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Instituto de Biociências, Letras e Ciencias Exatas- Campus de São José do Rio Preto

PAULO HENRIQUE REIS SANTANA

HILBERT NUMBERS, INVARIANT ALGEBRAIC CURVES AND POLYCYCLES

São José do Rio Preto

2026

PAULO HENRIQUE REIS SANTANA

HILBERT NUMBERS, INVARIANT ALGEBRAIC CURVES AND POLYCYCLES

Tese apresentada à Universidade Estadual Paulista (UNESP), Instituto de Biociências, Letras e Ciencias Exatas, São José do Rio Preto, para obtenção do título de Doutor em Matemática.

Área de Concentração: Geometria e sistemas dinâmicos

Orientador: Prof. Dr. Claudio Aguinaldo Buzzi
Coorientador: Prof. Dr. Armengol Gasull

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IMPACTO POTENCIAL DESTA PESQUISA

Esta pesquisa contribui para o avanço da teoria qualitativa das equações diferenciais ordinárias, abordando questões centrais da segunda parte do 16º Problema de Hilbert, curvas algébricas invariantes e policiclos. Do ponto de vista científico, os resultados aprofundam a compreensão estrutural das equações estudadas, com impacto duradouro na organização conceitual da área. De forma indireta e no longo prazo, tais avanços fundamentam métodos empregados em modelagens dinâmicas na física, química, engenharia e outras ciências. A pesquisa apresenta forte inserção internacional, dialogando com problemas clássicos e contemporâneos, e reforça a presença da produção matemática nacional em periódicos especializados de alto nível. No plano educacional e cultural, contribui para a formação avançada em matemática e para a consolidação de uma tradição de pesquisa baseada em rigor, profundidade teórica e produção sustentável de conhecimento.

POTENTIAL IMPACT OF THIS RESEARCH

This research contributes to the advancement of the qualitative theory of ordinary differential equations by addressing central questions of the second part of Hilbert's 16th Problem, invariant algebraic curves, and polycycles. From a scientific standpoint, the results deepen the structural understanding of the equations under study, with lasting impact on the conceptual organization of the field. Indirectly and in the long term, these advances provide a theoretical foundation for methods employed in dynamical modeling in physics, chemistry, engineering, and other sciences. The research has strong international insertion, engaging with classical and contemporary problems and reinforcing the presence of the Brazilian mathematical research in high-level specialized journals. On the educational and cultural level, it contributes to advanced training in mathematics and to the consolidation of a research tradition grounded in rigor, theoretical depth, and sustainable knowledge production.

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À minha esposa Raquel, por todas as aventuras que vivemos juntos.

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“Meu objetivo é continuar minha vida como sempre foi, melhorar como profissional, aprender mais, muito mais, porque eu tenho muito o que aprender ainda [...] e melhorar como pessoa, sobretudo.” (Senna, [1])

RESUMO

Nesta tese abordamos a segunda parte do décimo sexto problema de Hilbert, provando que o número de Hilbert é estritamente crescente e realizado por campos de vetores estruturalmente estáveis e sem curvas algébricas invariantes. Obtemos também uma desigualdade relacionando o número de Hilbert com o número de Harnack, além de relacionar o primeiro com curvas algébrica invariantes. Em seguida, provamos que genericamente, tanto do ponto de vista topológico quanto probabilístico, um campo de vetores planar e polinomial não possui curvas algébricas invariantes. Por fim, estudamos a ciclicidade de policiclos de duas maneiras distintas: a quebra sistematizada do policiclo original em policiclos menores, gerando um ciclo limite a cada passo, e na perturbação do policiclo de maneira a mantê-lo persistente, mas alterando sua estabilidade sucessivamente.

Palavras-Chave: problema de Hilbert; número de Hilbert; ciclos limite; curvas algébricas invariantes; policiclos.

ABSTRACT

In this thesis we study the second part of Hilbert's 16th problem, proving that the Hilbert number is achieved by structurally stable vector fields without invariant algebraic curves. We also obtain a inequality relating the Hilbert and the Harnack numbers, and relate the former with invariant algebraic curves. Moreover, we prove that generically, both from a topological and a probabilistic point of view, a given planar polynomial vector field does not have invariant algebraic curves. Finally, we study the cyclicity of polycycles in a two-folded away: the systematic bifurcation of the polycycle into smaller ones, generating one limit cycle at each step, and by perturbating it in such a way that it remain persistent, but interchanging its stability.

Keywords: Hilbert's problems; Hilbert's number; limit cycles; invariant algebraic curves; polycycles.

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1 INTRODUCTION

Consider $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, and let $P,\ Q\colon \mathbb{K}^2 \to \mathbb{K}$ be functions of class C^k, $k \in \mathbb{N} \cup \{\infty, \omega\}$, with $\mathbb{N} = \{1, 2, \dots\}$, and C^ω denoting the family of *analytic functions*. A *planar C^k-differential system* is a system of ordinary differential equations of the form

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y), \tag{1.1}$$

with the dot denoting the derivative with respect to the independent variable $t \in \mathbb{K}$. To system (1.1) corresponds the C^k*-vector field*

$$X = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y} \tag{1.2}$$

in the phase plane $(x, y) \in \mathbb{K}^2$. Throughout most of this monograph we shall suppose $\mathbb{K} = \mathbb{R}$, and make no distinction between (1.1) and (1.2). Hence, unless explicitly stated, the reader may also adopt such assumptions. For simplicity, let $X = (P, Q)$ denote system (1.1). If P and Q are polynomials, then we say that X is a *polynomial vector field*, or a *polynomial system*. If we let $n = \max\{\deg P, \deg Q\}$, then we say that X is a polynomial system of *degree* n. Let $\mathcal{X}^n$ be the set of polynomial systems (1.1) of degree n. Given $X = (P, Q) \in \mathcal{X}^n$, let

$$P(x, y) = \sum_{i+j=0}^{n} a_{ij}x^iy^j, \quad Q(x, y) = \sum_{i+j=0}^{n} b_{ij}x^iy^j,$$

with a_{ij}, $b_{ij} \in \mathbb{K}$, and consider the bijective linear function $\Phi\colon \mathcal{X}^n \to \mathbb{K}^{(n+1)(n+2)}$ given by,

$$\Phi(X) = (a_{0,0}, a_{1,0}, \dots, a_{1,n-1}, a_{0,n}, b_{0,0}, b_{1,0}, \dots, b_{1,n-1}, b_{0,n}).$$

We endow $\mathcal{X}^n$ with the *coefficients topology*, induced by the metric $\rho\colon \mathcal{X}^n \times \mathcal{X}^n \to \mathbb{R}$ given by

$$\rho(X, Y) = ||\Phi(X) - \Phi(Y)||, \tag{1.3}$$

where $||\cdot||$ denotes the standard norm of $\mathbb{K}^{(n+1)(n+2)}$. When $n = 1$, system (1.1) is a *planar linear differential system*. This class of systems is already very-well understood, see for instance [7, Chapter 1]. On the other hand, for $n \geqslant 2$, the characteristics of these systems remain largely unknown. This last class of systems is known as *planar nonlinear differential systems*, or just *planar nonlinear systems*.

The scarcity of the knowledge on planar nonlinear systems can be illustrated by *Hilbert's list of problems*. More specifically, in his seminal address to the International Congress of Mathematicians in Paris, 1900, David Hilbert raised his famous list of problems for the 20th century [8], with the second part of the 16th problem being about the limit cycles of planar polynomial vector fields. Hilbert asks if there is a uniform upper bound for the number of limit cycles (i.e. isolated periodic orbits) of planar polynomial vector fields of degree n.

More precisely, given $X \in \mathcal{X}^n$ let $\pi(X) \in \mathbb{Z}_{\geqslant 0} \cup \{\infty\}$ be its number of limit cycles. The *Hilbert number* $\mathcal{H}(n) \in \mathbb{Z}_{\geqslant 0} \cup \{\infty\}$ is defined as

$$\mathcal{H}(n) = \sup\{\pi(X) \colon X \in \mathcal{X}^n\}. \tag{1.4}$$

Under this notation the second part of Hilbert's 16th problem consists in obtaining an upper bound for $\mathcal{H}(n)$. So far, the only known Hilbert number is $\mathcal{H}(1) = 0$. For the quadratic case $n = 2$, it is not known even if $\mathcal{H}(2) < \infty$. Over the decades progress has focused on achieving lower bounds for $\mathcal{H}(n)$. For small values of n, as far was we know, the best lower bounds so far are $\mathcal{H}(2) \geqslant 4$ [9, 10], $\mathcal{H}(3) \geqslant 13$ [11] and $\mathcal{H}(4) \geqslant 28$ [12]. In general, it is known that $\mathcal{H}(n)$ increases at least as fast as $O(n^2 \ln n)$ [13, 14, 15, 16].

In the first part of this monograph, namely Chapter 2, we approach Hilbert's problem in a different manner, aiming to obtain new characterizations, relations, and generalizations for $\mathcal{H}(n)$. To this end, we first observe that although the known lower bounds of $\mathcal{H}(n)$ are given by strictly increasing functions, this does not imply that $\mathcal{H}(n)$ itself is strictly increasing. In our first main result we prove this fact.

Theorem A. *Given $n \in \mathbb{N}$, it holds $\mathcal{H}(n+1) \geqslant \mathcal{H}(n) + 1$.*

In particular, it follows from Theorem A that if $\mathcal{H}(n_0) = \infty$ for some $n_0 \in \mathbb{N}$, then $\mathcal{H}(n) = \infty$ for every $n \geqslant n_0$. Roughly speaking, if we suppose $\mathcal{H}(n) < \infty$, then the proof of Theorem A follows by considering a system $X_n \in \mathcal{X}^n$ with $\mathcal{H}(n)$ limit cycles, and then perturbing it into a new system $X_{n+1} \in \mathcal{X}^{n+1}$ in such a way that the previous $\mathcal{H}(n)$ limit cycles persist, and at the same time that one extra limit cycle bifurcate.

The persistence of those $\mathcal{H}(n)$ previous limit cycles follows from our second main result, in which we prove that $\mathcal{H}(n)$ is achievable by *structurally stable vector fields with hyperbolic limit cycles only*. In this monograph we shall now dip into the notion of structural stability. We only remark that, roughly speaking, a vector field $X \in \mathcal{X}^n$ is structurally stable if its dynamics persist under small perturbations within $\mathcal{X}^n$. Moreover, if we let $\Sigma_h^n \subset \mathcal{X}^n$ denote the family of the structurally stable vector fields with hyperbolic limit cycles only, then it is open and dense. These two properties make the family of structurally stable vector fields *generic* (Peixoto [17, p. 1]) and *amenable to simple description* (Sotomayor [18, p. 1]). For more details we refer to [19]. Our second main result is as follows.

Theorem B. *For $n \in \mathbb{N}$, the following statements hold.*

(a) If $\mathcal{H}(n) < \infty$, then there is $X \in \Sigma_h^n$ such that $\pi(X) = \mathcal{H}(n)$.

(b) If $\mathcal{H}(n) = \infty$, then for each $k \in \mathbb{N}$ there is $X_k \in \Sigma_h^n$ such that $\pi(X_k) \geqslant k$.

Based on the intuition of Peixoto [17] and Sotomayor [18], we may say from Theorem B that *the Hilbert number is achieved by simple vector fields*. Theorems A and B, and also the folklore result $\mathcal{H}(n) \leqslant \aleph_0$, are proved in Section 2.1, which in turn is based on [3].

Inspired by the famous *Descartes' rule of signs*, we also consider a new definition for the Hilbert number, by looking to the number of *monomials* of X, instead of it degree. More precisely, we say that $X = (P, Q)$ has m monomials if the sum of the number of monomials of P and Q is equal to m. Let $\mathcal{M}_m$ be the family of planar polynomial vector fields with m monomials, independently of its degree. We define the *Hilbert monomial number* $\mathcal{H}^M(m) \in \mathbb{Z}_{\geqslant 0} \cup \{\infty\}$ as

$$\mathcal{H}^M(m) = \sup\{\pi(X) \colon X \in \mathcal{M}_m\}.$$

Very little is known about $\mathcal{H}^M(m)$. From Buzzi et al [20] it follows that $\mathcal{H}^M(m) = 0$ for $m \in \{1, 2, 3\}$, $\mathcal{H}^M(m) \geqslant m - 3$ for $m \geqslant 4$, and that there is a sequence of positive integer numbers $m_k \to \infty$ such that $\mathcal{H}^M(m_k) \geqslant N(m_k)$, with $N(m)$ of order $O(m \ln m)$.

In our third main result we improve these general lower bounds proving that $\mathcal{H}^M(m)$ increases at least with order $O(m^2)$.

Theorem C. *If* $m \geqslant 9$, *then* $\mathcal{H}^M(m) \geqslant \frac{1}{2}m^2 - 3m - 8$.

Similarly to the usual Hilbert number, the general lower bounds obtained in Theorem C can be enhanced for small values of m, by use of a specific argumentation for each m. This allowed our fourth main theorem.

Theorem D. *If* $m \in \{4, 5, 6\}$, *then* $\mathcal{H}^M(m) \geqslant 12$. *Moreover,* $\mathcal{H}^M(7) \geqslant 16$, $\mathcal{H}^M(8) \geqslant 20$, $\mathcal{H}^M(9) \geqslant 24$, *and* $\mathcal{H}^M(10) \geqslant 32$.

For a summary of the previous and new lower bounds, see Table 1. The lower bounds

Table 1 – Lower bounds of $\mathcal{H}^M(m)$. Recall that $\mathcal{H}^M(m) = 0$ for $m \leqslant 3$. Source: Table 1 of [4].

Monomials	New lower bounds	Previous lower bounds
4	12	1
5	12	3
6	12	4
7	16	5
8	20	5
9	24	24
10	32	24
$m \geqslant 11$	$\frac{1}{2}m^2 - 3m - 8$	$m - 3$
Asymptotic	$O(m^2)$	$O(m \ln m)$

provided by Theorem D solved a problem posed by Gasull [21, Problem 8th], 2021, about the minimal $m_0 \in \mathbb{N}$ satisfying $\mathcal{H}^M(m_0) > m_0$, i.e. for the *simplest vector field with more limit* *cycles than monomials*. On that time it was known that $4 \leqslant m_0 \leqslant 9$. From Theorem D it now follows $m_0 = 4$. Theorems C and D are proved in Section 2.2, which in turn is based on [4].

A second different approach regarding the Hilbert number can be obtained by considering the notion of *invariant algebraic curves*. Let $X = (P, Q)$ be a planar system (1.1) and γ one of its orbits. We say that γ is *algebraic* if it is contained in an algebraic set. That is, there is a polynomial $C \in \mathbb{R}[x, y]$ such that $\gamma \subset C^{-1}(\{0\})$. If we choose $C = 0$ to be irreducible, then this condition reads as

$$P\frac{\partial C}{\partial x} + Q\frac{\partial C}{\partial y} = KC, \tag{1.5}$$

with $K \in \mathbb{R}[x, y]$ known as the *cofactor* of C. It is not hard to see that if X has degree n, then the degree of K is at most $n - 1$.

In general, given $C \in \mathbb{C}[x, y]$ not necessarily irreducible, we say that $C = 0$ is an *invariant algebraic curve* for X if C satisfies (1.5) for some $K \in \mathbb{C}[x, y]$ of degree at most $n-1$. The term *invariant* follows from the fact that if C satisfies (1.5), then the set $C^{-1}(\{0\}) \cap \mathbb{R}^2$ is invariant by the solution of X, when nonempty. In particular, if γ is a limit cycle, then it is said to be an *algebraic limit cycle*. If the curve $C = 0$ containing γ is irreducible and has degree $c > 0$, then we say that γ has degree c.

For an account of known results on planar polynomial vector fields with algebraic invariant curves, see [22, Chapter 8] and the references therein. The interest on this subject goes back to the seminal works of Darboux [23] and Poincaré [24].

In our following main results, we study how many limit cycles a family of polynomial vector fields, with a prescribed invariant algebraic curve, can have. We obtain lower bounds for this maximum number of limit cycles depending on the original Hilbert number $\mathcal{H}(n)$ and its known lower bounds.

We now state our results. Given $C \in \mathbb{R}[x, y]$, a point $p \in \mathbb{R}^2$ is a *singular point* of C if $C(p) = 0$ and $\nabla C(p) = 0$. We say that C is *non-degenerated* if $C = 0$ has at most a finite number of singular points. If $C = 0$ does not have singular points, then it is *non-singular*. An *oval* of $C = 0$ is a bounded connected component of this set without singular points of C. Let $\mathcal{O}(C) \in \mathbb{Z}_{\geqslant 0}$ denote the number of ovals of $C = 0$.

Let $\mathcal{X}^n_C \subset \mathcal{X}^n$ be the set of planar polynomial vector fields of degree n having $C = 0$ as an invariant algebraic curve. For $n \in \mathbb{N}$ we define the *Hilbert-C number* as

$$\mathcal{H}_C(n) = \sup\{\pi(X) \colon X \in \mathcal{X}^n_C\}.$$

We say that $X \in \mathcal{X}^n_C$ is *non-degenerated* if it has at most finitely many singularities at $C = 0$. Let $\Sigma^n_C \subset \mathcal{X}^n_C$ denote the family of non-degenerated vector fields.

In our fifth main result we use the usual Hilbert numbers $\mathcal{H}(n)$ to provide a lower bound for $\mathcal{H}_C$ in such way that every oval of $C = 0$ is a hyperbolic limit cycle.

Theorem E. *Given $C \in \mathbb{R}[x,y]$ non-degenerated of degree $c > 0$ and $n \in \mathbb{Z}_{\geqslant 0}$, the following statements hold.*

(a) If $\mathcal{H}(n) < \infty$, then there is $X \in \Sigma_C^{n+c}$ such that every oval of $C = 0$ is a limit cycle and $\pi(X) \geqslant \mathcal{H}(n) + \mathcal{O}(C)$,

(b) If $\mathcal{H}(n) = \infty$, then for each $k \in \mathbb{N}$ there is $X_k \in \Sigma_C^{n+c}$ such that every oval of $C = 0$ is a limit cycle and $\pi(X_k) \geqslant k + \mathcal{O}(C)$.

Moreover, all limit cycles of X and X_k are hyperbolic. In particular,

$$\mathcal{H}_C(n+c) \geqslant \mathcal{H}(n) + \mathcal{O}(C),$$

and as a consequence, for n big enough, $\mathcal{H}_C(n)$ increases at least as fast as $O(n^2 \ln n)$.

We remark that if in Theorem E we look simply for $X \in \mathcal{X}_C^{n+c}$ satisfying $\pi(X) \geqslant \mathcal{H}(n)$, then the proof is much easier. Consider for instance the case $\mathcal{H}(n) < \infty$. Take $Y \in \mathcal{X}^n$ such that $\pi(Y) = \mathcal{H}(n)$ and such that none of its limit cycles intersects the curve $C = 0$. If we define $X = CY \in \mathcal{X}_C^{n+c}$, then $\pi(X) = \pi(Y)$. However, in this case the curve $C = 0$ is full of singularities of X, and thus X is degenerated.

The non-degenerated X presented by Theorem E, in addition with the "simplicity" granted from Theorem B, allowed the proof of a new inequality regarding the Hilbert number. To state it, given $m \in \mathbb{N}$, we consider the *Harnack number*

$$\operatorname{Har}(m) = \frac{1}{2}(m-1)(m-2) + \frac{1}{2}\big[1 + (-1)^m\big]. \tag{1.6}$$

Our sixth main result read as follows.

Theorem F. *Given n, $m \in \mathbb{N}$, it holds*

$$\mathcal{H}(n+m) \geqslant \mathcal{H}(n) + \operatorname{Har}(m). \tag{1.7}$$

Moreover the vector field of degree $n+m$ constructed to prove (1.7) *has $\mathcal{H}(n) + \operatorname{Har}(m)$ hyperbolic limit cycles and at least $\operatorname{Har}(m)$ of them are algebraic and of degree m.*

We remark that while usual Hilbert number (1.4) is related to the *second* part of Hilbert 16th, the Harnack number (1.6) is related with the *first* part of Hilbert's 16th problem. More precisely, aware of Harnack's work [25, 26] on the maximum number of ovals of a planar algebraic curve of degree n given by (1.6), in the first part of his 16th problem, Hilbert asks for the relative position of such curves. Therefore, Theorem F can be seen as a relation between the two parts of Hilbert 16th problem.

Moreover, we observe that (1.7) can be seen as a generalization of our first main result, Theorem A. In particular, we have from (1.7) that a linear increment in the degree of the vector field results, at least, in a quadratic increment on the Hilbert number. This raises the question of whether a constant increment in the degree results, at least, in a linear increment on the Hilber number. More precisely, we propose the following problem.

Problem 1. *Is there $k > 0$ such that $\mathcal{H}(n+1) \geqslant \mathcal{H}(n) + kn$ for every $n \in \mathbb{N}$?*

A potentially easier version of Problem 1 would be to ask for a constant increment higher than the one provided by Theorem A. In particular, we propose the following problem.

Problem 2. *Does it hold $\mathcal{H}(n+1) \geqslant \mathcal{H}(n) + 2$ for every $n \in \mathbb{N}$?*

As an application of Theorem E, we study the family of *Kolmogorov systems*, i.e. systems (1.1) for which both coordinate axes $x = 0$ and $y = 0$ are invariant. Such systems extend the classical Lotka–Volterra predator–prey equations and reads as

$$\dot{x} = xf(x,y), \quad \dot{y} = yg(x,y), \tag{1.8}$$

see [27] for details. Here we only remark that the model is well defined only in the first quadrant of $\mathbb{R}^2$. The application reads as follows.

Corollary 1. *Let $\mathcal{H}_K(n)$ denote the maximum number of limit cycles of* (1.8) *in the first quadrant, with f and g varying among all polynomials of degree at most $n-1$. Then $\mathcal{H}_K(n) \geqslant \mathcal{H}(n-1)$. Moreover, all the limit cycles can be chosen hyperbolic.*

As a second application of Theorem E, we study a model based on the work of Smith and Price [28], who introduced concepts of Game Theory into the Biology, coining the notion of *Evolutionary Stable Strategies* (ESS). Roughly speaking, given a game between two or more players (modeling for example a conflict between different species of animals), an ESS is a strategy such that if most of the players follows it, then there is no disruptive strategy that would give higher advantages for the other players. That is, the best strategy for the other players is to also follow the ESS. In other words [29], an ESS is a "uninvadable" state of the population, meaning that if the populations are already in an ESS, then any deviant behavior will eventually disappear under natural selection.

Taylor and Jonker [30] introduced concepts of Ordinary Differential Equations (ODE) into the field of ESS in such a way that the game is now modeled by a system of ODE, see also [31]. In particular, if the game has two players then the models result in a planar polynomial vector field. Under these concepts, Hofbauer et al [32] proved in the planar case (and Palm [33] in the higher dimensional case) that every ESS is an asymptotically stable singularity, with the converse not holding.

However as observed by Hofbauer et al [32], under the language of ODE there is nothing special distinguishing ESS from asymptotic stability. Hence, it would be more appropriate to study asymptotic stability rather than ESS. In particular, in this context a stable limit cycle can be interpreted as an *oscillating stable strategy*. That is, an “uninvadable” oscillating state of the population which eradicates any deviant behavior. The model reads as

$$\dot{x} = x(x-1)f(x,y), \quad \dot{y} = y(y-1)g(x,y), \tag{1.9}$$

with f and g polynomials of degree n, and is well defined on the closed square

$$Q = \{(x,y) \in \mathbb{R}^2 \colon 0 \leqslant x \leqslant 1,\, 0 \leqslant y \leqslant 1\}. \tag{1.10}$$

Its applications can be found in many research areas, such as Parental Investing, Biology, Genetics, Evolution, Ecology, Politics and Economics. See [34, Section 2] and the references therein. Our application to the model reads as follows.

Corollary 2. *Let $\mathcal{H}_\square(n)$ denote the maximum number of limit cycles of* (1.9) *inside the interior of the square* (1.10)*, with f and g varying among all polynomials of degree at most $n-2$. Then $\mathcal{H}_\square(n) \geqslant \mathcal{H}(n-2)$. Moreover, all the limit cycles can be chosen hyperbolic.*

The proofs of Theorems E and F, and Corollaries 1 and 2, can be found in Section 2.3, which in turn follows from [6]. For a study on the structural stability of system (1.9) for any $n \in \mathbb{N}$, we refer to [35]. For a classification of all such systems with $n = 1$, we refer to [36].

Back to invariant algebraic curves, given a vector field X, notice that the existence of such a curve implies some *rigidity* to X. This can be seen either by the fact that X must have some orbit contained in an algebraic locus, or by the fact that it must satisfies (1.5) for some polynomials C and K. Therefore, is seems natural to expect that a *generic* vector field admits no such curves. This idea is explored and proved correct in the second part of this monograph, namely Chapter 3. To state our results precisely, we shall make distinctions between the complex $\mathbb{K} = \mathbb{C}$ and real $\mathbb{K} = \mathbb{R}$ cases. To this end, let $\mathcal{X}^n(\mathbb{K})$ denote the set of polynomial systems (1.1) of degree n, defined over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$.

We now state our results. Given a topological space, a subset is *residual* if it contains the intersection of countably many open and dense subsets. All topological spaces in this monograph are *Baire spaces*, and thus every residual set will be dense [37, Chapter 8]. Let $A \subset \mathcal{X}^n(\mathbb{K})$ be a measurable set. We say that it has *full Lebesgue measure* if its complement has zero Lebesgue measure. Given $C \in \mathbb{C}[x,y]$ a (not necessarily irreducible) polynomial, let $\mathcal{X}^n_C(\mathbb{K})$ be the family of vector fields defined over $\mathbb{K}^2$, of degree n, and having C as an invariant algebraic curve. We say that C is *nodal* if all its singularities are of normal crossing type, i.e. at each singularity of C there are exactly two smooth branches of $C = 0$ intersecting transversely. Moreover, we say that C is nodal *taking the line at infinity into account* if $C \cdot L_\infty$ is a nodal curve in complex projective plane $\mathbb{CP}^2$, where $L_\infty = \mathbb{CP}^2 \setminus \mathbb{C}^2$. Let $\Upsilon^{d,C}_n(\mathbb{K}) \subset \mathcal{X}^n_C(\mathbb{K})$ (resp. $\Upsilon^{\infty,C}_n(\mathbb{K}) \subset \mathcal{X}^n_C(\mathbb{K})$) be the family of vector fields having no invariant algebraic curve of degree at most d (resp. of any degree), other than C. In the case $C = 1$, we simply omit C from the notation.

In our next main result we prove the existence of a finite collection of equations, depending on C and n, which we call *Camacho-Sad obstructions*, such that if a generic $X \in \mathcal{X}_C^n(\mathbb{K})$ has an invariant algebraic curve other than C, then it satisfies at least one of such equations. The precise definitions of the Camacho-Sad obstructions and the generic conditions are postponed to Chapter 3. Our next main result reads as follows.

Theorem G. *Let $C \in \mathbb{C}[x,y]$ be nodal curve taking the line at infinity into account, and let $n \in \mathbb{N}$. Suppose there exists $X_0 \in \mathcal{X}_C^n(\mathbb{C})$ with only Poincaré singularities, and such that X_0 does not satisfy any* Camacho-Sad obstructions*. Then:*

(a) $\Upsilon_n^{\infty,C}(\mathbb{C})$ *contains an open and dense set.*

Moreover, suppose that there exists $X_1 \in \mathcal{X}_C^n(\mathbb{R})$ with only simple singularities, such that X_1 does not satisfy any Camacho-Sad obstructions*. Then:*

(b) $\Upsilon_n^{\infty,C}(\mathbb{R})$ *is residual; and*

(c) *for each $d \geqslant 1$, $\Upsilon_n^{d,C}(\mathbb{R})$ contains an open and dense set.*

Moreover, they all have full Lebesgue measure.

A singularity of X is said to be of *Poincaré* (resp. *simple*) type if its associated eigenvalues $\lambda_1, \lambda_2 \in \mathbb{C}$ satisfy $\lambda_1\lambda_2 \neq 0$ and $\lambda_1/\lambda_2 \in \mathbb{C} \setminus \mathbb{R}_{\geqslant 0}$ (resp. $\lambda_1/\lambda_2 \in \mathbb{C} \setminus \mathbb{Q}_{\geqslant 0}$).

In other words, Theorem G states that if there is at least one complex vector field X_0 (resp. real vector field X_1) satisfying the generic condition of having only Poincaré singularities (resp. simple singularities), and not satisfying any Camacho-Sad obstruction, then the family of complex (resp. real) vector fields without invariant algebraic curves other than C, contains an open and dense (resp. residual) set. Moreover, in both cases such a family has full Lebesgue measure. In particular, we may say that the family of vector fields without invariant algebraic curves, other than the prescribed C, is *generic both in the topological and probabilistic sense*.

If we restricted our attention to $C = 1$ (i.e. having no prescribed invariant algebraic curve), then the existence of the *Jouanolou's foliation* [38, Chapter 4] implies the following result.

Theorem H. *For each $n \geqslant 2$, the following statements hold.*

(a) $\Upsilon_n^{\infty}(\mathbb{C})$ *contains an open and dense set;*

(b) $\Upsilon_n^{\infty}(\mathbb{R})$ *is residual; and*

(c) *for each $d \geqslant 1$, $\Upsilon_n^{d}(\mathbb{R})$ contains an open and dense set.*

Moreover, they all have full Lebesgue measure.

In other words, Theorem H states that there is an open and dense (resp. residual) subset of $\mathcal{X}^n(\mathbb{C})$ (resp. $\mathcal{X}^n(\mathbb{R})$) whose elements have no invariant algebraic curves. Moreover, both subsets has full Lebesgue measure. Such characterizations may allow us to say that *a generic planar polynomial vector field has no invariant algebraic curve*.

We must remark that Theorem H is not completely new. For example, we included statement (a) for completeness, although it was already established in [39, Theorem 5] and [40, Theorem 1.1], with the latter stated for *foliations* on $\mathbb{CP}^2$, rather than vector fields on $\mathbb{C}^2$. Moreover, a foliation-version of statement (b) can be obtained from [40, Theorem 1.1] in addition with some observations made in [41, Section 3.4]. From this version, a vector-field-version can be deduced. Statement (c) on the other hand, is the real and vector-field-version of the argument presented by Jouanolou at [38, Chapter 4], for foliations. However, to our knowledge, they have not been explicitly stated in the literature. Furthermore, despite the close parallels between foliations and vector fields, our inquiries revealed that the vector-field-formulation of Theorem H has not been widely acknowledged, as discussions with several researchers from various regions showed that it remains unfamiliar within our field. Finally, as far as we know, the claim about Lebesgue measure is new.

As an application of Theorem G, we study *Kolmogorov systems*, i.e. systems (1.8). We prove that, in general, the only invariant algebraic curves of a planar Kolmogorov system are the coordinate axes $x = 0$ and $y = 0$.

Theorem I. *For each $n \geqslant 2$, the following statements hold.*

(a) $\Upsilon_n^{\infty,xy}(\mathbb{C})$ *contains an open and dense set;*

(b) $\Upsilon_n^{\infty,xy}(\mathbb{R})$ *is residual; and*

(c) *for each $d \geqslant 1$, $\Upsilon_n^{d,xy}(\mathbb{R})$ contains an open and dense set.*

Moreover, they all have full Lebesgue measure.

As a direct consequence of Theorem I and Corollary 1, we have the following result.

Corollary 3. *For each n, it holds $\mathcal{H}_K(n) \geqslant \mathcal{H}(n-1)$ Moreover, all the limit cycles can be chosen hyperbolic, non-algebraic and in the first quadrant.*

Corollary 3 has applications in the Kolmogorov family, raising previous lower bounds in the literature, see Section 2.3, and also in the family of *chemical reaction systems* [42, 43], implying that its limit cycles can be suppose hyperbolic and non-algebraic, except perhaps by an arbitrarily small perturbation inside such family.

Let $\Lambda_n = \Sigma_h^n \cap \Upsilon_n^{\infty}(\mathbb{R}) \subset \mathcal{X}^n$ be the family of structurally stable polynomial systems (1.1) of degree n, without invariant algebraic curve. A direct combination of Theorems B and H reads as follows.

Theorem J. *For $n \in \mathbb{N}$, the following statements hold.*

(a) If $\mathcal{H}(n)<\infty$, then there is $\mathcal{X}\in\Lambda_n$ such that $\pi(\mathcal{X})=\mathcal{H}(n)$.

(b) If $\mathcal{H}(n)=\infty$, then for each $k\in\mathbb{N}$ there is $\mathcal{X}_k\in\Lambda_n$ such that $\pi(\mathcal{X}_k)\geqslant k$.

In other words, we may say from Theorem J that the Hilbert number is achieved by simple vector fields without invariant algebraic curves. In particular, *all the limit cycles can be supposed hyperbolic and non-algebraic*.

Finally, notice in Theorem H that since $\Upsilon_n^\infty(\mathbb{C})$ is open and dense, we have in the complex case that the family of vector fields with some invariant algebraic curve is not dense. The same property also holds in relation to $\Upsilon_n^d(\mathbb{R})$, for each $d\in\mathbb{N}$. On the other hand, the fact that $\Upsilon_n^\infty(\mathbb{R})$ is residual and of full measure does not prevent the set of real vector fields with at least one invariant algebraic curve to be dense. This rises a natural problem.

Problem 3. *Does $\Upsilon_n^\infty(\mathbb{R})$ contain an open and dense set?*

The affirmative answer of Problem 3 would prevent the set of planar polynomial vector fields with at least one invariant algebraic curve to be dense. Theorems G, H, I, J and Corollary 3 are proved in Chapter 3, which is based on [44].

There is a less general version of Hilbert's 16th, known as *Dulac problem*, where it is asked if polynomial systems (1.1) can have infinitely many limit cycles, i.e. if there is $X\in\mathcal{X}^n$ such that $\pi(X)=\infty$, for some $n\in\mathbb{N}$. Of course such problems are related. For example, if one proves $\mathcal{H}(n)<\infty$, then Dulac's problem is solved as a byproduct. Reciprocally, if for each $n\in\mathbb{N}$ one finds $X_n\in\mathcal{X}^n$ satisfying $\pi(X_n)=\infty$, then Hilbert's 16th problem is solved. Nevertheless, such problems are not equivalent. For example, if one proves $\pi(X)<\infty$ for every $X\in\mathcal{X}^n$ and every $n\in\mathbb{N}$, it does not prevent $\mathcal{H}(n)=\infty$, as we could have a sequence $\{X_k\}_{k\in\mathbb{N}}\subset\mathcal{X}^n$ such that $\pi(X_k)\to\infty$ and $\pi(X_k)<\infty$ for every $k\in\mathbb{N}$. So far it is only known that $\pi(X)<\infty$ for every $X\in\mathcal{X}^2$, see [45]. In this monograph we work with the possibility of $\pi(X)=\infty$ for some $X\in\mathcal{X}^n$, $n\geqslant 3$. We choose to do this because although Il'yashenko [46] and Écalle [47] independently claimed to have proved that this is impossible, it seems that some of their results started to be under discussion. For instance, in the recent work [48] a possible gap was found in Il'yashenko's proof. This monograph is not based on their finiteness results.

The attempts to solve Dulac's problem usually leads one to suppose the existence of infinitely many limit cycles of a given system X, trying to solve the problem by contradiction. It is then a consequence of the Poincaré-Bendixson Theorem that such limit cycles must accumulate either on a singularity, or on a *graphic* [49]. Therefore, a better understanding of the graphics of a vector field seems helpful to solve Dulac's problem. This is what we do in the third and final part of this monograph, namely Chapter 4, by studying *polycycles*, which is the kind of graphics where the supposed infinitely many limit cycles can accumulate. We remark that our next main results are usually stated for systems (1.1) of class C^∞ or C^ω, rather than polynomials.

We now state the results. A *graphic* Γ of a C^∞ system X is a compact, non-empty invariant subset which is a continuous (but not necessarily homeomorphic) image of $\mathbb{S}^1$ and consists of a finite number of isolated singularities $\{p_1,\dots,p_n\}$ (not necessarily distinct) and a compatibly set of distinct regular orbits $\{L_1,\dots,L_n\}$ such that p_i is the ω-limit of L_i. A *polycycle* is a graphic with a well defined first return map on one of its sides. A polycycle is *hyperbolic* if all its singularities are hyperbolic saddles. Let Γ^n denote a hyperbolic polycycle composed by the hyperbolic saddles $\{p_1,\dots,p_n\}$ (not necessarily distinct) and by the distinct regular orbits $\{L_1,\dots,L_n\}$, the sides of the polycycle, such that p_i is the ω-limit of L_i, see Figure 1.

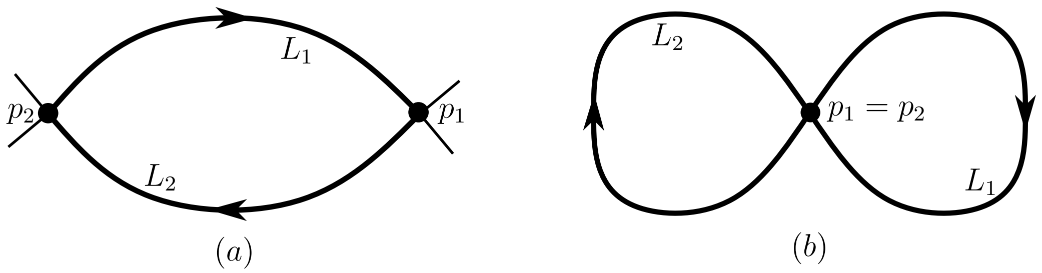
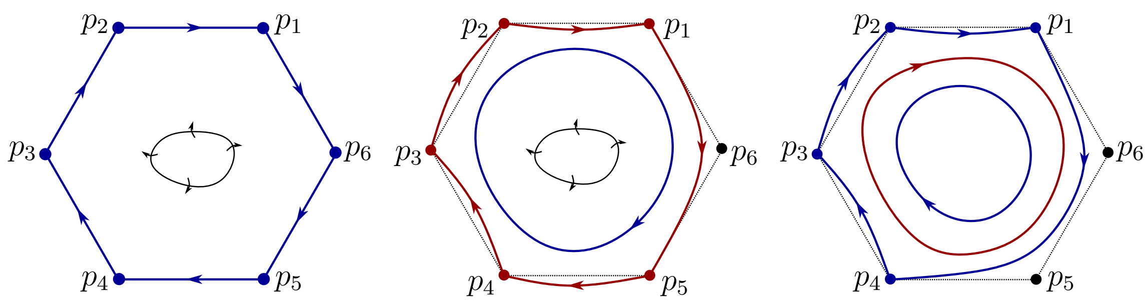


Figure 1 – Illustration of Γ^2, with (a) distinct and (b) non-distinct hyperbolic saddles. Source: Figure 1 of [2].

Roughly speaking, we say that Γ^n has *cyclicity* greater than or equal to k inside a family of vector fields containing X if it is possible to bifurcate at least k limit cycles from Γ^n for some arbitrarily small perturbation of X inside this family (a more rigorous definition is postponed to Section 4.1). Several authors have results computing exact cyclicities or upper bounds. For instance, Andronov and Leontovich [50] proved that if $n=1$ then generically the cyclicity of Γ^1 is at most one. For accessible and didactic versions of this result, we refer to Andronov et al [51, §29] or Kuznetsov [52, Section 6.2]. If $n=2$, Mourtada [53] proved that generically the cyclicity of Γ^2 is at most 2, and also found sufficient conditions for it to be exactly 2. For $n\in\{3,4\}$, Mourtada [54, 55] also proved similar generic results, showing also the striking result that when $n=4$ there are generic families with cyclicity 5. For more details, we refer to Roussarie [56, Chapter 5].

To obtain such cyclicities, usually the polycycle is broken. In our eleventh main result, we study how to break a given polycycle in such a way that we can “extract” as many limit cycle as possible. In a few words, the geometric idea consists in breaking a given polycycle Γ^n in “sub-polycycles” Γ^{n-1}, $\Gamma^{n-2},\dots$ by casting out its hyperbolic saddles one-by-one in such a way that at least one limit cycle bifurcates at each step, see Figure 8. More precisely, if at a given step the polycycles Γ^{n_0} and Γ^{n_0-1} have opposite stabilities, then it follows from the Poincaré-Bendixson Theorem (and some technical results) that at least one limit cycle of odd multiplicity bifurcates when we bifurcate from Γ^{n_0} to Γ^{n_0-1}. Moreover when casting out the hyperbolic saddles we do not need to follow the “canonical” indexation $\{p_1,\dots,p_6\}$, as in Figure 2. Rather at each step we can choose which singularity to expel in order to maximize the number of stability changes and thus the number of limit cycles. At Figure 2 for example, one could start the process by expelling p_4 and then p_1 and etc.



(a) Unperturbed. (b) First perturbation. (c) Second perturbation.

Figure 2 – Illustration of the bifurcation process. Blue means stable and red means unstable. Colors available in the online version. Source: Figure 2 of [2].

As we shall see in Proposition 17, any possible combination of stability changes is realizable by a *polynomial* vector field of degree n. Figure 2 for example is realizable by a polynomial system of degree 6. Hence, the possibility to choose the most convenient singularity to expel at each steep of the bifurcation process is very important in order to obtain better lower bounds for the number of limit cycles.

The precise statement of our main results is more technical and thus we postpone it to Chapter 4. However, the intuitive statement is as follows.

Theorem K. *Let X be a C^∞ system* (1.1) *with a hyperbolic polycycle Γ^n. Let $\Delta(\Gamma^n) \in \{0,\dots,n\}$ be the maximum number of stability changes allowed by any combination of sub-polycycles $\Gamma^n, \Gamma^{n-1}, \dots, \Gamma^1, \Gamma^0$. Then the cyclicity of Γ^n is at least $\Delta(\Gamma^n)$.*

As anticipated, it shall be proved in Proposition 17 that any possible combination of stability changes are realizable by polynomial vector fields of degree n. This fact, in addition with Theorem K results in the proof that for any $n \in \mathbb{N}$, $n \geqslant 3$, there is a *polynomial* $X \in \mathcal{X}^n$ in which we can apply the geometric reasoning given by Figure 2, with the perturbations inside $\mathcal{X}^n$ and arbitrarily small, in relation to the coefficients topology (1.3). More precisely, we have the following main result.

Theorem L. *For each $n \geqslant 3$ there is $X \in \mathcal{X}^n$ with a polycycle Γ^n with cyclicity at least n.*

Knowing that the cyclicity of a hyperbolic polycycle may be reduced to the number of zeros of equations of the form

$$\left(\dots\left((t^{r_1(\mu)} + b_1(\mu))^{r_2(\mu)} + b_2(\mu)\right)^{r_3(\mu)} \dots + b_{n-1}(\mu)\right)^{r_n(\mu)} + b_n(\mu) = \alpha(\mu)t, \tag{1.11}$$

where $\mu \in \mathbb{R}^N$ is a certain parameter [57], the family of vector fields given by Theorem L may be useful to prove that the zeros of (1.11) are realizable by limit cycles of polynomial vector fields of degree n. More details on this discussion are postponed to Section 4.1.7. Theorems K and L are proved in Section 4.1, which is based on [2].

As noticed, the strategy so far to bifurcate limit cycles from a given polycycle is heavily based on the idea of *breaking* the polycycle. In the final main results of this monograph, we explore the cyclicity of *persistent* polycycles, i.e. the act of bifurcating limit cycles from a polycycle without breaking it. In a few words, the main idea here is to treat the polycycle as a monodromic singularity, e.g. a non-degenerated weak focus, and to change its stability recursively without breaking it.

To this end, let Γ^n be a hyperbolic polycycle with saddles $\{p_1, \dots, p_n\}$, and let $\lambda_i^s < 0 < \lambda_i^u$ be the associated eigenvalues of p_i. The *hyperbolicity ratio* of p_i is the positive real number

$$\lambda_i = \frac{|\lambda_i^s|}{\lambda_i^u}.$$

The *graphic number* of Γ^n is the positive real number given by,

$$r(\Gamma^n) = \prod_{i=1}^{n} \lambda_i. \tag{1.12}$$

Cherkas [58] proved that if $r(\Gamma^n) \neq 1$, then Γ^n has a well defined stability. More precisely, if $r(\Gamma^n) > 1$, then Γ^n is stable (i.e. it attracts the orbits in the region where the first return map is defined). Similarly, if $r(\Gamma^n) < 1$, then Γ^n is unstable. Since $r(\Gamma^n)$ depends continuously on smooth perturbations, it follows that if $r(\Gamma^n) \neq 1$, then Γ^n has no change of stability for small perturbations that do not break the polycycle. Following Sotomayor [59, Section 2.2], when $r(\Gamma^n) \neq 1$ we say that Γ^n is *simple*.

We may say from the result of Cherkas that (1.12) acts like the trace of a non-degenerated weak focus, also denoted as the "zero" Lyapunov constant. Hence, to bifurcate limit cycles without breaking Γ^n we need to find the constant that acts like the first Lyapunov constant. In our next main results, we build upon the works of Marín and Villadelprat [60, 61, 62, 63] to do this.

Theorem M. *Let $\{X_\mu\}_{\mu\in\Lambda}$ be a C^∞-family of planar analytic vector fields having a persistent polycycle Γ^μ with hyperbolic saddles $p_1, \dots, p_n$. Then the first return map associated to Γ^μ is given by*

$$\mathscr{R}(s;\mu) = s^{r(\mu)}\big(A_{1,n} + \mathcal{F}_\ell^\infty(\mu_0)\big), \tag{1.13}$$

for any given $\ell \in \big(0, \min\{\Lambda_{i,n} \colon i \in \{0, \dots, n\}\}\big)$. Moreover, the following holds:

(a) $\mathrm{C}ycl(\Gamma^\mu, \mu_0) = 0$, *if* $r(\mu_0) \neq 1$*;*

(b) $\mathrm{C}ycl(\Gamma^\mu, \mu_0) \geqslant 1$, *if* $r(\mu_0) = 1$, $r(\mu) - 1$ *changes signs at* μ_0 *and* $\mathscr{R}(\cdot;\mu_0) \not\equiv Id$*;*

(c) $\mathrm{C}ycl(\Gamma^\mu, \mu_0) \leqslant 1$, *if* $A_{1,n}(\mu_0) \neq 1$*;*

(d) $\mathrm{C}ycl(\Gamma^\mu, \mu_0) \geqslant 2$, *if* $r(\mu_0) = A_{1,n}(\mu_0) = 1$, $r - 1$, $A_{1,n} - 1$ *are independent at* μ_0 *and* $\mathscr{R}(\cdot;\mu_0) \not\equiv Id$.

Here $\mathrm{C}\,ycl(\Gamma^{\mu}, \mu_0)$ denotes the cyclicity of the hyperbolic polycycle Γ^{μ_0} of the analytic vector field X_{μ_0}, while $\mathcal{F}_{\ell}^{\infty}(\mu_0)$ denotes a well-behaved reminder function, whose precise definition we postpone to Section 4.2. Roughly speaking, Theorem M provides an expression for the first return map of Γ^{μ}, and prove that the graphic number $r(\mu) = r(\Gamma^{\mu})$, and the constant $A_{1,n}$ (whose precise expression is postponed to Section 4.2) act like the zero and the first Lyapunov constants. Moreover, we can use Theorem M to extended Cherka's result by noticing from (1.13) that if $r(\mu_0) = 1$, then the stability of Γ^{μ_0} is given by then sign of $A_{1,n} - 1$.

Theorem N. *Let X be a planar C^{∞}-vector field having a polycycle Γ with hyperbolic saddles $p_1, \dots, p_n$ and graphic number $r = 1$. If $A_{1,n} < 1$ (resp. $A_{1,n} > 1$), then Γ is stable (resp. unstable).*

We remark that at Theorem M the hypothesis of each X_{μ} being analytic is only used to obtain the cyclicity of Γ. In particular, if each X_{μ} is C^{∞}, then (1.13) still holds. Hence, we have in Theorem N the hypothesis that X is C^{∞} rather than analytic.

In the final main result of this monograph, we generalize Theorem M, under a suitable hypothesis, to find the number that act likes the second Lyapunov constant.

Theorem O. *Let $\{X_{\mu}\}_{\mu\in\Lambda}$ be a smooth family of planar analytic vector fields having a persistent polycycle Γ^{μ} with hyperbolic saddles $p_1, \dots, p_m, p_{m+1}, \dots, p_n$. Let $\mu_0 \in \Lambda$ be such that $\lambda_i(\mu_0) < 1$ for $i \in \{1, \dots, m\}$ and $\lambda_i(\mu_0) > 1$ for $i \in \{m+1, \dots, n\}$. Then the first return map of Γ^{μ} is given by*

$$\mathscr{R}(s;\mu) = s^{r(\mu)}\big(A_{1,n} + \mathcal{A}s^{\Lambda_{0,m}} + \mathcal{F}_{\ell}^{\infty}(\mu_0)\big), \tag{1.14}$$

for any given $\ell \in (\Lambda_{0,m}^0, \min\{r(\mu_0), 2\Lambda_{0,m}^0, 1\})$. Moreover, the following holds:

(a) $\mathrm{C}\,ycl(\Gamma, \mu_0) \leqslant 2$, *if* $\mathcal{A}(\mu_0) \neq 0$;

(b) $\mathrm{C}\,ycl(\Gamma, \mu_0) \geqslant 3$ *if* $r(\mu_0) = A_{1,n}(\mu_0) = 1$, $\mathcal{A}(\mu_0) = 0$, $r-1,\ A_{1,n}-1,\ \mathcal{A}$ *are independent at* μ_0 *and* $\mathscr{R}(\cdot;\mu_0) \not\equiv Id$.

Again, the precise definition of the constants appearing in Theorem O are postponed to Section 4.2. We only remark that $\mathcal{A}$ plays the role of the second Lyapunov constant.

Similarly to Theorem N we notice that under the hypothesis of Theorem O, if $r(\mu_0) = 1$ and $A_{1,n} = 1$, then it follows from (1.14) that a sufficient condition to define the stability of Γ^{μ_0} can be obtained from the sign of $\mathcal{A} - 1$. Theorems M, N and O are proved in Section 4.2, which is based on [5].

Finally, we remark that all the results of this monograph are based on the following works.

- A. GASULL AND P. SANTANA, *A note on Hilbert 16th problem*, **Proc. Am. Math. Soc.** 153, No. 2, 669-677 (2025).

- A. GASULL AND P. SANTANA, *On a variant of Hilbert's 16th problem*, **Nonlinearity** 37, No. 12, Article ID 125012, 15 p. (2024).
- A. GASULL AND P. SANTANA, *Limit cycles and invariant algebraic curves*, **Commun. Pure Appl. Anal.** 28, 214-220 (2026).
- G. FAZOLI AND P. SANTANA, *Planar vector fields without invariant algebraic curves*, Preprint, arXiv:2511.13596 (2025).
- C. BUZZI, A. GASULL AND P. SANTANA, *On the cyclicity of hyperbolic polycycles*, **J. Differ. Equations** 429, 646-677 (2025).
- L. ARAKAKI AND P. SANTANA, *On the cyclicity of persistent hyperbolic polycycles*, **J. Dyn. Differ. Equations** (2025).

More precisely, Chapter 2 is based on the first three works, Chapter 3 follows from the fourth one, and Chapter 4 is derived from the last two. At the end, we draw some conclusions in Chapter 5.

5 CONCLUSION

There are two main conclusions that we would like to highlight from this monograph. The first one is that the Hilbert number is realizable by structurally stable systems. In particular, when considering a polynomial system satisfying Hilbert upper bound, one may supposed without loss of generality that every limit cycle is hyperbolic, see Theorem B.

This structurally stability allowed we to consider a system realizing the Hilbert number and then perturb it in any suitable way, in order to draw other results. For example, by embedding such a system into the space of systems of higher degree, we were able to prove that the Hilbert number is a strictly increasing function, whenever finite, see Theorem A.

It also allowed the embedding of such a system in the space of systems with a prescribed invariant algebraic curve in such a way that we can perturb every oval of this curve into an algebraic limit cycle, and at the same time that every previous limit cycle persist. Hence, the original Hilbert number can be used as an lower bound for the Hilbert number associated with the prescribed invariant algebraic curve, see Theorem E. This technique can be applied to write the associated Hilbert numbers of models governed by systems of differential equations in function of the original Hilbert number. See for example Corollary 1 for an application of in the family of Kolmogorov systems, and Corollary 2 for an application into a model of Game Theory. It was also used to draw a relation between Hilbert's and Harnack's numbers, see Corollary F.

The second main conclusion is the fact that generically speaking a planar polynomial system of differential equation does not have invariant algebraic curves. This genericity holds both from the topological and probabilistic point of view, i.e. there is a residual set of systems, which is also of total measure, whose elements does not have invariant algebraic curves. See Theorem H. Moreover, if we consider the family of systems with a prescribed invariant algebraic curve, then, in the same manner, generically this curve will be the only invariant algebraic curve. See Theorem G.

In particular, this second conclusion can be combined with the first one to obtain the fact that the Hilbert number is realizable by a structurally stable system without invariant algebraic curves, see Theorem J. Therefore, one may suppose without loss of generality that every limit cycle if hyperbolic and non-algebraic. In particular, for most conclusions and applications described above, the system can also be supposed without invariant algebraic curves and the limit cycles non-hyperbolic, see Corollary 3.

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