação mostrou que o controle passivo de vibrações em estruturas periódicas pode ser melhorado com o uso de molas não lineares.

Palavras-chave: estruturas periódicas, rigidez cúbica, método polinomial, método da perturbação; transmissibilidade.

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List of abbreviations and symbols

Designation	Explanation
Latin symbols	
A	Cross-section area
c	Viscous damping coefficient
C_1, C_2	Constants for the rod general solution
D_1, D_2	Rod dynamic stiffness matrix internal elements
D_d	Dynamic stiffness of the damping element
D_g	Generalized dynamic stiffness
$D(U)$	Dynamic stiffness of the nonlinear spring
$\mathbf{D}_{\text{rod}}$	Rod dynamic stiffness matrix
E	Young's modulus of the element
$f(t)$	Harmonic force applied to the free end of a spring
f_l	Harmonic force applied at the left side of the structure
F_l	Harmonic force amplitude at the left side of the structure
k	Quasi-longitudinal wavenumber
kl	Nondimensional <i>quasi</i> -longitudinal wavenumber
l	Length of the rod
N	Number of springs
T	Transmissibility
$\mathbf{T}_{\text{cell}}$	Transfer matrix of each cell
$\mathbf{T}_{\text{cell} - \text{i}}$	Transfer matrix of one cell for configuration (i)
$\mathbf{T}_{\text{cell} - \text{ii}}$	Transfer matrix of one cell for configuration (ii)

$u(t)$	Free-end nonlinear spring displacement
u_l	Displacement at the rod left end
u_r	Displacement at the rod right end
U	Rod general solution to harmonic excitation
U_l	Amplitude of the displacement at the left end
$\tilde{U}_j$	Normalized mode shapes for the rod attached to a nonlinear spring case
U_r	Amplitude of the displacement at the right end
s_1	Linear spring coefficient
s_3	Nonlinear spring coefficient

Designation	Explanation
Greek symbols	
β	Complex propagation constant
γ_1	Normalized linear spring coefficient of the spring attached to the rod
γ_3	Normalized nonlinear spring coefficient of the spring attached to the rod
δ	Delta Dirac function
ε	Perturbation parameter
λ	Longitudinal vibration wavelength
Λ	Roots of the characteristic polynomial equation obtained from the eigenvalue problem
ω	Harmonic force frequency
Ψ_l	Force and displacement state vector at the left boundary
Ψ_r	Force and displacement state vector at the right boundary
ρ	Density of the element
ξ	Damping ratio

Designation	Explanation
Abbreviations	
AM	Acoustic metamaterial
BC	Boundary conditions
EM	Elastic metamaterial
FEM	Finite Element Method
HBM1	First order Harmonic Balance Method
HSLDS	High-Static-Low-Dynamic Stiffness
LLC	Lateral Local Resonators
LR	Local Resonators
PC	Phononic Crystals
PGB	Photonic band gap
PWE	Phase Wave Expansion
SEM	Spectral element method
SMA	Shape Memory Allow

Chapter 1

Introduction

Overview

This chapter presents a brief motivation as well as the literature review on which this work is based on. Besides, the objectives, contributions to the knowledge, and the thesis outlines are also given. Finally, the organization of the thesis is presented.

1.1 Motivation

Vibration is the branch of engineering that studies the repetitive motion of mechanical systems. In some of these systems, such as strings of an acoustic guitar, vibrations are desired. On the other hand, in the case of the ground vibration produced by an earthquake, vibrations are unwanted (INMAN, 2008). In humans, occupational exposure to vibrations can lead to discomfort and fatigue, which reduces efficiency or even cause an accident. From machine parts to large structures, the continuous exposure to vibration can lead to excessive wear of bearings, the formation of cracks, loosening of fasteners, structural and mechanical failure. These situations increase the need for frequent maintenance, causing the operational cost to be increased (MEAD, 1999).

Vibration attenuation has a significant role in many engineering applications such as helicopters, airplanes, automobiles, industrial machinery, offshore platforms, buildings and railway vehicles (ZHOU; SHI, 2001; RAO, 2003; BRUNI et al., 2007; EL-SAYED;

BAUOMY, 2014; ZHANG; HAN; ZHANG, 2017; YANG et al., 2017). Different types of vibration attenuation using passive, semi-active and active techniques have been proposed through the last decades. Most of these techniques mainly involve the use of viscoelastic materials, dynamic absorbers, sensors and actuators devices.

Recent studies have demonstrated that vibration reduction can be obtained by using the concept of periodicity first related to the electromagnetic field (SHAMONINA; SOLYMAR, 2007). Structures built using the concept of periodicity are called periodic structures or metamaterials and involve the repetitive assembling of unit cell elements to form a whole structure (MEAD, 2009). This idea has already been applied, for example, in aircraft panels, railway tracks and truss structures (NARISSETTI, 2010). Depending on the design of the unit cell, periodic structures can be used to achieve frequency regions where the waves are highly attenuated or do not propagate. This feature provides the periodic structure effects similar to filtering properties (BRILLOUIN, 1953).

Over the past decades, the use of periodic structures among diverse ranges of science fields, from condensed matter physics to vibrations, has drawn the attention of researchers about its potential to solve a high number of engineering problems. In this context and inspired by the potentialities, this thesis proposes two different strategies to investigate the nonlinear harmonic response of vibration structures attached to nonlinear spring of the cubic type, analyzing the effect of periodic local nonlinearities on the dynamic behavior and wave propagation properties in waveguide structures. The first approach is rather analytical and constitutes a closed-form solution using the methodology to solve polynomial to solve the nonlinear stiffness matrix problem. Alternatively, a method based on a perturbation approach is proposed to calculate the nonlinear frequency responses, resulting in a cubic matrix equation that can be solved numerically. Based on the results of the numerical simulations, it is expected that both methodologies proposed to become a useful tool to investigate the nonlinear behavior of periodic structures. Finally, the effects of nonlinear springs on continuous mono-coupled periodic structures are investigated based on the concept of transmissibility of a single cell of finite structures are analyzed, showing the influence of nonlinear springs over the structure bandgaps.

1.2 Literature Review

1.2.1 Historical Background

A material or structure which exhibits any form of spatial periodicity on its internal geometry, boundary conditions, or constituent material phases can be considered as a periodic medium. According to Hussein, Leamy & Ruzzene (2014), when related to the field of acoustics and vibrations, the first studies related to these types of materials and structures can be connected to Newton’s attempt to describe the propagation of sound in the air and Rayleigh’s early study of continuous periodic systems.

A Photonic crystal is a periodic structure that affects the propagation of photons in a similar way to what occurs to electrons in a semiconductor. However, the interest of the scientific community has significantly grown after Yablonovitch (1987) and John (1987) proposed to describe the propagation of optical waves in a tridimensional periodic structure. The periodic spatial modulation of the structure opens up a forbidden gap, named photonic bandgap or stop bands (PBG), where the electromagnetic wave propagation is forbidden along with specific directions (VUKUSIC; SAMBLES, 2003). This feature allows PGB materials to become an interesting solution for many applications ranging from optical computing (JOANNOPOULOS; VILLENEUVE; FAN, 1997), tunable photonic circuits (MINGALEEV et al., 2004), more efficient laser (SRINIVASAN et al., 2003), fibers for optical communications (GOULD, 2002), among others. It is possible to find in the natural examples of PCs like the Opals stones, which is a mineral that consists of silica crystals responsible for the stone iridescence, or in the wings of the butterflies of the gender *morpho menelaus* in Figure 1.1, where the shining blue color results from the interference of the light in the “Christmas-tree” multilayer structure made of chitin and air in the wings (HAN et al., 2014).

Making use of a concept analogous to those used in Photonic Crystals, the Phononic Crystals (PCs) are also made of periodic distributions of inclusions embedded in a host matrix, designed to attenuate the vibration wave due to the destructive interference of incident and reflected waves inside the repeating subunit of the structure called Bragg



Figure 1.1: *Morpho menelaus*: butterfly which has its wings formed by a periodic porous network providing them (a) blue and (b) brown colors. Source: Han et al. (2014).

scattering phenomenon (DEYMIER, 2013). The work proposed by Sigalas & Economou (1992) is considered an important landmark by the community of mechanical wave propagation in periodic structures. The authors reported, for the first time, the appearance of frequency gaps of elastic waves propagating in systems composed of solid spheres placed periodically within a solid host structure.

It was in 1998 that the first bandgap for acoustic waves was measured experimentally. Sánchez-Pérez et al. (1998) presented the analysis of the acoustic transmission of a two-dimensional periodic array of hollow cylindrical bars mounted on a frame with square or triangular symmetry. The results showed that the attenuation intensity increases along with the number of elements considered in the whole structure. The work is based on Martínez-Sala et al. (1995) from the same authors. The structure which inspired their work is shown in Figure 1.2.

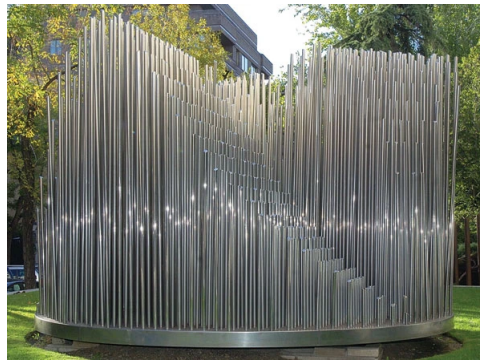


Figure 1.2: Eusebio Sempere's sculpture Órgano is an example of a phononic crystal and consists of an arrangement of cylindrical tubes, displayed at the Juan March Foundation in Madrid. Source: Thomas (2009).

Another landmark regarding the mechanical wave propagation was proposed by Liu et al. (2000). Their work presents the numerical and experimental investigation of a periodic structure built of lead spheres coated with rubber arranged in an epoxy structure with cubic configuration, as shown in Figure 1.3.

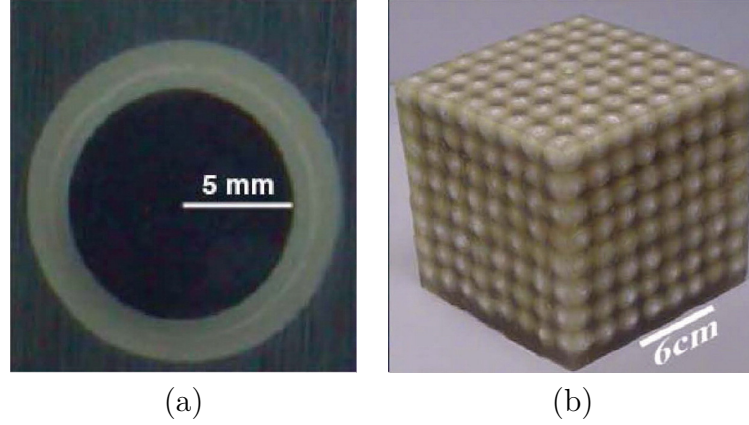


Figure 1.3: Mechanical metamaterial proposed by Liu et al. (2000): (a) Cross section of the coated lead sphere and (b) Unit cell of the metamaterial. Source: Liu et al. (2000).

Different from the Bragg scattering mechanism of phononic crystals, acoustic metamaterials (AM) and elastic metamaterials (EM) make use of a locally resonant mechanism (LR), which provides to the structure uncommon properties such as negative effective mass and effective modulus, and Poisson's ration (FANG et al., 2006; HUANG; SUN; HUANG, 2009; WANG et al., 2016). Band gaps at the low-frequency range are usually prohibitive for PCs since they require the subunits to have the same length of the propagating waves, which results in a very large structure (WEN et al., 2005). The existence of an acoustic metamaterial, with both negative effective density and bulk modulus, is proposed in Li & Chan (2004). Using a composite of soft rubber spheres suspended in water and based on Veselago's electromagnetic metamaterial analogy, they showed that the double negativity in acoustics is derived from low-frequency resonances but negative density and modulus are derived from a single resonance structure, being distinct from the electromagnetism in which negative permeability and permittivity happen due to different resonance mechanisms.

In Sheng et al. (2003), they have proposed a novel sonic bandgap material using locally resonant structures. The structure developed consists of a solid particle coated with a

silicone rubber layer mounted on an epoxy rectangular frame as shown in Figure 1.4. Theoretical and experimental results showed good agreement and showed that could be used in practical applications for effective sonic bandgap materials at a low-frequency range.

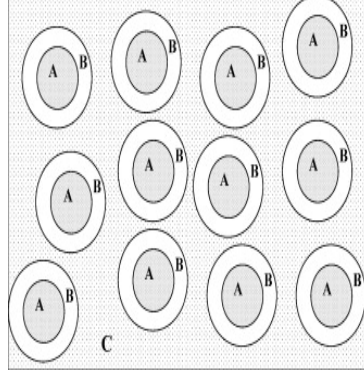


Figure 1.4: Epoxy frame filled with solid lead particles involved in silicon rubber layer. Source: Sheng et al. (2003).

Figure 1.5 shows a periodic structure for wave propagation in flexible structures proposed in Wang et al. (2016). The metamaterial consists of a lattice structure and auxiliary beams with embedded electromagnets, which behave as LR with tunable resonance frequencies by controlling the current in the electromagnets and provide an interesting opportunity for vibration isolation in flexible structures.

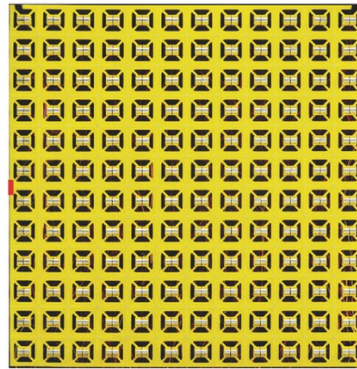


Figure 1.5: Structure with 12x12 array of unit cells, where each unit cell consists of a square frame, two pairs of beams, and two electromagnets. Source: Wang et al. (2016).

Chen & Wu (2016) demonstrated that stop band related to Bragg and Local Resonant mechanisms coexists and may not influence each other in a periodic truss beam carrying locally resonant oscillators. A new class of adaptive hybrid metamaterials by integrating

locally resonant passive structures with an array of piezoelectric PCs is proposed in (CHEN; HU; HUANG, 2017). By combining the negative effective mass density of the passive local resonators with the adjustable bending stiffness of the piezoelectric phononic crystals, they proposed a new concept design and approach for broadband wave and vibration manipulation and wave guiding applications.

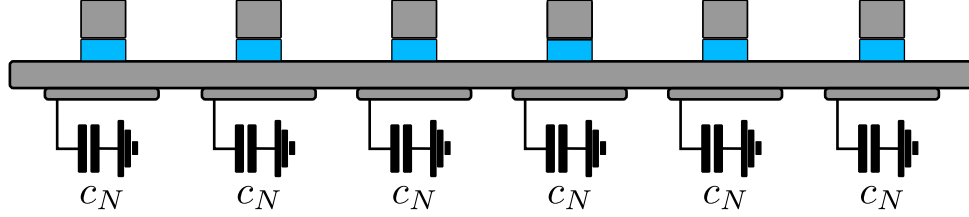


Figure 1.6: Schematic of the adaptive hybrid metamaterial beam. Source: Adapted from Chen & Wu (2016).

Over the past decades, the use of artificial structures and materials has proved their great potential. In engineering applications, there is a growing interest in using periodic structures for vibration problems such as in automotive, aerospace and industrial systems, among others (LU; FENG; CHEN, 2009).

Devices like turbines, compressors, or pumps, which are usually mounted upon supports like rods and beams, produce higher levels of noise and vibration along the receiving structure (SUTTON et al., 1997). In industrial applications, for example, it is crucial to reduce the vibration propagation in pipe conveying systems. Using the concept of PCs, Yu et al. (2008) investigated the flexural vibration in a pipe system conveying fluid and proposed a mechanism for vibration control using the Bragg and LR mechanisms.

For aeronautical applications, Szefti, Smith & Lesieutre (2003) proposed the noise reduction into the cockpit of the helicopter by using embedded vibration absorbers on periodic-layered elastomers and metal isolators, as shown in Figure 1.7. This combination has improved performance compared to the structure without the absorbers, extending the attenuation band of the periodic structure.

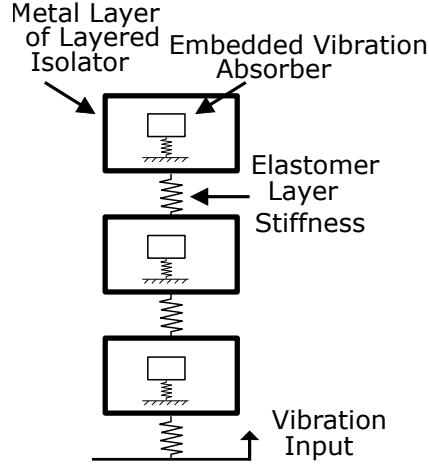


Figure 1.7: Periodic structure proposed for helicopter gearbox isolation. Source: Adapted from Szefi, Smith & Lesieutre (2003).

Singh, Pines & Baz (2004) proposed a strategy for active and passive vibration level reduction in a helicopter system, as shown in Figure 1.8. Periodic bars and beams of different geometries were used to reduce the vibration levels at high-frequency, and periodically attached piezoelectric actuators to improve the wave attenuation at the low-frequency.

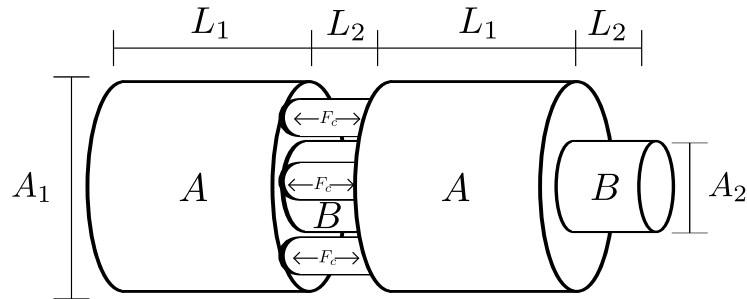


Figure 1.8: A periodic rod with piezoelectric actuators. Source: Adapted from Singh, Pines & Baz (2004).

In Figure 1.9, Asiri, Baz & Pines (2005) proposed the use of two different types of periodic structures of which consider geometric and material discontinuity to filter vibration propagation within critical frequency bands from the gearbox to the airframe of a helicopter. Within properly design, they showed that the amplitude of vibration propagating through the system could be attenuated and, the transmission of undesirable vibration and sound radiation in the helicopter cabin.

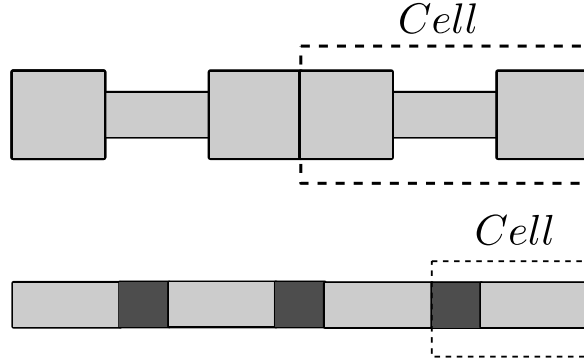


Figure 1.9: A periodic rod with piezoelectric actuators. Source: Adapted from Asiri, Baz & Pines (2005).

The investigation for more efficient solutions led researchers to the design of chiral configurations, where the appearance of band gaps due to negative equivalent mass and modulus can be tuned to increase the attenuation frequency region. Also, the geometry of the chiral configuration enables inertial translation and rotation, increasing the energy absorption from the host structure. In Liu et al. (2011), they proposed a chiral metal composite built of soft coated heavy cylinders for low-frequency band gaps and wave filtering applications as seen in Figure 1.10. They found that due to the useful properties of the chiral metal composite, such as negative mass density and negative elastic modulus, low-frequency band gaps could be achieved by adjusting the substructure parameters.

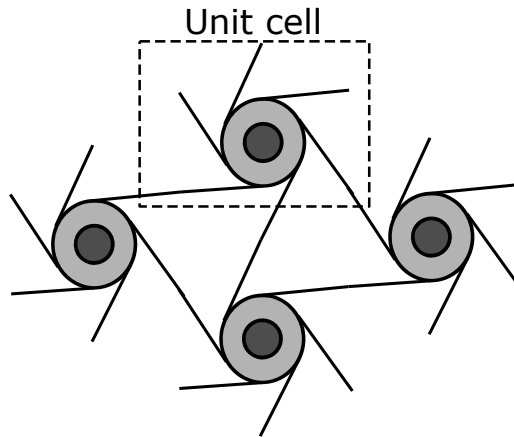


Figure 1.10: The geometry of a hexagonal chiral lattice. Source: Adapted from Liu et al. (2011).

A wave attenuation mechanism with negative and frequency-dependent Young's modulus is proposed in Huang & Sun (2011). The periodic system unit cell consists of a spring in its equilibrium state connected to a four-link structure, in turn, connected to

two microresonators, as shown in Figure 1.11. Numerical simulations demonstrated that acoustic metamaterials having simultaneously negative effective mass density and negative effective Young's modulus can be realized by proper design.

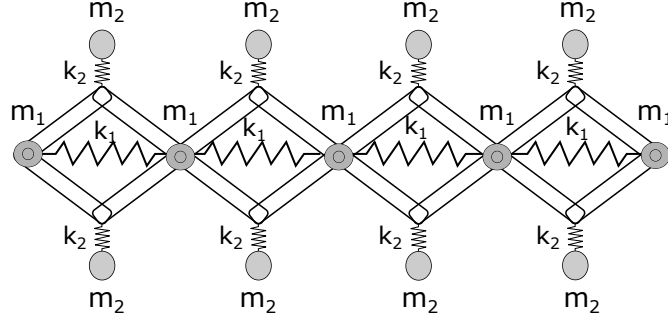


Figure 1.11: A 1D lattice model proposed in Huang & Sun (2011). Source: Adapted from Huang & Sun (2011).

A new metamaterial approach for broadband vibration reduction based on multi-frequency absorbers using two configurations of LR mechanisms is presented in Pai, Peng & Jiang (2014). The first consists of an isotropic beam attached to an array of mass-spring-damper subsystems located along the structure, providing the expansion of the attenuation zone band frequency where vibration cannot propagate as long as the mass of the absorber increases. The second configuration consists of a second and smaller mass-spring-damper subsystem connected to the first one as shown in Figure 1.12, resulting in the appearance of two attenuation zones with a smaller width than the single attenuation zone when using only one absorber. For an incoming wave with a frequency inside in one of the two stopbands, the absorbers are excited to vibrate in their optical modes, creating shear forces to stop the wave propagation.

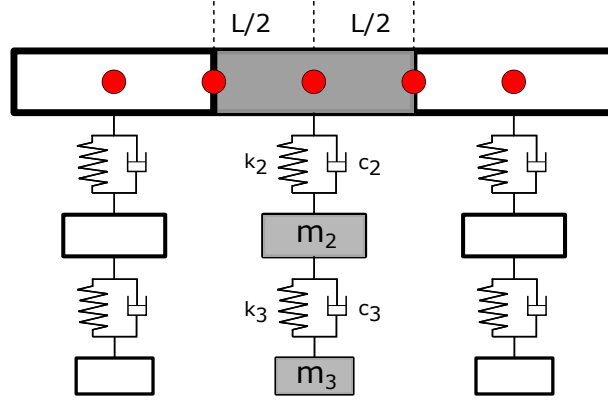


Figure 1.12: Acoustic metamaterial beam. Source: Adapted from Pai, Peng & Jiang (2014).

The use of periodic structures can provide enhancement on seismic isolation in the civil engineering field (CHENG; SHI, 2018). In Xiong, Shi & Xiang (2012), they demonstrated that, when exposed to not only horizontal but also vertical waves generated by earthquakes, the dynamic response of buildings could be significantly reduced when proper periodic foundations are considered. To mitigate seismic surface wave at low-frequency range, Lim, Reddy et al. (2019) proposed the use of periodically arranged built-up steel sections as resonators in single and multi-layered soil media. Results showed that the structure considered could be used as a passive isolation technique for seismic wave attenuation.

A similar strategy is presented in Wang, Sheng & Qin (2016). They showed using numerical simulations and experimental investigation, that longitudinal waves in an Euler-Bernoulli beam can be partially transformed into flexural waves and, consequently, block the wave propagation by using lateral local resonators (LLR). Huang, Lin & Tan (2016) studied the propagation and vibrational behavior of a metamaterial beam with LLR. They proposed a new prototype for in-plane wave and vibration mitigation using a metamaterial beam with lateral local resonators, and results were validated through numerical simulations experimental investigation. In Beli, Arruda & Ruzzene (2018), they proposed a new design for metamaterial beams and plates with interconnected local resonators. Numerical and experimental investigations of the considered structure showed the rotational and transverse splitting of the resonator chain vibration modes.

Naval and aerospace applications also make use of periodic structures for improved

dynamic response. In Song et al. (2015), they concluded that the use of periodic structures could efficiently reduce vibration propagation in flexible floating raft systems extensively used in ships and submarines. The schematic of the proposed structure is shown in Figure 1.13.

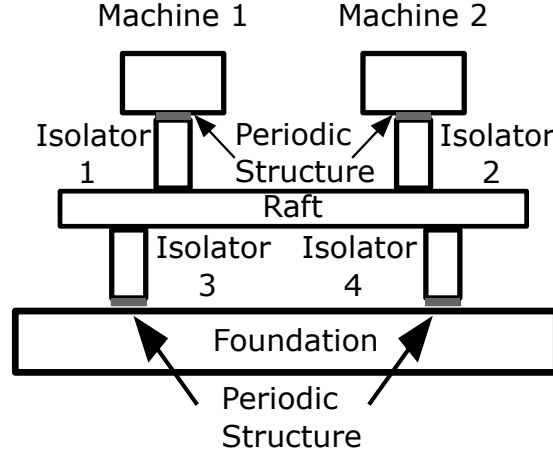


Figure 1.13: Model of the flexible floating raft system. Source: Adapted from Song et al. (2015).

Wang & Mak (2018) developed a practical design approach for the optimization of periodic structures applied to float raft systems using a genetic algorithm and the transfer matrix method. Such an approach demonstrated efficient results showing in broader stopband regions and reduces power flow through the periodic structure.

In satellite applications, standards waveguide filters offer narrow stopbands and moderate attenuation slopes. The classical approach, which consists of increasing the mass of the resonator, is not suitable for these types of applications. To overcome such drawbacks, Boria & Gimeno (2007) proposed the use of periodic waveguide resonators as a more compact and practical solution for these applications.

A periodic material-based vibration isolator to protect satellites during the launch is proposed in Liu et al. (2016). The structure consists of a periodic foundation built of aluminum and rubber supporting the satellite as demonstrated in Figure 1.14, and the results display good performance for multi-direction vibration isolation.

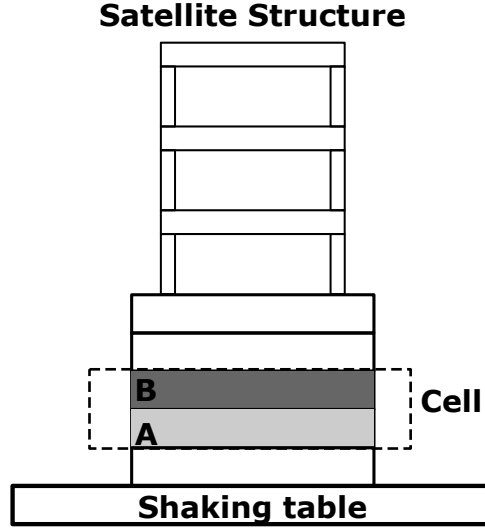


Figure 1.14: Satellite resting on periodic foundation isolator. Source: Adapted from Liu et al. (2016).

In practice, the stop bands provided by the use of periodic structures allow them to serve as filters, resonators, or waveguides. Figure 1.15 shows the wave propagation in a z-shaped waveguide formed in periodic structure when excited with a point source located on the left side edge (PAL; RUZZENE, 2017).

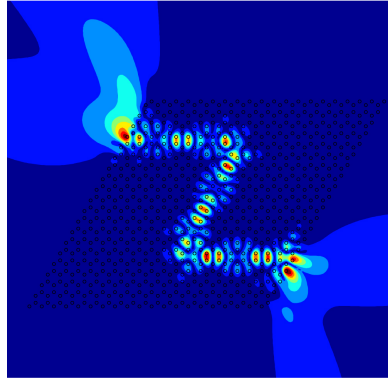


Figure 1.15: Wave propagation in a z-shaped interface. Source: Pal & Ruzzene (2017).

For linear periodic structures, the stopband regions depend on most of geometry, inclusions and elastic properties. However, the stop bands are likely to occur at frequencies higher than those often encountered in structural vibration problems. This condition makes them less relevant for controlling the vibration levels of structures in the low-frequency range. For low-frequency band gaps, a high relation between the inclusion mass and the host structure is necessary to increase the attenuation bandwidth in periodic LRs.

However, increasing the substructure mass can be prohibitive due to the high cost of the material or the substructure total mass can overpass the limit that the host structure can handle.

1.2.2 Usual methods to investigate the dynamic behavior and wave propagation of periodic structures

Here, some of the usual methods used to investigate the dynamic behavior and wave propagation of periodic structures found in the literature are briefly presented. The use of dispersion diagrams that relates the wavenumber and waveshapes to the angular frequency is usually employed when studying the wave propagation in periodic structures and has been employed in electromagnetic, acoustic, optical and elastic media such as phononic crystals and metamaterials (BRILLOUIN, 1946). Some methods have been used to compute the band structures by solving an eigenvalue problem. The eigenvalues provide the dispersion branches, while the eigenvectors provide the waveshapes representing the spatial distribution of displacements on the unit cell.

The plane wave expansion method (PWE) is a method to solve analytic equations to compute the band structures of periodic systems (LI, 1996). This method can be applied to solve different types of periodic structures, such as phononic crystals (CHARLES; BONELLO; GANOT, 2006; HOU; ASSOUAR, 2008), photonic crystals (EL-NAGGAR, 2012), and metamaterials (WANG et al., 2019). The PWE method makes use of the system periodicity and Floquet-Bloch's theorem to solve the wave equation (FLOQUET, 1883; BLOCH, 1928), to obtain the eigenvalue problem. One of the main advantages of the method is the easiness to be implemented. On the other hand, complex geometry or material mismatch can be a drawback for the method (MIRANDA JUNIOR, 2018).

The Finite Element Method (FEM) is a powerful and popular computation method to predict the frequency behavior of periodic structures. The FEM is a time-domain solution, where the equations are solved by using a numerical integration method, and can be applied to a large number of geometries, boundary conditions, and material variations for free and forced vibration (PHANI; WOODHOUSE; FLECK, 2006; HOU; ASSOUAR,

2008; SHARMA; SUN, 2016). However, the conventional finite element model requires a large number of degrees of freedom used to model each substructure, making the whole method prohibitive for the most complex and large structures (LEE, 2009).

An alternative approach to investigate structural dynamics, the spectral element method (SEM) is an analytical and discrete frequency-based approach, which can be used to achieve the dispersion relations in a complex form for infinite and forced finite systems by analyzing just one unit cell. The method offers high accuracy with low computational cost due to the smallness of the problem size when compared to the conventional FEM (WU; LI; WANG, 2013; BELI; SILVA; ARRUDA, 2018; WEN et al., 2020). The reason is that a geometrically and constitutively uniform structure can be considered as only one spectral element (LEE, 2001). Despite the aforementioned advantages, the method is restricted to simple geometries since exact wave solutions are not always available for complex and multi-dimensional problems.

The Wave Finite Element Method (WFE) has computational time reduced to design periodic structures when compared to the standard FE method (DUHAMEL; MACE; BRENNAN, 2006) because just one unit cell has to be meshed. Compared to the SFE method, the WFE has the advantage of being able to use conventional FE software packages to build the dynamic stiffness matrix of the system, which allows structures with complex geometries or material distributions to be analyzed with relative ease (MACE et al., 2005; MENCİK; ICHCHOU, 2007; ICHCHOU; MENCİK; ZHOU, 2009; MENCİK, 2014; SILVA; MENCİK; ARRUDA, 2016).

The transfer matrix method (TMM) can be efficiently used mainly for periodic one-dimensional structures. In general, most large space structures are built to have spatial periodicity for ease of assembly and construction. Once the transfer matrix for the element is obtained, solutions for the whole system can be readily achieved without requiring large computer time and cost (LEE, 2000; GRIFFITHS; STEINKE, 2001; YEH; CHEN, 2006; YU et al., 2012).

1.2.3 Nonlinear periodic structures

Nonlinear periodic structures may exhibit additional exciting properties, which enables enhanced wave transmission properties to the structure (WANG; LI; WANG, 2016). Sievers & Takeno (1988) reported a localization phenomenon occurring in pure anharmonic crystals of which the localized mode exhibits properties to be equivalent to the localized mode of a defective harmonic lattice. They showed that, although it takes more energy to produce and excite these localized vibrations, the intrinsic nonlinear modes are spatially localized and can appear at any lattice site. In Vakakis, King & Pearlstein (1994), they reported that forced localization occurs in a periodic nonlinear system without any structural disorders. They considered an undamped chain mass-spring system where the masses are attached to a rigid foundation by nonlinear springs. The key idea is that the resonance frequencies of a nonlinear periodic structure depend on the displacement levels, and as such, they can be shifted from low to high frequencies. This results in wide low-frequency bands without resonance phenomena, where the vibration levels can be well-attenuated.

Narisetti, Leamy & Ruzzene (2010) investigated the wave propagation in a discrete one-dimensional nonlinear periodic structure through perturbation analysis. Figure 1.16 shows the three configurations studied. Numerical simulations have been carried out and showed good agreement with the predictions using a perturbation approach. They found that nonlinearities in periodic systems may be applied to achieve amplitude-dependent dispersion properties and tunable acoustic devices.

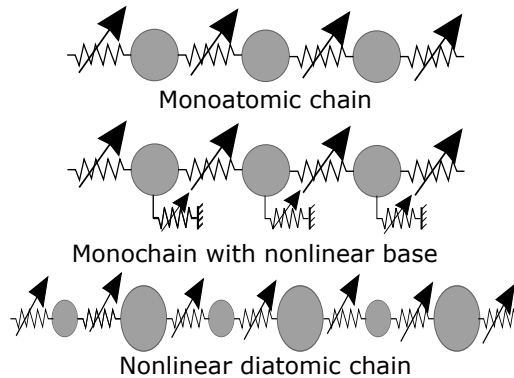


Figure 1.16: Three different configurations studied in Narisetti, Leamy & Ruzzene (2010) to investigate the wave propagation in discrete nonlinear periodic structures. Source: Adapted from Narisetti, Leamy & Ruzzene (2010).

Using a multiple time scales approach, Manktelow, Leamy & Ruzzene (2011) presented an extension of the analysis in Narisetti, Leamy & Ruzzene (2010) to include the propagation of multiple harmonic plane waves in the investigation of dispersion characteristics for wave interactions in a discrete monoatomic mass-spring chain with cubic stiffness. The continuation of the work is presented in Manktelow, Leamy & Ruzzene (2014). The perturbation approach is extended for application in the analysis for multi-degree of freedom and multidimensional systems. In Fronk & Leamy (2017), they proposed a higher-order multiple scales solution to predict the multi-harmonic plane wave propagation in cubically and quadratically nonlinear monoatomic and diatomic chains. High-order amplitude-dependent dispersion relationships were analytically and numerically investigated to provide information about quadratic and cubic terms in frequency/wavenumber shifts. In Sousa et al. (2018), they explored the attenuation capabilities of a locally resonant metamaterial, considering shape memory alloy (SMA) springs. The metamaterial exhibits a tunable bandgap by adequately changing the elastic modulus of the SMAs. Results for the SMA working inside the linear and nonlinear regime were compared, yielding additional dissipation, especially outside the band gaps range.

A new type of metamaterial beam is proposed in Zhou et al. (2017) with the addition of a local resonator with high-static-low-dynamic stiffness (HSLSD). The resonator is designed by combining a vertical spring with two oblique springs that provide negative effect stiffness in the vertical direction for a particular range of oscillation. The band structures demonstrated that band gaps could be translated to a much lower frequency than that of a conventional LR.

In contrast, studies on nonlinear stiffness elements connected to continuous structures were not explored extensively in the literature. Among the works made on this topic, some authors have published important works. In Gudmundson (1989), the harmonic balance method with truncated Fourier series to represent a beam supported by a nonlinear torsional spring is proposed. The case of a beam resting on a stiff spring of the Duffing type is discussed in Srinivasan (1966). He proposed a comparison between the Ritz and Fourier methods to calculate the force and free response of a simply supported

beam resting on a nonlinear spring.

The wave propagation in one-dimensional chains with attached nonlinear local oscillators by using analytical and numerical methods Lazarov & Jensen (2007) was investigated in Lazarov & Jensen (2007). They found that the position of the bandgap can be shifted by changing the oscillators nonlinearity level, which can allow the structure to be utilized in the design of adjustable filters in the lower frequency range.

Bernard et al. (2014) showed an experimental example of bifurcation induced filtering on mechanically coupled asymmetric nonlinear oscillators. The proposed configuration displayed an opportunity for amplitude-dependent filtering, where the low amplitude is allowed to be transmitted and high amplitude is attenuated, characteristic similar to a circuit breaker and applicability for systems with delicate components that are unable to undergo sustained exposure to large oscillations.

Recently, Tang, Brennan & Manconi (2018) computed the natural frequencies of uniform rods and beams with nonlinear boundaries using the phase-closure method. They concluded that a significant influence on the phase of reflection coefficient can be achieved by using softening springs, causing some instabilities, while the hardening spring has a small effect on the system.

In Lamarque, Savadkoobi & Charlemagne (2018), they proposed an experimental setup to investigate the behavior of a linear system coupled to a chain of nonlinear oscillators. Their experimental procedure consisted of a linear spring-mass-damper system attached to a set of eight lumped masses, each one connected by spring with linear and nonlinear restoring forces. According to the excitation amplitude, authors have found different regimes, such as periodic and strongly modulated responses.

In Casalotti, El-Borgi & Lacarbonara (2018), they investigated the case of a metamaterial consisting of a beam with embedded nonlinear absorbers. They initially studied the effects of a single cell, and further, using a path-following approach, they have determined the frequency response. Their work also proposed an optimization algorithm to improve the performance of the system. Chouvion (2019a), Chouvion (2019b) investigated the vibration of structures consisting of beam elements with localized nonlinearities and

nonlinear energy sink (NES).

The problem of a one-dimensional structure consisting of a lumped parameter masses with nonlinear vibration absorbers to reduce vibration without the addition of mass is discussed in Vasconcellos & Silveira (2020). The authors used numerical integration to obtain the frequency response functions discussing how the nonlinear stiffness parameters could be optimized based on sequential quadratic programming. In Campana et al. (2020), the study of the wave propagation in a periodic structure composed of a linear spring-mass chain with local Duffing nonlinear resonators is presented. Based on the perturbation method, they compared the results with the harmonic balance method discussed in Lazarov & Jensen (2007).

Among the few works presented here, it seems that efforts were devoted to understanding the effect of nonlinearity in periodic structures using nonlinear resonators (ROTHOS; VAKAKIS, 2009; KHAJEHTOURIAN; HUSSEIN, 2014; MANIMALA; SUN, 2016; FANG et al., 2017; CVETICANIN; ZUKOVIC; CVETICANIN, 2018; SHENG et al., 2021). Still, it appears that little was made in investigating the influence of nonlinear stiffness on other types of periodic structures presenting band gaps that are not related to local resonance.

1.3 Problem statement

Based upon and due to the intriguing properties of periodic structures related to vibration attenuation, in this work a strategy to describe the multi-mode response of linear continuous structures, i.e., structures whose dynamic response involves more than one contributing linear vibration mode, attached to nonlinear springs is proposed, as shown in Figure 1.17. The objective of this work is to investigate the harmonic response of vibrating structures attached to nonlinear springs of the cubic type. The proposed analysis aims at investigating the effect of periodic local nonlinearities on the dynamic behavior and wave propagation properties in waveguide structures. In order to solve the nonlinear dynamic stiffness matrix problem, a closed-form solution using the methodology to solve polynomial equations is proposed. This yields a scalar polynomial equation, which is well suited for accurately computing the nonlinear receptance functions at some point

in the structure made upon a small number of unit cells. Alternatively, a method based on the perturbation approach is proposed to calculate the nonlinear frequency responses, resulting in a cubic matrix equation that can be solved numerically. To explain the effect of nonlinear springs coupled to continuous periodic structures, a discussion based on the concept of transmissibility of a single cell of finite structures is made. Numerical simulations are carried out, which highlight the relevance of the approaches, which provides help to investigate the effects of nonlinear stiffness elements on the behavior of periodic structures and metamaterials and the effects on the band gaps.

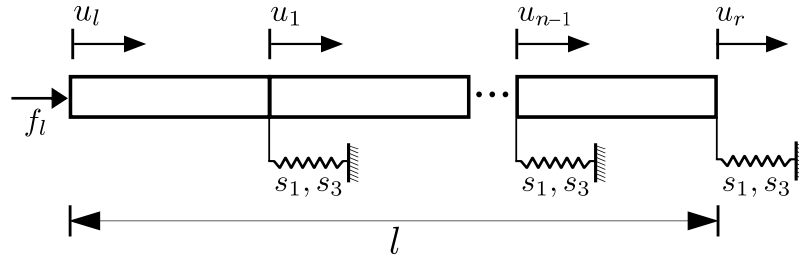


Figure 1.17: Schematic of a rod with an array of equally spaced nonlinear springs.

1.4 Objectives

The main objectives of this thesis are

- To investigate the harmonic response of vibrating structures attached to nonlinear springs of the cubic type;
- To analyze the effect of periodic local nonlinearities on the frequency response of finite continuous structures;
- Provide a physical insight on the variations of the resonance frequencies of continuous structures attached to a periodic array of nonlinear springs;
- Propose two approaches to solve periodic structures with multiple nonlinear springs;
- Investigate the effect of nonlinear stiffness elements on the behavior of periodic structures and the effect on band gaps based on the concept of transmissibility of a single cell of finite structure.

1.5 Thesis outline

Chapter 1 presents the historical background of this work, the literature review and the problem statement. Also, the objectives and the contribution achieved in this research are presented. The dynamic stiffness equations for the nonlinear spring using the first order Harmonic Balance Method (HBM1) and the influence of a nonlinear spring on the dynamic behavior of a rod is in Chapter 2 using a polynomial approach.

In Chapter 3, the investigation of the harmonic response of vibrating rods attached to an array of nonlinear springs is made. Two approaches are proposed and discussed in detail, the polynomial and perturbation approach. Numerical simulations are carried out and the dynamic behavior is compared to the case where only purely linear springs are considered.

Chapter 4 presents the effects of nonlinear springs on two different types of cell configurations of mono-coupled continuous periodic structures. The investigation is based on the concept of transmissibility of a single cell of finite structures. Numerical simulations are carried out using the Homotopy continuation algorithm for a system of algebraic nonlinear polynomials and the effects of linear and nonlinear springs are compared.

Finally, the general conclusions, original contributions and suggestions for future works are drawn in Chapter 5.

Chapter 2

The influence of the nonlinear stiffness on rod using the polynomial method approach

This chapter presents the investigation of the influence of a nonlinear spring on the dynamic behavior of a homogeneous rod using the polynomial method approach. Longitudinal motion is considered for the rod case, and free and forced vibrations are analyzed. In this chapter, the equation for the dynamic stiffness of the nonlinear spring is derived using the first order Harmonic Balance Method (HBM1). Transcendental equations are solved, and the comparison for the position of the roots for different values for the spring linear and nonlinear stiffness coefficient are investigated. Considerations about the damped case with both structural damping and with the introduction of a damping element are discussed using the polynomial approach.

2.1 Nonlinear spring model

In the present case, the springs are supposed to be clamped on one of its ends and coupled to the rod on the other end. The schematic of a clamped nonlinear spring subject to a force $f_e(t)$ is shown in Figure 2.1.

The assumption is made that each spring is governed by a cubic polynomial equation,

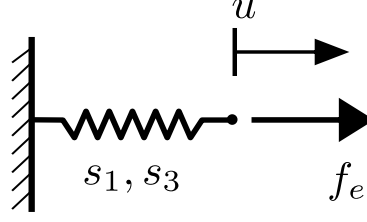


Figure 2.1: Schematic of a clamped nonlinear spring subject to a force $f_e = f_e(t)$ at its right end.

which links the restoring force $f = f(t)$ to the position $u = u(t)$ at the forced end. This means that there is a nonlinear relationship between the force applied to the spring and the displacement can be approximated as a series in u written as:

$$f \approx s_0 + s_1 u + s_2 u^2 + s_3 u^3 + s_4 u^4 \dots \quad (2.1)$$

If the spring has the same stiffness characteristic when under tension or compression, the spring is considered symmetric and Eq. (2.1) can be reduced considering only the exponents of u which are odd integers (KOVACIC; BRENNAN, 2011; CARRELLA et al., 2012; WORDEN; TOMLINSON, 2001; KARGARNOVIN et al., 2005; YOUNESIAN; MARJANI; ESMAILZADEH, 2013). For a series truncated at the first two terms, the force-deflection relationship in Eq. (2.1) can be written as

$$f \approx s_1 u + s_3 u^3 \quad (2.2)$$

where s_1 and s_3 denote linear and nonlinear stiffnesses terms, respectively. Two cases can be considered depending on whether $s_3 > 0$ or $s_3 < 0$. The first case, where $s_3 > 0$, is called hardening spring because the spring becomes stiffer as the displacement increases. For the case where $s_3 < 0$, the spring becomes softer as the displacement increases and is called softening spring. In both cases, the variation of the restoring force f against the displacement u can be plotted, as shown in Figure 2.2. Note that in Figure 2.2, for the

softening spring case, the equivalent stiffness of the system becomes negative for large values of the magnitude of the displacement u , and the system becomes statically unstable.

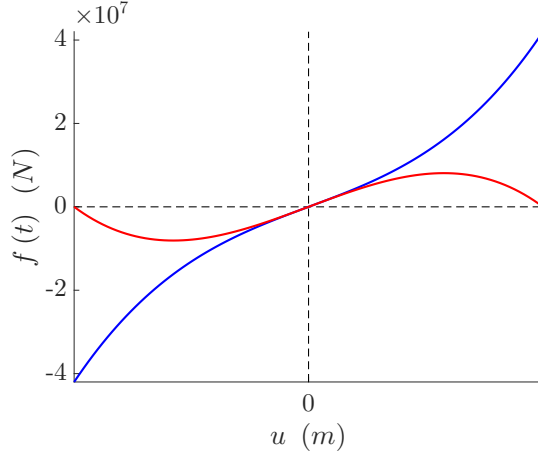


Figure 2.2: Variation of the restoring force f against the displacement u when $s_1 = 1$ N/m and $s_3 = 100$ N/m³ (blue curve), and when $s_1 = 1$ N/m and $s_3 = -100$ N/m³ (red curve).

Let us assume that the force is harmonic with frequency $\omega/2\pi$, i.e., $f_e = \hat{F}_e \cos(\omega t)$, and let us assume that the related harmonic displacement is expressed by $u = \hat{U} \cos(\omega t)$, say, by means of a first-order Fourier series expansion (HBM1 assumption) (MICKENS, 2001; HU; TANG, 2006). As a result, Eq. (2.2) can be rewritten as follows:

$$\hat{F}_e \cos(\omega t) = s_1 \hat{U} \cos(\omega t) + s_3 \hat{U}^3 \cos^3(\omega t) \quad (2.3)$$

An expression of the equivalent stiffness of the spring can be obtained by invoking the trigonometric relation $\cos^3(\omega t) = 3/4 \cos(\omega t) + 1/4 \cos(3\omega t)$. Therefore, Eq. (2.3) can be rewritten as

$$\hat{F}_e \cos(\omega t) = s_1 \hat{U} \cos(\omega t) + \frac{3}{4} s_3 \hat{U}^3 \cos(\omega t) + \frac{1}{4} s_3 \hat{U}^3 \cos(3\omega t) \quad (2.4)$$

An assumption of which considers that the contribution in energy of the third harmonic 3ω being small when compared to the first harmonic ω is made (TANG; BRENNAN; MANCONI, 2018). Therefore, neglecting the term associated to the third harmonic, the equivalent dynamic stiffness with the spring — namely, $D(\hat{U})$ — can be expressed as follows:

$$D(\hat{U}) = \frac{\hat{F}_e}{\hat{U}} = s_1 + \frac{3}{4}s_3\hat{U}^2. \quad (2.5)$$

The displacement $\hat{U}$ in Equation (2.5) can be replaced by the displacement of the structure considered at the position where the spring is attached.

2.2 The influence of the nonlinear stiffness on a rod

The dynamic behavior of a rod coupled to a nonlinear spring is analyzed first. A homogeneous linear rod vibrating in longitudinal motion is considered as shown in Figure 2.3, whose left end ($x = 0$) is subjected to a given prescribed force $f_l = f_l(t)$, and whose right end ($x = l$) is connected to a nonlinear spring with linear and nonlinear stiffness coefficients s_1 and s_3 , respectively. The displacements at the left and right ends of the rod are denoted by u_l and u_r , respectively.

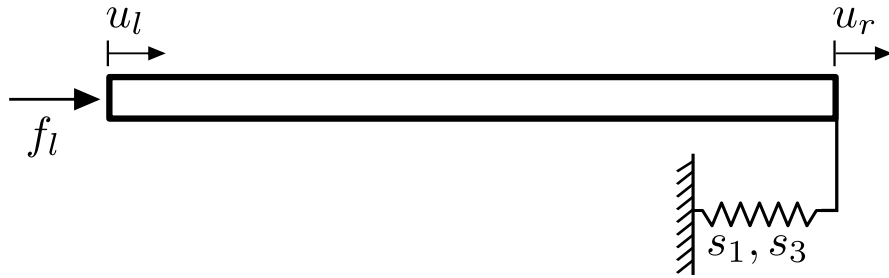


Figure 2.3: Schematic of a rod connected to a nonlinear spring.

Assuming that the rod is undamped, thus its local governing equation is given by (GRAFF, 1991):

$$u'' - \frac{\rho}{E}\ddot{u} = 0 \quad x \in]0, l[, \quad (2.6)$$

where, $u = u(x, t)$ is the longitudinal displacement of the rod, E is the Young's modulus, ρ is the density, and notations $\bullet''$ and $\ddot{\bullet}$ refer to the double space and double-time derivatives, respectively. The boundary conditions of the rod, at $x = 0$ and $x = l$, are expressed by:

$$-EAu'(0, t) = f_l \quad , \quad EAu'(l, t) = -f, \quad (2.7)$$

where A is the cross-sectional area of the rod, f is the restoring force applied by the spring

at the right end, see Eq. (2.2), and notation $\bullet'$ refers to the space derivative.

Let us assume that the prescribed force f_0 is harmonic, i.e., $f_l = \hat{F}_l \cos(\omega t)$, and let us seek a particular solution u of the form $u = u(x, t) = \hat{U}(x) \cos(\omega t)$ which corresponds to the HBM1 approximation explained earlier in Sec. 2.1. Hence, Eq. (2.6) yields:

$$\hat{U}'' + k^2 \hat{U} = 0 \quad x \in]0, l[\quad (2.8)$$

where $\hat{U} = \hat{U}(x)$ and $k = \omega \sqrt{\rho/E}$ is the *quasi*-longitudinal wavenumber, which represents the difference in phase between the propagating waves separated by unit length. The general solution of Eq. (2.8) is given by:

$$\hat{U} = C_1 \cos(kx) + C_2 \sin(kx) \quad (2.9)$$

where C_1 and C_2 are two constants whose determination follows from the analysis of the boundary conditions of the rod, at $x = 0$ and $x = l$. These are readily obtained by considering the boundary conditions in Eq.(2.7) which, in the frequency domain, are given by:

$$-EA\hat{U}'_l = \hat{F}_l \quad , \quad EA\hat{U}'_r = -s_1\hat{U}_r - \frac{3}{4}s_3\hat{U}_r^3, \quad (2.10)$$

where, $\hat{U}_l = \hat{U}(l)$ and $\hat{U}_r = \hat{U}(r)$ are the displacements at $x = 0$ and $x = l$, respectively.

2.2.1 Free vibration

The analysis of the free vibrations of the rod coupled to the nonlinear spring is proposed. In this framework, the left end of the rod is supposed to be free from the excitation source, i.e., $F_0 = 0$. The goal here is to investigate the influence of the local nonlinearity on the variation of the natural frequencies of the global system.

By considering Eq. (2.9) along with the boundary conditions (2.10) for $\hat{F}_l = 0$, the

following frequency equation can be obtained:

$$\tan(kl) = \frac{D(\hat{U}_r)}{EAk} = \frac{s_1}{EAk} + \frac{3}{4} \frac{s_3}{EAk} \hat{U}_r^2, \quad (2.11)$$

It is convenient to write the linear and nonlinear spring coefficients in a nondimensional form. Thereby, Eq. (2.11) can be rewritten as follows:

$$\tan(kl) = \frac{\gamma_1}{kl} + \frac{3}{4} \frac{\gamma_3}{kl} U_r^2, \quad (2.12)$$

where $\gamma_1 = s_1 l / EA$ and $\gamma_3 = s_3 l^3 / EA$ are dimensionless linear and nonlinear stiffnesses coefficients of the spring, and U_r is the displacement at $x = l$ divided by the length of the rod. For the linear case with $\gamma_3 = 0$, the frequency equation reduces to

$$\tan(kl) = \frac{\gamma_1}{kl} \quad (2.13)$$

Equation (2.13) is a transcendental equation, of which can be solved numerically. The solution of left and right-side hands terms for three different values of γ_1 are shown in Figure (2.4). It is possible to observe the shift of the roots from the free-free, $kl = n\pi/l$, to the free-fixed condition, $kl = (2n - 1)\pi/l$, for $n = 1, 2, \dots, j$, as the value of γ_1 increases. This means the system no longer behaves as a rod attached to a spring element, but as a clamped rod when the value of γ_1 increases. Also, it is possible to observe the decreasing influence of the spring element over the natural frequencies when the value of kl increases due to the asymptotic behavior of the systems for the different values of γ_1 .

Figure 2.5 shows the influence of the spring nonlinearity over the natural frequencies of a rod element, where both hardening and softening spring cases are analyzed. Considering the frequency equation for the hardening stiffness with $\gamma_3 > 0$ in Eq. (2.12), it can be observed that the equivalent stiffness increases as the value of γ_3 and U_r increases, which is an effect similar to the linear case presented in Figure 2.4. For the softening case, i.e., when $\gamma_3 < 0$, the equivalent stiffness is reduced as the values of the nonlinear stiffness decreases or the amplitude U_l increases. When related to the first mode of vibration, a critical value

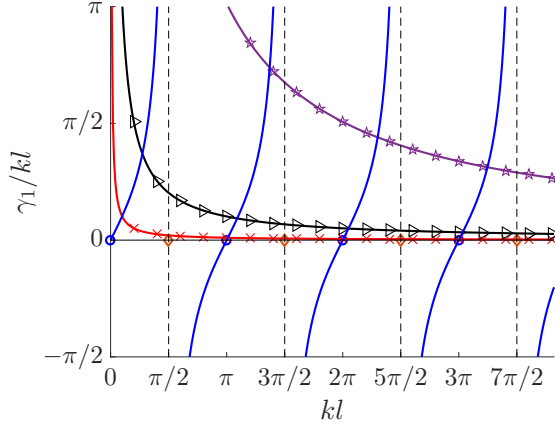


Figure 2.4: The solution of Equation (2.13) for $\gamma_1 = 0.1$ ($\times$), $\gamma_1 = 1$ ($\triangleright$), $\gamma_1 = 10$ ($\star$). Free-free natural frequencies ($\circ$), free-fixed natural frequencies ($\diamond$) and $\tan(kl)$ is represented by the continuous line ($-$).

for the nonlinear stiffness term occurs when $\gamma_3 = -4\gamma_1/3U_r^2$, which means, for a value of γ_3 smaller than the mentioned earlier, the equivalent stiffness becomes negative and the system becomes statically unstable. Equivalent behavior is achieved if the amplitude of displacement of the tip attached to the spring reaches the value of $U_r \geq \sqrt{-4\gamma_1/3\gamma_3}$.

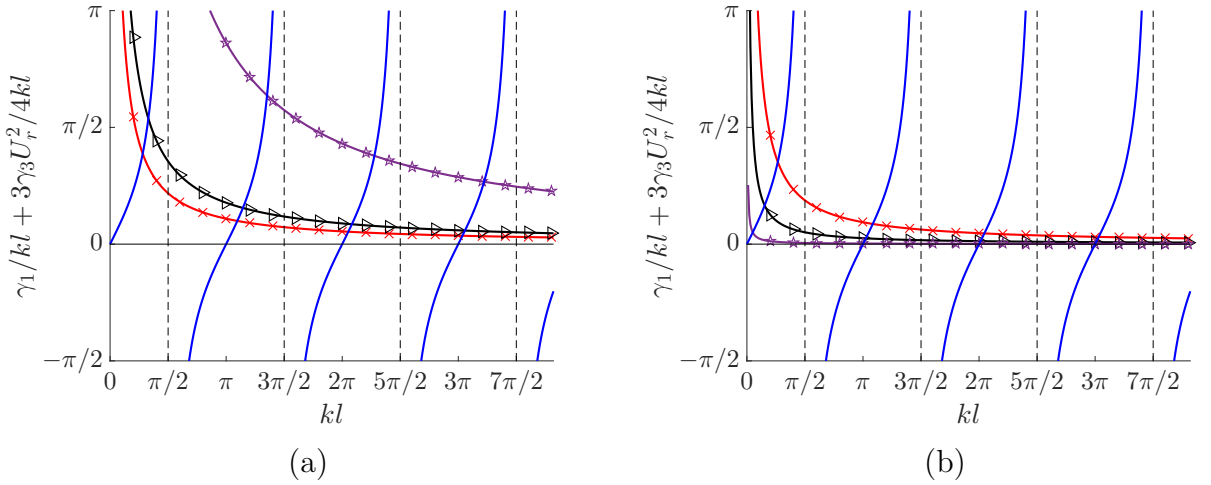


Figure 2.5: The solution of Equation (2.12) for $\gamma_1 = 1$ and $U_r = 0.1$: (a) Hardening spring for $\gamma_3 = 10$ ($\times$), $\gamma_3 = 100$ ($\triangleright$), $\gamma_3 = 1000$ ($\star$); (b) Softening spring for $\gamma_3 = -10$ ($\times$), $\gamma_3 = -100$ ($\triangleright$), $\gamma_3 = -130$ ($\star$). The continuous lines ($-$) represents $\tan(kl)$.

By plotting the solution of Equation (2.12) against the displacement U_r , which can be solved using a root-finding algorithm such as Levenberg-Marquardt, which can be accessed using the function `fsolve` in Matlab or Python-Scipy, yields the variation of the natural frequencies, commonly referred to as “backbone curves”.

Figure 2.6 shows the backbone curves of the first two modes for the hardening spring case $\gamma_3 > 0$. It is possible to observe that the natural frequencies of the rod attached to a nonlinear spring converge to those of the free-fixed rod when the displacement U_r increases. The solutions $k_j l$ lead to the natural frequencies ω_j of the system, which in the present case depend on the displacement U_r .

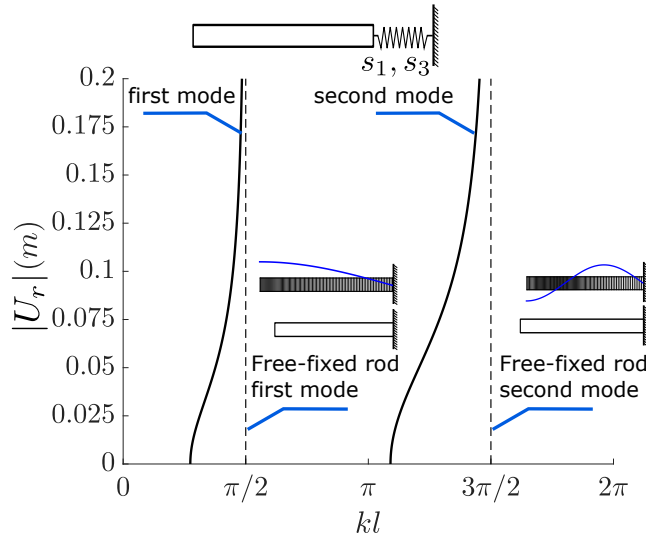


Figure 2.6: Backbone curves for the first two vibration modes of the rod when $\gamma_1 = 1$ and $\gamma_3 = 1000$. The dashed lines correspond to the natural frequencies of a free-fixed rod.

2.2.2 Forced vibrations

The forced response of the rod, whose right end is subjected to a harmonic force $f_l(t) = \hat{F}_l \cos(\omega t)$, as shown in Figure 2.3, is analyzed in this section. Emphasis is placed on assessing the effect of the nonlinear spring on the variation of the receptance functions.

By considering the boundary conditions (2.10) along with the general solution of the dynamic equilibrium equation (2.9), the following matrix equation can be obtained (FAHY;

WALKER, 2004) as in Eq. 2.14. The derivation of Eq. 2.14 is described in appendix A.

$$\begin{bmatrix} D_1 & D_2 \\ D_2 & D_1 + D(U_r) \end{bmatrix} \begin{bmatrix} U_l \\ U_r \end{bmatrix} = \begin{bmatrix} F_l \\ 0 \end{bmatrix}, \quad (2.14)$$

where $D_1 = \cot(kl)$, $D_2 = -1/\sin(kl)$, $U_l = \hat{U}_l/l$, $U_r = \hat{U}_r/l$, $D(U_r) = \gamma_1 + (3/4)\gamma_3 U_r^2$, and $F_l = \tilde{F}_0/k$, where $\tilde{F}_l = \hat{F}_l/EA$. Solving Eq. (2.14) yields the following receptances:

$$\frac{U_r}{F_l} = \frac{1}{D(U_r) \cos(kl) - \sin(kl)} \quad (2.15)$$

$$\frac{U_l}{F_l} = \frac{\cos(kl) + \sin(kl) D(U_r)}{D(U_r) \cos(kl) - \sin(kl)} \quad (2.16)$$

In particular, by considering Eq. (2.15) along with Eq. (2.5) for $U = U_r$ yields the following cubic equation for the displacement U_r :

$$\frac{3}{4} \frac{\gamma_3}{kl} \cos(kl) U_r^3 + \left[\frac{\gamma_1}{kl} \cos(kl) - \sin(kl) \right] U_r - F_l = 0, \quad (2.17)$$

which can be further simplified as follows:

$$U_r^3 + pU_r + q = 0, \quad (2.18)$$

where $p = 4[\gamma_1 - kl \tan(kl)]/3\gamma_3$ and $q = 4klF_l \sec(kl)/3\gamma_3$.

Equation (2.18) is a simple cubic equation that can be solved using appropriate solvers. Especially, the discriminant is given by $\Delta = -4p^3 - 27q^2$ and provides useful information about the roots of Eq. (2.18). The three possible determinant values are: $\Delta < 0$, $\Delta > 0$ and $\Delta = 0$. It is important to notice that the roots of the cubic equation represent the displacement amplitudes of a mechanical system, which means they should have real values. In this case, the complex solutions present no physical meaning. Therefore, in the polynomial methodology approach, only real roots are to be considered. For instance, the case $\Delta < 0$ means a single real root and a pair of complex conjugate roots, which are not physically consistent because the roots of the cubic equation represent the displacement

amplitude of a mechanical system. This yields region where the displacement U_r is uniquely defined for every frequency considered. In contrast, the case $\Delta > 0$ yields three real roots, and therefore, a frequency region where the displacement share three different values. For the case when $\Delta = 0$, the solution of Eq. (2.18) will present multiple real and with equal values roots. In the polynomial methodology approach, only real roots are to be considered.

The receptance functions $kl \mapsto |U_r|$ are plotted in Figures 2.7 when $\gamma_3 > 0$ (hardening spring). Also, the receptance function of the linear system — i.e., when $\gamma_3 = 0$ — is shown. As expected, when $\gamma_3 \neq 0$, several frequency bands occur (blue shaded areas) where the displacement share three different values. In addition, the backbone curves issued from Eq. (2.12) are displayed in red lines, which show that the natural frequencies of the system increase for the hardening spring case when U_l increases. Note that since damping effects are not taken into account, the resonance peaks of the system reach infinite values.

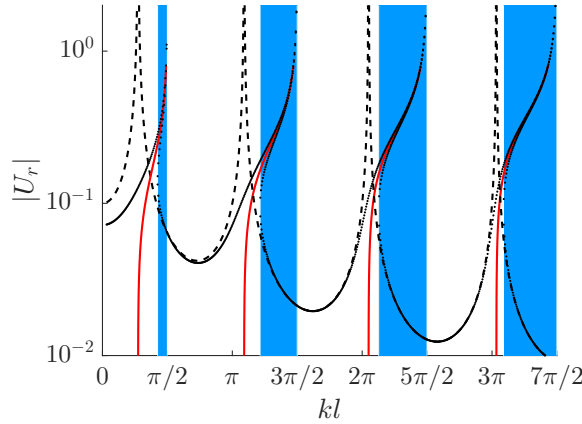


Figure 2.7: Receptance function $kl \mapsto |U_r|$ when $\gamma_1 = 1$, and $\tilde{F}_l = 0.1$ for $\gamma_3 = 100$. The blue shaded areas highlight the frequency bands where $\Delta > 0$ and the dashed curves shows the receptance function for the linear spring case when $\gamma_3 = 0$. The backbone curves are represented by the red lines.

To highlight the accuracy of the proposed approach, a comparison with a continuation method provided by the Python-based Dynamical Systems Toolkit *PyDSTool* (CLEWLEY et al., 2007) is offered as presented in Figure 2.8(a) for the hardening spring case. It is shown that the receptance function issued from the polynomial method closely matches the solution of the continuation method.

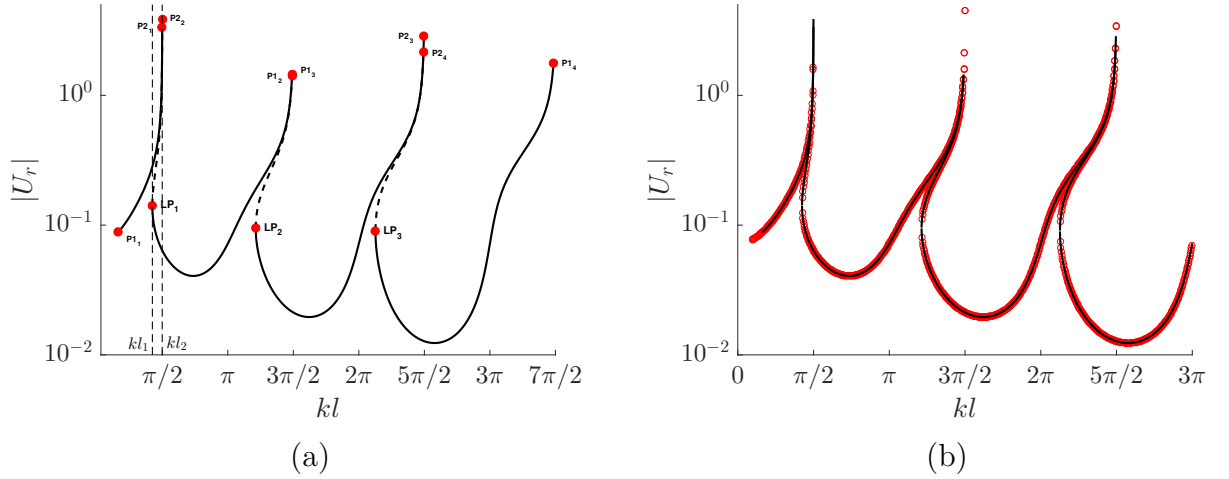


Figure 2.8: Receptance function $kl \mapsto |U_r|$ for a rod with one nonlinear springs: (a) Continuation method; (b) Comparison between continuation method (black line) and polynomial method (red dots).

Also, a bifurcation analysis can be proposed via the continuation method and the *PyDSTool* environment as shown in Figure 2.8(a). The solid and dashed lines correspond to stable and unstable branches, respectively, with beginning and ending points denoted by $P1_i$, $P2_i$, and LP_i ($i = 1, \dots, 4$). Around the first resonance in in Figure 2.8(a), there is a frequency band defined as $[kl_1, kl_2]$ referred to as interval of bistability (NAYFEH; BALACHANDRAN, 2008), and which is represented in blue shaded areas in Fig. 2.7, where kl_1 and kl_2 are the initial and final frequencies of the bistability region, respectively. The physical behavior of the system can be explained as follows. As the frequency term kl gradually increases from zero, the receptance follows the curve $P1_1 - P2_1$ until the critical value $kl = kl_2$ is reached. For this frequency, the stable and unstable branches converge to a same solution. The fact that the stable and unstable branches are close to $kl = kl_2$ follows from the assumption that the damping effects are not taken into account. As the frequency continues to increase, the receptance “jumps” from the stable branch $P1_1 - P2_1$ to the stable branch $LP_1 - P1_2$, and so on for the following resonances.

2.3 Chapter remarks

A polynomial methodology for closed-form solutions to investigate the nonlinear behavior of continuous structures is proposed. For simplicity of the methodology explanation,

the case of a homogeneous rod attached to a nonlinear spring was considered and discussed in detail. The assumption made is that nonlinear spring is governed by a cubic polynomial equation, which links the restoring force to the position at the forced end of the spring. The spring was considered symmetric and the restoring force approximated as a cubic polynomial with just odd terms. An assumption of which involves neglecting the third harmonic terms compared to the first harmonic ones is made and the dynamic stiffness of the nonlinear spring was derived using the first order Harmonic Balance Method (HBM1).

The free vibration case was analyzed to investigate the influence of the local nonlinearity on the variation of the natural frequencies for the hardening spring case. Backbone curves and mode shapes were drawn and it was found that for the hardening spring case, the natural frequencies of the rod, coupled to the spring, converge to those of the fixed-free rod when the amplitude increases.

The harmonically forced vibration case was analyzed next. The derivation of the equations using the polynomial approach was presented in detail for the case of a rod attached to a nonlinear spring. The polynomial approach makes use of closed-form solutions of the equation of motion of the rod attached to a nonlinear spring, which leads to scalar cubic polynomial equations. Receptance curves for both hardening and softening spring cases were presented and the regions where several stable and unstable bands occur, i.e., where the displacement share three different values called bi-stability regions were discussed in detail. To highlight the accuracy of the proposed approach, a comparison with a continuation method was proposed for the hardening spring case. It was shown that the receptance curves issued from the polynomial method closely match the solution of the continuation method.

Chapter 3

The case with multiple springs

This chapter proposes the investigation of the harmonic response of vibrating rods with an array of nonlinear springs. Two approaches are proposed. The first makes use of the polynomial approach presented in Chapter 2, which is well suited for accurately computing the receptance functions at some point of the rod. The second approach makes use of a perturbation method and a model reduction technique invoking a subspace projection, which consists of expanding the displacements of the rod on a reduced and finite basis of vibration modes. Furthermore, the influence of the spring attachment position on the dynamic response of the system is analyzed with and without damping. Numerical simulations are carried out and the dynamic behavior is compared to the case where only purely linear springs are considered.

3.1 The case with multiple cells using the polynomial methodology

It is possible to extend the previous analysis to consider the case where a rod is attached to multiple springs. For the sake of simplicity, the assumption of equally spaced springs is made, i.e., they divide a rod of length l into several undamped rod elements of the same length l/N where N is the number of springs considered, and that they share the equal stiffnesses s_1 and s_3 . Let us begin with a rod coupled to two nonlinear springs, as shown

in Figure 3.1.

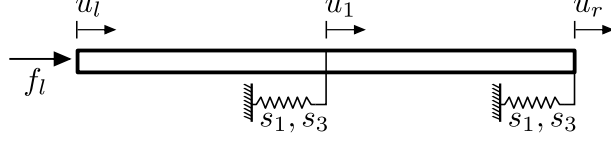


Figure 3.1: Schematic of a rod coupled to two nonlinear springs.

Again, let us assume that the left end of the rod is subject to a harmonic force $f_l = \hat{F}_l \cos(\omega t)$. Following the HBM1 assumption in Section 2.1, the displacements u_l , u_1 and u_r at the attachment points of the springs are expressed by $u_l = \hat{U}_l \cos(\omega t)$, $u_1 = \hat{U}_1 \cos(\omega t)$ and $u_r = \hat{U}_r \cos(\omega t)$. The dynamic stiffness matrix of the assembly involving two rod elements and two springs is built by considering a finite element procedure (ZIENKIEWICZ; TAYLOR, 2000), i.e.:

$$\begin{bmatrix} D_1 & D_2 & 0 \\ D_2 & 2D_1 + D(U_1) & D_2 \\ 0 & D_2 & D_1 + D(U_r) \end{bmatrix} \begin{bmatrix} U_l \\ U_1 \\ U_r \end{bmatrix} = \begin{bmatrix} F_l \\ 0 \\ 0 \end{bmatrix}, \quad (3.1)$$

where, $D_1 = \cot(kl/2)$ and $D_2 = -1/\sin(kl/2)$, $U_0 = \hat{U}_l/l$, $U_1 = \hat{U}_1/l$, $U_r = \hat{U}_r/l$, $F_l = \tilde{F}_l/k$, with $\tilde{F}_l = \hat{F}_l/ES$. The equivalent nondimensional stiffnesses of the springs are represented by $D(U_1) = \gamma_1 + (3/4)\gamma_3 U_1^2$, $D(U_r) = \gamma_1 + (3/4)\gamma_3 U_r^2$. Expressing the first row of the matrix system (3.1) gives:

$$D_1 U_l = -D_2 U_1 - F_l \quad (3.2)$$

Also, the last row gives:

$$D_2 U_1 = (D(U_r) + D_1) U_r \quad (3.3)$$

and replacing Equation (3.3) into Equation (3.2) yields

$$U_l = \frac{(D(U_r) + D_1) U_r + F_l}{D_1} \quad (3.4)$$

Introducing Eqs. (3.3) and (3.4) into the second row of Equation (3.1) finally leads to a

polynomial equation for the variable U_l :

$$\alpha_9 U_r^9 + \alpha_7 U_r^7 + \alpha_5 U_r^5 + \alpha_3 U_r^3 + \alpha_1 U_r + \alpha_0 = 0 \quad (3.5)$$

where α_0 , α_1 , α_3 , α_5 , α_7 and α_9 are frequency-dependent coefficients whose expressions are given as:

$$\begin{aligned} \alpha_0 &= \frac{\tilde{F}_l D_2}{kl D_1} \\ \alpha_1 &= \frac{D_2 \gamma_1}{D_1 kl} - \frac{2D_1 \gamma_1}{D_2 kl} - \frac{1}{D_2} \frac{\gamma_1^2}{(kl)^2} + 2D_2 - \frac{D_1^2}{D_2} \\ \alpha_3 &= \frac{3D_2 \gamma_3}{4D_1 kl} - \frac{3D_1 \gamma_3}{4D_2 kl} - \frac{3D_1^3 \gamma_3}{4D_2^3 kl} - \frac{3}{4D_2} \frac{\gamma_1 \gamma_3}{(kl)^2} \\ &\quad - \frac{9D_1^2 \gamma_1 \gamma_3}{4D_2^3 (kl)^2} - \frac{9D_1 \gamma_1^2 \gamma_3}{4D_2^3 (kl)^3} - \frac{3}{4D_2^3} \frac{\gamma_1^3 \gamma_3}{(kl)^4} \\ \alpha_5 &= -\frac{27D_1^2 \gamma_3^2}{16D_2^3 (kl)^2} - \frac{27D_1 \gamma_1 \gamma_3^2}{8D_2^3 (kl)^3} - \frac{27}{16D_2^3} \frac{\gamma_1^2 \gamma_3^2}{(kl)^4} \\ \alpha_7 &= -\frac{81D_1 \gamma_3^3}{64D_2^3 (kl)^3} - \frac{81}{64D_2^3} \frac{\gamma_1 \gamma_3^3}{(kl)^4} \\ \alpha_9 &= -\frac{81D_1 \gamma_3^4}{256D_2^3 (kl)^4} \end{aligned}$$

Equation (3.5) has many roots. However, only a few are real, i.e., have physical meaning, the rest of the roots being complex. Again, the receptance function of the system can be plotted, as shown in Figure 3.2. One of the key advantages of the proposed approach is that it can represent the multi-mode / multi-resonance behavior of the rod, i.e., the fact that several linear modes of the rod contribute to its dynamic response at the same frequency, and therefore, that several resonance peaks are observed over a wide frequency band. The analysis of a rod attached to an arbitrary number N of nonlinear springs involves a polynomial equation of order 3^N whose resolution can be cumbersome. For instance, a rod with three and four springs involves polynomials of order 27 and 81, respectively. In these cases, an analytical determination of the coefficients of the polynomial equation does not appear to be straightforward due to the higher number of coefficients. To solve this issue, a computer algebra software, like the Python Library *Sympy*, can be used to

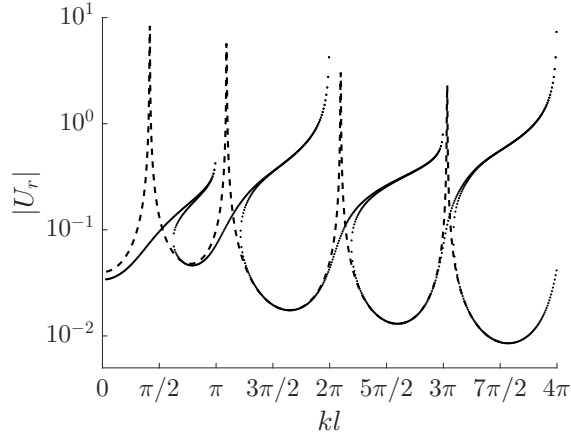


Figure 3.2: Receptance function $kl \mapsto |U_r|$ for a rod with two nonlinear springs, when $\gamma_1 = 1$, $\gamma_3 = 100$ and $\tilde{F}_l = 0.1$; the dashed curve shows the receptance function when $\gamma_3 = 0$ (linear case).

express the polynomial equation for U_r analytically. Also, the resolution of the polynomial equation, and further the computation of the receptances of the system, can be achieved by using a root-finding algorithm such as Levenberg-Marquardt, which can be accessed using the function `fsolve` in Matlab or Python-Scipy.

The receptance functions of the rod with three and four nonlinear springs are shown in Figure 3.3. In particular, it can be observed that the resonance peaks merge, for parts of them, and converge to the natural frequencies of the rod which is clamped on one or several points (local modes), i.e., when some springs are becoming infinitely stiff. This behavior results in a frequency shift of the resonance peaks to the high-frequency range, which appears to be a nice feature for passive control of nonlinear systems, in the sense that this creates wide frequency bands between resonances with low displacement levels. For instance, when $N = 4$, a wide low-frequency band corresponding to $kl < 12$ occurs before the first resonance peak. One may observe slight numerical dispersion errors in the vicinity of the resonance peaks in Figure 3.3 (b), due to the finite precision of the root-finding algorithm, which cannot distinguish the roots with high multiplicity (GALASSI et al., 2017). The proposed approach suffers from a severe computational burden when the number of springs increases. For instance, the cases with five and six springs would involve polynomial equations of orders 243 and 729, respectively. Especially, the most cumbersome issue concerns the “symbolic” calculation of the polynomial coefficients $(\alpha_0, \alpha_1, \alpha_3, \dots)$,

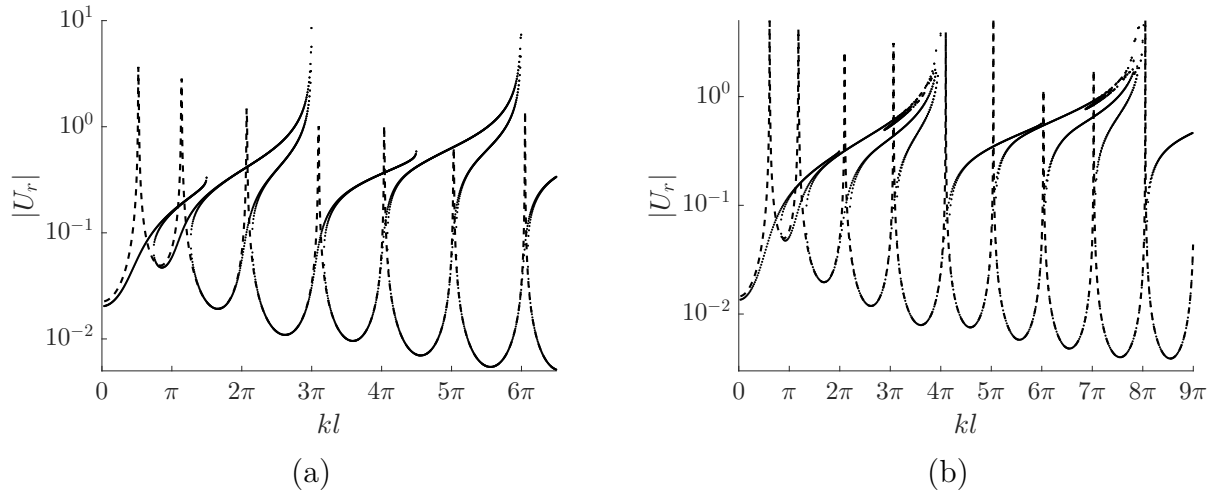


Figure 3.3: Receptance function $kl \mapsto |U_r|$ for a rod with (a) three nonlinear springs and (b) four nonlinear springs, when $\gamma_1 = 1$, $\gamma_3 = 100$ and $\hat{F}_l = 0.1$; the dashed curve shows the receptance function when $\gamma_3 = 0$ (linear case).

which is hardly handled via parallel computation, and which requires high memory storage. For the sake of clarity, the computational times involved with one, two, three and four springs, and an estimative for five and six springs are shown in Figure 3.4. All numerical simulations were obtained using a laptop equipped with a Core *i7* – 4700 CPU 3.40 Ghz $\times 4$, 16 GB RAM processor.

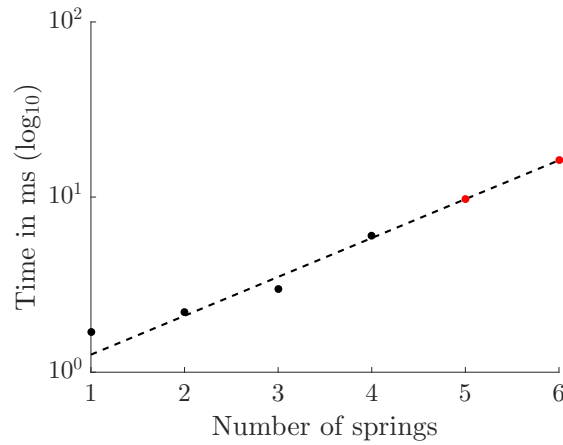


Figure 3.4: Computational times involved with one, two, three and four springs (black dots), and estimated time for five and six springs (red dots).

3.2 The influence of nonlinear stiffness on a rod using the perturbation method

To overcome the computational burden of the polynomial method, an alternative approach to investigate the nonlinear behavior for the rod attached to multiple nonlinear springs is proposed. This approach makes use of a model reduction technique, which used a subspace projecting considering a finite basis of linear modes, and a perturbation method approach (NARISSETTI; LEAMY; RUZZENE, 2010). Numerical simulations are carried out and the comparison with the polynomial approach is made.

3.2.1 Forced vibrations of a rod with multiple springs

As in Chapter 2, let us consider a rod of length l which is subject to a harmonic force $f_l = \hat{F}_l \cos(\omega t)$ on its left end (Figure 3.5), and which is coupled to N nonlinear springs located at several abscissa x_s . Although not mandatory, the springs are supposed to be equally spaced and share the same properties. Also, for the sake of simplicity, it will be assumed that the rod is undamped.

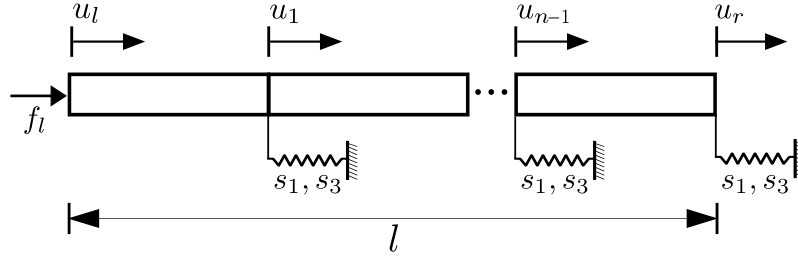


Figure 3.5: Schematic of a rod with an array of equally spaced nonlinear springs.

Let us denote by $u = u(x, t)$ the longitudinal displacement of the rod, and by $s_1 u(x_s, t) + \epsilon s_3 u(x_s, t)^3$ the restoring force for a spring at position x_s , where $\epsilon s_3 u(x_s, t)^3$ is supposed to be a small perturbation. In this case, the cubic stiffness will be denoted by ϵs_3 , instead of s_3 . The proposed method starts by considering the global boundary value problem of the rod, i.e.:

$$\rho A \ddot{u} - EA u'' + \sum_{s=1}^N [s_1 u + \epsilon s_3 u^3] \delta_{x-x_s} = 0, \quad x \in]0, l[, \quad (3.6)$$

$$EA u'(0, t) = -\hat{F}_l \cos(\omega t), \quad (3.7)$$

$$u(l, t) = 0, \quad (3.8)$$

where δ_{x-x_s} is a Dirac Delta function, defined by

$$\int_0^l g(x) \delta_{x-x_s} dx = g(x_s) \quad (3.9)$$

for any function $g(x)$. The integral/weak form of the boundary value problem can be obtained as follows. Let us (left) multiply both sides of Equation (3.6) by a test function $\delta u = \delta u(x)$, defined so that $\delta u(0) = 0$, and integrate the resulting equation over the space domain $]0, l[$:

$$\rho A \int_0^l \delta u \ddot{u} dx - EA \int_0^l \delta u u'' dx + \sum_{s=1}^N \delta u_s [s_1 u_s + \epsilon s_3 u_s^3] = 0 \quad (3.10)$$

where $u_s = u(x_s, t)$ and $\delta u_s = \delta u(x_s)$. Integrating the second integral term in Equation (3.10) by parts

$$-EA \int_0^l \delta u u'' dx = -EA \delta u(l) u'(l, t) + EA \delta u(0) u'(0, t) + EA \int_0^l \delta u' u' dx \quad (3.11)$$

and considering the boundary condition (3.8), as well as the boundary condition $\delta u(l) = 0$, yields

$$-EA \int_0^l \delta u u'' dx = -\delta u(0) F_0 \cos(\omega t) + EA \int_0^l \delta u' u' dx \quad (3.12)$$

As a result, Equation (3.10) can be rewritten as follows:

$$\rho A \int_0^l \delta u \ddot{u} dx + EA \int_0^l \delta u' u' dx + \sum_{s=1}^N \delta u_s [s_1 u_s + \epsilon s_3 u_s^3] = \delta u_0 F_l \cos(\omega t) \quad (3.13)$$

where $\delta u_0 = \delta u(0)$. Equation (3.13) is the weak form of the boundary value problem

(3.6)-(3.8). It is well suited to find approximate solutions of the boundary value problem via subspace projection. The projection basis considered here is that of the linear vibration modes of the free-free uncoupled rod, $X_j = X_j(x)$ for $j = 0, 1, 2, \dots$, defined by:

$$X_0 = \sqrt{\frac{1}{\rho Al}} \quad \text{and} \quad X_j = \sqrt{\frac{2}{\rho Al}} \cos\left(\frac{j\pi x}{l}\right) \text{ for } j \geq 1 \quad (3.14)$$

with the properties that:

$$\rho A \int_0^l X_j X_k dx = \delta_{jk} \quad , \quad EA \int_0^l X_j' X_k' dx = (\omega_j^0)^2 \delta_{jk} \quad (3.15)$$

where δ_{jk} denotes the Kronecker symbol, and ω_j^0 denotes the natural frequencies of the corresponding linear system:

$$\omega_j^0 = \frac{j\pi}{l} \sqrt{\frac{E}{\rho}} \quad \text{for } j \geq 0 \quad (3.16)$$

The proposed strategy consists in expressing the displacement u of the coupled rod using a finite set of $m + 1$ low-order modes $(X_0, X_1, \dots, X_m)$, i.e.:

$$u \approx \sum_{k=0}^m X_k \alpha_k \quad (3.17)$$

where $\alpha_k = \alpha_k(t)$ denotes mode amplitudes. By considering Equation (3.17) and a test function $\delta u = X_j$, and by considering the properties (3.15), the integral form (3.13) yields:

$$\ddot{\alpha}_j + (\omega_j^0)^2 \alpha_j + s_1 \sum_{s=1}^N X_{js} \left[\sum_{k=0}^m X_{ks} \alpha_k \right] = F_j^{\text{mod}} \cos(\omega t) - \epsilon s_3 \sum_{s=1}^N X_{js} \left[\sum_{k=0}^m X_{ks} \alpha_k \right]^3 \quad (3.18)$$

where $X_{js} = X_j(x_s)$ and $X_{ks} = X_k(x_s)$. Also, F_j^{mod} is defined by:

$$F_0^{\text{mod}} = X_0(0) \hat{F}_l = \sqrt{\frac{1}{\rho Al}} \hat{F}_l \quad (3.19)$$

$$F_j^{\text{mod}} = X_j(0) \hat{F}_l = (-1)^j \sqrt{\frac{2}{\rho Al}} \hat{F}_l \quad \text{for } j \geq 1 \quad (3.20)$$

Equation. (3.18) is a time differential nonlinear equation in the variable α_j , where coupling

terms occur between modes due to coupling with the springs. For simplification, it is supposed that the mode amplitudes α_j and the natural frequency ω_j^0 can be expanded at first-order by considering a small perturbation in the same way as the restoring forces of the strings:

$$\alpha_j \approx \alpha_j^0 + \epsilon \alpha_j^1 \quad \text{and} \quad (\omega_j^0)^2 \approx \omega_j^2 - \epsilon \lambda_j^1 \quad (3.21)$$

Introducing Equation (3.21) in Equation (3.18), and equating the terms of identical power of ϵ to zero gives:

- Power 0 (ϵ^0):

$$\ddot{\alpha}_j^0 + \omega_j^2 \alpha_j^0 + s_1 \sum_{s=1}^N X_{js} \left[\sum_{k=0}^m X_{ks} \alpha_k^0 \right] = F_j^{\text{mod}} \cos(\omega t) \quad (3.22)$$

- Power 1 (ϵ^1):

$$\ddot{\alpha}_j^1 + \omega_j^2 \alpha_j^1 + s_1 \sum_{s=1}^N X_{js} \left[\sum_{k=0}^m X_{ks} \alpha_k^1 \right] = \lambda_j^1 \alpha_j^0 - s_3 \sum_{s=1}^N X_{js} \left[\sum_{k=0}^m X_{ks} \alpha_k^0 \right]^3 \quad (3.23)$$

Introduce the following $(m+1) \times 1$ vectors $\boldsymbol{\alpha}^0$, $\boldsymbol{\alpha}^1$ and $\mathbf{F}^{\text{mod}}$, and the $(m+1) \times (m+1)$ impedance matrix $\mathbf{Z}^{\text{mod}}$:

$$\begin{aligned} \boldsymbol{\alpha}^0 &= \begin{bmatrix} \alpha_0^0 \\ \vdots \\ \alpha_m^0 \end{bmatrix}, \quad \boldsymbol{\alpha}^1 = \begin{bmatrix} \alpha_0^1 \\ \vdots \\ \alpha_m^1 \end{bmatrix}, \quad \mathbf{F}^{\text{mod}} = \begin{bmatrix} F_0 \\ \vdots \\ F_m \end{bmatrix}, \\ \mathbf{Z}^{\text{mod}} &= s_1 \sum_{s=1}^N \begin{bmatrix} X_{0s}X_{0s} & \cdots & X_{0s}X_{ms} \\ \vdots & \ddots & \vdots \\ X_{ms}X_{0s} & \cdots & X_{ms}X_{ms} \end{bmatrix} \end{aligned} \quad (3.24)$$

Thus, it is possible to rewrite Equation (3.22) in matrix form, i.e.:

$$\ddot{\boldsymbol{\alpha}}^0 + \left(\text{diag} \{ \omega_j^2 \}_{j=0}^m + \mathbf{Z}^{\text{mod}} \right) \boldsymbol{\alpha}^0 = \mathbf{F}^{\text{mod}} \cos(\omega t) \quad (3.25)$$

The stationary solution of Equation (3.25) can be sought in the form $\boldsymbol{\alpha}^0 = \underline{\boldsymbol{\alpha}}^0 \cos(\omega t)$ where $\underline{\boldsymbol{\alpha}}^0 = \underline{\boldsymbol{\alpha}}^0(\omega) = [\underline{\alpha}_0^0 \cdots \underline{\alpha}_m^0]^T$. This yields:

$$\left[-\omega^2 \mathbf{I}_{m+1} + \text{diag} \{ \omega_j^2 \}_{j=0}^m + \mathbf{Z}^{\text{mod}} \right] \underline{\boldsymbol{\alpha}}^0 = \mathbf{F}^{\text{mod}}, \quad (3.26)$$

where $\mathbf{I}_{m+1}$ denotes the $(m+1) \times (m+1)$ identity matrix. Also, in Equation (3.23), α_k^0 is to be expressed as $\underline{\alpha}_k^0 \cos(\omega t)$. Therefore, by considering the following trigonometric relation

$$\cos(\omega t)^3 = \frac{3}{4} \cos(\omega t) + \frac{1}{4} \cos(3\omega t) \quad (3.27)$$

Equation (3.23) leads to:

$$\begin{aligned} \ddot{\boldsymbol{\alpha}}^1 + \left(\text{diag}\{\omega_j^2\}_{j=0}^m + \mathbf{Z}^{\text{mod}} \right) \boldsymbol{\alpha}^1 &= \text{diag}\{\lambda_j^1\}_{j=0}^m \underline{\boldsymbol{\alpha}}^0 \cos(\omega t) \\ &+ \mathbf{F}^{\text{nl}}(\underline{\boldsymbol{\alpha}}^0) \left(\frac{3}{4} \cos(\omega t) + \frac{1}{4} \cos(3\omega t) \right) \end{aligned} \quad (3.28)$$

where

$$\mathbf{F}^{\text{nl}}(\underline{\boldsymbol{\alpha}}^0) = -s_3 \sum_{s=1}^N \begin{bmatrix} X_{0s} (\mathbf{X}_s \underline{\boldsymbol{\alpha}}^0)^3 \\ \vdots \\ X_{ms} (\mathbf{X}_s \underline{\boldsymbol{\alpha}}^0)^3 \end{bmatrix} \quad (3.29)$$

and $\mathbf{X}_s = [X_{0s} \cdots X_{ms}]$. To derive Equation (3.29), the following equality has been considered:

$$\sum_{j=0}^m X_{js} \alpha_j^0 = \mathbf{X}_s \boldsymbol{\alpha}^0 \quad (3.30)$$

The secular term in Equation (3.28) can be identified as follows:

$$\left(\text{diag}\{\lambda_j^1\}_{j=1}^m \underline{\boldsymbol{\alpha}}^0 + \frac{3}{4} \mathbf{F}^{\text{nl}}(\underline{\boldsymbol{\alpha}}^0) \right) \cos(\omega t) \quad (3.31)$$

This term represents an artifact introduced by the perturbation method, and can be the source of large values of $\boldsymbol{\alpha}^1$ when the excitation frequency ω tends to the resonance frequencies of the system. This is, of course, incompatible with the assumption that $\epsilon \boldsymbol{\alpha}^1$ is small compared to $\boldsymbol{\alpha}^0$ (MEIROVITCH, 2010). Hence, this term has to be canceled, i.e.,

$$\text{diag}\{\lambda_j^1\}_{j=1}^m \underline{\boldsymbol{\alpha}}^0 = -\frac{3}{4} \mathbf{F}^{\text{nl}}(\underline{\boldsymbol{\alpha}}^0) \quad (3.32)$$

Introducing Equation (3.32) into Equation (3.26) yields the following system of cubic

equations for the variables $\underline{\alpha}_j^0$:

$$\left[-\omega^2 \mathbf{I}_{m+1} + \text{diag}\{(\omega_j^0)^2\}_{j=0}^m + \mathbf{Z}^{\text{mod}} \right] \underline{\alpha}^0 - \varepsilon \frac{3}{4} \mathbf{F}^{\text{nl}}(\underline{\alpha}^0) = \mathbf{F}^{\text{mod}} \quad (3.33)$$

The harmonic response of the system, for the harmonic ω , follows by expressing the displacement vector $U = U(x, \omega)$:

$$U \approx \sum_{j=0}^m X_j \underline{\alpha}_j^0 \quad (3.34)$$

Compared to the polynomial method presented, the perturbation method based on Equation (3.33) is advantageous for two reasons: (1) the coefficients of Equation 3.33 can be easily computed, whatever the number of springs considered; (2) Equation 3.33 is a cubic equation, i.e., with a small degree compared to the equations involved in Chapter 2, which can be well handled by appropriate solvers. However, the perturbation method may suffer from two issues: (1) it involves a matrix equation with nonlinear terms, instead of a scalar equation used on the polynomial method; (2) it makes use of two approximations, which come from the subspace projection, with the use of the expression of the displacement with a reduced number of modes, and the perturbation method itself using the first-order expansion in the present case. The nonlinear matrix equation (3.33) is cubic and can be solved via an appropriate nonlinear solver. Attention must be paid to the selection of the initial guess for $\underline{\alpha}_j^0$ to ensure convergence to find all the roots of the equation at each frequency considered. In fact, root-finding algorithms are likely to be highly sensitive to the choice of the initial guess. The proposed strategy consists of using a real solution U_r of Equation (2.18), i.e., which is issued from the polynomial method for a rod coupled to one spring, and dividing U_l by the mode shape $X_j(0)$ to obtain a coarse approximation of the mode amplitude $\underline{\alpha}_j^0$. Hence, the following initial guess is proposed:

$$\text{Initial guess for the vector } \underline{\alpha}^0 \rightarrow \begin{bmatrix} U_r/X_0(0) \\ \vdots \\ U_r/X_m(0) \end{bmatrix} \quad (3.35)$$

Once a root for $\underline{\alpha}^0$ has been identified, it can be used as a second initial guess to find the second root, and so on for the third and subsequent roots.

3.3 Numerical results

The receptance functions $kl \mapsto |U_r|$ for an undamped rod with one or several nonlinear springs are computed and compared to those issued from the polynomial method. The material and geometrical parameters of the rod are: Young's modulus of 210 GPa, density of 7800 kg/m³, length of $l = 1$ m and cross-sectional area of $A = 0.001$ m². For comparison purpose, the parameters of the springs are selected so that they yield the same values as the normalized stiffness coefficients γ_1 and γ_3 used in Chapter 2, i.e.: linear stiffness $s_1 = 1 \times (EA/l)$ and cubic stiffness $\epsilon s_3 = 100 \times (EA/l^3)$ (hardening spring case). Also, the right end of the rod is subjected to a harmonic force $\hat{F}_l \cos(\omega t)$ of amplitude $\hat{F}_l = 0.1 \times (EA)$.

3.3.1 Rod with a nonlinear spring

The analysis of the dynamic behavior of the rod with one nonlinear spring at the right end $x = l$ is proposed. The receptance $|U_r| \mapsto kl$, where U_r is the nondimensional displacement at the right end is computed with the perturbation method and compared to the solution issued from the polynomial method. In this case, the number of vibration modes considered is $m + 1 = 100$, which means that the size of the nonlinear matrix equation (3.33) is 100×100 . Once Equation (3.33) has been solved, the displacement U_r is retrieved by considering Equation (3.34). The results are shown in Figure 3.6. It is shown that the solution issued from the perturbation method is in good agreement with the polynomial method. Especially, the frequency behavior of the receptance function in the vicinity of the resonance frequencies is well captured. In this case, the errors made at describing the resonance frequencies of the system are about 0.1%.

Attention must be paid to the choice of the number of vibration modes ($m + 1$) to describe the displacement of the rod. In linear elasticity, it is common to select the modes whose eigenfrequencies are less than two times, or three times, the maximum frequency of

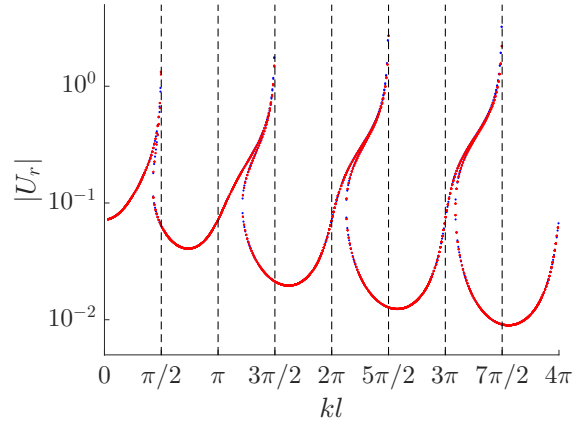


Figure 3.6: Receptance function $kl \mapsto |U_r|$ for a rod with one nonlinear spring: (blue dots) polynomial method; (red dots) perturbation method.

the frequency band considered. The study of nonlinear systems is more complicated in the sense that it requires a larger number of linear vibration modes to accurately describe the complex “nonlinear” frequency behavior around resonances. For instance, Figure 3.7 shows the receptance functions obtained with $m + 1 = 10$ modes. In this case, errors occur around resonances and significantly increase as the resonance order does. The selection of the number of modes is not an easy task, and one may recommend carrying out a convergence analysis of the perturbation method, e.g., by computing and analyzing the displacement at some frequencies close to the maximum frequency of the frequency band considered.

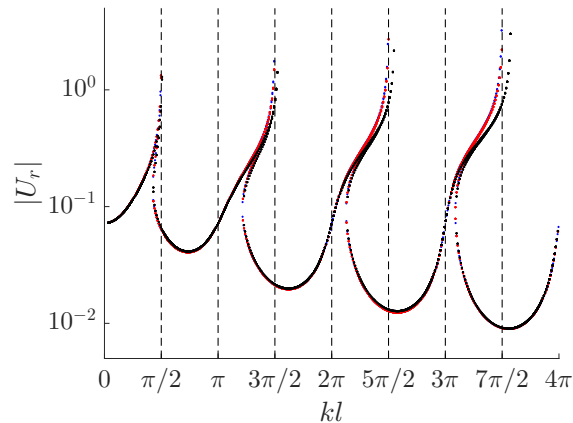


Figure 3.7: Receptance function $kl \mapsto |U_r|$ for a rod with one nonlinear spring: (blue dots) polynomial method; (red dots) perturbation method with 100 modes; (black line) perturbation method with 10 modes.

3.3.2 Multiple springs

Results about a rod with two equally spaced springs are shown in Figure 3.8. Again, the perturbation method is proved to be accurate, provided that the number of vibration modes retained to describe the displacement of the structure is high enough. In this case, considering a low number of modes appears to be dramatic for describing the receptance around resonances, hence leading to a wrong understanding of the dynamic behavior of the structure.

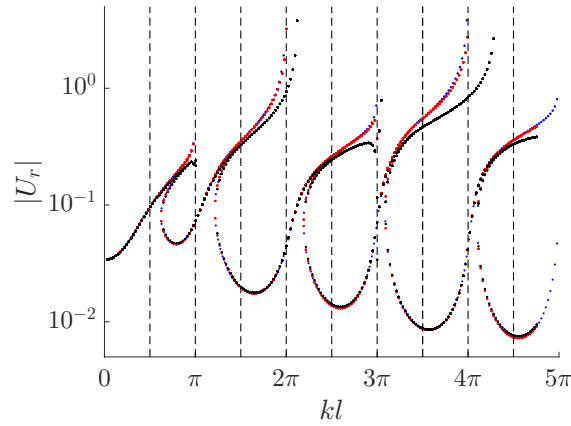


Figure 3.8: Receptance function $kl \mapsto |U_r|$ for a rod with two nonlinear springs: (blue dots) polynomial method; (red dots) perturbation method with 100 modes; (black dots) perturbation method with 10 modes.

Comparisons are carried out when three and four equally spaced nonlinear springs are dealt with as shown in Figures 3.9(a) and 3.9(b), which further highlight the accuracy of the perturbation method. Additional simulations are undertaken for rods with five and six nonlinear springs as shown in Figures 3.10(a) and 3.10(b). It is seen that the resonance frequencies of the system are shifted to high frequencies when the number of springs increases, which reveals an interesting feature for the passive control of vibrating structures. To meet the convergence criteria of the perturbation method, since wider frequency bands are analyzed, a higher number of modes have been retained, i.e., $m + 1 = 200$ modes for the rod with three springs and four springs, $m + 1 = 300$ for the rod with five springs, and $m + 1 = 400$ for the rod with six springs. Again, slight numerical dispersion errors can be observed close to the resonances. The interesting feature of the perturbation method is that it can handle, in theory, rods with an arbitrary number N of springs. In contrast, the

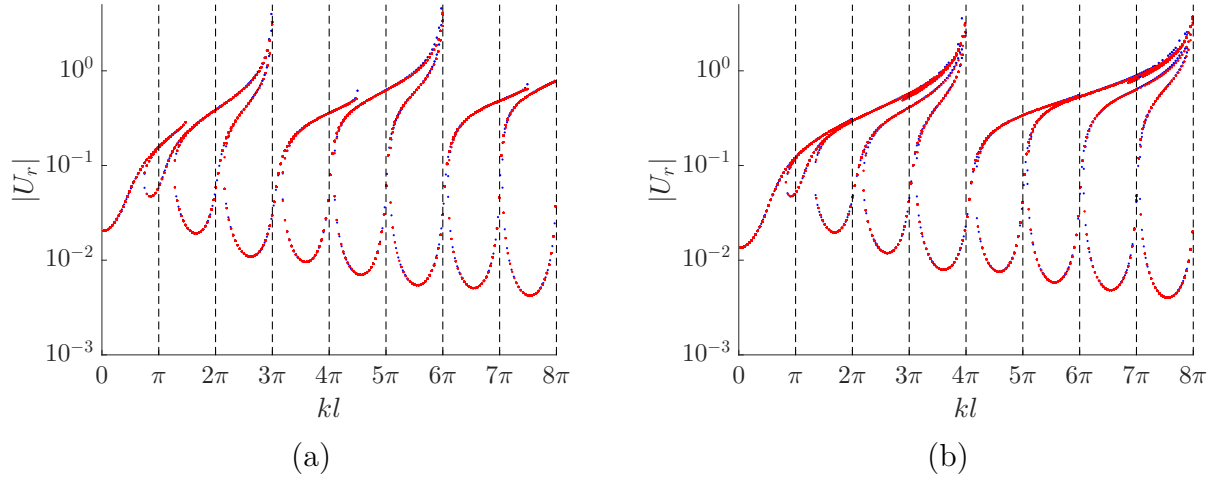


Figure 3.9: Receptance function $kl \mapsto |U_r|$ for a rod with (a) three, and (b) four nonlinear springs: (blue dots) polynomial method; (red dots) perturbation method.

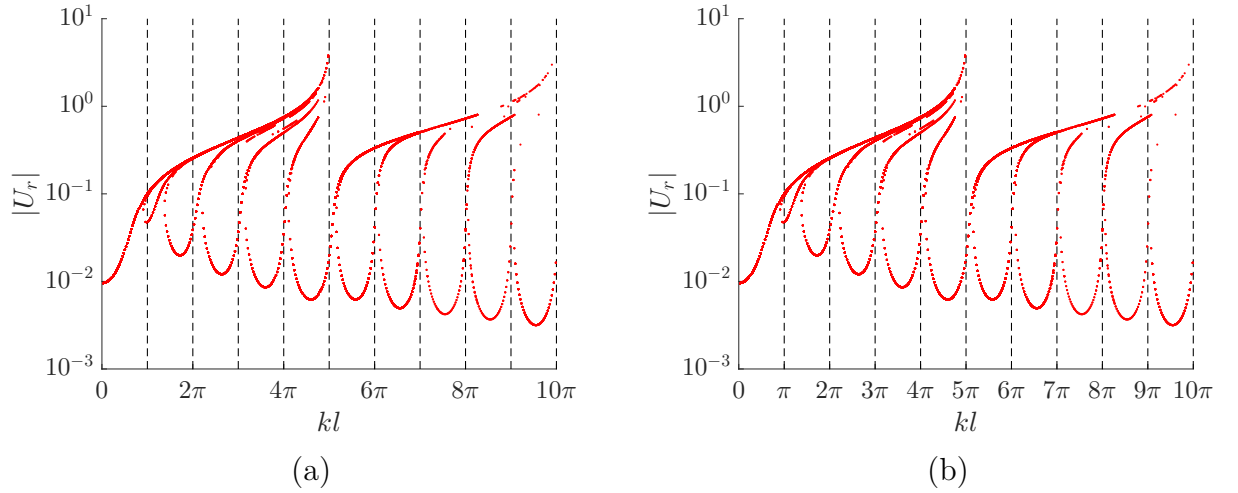


Figure 3.10: Receptance function $kl \mapsto |U_r|$ for a rod with (a) five and (b) six nonlinear springs using the perturbation method.

polynomial method appears to be limited when the number of springs exceeds five (see Section 3.1). For the sake of clarity, the computational times involved in both approaches are shown in Figure 3.11 and also in Table 3.1.

Table 3.1: Computational times involved for polynomial and perturbation approaches.

Method	Time in ms ($\log_{10}$)					
	$N=1$	$N=2$	$N=3$	$N=4$	$N=5$	$N=6$
Polynomial	1.68	2.15	2.99	6.04	9.75	16.23
Perturbation	6.29	6.70	7.76	7.85	8.40	8.87

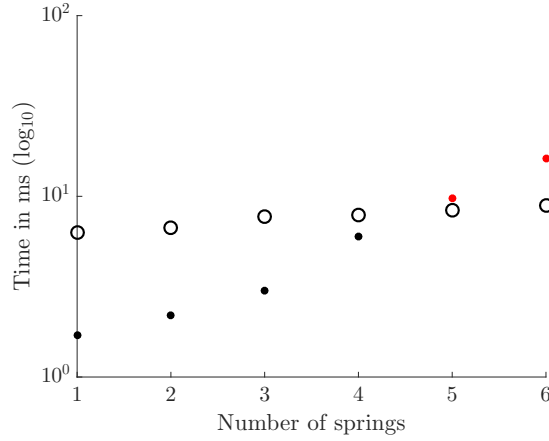


Figure 3.11: Computational times involved for one to six springs: (black circles) perturbation method; (black dots) polynomial method. The estimated times are highlighted in red for the polynomial method with $N = 5$ and $N = 6$ are highlighted in red.

It is seen that the perturbation method outperforms the polynomial method as soon as $N \geq 5$. The rate of growth of the computational times, concerning the number of springs, is much less compared to the polynomial method. For instance, the perturbation method is about $10^{8.40}/10^{9.75} = 10^{-1.35}$ and $10^{8.87}/10^{16.23} = 10^{-7.36}$ less cumbersome than the polynomial to compute the receptance function of the rod for $N = 5$ and $N = 6$, respectively.

Further insight about the receptance functions shows that, when the number of springs N is greater than 2, the resonances of the system occur when $kl = jN(\pi/2)$ for $j = 1, 2, \dots$. This means that for large-amplitude U_l , the system behaves like a fixed-fixed rod of length l/N . This is explained because the stiffness of the nonlinear hardening springs increases with U_l , causing them to behave like fixed boundaries. This also means that, for large N , the resonance peaks shift to the high-frequency range as remarked earlier with the polynomial method. This seems interesting in that the vibration levels of the structure can be reduced at low-frequency.

3.3.3 The influence of the spring attachment position

The influence of the spring attachment position along the rod on the dynamic behavior is taken into consideration for different configurations, with both linear and nonlinear spring cases. All the springs have the same linear and nonlinear coefficient parameters,

$\gamma_1 = 1$, $\gamma_3 = 100$. The configurations 1 to 3 displayed in Figure 3.12 considers the cases with one spring up to two springs, and the attachment positions are described in Table 3.2.

Table 3.2: System configurations taking into account different positions for the spring.

	Spring position		
Configuration 1	–	–	$x = l$
Configuration 2	–	$x = l/2$	–
Configuration 3	–	$x = l/2$	$x = l$

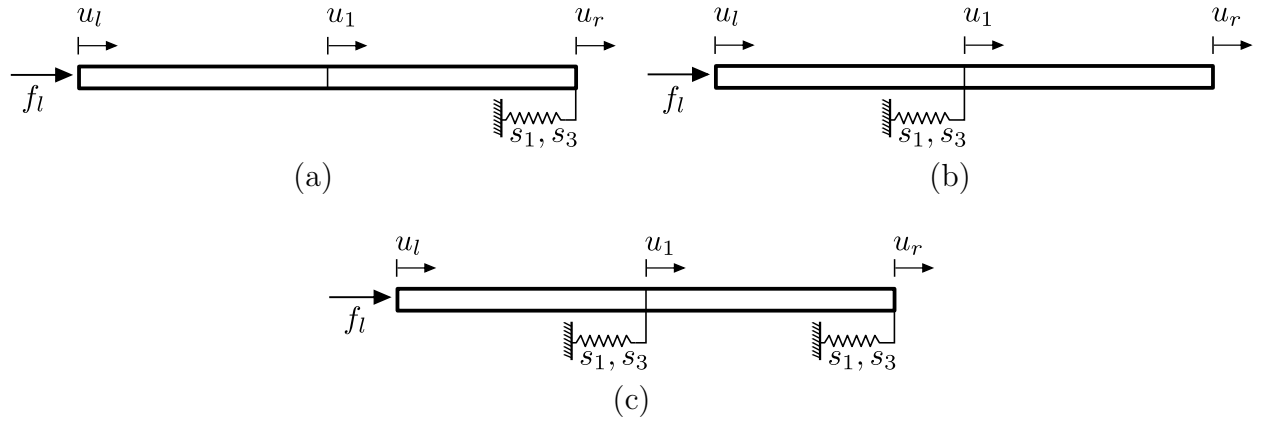


Figure 3.12: Schematic diagram of a rod attached nonlinear springs for the configurations shown in Table 3.2 for configurations: (a) 1, (b) 2 and (c) 3.

Figure 3.13 shows the comparison between the receptances curves for configurations 1, 2, and 3 considering both linear and nonlinear springs. It is possible to observe that the FRFs (Frequency Response Functions) are strongly influenced by the number of nonlinear springs and their coupling positions.

Configuration 1 presents the results already discussed in the previous section, where attach the spring at one of the rod ends, which cause the resonance peaks to appear close to the frequencies of a free-fixed rod, as long as the amplitude of U_r increases. Configuration 2 represents the case where a nonlinear spring is attached at $x = l/2$. The response curve for this configuration displays the coexistence of both linear and nonlinear behaviors. As kl increases, the frequency response peaks tend to shift to the clamped-free case with $l/2$, except at the resonance frequencies of the free-free rod where the attachment position of the spring represents a node point, and the influence of the nonlinear term of the spring can be neglected due to the small vibration amplitude at the node point. A rod with

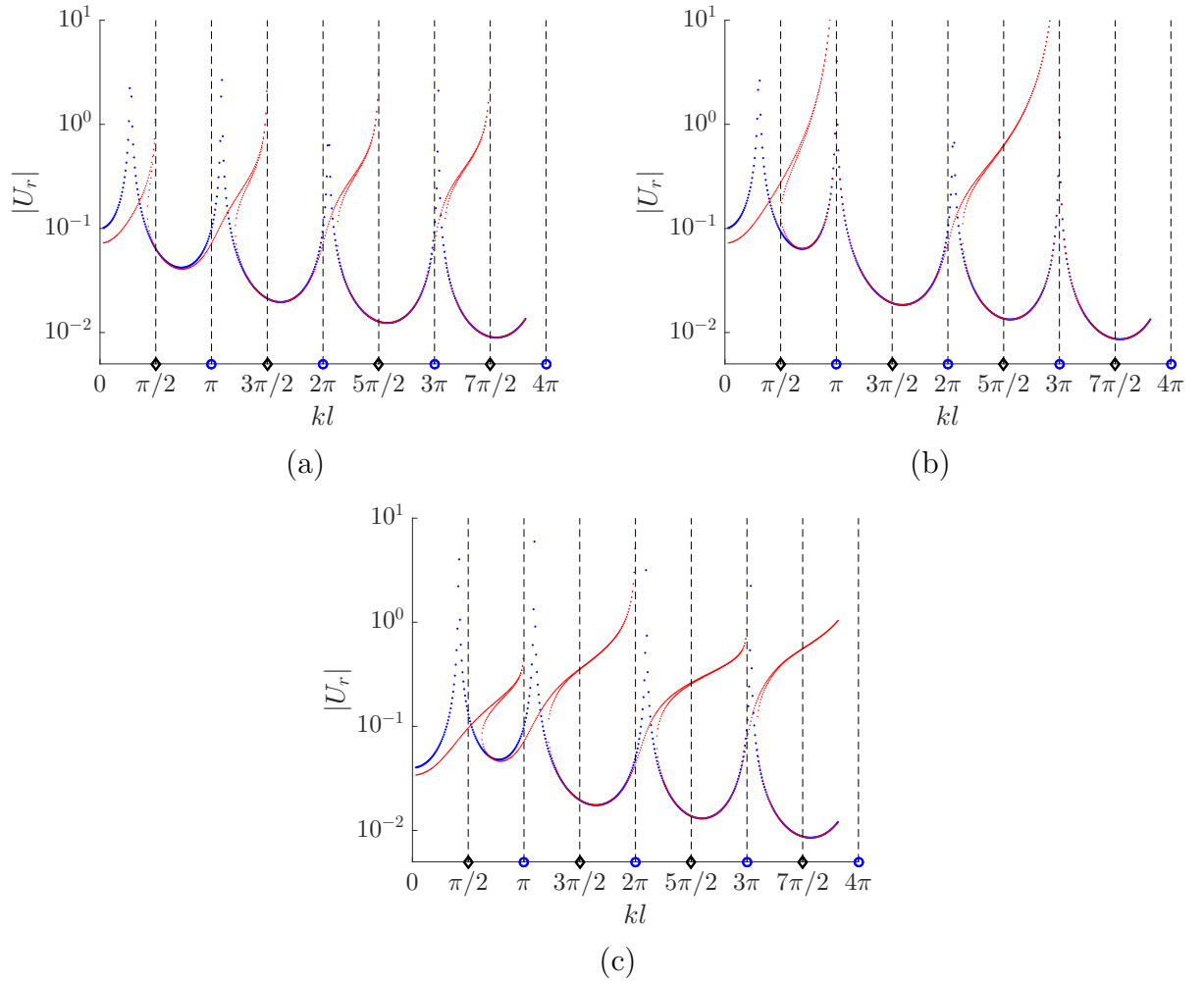


Figure 3.13: Frequency response curves for the configurations shown in Table 3.2: (a) Configuration 1; (b) Configuration 2 and (c) Configuration 3. The blue curves represents the linear case.

two springs attached at $x = l/2$ and $x = l$ is the case represented in configuration 3. In this configuration, the frequency response curves display the resonance peaks to shift to the clamped-free case for a rod with length $l/2$ when kl approximates to the frequencies where the position $x = l/2$ represents a node point. Also, for the frequencies where both positions $x = l/2$ and $x = l$ show large amplitudes, it is possible to observe the frequency peaks shifting to the resonances of the clamped-clamped case for a rod with length $l/2$.

Additional configurations considering different attachment position and the comparisons between linear and nonlinear springs are shown in Table 3.3 and their dynamic response presented in Figure 3.14.

It is possible to observe that the number of nonlinear springs considered and their

Table 3.3: System configuration for different spring attachment positions.

	Spring attachment position		
Configuration 4	$x = l/3$	—	—
Configuration 5	—	$x = 2l/3$	—
Configuration 6	$x = l/3$	—	$x = 2l/3$
Configuration 7	$x = l/3$	—	$x = l$
Configuration 8	—	$x = 2l/3$	$x = l$

attachment position have a large influence over the dynamic response of the system. Their contribution to the dynamic behavior is directly related to the amplitude of the displacement at the attachment position. Therefore, the change of the attachment position along with the structure accordingly to the amount of amplitude of vibration of that specific point can be an interesting strategy to control vibrations passively.

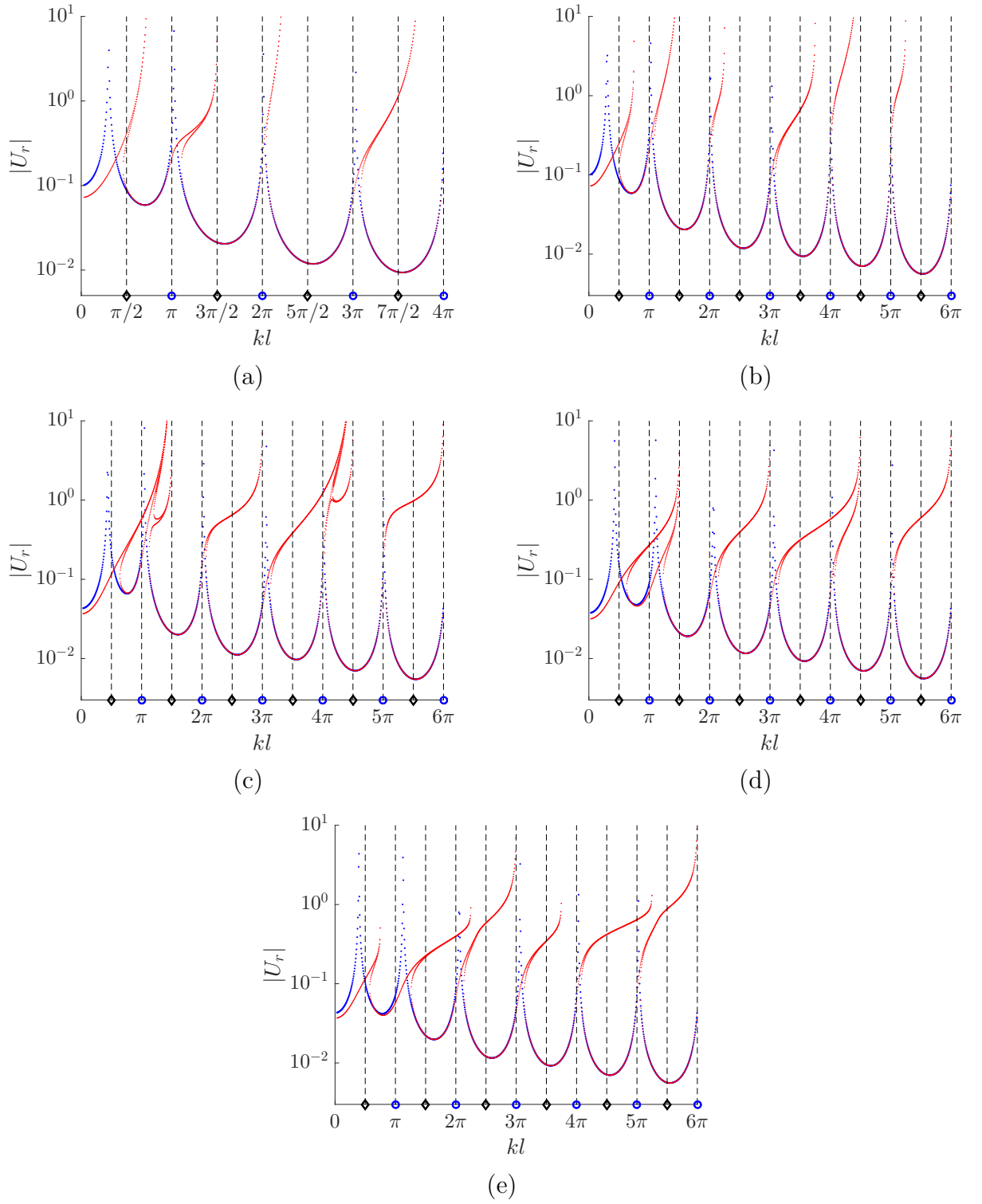


Figure 3.14: Schematic diagram of a rod attached nonlinear springs for the configurations shown in Table 3.3: (a) Configuration 4, (b) Configuration 5, (c) Configuration 6, (d) Configuration 7 and (e) Configuration 8.

3.4 Chapter remarks

The harmonic behavior of vibrating rods with an array of nonlinear springs has been investigated. The polynomial approach and the perturbation approach are proposed to predicting the multi-mode behavior of the structure, over wide frequency bands where several resonances can be observed. The polynomial method makes use of closed-form solutions of the equation of motion of the rod attached to nonlinear springs, which leads to scalar polynomial equations. Although efficient for modeling rods with a few numbers of springs, the method becomes cumbersome from the numerical point of view as soon as the number of springs exceeds a certain threshold.

On the other hand, the perturbation method invokes a subspace projection using reduced mode bases. This yields a cubic matrix equation which can be solved using appropriate solvers. Numerical simulations have been carried out which highlight the accuracy of the perturbation method through comparisons with the polynomial method. The receptance functions for the rod coupled with one to six nonlinear springs have been computed on wide frequency bands highlighting several resonance frequencies. It has been shown that, when the number of springs increases, the resonance frequencies of the system can be shifted to the high-frequency range. In this sense, wide low-frequency bands can be created where the vibration levels of the rod remain low compared to the linear case. This opens interesting prospects about vibration control of structures, e.g., periodic structures with strong bandgap properties.

The influence of the spring attachment position along the rod on the dynamic behavior is taken into consideration for different configurations, with both linear and nonlinear spring cases. It is shown that the attachment position of the spring greatly affects the dynamic behavior of the structure.

Chapter 4

The influence of nonlinear stiffness on the transmissibility of mono-coupled continuous periodic structures

In this chapter, the effects of nonlinear springs on continuous mono-coupled periodic structures are presented based on the concept of transmissibility of a single cell of finite structures. The longitudinal vibration of rod structures with nonlinear stiffness elements attached in the rod's specific locations is considered. The stiffness nonlinearity is represented by a dynamic stiffness function obtained using the first-order harmonic balance method, extending the concept presented in Chapter 2 when damping mechanisms are incorporated into the structure. The effects on the transmissibility of the periodic structure are analyzed considering two types of cell configurations.

4.1 Fundamental concepts and methodology

The concepts for analyzing mono-coupled periodic systems with nonlinear stiffness attachments are presented in this section. A unit cell of the system chosen as an example to illustrate such structure behavior consists of a homogeneous rod with repeated nonlinear

springs attached. The complete structure is built upon repeated elements called unit cells as shown in Figure 4.1. The nonlinear spring, the viscous damper, and the complex displacement amplitudes are represented by D , c and U_n . The goal is to understand the behavior of the displacement amplitudes, from left to right, along with the structure when a harmonic force with amplitude F_l is applied on the left side of the structure.

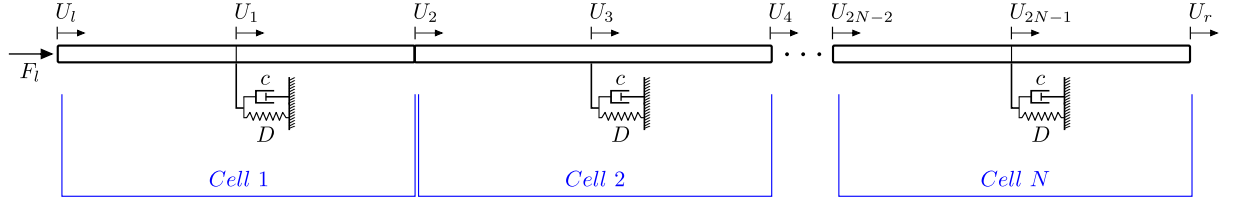


Figure 4.1: Periodic rod structure with many supports consisting of nonlinear springs and viscous dampers.

The two cell configurations analyzed are shown in Fig. 4.2 and are referred as cell configurations (i) and (ii). For configuration (i), the spring-damper element is attached in parallel arrangement with the rod sections. On the other hand, in configuration (ii), the spring-damper elements are assembled in series with the two-rod sections. The total cell length is given by $l_{cell} = 2l$.

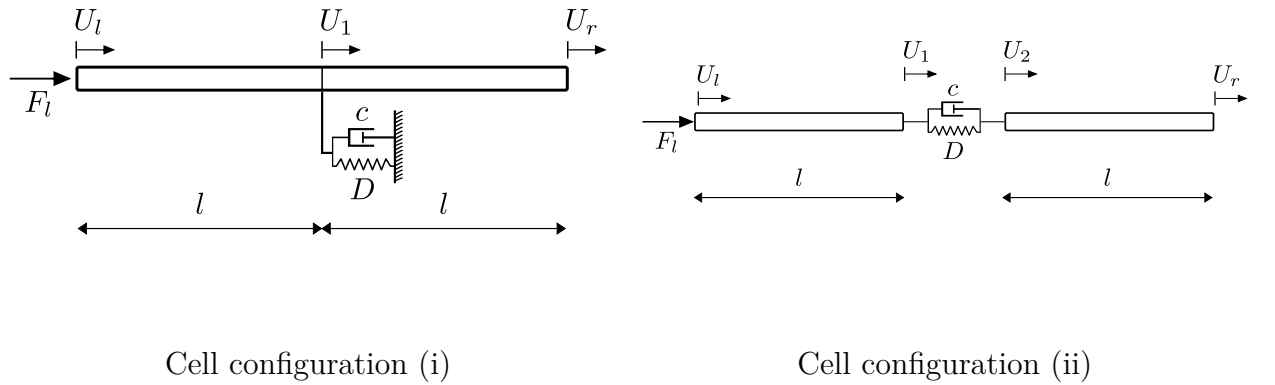


Figure 4.2: Two types of cell configuration for a periodic structure.

When harmonic motion is assumed, the longitudinal vibration of each rod section can be modeled according to the equation of motion and the force-displacement relationship,

respectively, given by

$$U''(x) + k^2 U(x) = 0 \quad (4.1a)$$

$$F = EA U'(x) \quad (4.1b)$$

where, $U(x)$ is a complex displacement amplitude, x is the position along the rod's length, $k = \omega \sqrt{\rho/E}$ is the quasi-longitudinal vibration wavenumber, ω is the angular frequency, E is the material Young's modulus, ρ is the material density, A is the cross section area and F_l is the external axial force applied. The prime (') denotes the derivative with respect to x .

4.2 Transfer matrix representation

A review of the transfer matrix representation is presented for the cases where only linear springs are considered. The transfer matrix relates the vectors of input forces and displacements at the output terminals on each side of the single cell, written as

$$\Psi_r = \mathbf{T}_{\text{cell}} \Psi_l \quad (4.2)$$

where $\Psi = [F \ U]^T$ is the state vector defining the forces and displacements at the boundaries of each cell. The subscripts l and r means left and right, respectively. The transfer matrix $\mathbf{T}_{\text{cell}}$ is calculated considering each rod section and the set of spring and viscous damper, for each configuration, is defined in (RUBIN, 1967) as

$$\mathbf{T}_{\text{cell} - i} = \mathbf{T}_{\text{rod}} \mathbf{T}_i \mathbf{T}_{\text{rod}} \quad (4.3a)$$

$$\mathbf{T}_{\text{cell} - ii} = \mathbf{T}_{\text{rod}} \mathbf{T}_{ii} \mathbf{T}_{\text{rod}} \quad (4.3b)$$

where

$$\mathbf{T}_{\text{rod}} = \begin{bmatrix} \cos(kl) & E Ak \sin(kl) \\ -\sin(kl)/EAk & \cos(kl) \end{bmatrix} \quad (4.4)$$

and,

$$\mathbf{T}_{\text{i}} = \begin{bmatrix} 1 & -D_a \\ 0 & 1 \end{bmatrix}, \quad \mathbf{T}_{\text{ii}} = \begin{bmatrix} 1 & 0 \\ -1/D_a & 1 \end{bmatrix} \quad (4.5)$$

The term $D_a = s_1 + j\omega c$ is the dynamic stiffness representing the set of spring, with linear stiffness coefficient s_1 and the viscous damper with coefficient c , attached to the rod sections and j is the imaginary unit.

4.3 Transmissibility and dispersion diagram

The transmissibility of a single cell is defined as the ratio between the displacement at the right and left-hand side when a force is applied on the left-hand side. This ratio can be written as

$$T = \frac{U_r}{U_l} = \frac{1}{\mathbf{T}_{\text{cell}}(1, 1)} \quad (4.6)$$

For a symmetric cell, the transmissibility depends only on the term of the first line and first column of the transfer matrix defined in Eq. (4.2) as $\mathbf{T}_{\text{cell}}(1, 1)$. The frequency range where the magnitude of the transmissibility is smaller than one corresponds to the frequency region where the displacement from the left side to the right side of the structure is attenuated. These frequency regions are called attenuation bands. It occurs that for periodic systems consisting of symmetric cells, these ranges of attenuation correspond to the band gaps (MEAD, 1975), which can be estimated from the dispersion diagram of an infinite structure by analyzing a single cell element. In this case, the characteristic polynomial obtained from the eigenvalue problem is defined by the transfer matrices given in Eqs. (4.3) to (4.6) can be written in the form

$$\Lambda^2 - \frac{2}{T}\Lambda + 1 = 0 \quad (4.7)$$

where, T is the transmissibility given by Eq. (4.6). The solution of Eq. (4.7) allow the calculation of the two eigenvalues

$$\Lambda_{1,2} = \frac{1}{T} \pm \sqrt{\frac{1}{T^2} - 1} \quad (4.8)$$

For the case where damping is neglected, T is a real-valued function. In this case, by inspecting Eq. (4.8), it is possible to understand that if the transmissibility $|T| \leq 1$, the two eigenvalues are also real, corresponding to an attenuation band, and when $|T| > 1$, the two eigenvalues become complex pair, corresponding to a range where waves can propagate without attenuation. This relationship between the eigenvalues and the transmissibility provides insight into the behavior of the muti-cell structure based on the analysis of a single cell element and is used as the basis for the analysis in the following sections.

4.4 Representation of the nonlinear cubic spring using the harmonic balance method

One of the methods that can be used to represent the behavior of nonlinear stiffness elements is the first-order harmonic balance method. This is done by obtaining a displacement-dependent dynamic stiffness function that represents the forces on the spring. The dynamic stiffness function depends on whether damping is considered or not. If damping is neglected, the dynamic stiffness is a real function, such that the spring force can be written using a polynomial function in time, as

$$f_s(t) = s_1 z(t) + s_3 z(t)^3 \quad (4.9)$$

where, f_s is the restoring force on the spring, s_1 and s_3 are stiffness coefficients and z is the relative displacement between the spring terminals. In the case of harmonic motion of the form $z(t) = Z \cos(\omega t)$, where ω is the frequency of excitation, the spring force can be written as $f(t) = F_s \cos(\omega t)$, such that

$$F_s \cos(\omega t) = s_1 Z \cos(\omega t) + s_3 Z^3 \cos^3(\omega t) \quad (4.10)$$

The last term on Eq. (4.10) can be written in terms of a trigonometric identity as $\cos^3(\omega t) = 3/4 \cos(\omega t) + 1/4 \cos(3\omega t)$. Hence, the equivalent dynamic stiffness of the spring can be expressed as follows

$$F_s \cos(\omega t) = s_1 Z \cos(\omega t) + s_3 Z^3 \left(\frac{3}{4} \cos(\omega t) + \frac{1}{4} \cos(3(\omega t)) \right) \quad (4.11)$$

The assumption in this problem is that the high order harmonic term $\left(\frac{1}{4} \cos(3(\omega t)) \right)$ can be neglected in comparison with the other terms. The dynamic stiffness function is found by dividing both sides of Eq. (4.11) by the relative displacement Z , in order to obtain

$$D = \frac{F_s}{Z} = s_1 + \frac{3}{4} Z^2 \quad (4.12)$$

The dynamic stiffness function in Eq. (4.12) is only valid if Z is real. In the presence of damping mechanisms, $Z = \text{Re}\{Z\} - i\text{Im}\{Z\}$ becomes complex. Therefore, the real and imaginary parts of the nonlinear stiffness force need to be written according to Tatzko and Jahn (TATZKO; JAHN, 2019) as

$$\text{Re}\{F_s\} = s_3 \frac{3}{4} \left(\text{Re}\{Z\}^3 + \text{Re}\{Z\} \text{Im}\{Z\}^2 \right) \quad (4.13a)$$

$$\text{Im}\{F_s\} = s_3 \frac{3}{4} \left(\text{Im}\{Z\}^3 + \text{Im}\{Z\} \text{Re}\{Z\}^2 \right) \quad (4.13b)$$

4.5 Linear stiffness case analysis

The discussion effect of purely linear stiffness is discussed before analyzing the influence of the nonlinear stiffness on the dynamics of a single cell. Considering only linear stiffness, the transmissibilities of each cell configuration shown in Fig. 4.3 are calculated from Eq.

(4.6) as

$$T_i = \frac{1}{\cos(4\pi l/\lambda) + \frac{\gamma_1 \lambda}{4\pi l} \sin(4\pi l/\lambda)} \quad (4.14a)$$

$$T_{ii} = \frac{1}{\cos(4\pi l/\lambda) - \frac{\pi l}{\gamma_1 \lambda} \sin(4\pi l/\lambda)} \quad (4.14b)$$

which were obtained by neglecting the nonlinear stiffness and the viscous damping terms. The parameters in Eq. (4.14) are written using non-dimensional variables, by considering $\gamma_1 = s_1 l / EA$ as a nondimensional term which relates the spring stiffness with the rod's axial stiffness. The term λ is the longitudinal vibration wavelength related to the wavenumber as $\lambda = 2\pi/k$.

The main results for the linear analyzis are shown in Figs. 4.3, which are plotted as a function of the nondimensional frequency l/λ . Apart from the dispersion diagram results, a small structural damping was considered to avoid sharp resonance peaks with a loss factor $\eta = 0.01$.

On the top of Fig. 4.3, subfigures (a) and (c) shows the dispersion diagram based on the roots of the characteristic Equation (4.8). The results are plotted considering that $\Lambda = \Lambda_1 = \Lambda_2^{-1}$, in which the attenuation within the cell can be evaluated as

$$\text{Attenuation} = |\text{Re} \{ \log(\Lambda) / j l \}| \quad (4.15)$$

and the phase change within the cell

$$\text{Phase} = -|\text{Im} \{ \log(\Lambda) / j l \}| \quad (4.16)$$

Comparing the dispersion diagrams, it can be observed that the cell configuration (i) presents a bandgap at lower frequencies, while for configuration (ii) there is no band gap at lower frequencies. Also, in configuration (i), the attenuation of the band gaps decreases as the frequency increases, on the other hand, it is the opposite behavior in case (ii).

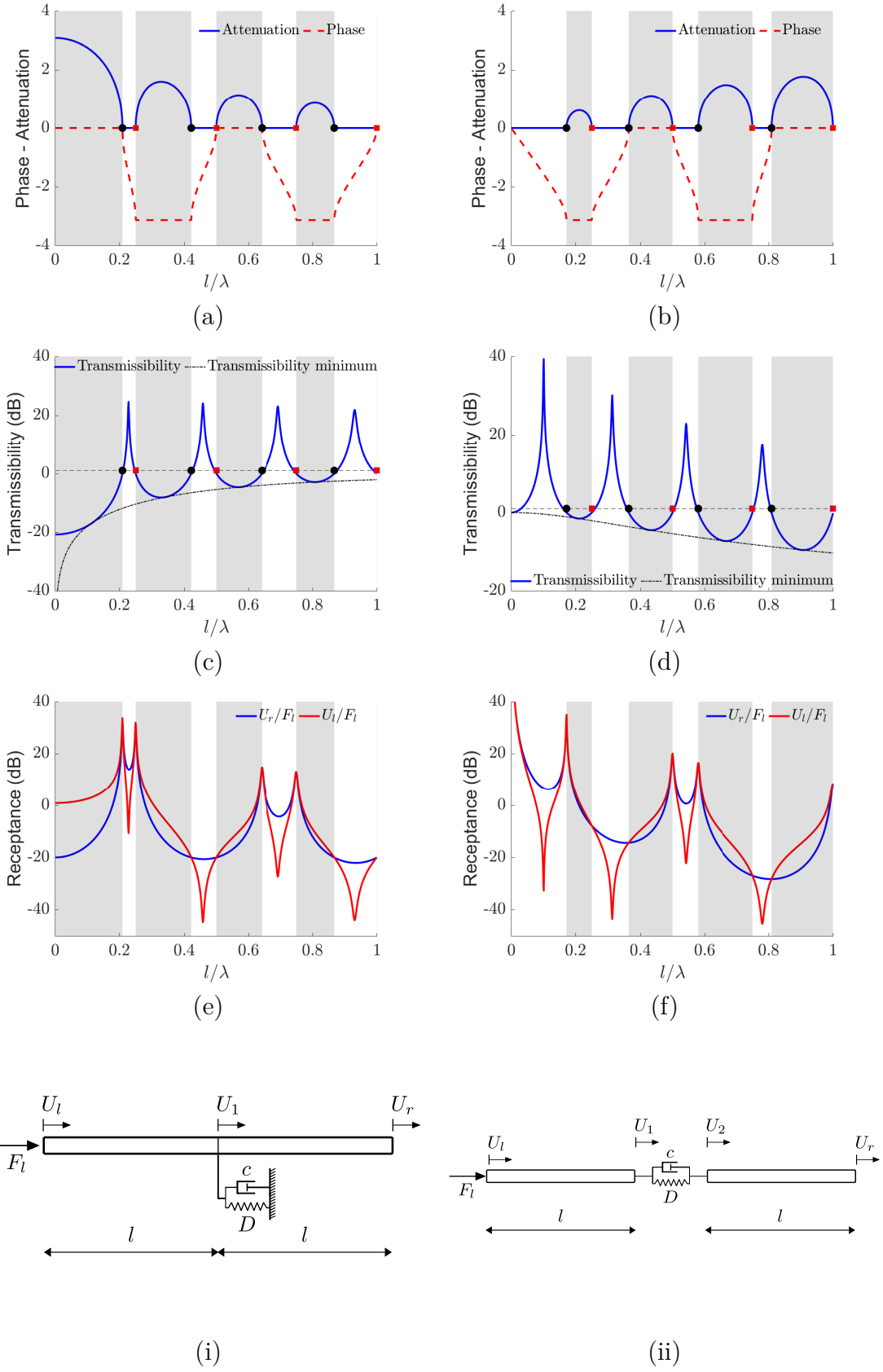


Figure 4.3: The behavior of a single cell for two different arrangements: (a) and (b) the dispersion diagram; (c) and (d) magnitude of the transmissibility (U_r/U_l); and (e) and (f) the magnitude of the receptance functions (dB ref. 1 m/N). For configuration (i), $\gamma_1 = 10$ and for configuration (ii), $\gamma_1 = 1$.

Figures 4.3c and 4.3d shows the magnitude of the transmissibility given by Eqs. 4.14. Observing these expressions and the results, when γ_1 increases, $|T_i|$ reduces and $|T_{ii}|$ increases. Also important to note, the frequency ranges where the transmissibility are smaller than one $|T| < 1$ correspond to the ranges in frequency range where the attenuation in the dispersion diagrams is greater than one, corresponding to the shaded areas. Is it also clear that, increasing the spring stiffness γ_1 favors the attenuation in configuration (i), while in configuration (ii) it favors the propagation. The transmissibility functions in Fig. 4.3 present other information. For instance, the black dashed line that follows the valleys define the minimum value of the transmissibility, which are given by

$$T_i = \left(1 + \left(\frac{\gamma_1 \lambda}{4\pi l} \right)^2 \right)^{-\frac{1}{2}} \quad (4.17a)$$

$$T_{ii} = \left(1 + \left(\frac{\pi l}{\gamma_1 \lambda} \right)^2 \right)^{-\frac{1}{2}} \quad (4.17b)$$

Other important details on Figs. 4.3(c-d) are the frequencies marked by the black circles and red squares. These frequencies defined values of frequency at which the transmissibility equals one ($|T| = 1$). The red square marks highlight fixed frequencies for both cases. This means that the transmissibility is not influenced by any mechanical element attached to the middle of each cell, these frequencies are defined as

$$\frac{l}{\lambda} = \frac{n}{4}, \quad n = 1, 2, 3, \dots \quad (4.18)$$

To find the frequencies defined by the black circles, it is necessary to solve the following

equation,

$$\tan(2\pi l/\lambda) = \alpha, \quad \text{for configuration i} \quad (4.19a)$$

$$\tan(2\pi l/\lambda) = -\alpha, \quad \text{for configuration i} \quad (4.19b)$$

where,

$$\alpha = \frac{\frac{\gamma_1 \lambda}{2\pi l}}{1 - \left(\frac{\gamma_1 \lambda}{4\pi l}\right)^2} \quad (4.20)$$

One of the most important aspects of the frequencies defined by the black circles is their correspondence in the receptances functions shown in Fig. 4.3. Some of these frequencies are related to the resonance frequencies. In order to highlight this behavior, Figures 4.4 and 4.5 shows the results of the behavior of two and three unit cell structure for configurations (i) and (ii), respectively.

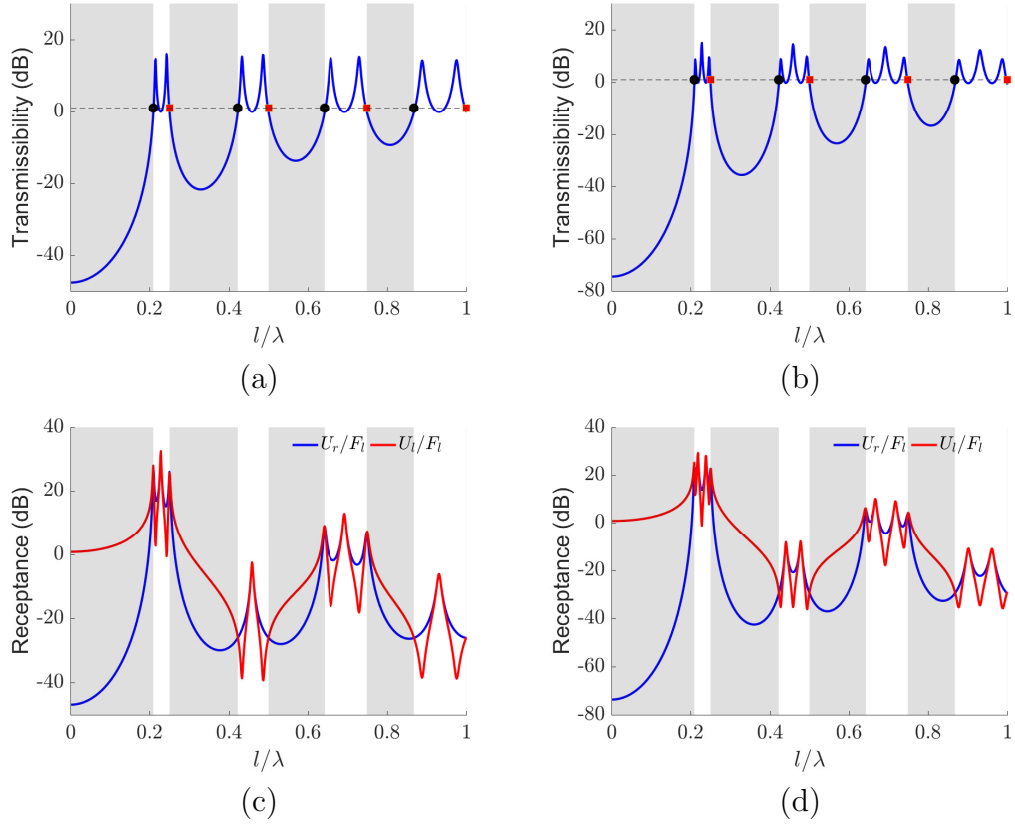


Figure 4.4: The behavior of a two and three unit cell structure for configuration (i) arrangement with $\gamma_1 = 10$: (a) and (b) magnitude of the transmissibility (U_r/U_l); and (c) and (d) the magnitude of the receptance functions (dB ref. 1 m/N). The left side represents the results for two cells, and the right side represents the results for three cells.

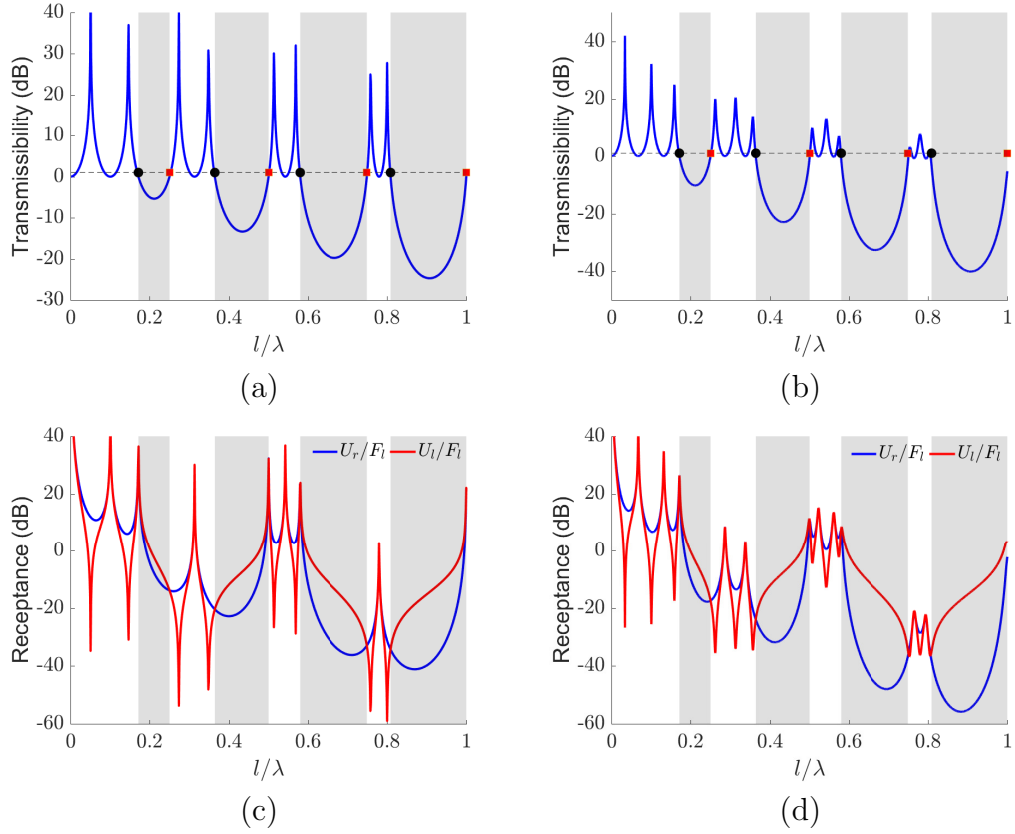


Figure 4.5: The behavior of a two and three unit cell structure for configuration (ii) arrangement with $\gamma_1 = 1$: (a) and (b) magnitude of the transmissibility (U_r/U_l); and (c) and (d) the magnitude of the receptance functions (dB ref. 1 m/N). The left side represents the results for two cells, and the right side represents the results for three cells.

It is possible to observe in Figures 4.3, 4.4 and 4.5 that, the position and the width of the band gaps, i.e., the regions where $|T| < 1$ and highlighted by the grey shaded areas, are not influenced by the number of unit cells in the structure when only linear springs are considered.

Considering the type of cubic type stiffness discussed here, it is clear that nonlinearity should only be important at a high amplitude of motion, which tends to occur at resonance frequencies. Therefore, the nonlinear spring would not affect the frequencies defined by the red squares in Figs 4.3 to 4.5, and will only be important for the frequencies defined in the black circles that correspond to a resonance frequency.

4.6 The effect of nonlinear springs

The influence of nonlinear spring on cell transmissibility is modeled and simulated in this section. To calculate the transmissibility, the transfer matrix method described previously cannot be used because of the dependence of internal degrees of freedom.

The procedure involves in writing the equations considering the force equilibrium at each cell node. The function representing the nonlinear stiffness is obtained by the harmonic balance method described in Section 4.4. After these equations are obtained, they need to be split into real and imaginary parts, which will double the number of equations.

The numerical solution of these equations can be obtained using different strategies. For instance, using a root-finding algorithm such as the Levenberg-Marquardt method, which can be accessed using the function `fsolve` in Matlab or Python-Scipy as presented in Chapter 3. As discussed, one of the issues about using the `fsolve` function is that it requires initial guesses that need to be given for each frequency of interest. In some cases, multiple starting guesses are necessary to obtain all the roots of the system. In this chapter, a different strategy is proposed using the Homotopy continuation algorithm for a system of algebraic nonlinear polynomials using the package Bertini (BATES et al., 2013). The advantage of the method proposed in Bertini is that it can find isolated solutions and controls the numerical processes internally.

Based on the discussion developed in Section 4.5, two conditions were considered for the simulations. For the cell configuration (i), a hardening type spring was used with positive coefficients defining the force deflection equations ($s_1 > 0$, $s_3 = 0$). For the cell configuration (ii), a hardening and a softening spring were used in the numerical simulations, with coefficients $s_1 > 0$, $s_3 > 0$ and $s_1 > 0$, $s_3 < 0$. To make the results more general, the numerical simulations were evaluated using nondimensional quantities with the following relationships

$$\gamma_1 = \frac{s_1 l}{EA}, \quad \gamma_3 = \frac{s_3 l^3}{EA}, \quad \xi = \frac{c}{A\sqrt{E\rho}}, \quad \hat{F}_l = \frac{F_l}{EA} \quad (4.21)$$

In all the numerical simulations a loss factor η was used and incorporated in the models by making the Young's Modulus complex $E = E_0 (1 + j\eta)$.

4.6.1 The influence of nonlinearity on cell configuration (i)

For the cell configuration (i), the force equilibrium at nodes U_l , U_1 and U_r are defined, respectively, as

$$D_1 U_l + D_2 U_1 = F_l \quad (4.22a)$$

$$D_2 U_l + (2D_1 + D_a) + D_2 U_r + F_{NL} = 0 \quad (4.22b)$$

$$D_2 U_1 + D_a U_r = 0 \quad (4.22c)$$

where $D_1 = E A k \cot(kl)$, $D_2 = -E A k \csc(kl)$, $D_a = s_1 - j\omega c$ and F_{NL} is the force component due to the nonlinear spring term which depends on the displacement U_1 .

Equation (4.22) is defined in terms of complex variables and are expanded into six equations containing the real and imaginary components as for instance

$$D_1 = D_1^{(r)} - jD_1^{(i)}, \quad U_1 = U_1^{(r)} - jU_1^{(i)} \quad (4.23a)$$

$$D_2 = D_2^{(r)} - jD_2^{(i)}, \quad U_r = U_r^{(r)} - jU_r^{(i)} \quad (4.23b)$$

Replacing the Eq. (4.23) into Eq. (4.22), the real and imaginary force equilibrium equations are defined as

$$D_2^{(r)}U_1^{(r)} - D_2^{(i)}U_1^{(i)} + D_1^{(r)}U_l^{(r)} - D_1^{(i)}U_l^{(i)} = F_l \quad (4.24a)$$

$$-D_2^{(i)}U_1^{(r)} - D_2^{(r)}U_1^{(i)} - D_1^{(i)}U_l^{(r)} - D_1^{(r)}U_l^{(i)} = 0 \quad (4.24b)$$

$$\begin{aligned} & D_2^{(r)}U_1^{(r)} - D_2^{(i)}U_1^{(i)} + D_a^{(r)}U_1^{(r)} + 2D_1^{(r)}U_1^{(r)} \\ & - D_a^{(i)}U_1^{(i)} - 2D_1^{(i)}U_1^{(i)} + D_2^{(r)}U_l^{(r)} - D_2^{(i)}U_l^{(i)} + F_{NL}^{(r)} = 0 \end{aligned} \quad (4.24c)$$

$$\begin{aligned} & -D_2^{(i)}U_r^{(r)} - D_2^{(r)}U_r^{(i)} - D_a^{(i)} - 2D_1^{(i)}U_1^{(r)} \\ & - D_a^{(r)}U_1^{(i)} - 2D_1^{(r)}U_1^{(i)} - D_2^{(i)}U_l^{(r)} - D_2^{(r)}U_l^{(i)} - F_{NL}^{(i)} = 0 \end{aligned} \quad (4.24d)$$

$$D_1^{(r)}U_r^{(r)} - D_1^{(i)}U_r^{(i)} + D_2^{(r)}U_1^{(r)} - D_2^{(i)}U_1^{(i)} = 0 \quad (4.24e)$$

$$-D_1^{(i)}U_r^{(r)} - D_1^{(r)}U_r^{(i)} - D_2^{(i)}U_1^{(r)} - D_2^{(r)}U_1^{(i)} = 0 \quad (4.24f)$$

where

$$D_a^{(r)} = s_1, \quad D_a^{(i)} = \omega c \quad (4.25)$$

and

$$F_{NL}^{(r)} = \frac{3}{4}s_3 \left(U_1^{(r)3} + U_1^{(r)}U_1^{(i)2} \right) \quad (4.26a)$$

$$F_{NL}^{(i)} = \frac{3}{4}s_3 \left(U_1^{(i)3} + U_1^{(i)}U_1^{(r)2} \right) \quad (4.26b)$$

The results obtained from the numerical simulation and the comparison with the linear case are shown in Fig. 4.6, which represents the magnitude of the transmissibility plotted in dB. At the bottom, Fig 4.6 shows the receptance for the magnitude displacements at

each end of the cell, where results were also plotted in dB (ref. 1 m/N).

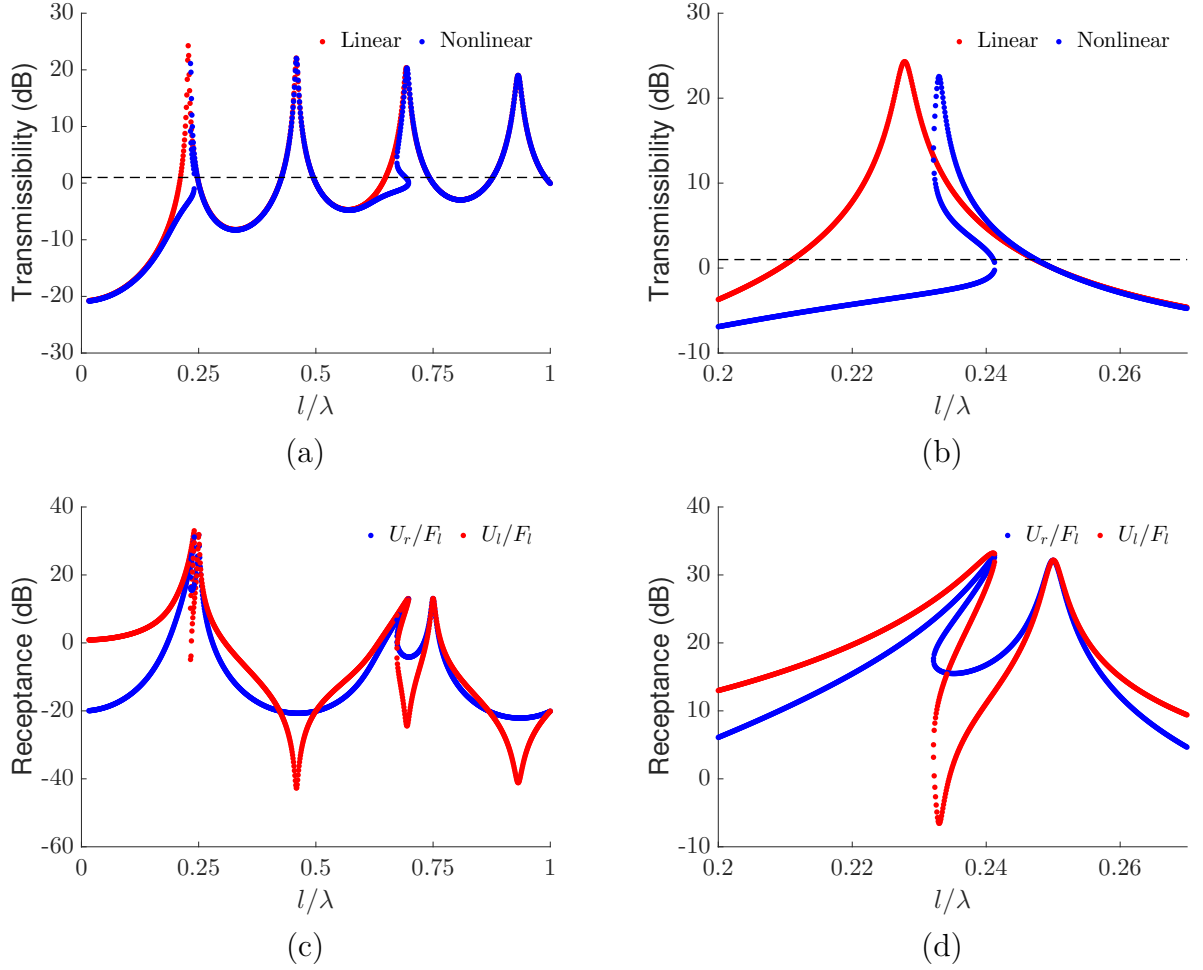


Figure 4.6: The behavior of a single cell for configuration (i) with a nonlinear spring for $\gamma_1 = 10$, $\gamma_3 = 10$, $\hat{F}_l = 1$, $\xi = 0.1$ and $\eta = 0.01$:(a) the magnitude of the transmissibility in dB, (c) the magnitude of the receptance functions in dB (dB ref. 1 m/N). Sub-figures (b) and (d) are details of the results in sub-figures (a) and (c), respectively.

The effect of the hardening spring on some resonance peaks is a known behavior, with the resonance peaks bending to higher frequencies. This effect is clearly seen in the detailed Fig 4.6(d). There are two resonance peaks at the nondimensional frequencies $l/\lambda = 1/4$ and $l/\lambda = 3/4$ that are not affected by the nonlinearity. This was discussed in section 4.5, which correspond to the rod fixed frequencies. The effect of the nonlinear hardening spring of this cell configuration is shown in the detail of Fig. 4.6(b), with the enlargement of the attenuation bandwidth. This occurs at two attenuation bands for the frequency range shown. Also as discussed before, neither the valleys nor the peaks in the transmissibility function are affected by the nonlinearity, since these frequencies are related

to low amplitude displacements. The same behavior is observed in Figures 4.7 and 4.8, which shows the results for a structure built of two and three-unit cells for arrangement (i). The system of equations defining the force equilibrium at the nodes for both two and three-unit cells structure is described in the Appendix C.

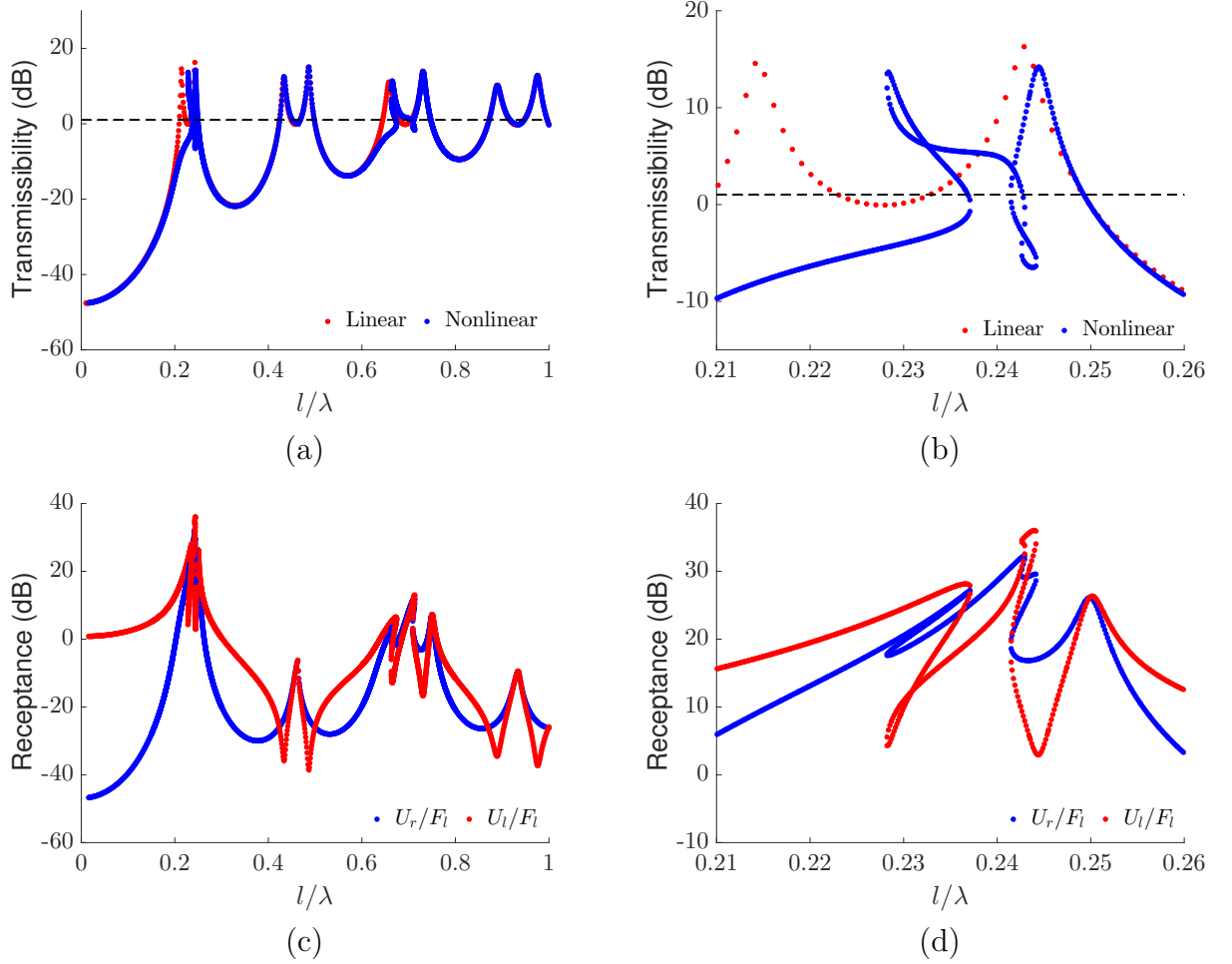


Figure 4.7: The behavior of a two-unit cells structure for configuration (i) with a non-linear spring with $\gamma_1 = 10$, $\gamma_3 = 10$, $\hat{F}_l = 1$, $\xi = 0.1$ and $\eta = 0.01$:(a) the magnitude of the transmissibility in dB, (c) the magnitude of the receptance functions in dB (dB ref. 1 m/N). Sub-figures (b) and (d) are details of the results in sub-figures (a) and (c), respectively.

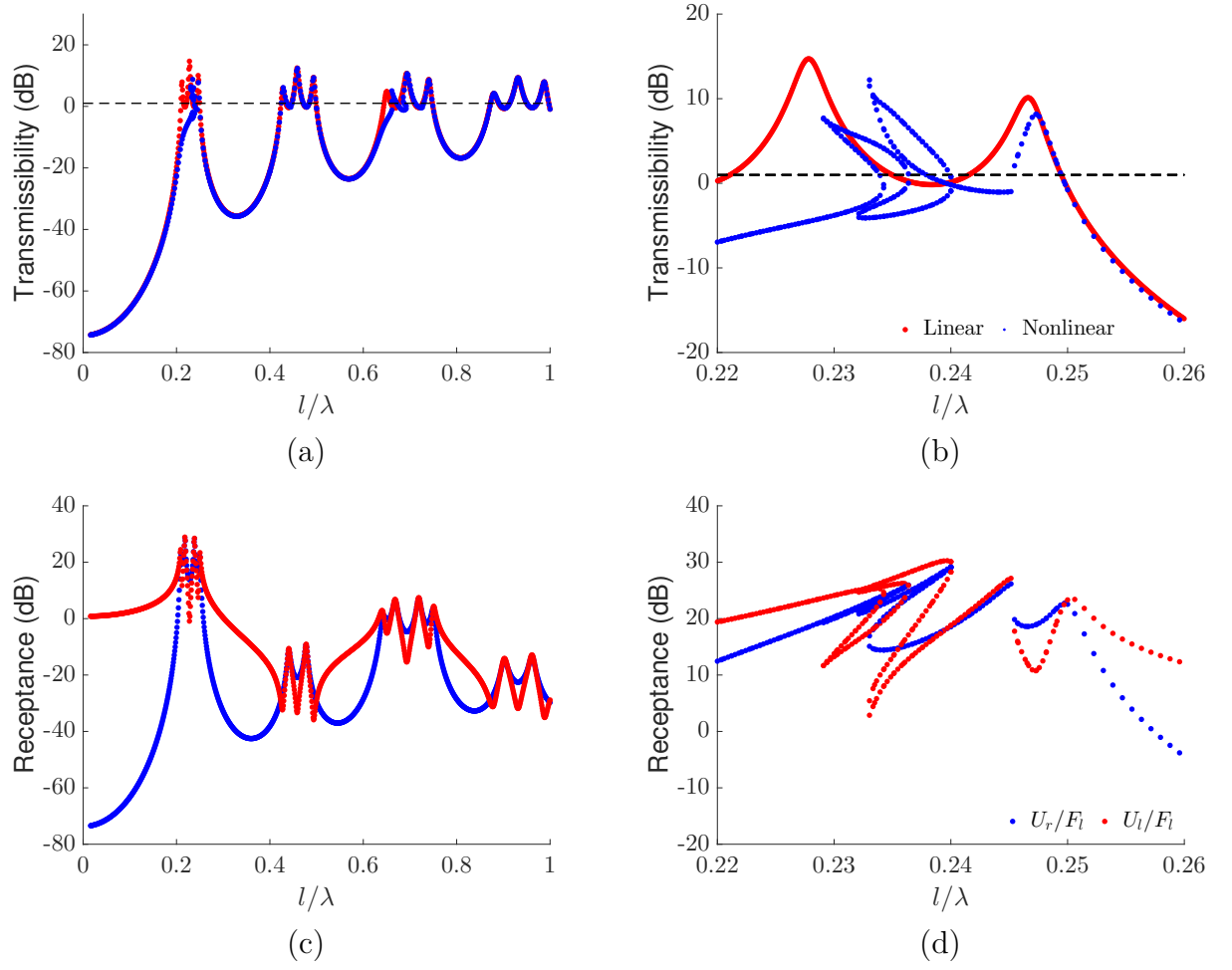


Figure 4.8: The behavior of a three-unit cells structure for configuration (i) with a nonlinear spring with $\gamma_1 = 10$, $\gamma_3 = 10$, $\hat{F}_l = 1$, $\xi = 0.1$ and $\eta = 0.01$: (a) the magnitude of the transmissibility in dB, (c) the magnitude of the receptance functions in dB (dB ref. 1 m/N). Sub-figures (b) and (d) are details of the results in sub-figures (a) and (c), respectively.

4.6.2 The influence of nonlinearity on cell configuration (ii)

For the cell configuration (ii), there are four equations defining the force equilibrium at nodes, U_l , U_1 , U_2 and U_r . The equations are written as

$$D_1 U_l + D_2 U_1 = F_l \quad (4.27a)$$

$$D_2 U_l + D_1 U_1 + D_a (U_1 - U_2) + F_{NL} = 0 \quad (4.27b)$$

$$D_1 U_2 + D_2 U_r - D_a (U_1 - U_2) - F_{NL} = 0 \quad (4.27c)$$

$$D_2 U_2 + D_1 U_r = 0 \quad (4.27d)$$

where

$$D_1 = D_1^{(r)} - j D_1^{(i)}, \quad U_1 = U_1^{(r)} - j U_1^{(i)} \quad (4.28a)$$

$$D_2 = D_2^{(r)} - j D_2^{(i)}, \quad U_2 = U_2^{(r)} - j U_2^{(i)} \quad (4.28b)$$

$$U_r = U_r^{(r)} - j U_r^{(i)} \quad (4.28c)$$

In this configuration, the term defining F_{NL} now depends on the relative displacement $U_1 - U_2$ and is written as

$$F_{NL}^{(r)} = \frac{3}{4} s_3 \left(\left(U_1^{(r)} - U_2^{(r)} \right)^3 + \left(U_1^{(r)} - U_2^{(r)} \right) \left(U_1^{(i)} - U_2^{(i)} \right)^2 \right) \quad (4.29a)$$

$$F_{NL}^{(i)} = \frac{3}{4} s_3 \left(\left(U_1^{(i)} - U_2^{(i)} \right)^3 + \left(U_1^{(i)} - U_2^{(i)} \right) \left(U_1^{(r)} - U_2^{(r)} \right)^2 \right) \quad (4.29b)$$

Therefore, replacing Eqs. (4.29) and (4.28) into Eq. (4.27) the expanded equations in real and imaginary parts are given as

$$D_2^{(r)}U_1^{(r)} - D_2^{(i)}U_1^{(i)} + D_1^{(r)}U_l^{(r)} - D_1^{(i)}U_l^{(i)} = F_l \quad (4.30a)$$

$$-D_2^{(i)}U_1^{(r)} - D_2^{(r)}U_1^{(i)} - D_1^{(i)}U_l^{(r)} - D_1^{(r)}U_l^{(i)} = 0 \quad (4.30b)$$

$$\begin{aligned} & -D_a^{(r)}U_2^{(r)} + D_l^{(i)}U_b^{(i)} + D_a^{(r)}U_1^{(r)} + D_1^{(r)}U_1^{(r)} \\ & -D_a^{(i)}U_1^{(i)} - D_1^{(i)}U_1^{(i)} + D_2^{(r)}U_l^{(r)} - D_2^{(i)}U_l^{(i)} + F_{NL}^{(r)} = 0 \end{aligned} \quad (4.30c)$$

$$\begin{aligned} & D_l^{(i)}U_2^{(r)} + D_l^{(r)}U_2^{(i)} - D_a^{(i)}U_1^{(r)} - D_1^{(i)}U_1^{(r)} \\ & -D_a^{(r)}U_1^{(i)} - D_1^{(r)}U_1^{(i)} - D_2^{(i)}U_l^{(r)} - D_2^{(r)}U_l^{(i)} - F_{NL}^{(i)} = 0 \end{aligned} \quad (4.30d)$$

$$\begin{aligned} & D_2^{(r)}U_r^{(r)} - D_2^{(i)}U_r^{(i)} + D_a^{(r)}U_2^{(r)} + D_1^{(r)}U_2^{(r)} \\ & -D_a^{(i)}U_2^{(i)} - D_1^{(i)}U_2^{(i)} - D_a^{(r)}U_1^{(r)} + D_l^{(i)}U_1^{(i)} - F_{NL}^{(r)} = 0 \end{aligned} \quad (4.30e)$$

$$\begin{aligned} & -D_2^{(i)}U_r^{(r)} - D_2^{(r)}U_r^{(i)} - D_a^{(i)}U_2^{(r)} - D_1^{(i)}U_2^{(r)} \\ & -D_a^{(r)}U_2^{(i)} - D_1^{(r)}U_2^{(i)} + D_a^{(i)}U_1^{(r)} + D_a^{(r)}U_1^{(i)} + F_{NL}^{(r)} = 0 \end{aligned} \quad (4.30f)$$

$$D_1^{(r)}U_r^{(r)} - D_1^{(i)}U_r^{(i)} + D_2^{(r)}U_2^{(r)} - D_2^{(i)}U_2^{(i)} = 0 \quad (4.30g)$$

$$-D_1^{(i)}U_r^{(r)} - D_1^{(r)}U_r^{(i)} - D_2^{(i)}U_2^{(r)} - D_2^{(r)}U_2^{(i)} = 0 \quad (4.30h)$$

Numerical simulations results for this cell configuration are shown in Figs. 4.9 and 4.10, where the results were compared with the linear case.

The effects on the transmissibility and receptance curves for $\gamma_3 > 0$ is shown in Fig. 4.9. It is possible to observe that the hardening stiffness can increase the transmissibility of the system and narrow the attenuation band when compared to the linear case for the configuration (ii). This effect is analogous to enhance the connection between the two-rod sections, which improves the propagation of the wave through the structure. Sub-figures (b) and (c) show in detail the results in sub-figures (a) and (c), respectively.

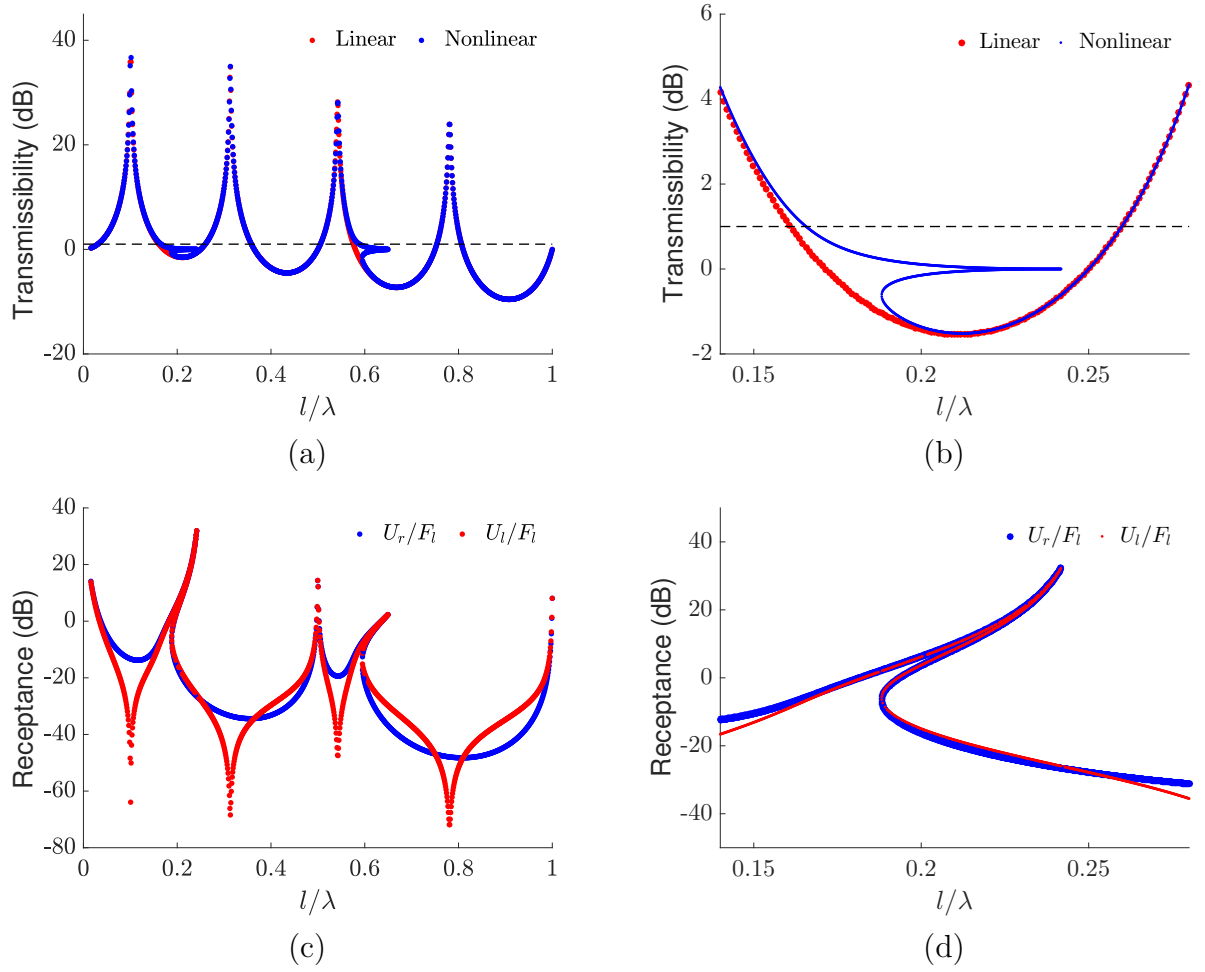


Figure 4.9: The behavior of a single cell for configuration (ii) with a nonlinear spring for $\gamma_1 = 1$, $\gamma_3 = 1$, $\hat{F}_l = 0.1$, $\xi = 0.0108$ and $\eta = 0.001$: (a) the magnitude of the transmissibility in dB, (c) the magnitude of the receptance functions in dB (dB ref. 1 m/N). Sub-figures (b) and (d) are details of the results in sub-figures (a) and (c), respectively.

The results for a $\gamma_3 < 0$ is shown in Fig. 4.10. The effect of softening nonlinearity on the low-frequency range is shown in subfigures (a) and (c) and it is possible to observe the appearance of multiple response amplitudes. Therefore, it is not clear the real benefit of using a softening spring for this cell configuration since the low-frequency response can be small or large, depending on the initial conditions and other factors controlling the stability of the periodic response.

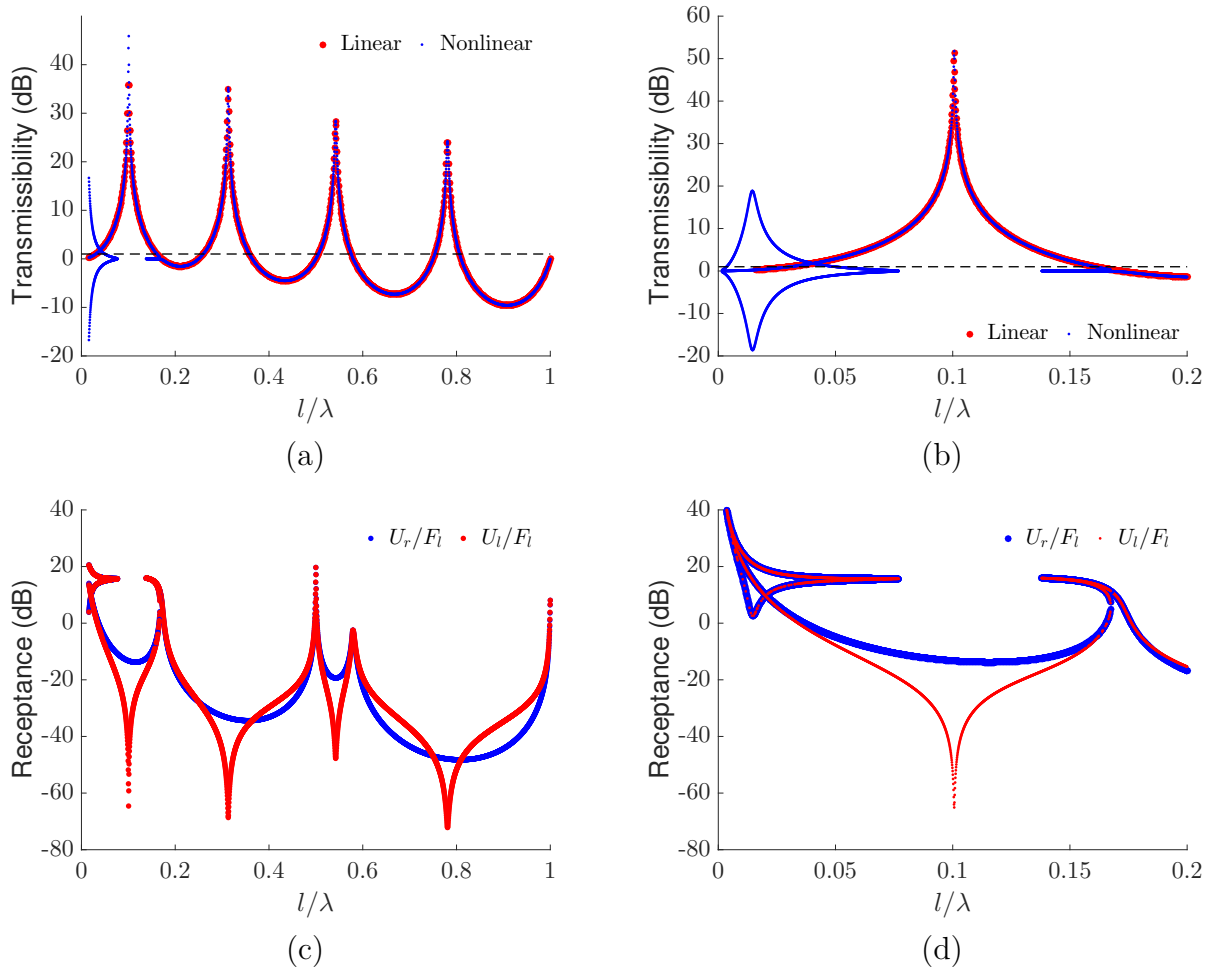


Figure 4.10: The behavior of a single cell for configuration (ii) with a nonlinear spring for $\gamma_1 = 1$, $\gamma_3 = -0.01$, $\hat{F}_l = 0.1$, $\xi = 0.0108$ and $\eta = 0.001$:(a) the magnitude of the transmissibility in dB, (c) the magnitude of the receptance functions in dB (dB ref. 1 m/N). Sub-figures (b) and (d) are details of the results in sub-figures (a) and (c), respectively.

Numerical results for a structure built of two-unit cells for configuration (ii) considering a softening spring is shown in Figure 4.11. It is shown that the nonlinear springs only have a strong influence at the low-frequency range with the appearance of multiple response amplitudes. Similar to what was found in the analysis of the single-cell structure, the response can be large or small, becoming necessary to perform the stability analysis to determine the real benefit for this configuration arrangement.

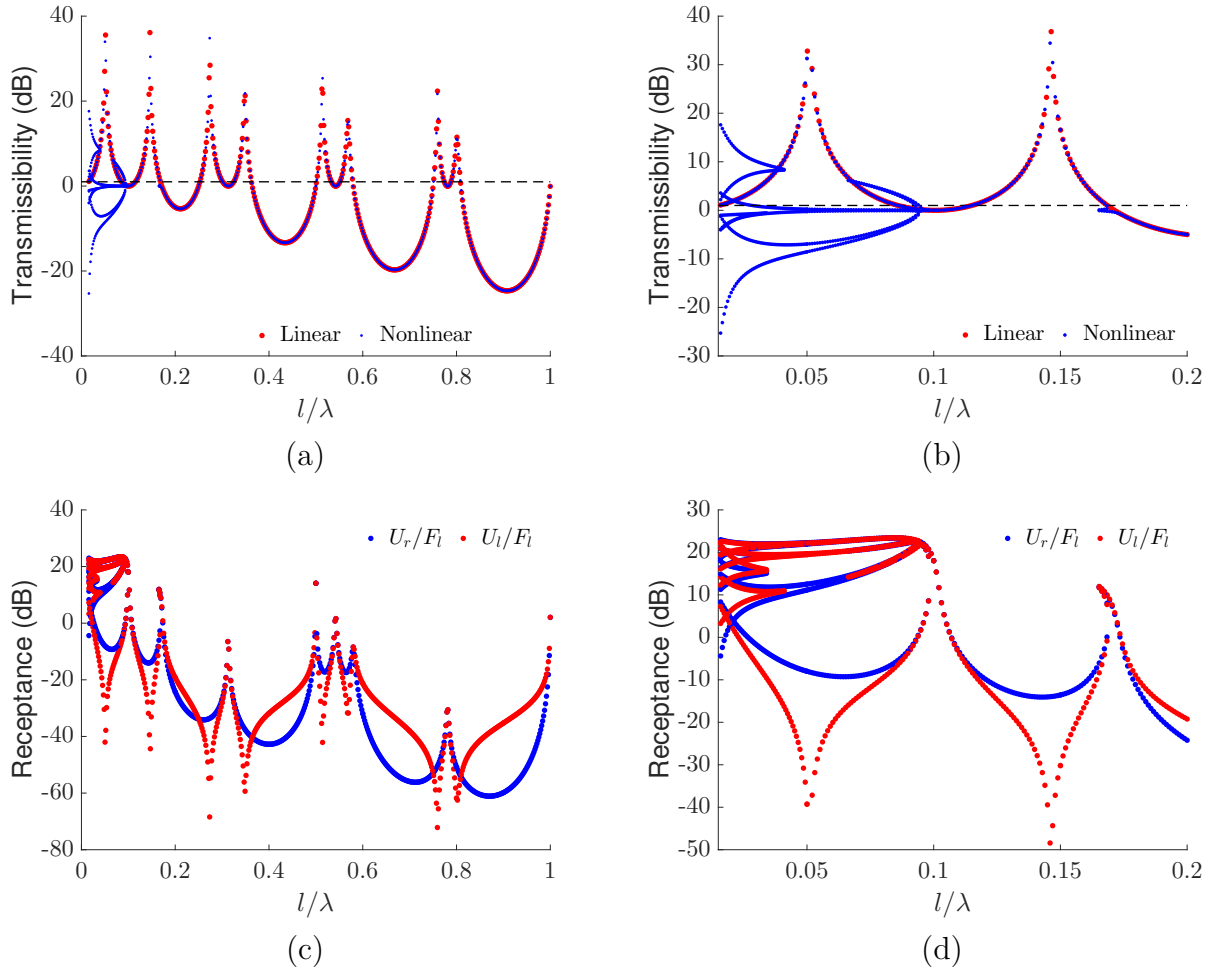


Figure 4.11: The behavior of a two-unit cells for configuration (ii) with a nonlinear spring for $\gamma_1 = 1$, $\gamma_3 = -0.01$, $\hat{F}_l = 0.1$, $\xi = 0.0108$ and $\eta = 0.001$: (a) the magnitude of the transmissibility in dB, (c) the magnitude of the receptance functions in dB (dB ref. 1 m/N). Sub-figures (b) and (d) are details of the results in sub-figures (a) and (c), respectively.

4.7 Chapter remarks

This chapter presented the discussion on the use of nonlinear cubic stiffness elements attached to a mono-coupled periodic structure. Two cases have been considered. In the first, the nonlinearity was attached in a parallel configuration at the middle of two-rod sections. In the second case, the nonlinearity was included in a series configuration with two-rod sections.

Initially, both system configurations were studied considering the linear case to observe the effects of a linear spring on the bandwidth and attenuation frequencies of a single cell. This was done considering the concept of transmissibility, which is directly related to

the dispersion diagram of an infinite structure. Later, the influence of cubic nonlinearity was investigated. The influence of this type of nonlinearity can enlarge the attenuation bandwidth, but with the cost of having regions in frequency where jumps can occur. This type of nonlinearity does not affect the maximum attenuation since it occurs at a frequency range where the displacements tend to be small.

For a structure with cell configuration (i), it makes more sense to use a hardening type spring since it increases the frequency range of attenuation. For the structure with cell configuration (ii), it is better to have a softening type nonlinear spring. It can also increase the frequency range of attenuation, especially at lower frequencies, but with the cost of having regions in the frequency response where jumps can occur.

In terms of engineering applications, the use of nonlinear springs to increase the Bragg-type band gaps may be very limited. The vibration amplitude in rods tends to be very small, providing a challenge to design springs that work in the nonlinear regime at higher frequencies. Nevertheless, this investigation has contributed to understanding what can be done to improve vibration control on a periodic structure by analyzing a single cell element.

Chapter 5

Conclusions

This work investigated the dynamic behavior of vibrating rods attached to an array of nonlinear springs of the cubic type. The proposed analysis aims at investigating the effect of periodic local nonlinearities on the dynamic behavior of wave propagation properties of the waveguide structure studied.

In Chapter 2, a polynomial methodology, which leads to a scalar polynomial equation, is used to obtain the closed-form solutions to investigate the nonlinear behavior of a homogeneous rod attached to a cubic-type nonlinear spring. The dynamic stiffness of the nonlinear spring was derived making use of the first order Harmonic Balance Method. Numerical simulations were carried out and the free and forced vibration cases were discussed. It was shown that the natural frequencies of the free-free rod converge to those of a free-fixed rod when the amplitude increases. An analogous case is found for the softening spring case, where the natural frequencies converge to previous free-fixed roots. Receptance curves for both hardening and softening springs were discussed and a comparison with a continuation method was carried out to highlight the accuracy of the proposed approach. The polynomial approach was about 200 times less expensive, in terms of CPU times than the numerical continuation.

In Chapter 3, the harmonic behavior of vibration rods with an array of nonlinear springs has been investigated. Two different approaches to predict the multi-mode behavior of a rod attached to an array of nonlinear springs were used, the polynomial and perturbation approach. The polynomial method makes use of closed-form solutions of the equation

of motion of the rod attached to nonlinear springs, which leads to scalar polynomial equations. The method proved to be efficient till the number of springs exceeds a certain threshold. The second approach makes use of the perturbation method and a model reduction technique invoking a subspace projection. This yields a cubic matrix equation that can be solved using appropriate solvers. Numerical simulations have been carried out which highlight the accuracy of the perturbation method through comparisons with the polynomial method. Receptance functions for the rod coupled with one to six nonlinear springs have been computed on wide frequency bands highlighting several resonance frequencies using both approaches. It has been shown that, when the number of springs increases, the resonance frequencies of the system can be shifted to the high-frequency range. In this sense, wide low-frequency bands can be created where the vibration levels of the rod remain low compared to the linear case. This opens interesting prospects about vibration control of structures, e.g., periodic structures with strong bandgap properties. The influence of the spring attachment position along the rod on the dynamic behavior is taken into consideration for different configurations, with both linear and nonlinear spring cases. It is shown that the attachment position of the spring greatly affects the dynamic behavior of the structure.

Chapter 4 presented the study about the effects of nonlinear stiffness on mono-coupled periodic structures based on the concept of transmissibility of a single cell of the finite structure. The longitudinal vibration of a periodic structure built upon a rod element attached to nonlinear stiffness elements of hardening and softening types at specific rod location was considered for two system configurations. Both arrangement configurations were studied considering the linear case to observe the effects of a linear spring on the bandwidth and attenuation frequencies of a single cell. The concept of transmissibility was applied, which is directly related to the dispersion diagram of an infinite structure. Later, the influence of cubic nonlinearity was investigated. For configuration (i), the hardening type spring increased the frequency range of the attenuation band. On the other hand, for the structure with cell configuration (ii), the best response arrived from the use of a softening spring, increasing the frequency range of attenuation, especially

at lower frequencies, but with a cost of having regions in the frequency response where jumps can occur. The investigation showed that this sort of nonlinearity does not affect the maximum attenuation since these occur at a frequency range where the displacements tend to be small.

In terms of engineering applications, the use of nonlinear springs to increase the Bragg-type band gaps may be very limited. The vibration amplitude in rods tends to be very small, which increases the challenge to design springs that work in the nonlinear regime at higher frequencies. Nevertheless, this investigation has contributed to the understanding of what can be done to improve vibration control on a periodic structure by analyzing a single and multiple cell elements.

5.1 Contributions to knowledge

The following presents some of the contributions of this work concerning the investigation of the dynamic behavior of periodic structures

- The development of two approaches to calculate the nonlinear response of finite continuous nonlinear structures;
- A physical insight of frequency variations of the resonance frequencies of continuous structures attached to nonlinear springs;
- The discussion about the effect of nonlinear springs coupled to periodic structures based on the concept of transmissibility of a single cell of finite structure;
- The strategies proposed could help to investigate the nonlinear stiffness elements on the behavior of periodic structures and metamaterials and the effect on band gaps.

5.2 Future works suggestions

In this work, the main objective was to proposed useful tools to investigate the nonlinear dynamic behavior and wave propagation properties of continuous structures attached to

nonlinear springs. During the development of this thesis, complementary investigation opportunities have been identified, which can also be helpful to design periodic structures. Thus, some of the suggestions are listed below:

- Adapt the methodologies proposed in this thesis to the design of periodic beam structures;
- Stability analysis of the complex behavior due to the introduction of the nonlinearity;
- Experimental validation of the proposed approaches, using simple benchmark structures, could also be carried out;
- Include the analysis of more complicated periodic structures like those modeled with finite elements.

5.3 Publications and scientific events

1 Article in preparation for future submission based on Chapter 4

D.R. Santo; J.-M. Mencik; P.J.P. Gonçalves and M.J. Brennan

Title: The influence of nonlinear stiffness on the transmissibility of continuous mono-coupled periodic structures from the perspective of a single cell

2 D.R. Santo; J.-M. Mencik and P.J.P. Gonçalves

Title: On the multi-mode behavior of vibrating rods attached to nonlinear springs, *Nonlinear Dynamics*, Springer, 2020. <https://doi.org/10.1007/s11071-020-05647-x>

3 XIII Postgraduation program seminars in Mechanical Engineering, November 28 2019, presented the work: On the multi-mode behavior of vibrating rods attached to nonlinear springs, oral presentation, Bauru / SP.

4 Séminaire du LaMé, October 10 2019, presented the work: Dynamic behavior of vibrating rods coupled to nonlinear springs, oral presentation, Tours, France.

- 5 D.R. Santo; J.-M. Mencik; B. Tang, P.J.P.Gonçalves and M.J. Brennan, On the dynamic behavior of rods and beams with nonlinear boundary stiffness In: CIL-AMCE 2018 - XXXIX Ibero-Latin American Congress on Computational Methods in Engineering, 2018, Paris/Compiègne, France.
- 6 Nucleus of Dynamics in Engineering seminars, October 2017, presented the work: Dynamics of an Euler-Bernoulli metamaterial beam, oral presentation, Bauru / SP.
- 7 XXI Postgraduation program seminars in Mechanical Engineering, October 2017, presented the work: Dynamics of an Euler-Bernoulli metamaterial beam, oral presentation, Bauru / SP.

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Appendix A

The dynamic stiffness equation derivation

A homogeneous linear rod vibrating in longitudinal motion is considered as shown in Figure A.1.

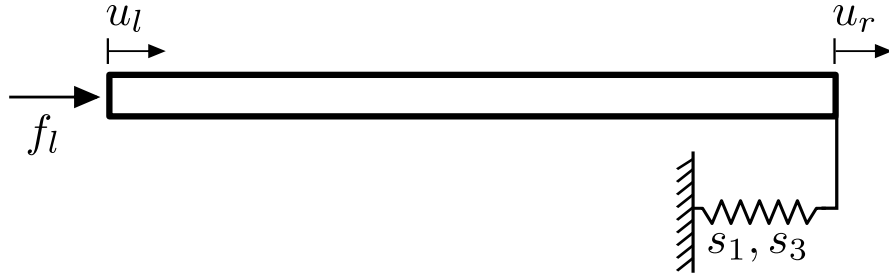


Figure A.1: Schematic of a rod connected to a nonlinear spring at the right end.

The solution of the problem presented in Fig. A.1 has a solution to harmonic excitation written as

$$U(x) = C_1 \cos(kx) + C_2 \sin(kx) \quad (\text{A.1})$$

The boundary conditions of the rod, at the left($x = 0$) and right end ($x = l$), are

expressed by

$$-EAU'_l = F_l \quad (\text{A.2a})$$

$$EAU'_r = F_r - D_a U_r \quad (\text{A.2b})$$

From Eq. A.2a it is possible to obtain

$$C_2 = -\frac{F_l}{EAk} \quad (\text{A.3})$$

From Eq. A.2b

$$EAk [-C_1 \sin(kl) + C_2 \cos(kl)] = F_r - D_a U_r \quad (\text{A.4})$$

It is possible to find C_1 by substituting Eq. A.3 into Eq. A.4

$$C_1 = \frac{D_a U_r - F_r - F_l \cos(kl)}{EAk \sin(kl)} \quad (\text{A.5})$$

Then, the general solution it is written by replacing Eqs. A.3 and A.5 into Eq. A.1

$$U(x) = \left[\frac{D_a U_r - F_r - F_l \cos(kl)}{EAk \sin(kl)} \right] \cos(kx) - \left[\frac{F_l}{EAk} \right] \sin(kx) \quad (\text{A.6})$$

To build the dynamic stiffness matrix equation, one should consider the solution of Eq. A.6 at the position $x = 0$, which yields

$$F_r = D_a U_r - F_l \cos(kl) - U_l EAk \sin(kl) \quad (\text{A.7})$$

Similarly, Eq. A.6 at the position $x = l$ can be written as

$$F_l = D_a U_r \cos(kl) - F_r \cos(kl) - U_r EAk \sin(kl) \quad (\text{A.8})$$

Replacing Eq. A.7 into Eq. A.8 and solving for F_l , yields

$$F_l = E A k \frac{\cos(kl)}{\sin(kl)} U_l - \frac{E A k}{\sin(kl)} U_r \quad (\text{A.9})$$

Substituting Eq. A.9 into Eq. A.7 and solving for F_l , yields

$$F_r = D_a U_r + E A k \frac{\cos(kl)}{\sin(kl)} U_r - \frac{E A k}{\sin(kl)} \quad (\text{A.10})$$

Being $D_1 = E A k \cot(kl)$ and $D_2 = -E A k \csc(kl)$, Eqs. A.9 and A.10 can be rewritten as

$$F_l = D_1 U_l + D_2 U_r \quad (\text{A.11a})$$

$$F_r = D_2 U_l + (D_1 + D_a) U_r \quad (\text{A.11b})$$

Therefore, the dynamic stiffness matrix equation is written as

$$\begin{bmatrix} F_l \\ F_r \end{bmatrix} = \begin{bmatrix} D_1 & D_2 \\ D_2 & D_1 + D_a \end{bmatrix} \begin{bmatrix} U_l \\ U_r \end{bmatrix} \quad (\text{A.12})$$

Appendix B

Normalized modes for a rod connected to a nonlinear spring at the right end

A homogeneous linear rod vibrating in longitudinal motion is considered as shown in Figure A.1.

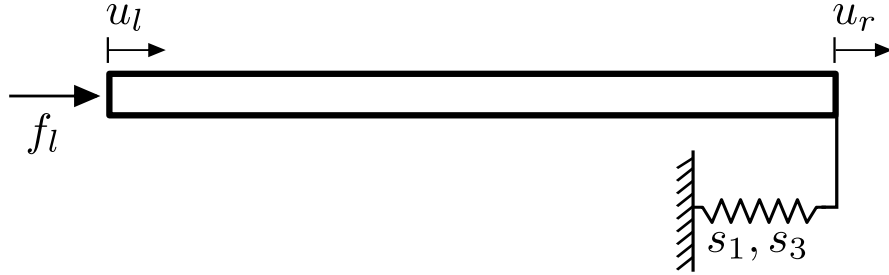


Figure B.1: Schematic of a rod connected to a nonlinear spring at the right end.

The solution of the problem presented in Fig. B.1 has a solution to harmonic excitation written as

$$U(x) = C_1 \cos(kx) + C_2 \sin(kx) \quad (\text{B.1})$$

The boundary conditions of the rod, at the left ($x = 0$) and right end ($x = l$), are

expressed by

$$-EAU'_l = F_l \quad (\text{B.2a})$$

$$EAU'_r = F_r - D_a U_r \quad (\text{B.2b})$$

From Eq. B.2b and considering $F_r = 0$

$$C_2 = \frac{D_a U_r}{EAk \cos(kl)} + C_1 \frac{\sin(kl)}{\cos(kl)} \quad (\text{B.3})$$

Replacing the general solution at the position $x = l$ in Eq. B.1 as

$$U_r = C_1 \cos(kl) + C_2 \sin(kl) \quad (\text{B.4})$$

into B.3, and solving for C_2 , yields

$$C_2 = C_1 \left[\frac{D_a \cos(kl) + EAk \sin(kl)}{EAk \cos(kl) - D_a \sin(kl)} \right] \quad (\text{B.5})$$

Substituting Eq. B.5 into Eq. B.1

$$U(x) = C_1 \cos(kx) + C_1 \left[\frac{D_a \cos(kl) + EAk \sin(kl)}{EAk \cos(kl) - D_a \sin(kl)} \right] \sin(kx) \quad (\text{B.6})$$

By dividing Eq. B.6 for C_1 , yields the normalized modes

$$\tilde{U}_j(x) = \cos(k_j x) + \left[\frac{D_a \cos(k_j l) + EAk_j \sin(k_j l)}{EAk_j \cos(k_j l) - D_a \sin(k_j l)} \right] \sin(k_j x) \quad (\text{B.7})$$

for

$$k_j = \frac{(2j-1)\pi}{2} \quad (\text{B.8})$$

The four first mode shapes as a function of the displacement U_l are represented in Figure B.2.

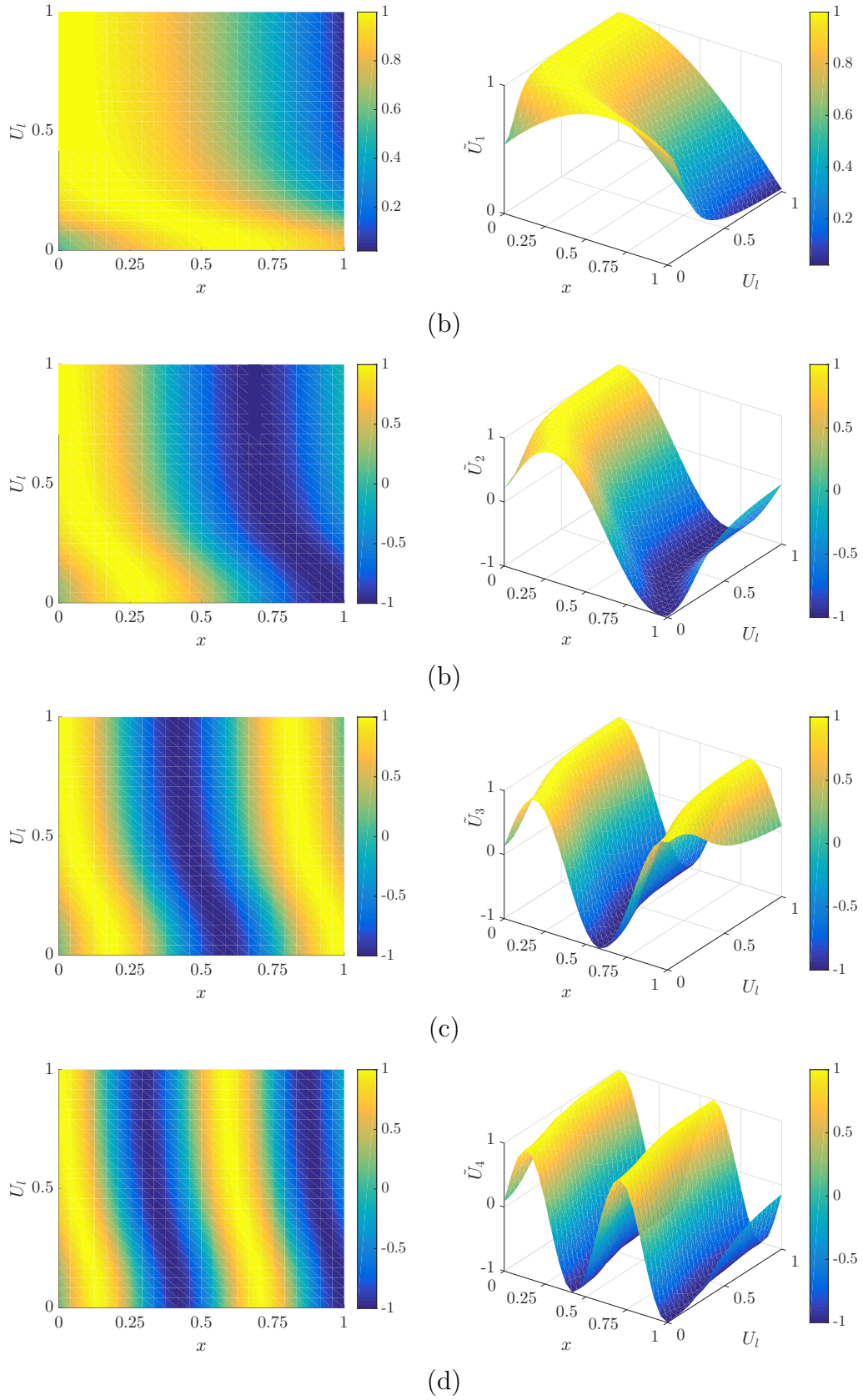


Figure B.2: Normalized mode shapes $\tilde{U}_j$ as a function of the displacement U_l for $s_1 = 1$ and $s_3 = 100$:(a) $j = 1$, (b) $j = 2$, (c) $j = 3$ and (d) $j = 4$. The color bar represents the normalized mode shapes amplitudes.

Appendix C

The force equilibrium at nodes with a two and three unit cell structure for configurations (i) and (ii)

C.1 Configuration (i) for a structure built of two unit cells

For the cell configuration (i) and a structure built of two unit cells, the force equilibrium at nodes U_l , U_1 , U_2 , U_3 and U_r are defined, respectively, as

$$F_l = D_1 U_l + D_2 U_1 \quad (\text{C.1a})$$

$$0 = D_2 U_l + (2D_1 + D_a) U_1 + D_2 U_2 + F_{NL_1} \quad (\text{C.1b})$$

$$0 = D_2 U_1 + 2D_1 U_2 + D_2 U_3 \quad (\text{C.1c})$$

$$0 = D_2 U_2 + (2D_1 + D_a) U_3 + D_2 U_r + F_{NL_3} \quad (\text{C.1d})$$

$$0 = D_2 U_3 + D_1 U_r \quad (\text{C.1e})$$

where $D_1 = E Ak \cot(kl)$, $D_2 = -E Ak \csc(kl)$, $D_a = s_1 - j\omega c$. The the force component due to the nonlinear spring term which depends on the displacement U_1 and U_3 are

$F_{NL_1} = (3/4) s_3 U_1^3$ and $F_{NL_3} = (3/4) s_3 U_3^3$ respectively.

Equations (C.1) are defined in terms of complex variables and can be expanded using real and imaginary components as for instance

$$D_1 = D_1^{(r)} - jD_1^{(i)}, \quad U_1 = U_1^{(r)} - jU_1^{(i)} \quad (\text{C.2a})$$

$$D_2 = D_2^{(r)} - jD_2^{(i)}, \quad U_2 = U_2^{(r)} - jU_2^{(i)} \quad (\text{C.2b})$$

$$U_3 = U_3^{(r)} - jU_3^{(i)}, \quad U_r = U_r^{(r)} - jU_r^{(i)} \quad (\text{C.2c})$$

Substituting Eqs. (C.2) into Eqs. (C.1), the real and imaginary force equilibrium equations are defined in Eqs. (C.3) to (C.7) as

$$D_2^{(r)} U_1^{(r)} - D_2^{(i)} U_1^{(i)} + D_1^{(r)} U_l^{(r)} - D_1^{(i)} U_l^{(i)} = F_l \quad (\text{C.3a})$$

$$-D_2^{(i)} U_1^{(r)} - D_2^{(r)} U_1^{(i)} - D_1^{(i)} U_l^{(r)} - D_1^{(r)} U_l^{(i)} = 0 \quad (\text{C.3b})$$

$$\begin{aligned} & D_2^{(r)} U_2^{(r)} - D_2^{(i)} U_2^{(i)} + D_a^{(r)} U_1^{(r)} + 2D_1^{(r)} U_1^{(r)} \\ & - D_a^{(i)} U_1^{(i)} - 2D_1^{(i)} U_1^{(i)} + D_2^{(r)} U_l^{(r)} - D_2^{(i)} U_l^{(i)} + F_{NL_1}^{(r)} = 0 \end{aligned} \quad (\text{C.4a})$$

$$\begin{aligned} & -D_2^{(i)} U_2^{(r)} - D_2^{(r)} U_2^{(i)} - D_a^{(i)} U_1^{(r)} - 2D_1^{(i)} U_1^{(r)} \\ & - D_a^{(r)} U_1^{(i)} - 2D_1^{(r)} U_1^{(i)} - D_2^{(i)} U_l^{(r)} - D_2^{(r)} U_l^{(i)} - F_{NL_1}^{(i)} = 0 \end{aligned} \quad (\text{C.4b})$$

$$\begin{aligned} & D_2^{(r)} U_3^{(r)} - D_2^{(i)} U_3^{(i)} + D_1^{(r)} U_2^{(r)} - 2D_1^{(i)} U_2^{(i)} + D_2^{(r)} U_1^{(r)} - D_2^{(i)} U_1^{(i)} = 0 \\ & -D_2^{(i)} U_3^{(r)} - D_2^{(r)} U_3^{(i)} - 2D_1^{(i)} U_2^{(r)} - 2D_1^{(r)} U_2^{(i)} - D_2^{(i)} U_1^{(r)} - D_2^{(r)} U_1^{(i)} = 0 \end{aligned} \quad (\text{C.5a})$$

$$\begin{aligned}
& D_2^{(r)}U_r^{(r)} - D_2^{(i)}U_r^{(i)} + D_a^{(r)}U_3^{(r)} + 2D_1^{(i)}U_3^{(r)} \\
& - D_a^{(i)}U_3^{(i)} - 2D_1^{(i)}U_3^{(i)} + D_2^{(r)}U_2^{(r)} - D_2^{(i)}U_2^{(i)} + F_{NL_3}^{(r)} = 0
\end{aligned} \tag{C.6a}$$

$$\begin{aligned}
& -D_2^{(i)}U_r^{(r)} - D_2^{(r)}U_r^{(i)} - D_a^{(i)}U_3^{(r)} - 2D_1^{(r)}U_3^{(r)} \\
& - D_a^{(r)}U_3^{(i)} - 2D_1^{(r)}U_3^{(i)} - D_2^{(i)}U_2^{(r)} - D_2^{(r)}U_2^{(i)} - F_{NL_3}^{(i)} = 0
\end{aligned} \tag{C.6b}$$

$$D_1^{(r)}U_r^{(r)} - D_1^{(i)}U_r^{(i)} + D_2^{(r)}U_3^{(r)} - D_2^{(i)}U_3^{(i)} = 0 \tag{C.7a}$$

$$-D_1^{(i)}U_r^{(r)} - D_1^{(r)}U_r^{(i)} - D_2^{(i)}U_3^{(r)} - D_2^{(r)}U_3^{(i)} = 0 \tag{C.7b}$$

where

$$D_a^{(r)} = s_1, \quad D_a^{(i)} = \omega c \tag{C.8}$$

and

$$F_{NL_1}^{(r)} = \frac{3}{4}s_3 \left(U_1^{(r)3} + U_1^{(r)}U_1^{(i)2} \right), \quad F_{NL_1}^{(i)} = \frac{3}{4}s_3 \left(U_1^{(i)3} + U_1^{(i)}U_1^{(r)2} \right) \tag{C.9a}$$

$$F_{NL_3}^{(r)} = \frac{3}{4}s_3 \left(U_3^{(r)3} + U_3^{(r)}U_3^{(i)2} \right), \quad F_{NL_3}^{(i)} = \frac{3}{4}s_3 \left(U_3^{(i)3} + U_3^{(i)}U_3^{(r)2} \right) \tag{C.9b}$$

C.2 Configuration (i) for a structure built of three unit cells

For the cell configuration (i) and a structure built of three unit cells, the force equilibrium at nodes U_l , U_1 , U_2 , U_3 , U_4 , U_5 and U_r are defined, respectively, as

$$F_l = D_1 U_l + D_2 U_1 \quad (\text{C.10a})$$

$$0 = D_2 U_l + (2D_1 + D_a) U_1 + D_2 U_2 + F_{NL_1} \quad (\text{C.10b})$$

$$0 = D_2 U_1 + 2D_1 U_2 + D_2 U_3 \quad (\text{C.10c})$$

$$0 = D_2 U_2 + (2D_1 + D_a) U_3 + D_2 U_4 + F_{NL_3} \quad (\text{C.10d})$$

$$0 = D_2 U_3 + 2D_1 U_4 + D_2 U_5 \quad (\text{C.10e})$$

$$0 = D_2 U_4 + (2D_1 + D_a) U_5 + D_2 U_r + F_{NL_5} \quad (\text{C.10f})$$

$$0 = D_2 U_5 + D_1 U_r \quad (\text{C.10g})$$

where $D_1 = E A k \cot(kl)$, $D_2 = -E A k \csc(kl)$, $D_a = s_1 - j\omega c$. The the force component due to the nonlinear spring term which depends on the displacement U_1 , U_3 and U_5 are $F_{NL_1} = (3/4) s_3 U_1^3$, $F_{NL_3} = (3/4) s_3 U_3^3$ and $F_{NL_5} = (3/4) s_3 U_5^3$ respectively.

Equations (C.10) are defined in terms of complex variables and can be expanded using real and imaginary components as for instance

$$D_1 = D_1^{(r)} - jD_1^{(i)}, \quad U_1 = U_1^{(r)} - jU_1^{(i)} \quad (\text{C.11a})$$

$$D_2 = D_2^{(r)} - jD_2^{(i)}, \quad U_2 = U_2^{(r)} - jU_2^{(i)} \quad (\text{C.11b})$$

$$U_3 = U_3^{(r)} - jU_3^{(i)}, \quad U_4 = U_4^{(r)} - jU_4^{(i)} \quad (\text{C.11c})$$

$$U_5 = U_5^{(r)} - jU_5^{(i)}, \quad U_r = U_r^{(r)} - jU_r^{(i)} \quad (\text{C.11d})$$

Substituting Eqs. (C.11) into Eqs. (C.10), the real and imaginary force equilibrium

equations are defined in Eqs. (C.12) to (C.18) as

$$D_2^{(r)}U_1^{(r)} - D_2^{(i)}U_1^{(i)} + D_1^{(r)}U_l^{(r)} - D_1^{(i)}U_l^{(i)} = F_l \quad (\text{C.12a})$$

$$-D_2^{(i)}U_1^{(r)} - D_2^{(r)}U_1^{(i)} - D_1^{(i)}U_l^{(r)} - D_1^{(r)}U_l^{(i)} = 0 \quad (\text{C.12b})$$

$$\begin{aligned} & D_2^{(r)}U_2^{(r)} - D_2^{(i)}U_2^{(i)} + D_a^{(r)}U_1^{(r)} + 2D_1^{(r)}U_1^{(r)} \\ & - D_a^{(i)}U_1^{(i)} - 2D_1^{(i)}U_1^{(i)} + D_2^{(r)}U_l^{(r)} - D_2^{(i)}U_l^{(i)} + F_{NL_1}^{(r)} = 0 \end{aligned} \quad (\text{C.13a})$$

$$\begin{aligned} & -D_2^{(i)}U_2^{(r)} - D_2^{(r)}U_2^{(i)} - D_l^{(i)}U_1^{(r)} - 2D_1^{(i)}U_1^{(r)} \\ & - D_a^{(r)}U_1^{(i)} - 2D_1^{(r)}U_1^{(i)} - D_2^{(i)}U_l^{(r)} - D_2^{(r)}U_l^{(i)} - F_{NL_1}^{(i)} = 0 \end{aligned} \quad (\text{C.13b})$$

$$\begin{aligned} & D_2^{(r)}U_3^{(r)} - D_2^{(i)}U_3^{(i)} + 2D_1^{(r)}U_2^{(r)} - 2D_1^{(i)}U_2^{(i)} \\ & + D_2^{(r)}U_1^{(r)} - D_2^{(i)}U_1^{(i)} = 0 \end{aligned} \quad (\text{C.14a})$$

$$\begin{aligned} & -D_2^{(i)}U_3^{(r)} - D_2^{(r)}U_3^{(i)} - 2D_1^{(i)}U_2^{(r)} - 2D_1^{(r)}U_2^{(i)} \\ & - D_2^{(i)}U_1^{(r)} - D_2^{(r)}U_1^{(i)} = 0 \end{aligned} \quad (\text{C.14b})$$

$$\begin{aligned} & D_2^{(r)}U_4^{(r)} - D_2^{(i)}U_4^{(i)} + D_a^{(r)}U_3^{(r)} + 2D_1^{(r)}U_3^{(r)} \\ & + D_a^{(i)}U_3^{(i)} - 2D_1^{(i)}U_3^{(i)} + D_2^{(r)}U_2^{(r)} - D_2^{(i)}U_2^{(i)} + F_{NL_3}^{(r)} = 0 \end{aligned} \quad (\text{C.15a})$$

$$\begin{aligned} & -D_2^{(i)}U_4^{(r)} - D_2^{(r)}U_4^{(i)} - D_a^{(i)}U_3^{(r)} - 2D_1^{(i)}U_3^{(r)} \\ & - D_a^{(r)}U_3^{(i)} - 2D_1^{(r)}U_3^{(i)} - D_2^{(i)}U_2^{(r)} - D_2^{(r)}U_2^{(i)} - F_{NL_3}^{(i)} = 0 \end{aligned} \quad (\text{C.15b})$$

$$\begin{aligned}
& D_2^{(r)}U_5^{(r)} - D_2^{(i)}U_5^{(i)} + 2D_1^{(r)}U_4^{(r)} - 2D_1^{(i)}U_4^{(i)} \\
& + D_2^{(r)}U_3^{(r)} - D_2^{(i)}U_3^{(i)} = 0
\end{aligned} \tag{C.16a}$$

$$\begin{aligned}
& -D_2^{(i)}U_5^{(r)} - D_2^{(r)}U_5^{(i)} - 2D_1^{(i)}U_4^{(r)} - 2D_1^{(r)}U_4^{(i)} \\
& -D_2^{(i)}U_3^{(r)} - D_2^{(r)}U_3^{(i)} = 0
\end{aligned} \tag{C.16b}$$

$$\begin{aligned}
& D_2^{(r)}U_r^{(r)} - D_2^{(i)}U_r^{(i)} + D_a^{(r)}U_5^{(r)} + 2D_1^{(r)}U_5^{(r)} \\
& -D_a^{(i)}U_5^{(i)} - 2D_1^{(i)}U_5^{(i)} + D_2^{(r)}U_4^{(r)} - D_2^{(i)}U_4^{(i)} + F_{NL_5}^{(r)} = 0
\end{aligned} \tag{C.17a}$$

$$\begin{aligned}
& -D_2^{(i)}U_r^{(r)} - D_2^{(r)}U_r^{(i)} - D_a^{(i)}U_5^{(r)} - 2D_1^{(i)}U_5^{(r)} \\
& -D_a^{(r)}U_5^{(i)} - 2D_1^{(r)}U_5^{(i)} - D_2^{(i)}U_4^{(r)} - D_2^{(r)}U_4^{(i)} - F_{NL_3}^{(i)} = 0
\end{aligned} \tag{C.17b}$$

$$D_1^{(r)}U_r^{(r)} - D_1^{(i)}U_r^{(i)} + D_2^{(r)}U_5^{(r)} - D_2^{(i)}U_5^{(i)} = 0 \tag{C.18a}$$

$$-D_1^{(i)}U_r^{(r)} - D_1^{(r)}U_r^{(i)} - D_2^{(i)}U_5^{(r)} - D_2^{(r)}U_5^{(i)} = 0 \tag{C.18b}$$

where

$$D_a^{(r)} = s_1, \quad D_a^{(i)} = \omega c \tag{C.19}$$

and

$$F_{NL_1}^{(r)} = \frac{3}{4}s_3 \left(U_1^{(r)3} + U_1^{(r)}U_1^{(i)2} \right), \quad F_{NL_1}^{(i)} = \frac{3}{4}s_3 \left(U_1^{(i)3} + U_1^{(i)}U_1^{(r)2} \right) \tag{C.20a}$$

$$F_{NL_3}^{(r)} = \frac{3}{4}s_3 \left(U_3^{(r)3} + U_3^{(r)}U_3^{(i)2} \right), \quad F_{NL_3}^{(i)} = \frac{3}{4}s_3 \left(U_3^{(i)3} + U_3^{(i)}U_3^{(r)2} \right) \tag{C.20b}$$

$$F_{NL_5}^{(r)} = \frac{3}{4}s_3 \left(U_5^{(r)3} + U_5^{(r)}U_5^{(i)2} \right), \quad F_{NL_5}^{(i)} = \frac{3}{4}s_3 \left(U_5^{(i)3} + U_5^{(i)}U_5^{(r)2} \right) \tag{C.20c}$$

C.3 Configuration (ii) for a structure built of two unit cells

For the cell configuration (i) and a structure built of two unit cells, the force equilibrium at nodes U_l , U_1 , U_2 and U_r are defined, respectively, as

$$F_l = D_1 U_l + D_2 U_1 \quad (\text{C.21a})$$

$$0 = D_2 U_l + D_1 U_1 + D_a (U_1 - U_2) + F_{NL_A} \quad (\text{C.21b})$$

$$0 = D_1 U_2 + D_2 U_r - D_a (U_1 - U_2) - F_{NL_A} \quad (\text{C.21c})$$

$$0 = D_2 U_2 + 2D_1 U_3 + D_2 U_4 \quad (\text{C.21d})$$

$$0 = D_2 U_3 + D_1 U_4 + D_a (U_4 - U_5) + F_{NL_B} \quad (\text{C.21e})$$

$$0 = D_1 U_5 + D_2 U_r - D_a (U_4 - U_5) - F_{NL_B} \quad (\text{C.21f})$$

$$0 = D_2 U_5 + D_1 U_r \quad (\text{C.21g})$$

where $D_1 = E A k \cot(kl)$, $D_2 = -E A k \csc(kl)$, $D_a = s_1 - j\omega c$. The the force component due to the nonlinear spring term which depends on the displacements U_1 and U_2 , and U_4 and U_5 are $F_{NL_A} = (3/4) s_3 (U_1 - U_2)^3$ and $F_{NL_B} = (3/4) s_3 (U_4 - U_5)^3$, respectively.

Equations (C.21) are defined in terms of complex variables and can be expanded using real and imaginary components as for instance

$$D_1 = D_1^{(r)} - jD_1^{(i)}, \quad U_1 = U_1^{(r)} - jU_1^{(i)} \quad (\text{C.22a})$$

$$D_2 = D_2^{(r)} - jD_2^{(i)}, \quad U_2 = U_2^{(r)} - jU_2^{(i)} \quad (\text{C.22b})$$

$$U_3 = U_3^{(r)} - jU_3^{(i)}, \quad U_4 = U_4^{(r)} - jU_4^{(i)} \quad (\text{C.22c})$$

$$U_5 = U_5^{(r)} - jU_5^{(i)}, \quad U_r = U_r^{(r)} - jU_r^{(i)} \quad (\text{C.22d})$$

Substituting Eqs. (C.22) into Eqs. (C.21), the real and imaginary force equilibrium equations are defined in Eqs. (C.23) to (C.29) as

$$D_2^{(r)}U_1^{(r)} - D_2^{(i)}U_1^{(i)} + D_1^{(r)}U_l^{(r)} - D_1^{(i)}U_l^{(i)} = F_l \quad (\text{C.23a})$$

$$-D_2^{(i)}U_1^{(r)} - D_2^{(r)}U_1^{(i)} - D_1^{(i)}U_l^{(r)} - D_1^{(r)}U_l^{(i)} = 0 \quad (\text{C.23b})$$

$$\begin{aligned} & -D_a^{(r)}U_2^{(r)} + D_a^{(i)}U_2^{(i)} + D_a^{(r)}U_1^{(r)} + D_1^{(r)}U_1^{(r)} \\ & -D_a^{(i)}U_1^{(i)} - D_1^{(i)}U_1^{(i)} + D_2^{(r)}U_l^{(r)} - D_2^{(i)}U_l^{(i)} + F_{NL_A}^{(r)} = 0 \end{aligned} \quad (\text{C.24a})$$

$$\begin{aligned} & D_a^{(i)}U_2^{(r)} + D_a^{(r)}U_2^{(i)} - D_a^{(i)}U_1^{(r)} - D_a^{(r)}U_1^{(r)} \\ & -D_a^{(r)}U_1^{(i)} - D_1^{(r)}U_1^{(i)} - D_2^{(i)}U_l^{(r)} - D_2^{(r)}U_l^{(i)} - F_{NL_A}^{(i)} = 0 \end{aligned} \quad (\text{C.24b})$$

$$\begin{aligned} & D_2^{(r)}U_3^{(r)} - D_2^{(i)}U_3^{(i)} + D_a^{(r)}U_2^{(r)} + D_1^{(r)}U_2^{(r)} \\ & -D_a^{(i)}U_2^{(i)} - D_1^{(i)}U_2^{(i)} - D_a^{(r)}U_1^{(r)} + D_a^{(i)}U_1^{(i)} - F_{NL_A}^{(r)} = 0 \end{aligned} \quad (\text{C.25a})$$

$$\begin{aligned} & -D_2^{(i)}U_3^{(r)} - D_2^{(r)}U_3^{(i)} - D_a^{(i)}U_2^{(r)} - D_1^{(i)}U_2^{(r)} \\ & -D_a^{(r)}U_2^{(i)} - D_1^{(r)}U_2^{(i)} + D_a^{(i)}U_1^{(r)} + D_a^{(r)}U_1^{(i)} + F_{NL_A}^{(i)} = 0 \end{aligned} \quad (\text{C.25b})$$

$$D_2^{(r)}U_4^{(r)} - D_2^{(i)}U_4^{(i)} + 2D_1^{(r)}U_3^{(r)} - 2D_1^{(i)}U_3^{(i)} + D_2^{(r)}U_2^{(r)} - D_2^{(i)}U_2^{(i)} = 0 \quad (\text{C.26a})$$

$$-D_2^{(i)}U_4^{(r)} - D_2^{(r)}U_4^{(i)} - 2D_1^{(i)}U_3^{(r)} - 2D_1^{(r)}U_3^{(i)} - D_2^{(i)}U_2^{(r)} - D_2^{(r)}U_2^{(i)} = 0 \quad (\text{C.26b})$$

$$-D_a^{(r)}U_5^{(r)} + D_a^{(i)}U_5^{(i)} + D_a^{(r)}U_4^{(r)} + D_1^{(r)}U_4^{(r)} - D_a^{(i)}U_4^{(i)} - D_1^{(i)}U_4^{(i)} + D_2^{(r)}U_3^{(r)} - D_2^{(i)}U_3^{(i)} + F_{NL_B}^{(r)} = 0 \quad (\text{C.27a})$$

$$D_a^{(i)}U_5^{(r)} + D_a^{(r)}U_5^{(i)} - D_a^{(i)}U_4^{(r)} - D_1^{(i)}U_4^{(r)} - D_a^{(r)}U_4^{(i)} - D_1^{(r)}U_4^{(i)} - D_2^{(i)}U_3^{(r)} - D_2^{(r)}U_3^{(i)} - F_{NL_B}^{(i)} = 0 \quad (\text{C.27b})$$

$$D_2^{(r)}U_r^{(r)} - D_2^{(i)}U_r^{(i)} + D_a^{(r)}U_5^{(i)} + D_1^{(r)}U_5^{(r)} - D_a^{(i)}U_5^{(i)} - D_1^{(i)}U_5^{(i)} - D_a^{(r)}U_4^{(r)} + D_a^{(i)}U_4^{(i)} - F_{NL_b}^{(r)} = 0 \quad (\text{C.28a})$$

$$-D_2^{(i)}U_r^{(r)} - D_2^{(r)}U_r^{(i)} - D_a^{(i)}U_5^{(r)} - D_1^{(i)}U_5^{(r)} - D_a^{(r)}U_5^{(i)} - D_1^{(r)}U_5^{(i)} + D_a^{(i)}U_4^{(r)} + D_a^{(r)}U_4^{(i)} + F_{NL_b}^{(i)} = 0 \quad (\text{C.28b})$$

$$D_1^{(r)}U_r^{(r)} - D_1^{(i)}U_r^{(i)} + D_2^{(r)}U_5^{(r)} - D_2^{(i)}U_5^{(i)} = 0 \quad (\text{C.29a})$$

$$-D_1^{(i)}U_r^{(r)} - D_1^{(r)}U_r^{(i)} - D_2^{(i)}U_5^{(r)} - D_2^{(r)}U_5^{(i)} = 0 \quad (\text{C.29b})$$

where

$$D_a^{(r)} = s_1, \quad D_a^{(i)} = \omega c \quad (\text{C.30})$$

and

$$F_{NL_A}^{(r)} = \frac{3}{4}s_3 \left(\left(U_1^{(r)} - U_2^{(2)} \right)^3 + \left(U_1^{(r)} - U_{(2)}^{(r)} \right) \left(U_1^{(i)} - U_2^{(i)} \right)^2 \right) \quad (\text{C.31a})$$

$$F_{NL_A}^{(i)} = \frac{3}{4}s_3 \left(\left(U_1^{(i)} - U_2^{(i)} \right)^3 + \left(U_1^{(i)} - U_{(2)}^{(i)} \right) \left(U_1^{(r)} - U_2^{(r)} \right)^2 \right) \quad (\text{C.31b})$$

$$F_{NL_B}^{(r)} = \frac{3}{4}s_3 \left(\left(U_4^{(r)} - U_5^{(2)} \right)^3 + \left(U_4^{(r)} - U_{(5)}^{(r)} \right) \left(U_4^{(i)} - U_5^{(i)} \right)^2 \right) \quad (\text{C.31c})$$

$$F_{NL_B}^{(i)} = \frac{3}{4}s_3 \left(\left(U_4^{(i)} - U_5^{(i)} \right)^3 + \left(U_4^{(i)} - U_{(5)}^{(i)} \right) \left(U_4^{(r)} - U_5^{(r)} \right)^2 \right) \quad (\text{C.31d})$$