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On the BRST cohomology of Lie superalgebras

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ON THE BRST COHOMOLOGY

OF LIE SUPERALGEBRAS

Dissertação de Mestrado apresentada ao Instituto de Física Teórica do Câmpus de São Paulo, da Universidade Estadual Paulista "Júlio de Mesquita Filho", como parte dos requisitos para obtenção do título Mestre em Ciências, área: Física.

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Aos meus sobrinhos,
de sangue e coração

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B de baixinho,
[...]
R de riacho,
S, saudade,
T de Terra!

— Xuxa, *Abecedário da Xuxa*

Resumo

Nas últimas décadas, houve um interesse expressivo na dualidade entre duas teorias distintas: uma teoria de cordas do Tipo IIB no espaço $\mathrm{AdS}_5 \times \mathbb{S}^5$, e uma teoria de Yang–Mills supersimétrica com $\mathcal{N} = 4$ em um espaço plano 4-dimensional. No nível de simetrias, o supergrupo de Lie $\mathrm{PSU}(2,2|4)$ contém as superisometrias da primeira teoria, enquanto serve de grupo superconforme da segunda. Através da dualidade, é possível comparar as funções de correlação dessas teorias. Olhando para o produto de operadores locais, pode-se perguntar em qual representação o seu produto se transforma. É, então, necessário lidar com os espaços de cohomologia do grupo de simetrias.

Nesta dissertação, eu apresento os primeiros passos para se levar a cabo tal estudo. Eu começo definindo a cohomologia de um complexo de cocadeias, as noções de graduação e filtração dos mesmos, e o delineamento da construção de uma sequência espectral advinda de uma filtração. A cohomologia de álgebra de Lie é revisada, com atenção às formulações de Chevalley–Eilenberg e BRST. Em seguida, as superalgebras de Lie são definidas, e a abordagem BRST é reinterpretada através de um isomorfismo entre o complexo de cocadeias de Chevalley–Eilenberg e a álgebra de polinômios em coordenadas fantasmas. A cohomologia em coeficientes triviais da superálgebra de Lie linear geral $\mathfrak{gl}(m|n)$ é então calculada em detalhe no formalismo BRST por meio da sua sequência espectral de Hochschild–Serre.

Palavras Chaves: Cohomologia BRST; Superalgebras de Lie; Sequências espectrais; Álgebras de Lie.

Áreas do conhecimento: Física; Física Matemática.

Abstract

In the last decades, there was an expressive interest on a duality between two distinct theories: a Type IIB string theory on $\mathrm{AdS}_5 \times \mathbb{S}^5$ space, and an $\mathcal{N} = 4$ supersymmetric Yang–Mills theory in 4-dimensional flat space. On the level of symmetries, the Lie supergroup $\mathrm{PSU}(2,2|4)$ contains the superisometries of the former, while being the superconformal group of the latter. Through the duality, one is capable of comparing correlation functions of those theories. When looking at the product of local operators, one may ask in what representation their product transforms. One then has to deal with the cohomology spaces of the group of symmetries.

In this thesis, I present the first steps one might take towards that. I start by defining the cohomology of a cochain complex, the notions of a grading and a filtration on it, and outline the construction of a spectral sequence from a filtration. The Lie algebra cohomology is reviewed, with attention to the Chevalley–Eilenberg and the BRST formulations. Next, Lie superalgebras are defined, and the BRST approach is reinterpreted via an isomorphism between the Chevalley–Eilenberg cochain complex and the algebra of polynomials in ghost coordinates. The cohomology in trivial coefficients of the general linear Lie superalgebra $\mathfrak{gl}(m|n)$ is then computed in detail in the BRST formalism through its Hochschild–Serre spectral sequence.

Keywords: BRST cohomology; Lie superalgebras; Spectral sequences; Lie algebras.

Areas of Knowledge: Physics; Mathematical Physics.

Contents

Chapter 1

Motivation

Anti-de Sitter space

In 1917, Willem de Sitter (1872–1934) published his solutions to Einstein's equations containing a cosmological constant Λ. The family of solutions with $\Lambda > 0$ is known today as the *de Sitter space*, while solutions with a negative cosmological constant, $\Lambda < 0$, are called *anti-de Sitter space*. Let AdS_{d+1} denote the $(d+1)$-dimensional anti-de Sitter space; it can be understood as a hyperboloid,[1]

$$(X_0)^2 + (X_{d+1})^2 - \delta^{ij} X_i X_j = R^2,$$

embedded in $(d+2)$-dimensional flat space $\mathbb{R}^{2,d}$ with line element

$$\mathrm{d}s^2 = -\,\mathrm{d}X_0^2 - \mathrm{d}X_{d+1}^2 + \delta^{ij}\,\mathrm{d}X_i\,\mathrm{d}X_j.$$

AdS_{d+1} is the maximally symmetric space with constant negative cosmological constant $\Lambda \propto -\frac{1}{R^2}$. The constant R is called the *radius* of AdS space. Being embedded in $\mathbb{R}^{2,d}$, the AdS_{d+1} space has $\mathrm{SO}(2,d)$ as isometry group.

In a suitable coordinate system, one can rewrite the line element for AdS_{d+1} as

$$\mathrm{d}s^2 = R^2\Big(-\cosh^2\!\rho\;\mathrm{d}\tau^2 + \mathrm{d}\rho^2 + \sinh^2\!\rho\;\mathrm{d}\Omega_{d-1}^2\Big), \tag{1.1}$$

where $\mathrm{d}\Omega_{d-1}^2$ is the line element of the $(d-1)$-dimensional sphere $\mathbb{S}^{d-1} \subset \mathbb{R}^d$, and $(\tau,\rho,\Omega) \in [0,2\pi) \times \mathbb{R}_+ \times \mathbb{S}^{d-1}$ are called *global coordinates*. Figure 1 shows pictorially how AdS_{d+1} can be visualized as a hyperboloid; each point in the 2-dimensional hyperboloid corresponds to a $(d-1)$-dimensional sphere $\mathbb{S}^{d-1}$. Note however that, strictly speaking, AdS space corresponds only to half the hyperboloid, for which $\rho \geqslant 0$ (where $\rho = 0$ corresponds to the cross-section with the smallest radius).

[1] Einstein summation convention will be assumed throughout the text, for all kinds of indices.

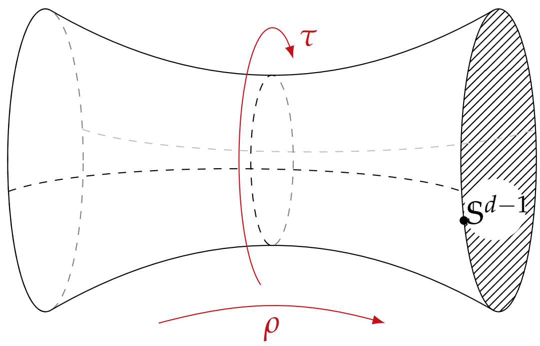


Figure 1: $(d+1)$-dimensional anti-de Sitter space can be visualized as a $(d+1)$-dimensional hyperboloid embedded in $\mathbb{R}^{2,d}$. Each point in the 2-dimensional hyperboloid above, parametrized by (τ, ρ), corresponds to a $(d-1)$-dimensional sphere $\mathbb{S}^{d-1}$.

By requiring that $\tan\theta = \sinh\rho$, one can rewrite (1.1) as

$$\mathrm{d}s^2 = \frac{R^2}{\cos^2\theta}\Big(-\,\mathrm{d}\tau^2 + \mathrm{d}\theta^2 + \sin^2\theta\ \mathrm{d}\Omega_{d-1}^2\Big), \tag{1.2}$$

with $(\tau, \theta, \Omega_i) \in [0, \pi) \times [0, \frac{\pi}{2}) \times \mathbb{S}^{d-1}$. To avoid closed time-like curves — which are easy to see in Figure 1 —, one has to unwrap τ to $\mathbb{R}$, obtaining the *universal cover* of AdS_{d+1} (AHARONY *et al.*, 2000). Then the topology of AdS_{d+1} is the same as that of $\mathbb{R} \times [0, \frac{\pi}{2}) \times \mathbb{S}^{d-1}$.

Conformal symmetries

An *isometry* on a (pseudo-)Riemannian manifold M is defined as a transformation $M \longrightarrow M$ that preserves distances; consequently, it also preserves angles. The isometries of the d-dimensional Minkowski spacetime are encoded in the Poincaré group, defined as the semidirect product of the Lorentz group (containing spatial rotations and boosts) and the translations in $\mathbb{R}^{1,d-1}$.

It is natural to ask if one can enlarge the Poincaré group, i.e. find a larger Lie group that contains Poincaré's as a subgroup. One such extension is by including *conformal transformations*, i.e. transformations that preserve angles, but not necessarily distances. An example is the scaling transformation $x \longmapsto sx$, for $s \neq 0$. On the line element, a conformal transformation acts as

$$\mathrm{d}s^2 \longmapsto \mathrm{e}^{2\sigma(x)}\,\mathrm{d}s^2, \tag{1.3}$$

for a certain function $\sigma(x)$ of spacetime.

The *conformal group* is the minimal group containing:

i. the Poincaré group,

ii. the scaling transformation $x \longmapsto sx$, and

iii. the "special conformal transformations", which are obtained as the composition of an inversion[2] $x \longmapsto x/\|x\|^2$, a translation and another inversion — thus, something like a "translation at infinity".

The Lie algebra of the conformal group, named the *conformal algebra*, is isomorphic to the special orthogonal Lie algebra $\mathfrak{so}(2,d)$ (AHARONY *et al.*, 2000).

Consider the d-dimensional Minkowski spacetime, $\mathbb{R}^{1,d-1}$. Its line element in spherical coordinates reads as usual and, after a series of coordinate transformations, as shown by AHARONY *et al.* (2000), it is possible to rewrite it as

$$\mathrm{d}s^2 = \frac{1}{4\cos^2\tilde{u}_+ \cos^2\tilde{u}_-}\Big(-\mathrm{d}\tau^2 + \mathrm{d}\psi^2 + \sin^2\psi\ \mathrm{d}\Omega^2_{d-2}\Big), \tag{1.4}$$

for $(\tau, \psi, \Omega) \in \mathbb{R} \times [0, \pi) \times \mathbb{S}^{d-2}$.

Compactifying Minkowski spacetime turns it into[3] $\mathbb{R} \times \mathbb{S}^{d-1}$. After that, one is allowed to assume conformal symmetry; in particular, transformation (1.3) on (1.4) shows that the metric on Minkowski spacetime is conformally equivalent (i.e. related by a conformal transformation) to

$$\mathrm{d}s'^2 = -\mathrm{d}\tau^2 + \underbrace{\mathrm{d}\psi^2 + \sin^2\psi\ \mathrm{d}\Omega^2_{d-2}}_{\mathrm{d}\Omega^2_{d-1}}. \tag{1.5}$$

For $\theta \to \frac{\pi}{2}$ on the AdS_{d+1} side, one has $\rho \to \infty$, for ρ as in (1.1). Thus, $\theta = \frac{\pi}{2}$ can be understood as the boundary of AdS_{d+1}. Such boundary has coordinates $(\tau, \Omega) \in \mathbb{R} \times \mathbb{S}^{d-1}$, which has same topology of compactified Minkowski spacetime and, furthermore, their metrics are conformally equivalent. That is the reason people say that the boundary of AdS_{d+1} space is conformally equivalent to the compactified d-dimensional Minkowski spacetime.

[2] For this transformation to be well-defined everywhere, one has to "compactify" Minkowski spacetime by adding "points at infinity" (only in space, not in time).

[3] More precisely, consider only a spacial slice $\mathbb{R}^{d-1}$ of the d-dimensional Minkowski spacetime. For some point $P \in \mathbb{S}^{d-1}$, the punctured sphere $\mathbb{S}^{d-1}\backslash\{P\}$ can be stereographically projected on the euclidean space $\mathbb{R}^{d-1}$, and such projection is a diffeomorphism. Consequently, by adding a point $\tilde{P}$ "at infinity" of $\mathbb{R}^{d-1}$, one realizes that $\mathbb{S}^{d-1}$ is diffeomorphic to $\mathbb{R}^{d-1} \cup \{\tilde{P}\}$. Consequently, the compactified d-dimensional Minkowski spacetime is diffeomorphic to $\mathbb{R} \times \mathbb{S}^{d-1}$.

Superconformal symmetries

One might not yet be satisfied with the enlargement of the Poincaré group by the conformal group, and may wish to push this further. One would then be glad to learn that it is possible if one includes *supersymmetries*, i.e. symmetries generated by fermionic operators that anti-commute with translations (AHARONY *et al.*, 2000). However, there is a catch: that only works for some dimensions (d) and some numbers of supersymmetry operators ($\mathcal{N}$).

One example is the $d = 4$, $\mathcal{N} = 4$ supersymmetric Yang–Mills theory (SYM, for short). The theory consists of the gauge field A_μ, four Weyl fermions ψ_a, and six scalar fields ϕ^I. From D'HOKER–FREEDMAN (2004), the action for $\mathcal{N} = 4$ SYM theory in d dimensions reads as

$$S_{\text{SYM}}[A,\psi,\phi] = \int \mathrm{d}^d x \ \operatorname{tr}\Bigg(-\frac{1}{2g_{\text{YM}}^2} F^2 + \frac{\theta_I}{8\pi^2} F\tilde{F} - \mathrm{i}\bar{\psi}\not{D}\psi - (D\phi)^2 + \\ + g_{\text{YM}}\Big(C_I^{ab}\psi_a[\phi_I,\psi_b] + \bar{C}_I^{ab}\bar{\psi}_a[\phi_I,\bar{\psi}_b]\Big) + \frac{g_{\text{YM}}^2}{2}\sum_{i,j}[\phi^I,\phi^J]^2 \Bigg).$$

The first term, $F^2 \equiv F_{\mu\nu}F^{\mu\nu}$, is the kinetic term for the gauge bosons, where F is the curvature of A, and g_{YM} is the Yang–Mills coupling constant. The second is the θ-term, containing the instanton angle θ, associated to a topological contribution. The third term, $\bar{\psi}\not{D}\psi \equiv \bar{\psi}^a\sigma^\mu D_\mu\psi_a$, is the fermionic kinetic term and coupling to the gauge field (encoded in the gauge covariant derivative D_μ), with σ^μ the Pauli matrix 4-vector. The fourth term, $(D\phi)^2 \equiv \delta_{IJ}(D_\mu\phi^I)(D^\mu\phi^J)$, is the scalar kinetic term and coupling to the gauge field. The fifth term, $C_I^{ab}\psi_a[\phi^I,\psi_b]$ and its conjugate, encodes the Yukawa coupling of the fermions with the scalar fields. The sixth term, $[\phi,\phi]^2$, is the scalar potential.

A noteworthy feature of $d = 4$, $\mathcal{N} = 4$ SYM theory is that it preserves the superconformal symmetry after quantization. The Poincaré and conformal symmetries form the conformal group $\mathrm{SO}(2,4) \sim \mathrm{SU}(2,2)$ which, upon extension by the supersymmetries, becomes the Lie *supergroup* $\mathrm{PSU}(2,2|4)$.

A supergroup is a group whose elements, which in this case are physical symmetry transformations, are separated into different *statistics*: the slash in $(2,2|4)$ separates the usual symmetries (which can be thought of being *even*, as they preserve statistics of the fields) from the $\mathcal{N} = 4$ supersymmetries (which flip statistics of the fields, thus being *odd*); the comma indicates the signature of the metric (e.g. the d-dimensional Minkowski spacetime is $\mathbb{R}^{1,d-1}$).

The correspondence

Originally conjectured by Maldacena in 1997, there is a duality between an $\mathcal{N} = 4$ supersymmetric Yang–Mills theory on 4-dimensional flat space (the gauge side) and a Type IIB superstring theory on $\mathrm{AdS}_5 \times \mathbb{S}^5$ space (the gravity side). Such duality is known as the *AdS/CFT correspondence* (MALDACENA, 1999).

The parameters defining $d = 4$, $\mathcal{N} = 4$ SYM, namely the gauge group $\mathrm{SU}(N)$ and the coupling constant g_{YM}, are related to the string length $\ell_\mathrm{s} \equiv \sqrt{\alpha'}$ (where α' is the inverse string tension), the string coupling g_s and the AdS radius R via

$$g_{\mathrm{YM}}^2 = \pi g_\mathrm{s} \qquad \text{and} \qquad g_{\mathrm{YM}}^2 N = \frac{R^4}{(\alpha')^2}.$$

As pointed out by WITTEN (1998), in the large N limit with $g_{\mathrm{YM}}^2 N$ large, but fixed, the string theory side turns out to be weakly coupled, to which supergravity ("sugra", for short) becomes a good approximation. That is an example of a gauge/gravity duality. The common vocabulary is that the supergravity theory *lives in the bulk* (of $\mathrm{AdS}_5 \times \mathbb{S}^5$ space), while SYM theory *lives at the boundary* — which is conformally Minkowski space,[4] as already discussed.

Given a field ϕ_0 on the boundary, there exists (under some conditions) a unique extension ϕ of ϕ_0 to the bulk (i.e. AdS_{d+1}) satisfying the field equations of supergravity in the bulk. As an example, for a massless scalar field ϕ with restriction ϕ_0 on the boundary, let $Z_{\mathrm{sugra}}(\phi_0)$ be the partition function for supergravity with boundary condition ϕ_0. Then the classical approximation to supergravity gives

$$Z_{\mathrm{sugra}}(\phi_0) \approx \exp\big(-I_{\mathrm{sugra}}(\phi)\big),$$

where I_{sugra} is the classical action for supergravity.

On the boundary ("bdry", for maths expressions), ϕ_0 couples to a conformal field $\mathcal{O}$ via a coupling $\int_{\mathrm{bdry}} \phi_0\, \mathcal{O}$ — i.e. ϕ_0 serves as a "source" for $\mathcal{O}$ in the context of generating functionals. Then, the correspondence between the CFT in the boundary and the supergravity on the bulk is, as in WITTEN (1998),

$$\left\langle \exp\left(\int_{\mathrm{bdry}} \phi_0\, \mathcal{O}\right)\right\rangle_{\mathrm{CFT}} = Z_{\mathrm{sugra}}(\phi_0).$$

[4] Strictly speaking, the boundary of $\mathrm{AdS}_5 \times \mathbb{S}^5$ space is $\partial(\mathrm{AdS}_5) \times \varnothing$, as the 5-dimensional sphere $\mathbb{S}^5 \subset \mathbb{R}^6$ has no boundary.

Reciprocally, a "nice enough" metric on the bulk induces a conformal structure on the boundary (WITTEN, 1998). If $Z_{\text{CFT}}(h)$ denotes the partition function of the CFT (i.e. SYM on AdS) with a conformal structure h, the correspondence can be understood as

$$Z_{\text{CFT}}(h) = Z_{\text{sugra}}(h),$$

where $Z_{\text{sugra}}(h)$ is the partition function for supergravity obtained by integrating over "nice enough" metrics that induce the conformal structure h on the boundary. Again, in the classical approximation to supergravity, one can find a metric g that solves the field equations and the boundary conditions, and set

$$Z_{\text{sugra}}(h) \approx \exp\big(-I_{\text{sugra}}(h)\big).$$

Having gauge-invariant local operators $\mathcal{O}_i$ (e.g. $\operatorname{tr} F^2$) in the gauge theory side (i.e. SYM on the boundary), one is interested in computing correlation functions

$$\big\langle \mathcal{O}_1(x_1)\, \mathcal{O}_2(x_2)\, \cdots\, \mathcal{O}_n(x_n) \big\rangle_{\text{CFT}}$$

that can be translated as computations of supergravity in the bulk (i.e. on AdS) through the correspondence.

Representation of operators

As it is discussed at the start of Chapter 3, gauge theories possess a special kind of symmetry, the BRST symmetry, infinitesimally generated by a fermionic nilpotent operator Q. Physical states must be in the kernel[5] of Q, up to trivial contributions from its image; hence one has to consider the cohomology of Q.

If the *ghost number* (i.e. the power in ghost fields) of the operator $V_1 V_2$ is such that the corresponding cohomology of the symmetry group G (PSU, for AdS/CFT) is zero, then $V_1 V_2$ is Q-exact. That means that there exists an operator $\widetilde{V}$ with one less ghost than $V_1 V_2$, such that

$$Q\widetilde{V} = V_1 V_2. \tag{1.6}$$

[5] If S is a (full) symmetry of the theory, then it does not change physical states, namely $S\psi = \psi$. Now, if s is the infinitesimal generator of S, then

$$S\psi = e^s \psi = (1 + s + \cdots)\psi = \psi + s\psi + \cdots,$$

from which one concludes that $s\psi = 0$. In mathematical terms, an object in the stabilizer of $S \in G$ is necessarily in the kernel of $s \in \mathfrak{g}$.

Suppose that the operators V_1 and V_2 transform in the representations R_1 and R_2, respectively. While Q is G-invariant, and $V_1 V_2$ transform in the representation $R_1 \otimes R_2$, it is not generally true that $\widetilde{V}$ transform in $R_1 \otimes R_2$ too. In fact, it can transform in a representation $\widetilde{R}$ fitting in the diagram

$$0 \longrightarrow A \xrightarrow{\subseteq} \widetilde{R} \xrightarrow{Q} R_1 \otimes R_2 \longrightarrow 0 \tag{1.7}$$

in a way that the map $A \longrightarrow \widetilde{R}$ is injective, the map $Q\colon \widetilde{R} \longrightarrow R_1 \otimes R_2$ is surjective, and $Q(A) = 0$. In that case, $\widetilde{R}$ is said to be an *extension* of $R_1 \otimes R_2$ by A. Intuitively, $\widetilde{R}$ is an extension of $R_1 \otimes R_2$ by something Q-exact, hence being washed away upon the action of Q on $\widetilde{V}$ in (1.6) (MIKHAILOV, 2025).

Motivated by the AdS/CFT correspondence, it is of interest to look for the cohomology of the Lie supergroup $\mathrm{PSU}(2,2|4)$ — which serves as the super-isometries of $\mathrm{AdS}_5 \times \mathbb{S}^5$ on the (super)gravity side, and as the superconformal group of $d = 4$, $\mathcal{N} = 4$ supersymmetric Yang–Mills theory on the gauge side.

As discussed for Lie groups in Chapter 3, the cohomology of a Lie group can be understood through the cohomology of its Lie algebra. The Lie superalgebra of $\mathrm{PSU}(2,2|4)$, which is denoted by $\mathfrak{psu}(2,2|4)$, is defined in Chapter 4.

To understand $\mathfrak{psu}(n|n)$, it is better first to start with the *general linear* Lie superalgebra $\mathfrak{gl}(m|n)$ with complex coefficients. Its elements are the transformations of a vector space containing m bosons and n fermions (i.e. $\mathbb{C}^m \oplus \mathbb{C}^n$). Thus, the bosonic transformations in $\mathfrak{gl}(m|n)$ are those of $\mathfrak{gl}(m) \oplus \mathfrak{gl}(n)$, while the fermionic transformations are the ones sending $\mathbb{C}^m$ into $\mathbb{C}^n$ and vice-versa.

As in the usual Lie algebra case, the *special linear* Lie superalgebra $\mathfrak{sl}(m|n)$ is defined as the kernel of a supertrace map $x \longmapsto \mathrm{Str}(x)$. For $m = n$, there remains a 1-dimensional centre, generated by the identity matrix in $\mathfrak{gl}(m) \oplus \mathfrak{gl}(m)$; the quotient of $\mathfrak{sl}(m|m)$ by this centre is the definition of the *projective special linear* Lie superalgebra $\mathfrak{psl}(m|m)$ (V. KAČ, 1977a). The definitions of the *special unitary* ($\mathfrak{su}$) and the *projective special unitary* ($\mathfrak{psu}$) Lie superalgebras are a bit more complicated, and are deferred to Chapter 4.

Due to time constraints and to better understand what has been done previously, we decided to keep this thesis as a review of spectral sequences and Lie superalgebras, focusing on the explicit computation of the cohomology of $\mathfrak{gl}(m|n)$ presentend shortly in FUKS (1986).

Structure of this thesis

This thesis is structured as follows. In Chapter 2, I present the abstract machinery of homological algebra, such as cochain complexes, cohomology groups and spectral sequences. The reader who is familiarized with the topic might perceive upfront the reason for each of those topics. The reader who is not familiar, however, might struggle, and I sympathise with them — because I did too at the start of this project; unfortunately, in the latter case I have to ask for patience and perseverance.

In Chapter 3, the Lie algebra cohomology theory is laid out. There, the idea of central extensions is defined, and its precise relation to cohomology is stated. The BRST formulation is introduced via ghost coordinates, and the Lie algebra $\mathfrak{gl}(n)$ is discussed as example. By the end, the Hochschild–Serre spectral sequence is shown, following up the construction started in Chapter 2, now in the context of Lie algebras.

At the start of Chapters 2 and 3, I tried to give a historical context to the subject, to situate the reader with topics they might have heard about. That digression follows closely the review articles by Weibel (1999) and Massey (1999), to where I refer the reader to find an extensive list of historical references.

Finally, in Chapter 4, I present the notion of Lie superalgebras. As one might expect, they are related to Lie supergroups — such as $\mathrm{PSU}(2,2|4)$, containing the symmetries of $d=4$, $\mathcal{N}=4$ SYM theory — in a similar fashion as Lie algebras are to Lie groups. I do not give an explicit definition of supergroups, as they are not directly the focus of this thesis. After the basic definitions, I show two useful results: the first is how one can define a Lie superalgebra structure on a super vector space out of an odd quadratic nilpotent vector field; the second is how the cochain complex of a Lie superalgebra can be understood as the algebra of polynomials in ghost coordinates. Those results put together give a clear explanation of what the BRST technique means and how it works. Following that, I work out the details of the computation of the cohomology of $\mathfrak{gl}(m|n)$ via the Hochschild–Serre spectral sequence given by Fuks (1986).

Notation

As already stated, Einstein summation convention is to be understood for all indices. Definitions shall be indicated by a colon, e.g. $A := B$. Equivalent notations are indicated by the triple bar, e.g. $A \equiv B$. Isomorphisms are denoted by a tilde above the equal sign, e.g. $A \cong B$. Writing an index outside curly brackets means that it shall run in its index set, viz.

$$\{a_k\}_k \equiv \{a_k \mid k \in K\}.$$

After the first few pages, I stop using parentheses to indicate the action of a morphism on an object — e.g. $fa \equiv f(a)$ — and I indicate composition of maps by juxtaposition wherever possible — e.g. $fg \equiv f \circ g$.

Chapter 2

An introduction to homological algebra

The history of homological algebra is a theatre piece deeply rooted in algebraic topology and played by many mathematicians. Its prelude starts in the 1850's with the works of Bernhard Riemann (1826–1866) and Enrico Betti (1823–1892), who were interested in the "connectedness numbers" of a manifold — which are related to today's notion of *Betti number* b_k, the dimension of its k^{th} (co)homology space. By the 1890's, Henri Poincaré (1854–1912) proved his Duality Theorem — which, in today's language, is an equality between b_k and $b_{\dim(M)-k}$ of a manifold M — and established the simplicial homology of triangulated manifolds. According to WEIBEL (1999), the first ever chain complex can be found in Poincaré's work in the context of simplices of an oriented polyhedron.

Until the 1920's, topologists remained interested only in the Betti numbers and torsion coefficients (introduced by Poincaré) of spaces. That changed in 1925 when Emmy Noether (1882–1935) pointed out that homology was an abelian group, rather than just numbers or coefficients. Heinz Hopf (1894–1971) and Pavel Alexandrov (1896–1982) inspired Walther Mayer (1887–1948) to build an algebraic approach to complexes and homology in 1929.

Between 1925 and 1935, many mathematicians worked on building homology theories; amongst them were Pavel Alexandrov, Solomon Lefschetz (1884–1972), James Alexander (1888–1971), Eduard Čech (1893–1960), Andrey Kolmogorov (1903–1987) and, possibly one of the most famous, Georges de Rham (1903–1990).

After Élie Cartan (1869–1951) introduced the notion of a cochain complex of differential forms on a smooth manifold in 1928, he made a conjecture about the Betti numbers of that manifold. De Rham realized in 1929 that he could solve that conjecture using a triangulation on M and a bilinear map taking cycles of the triangulation and closed forms. The quotient of closed k-forms by exact k-forms, H^k_{dR}, was precisely the k^{th} cohomology of Cartan's original complex — which is now referred as the "de Rham cohomology", even though de Rham did everything in homology terms, because cohomology had not yet been invented!

In fact, cohomology was introduced in 1935 at the 1st International Topological

Conference, in Moscow. Alexander (1936) and Kolmogorov (1936) presented their independent results back-to-back. Surprisingly, both had reached the same conclusions: the discovery of cohomology and the existence of a cup product, later corrected by Čech and Hassler Whitney (1907–1989), in 1936 and 1938 respectively.

In 1952, Samuel Eilenberg (1913–1998) and Norman Steenrod (1910–1971) set axioms for the treatment of homology theory and rederived the whole theory for finite complexes, pointing out that singular homology and Čech homology satisfy their axioms and hence they must agree on all finite complexes. The collaboration of Henri Cartan and Eilenberg, devoted to rewrite the foundations of homology and cohomology theories that were around, culminated in their book (H. CARTAN–EILENBERG, 1956), where they first used the name "homological algebra".

While serving in the French army during World War II, Jean Leray (1906–1998) was captured by German troops and held prisoner from 1940 to 1945. Imprisoned in Austria, he founded a university and taught a course on algebraic topology; by the end of imprisonment, he had invented sheaves, sheaf cohomology and, most importantly to this thesis, spectral sequences (LERAY, 1946).

Jean-Louis Koszul (1921–2018) found the algebraic properties of spectral sequences in 1947 using a suggestion by Henri Cartan (1904–2008) on focusing on a filtered chain complex, and his language became standard.

Structure of this chapter

In this Chapter, I present the basics of homological algebra following mainly the works of LIMA (2009), HATCHER (2002, 2004), H. CARTAN–EILENBERG (1956), GELFAND–MANIN (1999), and WEIBEL (1994). The purpose of each Section shall become clearer in Chapters 3 and 4.

In Section 2.1, the notion of chain and cochain complexes is introduced, as well as the definitions of cohomology groups of a complex, subcomplexes and their relative cohomology, the concept of complex morphism and, very importantly, how the latter induces a homomorphism in cohomology.

Section 2.2 defines exact sequences as the one in (1.7) and their relation with complexes, ending with the remarkable Theorem 1 that is used latter in this Chapter. With a little detour, graded modules are presented in Section 2.3, and it is shown how a complex can be seen as a differential graded module.

Finally, Section 2.4 starts with the meaning of a filtration and outlines the construction of a spectral sequence out of it. The Hochschild–Serre spectral sequence presented in Chapter 3 follows this construction and is used in Chapter 4.

2.1 Cochain complexes

In this section and the next, I will mainly follow the books of LIMA (2009), HATCHER (2002), and GELFAND–MANIN (1999). For all purposes, k can be taken as an integer.

Definition 2.1. A **chain complex** $C_\bullet \equiv \{C_k, d_k\}_k$ is a sequence of abelian groups $\{C_k\}_k$ and homomorphisms $\{d_k\}_k$,

$$C_\bullet : \quad \cdots \xrightarrow{d_{k+1}} C_k \xrightarrow{d_k} C_{k-1} \xrightarrow{d_{k-1}} \cdots,$$

such that $d_k \circ d_{k+1} = 0$ for every index k. A **cochain complex** $C^\bullet \equiv \{C^k, d^k\}_k$ is also a sequence of abelian groups and homomorphisms,

$$C^\bullet : \quad \cdots \xrightarrow{d^{k-1}} C^k \xrightarrow{d^k} C^{k+1} \xrightarrow{d^{k+1}} \cdots, \tag{2.1}$$

such that $d^k \circ d^{k-1} = 0$ for all k.

A helpful way to visualize the requirement that $d^k \circ d^{k-1} = 0$ is shown in Figure 2. Each group C^k is represented as a gray blob, with a dot indicating its neutral element; the reddish blobs represent the image of the differentials. All elements in C^{k-1} are inevitably mapped by $d^k \circ d^{k-1}$ to the neutral element of C^{k+1}.

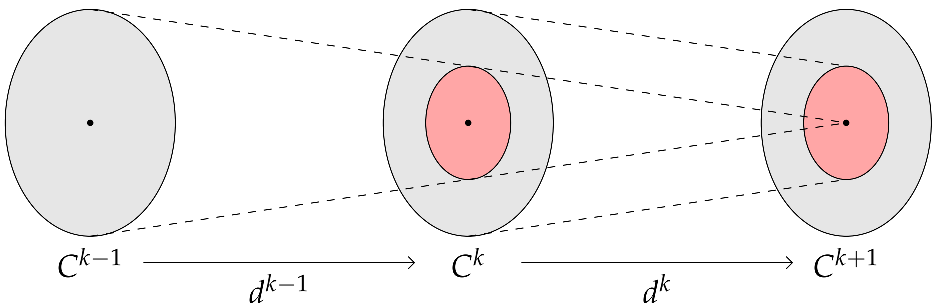


Figure 2: Pictorial representation of the requirement that $d^k \circ d^{k-1} = 0$ on a cochain complex $C^\bullet$.

A common terminology is the following: elements of C^k are called ***k*-cochains**, and the homomorphisms d^k are collectively called **coboundary maps**; similarly, elements of C_k are ***k*-chains** and the homomorphisms d_k are **boundary maps**. It is common to simply refer to both d^k and d_k as the **differential maps**.

For the cochain complex $C^\bullet$ in (2.1), define the subgroups

$$Z^k(C^\bullet) := \ker d^k, \tag{2.2a}$$
$$B^k(C^\bullet) := \operatorname{im} d^{k-1}; \tag{2.2b}$$

the elements of $Z^k(C^\bullet)$ are called ***k*-cocycles** of $C^\bullet$, and the elements of $B^k(C^\bullet)$ are called ***k*-coboundaries** of $C^\bullet$. For the chain complex $C_\bullet$, simply remove the prefix "co", lower the indices, and change d^{k-1} to d_{k+1} in the definition of $B_k(C_\bullet)$.

From a purely algebraic perspective, the distinction between chain and cochain complexes is simply a question of taste: by reversing the enumeration, namely doing $C^k = C_{-k}$ and $d^k = d_{-k}$, a chain complex is turned into a cochain complex and vice-versa. For this reason, in this chapter I talk only about cochains complexes. For specific constructions, however, one of them may seem more natural than the other (HATCHER, 2002).

Since $d^k \circ d^{k-1} = 0$, all elements in the image of d^{k-1} are in the kernel of d^k, i.e. $\operatorname{im} d^{k-1} \subseteq \ker d^k$, as it is represented in Figure 3. Thus every k-coboundary (in the smaller blob in light red) is a k-cocycle (in the darker, larger blob). The converse, however, is not always true, and is what motivates the following definition.

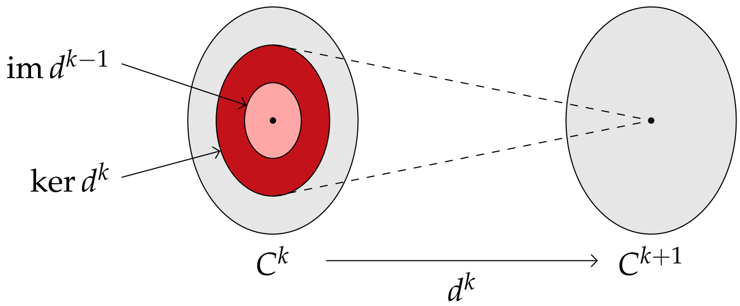


Figure 3: From the requirement that $d^k \circ d^{k-1} = 0$, one concludes that all k-coboundaries are k-cocyles. The converse, however, is not generally true.

Definition 2.2. The k^{th} **cohomology group**[6] of the cochain complex $C^\bullet$ is

$$H^k(C^\bullet) := \frac{Z^k(C^\bullet)}{B^k(C^\bullet)} = \frac{\ker d^k}{\operatorname{im} d^{k-1}}, \tag{2.3}$$

[6]Note that this is a group because it is defined as a quotient of subgroups of C^k, which is abelian. However, in specific applications (e.g. de Rham complexes), cohomology groups are actually linear spaces; but it is common nonetheless to still call them "groups".

whose elements are the **cohomology classes**

$$[z] := z + B^k(C^\bullet) = \{z + d^{k-1}x \mid x \in C^{k-1}\}.$$

For $z, z' \in C^k$, it holds that $[z] = [z']$ if and only if $z' - z = d^{k-1}x$ for some $x \in C^{k-1}$; one then says that the cochains z and z' are **cohomologous**.

A notion that must be introduced is that of a relative cohomology, which appears when one can consider subgroups of a cochain complex that behave "nicely" with respect to the coboundary map. In those cases, the cochains can be taken in the quotient group, which ends up simplifying them — but empowering one with easier calculations!

Definition 2.3. The cochain complex $B^\bullet$ is called a **cochain subcomplex** of $C^\bullet$ if, for every k, the group B^k is a subgroup of C^k, the coboundary map of $C^\bullet$ "restricts well" to $B^\bullet$ — i.e. $d^{k-1}B^{k-1} \subseteq B^k$ and ${d_B}^k = d^k|_{B^k}$.

For each index k, consider the quotient $\overline{C}^k := C^k/B^k$ and the quotient map $J^k \colon C^k \longrightarrow \overline{C}^k$. There is a unique map $\overline{d}^k \colon \overline{C}^k \longrightarrow \overline{C}^{k+1}$ making the diagram

$$\begin{tikzcd} C^k \arrow[r, "d^k"] \arrow[d, "J^k"'] & C^{k+1} \arrow[d, "J^{k+1}"] \\ \overline{C}^k \arrow[r, "\overline{d}^k"'] & \overline{C}^{k+1} \end{tikzcd} \tag{2.4}$$

commutative (LIMA, 2009). By definition, $\overline{d}(Jc) := J(dc)$ for every cochain c. Then $\overline{d}^k \circ \overline{d}^{k-1} = 0$, and $\overline{C}^\bullet \equiv \{\overline{C}^k, \overline{d}^k\}_k$ is a cochain complex.

Definition 2.4. The cochain complex $\overline{C}^\bullet$ is called the **quotient** of $C^\bullet$ by $B^\bullet$; its cohomology is called the **relative cohomology** of the pair $(C^\bullet, B^\bullet)$, denoted $H(C^\bullet, B^\bullet)$.

Note that every cochain of $B^\bullet$ is trivial in $\overline{C}^\bullet$; thus, in way, cochains of $\overline{C}^\bullet$ are simpler than those of $C^\bullet$. More precisely, if y is a k-cochain of $B^\bullet$, then it becomes the trivial k-cochain in $\overline{C}^\bullet$. Moreover, if a cochain in $C^\bullet$ is of the form xy, with $y \in B^k$ and $x \in C^k \backslash B^k$, then xy is mapped by J to the quotient class of x in $\overline{C}^k$. Once again, y is regarded as the trivial cochain in $\overline{C}^k$, and the differential $\overline{d}^k$ acts only on x, leaving y unchanged. That is why the relative cohomology $H(C^\bullet, B^\bullet)$ is also referred as the *cohomology of $C^\bullet$ with coefficients in $B^\bullet$*.

An important result that will be used later comes from the notion of morphism of complexes.

Definition 2.5. A **cochain complex morphism** $f\colon C^\bullet \longrightarrow D^\bullet$ is a family of homomorphisms $\{f^k\colon C^k \longrightarrow D^k\}_k$ that commute with the differentials, i.e.

$$f^{k+1} \circ d_C^k = d_D^k \circ f^k, \qquad \text{for every } k.$$

As it is customary in homological algebra and, more generally, in branches of mathematics strongly rooted in category theory, one draws diagrams instead of writing down expressions for commutations of maps. In the definition above, this can be expressed as requiring that the square

$$\begin{tikzcd} C^k \arrow[r, "d_C^k"] \arrow[d, "f^k"'] & C^{k+1} \arrow[d, "f^{k+1}"] \\ D^k \arrow[r, "d_D^k"'] & D^{k+1} \end{tikzcd}$$

commutes. For the rest of the text, every square in a diagram will be assumed to be commutative unless explicitly stated.

Note that any cochain complex morphism f maps k-cocycles of $C^\bullet$ into k-cocycles of $D^\bullet$. Indeed, if $x \in Z^k(C^\bullet)$, then

$$d_D^k(f^k(x)) = f^{k+1}(d_C^k x) = 0,$$

thus $f^k(x) \in Z^k(D^\bullet)$. Similarly, f maps $B^k(C^\bullet)$ onto $B^k(D^\bullet)$. For this reason, one has the following important result.

Proposition 1. *For each k, the cochain complex morphism $f\colon C^\bullet \longrightarrow D^\bullet$ induces a natural homomorphism,*

$$\begin{aligned} (f^k)^*\colon \quad H^k(C^\bullet) &\longrightarrow H^k(D^\bullet) \\ [z] &\longmapsto [f^k(z)], \qquad z \in Z^k(C^\bullet). \end{aligned} \tag{2.5}$$

Thus, $f^* \equiv H^\bullet(f) \equiv \{(f^k)^*\}_k$ is called a **morphism of cohomology***. It is "natural" in the sense that, if $g\colon B^\bullet \longrightarrow C^\bullet$ is another cochain complex morphism, then the composition $f \circ g\colon B^\bullet \longrightarrow D^\bullet$ induces the morphism of cohomology $(f \circ g)_* = f_* \circ g_*$.*

2.2 Exact sequences

Definition 2.6. A sequence of homomorphisms

$$\cdots \longrightarrow C^{k-1} \xrightarrow{f^{k-1}} C^k \xrightarrow{f^K} C^{k+1} \longrightarrow \cdots$$

is said to be **exact in the term C^k** if $\operatorname{im} f^{k-1} = \ker f^k$. If it is exact in all its terms, then it is said to be an **exact sequence**.

Example 2.1. Consider $\{C^k, f^k\}_k$ an exact sequence.

a. The homomorphism f^k is injective if and only if (iff) $f^{k+1} = 0$. Indeed, f^k is injective iff $\ker f^k = \{0\}$; if the sequence is exact, one must have $\operatorname{im} f^{k+1} = \ker f^k = \{0\}$, thus $f^{k+1} = 0$. In particular, the sequence

$$0 \xrightarrow{\phi_K} K \xrightarrow{i} L$$

is exact iff i is injective; in this case, I will indicate i with a hooked arrow, $\hookrightarrow$. Furthermore, that exact sequence is a complex, because $\operatorname{im} \phi_K = \{0_K\} \subseteq \ker i$ independently of the map i.

b. The map f^{k+1} is surjective iff $f^k = 0$. Indeed, for f^{k+1} to be surjective, one must have $\operatorname{im} f^{k+1} = C^k$, and the exactness of the sequence yields $\ker f^k = C^k$, thus $f^k = 0$. In particular, the sequence

$$L \xrightarrow{j} M \xrightarrow{\psi} 0$$

is exact iff j is surjective; in that case, I will indicate j with a double-headed arrow, $\twoheadrightarrow$. Furthermore, such exact sequence is a complex, because $\operatorname{im} j = M = \ker \psi$.

c. Considering the previous remarks, one can conclude that the sequence

$$0 \longrightarrow L \xrightarrow{f} M \longrightarrow 0$$

is exact iff f is an isomorphism.

d. The sequence

$$0 \longrightarrow K \xrightarrow{i} L \xrightarrow{j} M \longrightarrow 0$$

is called a **short sequence**; it is exact if i is injective, j is surjective, and $\operatorname{im} i = \ker j$. The prototypical example of a short exact sequence is

$$0 \longrightarrow K \xhookrightarrow{i} L \xrightarrow{j} L/K \longrightarrow 0, \tag{2.6}$$

where K is a subgroup of L, $i\colon K \hookrightarrow L$ is the inclusion map, and $j\colon L \twoheadrightarrow L/K$ is the quotient map. ▷

Take three complexes $B^\bullet$, $C^\bullet$ and $D^\bullet$ and let $f\colon B^\bullet \longrightarrow C^\bullet$ and $g\colon C^\bullet \longrightarrow D^\bullet$ be complex morphisms. Consider the *sequence of complexes and morphisms*

$$0 \longrightarrow B^\bullet \xrightarrow{f} C^\bullet \xrightarrow{g} D^\bullet \longrightarrow 0, \tag{2.7}$$

which can be diagrammatically put in a mesh,

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & B^{k-1} \arrow[r, "f"] \arrow[d, "d_B"] & C^{k-1} \arrow[r, "g"] \arrow[d, "d_C"] & D^{k-1} \arrow[r] \arrow[d, "d_D"] & 0 \\
0 \arrow[r] & B^{k} \arrow[r, "f"] \arrow[d, "d_B"] & C^{k} \arrow[r, "g"] \arrow[d, "d_C"] & D^{k} \arrow[r] \arrow[d, "d_D"] & 0 \\
0 \arrow[r] & B^{k+1} \arrow[r, "f"] \arrow[d] & C^{k+1} \arrow[r, "g"] \arrow[d] & D^{k+1} \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

For simplicity, I will omit the indices of f, g and d.

Definition 2.7. The sequence of complexes and morphisms (2.7) is said to be an **exact triple** (or simply a **short exact sequence**) if, for each k, the short sequence

$$0 \longrightarrow B^k \xrightarrow{f^k} C^k \xrightarrow{g^k} D^k \longrightarrow 0 \tag{2.8}$$

is exact.

The exactness of the short sequence (2.8) implies that it is a cochain complex, thus $g^k \circ f^k = 0$. For it to be an exact sequence, f^k must be injective, g^k must be surjective, and $\operatorname{im} f^k = \ker g^k$ (recall Example 2.1.d.).

The next theorem will be fundamental in the construction of spectral sequences in the following sections. Its proof is rather long and does not add value to the main goal of this thesis; but, for completion, it is shown in Chapter A.

Theorem 1 (Eilenberg, Steenrod). *Let* (2.7) *be a short exact sequence. For each k, there exists a homomorphism $\delta^k\colon H^k(D^\bullet) \longrightarrow H^{k+1}(B^\bullet)$ such that the cohomology sequence*

$$\cdots \longrightarrow H^k(B^\bullet) \xrightarrow{f^*} H^k(C^\bullet) \xrightarrow{g^*} H^k(D^\bullet) \xrightarrow{\delta^k} H^{k+1}(B^\bullet) \longrightarrow \cdots$$

is exact.

One says that the map $\delta\colon H(D^\bullet) \longrightarrow H(B^\bullet)$ is the **connecting homomorphism**, and it can schematically be written as

$$\delta\colon [y] \longmapsto [f^{-1} d_C\, g^{-1} y]. \tag{2.9}$$

Consider a cochain complex $C^\bullet$, a cochain subcomplex $B^\bullet$, and the quotient $\overline{C}^\bullet$, as in Definition 2.4. For each k, the short sequence

$$0 \longrightarrow B^k \xhookrightarrow{i^k} C^k \xrightarrow{j^k} \overline{C}^k \longrightarrow 0$$

is exact (recall Example 2.1.d.), hence

$$0 \longrightarrow B^\bullet \xhookrightarrow{i} C^\bullet \xrightarrow{j} \overline{C}^\bullet \longrightarrow 0$$

is an exact triple. Theorem 1 then guarantees that it induces the long exact sequence in cohomology

$$\cdots \longrightarrow H^k(B^\bullet) \xrightarrow{i^*} H^k(C^\bullet) \xrightarrow{j^*} H^k(C^\bullet, B^\bullet) \xrightarrow{\delta} H^{k+1}(B^\bullet) \longrightarrow \cdots. \tag{2.10}$$

Up to cohomology classes, the connecting homomorphism is given by $i^{-1} d_C\, j^{-1}$.

A natural question is: what if one considers a cochain subcomplex of $B^\bullet$? Or, more generally, what if one considers a nested sequence of subcomplexes $C^\bullet \supseteq B_1^\bullet \supseteq B_2^\bullet \supseteq \cdots \supseteq B_p^\bullet \supseteq \cdots$? That is the motivation to introduce the notion of a *filtration* in Section 2.4. But, before that, let me introduce another concept that shall play a part later too.

2.3 Graded modules

An important notion, both to the study of spectral sequences later this chapter and to the definition of superalgebras in Chapter 4, is that of a *grading*. Instead of considering abelian groups, consider modules.[7] For this section, I will follow the definitions and statements in the book of H. CARTAN–EILENBERG (1956).

Definition 2.8. Let I be an abelian group. An ***I*-grading** on a module A is a family of submodules of A labelled by I, $\{A_i\}_i$, whose direct sum is equal to A, i.e.

$$A = \bigoplus_{i \in I} A_i.$$

Elements of A_i are called **homogeneous of degree *i***; the degree of a homogeneous element $a \in A$ shall be denoted by $\deg(a)$; whenever $\deg(a)$ is written, it is assumed that a is homogeneous.

Let B be a submodule of A such that $B = \bigoplus_i B_i$, where $B_i = B \cap A_i$. The quotient A/B has an induced I-grading defined by

$$(A/B)_i \coloneqq A_i/B_i. \tag{2.11}$$

Definition 2.9. A linear map $f\colon A \longrightarrow B$, between two I-graded modules A and B, is called an ***I*-homomorphism** if it preserves the I-grading, i.e. for each $i \in I$, there is $j \in I$ such that $f(A_i) \subseteq B_j$. An I-homomorphism is **of degree *k*** if $f(A_i) \subseteq B_{i+k}$ for every $i \in I$.

Example 2.2. Let $A = \bigoplus_{i \in I} A_i$ be an I-graded module. Then, the associative algebra $\mathrm{End}(A)$, which contains strictly all the I-endomorphisms of A, is equipped with the induced I-grading $\mathrm{End}(A) = \bigoplus_{i \in I} \mathrm{End}_i(A)$, where

$$\mathrm{End}_i(A) = \{f \in \mathrm{End}(A) \mid f(A_j) \subseteq A_{i+j}\}$$

is the subspace of all I-homomorphisms of degree i. ▷

[7] A vector space has an underlying field which defines the product by a scalar; a module is analogous, but one has a ring instead of a field. Loosely speaking, a ring is like a field, but one does not ask for the existence of multiplicative inverses. Therefore every field is a ring, and every vector space is a module. For all purposes, one can consider only vector spaces, as those are the ones to be used in Chapter 3.

Consider a cochain complex $C^\bullet$ where each abelian group C^k is in fact a module. Then it is immediate to see that

$$C := \bigoplus_{k\in\mathbb{Z}} C^k \tag{2.12}$$

is a $\mathbb{Z}$-graded module. If one defines a map $d\colon C \longrightarrow C$ to be such that it acts as the k^{th} coboundary operator d^k on each homogeneous component C^k, then one concludes that the homomorphism d is nilpotent of degree 1. A module like (2.12) equipped with a nilpotent homomorphism of degree 1 is sometimes called a **differential graded module**. Cocycles and coboundaries can then be referred as d-closed and d-exact elements of C, respectively.

2.4 Spectral sequence of a filtered complex

Going back to the idea presented at the very end of Section 2.2, one shall consider a nested sequence of subcomplexes or, more generally, subspaces. For this section, I follow mostly the definitions of HATCHER (2002, 2004).

Definition 2.10. Consider a space X. A (decreasing) **filtration** of X is a family of subspaces $\{F^pX \subseteq X\}_p$ such that $F^{p+1}X \subseteq F^pX$ for every $p \in \mathbb{Z}$ and $X = \bigcup_p F^pX$.

For the purpose of this thesis, one should consider X to be a differential graded module like (2.12), and p to range only on $\mathbb{N}_0$; for convenience, let $F^{+\infty}C := 0$. Explicitly,

$$C =: F^0C \supseteq F^1C \supseteq \cdots \supseteq F^pC \supseteq F^{p+1}C \supseteq \cdots \supseteq F^{+\infty}C := 0. \tag{2.13}$$

Furthermore, one should require that the differential d preserves the filtration on C in the sense that

$$d(F^pC) \subseteq F^pC, \qquad \text{for all } p. \tag{2.14}$$

A complex as in (2.12), with a filtration like above, is called a **filtered complex**. Each of the *filtering subspaces* F^pC inherits a grading from (2.12), namely

$$F^pC = \bigoplus_k F^pC^k;$$

elements of F^pC^k are homogeneous of degree k and are in the p^{th} filtration stage of the filtered complex C.

Now I proceed with the construction of a *spectral sequence* for a decreasing filtration of C. Consider, for each p, the short exact sequence

$$0 \longrightarrow F^{p+1}C \xhookrightarrow{i} F^pC \xrightarrow{j} F^pC/F^{p+1}C \longrightarrow 0;$$

the map i is the inclusion, and j is the quotient map.

Let

$$E_0^{p,q} := \frac{F^pC^{p+q}}{F^{p+1}C^{p+q}}; \tag{2.15}$$

note that, while the differential $d\colon C \longrightarrow C$ preserves the filtration, it increases the $\mathbb{Z}$-grading of C, thus $d\colon F^pC^{p+q} \longrightarrow F^pC^{p+q+1}$. As done in (2.4), there is a unique map

$$d_0\colon E_0^{p,q} \longrightarrow E_0^{p,q+1}$$

making commutative the diagram

$$\begin{tikzcd} F^pC^{p+q} \arrow[r, two heads, "j"] \arrow[d, "d"'] & E_0^{p,q} \arrow[d, "d_0"] \\ F^pC^{p+q+1} \arrow[r, two heads, "j"'] & E_0^{p+1,q}. \end{tikzcd} \tag{2.16}$$

Such map d_0 induced on the quotient is nilpotent, as seen at the end of Section 2.1.

Being d_0 nilpotent, it makes sense to consider its cohomology. Let

$$E_1^{p,k-p} := \frac{\ker d_0^{p,k-p}}{\operatorname{im} d_0^{p-1,k-p}} = H^k(F^pC, F^{p+1}C) \tag{2.17}$$

be the relative cohomology of the pair $(F^pC, F^{p+1}C)$. According to Theorem 1, the above short exact sequence induces a long exact sequence in cohomology, which shall be depicted as

$$\begin{tikzcd} \cdots \arrow[r] & H^k(F^{p+1}C) \arrow[d, "i^*"] & & \\ & H^k(F^pC) \arrow[r, "j^*"] & E_1^{p,k-p} \arrow[r, "\delta"] & H^{k+1}(F^{p+1}C) \arrow[d, "i^*"] & \\ & & & H^{k+1}(F^pC) \arrow[r] & \cdots. \end{tikzcd}$$

Note that the domains of i^* above — namely the spaces $H^k(F^{p+1}C)$ and $H^{k+1}(F^{p+1}C)$, which pictorially are the "upper edges" of the ladder — will appear in the analogous long exact sequence induced by the short exact sequence

$$0 \longrightarrow F^{p+2}C \xhookrightarrow{i} F^{p+1}C \xrightarrow{j} F^{p+1}C/F^{p+2}C \longrightarrow 0,$$

but now as "lower edges" (i.e. as codomains of the i^* map). Thus, one can "sewn" both long exact sequences together and keep doing it so, building an infinite mesh. Below, one can see the two aforementioned long exact sequences in black, with the remaining parts of the mesh completed in gray:

$$\begin{tikzcd}
 & \vdots \arrow[d] & & \vdots \arrow[d] & & \\
\cdots \arrow[r] & H^k(F^{p+2}C) \arrow[r, "j^*"] \arrow[d, "i^*"] & E_1^{p+2,k-p-2} \arrow[r, "\delta"] & H^{k+1}(F^{p+3}C) \arrow[r, "j^*"] \arrow[d, "i^*"] & E_1^{p+3,k-p-2} \arrow[r] & \cdots \\
\cdots \arrow[r] & H^k(F^{p+1}C) \arrow[r, "j^*"] \arrow[d, "i^*"] & E_1^{p+1,k-p-1} \arrow[r, "\delta"] & H^{k+1}(F^{p+2}C) \arrow[r, "j^*"] \arrow[d, "i^*"] & E_1^{p+2,k-p-1} \arrow[r] & \cdots \\
\cdots \arrow[r] & H^k(F^{p}C) \arrow[r, "j^*"] \arrow[d, "i^*"] & E_1^{p,k-p} \arrow[r, "\delta"] & H^{k+1}(F^{p+1}C) \arrow[r, "j^*"] \arrow[d, "i^*"] & E_1^{p+1,k-p} \arrow[r] & \cdots \\
\cdots \arrow[r] & H^k(F^{p-1}C) \arrow[r, "j^*"] \arrow[d] & E_1^{p-1,k-p+1} \arrow[r, "\delta"] & H^{k+1}(F^{p}C) \arrow[r, "j^*"] \arrow[d] & E_1^{p,k-p+1} \arrow[r] & \cdots \\
 & \vdots & & \vdots & &
\end{tikzcd} \tag{2.18}$$

Look closely at the $H^{k+1}(F^{p+1}C)$ spot, viz.

$$\begin{tikzcd}
 & H^{k+1}(F^{p+2}C) \arrow[d, "i^*"] \arrow[dr, bend left] & \\
E_1^{p,k-p} \arrow[r, "\delta"] \arrow[dr, bend left] & H^{k+1}(F^{p+1}C) \arrow[r, "j^*"'] \arrow[d, "i^*"] & E_1^{p+1,k-p} \\
 & H^{k+1}(F^{p}C); &
\end{tikzcd}$$

in this patch, the $j^* \circ i^*$ map (in red, going north-to-east) and the $i^* \circ \delta$ map (in blue, going west-to-south) are zero, because they each come from a long exact sequence. However, the δ and the j^* maps above come from different sequences, thus one might be interested in their composition, viz.

$$d_1 := j^* \circ \delta \colon E_1^{p,q} \longrightarrow E_1^{p+1,q}. \tag{2.19}$$

If one draws the next patch of the mesh of spaces, one will see that

$$d_1 \circ d_1 = j^* \circ (\delta \circ j^*) \circ \delta = 0,$$

because the $(\delta \circ j^*)$ part is inside a long exact sequence, hence it is zero. Therefore, this map d_1 is nilpotent, allowing one to define E_2 as its cohomology, i.e.

$$E_2^{p,q} := \frac{\ker d_1^{p,q}}{\operatorname{im} d_1^{p-1,q}}. \tag{2.20}$$

Going back to the mesh (2.18), consider now the patch

$$\begin{tikzcd}
 & H^{k+1}(F^{p+2}C) \arrow[r, "j^*"] \arrow[d, "i^*"'] \arrow[dr, bend right, red] & E_1^{p+2,k-p-1} \\
E_1^{p,k-p} \arrow[r, "\delta"] & H^{k+1}(F^{p+1}C) \arrow[r, "j^*"'] & E_1^{p+1,k-p};
\end{tikzcd}$$

if $a \in \ker d_1^{p,k-p} \subseteq E_1^{p,k-p}$ (on the lower left corner of the patch above), it means by definition that $j^*(\delta a) = 0$. Consequently, by exactness of the red path above,

$$\delta a \in \ker j^* = \operatorname{im} i^* \subseteq H^{k+1}(F^{p+1}C).$$

Let $b \in H^{k+1}(F^{p+2}C)$ be such that $i^* b = \delta a$, and act on b with j^*, obtaining an element of $E_1^{p+2,k-p-1}$ (on the top left corner of the patch above).

If $[a] \in E_2^{p,k-p}$ is the cohomology class of a with respect to d_1, define a map

$$\begin{aligned} d_2 \colon E_2^{p,q} &\longrightarrow E_2^{p+2,q-1} \\ [a] &\longmapsto [j^* b], \qquad \text{where} \quad i^* b = \delta a. \end{aligned}$$

Syntactically, it means that

$$d_2 = [\cdot] \circ \left(j^* \, (i^*)^{-1} \, \delta \right) \circ [\cdot]^{-1}, \tag{2.21}$$

where $[\cdot]$ or $[\cdot]^{-1}$ indicates that one has to take the cohomology class (with respect to d_1) or one of its representatives, respectively. The map (2.21) is well-defined as it does not depend on the representatives taken in each class (HATCHER, 2004).

Furthermore, d_2 is also nilpotent, because

$$d_2 \circ d_2 = [\cdot] \circ \left(j^* \, (i^*)^{-1} \, {\color{red}(\delta \, j^*)} \, (i^*)^{-1} \, \delta \right) \circ [\cdot]^{-1}$$

and the red term $\delta \circ j^*$ is zero, since it lives inside a long exact sequence. Hence, one can define

$$E_3^{p,q} := \frac{\ker d_2^{p,q}}{\operatorname{im} d_2^{p-2,q+1}}.$$

For this $E_3^{p,q}$ spaces, things now get a bit more abstract and complicated to represent, since an element of $E_3^{p,q}$ is a cohomology class (with respect to d_2) of cohomology classes (with respect to d_1) — because $\ker d_2^{p,q} \subseteq E_2^{p,q}$ is already a cohomology of $d_1^{p,q}$. With a grain of salt, one can still consider the mesh (2.18), but keeping track of the several quotients made along the way.

In that spirit, look at the patch

$$\begin{tikzcd}
 & H^{k+1}(F^{p+3}C) \arrow[r, "j^*"] \arrow[d, "i^*"] \arrow[dr, bend right, red] & E_1^{p+3,k-p-2} \\
 & H^{k+1}(F^{p+2}C) \arrow[r, "j^*"'] \arrow[d, "i^*"] & E_1^{p+2,k-p-1} \\
E_1^{p,k-p} \arrow[r, "\delta"] & H^{k+1}(F^{p+1}C) &
\end{tikzcd}$$

and consider a cocycle $a \in \ker d_2^{p,k-p} \subseteq E_2^{p-k}$ (on the bottom left corner, up to quotients). By definition of d_2, one has $j^* \, (i^*)^{-1} \, \delta a = 0$. Thus, from the exactness of the red path, one has

$$(i^*)^{-1} \, \delta a \in \ker j^* = \operatorname{im} i^* \subseteq H^{k+1}(F^{p+3}C).$$

Let $b \in H^{k+1}(F^{p+3}C)$ be such that $i^*b = (i^*)^{-1}\,\delta a$ — or, more precisely, be such that $i^*\,i^*b$ (going down twice) equals δa — and define

$$\begin{aligned} d_3 \colon E_3^{p,q} &\longrightarrow E_3^{p+3,q-2} \\ [a] &\longmapsto [j^*b], \qquad \text{where} \quad i^*\,i^*b = \delta a. \end{aligned}$$

Syntactically,

$$d_3 = [\cdot] \circ \left(j^*\,(i^*)^{-1}\,(i^*)^{-1}\,\delta \right) \circ [\cdot]^{-1}, \tag{2.22}$$

which again is nilpotent because $d_3 \circ d_3$ contains a term $\delta \circ j^* = 0$ inside the same long exact sequence.

This can be done iteratively. The space $E_r^{p,q}$ will be the cohomology space of the differential d_{r-1} at the (p,q) spot — i.e. the kernel of the differential that *starts at* $E_{r-1}^{p,q}$, quotiented by the image of the differential that *takes to* $E_{r-1}^{p,q}$. The next differential (d_r) will be defined analogously: with respect to the mesh (2.18), start at $E_{r-1}^{p,q}$, move to the right with δ, move $(r-1)$ times upwards with $(i^*)^{-1}$, and move again to the right with j^*. Thus, it is a map

$$d_r \colon E_r^{p,q} \longrightarrow E_r^{p+r,q-r+1} \tag{2.23}$$

which can be written syntactically as

$$d_r = [\cdot] \circ \left(j^*\,\underbrace{(i^*)^{-1} \cdots (i^*)^{-1}}_{(r-1)\ \text{times}}\,\delta \right) \circ [\cdot]^{-1}. \tag{2.24}$$

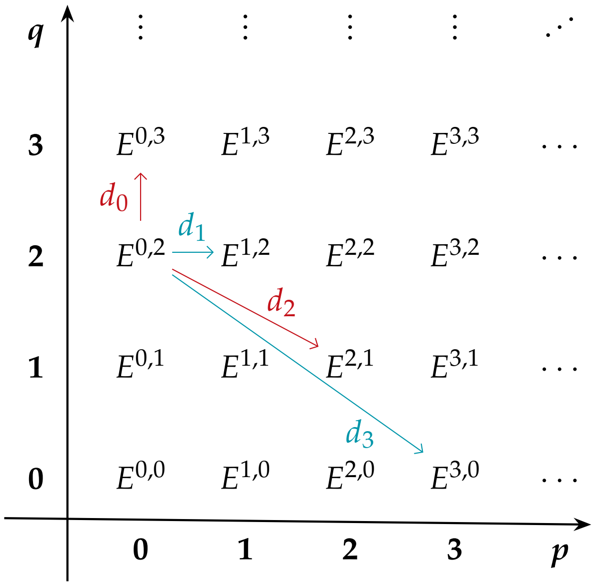
This is roughly the construction of the **spectral sequence** from a filtration of C. Each

$$E_r = \bigoplus_{p,q} E_r^{p,q}$$

is called the ***r*^th^ page** (or r^{th} sheet, as in a book) of the spectral sequence. This procedure shall be used later in Chapter 4 for the calculation of the cohomology of Lie superalgebras, starting with the Hochschild–Serre spectral sequence presented in Chapter 3.

A useful way of visualizing a spectral sequence is the following. For each $r \in \mathbb{N}_0$, consider a pictorial representation of the r^{th} page on the lattice $\mathbb{Z}^2$, with $E_r^{p,q}$ at the spot (p,q) on the lattice.

For the purposes of this thesis, p and q shall each be taken only in $\mathbb{N}_0$, forming a *first quadrant* lattice — i.e. every spot outside the sublattice ${\mathbb{N}_0}^2 \subset \mathbb{Z}^2$ corresponds to the trivial module. Below one can find the lattice with the first few differentials whose domain in $E_r^{0,2}$.



It is not hard to see that, for sufficiently large r, some $d_r^{p,q}$ will be trivially zero — either because they start or end outside of the first quadrant. For example, consider the second differential, $d_2\colon E_2^{p,q} \longrightarrow E_2^{p+2,q-1}$. At the spots with $q = 0$, the map d_2 is trivial, because $E_2^{p+2,-1}$ is trivial. Consequently, the kernel of $d_2^{p,0}$ is the whole space $E_2^{p,0}$, thus $E_3^{p,0} \cong E_2^{p,0}$. This will be so for every following differential.

More generally,

$$r > \max\{p, q+1\} \qquad \Longrightarrow \qquad E_r^{p,q} \cong E_{r+1}^{p,q} \cong \cdots =: E_\infty^{p,q}.$$

Therefore, one can see $E_\infty^{p,q}$ as the *stabilization* of the spot (p,q) after a large enough number of iterations. The differential $d_r\colon E_r^{0,r-1} \longrightarrow E_r^{r,0}$, which is the last (at least in the general scenario) non-trivial differential acting on the spot $(0, r-1)$, is called **transgression**.

As stated before, this construction corresponds to a spectral sequence. For completion, I present next the formal definition of that concept by adapting the definition in WEIBEL (1994). One will quickly realize that the above steps indeed lead to a spectral sequence.

Definition 2.11. A **cohomological spectral sequence** E (in an abelian category $\mathcal{A}$), starting with E_a, consists of the following data:

i. A family $\{E_r^{p,q}\}_{p,q,r}$ of abelian groups defined for all integers p, q and $r \geqslant a$;

ii. Homomorphisms $d_r^{p,q}\colon E_r^{p,q} \longrightarrow E_r^{p+r,q-r+1}$ that are differentials, meaning that $d_r \circ d_r = 0$ — hence the "lines of slope $-(r-1)/r$" in the lattice $E_r^{\bullet,\bullet}$ form cochain complexes;

iii. Isomorphisms between $E_{r+1}^{p,q}$ and the cohomology of $E_r^{\bullet,\bullet}$ at the spot (p,q), viz.

$$E_{r+1}^{p,q} \cong \frac{\ker d_r^{p,q}}{\operatorname{im} d_r^{p-r,q+r-1}}.$$

For the construction previously presented, the isomorphisms in iii. are actually equalities. In addition, there are the following notions of convergence of spectral sequences. One shall note that the case in hand will be that of a *bounded* and *regular* spectral sequence.

Definition 2.12. Let E be a spectral sequence.

a. Let the **total degree** of the space $E_r^{p,q}$ be $p+q$.

b. The spectral sequence is said to be **bounded** if, for each n, there are only finitely many spaces of total degree n.

c. The (bounded) spectral sequence is said to **converge** to $H^\bullet$ if there is a finite (decreasing) filtration $H^k = F^t H^k \supseteq F^{t+1} H^k \supseteq \cdots \supseteq F^p H^k \supseteq \cdots \supseteq F^{s-1} H^k \supseteq F^s H^k = 0$ such that

$$E_\infty^{p,q} \cong F^p H^{p+q} / F^{p+1} H^{p+q}.$$

d. The spectral sequence E is said to **abut to** $E_\infty^{p,q}$ if, for every p and q, there exists $r_0 = r_0(p,q)$ such that $E_r^{p,q} \cong E_{r_0}^{p,q}$ for every $r \geqslant r_0$. Then $E_\infty^{p,q} := E_{r_0}^{p,q}$ is the **limiting term** at (p,q).

e. The spectral sequence E is **regular** or it **degenerates at** r_0 if the differentials $d_r^{p,q}$ are trivial for every $r \geqslant r_0$.

Chapter 3

Cohomology of Lie algebras

After the first acts on the history of homological algebra, in 1929 Georges de Rham solved Élie Cartan's conjecture — that the Betti number b_k of a smooth manifold M is the maximum number of closed k-forms $\omega_i \in \Omega^k(M)$ such that no linear combination $\omega = \sum_i \lambda_i \omega_i$ is exact — and constructed what is now known as the de Rham cohomology.

In 1935, Lev Pontryagin (1908–1988) calculated the Betti numbers for classical Lie groups; also in 1935, Richard Brauer (1901–1977) computed the ring structure of homology for Lie groups using the theory introduced by de Rham.

Eilenberg and Saunders Mac Lane (1909–2005), Hopf, Beno Eckmann (1917–2008), and Hans Freudenthal (1905–1990) — with their publications in 1943, 1944, 1945 and 1946, respectively — contributed to the discovery of group homology and cohomology, restricted to groups which were fundamental groups of familiar topological spaces. The last one, by Freudenthal, was possibly smuggled out of the Netherlands in 1944 (WEIBEL, 1999).

Roger Lyndon (1917–1988), in his thesis in 1948, discovered a way of calculating the cohomology of a group when knowing about a normal subgroup (which, in the Lie algebra scenario, means knowing of an ideal). Jean-Pierre Serre (b. 1926) noticed two years later, in 1950, that Lyndon's technique was that of a spectral sequence, completing the work with Gerhard Hochschild (1915–2010) in 1953.

In 1930, Élie Cartan proved that every connected Lie group is diffeomorphic to the product of a compact Lie group and an Eucliden space $\mathbb{R}^n$, which restricted the study of Lie group cohomology to compact Lie groups — and that was solved in 1935 by Pontryagin and Brauer. Élie Cartan and de Rham noted that the de Rham (co)homology of a group G could be computed using left-invariant differential forms, which in turn clarified that the Lie algebra $\mathfrak{g}$ of a Lie group G is enough to define the cohomology of G.

Claude Chevalley (1909–1984) and Samuel Eilenberg used that to define the cohomology of any Lie algebra independently of associated Lie groups in 1948.

They showed that the algebra of left-invariant differential forms on a Lie group is isomorphic to the dual algebra $\mathrm{CE}(\mathfrak{g})$ of the exterior algebra $\Lambda(\mathfrak{g})$.

Translating the de Rham differential into this language, they defined a differential d_{CE} turning $\mathrm{CE}(\mathfrak{g})$ into a differential graded algebra. Wanting to study the cohomology of homogeneous spaces G/H of G, Chevalley and Eilenberg defined the cohomology with respect to a representation ρ of $\mathfrak{g}$ on a vector space V, concluding that central extensions of $\mathfrak{g}$ by V are in 1-to-1 correspondence with elements of the 2nd cohomology group, $H^2_\rho(\mathfrak{g}, V)$.

In 1936, J. H. C. Whitehead (1904–1960) published his lemmas about inner representations and the famous result that, for a simple representation of $\mathfrak{g}$ on $V \neq \mathbb{K}$, then $H^q(\mathfrak{g}, V) = 0$ for all q — rendering interesting only the cohomology groups of $\mathfrak{g}$ which have trivial coefficients.

In 1955, Henri Cartan published notes of his algebraic topology seminar held in 1948/1949 in which he covered Jean Leray's theory of sheaves and spectral sequences. That material was later used by Hochschild and Serre to redo Lyndon's spectral sequence in the context of Lie algebras — creating what is now known as the Hochschild–Serre spectral sequence, discussed in Section 3.4.

BRST quantization

Ghosts arise in quantum field theory as a way to keep a local description of quantum gauge theories in terms of elementary processes, involving free propagators and local vertices, and also ensuring that the theory remains unitary and independent of gauge choice (HENNEAUX–TEITELBOIM, 1992). Ludvig Faddeev (1934–2017) and Viktor Popov (1937–1994) showed in 1967 that the ghosts appear as an artifact when integrating over equivalence classes of gauge field histories, introducing what is known as the Faddeev–Popov procedure.

Carlo Becchi (b. 1939), Alain Rouet (b. 1942) and Raymond Stora (1930–2015), in a series of works starting in 1974, and Igor Tyutin (b. 1940), in his preprint from 1975, introduced what is now known as the *BRST symmetry*, leading the ghost fields to a more important role. Such symmetry mixes physical and ghost fields, making it necessary to consider all those fields as part of a single object.

The idea of the BRST symmetry is to consider a fermionic rigid symmetry (fermionic because it flips statistics, rigid because it acts in the same way everywhere, as opposed to local gauge symmetries). The infinitesimal generator Q of this symmetry is nilpotent, i.e. $Q^2 = 0$, leading one to consider its cohomology.

The original gauge invariance is restored by going to cohomology, i.e.

$$H^0(Q) = \{\text{gauge invariant functions}\}.$$

More generally, physical states are Q-closed, modulo Q-exact states (recall footnote 5).

Any gauge symmetry by a Lie group G allows for the construction of a nilpotent operator such as Q. This leads to simplifications, such as one does not need to reduce the phase space for the theory. The useful relation to this thesis is that the generator of the BRST symmetry matches the Chevalley–Eilenberg differential in some sense (to be clarified in Chapter 4), hence it can be used to compute the Chevalley–Eilenberg cohomology of $\mathfrak{g}$.

Structure of this chapter

This chapter is devoted to define Lie algebra cohomology, through cochains in Section 3.1, differential forms à la Chevalley–Eilenberg in Section 3.1.2, and polynomials à la BRST in Section 3.2. The meaning of lower order cohomology groups are given, especially the relation with central extensions in Section 3.1.1, which motivate this thesis from (1.7). The Lie algebra $\mathfrak{gl}(n)$ is discussed as an example, laying out the results which are used later in Chapter 4. The Hochschild–Serre spectral sequence is presented at the end, in Section 3.4.

3.1 Definitions

Henceforward, let $\mathfrak{g}$ be a finite-dimensional Lie algebra, and $\rho\colon \mathfrak{g} \longrightarrow \mathrm{End}(V)$ be a representation of $\mathfrak{g}$ on the (finite-dimensional) vector space V. The field $\mathbb{K}$ over which $\mathfrak{g}$ and V are defined shall be taken to have characteristic zero.

Definition 3.1. A ***V*-valued *k*-cochain** ω of $\mathfrak{g}$ is a skew-symmetric k-linear map

$$\omega\colon \mathfrak{g}^{\times k} \longrightarrow V. \tag{3.1}$$

The space[8] of all V-valued k-cochains of $\mathfrak{g}$ will be denoted by

$$C^k(\mathfrak{g}, V) \equiv \mathrm{Hom}\big(\Lambda^k(\mathfrak{g}), V\big).$$

[8] Which is an abelian group under point-wise addition

Define the operator

$$d^k \colon C^k(\mathfrak{g}, V) \longrightarrow C^{k+1}(\mathfrak{g}, V)$$

acting on a V-valued k-cochain $\omega \in C^k(\mathfrak{g}, V)$ as

$$\begin{aligned}(d^k\omega)(x_1, \dots, x_{k+1}) := \sum_{i=1}^{k+1} (-1)^{i+1} \rho(x_i)\, \omega\big(x_1, \dots, \widehat{x}_i, \dots, x_{k+1}\big) + \\ + \sum_{\substack{i,j=1 \\ i<j}}^{k+1} (-1)^{i+j} \omega\big([x_i, x_j], x_1, \dots, \widehat{x}_i, \dots, \widehat{x}_j, \dots, x_{k+1}\big),\end{aligned} \tag{3.2}$$

where $x_1, \dots, x_{k+1} \in \mathfrak{g}$, and $\widehat{\cdot}$ indicates the removal of that entry. This is a well-defined map (AZCÁRRAGA–IZQUIERDO, 1995).

The nilpotency of d (i.e. $d^{k+1} \circ d^k = 0$) implies that

$$C^0(\mathfrak{g}, V) \xrightarrow{\ d\ } C^1(\mathfrak{g}, V) \xrightarrow{\ d\ } \cdots \xrightarrow{\ d\ } C^k(\mathfrak{g}, V) \xrightarrow{\ d\ } C^{k+1}(\mathfrak{g}, V) \longrightarrow \cdots$$

is a cochain complex. The k^{th} cohomology space of this cochain complex is denoted by $H^k_\rho(\mathfrak{g}, V)$, making explicit the dependence on the representation ρ. If V is the field $\mathbb{K}$ over which $\mathfrak{g}$ is defined (and ρ trivial), one then writes $H^k(\mathfrak{g}) \equiv H^k_0(\mathfrak{g}, \mathbb{K})$.

Example 3.1 (0^{th} cohomology group)**.** The 0-cochains are, by definition, constant mappings from $\mathfrak{g}$ to V, thus $C^0(\mathfrak{g}, V) = V$ and a 0-cochain is a vector $v \in V$. If it is a 0-cocycle (i.e. an element in the kernel of d^0), then

$$0 = (d^0 v)(x) = \rho(x)\, v, \qquad \forall x \in \mathfrak{g},$$

i.e. v is an invariant element of V under the action of ρ. Thus

$$H^0_\rho(\mathfrak{g}, V) = \big\{\text{invariants in } V \text{ under } \rho\big\}.$$

In particular, $H^0(\mathfrak{g}) = \mathbb{K}$. ▷

Example 3.2 (1^{st} cohomology group)**.** From the previous example, a 1-cochain ω is a 1-coboundary if there exists a 0-cochain v such that $d^0 v = \omega$, i.e.

$$\omega(x) = \rho(x)\, v, \qquad \forall x \in \mathfrak{g}.$$

Now, a 1-cochain ω is a 1-cocycle if

$$0 = (d^1\omega)(x_1, x_2) = \rho(x_1)\,\omega(x_2) - \rho(x_2)\,\omega(x_1) - \omega\big([x_1, x_2]\big), \qquad \forall x_1, x_2 \in \mathfrak{g}.$$

In particular, consider $V = \mathbb{K}$ and $\rho = 0$. Then ω is a coboundary if $\omega = 0$, while the cocycle condition above becomes $\omega\big([x, x']\big) = 0$ for all $x, x' \in \mathfrak{g}$. Consequently, 1-cocycles are linear mappings vanishing on $[\mathfrak{g}, \mathfrak{g}]$, therefore

$$H^1(\mathfrak{g}) = \big(\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]\big)^*.$$

If $\mathfrak{g}$ is semisimple, then $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$, which implies that $H^1(\mathfrak{g}) = 0$. ▷

As claimed in Chapter 1, the 2nd cohomology group of a Lie algebra encodes information about its central extensions. This is a rather long discussion, which deviates from the main purpose of this thesis; thus I shall present only the definition of a central extension and state the main result, referring the reader to the book of Azcárraga–Izquierdo (1995) for more details.

3.1.1 Central extensions

Definition 3.2. Let $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$ be Lie algebras, and let $\mathfrak{a}$ be an abelian Lie algebra. One says that $\tilde{\mathfrak{g}}$ is a **central extension** of $\mathfrak{g}$ by $\mathfrak{a}$ if there is an exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \mathfrak{a} \arrow[r, hook, "i"] & \tilde{\mathfrak{g}} \arrow[r, bend left, "\pi"] & \mathfrak{g} \arrow[l, bend left, dashed, "\tau"] \arrow[r] & 0. \end{tikzcd} \tag{3.3}$$

Comparing (3.3) with the prototypical short exact sequence (2.6), one sees that $\mathfrak{a} \equiv i(\mathfrak{a})$ is an ideal of $\tilde{\mathfrak{g}}$, and that $\mathfrak{g} \cong \tilde{\mathfrak{g}}/\mathfrak{a}$. The *trivializing map*

$$\begin{aligned} \tau\colon \mathfrak{g} \cong \ \tilde{\mathfrak{g}}/\mathfrak{a} &\longrightarrow \tilde{\mathfrak{g}} \\ \tilde{x} + \mathfrak{a} &\longmapsto \tilde{x} \end{aligned}$$

is such that $\pi \circ \tau = \mathrm{id}_{\mathfrak{g}}$. Obviously it is not unique: in each equivalence class $\tilde{x} + \mathfrak{a}$, one can pick another representative $\tilde{x}'$ and set $\tau(\tilde{x} + \mathfrak{a}) = \tilde{x}'$, as long as $\tilde{x} - \tilde{x}' \in \mathfrak{a}$.

Starting with a Lie algebra $\mathfrak{g}$, suppose one knows a pair $(\tilde{\mathfrak{g}}, \mathfrak{a})$ making (3.3) an exact sequence. One can define a representation ρ of $\mathfrak{g}$ on $\mathfrak{a}$ by requiring that

$$\rho(x)\,y = \big[\tau(x), y\big]_{\tilde{\mathfrak{g}}}, \qquad \forall\, x \in \mathfrak{g},\ y \in \mathfrak{a}.$$

This is well-defined because two different choices of $\tau(x) = \pi^{-1}(x) \in \tilde{\mathfrak{g}}$ differ by an element of $\mathfrak{a}$, which was assumed to be abelian.

Definition 3.3. Two central extensions $\tilde{\mathfrak{g}}$ and $\tilde{\mathfrak{g}}'$ of $\mathfrak{g}$ by $\mathfrak{a}$ are **equivalent** if there exists a Lie algebra isomorphism f such that the diagram below is commutative:

$$\begin{tikzcd}
 & & \tilde{\mathfrak{g}} \arrow[dr, "\pi"] \arrow[dd, "f"] & & \\
0 \arrow[r] & \mathfrak{a} \arrow[ur, "i"] \arrow[dr, "i'"'] & & \mathfrak{g} \arrow[r] & 0. \\
 & & \tilde{\mathfrak{g}}' \arrow[ur, "\pi'"'] & &
\end{tikzcd}$$

Equivalent extensions necessarily lead to the same representation ρ.

Theorem 2 (Chevalley, Eilenberg)**.** *Given representation ρ of the Lie algebra $\mathfrak{g}$ on $\mathfrak{a}$, the classes of equivalent central extensions $\tilde{\mathfrak{g}}$ of $\mathfrak{g}$ by the abelian algebra $\mathfrak{a}$ are in 1-to-1 correspondence with the elements of the 2^{nd} cohomology group $H^2_\rho(\mathfrak{g}, \mathfrak{a})$.*

3.1.2 The Chevalley–Eilenberg formulation

Let $\mathfrak{g}$ be a real Lie algebra. If G is a connected Lie group of which $\mathfrak{g}$ is the Lie algebra, then $\mathfrak{g}$ can be understood as the tangent space at the neutral element of G, i.e. $\mathfrak{g} \cong T_e G$. Consequently, a real-valued k-cochain of $\mathfrak{g}$ can be understood as the image of a differential k-form on G, i.e. as $\omega(e)$. The Lie algebra element $x_i \in \mathfrak{g}$ in (3.2) is the value at $e \in G$ of a vector field X_i defined on G.

Extending the values of ω to the rest of G by requiring that

$$\omega(g) = (L_{g^{-1}})^*(e)\, \omega(e),$$

one obtains that ω is a left-invariant differential form (AZCÁRRAGA–IZQUIERDO, 1995). Above, $(L_{g^{-1}})^*$ is the pullback of the left multiplication map by g^{-1} on G, viz. $L_{g^{-1}} \colon a \longmapsto g^{-1}a$. The reason why one wants a left-invariant differential form is that its differential becomes

$$\mathrm{d}\omega(X_1, \ldots, X_{k+1}) = \sum_{\substack{i,j=1 \\ i<j}}^{k+1} (-1)^{i+j} \omega\big([X_i, X_j], X_1, \ldots, \widehat{X}_i, \ldots, \widehat{X}_j, \ldots, X_{k+1}\big),$$

for $X_1, \ldots, X_{k+1}$ left-invariant vector fields on G. The above expression matches the definition of $d^k\omega$ in (3.2).

The discussion can be summarized as: the exterior derivative acting on left-invariant k-forms on G is the same as the differential operator d^k acting on $\mathbb{K}$-valued k-cochains of $\mathfrak{g}$ (AZCÁRRAGA–IZQUIERDO, 1995).

In the formulation by Chevalley and Eilenberg, cochains of $\mathfrak{g}$ can be defined intrinsically, without depending on its associated Lie group. Thus, they are elements of $\Lambda^k(\mathfrak{g}^*)$, and the **Chevalley–Eilenberg cochain complex** of a Lie algebra $\mathfrak{g}$ with trivial coefficients is

$$\mathrm{CE}^\bullet(\mathfrak{g}) := \Lambda^\bullet(\mathfrak{g}^*), \tag{3.4}$$

or yet

$$\Lambda^0(\mathfrak{g}^*) \xrightarrow{d} \Lambda^1(\mathfrak{g}^*) \xrightarrow{d} \cdots \xrightarrow{d} \Lambda^k(\mathfrak{g}^*) \xrightarrow{d} \Lambda^{k+1}(\mathfrak{g}^*) \xrightarrow{d} \cdots .$$

If one denotes by $H^k_{\mathrm{CE}}(\mathfrak{g})$ the k^{th} cohomology of the Chevalley–Eilenberg complex (3.4), one has:

Theorem 3. *Let $\mathfrak{g}$ be the Lie algebra of a Lie group G. Then, the k^{th} cohomology space of $\mathfrak{g}$ with the trivial representation, $H^k_0(\mathfrak{g}, \mathbb{K})$, is isomorphic to the k^{th} Chevalley–Eilenberg cohomology space, $H^k_{\mathrm{CE}}(\mathfrak{g})$.*

For V-valued cochains, one simply has $\mathrm{CE}^\bullet(\mathfrak{g}, V) := \mathrm{Hom}(\Lambda^\bullet(\mathfrak{g}), V)$.

A careful reader might have noticed by now that, by discussing Lie algebra cohomology, I am running on a tangent about the original motivation described in Chapter 1; namely, that of the cohomology of a Lie (super)group — particularly, that of $\mathrm{PSU}(2,2|4)$.

Although already mentioned at the historical digression at the beginning of this Chapter, this is a good point to explain in more detail the relation between Lie group and Lie algebra cohomologies. Firstly, one should consider the following theorem (CHEVALLEY–EILENBERG, 1948).

Theorem 4 (Chevalley, Eilenberg)**.** *Let G be a compact and connected Lie group. Then every de Rham cohomology class of differential forms on G contains precisely one bi-invariant (i.e. both left- and right-invariant) form. The bi-invariant forms span a ring isomorphic to the cohomology ring $H_{\mathrm{dR}}(G)$.*

The immediate consequence of this Theorem is that the de Rham cohomology of a compact and connected Lie group G is isomorphic to the Chevalley–Eilenberg cohomology of its Lie algebra $\mathfrak{g}$.

Theorem 4 is nevertheless not applicable to the case in hand, as the Lie algebra $\mathfrak{gl}(n)$ — which shall be generalized to the super- case in Chapter 4 — is not compact. There is still hope because of the following (NLAB AUTHORS, 2025; VAN EST, 1953), as it contains the cases of interest.

Theorem 5 (van Est)**.** *Let G be a contractible topological space, and $\mathfrak{g}$ its Lie algebra. Then the Lie group cohomology of G coincides with the Lie algebra cohomology of $\mathfrak{g}$.*

3.2 The BRST formulation

In this section, I present the BRST formulation to Lie algebra cohomology, which is of course related to the previous definitions, and originates in the context of quantization of gauge theories by Becchi, Rouet, Stora, and Tyutin. This approach resurfaces later in Chapter 4 in the context of Lie superalgebras.

As already discussed in the Chevalley–Eilenberg formulation, a scalar-valued k-cochain on the Lie algebra $\mathfrak{g}$ can be seen as a left-invariant differential k-form on the associated Lie group G. Choosing local coordinates $\{q^i\}_i$ on G (as a manifold), one can write

$$\omega = \omega_{i_1 i_2 \cdots i_k}\, \mathrm{d}q^{i_1} \wedge \mathrm{d}q^{i_2} \wedge \cdots \wedge \mathrm{d}q^{i_k}.$$

Each $\mathrm{d}q^i$ is a section of the cotangent bundle T^*G, whose fibres look like $\mathfrak{g}^*$ at each point. Thus, if $\{e_i \equiv \partial/\partial q^i|_e\}_i$ is the coordinate basis of $\mathfrak{g} \cong T_eG$, and $\{c^i\}_i$ is the dual basis of $\mathfrak{g}^*$, one can write

$$\omega = \omega_{i_1 i_2 \cdots i_k}\, c^{i_1} \wedge c^{i_2} \wedge \cdots \wedge c^{i_k}.$$

One often drops out the wedge products and require that c^i are anti-commuting objects,[9] viz.

$$c_i\, c_j = -c_j\, c_i, \qquad \forall i, j \in \{1, \ldots, n\}. \tag{3.5}$$

As elements of $\mathfrak{g}^*$, each c^i is a linear functional of $\mathfrak{g}$. Even more, they are the coordinate functions with respect to the basis$\{e_i\}_i$, thus the k-cochain

$$\omega_{\mathrm{BRST}} = \omega_{i_1 i_2 \cdots i_k}\, c^{i_1} c^{i_2} \cdots c^{i_k} \tag{3.6}$$

can be understood as a k^{th} order polynomial in $\mathfrak{g}$, with the anti-commutation (3.5) inherited from the wedge product.

[9] Sometimes called (Faddeev–Popov) *ghosts*, but that terminology shall be saved for Chapter 4.

More generally, assume one starts with n anti-commuting variables like (3.5), and let V be the vector space spanned by them. Let ρ be a representation of $\mathfrak{g}$ on V. On the space of functions on V, define the operator

$$Q := c^i \rho(e_i) - f_{ij}^k c^i c^j \frac{\partial}{\partial c^k}, \tag{3.7}$$

where f_{ij}^k are the structure constants of $\mathfrak{g}$ with respect to the basis $\{e_i\}_i$. The operator Q, which is nilpotent (AZCÁRRAGA–IZQUIERDO, 1995), is called the **BRST differential** of $\mathfrak{g}$. The cochains on which Q acts are defined as in (3.6), where

$$\omega_{i_1 i_2 \cdots i_k} := \omega(e_{i_1}, e_{i_2}, \dots, e_{i_k})$$

are totally anti-symmetric coefficients.

As one might expect, the following holds (AZCÁRRAGA–IZQUIERDO, 1995).

Theorem 6. *The BRST differential acts on BRST k-cochains as*

$$Q\omega_{\text{BRST}} = \frac{1}{k+1}(d^k \omega)_{\text{BRST}},$$

therefore d and Q generate the same Lie algebra cohomology.

Note that, for a trivial action ρ, one has

$$Qc^k = -f_{ij}^k c^i c^j.$$

This shows that Q is (minus) the adjoint of the Lie bracket on $\mathfrak{g}$. More precisely, the Lie bracket is a bilinear map $\mu\colon \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ that acts on basis elements as

$$\mu(e_i, e_j) \equiv [e_i, e_j] = f_{ij}^k e_k.$$

There exists a bilinear map $\mu^*\colon \mathfrak{g}^* \longrightarrow \mathfrak{g}^* \times \mathfrak{g}^*$, the *adjoint (transpose, dual) map* of μ, satisfying

$$\langle \xi, \mu(x,y) \rangle_{\mathfrak{g}} = \langle \mu^*(\xi), (x,y) \rangle_{\mathfrak{g}\times\mathfrak{g}}, \qquad \forall\, x, y \in \mathfrak{g},\ \forall\, \xi \in \mathfrak{g}^*,$$

where $\langle \cdot, \cdot \rangle_V$ is the natural pairing of a vector space V with its dual V^*. Its not hard to see that $Q = -\mu^*$.

3.3 The cohomology of $\mathfrak{gl}(n)$

In this section, I want to briefly show the calculation of the cohomology of the Lie algebra that appears in the next chapter. For that, recall that $\mathfrak{gl}(n)$ is the Lie algebra containing all $n \times n$ matrices over $\mathbb{C}$.

Let $\mathcal{B} := \{e_i{}^j\}_{i,j}$ be the canonical basis of $\mathfrak{gl}(n)$, where $e_i{}^j$ is the matrix with 0s everywhere except at the $(i,j)^{\text{th}}$ entry, which is 1; more precisely,

$$[e_i{}^j]_k{}^\ell = \delta_{ik}\, \delta^{j\ell}.$$

It is not difficult to see that the product of two matrix elements is non-zero only if the inner indices match, i.e.

$$e_i{}^j\, e_k{}^\ell = \delta_k{}^j\, e_i{}^\ell. \tag{3.8}$$

The Lie bracket of $\mathfrak{gl}(n)$ is the commutator $[x, y] := xy - yx$; hence, the structure constants of $\mathfrak{gl}(n)$ with respect to the canonical basis $\mathcal{B}$ are

$$f^{(i,\ell)}_{(i,j)(k,\ell)} = \delta_k{}^j.$$

Consequently, the BRST differential for the trivial representation of $\mathfrak{gl}(n)$ becomes

$$Q = -c^i{}_j\, c^j{}_k\, \frac{\partial}{\partial c^i{}_k},$$

where $c^i{}_j$ are the anti-commuting variables (3.5). In matrix form, one can write

$$Qc = -c^2. \tag{3.9}$$

From Fuks (1986), one has the following result.

Theorem 7. *The cohomology of $\mathfrak{gl}(n)$ is the exterior algebra in n generators of odd degrees $1, 3, \ldots, 2n-1$. The generators are the cohomology classes of the cochains*

$$\phi_{2k-1} \colon (x_1, \ldots, x_{2k-1}) \longmapsto \sum_{\sigma \in \mathcal{S}_k} \operatorname{sgn}(\sigma)\, \operatorname{tr}\Big(x_{\sigma(1)}\, x_{\sigma(2)} \cdots x_{\sigma(2k-1)}\Big),$$

where $\mathcal{S}_k$ is the group of permutations of k elements.

Beware that Theorem 7 does not mean that the only non-trivial cohomology spaces are those in odd degrees. E.g., the cohomology of $\mathfrak{gl}(2)$ is

$$H^k(\mathfrak{gl}(2)) \cong \begin{cases} \mathbb{C}, & \text{if } k \in \{0,1,3,4\}, \\ 0, & \text{else,} \end{cases} \tag{3.10}$$

because the 4th cohomology space is spanned by the cohomology class of $\phi_1 \wedge \phi_3$.

The cochain ϕ_k acting on the basis elements ${e_i}^j \in \mathfrak{gl}(n)$ yields

$$\phi_k\left({e_{i_1}}^{j_1}, \cdots, {e_{i_k}}^{j_k}\right) = \sum_{\sigma \in \mathcal{S}_k} \operatorname{sgn}(\sigma)\, \delta^{\sigma(j_1)}_{\sigma(i_2)} \delta^{\sigma(j_2)}_{\sigma(i_3)} \cdots \delta^{\sigma(j_{k-1})}_{\sigma(i_k)} \delta^{\sigma(j_k)}_{\sigma(i_1)};$$

thus, its associated BRST cochain is the polynomial $\varphi_k = 2k \operatorname{tr}(c^k)$, where c^k is the the k^{th} power of the matrix whose $(i,j)^{\text{th}}$ entry is the anti-commuting variable ${c^i}_j$ dual to the basis element ${e_i}^j$.

3.4 The Hochschild–Serre spectral sequence

In this section, I want to introduce a useful tool in the computation of Lie algebra cohomology: the Hochschild–Serre spectral sequence. Rather than proving its existence, convergence etc., I shall enunciate it and digress a little, following FUKS (1986) and WEIBEL (1994).

Theorem 8 (Hochschild, Serre)**.** *Let $\mathfrak{g}$ be a Lie algebra, $\mathfrak{h}$ be a Lie subalgebra of $\mathfrak{g}$, and V a vector space. Then, there exists a spectral sequence $\{E_r, d_r\}_r$ with the following properties:*

i. Its first page is $E_1^{p,q} = H^q\left(\mathfrak{h}; C^p(\mathfrak{g}, \mathfrak{h}; V)\right)$ — i.e. the q^{th} cohomology space of $\mathfrak{h}$ with coefficients in the group of V-valued p-cochains of $\mathfrak{g}$ relative to $\mathfrak{h}$;

ii. Its second page has $E_2^{p,0} = H^p(\mathfrak{g}, \mathfrak{h}; V)$;

iii. If $\mathfrak{h}$ is an ideal of $\mathfrak{g}$, then $E_2^{p,q} = H^p\left(\mathfrak{g}/\mathfrak{h}; H^q(\mathfrak{h}; V)\right)$;

iv. It converges to $H^{p+q}(\mathfrak{g}, V)$, i.e.

$$H^k(\mathfrak{g}, V) = \bigoplus_{\substack{p,q \\ p+q=k}} E_\infty^{p,q}.$$

As one can expect from what was presented in Section 2.4, the Hochschild–Serre spectral sequence arises from a filtration in the cochains of $\mathfrak{g}$, and is based on the short exact sequence

$$0 \longrightarrow \mathfrak{h} \hookrightarrow \mathfrak{g} \twoheadrightarrow \mathfrak{g}/\mathfrak{h} \longrightarrow 0.$$

The filtration is the following:

$$F^p C^{p+q}(\mathfrak{g}, V) := \left\{ \omega \in C^{p+q}(\mathfrak{g}, V) \,\middle|\, \omega(g_1, \ldots, g_{p+q}) = 0, \quad \forall g_1, \ldots, g_{q+1} \in \mathfrak{h} \right\},$$

i.e. it strictly contains the $(p+q)$-cochains that vanish whenever $(q+1)$ of the entries are in $\mathfrak{h}$. In Chevalley–Eilenberg terms, one can write

$$F^p C^{p+q}(\mathfrak{g}) \cong \bigoplus_{k=0}^{q} \left(\Lambda^{p+k} (\mathfrak{g}/\mathfrak{h})^* \right) \wedge \left(\Lambda^{q-k}(\mathfrak{h}^*) \right); \tag{3.11}$$

of course, a cochain in that space yields zero when acting on $(q+1)$ elements of $\mathfrak{h}$.

In the BRST formulation, elements of $\mathfrak{h}^*$ and $(\mathfrak{g}/\mathfrak{h})^*$ become anti-commuting variables, and the cochain ω becomes a polynomial in those variables. Thus, in the BRST case, an element of $F^p C^{p+q}$ is a polynomial with *at most* q "ghost" coordinates on $\mathfrak{h}$ or, equivalently, with *at least* p "ghost" coordinates on $\mathfrak{g}/\mathfrak{h}$.

The filtration (3.11) preserves the differential (3.2). It is not difficult to see that it also is a decreasing filtration. Indeed, by simply rearranging indices, one has

$$F^{p+1} C^{p+q}(\mathfrak{g}) \cong \bigoplus_{k=1}^{q} \left(\Lambda^{p+k} (\mathfrak{g}/\mathfrak{h})^* \right) \wedge \left(\Lambda^{q-k}(\mathfrak{h}^*) \right);$$

consequently, by splitting the direct sum on the $k = 0$ term, the space $F^p C^{p+q}$ reads as

$$F^p C^{p+q}(\mathfrak{g}) \cong \left(\Lambda^{p} (\mathfrak{g}/\mathfrak{h})^* \right) \wedge \left(\Lambda^{q}(\mathfrak{h}^*) \right) \oplus \underbrace{\bigoplus_{k=1}^{q} \left(\Lambda^{p+k} (\mathfrak{g}/\mathfrak{h})^* \right) \wedge \left(\Lambda^{q-k}(\mathfrak{h}^*) \right)}_{F^{p+1}C^{p+q}}.$$

Thus, obviously

$$F^{p+1} C^{p+q} \subseteq F^p C^{p+q}, \qquad \forall p, q.$$

For completion, note that $F^0 C^p = C^p$ and $F^{p+1} C^p = 0$.

Additionally, the 0th page of the spectral sequence coming from the filtration (3.11) is, according to (2.15) and the remark above,

$$E_0^{p,q} = \frac{F^p C^{p+q}}{F^{p+1} C^{p+q}} \cong \left(\Lambda^p (\mathfrak{g}/\mathfrak{h})^*\right) \wedge \left(\Lambda^q (\mathfrak{h}^*)\right).$$

As it will become clearer with the example of Lie superalgebras, the differential d_0 on the 0th page is simply the restriction of the BRST differential to cochains of $\mathfrak{h}$. Consequently, it is reasonable to see that the 1st page should look like

$$E_0^{p,q} \cong \left(\Lambda^p (\mathfrak{g}, \mathfrak{h})^*\right) \otimes H^q(\mathfrak{h}) \cong H^q\left(\mathfrak{h}; \Lambda^p(\mathfrak{g}, \mathfrak{h})^*\right),$$

as stated by Theorem 8. For a detailed and rigorous proof of Theorem 8, I refer the reader to the book by FUKS (1986).

Chapter 4

Lie superalgebras

Dealing with strong interaction processes, Jean-Loup Gervais (1936–2020) and Bunji Sakita (1930–2002) wrote down in 1971 the action of a theory with a surprising feature: under appropriate conditions, it had a symmetry interchanging the bosonic and fermionic fields, which is nowadays called *supersymmetry* (WEINBERG, 2000). Before that, in 1966, Hironari Miyazawa (1927–2023) had already proposed a version of supersymmetry in the context of hadronic physics.

In 1974, Julius Wess (1934–2007) and Bruno Zumino (1923–2014) built quantum field theories in 4 dimensions including supersymmetry. However, a few years earlier in 1971, Yuri Gol'fand (1922–1994) and Evgeny Likhtman (b. 1946) had already published papers in the Soviet Union extending the Poincaré algebra with supersymmetries and constructing a 4-dimensional supersymmetric field theory.

On the mathematics side, Felix Berezin (1931–1980) pioneered the *supermathematics*, studying superspaces, supergroups etc., and many of his works were edited by Alexandr Kirillov (b. 1936) and published posthumously in 1987 under Berezin's name. In 1977, Victor Kač (b. 1943) published his classification of complex Lie superalgebras, which is one of the main references for this Chapter.

Structure of this chapter

Section 4.1 lays out the basic terminology and definitions of linear superspace, flipped statistics space and Lie superalgebra. In Section 4.2, it is shown how a Lie superalgebra structure on a linear superspace defines an odd quadratic nilpotent vector field on the flipped statistics space; in Section 4.3, it is shown that the cohomology of such vector field is equivalent to the Chevalley–Eilenberg cohomology, revisiting the BRST approach presented in Section 3.2. In Section 4.4, the computation of the cohomology of Lie superalgebras is done following FUKS (1986), focusing on $\mathfrak{gl}(m|n)$. Lastly, in Section 4.5, the definitions of $\mathfrak{sl}(m|n)$, $\mathfrak{psl}(m|m)$, $\mathfrak{su}(m|n)$ and $\mathfrak{psu}(m|m)$ are given, as well as future directions to this project.

4.1 Definitions

For this section, I follow BEREZIN–G. KATS (1970) and V. KAČ (1977a,b). Recall the discussion presented in Section 2.3 about graded modules; a superspace is a particular example, as seen below.

Definition 4.1. A **linear superspace** (or *super vector space*) is a $\mathbb{Z}_2$-graded vector space; similarly, a **superalgebra** is a $\mathbb{Z}_2$-graded algebra. The $\mathbb{Z}_2$-degree is called **statistics** (or sometimes *parity*); elements of V_0 are called **bosonic** (or *even*), whilst those of V_1 are called **fermionic** (or *odd*).

An immediate and concrete example is that any vector space V (without grading) can be regarded as a linear superspace by taking $V_0 = V$ and V_1 to be trivial. Similarly for an algebra.

Example 4.1. Following up on Example 2.2, for a superspace $V = V_0 \oplus V_1$, one has the endomorphism superalgebra $\mathrm{End}(V) = \mathrm{End}_0(V) \oplus \mathrm{End}_1(V)$, where

$$\mathrm{End}_0(V) = \mathrm{End}(V_0) \oplus \mathrm{End}(V_1), \tag{4.1a}$$

$$\mathrm{End}_1(V) = \mathrm{Hom}(V_0, V_1) \oplus \mathrm{Hom}(V_1, V_0), \tag{4.1b}$$

i.e. $\mathrm{End}_0(V)$ contains the endomorphisms preserving the statistics (sending bosons to bosons, and fermions to fermions), and $\mathrm{End}_1(V)$ contains the endomorphisms flipping the statistics (sending bosons to fermions, and vice-versa). ▷

Definition 4.2. Let V be a linear superspace. The **flipped statistics superspace** (or *parity reversed superspace*) ΠV is a linear superspace where $(\Pi V)_0 = V_1$ and $(\Pi V)_1 = V_0$ — i.e. bosons in V are fermions in ΠV, and vice-versa. Elements of ΠV may be called **ghosts** due to the flipped statistics with respect to V.

Direct and semidirect products of superalgebras are defined in the usual way; the tensor product, however, is a bit more involved. Let A and B be superalgebras; their **tensor product** $C \equiv A \otimes B$ is the superalgebra whose space is the usual tensor product of the spaces of A and B, with the induced $\mathbb{Z}_2$-grading,

$$\deg_C(a \otimes b) := \deg_A(a) + \deg_B(b), \qquad a \in A, b \in B, \tag{4.2}$$

and the product defined by

$$(a_1 \otimes b_1)(a_2 \otimes b_2) := (-1)^{(\deg a_2)(\deg b_1)}\, a_1 a_2 \otimes b_1 b_2, \qquad a_i \in A,\ b_i \in B. \tag{4.3}$$

Associativity of superalgebras is defined similiarly as for ordinary algebras. For an associative superalgebra A, the *graded Leibniz rule* holds:

$$[a, bc] = [a, b]\, c + (-1)^{(\deg a)(\deg b)}\, b\, [a, c], \qquad \forall a, b, c \in A. \tag{4.4}$$

Definition 4.3. A **Lie superalgebra** is a linear superspace $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ with a binary operation $[\,\cdot,\cdot]$, called the **Lie superbracket**, satisfying, for all $a, b, c \in A$,

$$[a, b] = -(-1)^{\deg(a)\deg(b)}[b, a] \qquad \text{(graded anti-commutativity)}, \tag{4.5}$$

$$\big[a, [b, c]\big] = \big[[a, b], c\big] + (-1)^{\deg(a)\deg(b)}\big[b, [a, c]\big] \quad \text{(graded Jacobi identity)}. \tag{4.6}$$

Firstly, note that $\mathfrak{g}_0$ alone has to be an ordinary Lie algebra. Indeed, the graded anti-commutativity (4.5) defines an anti-symmetric bilinear map on $\mathfrak{g}_0$ (which turns out to be the usual Lie bracket), and the graded Jacobi identity (4.6) restricted to $\mathfrak{g}_0$ yields the ordinary identity

$$\big[a, [b, c]\big] + \big[c, [a, b]\big] + \big[b, [c, a]\big] = 0.$$

On the other hand, the restriction of the Lie superbracket of $\mathfrak{g}$ to $\mathfrak{g}_1$ is a symmetric bilinear map.

Example 4.2. If A is an associative superalgebra, then the **supercommutator**

$$[a, b] = ab - (-1)^{(\deg a)(\deg b)}\, ba \tag{4.7}$$

turns A into a Lie superalgebra. Restricted to $\mathfrak{g}_1$, the supercommutator yields

$$[a, b] = ab + ba, \qquad \forall a, b \in \mathfrak{g}_1,$$

thus being an *anti-commutator* on $\mathfrak{g}_1$. This justifies the name "fermionic" to homogeneous elements of degree 1, since fermions have to be dealt with anti-commutators in quantum field theory. ▷

Example 4.3. As from Example 4.2, the endomorphism superalgebra $\mathrm{End}(V)$ becomes a Lie superalgebra when equipped with the supercommutator. For the linear superspace V with $V_0 = \mathbb{K}^m$ and $V_1 = \mathbb{K}^n$, one writes $\mathfrak{gl}(m|n) := \mathrm{End}(V)$; it is called the **general linear** Lie superalgebra. ▷

Standard algebraic notions are defined analogously to ordinary algebras, e.g. subsuperalgebras, associativity, commutativity, simplicity, and nilpotency.

Structure constants

Given a Lie superalgebra $\mathfrak{g}$, pick a homogeneous basis $\mathcal{B} = \{e_A\}_A$. Let a capital Latin index $(A, B, C, \ldots)$ stand for a general element of the basis; a small Latin index $(a, b, c, \ldots)$ shall be used to indicate bosonic elements of $\mathcal{B}$, while a small Greek index $(\alpha, \beta, \gamma, \ldots)$ shall be used to represent fermionic elements of $\mathcal{B}$.

Define the **structure constants** f^A_{BC} of $\mathfrak{g}$ with respect to the basis $\mathcal{B}$ by

$$[e_B, e_C] = f^A_{BC}\, e_A. \tag{4.8}$$

Since the superbracket preserves the parity, one knows that $e_a\, e_b$ and $e_\alpha\, e_\beta$ are bosonic, and $e_a\, e_\alpha$ is fermionic. Thus, the only nonzero structure constants are of the form f^a_{bc}, $f^a_{\alpha\beta}$ and $f^\alpha_{a\beta}$. More compactly, if $\bar{A} \equiv \deg(e_A)$, the only nonzero structure constants f^A_{BC} are those for which

$$\bar{A} + \bar{B} + \bar{C} = 0 \mod 2. \tag{4.9}$$

From the graded anti-commutativity (4.5), one realises that

$$f^C_{AB} = -(-1)^{\bar{A}\bar{B}} f^C_{BA}, \tag{4.10}$$

while one can rewrite the graded Jacobi identity (4.6) as

$$f^A_{BC} f^C_{DE} - f^A_{CE} f^C_{BD} - (-1)^{\bar{B}\bar{D}} f^A_{DC} f^C_{BE} = 0. \tag{4.11}$$

Example 4.4. Consider the general linear Lie superalgebra $\mathfrak{gl}(m|n)$. A general element of it has the form

$$X = \begin{pmatrix} a & \alpha \\ \beta & b \end{pmatrix}, \tag{4.12}$$

with

$$a \in \mathfrak{gl}(m), \qquad b \in \mathfrak{gl}(n), \qquad \alpha \in \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^m), \qquad \beta \in \mathrm{Hom}(\mathbb{C}^m, \mathbb{C}^n).$$

Let me pick a basis that resembles the canonical basis of $\mathfrak{gl}(m+n)$ in the following way. Consider that lower Latin indices $i, j, \ldots$ range in $\{1, \ldots, m\}$, and capital Latin indices $I, J, \ldots$ range in $\{1, \ldots, n\}$. Take $\{e_i{}^j\}_{i,j}$ and $\{e_I{}^J\}_{I,J}$ to be the canonical bases of $\mathfrak{gl}(m)$ and $\mathfrak{gl}(n)$, respectively; let $\{\varepsilon_i{}^J\}_{i,J}$ and $\{\varepsilon_I{}^j\}_{I,j}$ be the canonical bases of $\mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^m)$ and $\mathrm{Hom}(\mathbb{C}^m, \mathbb{C}^n)$, respectively.

In this way, it is possible to write

$$\begin{aligned}
a &= a^i{}_j\, e_i{}^j \ \in \mathfrak{gl}(m),\\
b &= b^I{}_J\, e_I{}^J \in \mathfrak{gl}(n),\\
\alpha &= \alpha^i{}_J\, \varepsilon_i{}^J \in \mathrm{Hom}(\mathbb{C}^n,\mathbb{C}^m),\\
\beta &= \beta^I{}_j\, \varepsilon_I{}^j \in \mathrm{Hom}(\mathbb{C}^m;\mathbb{C}^n);
\end{aligned}$$

hence $a^i{}_j$ and $b^I{}_J$ are the coordinates on $\mathfrak{gl}(m|n)_0$, whereas $\alpha^i{}_J$ and $\beta^I{}_j$ are the coordinates on $\mathfrak{gl}(m|n)_1$. Following the previous convention, write $A = (i, j)$ and $A = (I, J)$ for bosons, while $A = (i, J)$ and $A = (I, j)$ for fermions.

As in (3.8), once again

$$e_i{}^j e_k{}^\ell = \delta_k{}^j\, e_i{}^\ell.$$

Using the supercommutator (4.7), one has

$$\left[e_i{}^j, e_k{}^\ell\right] = \delta_k{}^j\, e_i{}^\ell - \delta_i{}^\ell\, e_k{}^j,$$

thus

$$f^{(i,\ell)}_{(i,j)(k,\ell)} = \delta_k{}^j.$$

Similarly, apart from graded anti-symmetrization (4.10), one can find that the non-zero structure constants of $\mathfrak{gl}(m|n)$ with respect to the canonical basis $\mathcal{B}$ are

$$f^{(j,\ell)}_{(j,i)(k,\ell)} = f^{(I,J)}_{(I,k)(i,J)} = f^{(I,j)}_{(I,k)(i,j)} = f^{(j,I)}_{(j,k)(i,I)} = \delta_k{}^i, \tag{4.13a}$$

$$f^{(J,L)}_{(J,I)(K,L)} = f^{(i,j)}_{(i,K)(I,j)} = f^{(i,J)}_{(i,K)(I,J)} = f^{(J,i)}_{(J,K)(I,i)} = \delta_K{}^I. \tag{4.13b}$$

The structure constants are symmetric with respect to capitalized/uncapitalized indices, since both $e_i{}^j$ and $e_I{}^J$ are bosons on their own right etc. ▷

4.2 An odd quadratic nilpotent vector field

The goal of this section is to prove the following proposition, inspired by ALEXANDROV *et al.* (1997).

Proposition 2. *Let L be a linear superspace, and let ΠL be its parity reversed linear superspace. Then, any odd quadratic nilpotent vector field on ΠL defines a Lie superalgebra structure on L, and vice-versa.*

Proof. Let $\mathcal{B} := \{e_A\}_A$ be a basis of L. For a general element $\xi = \xi^A\, e_A \in L$, its coordinates ξ^A can be seen as coordinate functions on L which, in turn, are elements of the dual superspace L^* — whose homogeneous subspaces are $(L^*)_s = (L_s)^*$ for $s \in \mathbb{Z}_2$. Hence $\mathcal{B}^* := \{\xi^A\}_A$ is the dual basis of $\mathcal{B}$, and

$$\deg(\xi^A) = \deg(e_A) =: \bar{A}.$$

Let $\mathcal{B}^*_\Pi := \{c^A\}_A$ be the ghost version of $\mathcal{B}^*$, i.e. the basis of $(\Pi L)^* \cong \Pi(L^*)$. As such, each c^A is a coordinate function on ΠL. Then

$$\deg(c^A) = \bar{A} + 1 \mod 2.$$

Define the quadratic vector field

$$Q = g^A_{BC}\, c^B c^C \frac{\partial}{\partial c^A} \in \mathrm{Vect}(\Pi L), \tag{4.14}$$

where g^A_{BC} are constant coefficients. For Q to be an odd vector field, it has to be an element of $\mathrm{End}_1(\Pi L)$, which is equivalent to demanding that $\deg(c^A) + \deg(c^B) + \deg(c^C) = 1 \pmod 2$, or yet

$$\bar{A} + \bar{B} + \bar{C} = 0; \tag{4.15}$$

in other words, the only non-zero g^A_{BC} have to be the ones satisfying (4.15).

As a result, one can rewrite Q as

$$Q = \left(g^a_{bc}\, c^a c^b + g^a_{\alpha\beta}\, c^\alpha c^\beta\right) \frac{\partial}{\partial c^a} + \left(g^\alpha_{b\beta} + g^\alpha_{\beta b}\right) c^\beta c^b \, \frac{\partial}{\partial c^\alpha}, \tag{4.16}$$

where c^a are fermionic ghosts, and c^α are bosonic ghosts. By splitting g into its symmetric and anti-symmetric parts (with respect to the lower indices), one can consider only the anti-symmetric part of g^a_{bc} and the symmetric parts of $g^a_{\alpha\beta}$ and $g^\alpha_{b\beta}$. Henceforth, one may assume without loss of generality that

$$g^a_{bc} = -g^a_{cb}, \qquad g^a_{\beta\gamma} = g^a_{\gamma\beta}, \qquad g^\alpha_{b\beta} = g^\alpha_{\beta b}. \tag{4.17}$$

For any (super)manifold M, there is a natural definition of Lie (super)bracket on $\mathrm{Vect}(M)$ (VARADARAJAN, 2004). Explicitly, for homogeneous $U, V \in \mathrm{Vect}(M)$, one has

$$[U, V] := \left(U^A\, \partial_A V^B - (-1)^{\deg(U)\deg(V)} V^A\, \partial_A U^B\right) \partial_B. \tag{4.18}$$

Given an element $\xi \in L$, with coordinates ξ^A, define the constant vector field

$$\iota_\xi := \xi^A \, \partial_A \in \mathrm{Vect}(\Pi L), \tag{4.19}$$

with $\partial_A \equiv \partial/\partial c^A$. The Lie superbracket of ι_ξ with Q is

$$\begin{aligned}[\iota_\xi, Q] = \left(\xi^A \partial_A Q^B + Q^A \cancelto{0}{\partial_A \xi^B}\right) \partial_B &= \xi^A \left(g^B_{AD} + (-1)^{(1+\bar{A})(1+\bar{D})} g^B_{DA}\right) c^D \, \partial_B \\ &= \xi^A \left(g^B_{AD} - (-1)^{\bar{A}\bar{D}} (-1)^{\bar{A}+\bar{D}} g^B_{DA}\right) c^D \, \partial_B .\end{aligned}$$

For $\eta \in L$, one can go further and compute

$$\begin{aligned}[[\iota_\xi, Q], \iota_\eta] &= \left([\iota_\xi, Q]^A \cancelto{0}{\partial_A \eta^B} - \eta^A \partial_A [\iota_\xi, Q]^B\right) \partial_B \\ &= -(-1)^{\bar{A}} \xi^A \eta^D \left(g^B_{AD} - (-1)^{\bar{A}\bar{D}} (-1)^{\bar{A}+\bar{D}} g^B_{DA}\right) \partial_B .\end{aligned}$$

Thus, for $\xi, \eta \in L$, define an element $\zeta \in L$ whose coordinates in the basis $\{e_A\}_A$ are

$$\zeta^B := -(-1)^{\bar{A}} \, \xi^A \eta^D \left(g^B_{AD} - (-1)^{\bar{A}\bar{D}} (-1)^{\bar{A}+\bar{D}} g^B_{DA}\right). \tag{4.20}$$

One has, then, a bilinear map $L \times L \longrightarrow L$, denoted by square brackets $[\cdot, \cdot]$, sending the pair (ξ, η) to ζ. Furthermore, define

$$f^B_{AD} := -(-1)^{\bar{A}} \left(g^B_{AD} - (-1)^{\bar{A}\bar{D}} (-1)^{\bar{A}+\bar{D}} g^B_{DA}\right)$$

so that

$$\zeta^B = [\xi, \eta]^B = f^B_{AD} \, \xi^A \eta^D. \tag{4.21}$$

It remains to see if f^B_{AD} are the structure constants of a Lie superalgebra.

Starting with the graded anti-commutation (4.5), one sees that

$$\begin{aligned} f^B_{DA} &= -(-1)^{\bar{D}} \left(g^B_{DA} - (-1)^{\bar{A}\bar{D}} (-1)^{\bar{A}+\bar{D}} g^B_{AD}\right) \\ &= -(-1)^{\bar{A}\bar{D}} \left(-(-1)^{\bar{A}} \left(g^B_{AD} - (-1)^{\bar{A}\bar{D}} (-1)^{\bar{A}+\bar{D}} g^B_{DA}\right)\right) \\ &= -(-1)^{\bar{A}\bar{D}} \, f^B_{AD}, \end{aligned}$$

as wanted — where one has to recall that $\bar{D} + \bar{D} = 0 \pmod 2$.

Before moving on to the graded Jacobi identity, note that the only non-zero f^A_{BC} are those satisfying (4.15). Using the symmetries (4.17), one concludes that

$$f^a_{bc} = -g^a_{bc}, \qquad f^a_{\beta\gamma} = g^a_{\beta\gamma}, \qquad f^\alpha_{b\beta} = -g^\alpha_{b\beta},$$

by recalling that, if c^a is a fermionic ghost, then ξ^a is a boson, thus $\bar{a} \equiv \deg(\xi^a) = 0$ — thus $\bar{\alpha} = 1$. This shows that one can reduce the expression of f to[10]

$$f^A_{BC} = -(-1)^{\bar{B}} g^A_{BC} \qquad \Longleftrightarrow \qquad g^A_{BC} = -(-1)^{\bar{B}} f^A_{BC}. \tag{4.22}$$

Henceforth, one can write the quadratic odd vector field Q as

$$Q = -(-1)^{\bar{B}} f^A_{BC}\, c^B c^C \,\frac{\partial}{\partial c^A}. \tag{4.23}$$

Now, to check that the coefficients f^A_{BC} satisfy the graded Jacobi identity, one has to impose that Q is nilpotent. Particularly, take the function $\phi = c^A$ (i.e. ϕ extracts the A^{th} coordinate of each point $c \in \Pi L$); then

$$\begin{aligned} Q^2 c^A &= (-1)^{\bar{E}} f^D_{EF}\, c^E c^F\, \partial_D \left((-1)^{\bar{B}} f^A_{BC}\, c^B c^C \right) \\ &= (-1)^{\bar{F}} f^D_{EF} \underbrace{\left(f^A_{DB} - (-1)^{\bar{D}\bar{B}} f^A_{BD} \right)}_{2 f^A_{DB}} c^E c^F c^B, \end{aligned}$$

thus

$$Q^2 c^A = 2\epsilon^A_{EFB}\, c^E c^F c^B, \qquad \text{where} \quad \epsilon^A_{EFB} \equiv (-1)^{\bar{F}} f^D_{EF} f^A_{DB}.$$

The nilpotency of Q dictates that the above function must be zero *everywhere* (i.e. at all points of ΠL); in particular,

$$\begin{aligned} 0 = 3 \times 0 &= \epsilon^A_{EFB} c^E c^F c^B + \epsilon^A_{FBE} c^F c^B c^E + \epsilon^A_{BEF} c^B c^E c^F \\ &= \left(\epsilon^A_{EFB} + (-1)^{(\bar{E}+1)(\bar{B}+\bar{F})} \epsilon^A_{FBE} + (-1)^{(\bar{B}+1)(\bar{E}+\bar{F})} \epsilon^A_{BEF} \right) c^E c^F c^B. \end{aligned}$$

[10]The same conclusion arises if one sees that

$$Qc^A = g^A_{BC}\, c^B c^C = (-1)^{(\bar{B}+1)(\bar{C}+1)} g^A_{BC}\, c^C c^B = g^A_{CB}\, c^C c^B,$$

where the second equality holds due to $c^B c^C = (-1)^{(\bar{B}+1)(\bar{C}+1)} c^C c^B$, while the third equality is simply a relabelling of indices from the first equality. From that, one concludes that it is possible to assume, without loss of generality, that $g^A_{CB} = -(-1)^{\bar{B}\bar{C}}(-1)^{\bar{B}+\bar{C}} g^A_{BC}$. This, in turn, implies that (4.20) becomes (4.22).

Being zero everywhere, the term in parenthesis must vanish. This amounts to

$$\begin{aligned}0 &= (-1)^{\bar{F}} f^D_{EF} f^A_{DB} + (-1)^{(\bar{E}+1)(\bar{B}+\bar{F})}(-1)^{\bar{B}} f^D_{FB} f^A_{DE} + (-1)^{(\bar{B}+1)(\bar{E}+\bar{F})}(-1)^{\bar{E}} f^D_{BE} f^A_{DF} \\ &= (-1)^{\bar{F}} \Big(f^D_{EF} f^A_{DB} - f^D_{FB} f^A_{ED} - (-1)^{\bar{B}\bar{F}} f^D_{EB} f^A_{DF} \Big),\end{aligned}$$

which is the condition (4.11) on structure constants induced by the graded Jacobi identity.

Therefore, the odd quadratic nilpotent vector field $Q \in \mathrm{Vect}(\Pi L)$ in (4.23) induces a Lie superbracket on L, since (4.21) can be rewritten as

$$f^A_{BC}\, \xi^B \eta^C\, e_A = \zeta = [\xi, \eta] = \xi^B \eta^C\, [e_B, e_C],$$

from which one defines the Lie superbracket on the basis $\mathcal{B} = \{e_A\}_A$ of L as

$$[e_B, e_C] = f^A_{BC}\, e_A.$$

It is trivial to see that everything done above is reversible, i.e. the Lie superalgebra structure on L defines an odd quadratic nilpotent vector field on ΠL. □

Obviously, one can prove Proposition 2 in a coordinate-free way. Let me do that for completion.

Proof (bis). Start by defining a bilinear map

$$\mu\colon L \times L \longrightarrow L \tag{4.24}$$

such that

$$\iota_{\mu(\xi,\eta)} = \big[[\iota_\xi, Q], \iota_\eta\big].$$

Note that the RHS above is a constant vector field on ΠL. Indeed, since ι_ξ and ι_η are constant vector fields, and Q is a quadratic vector field, the above has to be constant (one can convince oneself by looking at the coordinatized version presented earlier). Being a constant vector field, there must exist $\zeta \in L$ for which the RHS equals to ι_ζ. This shows that μ is well-defined. The proof of Proposition 2 is done by showing that μ is a Lie superbracket on L.

For the graded anti-commutativity (4.5), one must show that

$$\mu(\xi, \eta) \stackrel{?}{=} -(-1)^{(\deg \xi)(\det \eta)}\, \mu(\eta, \xi). \tag{4.25}$$

Using the properties (4.5) and (4.6) of the Lie superbracket on $\mathrm{Vect}(\Pi L)$ one gets

$$\begin{aligned}\iota_{\mu(\xi,\eta)} = \big[[\iota_\xi, Q], \iota_\eta\big] &\overset{(4.6)}{=} \big[\iota_\xi, [Q, \iota_\eta]\big] - (-1)^{(\deg \iota_\xi)(\deg Q)} \big[Q, \cancelto{0}{[\iota_\xi, \iota_\eta]}\big] \\ &\overset{(4.6)}{=} -(-1)^{(\deg Q)\deg(\iota_\eta)} \big[\iota_\xi, [\iota_\eta, Q]\big] \\ &\overset{(4.5)}{=} +(-1)^{(\deg Q)(\deg \iota_\eta)}(-1)^{(\deg \iota_\xi)(\deg [\iota_\eta, Q])} \big[[\iota_\eta, Q], \iota_\xi\big];\end{aligned}$$

the crossed Lie superbracket vanishes because each entry is a constant vector field.

Since Q is an odd vector field on ΠL, its degree is 1; furthermore, the degree of ι_ξ, as a vector field on ΠL, is $\deg(\xi) + 1$ — due to the flipping statistics. Then

$$\iota_{\mu(\xi,\eta)} = (-1)^{1+(\deg \xi)(\deg \eta)} \, \iota_{\mu(\eta,\xi)},$$

which proves that (4.25) is true.

Let $\zeta_i := \mu(\zeta_i, \eta_i)$ for $i \in \{1, 2, 3\}$. The graded Jacobi identity (4.6) reads as

$$\mu\big(\zeta_1, \mu(\zeta_2, \zeta_3)\big) - \mu\big(\mu(\zeta_1, \zeta_2), \zeta_3\big) \overset{?}{=} (-1)^{(\deg \zeta_1)(\deg \zeta_2)} \mu\big(\zeta_2, \mu(\zeta_1, \zeta_3)\big); \tag{4.26}$$

apart from the sign factor, each term corresponds to the respective vector field:

$$\iota_{\mu(\zeta_1,\mu(\zeta_2,\zeta_3))} = \Big[[\iota_{\zeta_1}, Q], \big[[\iota_{\zeta_2}, Q], \iota_{\zeta_3}\big]\Big], \tag{4.27a}$$

$$\iota_{\mu(\mu(\zeta_1,\zeta_2),\zeta_3)} = \Big[\big[\big[[\iota_{\zeta_1}, Q], \iota_{\zeta_2}\big], Q\big], \iota_{\zeta_3}\Big], \tag{4.27b}$$

$$\iota_{\mu(\zeta_2,\mu(\zeta_1,\zeta_3))} = \Big[[\iota_{\zeta_2}, Q], \big[[\iota_{\zeta_1}, Q], \iota_{\zeta_3}\big]\Big]. \tag{4.27c}$$

Making $A = [\iota_{\zeta_1}, Q]$, $B = [\iota_{\zeta_2}, Q]$ and $C = \iota_{\zeta_3}$, one can use the graded Jacobi identity (4.6) on $\mathrm{Vect}(\Pi L)$ to rewrite (4.27a) as

$$\iota_{\mu(\zeta_1,\mu(\zeta_2,\zeta_3))} = \Big[\big[[\iota_{\zeta_1}, Q], [\iota_{\zeta_2}, Q]\big], \iota_{\zeta_3}\Big] + (-1)^{(\deg \zeta_1)(\deg \zeta_2)} \Big[[\iota_{\zeta_2}, Q], \big[[\iota_{\zeta_1}, Q], \iota_{\zeta_3}\big]\Big].$$

Thus, the LHS of (4.26) corresponds to the vector field

$$\begin{aligned}\iota_{\mu(\zeta_1,\mu(\zeta_2,\zeta_3))} - \iota_{\mu(\mu(\zeta_1,\zeta_2),\zeta_3)} = (-1)^{(\deg \zeta_1)(\deg \zeta_2)} &\Big[[\iota_{\zeta_2}, Q], \big[[\iota_{\zeta_1}, Q], \iota_{\zeta_3}\big]\Big] + \\ &+ \Big[\Big(\big[[\iota_{\zeta_1}, Q], [\iota_{\zeta_2}, Q]\big] - \big[\big[[\iota_{\zeta_1}, Q], \iota_{\zeta_2}\big], Q\big]\Big), \iota_{\zeta_3}\Big];\end{aligned}$$

the first line of the RHS above is the desired result (compare it with (4.27c)); hence one has to show that the second line of the RHS above vanishes.

Indeed, it can be simplified with the graded Jacobi identity (4.6) by choosing $A = [\iota_{\zeta_1}, Q]$, $B = \iota_{\zeta_2}$ and $C = Q$, which yields

$$(-1)^{(\deg \zeta_1)(1+\deg \zeta_2)} \Big[\big[\iota_{\zeta_2}, \cancelto{0}{[[\iota_{\zeta_1}, Q], Q]}\big], \iota_{\zeta_3} \Big];$$

the fact that the crossed term is zero can be seen by noticing that, for any (super)vector field V, one has

$$\begin{aligned}
[[V,Q],Q] &\overset{(4.6)}{=} -(-1)^{(\deg V)} [Q,[V,Q]] + [V, \cancelto{0}{[Q,Q]}] \\
&\overset{(4.6)}{=} +(-1)^{(\deg V)} (-1)^{(\deg [V,Q])} [[V,Q],Q] = -[[V,Q],Q].
\end{aligned}$$

Consequently, (4.26) is true. This shows that μ is a Lie superbracket on L. □

It is worth mentioning that the differential Q used in this Section is precisely the BRST differential, now in the context of Lie superalgebras. What was done in Section 3.2 becomes much more transparent under this light: an ordinary Lie algebra, as previously discussed, is a Lie superalgebra with trivial fermionic part; consequently, the odd quadratic nilpotent differential on $\Pi\mathfrak{g}$ only has fermionic ghosts — which are precisely the anti-commuting variables defined in (3.5).

Having Example 4.4 in mind, the Lie superalgebra structure on $\mathfrak{gl}(m|n)$ — encoded in the structure constants (4.13) — defines an odd quadratic nilpotent vector field on $\Pi\mathfrak{gl}(m|n)$ which, with respect to the canonical basis, takes the form

$$\begin{aligned}
Q = &\left(-a^i{}_k a^k{}_j + \alpha^i{}_K \beta^K{}_j\right) \frac{\partial}{\partial a^i{}_j} + \left(-b^I{}_K b^K{}_J + \beta^I{}_k \alpha^k{}_J\right) \frac{\partial}{\partial b^I{}_J} + \\
&+ \left(-a^i{}_k \alpha^k{}_J + \alpha^i{}_K b^K{}_J\right) \frac{\partial}{\partial \alpha^i{}_J} + \left(\beta^I{}_k a^k{}_j - b^I{}_K \beta^K{}_j\right) \frac{\partial}{\partial \beta^I{}_j}.
\end{aligned} \tag{4.28}$$

In matrix form, one can write

$$Qa = -a^2 + \alpha\beta, \tag{4.29a}$$

$$Qb = -b^2 + \beta\alpha, \tag{4.29b}$$

$$Q\alpha = -a\alpha + \alpha b, \tag{4.29c}$$

$$Q\beta = -b\beta + \beta a. \tag{4.29d}$$

4.3 The algebra of polynomials of ghosts

Given a Lie superalgebra $\mathfrak{g}$ of finite dimension, define its Chevalley–Eilenberg algebra $\mathrm{CE}(\mathfrak{g})$ as in (3.4), i.e. as the exterior superalgebra of the dual of $\mathfrak{g}$,

$$\mathrm{CE}^{\bullet}(\mathfrak{g}) := \Lambda^{\bullet}(\mathfrak{g}^*), \tag{4.30}$$

equipped with a differential that is the dual of the Lie superbracket, i.e.

$$d := [\cdot,\cdot]^* \colon \mathfrak{g}^* \longrightarrow \mathfrak{g}^* \wedge \mathfrak{g}^*, \tag{4.31}$$

extended to $\Lambda^{\bullet}(\mathfrak{g}^*)$ by the graded Leibniz rule (4.4). Note that $\mathrm{CE}(\mathfrak{g})$ is $(\mathbb{N}_0 \times \mathbb{Z}_2)$-graded. The $\mathbb{N}_0$-grading is cohomological in $\Lambda^{\bullet}(\cdot)$, while the $\mathbb{Z}_2$-grading is the one of $\mathfrak{g}^*$ inherited from $\mathfrak{g}$.

A k-cocycle on $\mathfrak{g}$ with coefficients in the underlying field $\mathbb{K}$ is an element of $\mathrm{CE}(\mathfrak{g})$, of degree $(k,0)$, in the kernel of d. The **Lie superalgebra cohomology** of $\mathfrak{g}$ with trivial coefficients is the cohomology of its Chevalley–Eilenberg algebra, i.e.

$$H^{\bullet}(\mathfrak{g}) \equiv H^{\bullet}(\mathfrak{g};\mathbb{K}) := H^{\bullet}\big(\mathrm{CE}(\mathfrak{g})\big).$$

The goal of this section is to show the following (ALEXANDROV *et al.*, 1997).

Proposition 3. *The algebra of polynomials on $\Pi\mathfrak{g}$ with the odd quadratic nilpotent vector field Q is isomorphic to the Chevalley–Eilenberg complex of $\mathfrak{g}$ with differential* (4.31).

Proof. As from Definition 4.2, the flipped statistics space $\Pi\mathfrak{g}$ is the one whose even part is $\mathfrak{g}_1$ and whose odd part is $\mathfrak{g}_0$. That is to say there exists an isomorphism $\psi \colon \Pi\mathfrak{g} \longrightarrow \mathfrak{g}$, of fermionic statistics, such that both

$$\begin{aligned} \psi_0 &\colon (\Pi\mathfrak{g})_0 \longrightarrow \mathfrak{g}_1, \\ \psi_1 &\colon (\Pi\mathfrak{g})_1 \longrightarrow \mathfrak{g}_0, \end{aligned}$$

are isomorphisms of linear spaces.

Let $\{e_A\}_A$ be a homogeneous basis of $\mathfrak{g}$, and $\{\xi^A\}_A$ be its dual basis on $\mathfrak{g}^*$. The linear functionals $\xi^A \colon \mathfrak{g} \longrightarrow \mathbb{K}$ are the coordinate functions on $\mathfrak{g}$, thus they are the generators of the algebra of polynomials of $\mathfrak{g}$. Define $c^A \equiv \phi(\xi^A)$; clearly $\{c^A\}_A$ is a homogeneous basis of the algebra of polynomials of $\Pi\mathfrak{g}$, which shall be denoted $\mathrm{Pol}^{\bullet}(\Pi\mathfrak{g})$. It is also $(\mathbb{N}_0 \times \mathbb{Z}_2)$-graded, with the $\mathbb{Z}_2$-grading inherited from $\Pi\mathfrak{g}$, and the $\mathbb{N}_0$-grading being the power of each monomial.

Consider a $(\mathbb{N}_0 \times \mathbb{Z}_2)$-homomorphism $\Psi \colon \Lambda^\bullet(\mathfrak{g}^*) \longrightarrow \mathrm{Pol}^\bullet(\Pi\mathfrak{g})$ that acts as the identity on $\Lambda^0(\mathfrak{g}^*) = \mathbb{K}$, and acts as ψ on $\Lambda^1(\mathfrak{g}^*)$, i.e.

$$\Psi(x^A) = c^A.$$

To define Ψ on the rest of $\Lambda^\bullet(\mathfrak{g}^*)$, one has to recall that its product obeys

$$\xi^A \wedge \xi^B = -(-1)^{\bar{A}\bar{B}}\, \xi^B \wedge \xi^A; \tag{4.32}$$

indeed, following the Latin/Greek index convention of Example 4.4,

$$\begin{aligned}
\xi^a, \xi^b \in (\mathfrak{g}^*)_0 = (\mathfrak{g}_0)^* \qquad &\Longrightarrow \qquad \xi^a \wedge \xi^b = -\xi^b \wedge \xi^a;\\
\xi^\alpha, \xi^\beta \in (\mathfrak{g}^*)_1 = (\mathfrak{g}_1)^* \qquad &\Longrightarrow \qquad \xi^\alpha \wedge \xi^\beta = +\xi^\beta \wedge \xi^\alpha;\\
\xi^a \in (\mathfrak{g}_0)^*, \xi^\alpha \in (\mathfrak{g}_1)^* \qquad &\Longrightarrow \qquad \xi^a \wedge \xi^\alpha = -\xi^\alpha \wedge \xi^a.
\end{aligned}$$

However, the product of monomials in $\Pi\mathfrak{g}$ obeys

$$c^A c^B = (-1)^{(\bar{A}+1)(\bar{B}+1)}\, c^B c^A, \tag{4.33}$$

or, explicitly,

$$\begin{aligned}
c^a, c^b \in (\Pi\mathfrak{g})_0 \cong \mathfrak{g}_1 \qquad &\Longrightarrow \qquad c^a c^b = -c^b c^a;\\
c^\alpha, c^\beta \in (\Pi\mathfrak{g})_1 \cong \mathfrak{g}_0 \qquad &\Longrightarrow \qquad c^\alpha c^\beta = +c^\beta c^\alpha;\\
c^a \in (\Pi\mathfrak{g})_0, c^\alpha \in (\Pi\mathfrak{g})_1 \qquad &\Longrightarrow \qquad c^a c^\alpha = +c^\alpha c^a.
\end{aligned}$$

As one can see, there is a flipped sign in the fermion-boson case, which should somehow be accommodated by Ψ. For that reason, define Ψ to act on $\Lambda^2(\mathfrak{g}^*)$ as

$$\Phi(\xi^A \wedge \xi^B) = (-1)^{\bar{A}}\, c^A c^B. \tag{4.34}$$

Then,

$$\begin{aligned}
\Phi(\xi^B \wedge \xi^A) &\overset{(4.34)}{=} (-1)^{\bar{B}} c^B c^A\\
&\overset{(4.33)}{=} (-1)^{\bar{B}}(-1)^{(\bar{A}+1)(\bar{B}+1)} c^A c^B = -(-1)^{\bar{A}\bar{B}}(-1)^{\bar{A}} c^A c^B\\
&\qquad\qquad\qquad\qquad\qquad\qquad\quad = \Psi(-(-1)^{\bar{A}\bar{B}} \xi^A \wedge \xi^B),
\end{aligned}$$

which shows that Ψ preserves the graded products of the algebras.

As discussed at the start, Ψ sends the generators of one algebra into the generators of the other, thus it is bijective; it is clearly linear and preserves the product, therefore it is an isomorphism of algebras.

Aditionally, one has to check that the nilpotent operators in each superalgebra are preserved by Ψ, i.e. that the diagram below commutes:

$$\begin{tikzcd}
\mathrm{CE}^k(\mathfrak{g}) \arrow[r, "d"] \arrow[d, "\Psi"'] & \mathrm{CE}^{k+1}(\mathfrak{g}) \arrow[d, "\Psi"] \\
\mathrm{Pol}^k(\Pi\mathfrak{g}) \arrow[r, "Q"'] & \mathrm{Pol}^{k+1}(\Pi\mathfrak{g}).
\end{tikzcd}$$

On the basis $\{\xi^A\}_A$ of $\mathfrak{g}^*$, the differential (4.31) acts as

$$d\xi^A = f^A_{BC}\, \xi^B \wedge \xi^C,$$

where f^A_{BC} are the structure constants on $\mathfrak{g}$. Then, according to (4.23),

$$\Psi(d\xi^A) = f^A_{BC}\, \Phi(\xi^B \wedge \xi^C) = (-1)^{\bar{B}} f^A_{BC} c^B c^C.$$

On the other hand,

$$Q\,\Phi(\xi^A) = Qc^A = (-1)^{\bar{B}} f^A_{BC}\, c^B c^C.$$

Both d and Ψ are extended to $\Lambda^\bullet(\mathfrak{g}^*)$ by the graded Leibniz rule (4.4), therefore $\Psi \circ d = Q \circ \Psi$. □

This concludes that Ψ is an isomorphism between the Chevalley–Eilenberg cochain complex $\mathrm{CE}^\bullet(\mathfrak{g})$ and the complex of polynomials $\mathrm{Pol}^\bullet(\Pi\mathfrak{g})$ equipped with the odd nilpotent vector field Q. Therefore, their cohomology is the same:

$$H^\bullet\big(\mathrm{CE}(\mathfrak{g})\big) = H^\bullet\big(\mathrm{Pol}(\Pi\mathfrak{g})\big). \tag{4.35}$$

4.4 Cohomology of Lie superalgebras

Finally, one faces the challenge of explicitly calculating cohomologies of Lie superalgebras. Regarding the Hochschild–Serre spectral sequence presented in Section 2.4, the spectral sequence of a Lie superalgebra will also be $\mathbb{Z}_2$-graded (FUKS, 1986).

Note, however, that for any Lie superalgebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, one can pick the subalgebra $\mathfrak{h} = \mathfrak{g}_0$ of $\mathfrak{g}$, and realising that $\mathfrak{g}/\mathfrak{h} \cong \mathfrak{g}_1$. This is the most used case when dealing with the Hochschild–Serre spectral sequence for Lie superalgebras. Thus, one knows that there exists a spectral sequence with

$$\begin{aligned} E_1^{p,q} &= H^q\big(\mathfrak{g}_0; S^p(\mathfrak{g}_1{}^*)\big), \\ E_2^{p,q} &= H^p\big(\mathfrak{g}_1; H^q(\mathfrak{g}_0)\big). \end{aligned}$$

Hereafter, fix[11]

$$C \equiv \mathrm{Pol}(\Pi\mathfrak{g})$$

and introduce the filtration of Section 3.4 on it, viz.

$$F^pC := C \cap \mathcal{O}^p\big((\Pi\mathfrak{g})_0\big), \tag{4.36}$$

i.e. the subspace F^pC is spanned by all the monomials with *at least* p bosonic ghosts. For example, if γ is a coordinate function on $(\Pi\mathfrak{g})_0$, then $\gamma\gamma$ is an element of F^0C, F^1C and F^2C, but not an element of any F^pC for $p > 2$.

The filtration (4.36) is decreasing, with $F^0C = C$ and $F^{+\infty}C = 0$. Furthermore, the odd quadratic nilpotent vector field (4.23) preserves the filtration, as it either keeps the number of bosonic ghosts, as in Qc^α in (4.16), or increases it, as in Qc^a.

Let the 0^{th} page of the spectral sequence be as in (2.15), viz.

$$E_0^{p,q} = \frac{F^pC^{p+q}}{F^{p+1}C^{p+q}};$$

the elements of $E_0^{p,q}$ are the $(p+q)$-cochains with *exactly* p bosonic ghosts. The d_0 map makes the diagram (2.16) commutative, with the differential d replaced by Q.

For a more concrete calculation, set $\mathfrak{g} = \mathfrak{gl}(m|n)$ over the field $\mathbb{C}$. The goal of the rest of this section is to explain the following theorem.

Theorem 9 (Fuks)**.** *If $m \geqslant n$, then the natural inclusion*

$$\mathfrak{gl}(m) \hookrightarrow \mathfrak{gl}(m|n)_0 \subseteq \mathfrak{gl}(m|n)$$

induces an isomorphism in cohomology with trivial coefficients.

[11] Recall (2.12), which established the relation between a cochain complex and a graded module.

The proof of Theorem 9 found in FUKS (1986) is done using the Hochschild–Serre spectral sequence adapted to Lie superalgebras. In this thesis, I shall redo the proof by constructing the Hochschild–Serre spectral sequence from scratch and using the BRST approach to cohomology, by means of Propositions 2 and 3.

The decomposition (4.12) has two fermionic and two bosonic ghosts, viz.

$$
\begin{aligned}
(a, b) &\in \mathfrak{gl}(m|n)_0 = \mathfrak{gl}(m) \oplus \mathfrak{gl}(n), \\
(\beta, \alpha) &\in \mathfrak{gl}(m|n)_1 = \mathrm{Hom}(\mathbb{C}^m, \mathbb{C}^n) \oplus \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^m),
\end{aligned}
$$

respectively. To compute the cohomology of $\mathfrak{g}$, one has to consider cochains — or, by Proposition 3, ghost polynomials — that are index-free; thus, one needs to take traces of matrices.[12] Note that, independently of the dimensions, the bosonic ghosts α and β have to always come in pairs, e.g. $\mathrm{tr}(\alpha\beta)$, $\mathrm{tr}(\beta a \alpha)$, $\mathrm{tr}(\alpha b \beta)$ etc.

4.4.1 The general linear Lie superalgebra $\mathfrak{gl}(2|2)$

Let me start with a simple example: set $\mathfrak{g} = \mathfrak{gl}(2|2)$. This is the first interesting example, as the decompositions of $\mathfrak{gl}(1|1)$ and $\mathfrak{gl}(2|1)$ can be misleading, as they contain 1×1 matrices. A general element of $\mathfrak{gl}(2|2)$ has the form (4.12), with a, b, α and β are all 2×2 matrices.

The 0th page Since a is a 2×2 matrix, there are 4 fermionic ghosts $a^i{}_j$. Consequently, polynomials with 5 or more a-ghosts vanish trivially, because they contain at least one pair of the same ghost. Following the notation of FUKS (1986), the generators of the $q = 0$ column are the fermionic ghosts

$$
\varphi_1' \equiv \mathrm{tr}(a), \qquad \varphi_1'' \equiv \mathrm{tr}(b), \qquad \varphi_2' \equiv \mathrm{tr}(a^3), \qquad \varphi_2'' \equiv \mathrm{tr}(b^3). \tag{4.37}
$$

Being fermionic, one cannot repeat any of them in a monomial. Thus the non-zero terms in the $E_0^{p,0}$ column of the Hochschild–Serre spectral sequence of $\mathfrak{gl}(2|2)$ are[13]

$$
\begin{aligned}
E_0^{0,0} &= \mathbb{C}, \\
E_0^{1,0} &= \langle \mathrm{tr}(a), \mathrm{tr}(b) \rangle && \cong \mathbb{C}^2, \\
E_0^{2,0} &= \langle \mathrm{tr}(a)\,\mathrm{tr}(b) \rangle && \cong \mathbb{C}^1,
\end{aligned}
$$

[12] A quick explanation for wanting index-free polynomials is that the cohomology should be independent of a choice of basis, and the trace is basis-independent.

[13] The angle brackets $\langle \ldots \rangle$ mean *linear span* over $\mathbb{C}$ (implicitly assumed over $\mathbb{C}$).

$$\begin{aligned}
E_0^{3,0} &= \langle \operatorname{tr}(a^3), \operatorname{tr}(b^3) \rangle && \cong \mathbb{C}^2,\\
E_0^{4,0} &= \langle \operatorname{tr}(a^3)\operatorname{tr}(a), \operatorname{tr}(b^3)\operatorname{tr}(b), \operatorname{tr}(a^3)\operatorname{tr}(b), \operatorname{tr}(b^3)\operatorname{tr}(a) \rangle && \cong \mathbb{C}^4,\\
E_0^{5,0} &= \langle \operatorname{tr}(a^3)\operatorname{tr}(a)\operatorname{tr}(b), \operatorname{tr}(b^3)\operatorname{tr}(a)\operatorname{tr}(b) \rangle && \cong \mathbb{C}^2,\\
E_0^{6,0} &= \langle \operatorname{tr}(a^3)\operatorname{tr}(b^3) \rangle && \cong \mathbb{C}^1,\\
E_0^{7,0} &= \langle \operatorname{tr}(a^3)\operatorname{tr}(b^3)\operatorname{tr}(a), \operatorname{tr}(a^3)\operatorname{tr}(b^3)\operatorname{tr}(b) \rangle && \cong \mathbb{C}^2,\\
E_0^{8,0} &= \langle \operatorname{tr}(a^3)\operatorname{tr}(b^3)\operatorname{tr}(a)\operatorname{tr}(b) \rangle && \cong \mathbb{C}^1.
\end{aligned}$$

To have an index-free monomial with bosonic ghosts α and β, one has to put them in pairs to be able to take a trace. Thus, the columns with p odd are filled with trivial spaces. In the $p = 0$ row, the first non-trivial spaces are

$$E_0^{0,0} = \mathbb{C}, \qquad E_0^{0,2} = \langle \operatorname{tr}(\alpha\beta) \rangle, \qquad E_0^{0,4} = \langle \operatorname{tr}(\alpha\beta)^2, [\operatorname{tr}(\alpha\beta)]^2 \rangle, \qquad \ldots$$

There are, however, finitely many generators for the $E_0^{0,q}$ row — one for each non-trivial space up to $\ell := \min\{m, n\}$. One possible explanation can be found in FUKS (1986) by realizing that the ring of invariant polynomials of $\mathfrak{g}_1 \cong (\Pi\mathfrak{g})_0$ is generated by those polynomials. Another way of convincing oneself is by virtue of the Cayley–Hamilton theorem, presented below.

The Cayley–Hamilton theorem For a matrix $N \in \mathfrak{gl}(n)$, define its *characteristic polynomial* as

$$\pi_N(x) := \det(Ix - N),$$

where I is the identity matrix. The Cayley–Hamilton theorem states that N annihilates its own characteristic polynomial π_N. For $n = 2$, one has

$$\pi_N(x) = x^2 - \operatorname{tr}(N)\, x + \frac{1}{2}\Big([\operatorname{tr}(N)]^2 - \operatorname{tr}(N^2) \Big),$$

thus

$$\pi_N(N) = N^2 - \operatorname{tr}(N)\, N + \frac{1}{2}\Big([\operatorname{tr}(N)]^2 - \operatorname{tr}(N^2) \Big)\, I \overset{!}{=} 0;$$

multiplying by N and taking the trace, one has

$$\operatorname{tr}(N^3) = \frac{3}{2}\operatorname{tr}(N)\operatorname{tr}(N^2) - \frac{1}{2}[\operatorname{tr}(N)]^3,$$

therefore the trace of N^3 can be written in terms of $\operatorname{tr}(N)$ and $\operatorname{tr}(N^2)$. This can be done inductively for each power greater than n.

Applying that to $\mathrm{tr}(\alpha\beta)$ or $\mathrm{tr}(\beta\alpha)$ — depending whether $m < n$ or $n < m$, respectively —, one sees that higher powers can be decomposed by the Cayley–Hamilton theorem. Therefore, if $\ell := \min\{m, n\}$, the generators of the $E_0^{0,q}$ row of the Hochschild–Serre spectral sequence of $\mathfrak{gl}(m|n)$ are

$$\eta_1 \equiv \mathrm{tr}(\alpha\beta) \in E_0^{0,2}, \qquad \eta_2 \equiv \mathrm{tr}(\alpha\beta)^2 \in E_0^{0,4}, \qquad \ldots, \qquad \eta_\ell \equiv \mathrm{tr}(\alpha\beta)^\ell \in E_0^{0,2\ell}. \tag{4.38}$$

The 1st page Other spaces at the 0th page are more complicated. For example, the space $E_0^{1,2}$ contains both $\mathrm{tr}(a)\,\mathrm{tr}(\alpha\beta)$ and $\mathrm{tr}(b)\,\mathrm{tr}(\alpha\beta)$ — which can be seen as elements of the tensor product of $E_0^{1,0}$ and $E_0^{0,2}$ —, but it also contains $\mathrm{tr}(\beta a\alpha)$.

The map on the 0th page is induced by the quotient map $F^pC^{p+q} \twoheadrightarrow E_0^{p,q}$, which keeps only terms with exactly p bosonic ghosts. Thus, looking at (4.29), one concludes that d_0 acting on 1st order monomials is

$$d_0 a = -a^2, \qquad d_0 b = -b^2, \qquad d_0\alpha = Q\alpha, \qquad d_0\beta = Q\beta. \tag{4.39}$$

Note that every polynomial in the $q = 0$ column is in the kernel of d_0, because d_0 increases the number of a (or b) by 1, and the trace of an even power of a (or b) vanishes. Therefore,

$$E_1^{p,0} \cong E_0^{p,0}.$$

Similarly, note that

$$d_0\,\mathrm{tr}(\alpha\beta) = \mathrm{tr}\Big((d_0\alpha)\beta + \alpha(d_0\beta)\Big) = \mathrm{tr}\big(-a\alpha\beta + \alpha\beta a\big) = 0;$$

the analogue happens to all other elements of the row $p = 0$, thus

$$E_1^{0,q} \cong E_0^{0,q}.$$

Now take $\mathrm{tr}(\alpha a\beta) \in E_0^{1,2}$ mentioned before. It is not the kernel of d_0, because

$$d_0\,\mathrm{tr}(\beta a\alpha) = \mathrm{tr}\Big((d_0\beta)a\alpha + \beta(d_0 a)\alpha - \beta a(d_0\alpha)\Big) = \mathrm{tr}(\beta a^2\alpha).$$

More generally, all these mixed polynomials — i.e. that are not in $E_0^{p,0} \otimes E_0^{0,q} \subset E_0^{p,q}$ — are not in the kernel of d_0. Therefore,

$$E_1^{p,q} \cong E_0^{0,q} \otimes E_0^{p,0}. \tag{4.40}$$

As mentioned above, FUKS (1986) shows that $E_0^{p,0}$ is the ring of invariant cochains in $\mathfrak{g}_1$; the space $E_0^{0,q}$, on the other hand, is precisely the q^{th} cohomology of $\mathfrak{g}_0$. This is easy to see, since d_0 acting on fermionic ghosts is precisely the BRST differential of $\mathfrak{g}_0 = \mathfrak{gl}(m) \oplus \mathfrak{gl}(n)$, as one can see by comparing (3.9) and (4.39).

Note that one does not need to know what happens to all possible elements in the 1st page, but only with the generators in (4.37) and (4.38) — as all other cochains/polynomials are products of generators, and the differentials obey the graded Leibniz rule.

The 2nd page Back to (2.19), the differential on the 1st page is of degree $(1, 0)$. Since the 1st page is filled with 0s in the columns with odd p, every differential d_1 is trivial — either starting or ending at a trivial space. Consequently

$$E_2^{p,q} \cong E_1^{p,q}. \tag{4.41}$$

Pictorially, the dimensions of each space at the 2nd are displayed below; the spaces where the generators lie are depicted in red.

$$\begin{array}{c|cccccc}
q & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\
\mathbf{8} & 1 & 0 & 1 & 0 & 2 & \cdots \\
\mathbf{7} & 2 & 0 & 2 & 0 & 4 & \cdots \\
\mathbf{6} & 1 & 0 & 1 & 0 & 2 & \cdots \\
\mathbf{5} & 2 & 0 & 2 & 0 & 4 & \cdots \\
\mathbf{4} & 4 & 0 & 4 & 0 & 8 & \cdots \\
\mathbf{3} & {\color{red}2} & 0 & 2 & 0 & 4 & \cdots \\
\mathbf{2} & 1 & 0 & 1 & 0 & 2 & \cdots \\
\mathbf{1} & {\color{red}2} & 0 & 2 & 0 & 4 & \cdots \\
\mathbf{0} & 1 & 0 & {\color{red}1} & 0 & {\color{red}2} & \cdots \\
\hline
E_2 & \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} & p
\end{array}$$

The 3rd page The differential on the 2nd page is defined as $j^*(i^*)^{-1}\delta$, where i^* and j^* are the maps induced in cohomology by the inclusion map $i\colon F^{p+1}C^k \hookrightarrow F^pC^k$ and the quotient map $j\colon F^pC^k \twoheadrightarrow E_0^{p,k-p}$, respectively. The map δ is the connecting homomorphism from Theorem 1, which is syntactically given by

$$\delta\colon [y] \longmapsto [i^{-1}\,Q\,j^{-1}y].$$

Take $\varphi_1' = \operatorname{tr}(a) \in E_2^{0,1} \cong E_0^{0,1}$.[14] A cochain $j^{-1}\varphi_1'$ is one in F^0C^1 such that, when quotiented by F^1C^1, yields φ_1'. Since any choice is good (because δ is well-defined), one can take φ_1' itself. Then

$$Q\varphi_1' = Q\operatorname{tr}(a) = \operatorname{tr}(-a^2 + \alpha\beta) = \operatorname{tr}(\alpha\beta) = \eta_1 \in F^1C^2 \subset F^0C^2.$$

Next, $i^{-1}\eta_1$ is an element of F^0C^2 which becomes η_1 upon inclusion in F^1C^2; that element can be taken to be precisely η_1. Hence, $\delta[\varphi_1'] = [\eta_1] \in H^1(F^1C^2)$. Finally, applying $(i^*)^{-1}$ and j^*, one obtains the same thing (up to quotients of quotients etc.). Therefore

$$d_2\varphi_1' = \eta_1. \tag{4.42}$$

The result (4.42) is an example of the "transgression" that FUKS (1986) talks about. The exact same thing happens to $\varphi_2'' = \operatorname{tr}(b) \in E_0^{0,1}$. Consequently, one has

$$\operatorname{tr}(a) - \operatorname{tr}(b) \in \ker d_2^{0,1},$$

which in turn yields

$$E_3^{0,1} = \big\langle \operatorname{tr}(a) - \operatorname{tr}(b) \big\rangle.$$

Since d_2 is the transgressive differential at the spot $(0,1)$, one concludes that

$$E_\infty^{0,1} \cong \mathbb{C}. \tag{4.43}$$

For $\varphi_2' = \operatorname{tr}(a^3) \in E_0^{0,3}$, a very similar argument can be used by ignoring the i^{-1} and j^{-1} maps (due to the well-definedness of the connecting homomorphism). Then

$$Q\varphi_2' = \operatorname{tr}(\alpha\beta a^2) \in F^2C^4 \subset F^0C^4,$$

consequently

$$\delta\varphi_2' = \big[\operatorname{tr}(\alpha\beta a^2)\big] \in H(F^1C^4).$$

Now, when acting with j^* on $\delta\varphi_2'$, recall that j^* is the map induced in cohomology by the surjection $j\colon F^1C^4 \twoheadrightarrow F^1C^4/F^2C^4$ — i.e., the map that "kills" all terms with

[14] Recall that higher pages are the result of taking cohomologies of the previous ones. While $\operatorname{tr}(a)$ is a cochain in F^0C^1, it is a representative of an equivalence class in $E_0^{0,1}$ (equivalence up to contributions of higher orders in the bosonic ghosts), which then becomes a cohomology class in $E_1^{0,1}$, which again becomes a cohomology class in $E_2^{0,1}$. To avoid these bunch of "cohomology class of" and an overdense notation, let me make the identifications of higher pages with the 0^{th} one.

2 or more bosonic ghosts. Hence,

$$d_2\varphi_2' = j^*\,\delta\varphi_2' = 0. \tag{4.44}$$

The same thing will happen to $\varphi_2'' = \mathrm{tr}(b^3)$, therefore

$$E_3^{0,3} = \ker d_2^{0,3} = \langle \mathrm{tr}(a^3), \mathrm{tr}(b^3) \rangle. \tag{4.45}$$

It remains to see what happens to the bosonic generators (4.38) under d_2. A seen in (4.42), the generators $\varphi_1', \varphi_1'' \in E_1^{1,0}$ transgressed to η_1, therefore $\eta_1 \in \mathrm{im}\, d_2^{1,0}$. This means that $E_3^{0,2} = 0$; after that, the spot $(0,2)$ stabilizes, since all other incoming and outgoing differentials will be trivial. Therefore

$$E_\infty^{0,2} = 0. \tag{4.46}$$

Lastly, the generator $\eta_2 \equiv \mathrm{tr}(\alpha\beta)^2 \in E_0^{4,0}$ is mapped to 0 by Q, but it is not in the image of $d_2^{2,1}$, thus

$$E_3^{4,0} = \langle \mathrm{tr}(\alpha\beta)^2 \rangle. \tag{4.47}$$

Knowing that d_2 acts on the generators as (4.42), (4.44) and $d_2 \colon \eta_i \longmapsto 0$, one can deduce the image under d_2 of any other polynomial. E.g.

$$\begin{aligned}
E_0^{0,2} &\ni \quad \mathrm{tr}(b)\,\mathrm{tr}(a) \longmapsto \mathrm{tr}(\alpha\beta)\,\big(\mathrm{tr}(a) - \mathrm{tr}(b)\big),\\
E_0^{0,4} &\ni \quad \mathrm{tr}(a)\,\mathrm{tr}(a^3) \longmapsto \mathrm{tr}(\alpha\beta)\ \mathrm{tr}(a^3),\\
E_0^{2,1} &\ni \quad \mathrm{tr}(a)\,\mathrm{tr}(\alpha\beta) \longmapsto [\mathrm{tr}(\alpha\beta)]^2,\\
E_0^{2,3} &\ni \mathrm{tr}(a^3)\,\mathrm{tr}(\alpha\beta) \longmapsto 0.
\end{aligned}$$

Note, for example, that the space $E_2^{0,2}$, which is spanned by the cohomology class of $\mathrm{tr}(b)\,\mathrm{tr}(a)$, is not in the kernel of d_2; hence $E_3^{0,2} = 0$. Consequently, the generators of the 3$^{\text{rd}}$ page are

$$\varphi_2' \equiv \mathrm{tr}(a^3), \qquad \varphi_2'' \equiv \mathrm{tr}(b^3) \tag{4.48a}$$

$$\eta_2 \equiv \mathrm{tr}(\alpha\beta)^2, \qquad \chi_1 \equiv \varphi_1' - \varphi_1'' \equiv \mathrm{tr}(a) - \mathrm{tr}(b). \tag{4.48b}$$

Schematically, the dimensions of the spaces in the 3$^{\text{rd}}$ page are shown in the following diagram, again with generators in red.

q						
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\iddots$
8	0	0	0	0	0	$\cdots$
7	1	0	0	0	1	$\cdots$
6	1	0	0	0	1	$\cdots$
5	0	0	0	0	0	$\cdots$
4	2	0	0	0	2	$\cdots$
3	2	0	0	0	2	$\cdots$
2	0	0	0	0	0	$\cdots$
1	1	0	0	0	1	$\cdots$
0	1	0	0	0	1	$\cdots$
E_3	**0**	**1**	**2**	**3**	**4**	p

The 4th page From (2.22), the differential on the 3rd page is of the type $d_3\colon E_3^{p,q} \longrightarrow E_3^{p+3,q-2}$. Since p and $p+3$ have different parity, all d_3 differentials are trivial — either by starting or ending at a trivial space. Therefore

$$E_4^{p,q} \cong E_3^{p,q}. \tag{4.49}$$

The 5th page For the generators $\varphi_2', \varphi_2'' \in E_4^{0,3}$, the differential d_4 is transgressive, sending them down to $E_4^{4,0}$. To show that both φ_2' and φ_2'' transgress to $\eta_2 \equiv \operatorname{tr}(\alpha\beta)^2$ is tricky. FUKS (1986) invokes Weyl algebras, while I try a more brute approach.

The differential on the 4th page is $j^*\,(i^*)^{-1}(i^*)^{-1}(i^*)^{-1}\,\delta$. Back to Section 2.4, it means going up the red dashed path

$$\begin{tikzcd}
 & H^4(F^4C) \arrow[r, "j^*"] \arrow[d, "i^*"] & E_1^{4,0} \\
 & H^4(F^3C) \arrow[d, "i^*"] & \\
 & H^4(F^2C) \arrow[d, "i^*"] & \\
E_1^{0,3} \arrow[r, "\delta"] \arrow[rruuu, dashed, bend left, "d_4"] & H^4(F^1C). &
\end{tikzcd}$$

The cohomology classes of $\operatorname{tr}(a^3)$ and $\operatorname{tr}(b^3)$ start at the bottom left, at $E_4^{0,3}$ (recall one needs to take several quotients of E_1 to get to E_4). As seen before,

$$\delta\big[\operatorname{tr}(a^3)\big] = \big[\operatorname{tr}(a^2\alpha\beta)\big],$$

and $\operatorname{tr}(a^2\alpha\beta) \in F^2C \subset F^1C$. One needs to find a cohomology class in $H^4(F^4C)$ which, after descending thrice by i^*, reaches the cohomology class of $\operatorname{tr}(a^2\alpha\beta)$ in $H^4(F^1C)$. The claim is that the class of $\operatorname{tr}(\alpha\beta)^2 \in F^4C$ is a valid choice.

Note that $Q\operatorname{tr}(\alpha\beta)^2 = Q\operatorname{tr}(a^2\alpha\beta) = 0$, thus these polynomials are cohomologous in F^1C. Furthermore, as one already knows, $E_4^{4,0}$ is a 1-dimensional space spanned by the cohomology class of $\operatorname{tr}(\alpha\beta)^2$ — therefore, it can be the only non-trivial choice for the $(i^*)^{-1}(i^*)^{-1}(i^*)^{-1}$ path.

Hence

$$d_4 \colon \varphi_2', \varphi_2'' \longmapsto \eta_2,$$

and $\operatorname{tr}(a^3) - \operatorname{tr}(b^3) \in \ker d_4$, while $\eta_2 \in \operatorname{im} d_4$. This implies that

$$\begin{aligned} E_\infty^{0,3} &\cong E_5^{0,3} = \big\langle \operatorname{tr}(a^3) - \operatorname{tr}(b^3)\big\rangle, \\ E_\infty^{4,0} &\cong E_5^{4,0} = 0. \end{aligned}$$

Consequently, the generators of the 5^{th} page are

$$\begin{aligned} \chi_1 &\equiv \varphi_1' - \varphi_1'' \equiv \operatorname{tr}(a) - \operatorname{tr}(b), \\ \chi_2 &\equiv \varphi_2' - \varphi_2'' \equiv \operatorname{tr}(a^3) - \operatorname{tr}(b^3). \end{aligned}$$

The ∞^{th} page Since the generators of the 5^{th} page already went through transgression, they are no longer affected by higher differentials. Thus one can conclude that the non-trivial spaces at the ∞^{th} page are

$$\begin{aligned} E_\infty^{0,0} &= \mathbb{C}, \\ E_\infty^{0,1} &\cong \mathbb{C}, \\ E_\infty^{0,3} &\cong \mathbb{C}, \\ E_\infty^{0,4} &\cong \mathbb{C}, \end{aligned}$$

where the $E_\infty^{0,4}$ space is generated by the polynomial $\chi_1\chi_2$.

Finally, one obtains that the cohomology of $\mathfrak{gl}(2|2)$ is

$$H^k\big(\mathfrak{gl}(2|2)\big) \cong \begin{cases} \mathbb{C}, & \text{if } k \in \{0,1,3,4\}; \\ 0, & \text{else.} \end{cases}$$

This is isomorphic to the cohomology of $\mathfrak{gl}(2)$, as seen from (3.10).

4.4.2 The general linear Lie superalgebra $\mathfrak{gl}(m|n)$

After going through the example for $\mathfrak{gl}(2|2)$, it is quite easy to extend the technique to the general case of $\mathfrak{gl}(m|n)$. The 0^{th} page will have all index-free polynomials on the ghosts of $\mathfrak{gl}(m|n)$, on which the differential d_0 acts as in (4.39). Once again, d_0 restricted to the column $E_0^{0,q}$ acts as the BRST differential on $\mathfrak{gl}(m|n)_0$, yielding

$$E_1^{0,q} = H^q\big(\mathfrak{gl}(m) \oplus \mathfrak{gl}(n)\big),$$

which has generators

$$\begin{aligned} &\varphi_1' \in H^1\big(\mathfrak{gl}(m)\big) \subset E_1^{0,1}, \qquad \dots, \qquad \varphi_m' \in H^{2m-1}\big(\mathfrak{gl}(m)\big) \subset E_1^{0,2m-1}, && \text{(4.50a)}\\ &\varphi_1'' \in H^1\big(\mathfrak{gl}(n)\big) \subset E_1^{0,1}, \qquad \dots, \qquad \varphi_n'' \in H^{2n-1}\big(\mathfrak{gl}(n)\big) \subset E_1^{0,2n-1}. && \text{(4.50b)} \end{aligned}$$

Likewise, the row $E_0^{p,0}$ will be non-trivial only for even p; if $\ell := \min\{m,n\}$, the Cayley–Hamilton theorem implies that such row is generated by the polynomials

$$\eta_1 \equiv \operatorname{tr}(\alpha\beta) \in E_0^{2,0}, \qquad \dots, \qquad \eta_\ell \equiv \operatorname{tr}(\alpha\beta)^\ell \in E_0^{2\ell,0}. \tag{4.51}$$

The only polynomials in the kernel of d_0 are, again, those obtained by products and sums of the generators in (4.50) and (4.51). Consequently, as before,

$$E_1^{p,q} \cong E_0^{p,0} \otimes E_1^{0,q}. \tag{4.52}$$

Everything works out *mutatis mutandis*. The fermionic ghost generators (4.50) and the bosonic ghost generators (4.51) are in the kernels of all higher differentials d_r, but never in their image — except for the transgressive differentials

$$d_{2k}\colon E_{2k}^{0,2k-1} \longrightarrow E_{2k}^{2k,0}, \qquad \text{for } k \in \{1,\dots,\ell\},$$

which send both φ_k' and φ_k'' to η_k.

For $k > \min\{m,n\}$, however, only the fermionic generators of $\mathfrak{gl}(\max\{m,n\})$ are left on the $E_r^{0,q}$ column. Nevertheless, there is nothing at the row $E_r^{p,0}$ to which φ_k' can transgress — since all of them have been used in the transgressions of the generators of $\mathfrak{gl}(\min\{m,n\})$.

In this way, all the bosonic ghost generators η_k, for $k \leqslant \ell = \min\{m,n\}$, will eventually be in the image of a differential, whence being washed away for late enough pages. The fermionic ghost generators φ_k' and φ_k'' both transgress into η_k, yielding

$$\begin{aligned} E_\infty^{0,2k-1} \cong E_{2k+1}^{0,2k-1} &\cong \ker d_{2k}^{0,2k-1} \\ &= \langle \varphi_k' - \varphi_k'' \rangle, \qquad \text{for } k \leqslant \min\{m,n\}. \end{aligned}$$

In turn, for $k > \max\{m,n\}$, the fermionic ghost generators transgress to zero, as there is nothing left at the row $E_{2k}^{2k,0}$, hence

$$\begin{aligned} E_\infty^{0,2k-1} \cong E_{2k+1}^{0,2k-1} &\cong \ker d_{2k}^{0,2k-1} \\ &= \langle \varphi_k' \rangle, \qquad \text{for } k > \max\{m,n\}. \end{aligned}$$

Therefore, the ∞^{th} page of the Hochschild–Serre spectral sequence for the pair $\big(\mathfrak{gl}(m|n), \mathfrak{gl}(m|n)_0\big)$ has

$$E_\infty^{0,2k-1} \cong \mathbb{C}, \qquad \text{for } k \in \{1, \ldots, \max\{m,n\}\},$$

which implies that

$$H^\bullet\big(\mathfrak{gl}(m|n)\big) \cong H^\bullet\big(\mathfrak{gl}(\max\{m,n\})\big). \tag{4.53}$$

The symbol $\cong$ above deserves a remark. The cohomology of the Lie superalgebra $\mathfrak{gl}(m|n)$ inherits a $\mathbb{Z}_2$-grading, while the cohomology of $\mathfrak{gl}(m)$ is ungraded. Nevertheless, all the non-trivial spaces of the former are fermionic, thus one can ignore its $\mathbb{Z}_2$-grading and understand the isomorphism above in the natural way.

This concludes the proof of (the first part of) Theorem 9, as wanted, using the BRST approach to cohomology. Note that I did not use Theorem 8 to adapt the Hochschild–Serre spectral sequence, but rather I constructed it from scratch using the associated filtration (4.36).

4.5 Final remarks

After having understood the use of the Hochschild–Serre spectral sequence in the context of Lie superalgebras, the next step of this project would be to compute the cohomology of the *special linear* Lie superalgebra, $\mathfrak{sl}(m|m)$. Although FUKS (1986) claims its cohomology is isomorphic to that of $\mathfrak{sl}(m)$ when $m \geqslant n$, we believe that such computation has to be done more carefully.

While the special linear Lie algebra $\mathfrak{sl}(m)$ is defined as the kernel of the trace, the analogous happens with Lie superalgebras, by replacing the usual trace by the **supertrace**

$$\mathrm{Str}(X) := \mathrm{tr}(a) - \mathrm{tr}(b), \qquad X \in \mathfrak{gl}(m|n)$$

using the decomposition (4.12). As pointed out by V. KAČ (1977b), the bilinear map $(X, Y) \longmapsto \mathrm{Str}(XY)$ is consistent, invariant and supersymmetric[15]. Then one defines the **special linear Lie superalgebra** $\mathfrak{sl}(m|n)$ as the Lie subsuperalgebra of $\mathfrak{gl}(m|n)$ strictly containing all the supertraceless matrices.

From the previous sections, one knows that $\chi_1 \equiv \varphi_1' - \varphi_1''$ is the generator of the 1$^{\text{st}}$ cohomology space of $\mathfrak{gl}(m|n)$. In polynomial terms, one has that $\chi_1 \in E_1^{0,1}$ is precisely the cohomology class of $\mathrm{tr}(a) - \mathrm{tr}(b) = \mathrm{Str}(X)$. Consequently, the cohomology of $\mathfrak{sl}(m|n)$ will not have χ_1 among its generators.

Note that, when $m = n$, the special linear Lie superalgebra $\mathfrak{sl}(m|m)$ contains the identity matrix

$$I \equiv \begin{pmatrix} I_m & 0 \\ 0 & I_m \end{pmatrix} \in \mathfrak{gl}(m|m);$$

that is because $\mathrm{Str}(I) = \mathrm{tr}(I_m) - \mathrm{tr}(I_m) = 0$. Consequently, $\mathfrak{sl}(m|m)$ has a central ideal $\mathfrak{i} := \langle I \rangle$ spanned by the identity matrix I. Define the **projective special linear** Lie superalgebra as the quotient

$$\mathfrak{psl}(m|m) := \mathfrak{sl}(m|m)/\mathfrak{i}.$$

The next natural step is to compute its cohomology using the Hochschild–Serre spectral sequence, to understand its structure and its central extensions.

[15]Let $f\colon \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathbb{C}$ be a bilinear map. f is *consistent* if $f(X, Y) = 0$ for $X \in \mathfrak{g}_0$ and $Y \in \mathfrak{g}_1$. f is *invariant* if $f(X, [Y, Z]) = f([X, Y], Z)$ for all $X, Y, Z \in \mathfrak{g}$. And, most importantly, f is *supersymmetric* if $f(X, Y) = (-1)^{\deg(X)\deg(Y)} f(Y, X)$.

Nonetheless, the correspondence discussed in Chapter 1 between a theory of supergravity on $\mathrm{AdS}_5 \times \mathbb{S}^5$ and the $\mathcal{N} = 4$ supersymmetric Yang–Mills theory on d-dimensional Minkowski spacetime talks about the *projective special unitary* Lie superalgebra $\mathfrak{psu}(2,2|4)$.

To explicitly define it, one should first consider the **special unitary** Lie superalgebra $\mathfrak{sl}(m,p|n,q)$, whose homogeneous spaces are (V. Kač, 1977a)

$$\mathfrak{su}(m,p|n,q)_s := \left\{ X \in \mathfrak{sl}(m|n)_s \;\middle|\; S_{p,q}^{-1}\, X^\dagger\, S_{p,q} = -\mathrm{i}^s X \right\}, \tag{4.54}$$

where $s \in \mathbb{Z}_2$ is the supergrading of the Lie superalgebra, $X^\dagger \equiv \bar{X}^\mathsf{T}$ is the transpose conjugate of X, and

$$S_{p,q} := \left(\begin{array}{cc|cc} +I_p & 0 & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ 0 & -I_{m-p} & & \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{0}} & +I_q & 0 \\ & & 0 & -I_{n-q} \end{array}\right) \in \mathfrak{gl}(m|n).$$

For comparison, consider the usual special unitary Lie algebra, defined as

$$\mathfrak{su}(n) := \left\{ X \in \mathfrak{sl}(n) \;\middle|\; X^\dagger = -X \right\};$$

comparing it with (4.54), it is similar to the bosonic part of $\mathfrak{su}(m|n) \equiv \mathfrak{su}(m,0|n,0)$, only with the substitution of $\mathfrak{sl}(m)$ to $\mathfrak{sl}(m|n)_0 = \mathfrak{sl}(m) \oplus \mathfrak{sl}(n)$.

As for the usual Lie algebra case, $\mathfrak{su}(m,p|n,q)$ is a Lie superalgebra over the real numbers. That is important to remark because, when $m = n$, the identity matrix I is not an element of $\mathfrak{su}(m,p|m,q)$, but $\mathrm{i}I$ (the imaginary unit times the identity matrix) is. Consequently, the Lie superalgebra $\mathfrak{su}(m,p|m,q)$ also has a central ideal, this time

$$\mathfrak{i}' := \langle \mathrm{i}I \rangle_\mathbb{R}.$$

That yields to the definition of the **projective special unitary** Lie superalgebra

$$\mathfrak{psu}(m,p|m,q) := \mathfrak{su}(m,p|m,q)/\mathfrak{i}'.$$

Under the light of the AdS/CFT correspondence, it is of interest to consider the case when $m = 4$, $p = 2$ and $q = 0$, obtaining $\mathfrak{psu}(2,2|4) \equiv \mathfrak{psu}(2,2|4,0)$. For simplicity and for clarity, take the case $(1,1|2)$; the elements of $\mathfrak{sl}(2|2)$ and the matrix $S_{1,0}$ respectively look like

$$X = \begin{pmatrix} a & \alpha \\ \beta & b \end{pmatrix} \qquad \text{and} \qquad S_{1,0} = \begin{pmatrix} \widetilde{I} & 0 \\ 0 & I \end{pmatrix}, \qquad \text{where} \quad \widetilde{I} := \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix}$$

and $\mathrm{tr}(a) - \mathrm{tr}(b) = 0$. Thus an element of $\mathfrak{su}(1,1|2)$ in the above decomposition is such that

$$\begin{aligned} a^\dagger &= -\widetilde{I} a \widetilde{I}, & \beta &= \mathrm{i}\alpha^\dagger \widetilde{I}, \\ b^\dagger &= -b, & \mathrm{tr}(a) &= \mathrm{tr}(b). \end{aligned}$$

The generalization to $\mathfrak{psu}(2,2|4)$ is immediate.

After understanding the cohomology of $\mathfrak{psl}(m|m)$, this project shall move closer to its original motivation in physics by studying the cohomologies of $\mathfrak{su}(2,2|4)$ and $\mathfrak{psu}(2,2|4)$. From that, one can better understand the central extensions of $\mathfrak{psu}(2,2|4)$ through its 2^{nd} cohomology group, which gives direct information about representation of products of local operators in the computation of correlation functions (MIKHAILOV, 2025).

Acta est fabula plaudite

— Augustus Caesar, *apud* Suetonius

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Appendix A

The long exact sequence in cohomology from an exact triple

The goal of this Appendix is to prove Theorem 1. I present two proofs, one using the Snake Lemma, and the other by brute force using "diagram chasing".

A.1 Via Snake Lemma

Consider the homomorphism $f\colon L' \longrightarrow L$ between two abelian groups. As one knows, its *kernel* is the subgroup of the domain L' defined by the elements mapped by f to the neutral element of the codomain L. One can also define the *cokernel* of f as the quotient space $\operatorname{coker} f := L/\operatorname{im} f$.

Then, let me enunciate the following result (Weibel, 1994).

Snake Lemma. *Consider the diagram*

$$\begin{tikzcd} & L' \arrow[r] \arrow[d, "f"] & M' \arrow[r] \arrow[d, "g"] & N' \arrow[r] \arrow[d, "h"] & 0 \\ 0 \arrow[r] & L \arrow[r, "i"'] & M \arrow[r, "j"'] & N. \end{tikzcd} \tag{A.1}$$

If the rows are exact, then there exists an exact sequence

$$\begin{tikzcd} \ker f \arrow[r] & \ker g \arrow[r] & \ker h \arrow[dll, rounded corners, "\delta"] \\ \operatorname{coker} f \arrow[r] & \operatorname{coker} g \arrow[r] & \operatorname{coker} h, \end{tikzcd}$$

where δ is (schematically) defined by the formula

$$\delta[y] = \left[i^{-1} g\, j^{-1} y\right], \qquad y \in \ker h.$$

This is a famous Lemma in homological algebra, whose name is a playful reference to the shape of the connecting homomorphism δ — which is drawn in that way because each column of the original diagram (A.1) is extended as

$$\ker f \hookrightarrow L' \longrightarrow L \twoheadrightarrow \operatorname{coker} f.$$

The proof for the Snake Lemma is by diagram chasing, which is done directly for Theorem 1 in the next section of this Appendix. Under the light of the Snake Lemma, Theorem 1 is a consequence of considering the diagram

$$\begin{tikzcd}
 & B^k / \operatorname{im} d_B^{k-1} \arrow[r] \arrow[d, "\overline{d}_B^k"] & C^k / \operatorname{im} d_C^{k-1} \arrow[r] \arrow[d, "\overline{d}_C^k"] & D^k / \operatorname{im} d_D^{k-1} \arrow[r] \arrow[d, "\overline{d}_D^k"] & 0 \\
0 \arrow[r] & \ker d_B^{k+1} \arrow[r] & \ker d_C^{k+1} \arrow[r] & \ker d_D^{k+1},
\end{tikzcd}$$

where $\overline{d}^k$ is the unique homomorphism such that

$$\overline{d}^k \circ J^k = J^{k+1} \circ d^k,$$

for J the quotient map. This means that $\overline{d}^k[x^k] = [d^k x^k]$ for all $x^k \in C^k$. Thus $x^k \in \ker d^k$ if and only if $[x^k] \in \ker \overline{d}^k$; therefore

$$\ker \overline{d}^k = \frac{\ker d^k}{\operatorname{im} d^{k-1}} =: H^k.$$

Similarly, the cokernel of $\overline{d}^k$ is H^{k+1}. The Snake Lemma then guarantees that

$$\begin{tikzcd}
\ker \overline{d}_B^k \arrow[r] & \ker \overline{d}_C^k \arrow[r] & \ker \overline{d}_D^k \arrow[dll, rounded corners, "\delta"] \\
\operatorname{coker} \overline{d}_B^k \arrow[r] & \operatorname{coker} \overline{d}_C^k \arrow[r] & \operatorname{coker} \overline{d}_D^k
\end{tikzcd}$$

is an exact sequence. From the previous observations, one has

$$\begin{tikzcd}
H^k(B^\bullet) \arrow[r] & H^k(C^\bullet) \arrow[r] & H^k(D^\bullet) \arrow[dll, rounded corners, "\delta^k"] \\
H^{k+1}(B^\bullet) \arrow[r] & H^{k+1}(C^\bullet) \arrow[r] & H^{k+1}(D^\bullet).
\end{tikzcd}$$

A.2 Via diagram chasing

In this section, I sketch the proof of Theorem 1 by diagram chasing, following LIMA (2009). Trying to simplify things, I will use w, x and y for cochains of $B^\bullet$, $C^\bullet$ and $D^\bullet$, respectively; the superscript on each cochain shall represent its $\mathbb{Z}$-grading on the corresponding differential graded group.

Definition of δ^k. Considering the exact triple

$$0 \longrightarrow B^\bullet \overset{f}{\hookrightarrow} C^\bullet \overset{g}{\twoheadrightarrow} D^\bullet \longrightarrow 0,$$

let $y^k \in D^k$ be a cocycle and pick $x^k \in C^k$ such that $g^k x^k = y^k$ — which is possible because g^k must be surjective for the triple to be exact (Example 2.1.d.). Being y^k a cocycle, it is an element of $\ker d_D^k$; thus

$$0 = d_D^k\, y^k = d_D^k(g^k x^k) = g^{k+1}\, d_C^k x^k \qquad \text{because} \qquad \begin{tikzcd} C^k \arrow[r, "g^k"] \arrow[d, "d_C^k"'] & D^k \arrow[d, "d_D^k"] \\ C^{k+1} \arrow[r, "g^{k+1}"'] & D^{k+1}. \end{tikzcd}$$

Now, $d_C^k x^k$ is an element of $\ker g^{k+1} = \operatorname{im} f^{k+1}$, thus there must exist some cochain $w^{k+1} \in B^{k+1}$ such that $d_C^k x^k = f^{k+1} w^{k+1}$. Such w^{k+1} is unique, because f^{k+1} is injective. Obviously, $d_C^k x^k$ is in $\ker d_C^{k+1}$, thus

$$d_C^{k+1} f^{k+1} w^{k+1} = 0.$$

Such maps, namely f^{k+1} and d_C^{k+1}, fit into the following commuting diagram:

$$\begin{tikzcd} B^{k+1} \arrow[r, "f^{k+1}"] \arrow[d, "d_B^{k+1}"'] & C^{k+1} \arrow[d, "d_C^{k+1}"] \\ B^{k+2} \arrow[r, "f^{k+2}"'] & C^{k+2}. \end{tikzcd}$$

The element w^{k+1}, starting in the upper-left corner, goes to the lower-right corner through $d_C^{k+1} \circ f^{k+1}$ and reaches $0 \in C^{k+2}$. Since f^{k+2} is injective, it must be that $d_B^{k+1} w^{k+1} = 0$, i.e. w^{k+1} is a cocycle of B^{k+1}.

Finally, for each k define a homomorphism

$$\begin{aligned} \delta^k \colon H^k(D^\bullet) &\longrightarrow H^{k+1}(B^\bullet) \\ [y^k] &\longmapsto [w^{k+1}]. \end{aligned}$$

From this construction, it is reasonable to write symbolically

$$\delta^k[y^k] = [(f^{k+1})^{-1}\, d_C^k\, (g^k)^{-1} y^k]$$

or simply

$$\delta = [\cdot] \circ f^{-1} \circ d_C \circ g^{-1} \circ [\cdot]^{-1},$$

where $[\cdot]$ or $[\cdot]^{-1}$ represent taking the cohomology class or a representative of it, respectively. Even though f^{-1} and g^{-1} are not univocally defined, the cohomology classes are well-defined by the formula, as it is shown next. $\square$

Well-definedness of δ^k. Now, one has to show that the choices of representative y^k in the homology class $[y^k] \in H^k(D^\bullet)$ and cochain x^k such that $y^k = g^k x^k$ do not affect the value of $[w^{k+1}] \in H^{k+1}(B^\bullet)$.

The most general choice in the cohomology class $[y^k]$ would be a cochain of the form $\tilde{y}^k = y^k + d_D^{k-1} y^{k-1}$, for some $y^{k-1} \in D^{k-1}$. Again, g^{k-1} is surjective, thus there exists $x^{k-1} \in C^{k-1}$ such that $y^{k-1} = g^{k-1} x^{k-1}$, hence

$$\begin{aligned} \tilde{y}^k = y^k + d_D^{k-1} y^{k-1} &= y^k + d_D^{k-1}(g^{k-1} x^{k-1}) \\ &= y^k + g^k d_C^{k-1} x^{k-1}, \end{aligned}$$

where the second equality comes from $d_D^{k-1} \circ g^{k-1} = g^k \circ d_C^{k-1}$.

The most general choice of cochain $\tilde{x}^k \in C^k$ such that $\tilde{y}^k = g^k \tilde{x}^k$ would be

$$\tilde{x}^k = x^k + d_C^{k-1} x^{k-1} + f^k w^k$$

for some $w^k \in B^k$, which would yield

$$g^k \tilde{x}^k = g^k x^k + g^k\, d_C^{k-1} x^{k-1} + \cancelto{0}{g^k f^k} w^k = y^k + d_D^{k-1} g^{k-1} x^{k-1},$$

thus $g^k \tilde{x}^k = y^k + d_D^{k-1} y^{k-1}$ — which is, again, the most general form of an element in the cohomology class $[y^k]$.

Now,

$$d^k_C \tilde{x}^k = d^k_C x^k + \cancelto{0}{d^k_C d^{k-1}_C} x^{k-1} + d^k_C f^k w^k = d^k_C x^k + f^{k+1} d^k_B w^k;$$

recall that w^{k+1} was picked in B^{k+1} to be such that $d^k_C x^k = f^{k+1} w^{k+1}$, thus

$$d^k_C \tilde{x}^k = f^{k+1} w^{k+1} + f^{k+1} d^k_B w^k = f^{k+1}\left(w^{k+1} + d^k_B w^k\right).$$

With these new choices, one would have

$$\delta^k[y^k + d^{k-1}_D y^{k-1}] = [w^{k+1} + d^k_B w^k] = [w^{k+1}] = \delta^k[y^k],$$

hence the image of δ^k does not change with different choices of representatives. □

δ^k is a homomorphism. From $\delta^k[y^k] = \left[(f^{k+1})^{-1} d^k_C (g^k)^{-1} y^k\right]$, one sees that δ^k is defined in terms of homomorphisms, thus it inherits that nature from f, g, d_C and the quotients. □

The sequence in cohomology (below) is exact.

$$\cdots \longrightarrow H^k(B^\bullet) \xrightarrow{(f^k)^*} H^k(C^\bullet) \xrightarrow{(g^k)^*} H^k(D^\bullet) \xrightarrow{\delta^k} H^{k+1}(B^\bullet) \longrightarrow \cdots$$

For that, one has to show three similar (but independent) things:

* *The kernel of δ^k equals the image of $(g^k)^*$.* Firstly, take $[y^k] = (g^k)^*[x^k]$ in the image of $(g^k)^*$. From Proposition 1, one has $[y^k] = [g^k x^k]$ with $d^k_C x^k = 0$, thus

  $$\delta^k (g^k)^*[x^k] = \delta^k[y^k] = \left[(f^{k+1})^{-1}\, d^k_C x^k\right] = 0;$$

  thus $\operatorname{im}(g^k)^* \subseteq \ker \delta^k$.

  For the reversed inclusion, take $[y^k] \in \ker \delta^k$, i.e. $\delta^k[y^k] = [w^{k+1}] = 0$; then $w^{k+1} = d^k_B w^k$ for some $w^k \in B^k$. Let $x^k \in C^k$ be such that $g^k x^k = y^k$. Then, from the construction of δ^k, one knows that $d^k_C x^k = f^{k+1}\, w^{k+1} = f^{k+1}\, d^k_B w^k = d^k_C\, f^k w^k$; thus $d^k_C(x^k - f^k w^k) = 0$. Hence $\tilde{x}^k := x^k - f^k w^k$ is a cocycle in C^k with $g^k \tilde{x}^k = g^k x^k - g^k f^k w^k = g^k x^k$, thus

  $$(g^k)^*[\tilde{x}^k] = [g^k x^k] = [y^k],$$

  therefore $\ker \delta^k \subseteq \operatorname{im}(g^k)^*$.

* *The kernel of $(f^{k+1})^*$ equals the image of δ^k.* Note that $(f^{k+1})^* \circ \delta^k = 0$. Indeed, for any cohomology class $[y^k] \in H^k(D^\bullet)$, one has $\delta^k[y^k] = [w^{k+1}]$, where $w^{k+1} \in \ker d_B^{k+1}$, and $f^{k+1}w^{k+1} = d_C^k x^k$ and $g^k x^k = y^k$, by the construction of δ^k. Consequently,

$$(f^{k+1})^* \, \delta^k[y^k] = (f^{k+1})^*[w^{k+1}] = [f^{k+1}w^{k+1}] = [d_C^k x^k] = 0 \in H^{k+1}(C^\bullet),$$

hence $\operatorname{im} \delta^k \subseteq \ker(f^{k+1})^*$.

Now, if $[w^{k+1}] \in \ker(f^{k+1})^*$, i.e. $0 = (f^{k+1})^*[w^{k+1}] = [f^{k+1}w^{k+1}]$ for a certain cocycle $w^{k+1} \in Z^{k+1}(B^\bullet)$, then $f^{k+1}w^{k+1} = d_C^k x^k$ with $x^k \in C^k$. Take $y^k = g^k x^k$; then $\delta^k[y^k] = [w^{k+1}]$, which implies that $\ker(f^{k+1})^* \subseteq \operatorname{im} \delta^k$.

* *The kernel of $(g^k)^*$ equals the image of $(f^k)^*$.* Since $(g^k)^* \circ (f^k)^* = (g^k \circ f^k)^* = 0$ (recall Proposition 1), it is immediate to conclude that $\operatorname{im}(f^k)^* \subseteq \ker(g^k)^*$.

Now, take $[x^k] \in \ker(g^k)^* \subseteq H^k(C^\bullet)$; then $0 = (g^k)^*[x^k] = [g^k x^k] \in H^k(D^\bullet)$, which means there is $y^{k-1} \in D^{k-1}$ such that $g^k x^k = d_D^{k-1} y^{k-1}$. Since g is surjective, there is $x^{k-1} \in C^{k-1}$ such that $g^{k-1}x^{k-1} = y^{k-1}$. Thus,

$$g^k x^k = d_D^{k-1} y^{k-1} = d_D^{k-1} \, g^{k-1} x^{k-1} = g^k \, d_C^{k-1} x^{k-1},$$

hence $g^k(x^k - d_C^{k-1}x^{k-1}) = 0$. By exactness of the original cochain complex, there must be $w^k \in B^k$ such that $x^k - d_C^{k-1}x^{k-1} = f^k w^k$. Since $[x^k] \in H^k(C^\bullet)$ means that $x^k \in \ker d_C^k$, it follows that

$$f^{k+1} \, d_B^k w^k = d_C^k \, f^k w^k = d_C^k(x^k - d_C^{k-1}x^{k-1}) = 0;$$

since f^{k+1} is injective, one concludes that $d_B^k w^k = 0$, thus $w^k \in Z^k(B^\bullet)$ and $(f^k)^*[w^k] = [f^k w^k] = [x^k - d_C^{k-1}x^{k-1}] = [x^k]$, hence $\ker(g^k) \subseteq \operatorname{im}(f^k)^*$. □