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Citation: American Journal of Physics **57**, 359 (1989); doi: 10.1119/1.16021
View online: http://dx.doi.org/10.1119/1.16021
View Table of Contents: http://scitation.aip.org/content/aapt/journal/ajp/57/4?ver=pdfcov
Published by the American Association of Physics Teachers

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A remark on the gravitational field produced by an infinite straight string

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(Received 21 December 1987; accepted for publication 3 May 1988)

The results predicted by Newtonian gravity and general relativity are compared regarding the field produced by an infinite gauge string with constant density λ. A simple gedankenexperiment is suggested to stress the remarkable differences between these two theories. The existence of the usual Newtonian limit is discussed in this case.

I. INTRODUCTION

The development of general relativity (GR) has provided us with more than new and accurate results of old phenomena. It has provided us with a new conception of the world. The unification of the concepts of space and time in one single continuum space-time manifold in special relativity was improved in GR by the assumption that the curvature of this manifold was linked with the concept of gravitation. General relativity was then considered a more precise description of the gravitational interaction and the results of the three crucial tests of relativity (bending of light rays by massive bodies, gravitational red shift, and Mercury's perihelion precession have had excellent agreement with observed values). In spite of this, the tests above do not stress the qualitatively new results introduced by the theory. In other words, the tests could be qualitatively understood in the context of other theories. The bending of light rays by spherical massive bodies may be calculated with the Newtonian theory of gravitation plus the equivalence principle (but giving half the value as predicted by GR),[1] the gravitational red shift predicted by GR for weak fields may be derived[2] directly from the Newtonian gravitational potential if we assume conservation of energy and the quantum relation to the photon energy $E = h\nu$ (h is the Planck universal constant and ν is the frequency of the light wave) and, finally, it was shown in this Journal[3] that the assumption of the equivalence principle enables us to evaluate, just by using special relativity, an estimate of Mercury's perihelion precession, which is again half that predicted by GR.

In Sec. II of this article we propose a gedankenexperiment involving the attraction of a test mass by a gauge string. Remarkable qualitative differences arise between predictions of GR and the Newtonian theory of gravitation. Finally, in Sec. III, we discuss the Newtonian limit of this system.

II. THE STRING GRAVITATION FIELD

In this Journal, Helliwell and Konkowski[4] have published an article where a clear view of the topological effects of cosmic strings is given. Following their exposition,[4,5] we write in weak field approximation the space-time line element of an infinite gauge string[5] with linear mass density λ in cylindrical coordinates ρ, ϕ, and z assuming the string lies along the z axis, as

$$ds^2 = -dt^2 + d\rho^2 + (1-4\lambda)^2\rho^2\, d\phi^2 + dz^2, \qquad (1)$$

where we have used the natural units $c = G = 1$ and imposed $\lambda < \frac{1}{4}$ (hereafter, our discussion applies only to gauge strings). A suitable change of coordinate $\phi \rightarrow \phi' = (1-4\lambda)\phi$ leads (1) to the Minkowskian line element,

$$ds^2 = -dt^2 + d\rho^2 + \rho^2\, d\phi'^2 + dz^2. \qquad (2)$$

We cast (2) in a more suitable form to visualize the fact that the infinite string does not bend the space-time surrounding it: Making the following transformation,

$$\rho = (x^2+y^2)^{1/2}, \quad \phi' = \arctan y/x,$$

we get

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2. \quad (3)$$

Equation (3) means that we have found a single coordinate system in which gravitation (bending of space-time) vanishes at all space-time points (except at $\rho = 0$). In other words, GR tells us that such a string does not attract any test mass.[4]

On the other hand, Newtonian gravitation tells us that the infinite string will generate at each point with radial distance ρ from the string a potential V given by[6]

$$V(\rho) = -\int_{-\infty}^{+\infty} \frac{\lambda}{(z^2+\rho^2)^{1/2}}\,dz + \int_{-\infty}^{+\infty} \frac{\lambda}{(z^2+\rho_0^2)^{1/2}}\,dz,$$

$$V(\rho) = 2\lambda \ln(\rho/\rho_0), \quad (4)$$

where ρ_0 is an arbitrary constant. Thus a test mass m at radial distance $\bar{\rho}$ from the string would be attracted by a force

$$\mathbf{F} = -m\left(\frac{\partial V}{\partial \rho}\right)_{\rho=\bar{\rho}} = -\frac{2m\lambda}{\bar{\rho}}\,\mathbf{e}_\rho. \quad (5)$$

Result (5) may be appreciated more if we imagine an experiment to measure the attraction on a test mass. The Newtonian potential at a point (ρ, ϕ, z) produced by a string of length $2L$ lying along the z axis, with its center at the origin of the coordinate axis, is easily calculated to be

$$V = -\int_{-L}^{L} \frac{\lambda}{[\rho^2 + (z-z')^2]^{1/2}}\,dz' = -\lambda \ln\left(\frac{z+L+\sqrt{(z+L)^2+\rho^2}}{z-L+\sqrt{(z-L)^2+\rho^2}}\right), \quad (6)$$

where we have assumed that infinitely distant points have zero potential. So we may calculate the field $\mathbf{G}$ generated by this string,

$$\mathbf{G} = -\nabla V = (G_\rho, 0, G_z),$$

where

$$G_z = \lambda\left(\frac{1+(z+L)/\sqrt{(z+L)^2+\rho^2}}{z+L+\sqrt{(z+L)^2+\rho^2}} - \frac{1+(z-L)/\sqrt{(z-L)^2+\rho^2}}{z-L+\sqrt{(z-L)^2+\rho^2}}\right), \quad (7)$$

$$G_\rho = \lambda\rho\left(\frac{[(z+L)^2+\rho^2]^{-1/2}}{z+L+\sqrt{(z+L)^2+\rho^2}} - \frac{[(z-L)^2+\rho^2]^{-1/2}}{z-L+\sqrt{(z-L)^2+\rho^2}}\right). \quad (8)$$

The force of attraction that a particle located in the plane $z = 0$ feels as predicted by Newtonian theory must evidently be

$$F_z = 0 \quad \text{(by symmetry)},$$

$$F_\rho = mG_\rho = -2\lambda m/\rho(1+\rho^2/L^2)^{1/2}, \quad (9)$$

where ρ is the distance between the string's center and the particle with mass m. We see that F_ρ increases monotonically with L starting from zero (Fig. 1). On the other hand, GR predicts that the infinite string does not attract any particle. So if the attraction of the particle by the finite string of length $2L$ is specified by some measuring instrument in terms of a smooth function $F = F(L)$, then GR says that

$$\lim_{L\to\infty} F(L) = 0.$$

Obviously, we also have

$$\lim_{L\to 0} F(L) = 0,$$

and so two possible cases can be distinguished for the function $F(L)$. In the first one, $F(L)$ is identically zero while in the second there must be at least one L_0 for which $F(L_0) \neq 0$. We discard the former case in favor of the latter for the following reason. For a sufficiently small piece of string, spherical symmetry gives a good asymptotic approximation for the field. So for some $L_0 \ll 1$, a string generates a quasi-Schwarzschild space-time and it is well known that a test mass initially at rest in a Schwarzschild space-time moves toward the source of the field. Consequently, there is a nonvanishing contribution to the function F in this range of L. Since the infinite string in GR does not generate any gravitational field, it is clear that there will be values for L for which F is decreasing. Thus the comparison between GR and Newtonian theory can be summarized as in Figs. 1 and 2.

These remarkable aspects of GR can be attributed to the nonlinearities of the gravitational field as opposed to Newtonian theory: We cannot just add linearly infinitesimal contributions as we did when integrating (6). An infinitesimal piece of the string will give a nonzero gravitational field in relativistic gravitation, although the total contribution will "sum" up to zero. An interesting experiment can now be proposed. Just set a particle and a finite string in the configuration described above and add small pieces onto its extremities. One should notice, sooner or later, that the attraction felt by the particle should decrease. The obvious practical limitations are: (1) size of the string; (2) isolation

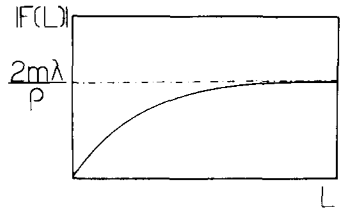

Fig. 1. The graph shows the Newtonian attraction force $|F(L)|$ on a test particle in the presence of a straight gauge string. The function $|F(L)|$ increases monotonically with L.

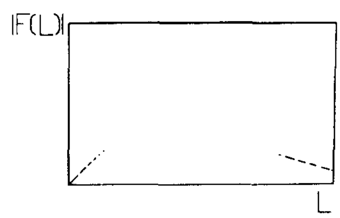

Fig. 2. The graph shows the attraction force $|F(L)|$ predicted by GR on a test particle in the presence of a straight gauge string in the asymptotic regions $L \ll 1$ and $L \to \infty$.

from other bodies; and (3) the string being sufficiently straight.

This effect goes against the intuition that in a region of space-time where the gravitational potential is weak GR must be at least qualitatively approximated by Newtonian theory. We are then motivated to study the Newtonian limit of GR in the special case of the infinite string.

III. THE NEWTONIAN LIMIT

We discuss now the existence of a Newtonian limit for the infinite string. The usual assumptions for this limit to exist can be found in many textbooks on this subject.[7] The condition that clocks can be synchronized on spatial regions of space-time is expressed by setting[8] (1) $g_{0i} = 0$; (2) the static field condition is $g_{\mu\nu,0} = 0$; (3) the elimination of relativistic kinematics effects of the test mass is expressed by $u^i = dx^i/ds \sim 0$, where u^i represents the velocity of the particle in space, and, finally, (4) the weak field condition $g_{\mu\nu,m} \ll 1$, that is, the field does not vary too quickly in space. Using the geodesic equation,

$$\frac{d^2x^\mu}{ds^2} + \Gamma^\mu_{\nu\sigma}\frac{dx^\nu}{ds}\frac{dx^\sigma}{ds} = 0,$$

and (1)–(3), we get

$$\frac{du_m}{dt} = -(\sqrt{-g_{00}})_{,m} \qquad (10)$$

where $\Gamma^\mu_{\nu\sigma}$ are the Christoffel symbols and the comma means partial derivative. The interpretation of g_{00} from (10) can be obtained by the following equation,

$$g^{mn}g_{00,mn} = 0 \qquad (11)$$

[in order to derive (11) we need Einstein's field equation[7] and condition (4)]. Writing the vacuum d'Alembert equation $\Box V = 0$ for the potential V in covariant form as

$$g^{\mu\nu}V_{;\mu\nu} = g^{\mu\nu}(V_{,\mu\nu} - \Gamma^\alpha_{\mu\nu}V_{,\alpha}), \qquad (12)$$

where the semicolon is the covariant derivative, we obtain in the weak field approximation (4) the relation

$$g^{mn}V_{,mn} = 0. \qquad (13)$$

We may choose the unit of time so that $-g_{00}$ is approximately 1. This means that we are studying a small deviation from Minkowskian space-time, where $g_{00} = -1$. Then from (11) and (13), we identify g_{00} and V as follows:

$$-g_{00} = 1 + 2(V - V_0), \qquad (14)$$

where V_0 is an arbitrary constant. Note that if we replace g_{00} in (11) by (14), we recover (13). It is often said that for weak fields $|V| \ll 1$. However, this statement assumes that the potential goes to zero at infinity. In general, this occurs only when the source of the field is bounded. In our case, such a condition is not satisfied unambiguously because the value of the potential depends on the arbitrary choice of ρ_0 in (4). A more general weak field condition is

$$|\nabla V| \ll 1. \qquad (15)$$

Every region in which (15) is satisfied must have a potential that does not differ too much from a certain value V_0. Expanding $(-g_{00})^{1/2}$ in the neighborhood of $V \sim V_0$ we get, from (14),

$$(-g_{00})^{1/2} = 1 + \frac{1}{[1 + 2(V - V_0)]^{1/2}_{V = V_0}} \times (V - V_0) + O[(V - V_0)^2].$$

Disregarding second-order terms in $V - V_0$, we obtain

$$(-g_{00})^{1/2} = 1 + (V - V_0). \qquad (16)$$

Substituting (16) into (10), we recover the famous Newtonian second law acceleration $= -\nabla V$, and we have obtained the so-called Newtonian limit.

Although g_{00} and V satisfy, respectively, (11) and (13) (that is, the same equation given by the Laplacian operator), they need not have the same functional solution because the integration constants are not known *a priori*. To see this more clearly, let us consider the solution of the Laplace equation

$$g^{mn}f_{,mn} = 0 \qquad (17)$$

in cylindrical coordinates

$$f(\rho) = A \ln \rho/\rho_0 + B, \qquad (18)$$

where A, B, and ρ_0 are arbitrary constants of integration. If we solve (11) for g_{00}, we must set $A = 0$ and $B = -1$ because g_{00} is exactly -1 for the string. On the other hand, if we solve (13) for V, we must set $A = 2\lambda$ and $B = 0$ to get the potential in the functional form given in (4). So, in this special case, we are not allowed to do the identification (14) since the left-hand side is a constant while the right-hand side is not. Notice that all conditions, (1)–(4), are satisfied by $g_{00} = -1$, but there is no way to recover from this the functional form of V.

We have seen that the infinite string plays the role of a counterexample where the identification (14) cannot be made. In view of the arguments presented, our conclusion is that it is not possible to determine a consistent, unambiguous Newtonian limit for a straight infinite gauge string. One consequence of this fact is that in any subregion of space-time (where the weak field approximation is valid), we will obtain qualitatively different predictions between GR and Newtonian theory, even if λ is chosen to be small.

IV. CONCLUSIONS

By imposing a special symmetry, we have obtained a model in which remarkable differences between Newtonian gravity and general relativity are made manifest. These differences can be attributed to the fact that GR is a nonlinear theory of space-time where symmetry configurations can be explored. Although the infinite gauge string is a somewhat pathological model, it is nevertheless an instructive example against which our intuition can be tested.

ACKNOWLEDGMENTS

We would like to acknowledge Professor R. Aldrovandi for discussions on the subject and FINEP, CNPq (GE), and FAPESP (GM) for financial support.

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Exercises in the synthesis of electrical impedances

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(Received 28 October 1987; accepted for publication 20 May 1988)

Two general circuits are developed for the synthesis of user-variable electrical impedances. One circuit permits user-variable enlargement of any existing impedance without affecting its phase characteristics. This circuit facilitates such devices as floating capacitors and inductors which are continuously variable over wide ranges. The other circuit permits simulation of user-variable floating impedances whose phase and frequency characteristics are widely subject to design objectives. In addition to all conventional impedance functions, it poses access to an infinite set of alternatives for the simulation of resistance and reactance functions of frequency. Included is a theoretical approach for predicting parasitic impedances reported in earlier works on similar devices known as gyrators.

I. INTRODUCTION

The effect of any passive two-terminal current path on the circuit in which it participates depends exclusively on the path's characteristic impedance. A few physical laws typically determine the path's resistance and reactance characteristics, but any alternative "manager" that enforces the essential relationship between voltage and current will affect the circuit identically. The synthetic impedance circuits presented here can perform in such management roles with unusual versatility.

These circuits' abilities to simulate capacitors and inductors that are gang-tunable and continuously variable over unlimited ranges pose interesting new options in signal filtering, from subaudio through the UHF bands. In addition, they possess theoretical prospects for simulating uncommon impedances such as frequency-dependent variable resistors and variable capacitors and inductors with uncommon reactance versus frequency relationships. Such prospects are influential in motivating student interest in the study of these exercises.

The theory presented here is approached exclusively from the perspective of phasor analysis. Students may mistakenly contrive that this limits applications of these devices to circuits whose signals are exclusively sinusoidal or periodic. The rubrics of Laplace and Fourier transform theory may be employed to project the phasor analyses into the time domain, to better establish students' awareness of the equivalence of these distinct perspectives. Alternatively, any specific synthesizer may be analyzed directly in the time domain with the loss of generality that the phasor analysis accommodates. Such analyses in the time domain may be suitable for presentation even at introductory levels of analog electronics.

Tellegen[1] developed the first circuit for simulating electrical inductances, naming his circuit the gyrator. Early analog gyrators, given minor alterations, could have (inductively) coupled any charge source to any autonomous terminal; that is, any terminal whose potential is independent of charge flows from the terminal. Despite this small (and herewith minimized) limitation, the formulated techniques of engineers in those days led their analyses to assume unnecessarily that the autonomous terminal must have a fixed (ground) potential. Riordan[2] freed the grounded terminal by introducing rather profound complications into the earlier circuits, complications which, for many purposes, were entirely unnecessary.

Sheahan and Orchard[3] and other authors cited problems with both the grounded inductor and the floating inductor devised by Riordan, but for their concentrations on filtering techniques they produced little substantial enlightenment on simpler or more versatile alternatives for the flotation of simulated inductors.

The author has discovered only one theory[4] that claims to simulate capacitors via analog circuits. The apparent unnoteworthiness of capacitive gyrators among engineering literature is explained here during analysis of the general impedance synthesizer (GIS), wherein a devastating problem with the simulation of capacitance via conventional techniques is predicted theoretically. This theoretical approach provides groundwork facilitating prediction and evaluation of parasitic impedances whose presence was frequently acknowledged in previous works on inductive gyrators.

Recently, a digital approach known as the switch-capacitor technique has been developed to simulate floating reactances, but these techniques are quite complicated compared to the simple tasks they perform, and their fre-

