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Covariant Quantization of the Superstring: Analysis of the RNS and Pure Spinor Formalisms and Perspectives on a New Construction

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COVARIANT QUANTIZATION OF THE SUPERSTRING: ANALYSIS OF THE RNS AND PURE SPINOR FORMALISMS AND PERSPECTIVES ON A NEW CONSTRUCTION

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Dedico este trabajo a mi familia:
Juan, Rosmery y Joel

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“Nothing is straightforward
until you do it”

N. B.

Resumo

Esta dissertação investiga duas abordagens complementares de folha de mundo para a teoria das supercordas: o formalismo de Ramond–Neveu–Schwarz (RNS) e o formalismo de espinores puros. Inicialmente, analisamos a estrutura da supercorda no formalismo RNS, com ênfase em sua ação, simetrias conformes e superconformes, e nos sistemas de fantasmas bc e $\beta\gamma$. A construção de operadores de vértice, as propriedades dos estados fundamentais e a implementação da projeção GSO são discutidas em detalhe, juntamente com seu papel na garantia da supersimetria no espaço-tempo. Em seguida, examinamos a prescrição para amplitudes de espalhamento no formalismo RNS, destacando a função dos operadores de mudança de picture e a relação com a anomalia da corrente de número fantasma.

Na segunda parte, exploramos o formalismo de espinores puros, destacando sua formulação manifestamente supersimétrica. Após apresentar a restrição do espinor puro e a decomposição dos espinores, estudamos as propriedades conformes dos fantasmas de espinor puro e a construção de operadores de vértice. A prescrição em nível árvore para amplitudes de espalhamento é analisada nesse contexto, com ênfase em suas vantagens em relação ao formalismo RNS.

Finalmente, introduzimos uma versão modificada do formalismo de espinores puros que surge da substituição dos campos do formalismo RNS por certos campos do formalismo de espinores puros. Essa construção possibilita a definição de operadores de vértice manifestamente supersimétricos e covariantes, e sugere uma carga BRST modificada com covariância explícita sob $U(5)$. Como exemplo específico, o operador de vértice do glúon sem massa foi analisado. Ao estender essa construção, tanto o glúon quanto o gluíno puderam ser descritos dentro de uma formulação em termos de supercampos, onde os oito graus de liberdade do gluíno foram naturalmente organizados em dois conjuntos de quatro, cada um associado a um *picture* distinto. Embora a formulação não esteja completamente detalhada nesta dissertação, os resultados obtidos indicam uma nova conexão entre as abordagens RNS e *pure spinor*.

Palavras Chaves: Formalismo RNS da Supercorda; Formalismo de espinores

puros; Teoria de Super-Yang-Mills; Amplitudes de espalhamento.

Áreas do conhecimento: Física matemática; Física das partículas elementais e campos; Teoría das cordas; Física de altas energias.

Abstract

This thesis investigates two complementary worldsheet approaches to superstring theory: the Ramond–Neveu–Schwarz (RNS) formalism and the pure spinor formalism. We first analyze the structure of the RNS superstring, focusing on its action, conformal and superconformal symmetries, and the associated bc and $\beta\gamma$ ghost systems. The construction of vertex operators, the properties of the ground states, and the implementation of the GSO projection are discussed in detail, together with their role in ensuring spacetime supersymmetry. We then examine the prescription for superstring scattering amplitudes, emphasizing the function of picture changing operators and the interplay with the ghost number current anomaly.

In the second part, we explore the pure spinor formalism, highlighting its manifestly supersymmetric formulation. After presenting the pure spinor constraint and the decomposition of spinors, we study the conformal properties of the pure spinor ghosts and the construction of vertex operators. The tree-level prescription for scattering amplitudes is analyzed within this framework, with particular emphasis on its advantages over the RNS approach.

Finally, we introduce a modified version of the pure spinor formalism that comes from the replacement of the RNS fields by some pure spinor fields. This construction enables the definition of manifestly supersymmetric and covariant vertex operators and suggests a modified BRST charge with explicit covariance under both $U(5)$. As a specific example, the massless gluon vertex operator was analyzed. By extending this construction, both the gluon and the gluino could be described within a superfield framework, where the eight gluino degrees of freedom were naturally arranged into two sets of four, each associated with a different picture. While the formulation is not fully detailed in this thesis, the results obtained indicate a novel link between the RNS and pure spinor approaches.

Keywords: RNS Superstring; Super-Yang-Mills; Pure spinor superstring; scattering amplitudes.

Knowledge areas: Mathematical Physics; String theory; Elementary Particle

Physics and Fields; High energy physics.

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Chapter 1

Introduction

Superstring theory has emerged over the past decades as one of the most promising frameworks for unifying quantum mechanics and general relativity. By postulating that the fundamental constituents of nature are not point particles but one-dimensional extended objects, it provides a consistent ultraviolet completion of perturbative quantum gravity and an elegant explanation of the rich spectrum of particles and interactions observed in high energy physics. Furthermore, its inherent connection to supersymmetry and higher-dimensional geometries has deepened our understanding of both mathematical structures and physical principles underlying fundamental interactions.

A central feature of string theory is its dependence on the choice of worldsheet description, which encodes the dynamics of strings through two-dimensional conformal field theories. Different worldsheet formalisms not only provide complementary insights into the theory but also give rise to distinct computational tools and conceptual frameworks. Among them, the Ramond–Neveu–Schwarz (RNS) formalism and the pure spinor formalism occupy a prominent position.

RNS Formalism and its Challenges

The RNS formalism was one of the earliest successful formulations of the superstring. It extends the bosonic string by incorporating worldsheet fermions and local supersymmetry, thereby producing a consistent spectrum that includes spacetime fermions after the Gliozzi–Scherk–Olive (GSO) projection. This projection plays a fundamental role in ensuring modular invariance and eliminating tachyonic instabilities, while simultaneously enforcing spacetime supersymmetry. The RNS approach provides a clear geometric picture, where conformal and superconformal symmetries of the worldsheet play a guiding role.

Despite these strengths, the RNS formalism suffers from intrinsic technical difficulties. The presence of worldsheet supersymmetry leads to the necessity of introducing the $\beta\gamma$ and bc ghost systems, and the construction of physical states requires the use of vertex operators in different ghost "pictures." Scattering amplitudes then rely on the insertion of picture changing operators (PCOs), which complicates computations and often obscures spacetime supersymmetry. While numerous prescriptions exist to address these challenges, the lack of manifest supersymmetry and the dependence on picture-changing procedures remain a central drawback of the RNS framework.

Pure Spinor Formalism

In contrast, the pure spinor formalism, developed by Berkovits and collaborators, provides a manifestly supersymmetric formulation of the superstring. The key idea is the introduction of bosonic spinor variables λ^α, subject to the pure spinor constraint $\lambda^\alpha \gamma^m_{\alpha\beta} \lambda^\beta = 0$. These variables serve as ghosts for a BRST-like operator that enforces the physical state conditions, thereby eliminating the need for picture changing operators and allowing scattering amplitude prescriptions that preserve spacetime supersymmetry at all stages of computation.

This approach has several remarkable advantages. It unifies the description of superstrings in various backgrounds, streamlines the computation of loop amplitudes, and has been successfully applied to both ten-dimensional super-Yang–Mills theory and type II supergravity. Moreover, its manifest supersymmetry makes it particularly appealing for exploring the structure of higher-loop corrections and dualities in string theory. Nevertheless, the pure spinor formalism introduces its own conceptual challenges, including the nontrivial geometry of the pure spinor space, the treatment of its constrained variables, and the construction of composite operators.

Towards a New Formalism

The RNS and pure spinor formalisms thus represent two complementary approaches: the former grounded in a traditional conformal field theory perspective with explicit worldsheet supersymmetry, and the latter offering manifest spacetime supersymmetry at the cost of introducing constrained ghost variables. A natural question arises: can one develop a hybrid formalism that combines the computational simplicity of pure spinors with structural elements inherited from the RNS ghosts?

In the final part of this work, we outline such a construction. It emerges from the replacement of RNS fermions by new pure spinor variables, and some similarity transformations. We analyze their supersymmetry charges, construct $U(5)$ and $SO(10)$–covariant operators, and provide preliminary steps towards a manifestly supersymmetric and covariant BRST operator.

Although the formulation is still under active development and is not fully detailed here, with a complete description for the scattering amplitude prescription of this new formalism left for future work, the preliminary results suggest a promising new bridge between the two formalisms. In particular, this approach may provide new perspectives on the calculation of superstring scattering amplitudes, potentially resolving some of the long-standing issues associated with picture changing while retaining the advantages of manifest supersymmetry.

This dissertation is organized as follows. In Chapter 2, we introduce the main ideas behind the RNS formalism, including its action, symmetries, and ghost sector based on [1–3]. Chapter 3 presents the construction of vertex operators and discusses the GSO projection, followed by the analysis of the scattering amplitude prescription and the role of picture changing operators. In Chapter 4 takes results from [4–6]. On this chapter we turn to the pure spinor formalism, reviewing its action, constraints, and ghost structure, as well as the construction of vertex operators and the tree-level prescription for scattering amplitudes. Finally, Chapter 5 presents a introdution for a new variant of the pure spinor formalism, with a specific example where we analyze the massless gluon vertex operator and extend the construction to a superfield description that simultaneously incorporates gluon and gluino degrees of freedom. Also some hints on the construction of the full covariant manifest supersymmetric BRST and supercharge operators is given, taking [7] as the main reference.

Part I

Ramond-Neveu-Schwarz formalism

Chapter 2

RNS superstring and Vertex Operators

The RNS (named after Ramond, Neveu and Schwarz) formalism establishes a mathematical framework to study superstring theory. The main objective of such formalism was the description of fermionic matter in the model, and also to get rid of tachyons described by bosonic string theory.

On this work we use $\eta^{mn} = \text{diag}(-1, +1, \ldots, +1)$ as the signature of the metric.

2.1 RNS superstring action and symmetries

We start by considering the following gauge fixed world-sheet supersymmetric action

$$S_{\text{matter}} = \frac{1}{2} \int d^2 z \left(\partial X^m \bar{\partial} X_m - \Psi^m \bar{\partial} \Psi_m - \tilde{\Psi}^m \partial \tilde{\Psi}_m \right), \tag{2.1}$$

where X^m, Ψ^m and $\tilde{\Psi}^m$ are D-dimensional spacetime vectors. These can be seen as scalar or spinor fields of the two-dimensional flat worldsheet with coordinates $(z, \bar{z})$. The equations of motion can be easily obtained:

$$\partial \bar{\partial} X^m(z, \bar{z}) = 0, \quad \bar{\partial} \Psi^m(z, \bar{z}) = 0, \quad \partial \tilde{\Psi}^m(z, \bar{z}) = 0. \tag{2.2}$$

Notice that from these equations we can conclude that $\partial X^m(z)$ and $\Psi^m(z)$ are holomorphic functions, while $\bar{\partial} X^m(\bar{z})$ and $\tilde{\Psi}^m(\bar{z})$ are antiholomorphic. Due to this distinction on holomorphic and antiholomorphic fields it is possible to describe each of them separately.

Grassmann spacetime vectors Ψ^m used in the description of the RNS superstring must also satisfy the open- or closed-string boundary conditions

2.1.1 Closed string case

If we work with cylinder coordinates, we require the periodicity condition $w \cong w + 2\pi$. There are two possible choices for the boundary conditions in Ψ^m

and its analogous antiholomorphic field;

$$\Psi^m(w+2\pi) = \exp(2\pi i\nu)\Psi^m(w), \tag{2.3}$$

$$\tilde{\Psi}^m(\overline{w}+2\pi) = \exp(-2\pi i\tilde{\nu})\tilde{\Psi}^m(\overline{w}). \tag{2.4}$$

where $\nu, \tilde{\nu} = 0, 1/2$ correspond to Neveu-Schwarz (NS) or Ramond (R) fields. This will lead us to separate the Hilbert space of states into four sectors defined by $(\nu, \tilde{\nu})$. They are denoted as NS-NS, NS-R, R-NS and R-R.

2.1.2 Open string case

By using the doubling trick[1] we only have to deal with the holomorphic Ψ^m field, since it is related with $\tilde{\Psi}^m$ in cylinder coordinates $\left(\sigma^1, \sigma^2\right)$ as a function with a extended range $\sigma^1 \in (0, 2\pi]$

$$\Psi^m\left(\sigma^1, \sigma^2\right) = \tilde{\Psi}^m\left(2\pi - \sigma^1, \sigma^2\right),\ \sigma^1 \in (\pi, 2\pi]. \tag{2.5}$$

Considering the condition that the surface term must vanish at the boundaries of $\sigma^1 \in (0, \pi]$ of the open string we have the following possible choices

$$\Psi^m\left(\sigma^1 = 0\right) = \exp(2\pi i\nu)\Psi^m\left(\sigma^1 = 0\right), \tag{2.6}$$

$$\Psi^m\left(\sigma^1 = \pi\right) = \exp(2\pi i\nu')\Psi^m\left(\sigma^1 = \pi\right). \tag{2.7}$$

We can renormalize Ψ^m to get rid of the ν' exponential factor, and we remain with two possible choices for ν, which define the NS $(\nu = 0)$ and R $(\nu = 1/2)$ sectors. Since the open string worldsheet has boundaries, it can be mapped to the upper half-plane of a complex plane, with the boundary corresponding to the real axis. This justifies restricting our attention to operator insertions and OPEs along the real line when working in such planar coordinates.

The closed string spectrum can essentially be understood as the combination of two open string spectra. Consequently, we will focus primarily on the open string case in the remainder of this work.

[1] It basically consists on relating holomorphic fields above the real line on the radial plane, and antiholomorphic fields that are below such line, as the complex conjugated of each other. A more complete description can be found on [1]

2.2 Conformal and superconformal symmetries

The system still has two kinds of residual symmetry. The first one is the conformal symmetry generated by the energy-momentum tensor[2]

$$T_{\text{matter}}(z) = -\frac{1}{2}\left[\partial X^m \partial X_m - \Psi^m \partial \Psi_m\right], \tag{2.8}$$

and also superconformal invariance, that mixes both commuting and anticommuting fields:

$$\delta\left[\partial X^m(z)\right] = \epsilon\partial[v(z)\Psi^m(z)], \quad \delta\Psi^m(z) = \epsilon v(z)\partial X^m(z), \tag{2.9}$$

where $v(z)$ is a holomorphic function. Such transformation is generated by

$$G_{\text{matter}}(z) = -\frac{1}{2}\Psi^m \partial X_m. \tag{2.10}$$

In bosonic string theory conformal transformations can also be seen as some transformation on the complex coordinates of the worldsheet by a holomorphic function $\delta z = \epsilon v(z)$. The same can be done for superconformal transformations: An infinitesimal conformal transformation is parametrized by a superfield $\mathbb{V}(z,\theta) = v_0(z) + \theta v_1(z)$ as

$$\delta z = v_0(z) + \frac{1}{2}\theta v_1(z) = \mathbb{V} + \theta\delta\theta, \tag{2.11}$$

$$\delta\theta = \frac{1}{2}[v_1(\theta) + \theta\partial v_0(z)] = \frac{1}{2}D\mathbb{V}. \tag{2.12}$$

It is important to mention that the composition of two superconformal transformations is a conformal transformation, as the reader can verify.

The conformal structure of the theory we are describing let us to define operator product expansions (OPEs) between the operator fields. We have

$$X^m(z)X^n(0) \sim -\eta^{mn} \ln|z|, \tag{2.13}$$

$$\Psi^m(z)\Psi^n(0) \sim \frac{\delta^{mn}}{z}, \tag{2.14}$$

and the remaining ones are non-singular expansions. It is important to emphasize

[2]Normal ordering of the product of operators is defined implicitly along this work. It will be made explicit if in some calculation is required.

that the XX OPE is defined on such way with z being on the real axis. This restriction is sufficient because, in open superstring theory, vertex operators are inserted at the boundary of the worldsheet, which can be mapped to the real axis in the upper half-plane formulation, making it the only region relevant for physical observables.

A useful tool to compute OPEs is the following prescription for the product of normal-ordered operators:

$$:F(X(z))::G(X(w)):=e^{\left[\int dz'dw'(-\ln|z'-w'|)\frac{\delta}{\delta X_F(z')}\cdot\frac{\delta}{\delta X_G(w')}\right]}:F(X(z))G(X(w)):, \tag{2.15}$$

where the integrals are over the real axis and the functional derivatives $\delta/\delta X_F$ and $\delta/\delta X_G$ only evaluate terms on $F(X)$ and $G(X)$ respectively. An analogous expresion can be written for any other pair of fields that contain singular terms on their OPEs. With this we can verify get some useful relations for our work

$$e^{ik_1\cdot X(z)}e^{ik_2\cdot X(0)}=|z|^{k_1\cdot k_2}:e^{ik_1\cdot X(z)}e^{ik_2\cdot X(0)}:, \tag{2.16}$$

$$\partial X_m(z)F(X(0))\sim-\frac{1}{z}\partial_m F(X(0)), \tag{2.17}$$

with $\partial_m=\partial/\partial X^m$.

There are some special operators $O(z)$ that transforms under a holomorphic change of coordinates $z\to f(z)$ as

$$\mathcal{O}(z)\to(f'(z))^h\mathcal{O}(f(z)). \tag{2.18}$$

We define them as primary or tensor operators with conformal weight h. This behavior is encoded in its operator product expansion (OPE) with the energy-momentum tensor, given by

$$T(z)\mathcal{O}(w)\sim\frac{h\,\mathcal{O}(w)}{(z-w)^2}+\frac{\partial\mathcal{O}(w)}{z-w}+\text{regular terms}. \tag{2.19}$$

This OPE defines the operator as primary and governs how it contributes to correlation functions and the realization of conformal symmetry on the worldsheet. Based on this it is possible to verify that (X,Ψ) have conformal weight $(h_X,h_\Psi)=(0,1/2)$

For the product of two primary operators A and B with conformal weight h_A

and h_B it is one can verify that

$$T(z)AB(w) \sim \frac{(h_A + h_B)}{(z-w)^2} AB(w) + \frac{1}{z-w}\partial[AB](w). \tag{2.20}$$

Also, it is easy to see that the derivative $\partial = \partial/\partial z$ of some tensor operator of weight h will be primary as well, with conformal weight $h+1$.

2.3 Ghost fields

Since the worldsheet metric and gravitino have been fixed to get action (2.1) the introduction of ghost fields is required to have a quantum description of the theory. We introduce two pairs of holomorphic ghost fields: (b, c) anticommuting pair appears because of conformal symmetry, and the (β, γ) commuting pair deals with superconformal transformations. The ghost contribution of the RNS action is

$$S_{\text{ghost}} = \int d^2z(b\bar{\partial}c + \beta\bar{\partial}\gamma). \tag{2.21}$$

Notice that such the ghost sector of the action has conformal invariance, since b, c, β and γ have weights $(2,0)$, $(0,-1)$, $(3/2,0)$ and $(0,-1/2)$ respectively. Superconformal symmetry is also a feature of the action. The generators are given by

$$T_{\text{ghost}} = c\partial b + 2(\partial c)b - \tfrac{1}{2}\gamma\partial\beta - \tfrac{3}{2}(\partial\gamma)\beta, \tag{2.22}$$

$$G_{\text{ghost}} = c\partial\beta + \frac{3}{2}(\partial c)\beta - 2b\gamma. \tag{2.23}$$

The same can be generalized for antiholomorphic ghost fields to study the closed string spectrum. In the open string case the antiholomorphic and holomorphic fields are related by using the doubling trick.

The ghost fields behave in the same way as a bc and $\beta\gamma$ conformal field theories, with parameters $\lambda_{bc} = 2$ and $\lambda_{\beta\gamma} = 3/2$. The OPEs are defined as

$$c(z)b(0) = b(0)c(z) \sim \frac{1}{z}, \tag{2.24}$$

$$\gamma(z)\beta(0) = -\beta(0)\gamma(z) \sim \frac{1}{z}. \tag{2.25}$$

Since we are looking for a quantum description of the matter-ghost system, we have to ensure that the theory does not have anomalies (so T must be a primary conformal operator). The central charge of $S_{\text{matter}} + S_{\text{ghost}}$ is $3D/2 - 15$. Therefore, to preserve conformal symmetry, we must impose $D = 10$ throughout this work.

2.4 Matter ground state and state operator correspondence

To describe the Hilbert space of the theory we can follow a similar procedure as in bosonic string theory. First it is possible to define the superconformal generators in terms of a Laurent series as

$$T(z) = \sum_n \frac{L_n}{z^{n+2}}, \; G(z) = \sum_r \frac{G_r}{z^{r+3/2}}, \tag{2.26}$$

with $r \in \mathbb{Z}$ for the R sector and $r \in \mathbb{Z} + 1/2$ for the NS sector. The algebra becomes

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{1}{12}c\left(m^3 - m\right)\delta_{m+n,0}, \tag{2.27}$$

$$\{G_r, G_s\} = 2L_{r+s} + \frac{1}{12}c\left(4r^3 - 1\right)\delta_{m+n,0}, \tag{2.28}$$

$$[L_m, G_r] = \frac{m-2r}{2}G_{m+r}, \tag{2.29}$$

and it is called the Ramond algebra for the r, s integer and the Neveu-Schwarz algebra for the r, s half integer. Terms that contain central charge factors can be dropped once we add the ghosts and make $D = 10$. Notice that (L_0, L_1, L_{-1}) form a closed SL_2 algebra.

It is possible to see that in string theory, the operator L_0 corresponds to the worldsheet Hamiltonian, generating translations in worldsheet time. Its eigenvalue determines the energy levels of string states and is directly related to their physical mass through the Virasoro constraints. This relation arises because L_0 generates time translations on the worldsheet cylinder.

We define the highest weight state $|h\rangle$ as the one that satisfies $L_{-n}|h\rangle = 0$ and $G_{-r}|h\rangle = 0$ for $n > 0, r \geq 0$ and $L_0|h\rangle = h|h\rangle$. We can build a full tower of states to define the Hilbert space by applying the raising operators L_{-n} and G_{-r} with $n, r > 0$.

In the same way, it is possible to describe matter fields in terms of Laurent series.

$$\partial X^m = -\frac{i}{2}\sum_n \frac{\alpha_n^m}{z^{n+1}},\ \Psi^m = \sum_r \frac{\Psi_r^m}{z^{r+1/2}}. \tag{2.30}$$

The coefficients (α_n^m, Ψ_n^m) will be promoted to operators. In addition, the zero mode of X^m (which will be denoted by X_0^m) is required to have a non zero commutator with the canonical momentum $\alpha_0^m \equiv P^m$ to have a complete description of the operator $X^m(z)$. It is possible to have the mode operators in terms a contour integral of the field. For example

$$\alpha_{-n}^m = 2i\oint z^{-n}\partial X(z),\ \Psi_{-s}^m = \oint z^{-s-1/2}\Psi^m(z), \tag{2.31}$$

where we denote $\oint = \oint \frac{dz}{2\pi i}$ and the contour encloses the position of the local operator, such as $z = 0$. Notice that for the Ramond sector the definition of the modes is complicated to deal with, since the integral encloses sectors with branch cuts due to the half-integer power of z.

The commutation relation of the coefficients can be obtained from the operator product expansion of the fields, obtaining the following expressions:

$$\left[\alpha_p^m, \alpha_q^n\right] = -p\eta^{mn}\delta_{p,-q}, \tag{2.32}$$

$$\left[\alpha_p^m, X^n\right] = -\eta^{mn}\delta_{p0}, \tag{2.33}$$

$$\{\Psi_r^m, \Psi_s^n\} = -\eta^{mn}\delta_{r,-s}, \tag{2.34}$$

and we can write the L_n and G_r modes of $T(z)$ and $G(z)$as

$$L_m = \frac{1}{2}\sum_{n\in\mathbb{Z}} :\alpha_{m-n}\cdot\alpha_n: +\frac{1}{4}\sum_{r\in\mathbb{Z}+\nu}(2r-m):\psi_{m-r}\cdot\psi_r: +A\delta_{m,0}, \tag{2.35}$$

$$G_r = \sum_{n\in\mathbb{Z}} \alpha_n\cdot\psi_{r-n}, \tag{2.36}$$

where : : in this context denotes creation-annihilation normal ordering, taking $(\alpha_{n>0}^m, \psi_{s>0}^m, P^m)$ and $(\alpha_{n<0}^m, \psi_{s<0}^m, X_0^m)$ as lowering and raising operators respectively. The normal ordering constant A takes the value 10/16 in the Ramond sector, while it vanishes in the Neveu–Schwarz sector.

The vacuum state of the theory $|\text{vacuum}\rangle$ is defined as a state that satisfies

$$\alpha_n^m|\text{vacuum}\rangle = 0,\ \psi_r^m|\text{vacuum}\rangle = 0,\ \text{for } n, r > 0. \tag{2.37}$$

So we can see that $|\text{vacuum}\rangle$ is a highest weight state of L_n modes. Notice that in the NS sector the index r can only have half-integer values, so there is no problem with the definition of the vacuum state. However, in the R sector we can see that ψ_0^m is a well-defined operator that maps vacuum states into itself. So, the ground state in the R sector is degenerate.

Redefining $\Psi_0^m = (2i)^{-1/2}\Gamma^m$ with itself, the Dirac gamma matrices algebra $\{\Gamma^m, \Gamma^n\} = 2\delta^{mn}$ is recovered. So $|\text{vacuum}\rangle$ in the Ramond sector belongs to a spacetime spinor representation of the Dirac algebra on $D = 10$. We can summarize this as

$$\begin{aligned} \text{NS sector vacuum:} &\quad |0\rangle \ \text{(scalar)} \\ \text{R sector vacuum:} &\quad |s\rangle \ \text{(10-dimensional spinor)} \end{aligned} \tag{2.38}$$

It is possible to reduce the 32-dimensional spinor representation into two Weyl representations $16 + 16'$. These two can be distinguished by their eigenvalue under the gamma matrix Γ_{11} (which is the 10-dimensional analog of well known 4-dimensional Γ_5 matrix. From now on we will denote it just as Γ).

By definition it is trivial that $\{\Gamma, \Psi_0^m\} = 0$. We can generalize this by introducing an operator that anticommutes with the full ψ^m. First, we can obtain the mode expansion of the Noether currents associated with Lorentz transformations for the anticommuting part of the superfield (in both R and NS sectors).

$$\Sigma^{mn} = \frac{i}{2} \sum_{r \in \mathbb{Z}+\nu} [\Psi_r^m, \Psi_{-r}^n] \,. \tag{2.39}$$

This is a generalization of the zero mode Lorentz generators. Now we define the world sheet fermion number as

$$F = \sum_{a=0}^{4} S_a, \tag{2.40}$$

where $S_a = i^{\delta_{a,0}} \Sigma^{2a,2a+1}$. We can use it do define the fermion parity operator Γ as

$$\Gamma = \exp(i\pi F) = (-1)^F, \tag{2.41}$$

and verify

$$\Gamma \Psi^m = (-1)^F \Psi^m = -\Psi^m (-1)^F = \Psi^m (-1)^{F+1}, \tag{2.42}$$

so the worldsheet fermion number is defined only $\bmod 2$. We can see that this operator anticommutes then with all the fermionic components of the superfields and commutes with its bosonic part. We can also define the parity of states in the NS sector by using the same operator. We will set the NS vacuum state to have a negative parity

$$(-1)^F |0\rangle_{\text{NS}} = -|0\rangle_{\text{NS}}, \tag{2.43}$$

Therefore, any product of even factors of ψ^m_r will have negative parity. This sector is denoted by NS−. States that contain an odd number of anticommuting components, such as $\Psi^\mu |0\rangle_{\text{NS}}$, posses positive parity and belong to NS+. On the other hand, the action of Γ on the R ground state is as follows

$$|s\rangle_{\text{R}} = \Gamma_{ss'} |s'\rangle_{\text{R}}, \tag{2.44}$$

and in the same way, spinors with positive and negative parity will compose the R+ and R− sectors, respectively. We will see this in more detail at the end of this section.

It is possible to map states of the theory to a set of local operators by using the state-operator correspondence[3]. We use this get operators that are related to the ground states we described. The NS ground state is related to the identity operator $|0\rangle_{\text{NS}} \equiv |1\rangle \cong 1$. We can verify that for for $n \geq 0$

$$\alpha^m_{-n} |1\rangle \cong 2i \oint z^{-m} \partial X(z) \cdot 1 = 0, \tag{2.45}$$

as we defined the vacuum state. The same can be verified for ψ^m_{-s} NS operators.

It is not trivial to get the operator associated to Ramond vacuum state $|\alpha\rangle_{\text{R}}$. To get it we start by doing a Wick rotation to define $\Psi^{10} \equiv i\Psi^0$ and we end up with a $SO(10)$ spacetime symmetry. We must break it by defining $U(5)$ quantities:

$$\psi^a = \frac{1}{\sqrt{2}} \left(\Psi^a + i\Psi^{a+5} \right), \; \psi_a = \frac{1}{\sqrt{2}} \left(\Psi^a - i\Psi^{a+5} \right). \tag{2.46}$$

where $a = 1, \ldots, 5$. If ψ^a transforms under the fundamental representation of $U(5)$ then ψ_a must transform in the antifundamental representation. Since these are fermionic fields obey a 2D CFT, they can be equivalently described in terms of bosonic fields. This procedure is known as bosonization, which allows fermions to

[3] A more extended discussion about this can be found on [1, 8].

be represented as exponentials of chiral bosons. So we can bosonize these fields as

$$\psi^a \cong e^{i\sigma_a},\ \psi_a \cong e^{-i\sigma_a}, \tag{2.47}$$

where σ_a are chiral scalar fields that follow the OPE $\sigma_a(z)\sigma_b(0) \sim -\delta_{ab} \ln z$. Also we must define the energy momentum of bosons as

$$T = -\frac{1}{2}\sum_{a=1}^{5} (\partial\sigma^a)^2\,, \tag{2.48}$$

and to verify that both theories are equivalent one can compute OPEs of the fermionic fields and their bosonized form, and also looking for the conformal weight of ψ^a and $e^{i\sigma}$ verifying that both are equal. Bosonization is helpful since we avoid dealing with branch cuts that appear on the Ramond sector.

We can write the 32 components of the spin field S in chiral basis as

$$S_{s_a}(z) \cong \exp(is\cdot\sigma(z)), \tag{2.49}$$

where $s_a \to (\pm 1/2, \dots, \pm 1/2)$, so the 32 components of S are described by the 2^5 possible ways of choosing eachs_a. One can verify that the 32 components of S have conformal weight $h = 5/8$. The Ramond sector ground state is given by

$$|s\rangle_{\mathrm{R}} = S\,|0\rangle_{\mathrm{NS}} \cong S(z). \tag{2.50}$$

The decomposition in terms of the parity can be easily done in this case. We obtain two sets of 16 component chiral S^α and antichiral S_α sectors, where these can be distinguished by looking to the coefficient $F = \mathrm{sign}\left(\prod_a s_a\right) = -1$ (chiral) or $+1$ (antichiral). Notice that F plays a similar role as the fermion number operator defined on (2.41) as it similarly distinguishes between the two Ramond sectors $R+$(for $F = -1$) and $R-$(for $F = +1$).

2.5 Ghost ground state

2.5.1 *bc* system

To analyze vacuum states of ghost systems we can follow the same procedure. The mode expansion for the fields

$$b(z) = \sum_{m=-\infty}^{\infty} \frac{b_n}{z^{n+2}}, \quad c(z) = \sum_{m=-\infty}^{\infty} \frac{c_n}{z^{n-1}}, \tag{2.51}$$

and from OPEs one can verify the commutation relations

$$\{c_m, b_n\} = \delta_{m,-n}. \tag{2.52}$$

One first guess could be to consider the ground state as $|1\rangle \cong 1$ as we did for the matter fields. Such state satisfies

$$b_n |1\rangle = 0, \; n \geq -1, \tag{2.53}$$

$$c_n |1\rangle = 0, \; n \geq 2, \tag{2.54}$$

but it does not get annihilated by all the positive modes of c ($c_1 |1\rangle \neq 0$).

On the other hand notice that the bc CFT has a similar commutation relation as a fermionic oscillator for the zero modes b_0, c_0. Therefore, if we want to define $|0\rangle_{bc}$ as $b_m |0\rangle_{bc} = c_m |0\rangle_{bc} = 0$ for $m \geq 1$ we would get a degenerate vacuum, where we denote the states as $|\uparrow\rangle$ and $|\downarrow\rangle$. These are related by

$$\begin{aligned} b_0 |\downarrow\rangle &= 0, \quad c_0 |\downarrow\rangle = |\uparrow\rangle, \\ b_0 |\uparrow\rangle &= |\downarrow\rangle, \quad c_0 |\uparrow\rangle = 0. \end{aligned} \tag{2.55}$$

As we will see during the computation of scattering amplitudes it required to define $|0\rangle_{bc} \equiv |\downarrow\rangle$ as the *main vacuum,* and we denote its degenerate partner as $c_0 |0\rangle_{bc}$.

We require to get the operator associated with $|0\rangle_{bc}$. This can be obtained as follows: Since we have $(b_{-1})^2 = 0$ it is possible to establish a relation between $|1\rangle$ and $|0\rangle_{bc}$:

$$|1\rangle = b_{-1} |0\rangle_{bc}. \tag{2.56}$$

This expresion is enough to fulfill the condition $b_n |1\rangle = 0$ for $n \geq -1$. By using the bc anticommutation relation we can also write $|0\rangle_{bc} = c_1 |1\rangle$ and we can write the operator as

$$|0\rangle_{bc} = c_1 |1\rangle \cong \oint z^{-1} c(z) \cdot 1 = c(0). \tag{2.57}$$

2.5.2 $\beta\gamma$ system

In the $\beta\gamma$ CFT it is complicated to get an operator for the vacuum state. We can see two problems, First, to get (2.56) on bc system it was important to consider that the b ghost is fermionic, so the square of its modes is zero. We can not use this fact on β, since it is a bosonic field. In principle this issue can be solved if we work with Dirac delta functions of the ghost fields, such that

$$\beta(z)\delta(\beta(0)) \sim O(z), \ \gamma(z)\delta(\gamma(0)) \sim O(z). \tag{2.58}$$

Notice that these OPEs are easily proposed if we use the property $\lim_{x\to y} f(x)\delta(f(y)) = 0$. This is enough to write the vacuum state of the NS sector as $\delta(\gamma(0))$ as we will see[4].

The second and more complicated problem arise when we require to deal with the R sector, where such description is more complicated because of the branch cuts. Because of this we can work with the same strategy we used to define the matter spin field (2.49): Bosonization.

It will be useful to study the bosonization of the general case of a bosonic ghost $\beta\gamma$ system with weight λ on β and $1-\lambda$ on γ. The energy-momentum tensor is given by

$$T = -\frac{1}{2}[\gamma\partial\beta + \beta\partial\gamma] + \left(\frac{1}{2} - \lambda\right)\partial(\beta\gamma). \tag{2.59}$$

In this case we must bosonize as $\gamma = \eta e^{\phi}$ and $\beta = \partial\xi e^{-\phi}$, with $\phi(z)\phi(0) \sim -\ln z$, and the $\xi\eta$ system play the role of a bc ghost system of weight $h=0$ and $h=1$ respectively. The conformal transformation generator is a sum from the $\xi\eta$ and a linear dilaton CFT.

$$T = -\eta\partial\xi - \frac{1}{2}\partial\phi\partial\phi + \left(\frac{1}{2} - \lambda\right)\partial^2\phi. \tag{2.60}$$

To verify that $\beta\gamma$ and $\phi\eta\xi$ systems are equivalent we can look to their conformal weights. The conformal weight of $e^{ik\phi}$ is $h = \left[k^2 + ik(1-2\lambda)\right]/2$, so $\gamma = \eta e^{\phi}$ has weight $h = 1-\lambda$ and $\beta = \partial\xi e^{-\phi}$ has weight $h = \lambda$, in the same way as the $\beta\gamma$

[4] The approach of the theory in terms of δ distributions is well described on [9]

system.

We go back to the $\lambda = 3/2$ $\beta\gamma$ system we were describing. The mode expansion is given by

$$\begin{aligned} \beta(z) &= \sum_{r\in\mathbb{Z}+\nu} \frac{\beta_r}{z^{n+3/2}}, \quad \gamma(z) = \sum_{r\in\mathbb{Z}+\nu} \frac{\gamma_r}{z^{n-1/2}}, \\ \beta_r &= \oint z^{n+1/2}\beta(z), \quad \gamma_r = \oint z^{n-3/2}\gamma(z). \end{aligned} \tag{2.61}$$

and the mode coefficients satisfy $[\gamma_s, \beta_r] = \delta_{s,-r}$. We define $|0\rangle_{\text{NS},\beta\gamma}$ and $|0\rangle_{\text{R},\beta\gamma}$ in the same way we did for the previous described systems

$$\begin{aligned} \beta_r |0\rangle_{\text{NS},\beta\gamma} = 0, \gamma_r |0\rangle_{\text{NS},\beta\gamma} = 0, \quad r \geq 1/2 \\ \beta_r |0\rangle_{\text{R},\beta\gamma} = 0, \gamma_r |0\rangle_{\text{R},\beta\gamma} = 0. \qquad r \geq 1 \end{aligned} \tag{2.62}$$

On the other hand by doing the OPEs with the identity operator we can verify that

$$\begin{aligned} \text{NS sector:} \quad & \beta_r |1\rangle = 0,\ r \geq -1/2, \\ & \gamma_r |1\rangle = 0,\ r \geq 3/2, \\ \text{R sector:} \quad & \beta_r |1\rangle = 0,\ r \geq 0, \\ & \gamma_r |1\rangle = 0,\ r \geq 2, \end{aligned} \tag{2.63}$$

and it is evident that $|1\rangle \neq |0\rangle_{\text{R/NS},\beta\gamma}$.

To obtain $\mathcal{O}_{\text{R/NS}}(z) \cong |0\rangle_{\text{R/NS},\beta\gamma}$ we begin by examining the NS sector, as the method for the Ramond sector follows in a similar manner. We are looking for some operator that behaves as

$$\beta_{r\geq 1/2} |0\rangle_{\text{NS},\beta\gamma} = \oint z^{r+1/2}\beta(z)\mathcal{O}_{\text{NS}}(0) = 0 \to \beta(z)\mathcal{O}_{\text{NS}}(0) \sim O\left(z^{-1}\right), \tag{2.64}$$

$$\gamma_{r\geq 1/2} |0\rangle_{\text{NS},\beta\gamma} = \oint z^{r-3/2}\gamma(z)\mathcal{O}_{\text{NS}}(0) = 0 \to \gamma(z)\mathcal{O}_{\text{NS}}(0) \sim O(z). \tag{2.65}$$

From (2.65) we see that we can would be able to choose $\mathcal{O}_{\text{NS}}(z) = \delta(\gamma(z))$ as we stated before. However, dealing with distributions of ghost fields is way more complicated. So we can start the bosonization process. Notice that we can define $\mathcal{O}_{\text{NS}}(z) = \delta(\gamma(z)) \cong e^{-\phi(z)}$ because it satisfies the required OPEs

$$\beta(z)\mathcal{O}_{\text{NS}}(0) \cong \partial\xi(z)e^{-\phi(z)}e^{-\phi(0)} \sim \frac{1}{|z|}\partial\xi : e^{-\phi(z)}e^{-\phi(0)} :, \tag{2.66}$$

$$\gamma(z)\mathcal{O}_{\text{NS}}(0) \cong \eta(z)e^{\phi(z)}e^{-\phi(0)} \sim |z|\partial\eta : e^{\phi(z)}e^{-\phi(0)} :, \tag{2.67}$$

where operator normal ordering has been written explicitly. Notice that in the same way we can write $\delta(\beta) = e^{\phi}$.

The procedure is analogous in the R sector, and we end up with the $\beta\gamma$ ground states in R/NS sectors

$$\mathcal{O}_{\text{NS}}(z) = e^{-\phi(z)},\ \mathcal{O}_{\text{R}}(z) = e^{-\phi(z)/2}. \tag{2.68}$$

2.6 Vertex operators

Vertex operators play a central role in string theory, as they provide the bridge between the worldsheet conformal field theory and spacetime physics as we compute scattering amplitudes. The approach we will use to compute the physical spectrum of the superstring is by BRST quantization. The BRST charge is given by

$$Q_{\text{RNS}} = \oint \left(cT^{\text{m,g}} - bc\partial c + \gamma\Psi \cdot \partial X - \gamma^2 b \right), \tag{2.69}$$

where $T^{\text{m,g}} = T^{\text{m}} + T^{\text{g}}$ is the sum of the energy momentum tensor of the matter and ghost fields. It is useful to write $\beta\gamma$ as a bosonized system

$$Q_{\text{RNS}} = \oint \left(cT^{\text{m,g}} - bc\partial c + \eta e^{\phi}\Psi \cdot \partial X - \eta\partial\eta e^{2\phi} b \right), \tag{2.70}$$

and

$$T^{\text{m,g}}(z) = -\frac{1}{2}\partial X^m \partial X_m + \frac{1}{2}\Psi^m \partial \Psi_m - b\partial c - \partial(bc) - \eta\partial\xi - \frac{1}{2}\partial\phi\partial\phi - \partial^2\phi. \tag{2.71}$$

So physical states will be described by local vertex operators $V(z)$ that satisfy $Q_{\text{RNS}} V = 0$. Notice that any vertex operator V is equivalent to $V + Q_{\text{RNS}}\Omega$. In fact, this can be seen as gauge transformations of some physical state. In open superstring theory, vertex operators are confined to the boundary of the worldsheet, which corresponds to the real axis when using planar coordinate representation.

Here we present some vertex operators that will be important in the remaining of this work.

2.6.1 Tachyon vertex operator

Take the case of the product of some matter state on the NS vacuum state $\rho(X)|1\rangle$ with vacuum states of the ghost fields as

$$|\rho\rangle = \rho(X)|0\rangle_{\rm NS} \otimes |0\rangle_{\rm bc} \otimes |0\rangle_{{\rm NS},\beta\gamma} \cong V_\rho(0) = c(0)e^{-\phi(0)}\rho(X(0)), \tag{2.72}$$

where ρ is some bosonic primary operator that is function of $X(z)$. Since this state is composed by the tensor product of ground states of matter and ghost sectors we will obtain the lowest value on the mass spectrum of the superstring.

We look for the conditions that ρ must obey to be a physical state. It is convenient to write[5]

$$Q_0 = \oint \left(cT^{{\rm m},\beta\gamma} + bc\partial c\right),\ Q_1 = \oint \eta e^{\phi}\Psi\cdot\partial X,\ Q_2 = -\oint \eta\partial\eta e^{2\phi}b. \tag{2.73}$$

where $T^{{\rm m},\beta\gamma}$ contains only the energy momentum tensor of matter and the bosonized $\beta\gamma$ system. So the first term

$$Q_0V(0) = \oint_{z=0} \left(cT^{\rm m,g} - bc\partial c\right)(z)c(0)e^{-\phi(0)}\rho(X(0)). \tag{2.74}$$

$$= \oint_{z=0}\left[\frac{1}{2z}\left[\partial^m\partial_m\rho(X(0)) + \rho(X(0))\right]c\partial c(0)e^{-\phi(0)} + \text{regular,}\right] \tag{2.75}$$

where $\partial_m = \partial/\partial X^m$. Then, to have $Q_0V = 0$ we must impose

$$\partial^m\partial_m\rho + \rho = 0, \tag{2.76}$$

$$m^2 = -k^2 = -1. \tag{2.77}$$

This is the field equation for a scalar field with negative squared mass. We say then that V describes tachyons. On the other hand Q_1V and Q_2V vanish since they do not contain singular terms on their OPEs:

$$Q_1V(0) = \oint \eta c(0)\Psi^m(z)\left[z : e^{\phi(z)}e^{-\phi(0)} :\right]\left[\frac{1}{z}\partial_m\rho(X(0)) + \text{regular}\right] = 0, \tag{2.78}$$

$$Q_2 = -\oint \eta\partial\eta\left(\frac{1}{z} + \text{regular}\right)z^2 : e^{2\phi}e^{-\phi(0)} : \rho(X(0)) = 0. \tag{2.79}$$

[5]The separation is done by considering the following: Q_i is said to have some i ghost number that comes from the number of η factors on each term. Adding operators with different ghost numbers is analogous to summing mathematical objects of different types, such as adding a 1-form to a 2-form, which is ill-defined. Because of this each Q_iV must vanish separately.

Tachyonic matter causes problems on the causality of the problem. As we will see at the end of this section this kind of states can be removed by using the GSO projection.

2.6.2 Massless bosonic vertex operator

To get states that correspond to higher values of the mass spectrum of the superstring one can add some other matter field with some conformal weight different than zero. This will change the field equation generated by Q_0V because of the cT term that Q_0 contains. For example, the description of massless bosons can be done by working with the following vertex operator

$$V_A(z) = c(z)e^{-\phi(z)}\Psi^m(z)A_m(X(z)). \tag{2.80}$$

with $A^m(z)$ being some bosonic vector coefficient. The action of Q_{RNS} on such operator gives the following results

$$\begin{aligned}
Q_0V_A(0) &= \oint_{z=0}\left[\frac{1}{2z}\partial^m\partial_m A^n(0)c\partial c(0)\Psi_n(0)e^{-\phi(0)} + \text{regular}\right],\\
Q_1V_A(0) &= \oint \eta c(0)z : e^{\phi(z)}e^{-\phi(0)} : \left(\frac{\eta^{mn}}{z} + \text{regular}\right)\left(\frac{1}{z}\partial_n A_m(X(0)) + \text{regular}\right),\\
&= \oint \eta c(0) : e^{\phi(z)}e^{-\phi(0)} : \left(\frac{1}{z}\partial_n A^n(X(0)) + \text{regular}\right),\\
Q_2V_A(0) &= \oint \eta\partial\eta z^2 : e^{2\phi(z)}e^{-\phi(0)} : \left(\frac{1}{z} + \text{regular}\right)\Psi^m(0)A_m(X(0)) = 0.
\end{aligned} \tag{2.81}$$

Thus, the physical conditions are

$$\partial^m\partial_m A^n(X) = 0,\ \partial_m A^m(X) = 0, \tag{2.82}$$

and the vertex operator describes photons in Lorentz gauge, so there must be nine independent components. Notice that not all the 10 components of $A^m(X)$ are physical. To see this consider the operator $P = ce^{-2\phi}\partial\xi\Omega(X)$ with $\partial^m\partial_m\Omega = 0$. One can verify that

$$QP = ce^{-\phi}\Psi^m\partial_m\Omega. \tag{2.83}$$

So the massless bosonic vertex operator V_A is equivalent to

$$V_A \cong V_A + QP = V_A(z) = ce^{-\phi}\Psi^m[A_m(X) + \partial_m\Omega(X)]. \tag{2.84}$$

Since QP is a exact state it is not considered a physical state. This can be seen also as a gauge transformation of A as $\delta A_m = \partial_m \Omega$, in the same way as in electromagnetism. We end up with a set of eight degrees of freedom for $A_m(X)$ and therefore eight different vertex operators.

2.6.3 Massless fermionic vertex operator

Let us now consider a state built from the ground states of the ghost system and the Ramond vacuum. As previously mentioned, the Ramond vacuum $|0\rangle_{\mathrm{R}}$ can be decomposed in a chiral and antichiral sectors. We will focus in the chiral sector. The 32×32 components of the Γ Dirac matrices can be decomposed in diagonal symmetric γ blocks of 16×16 components, such that these will act over the chiral and antichiral operators

$$\Gamma^m = \begin{pmatrix} 0 & (\gamma^m)^{\alpha\beta} \\ (\gamma^m)_{\alpha\beta} & 0 \end{pmatrix}. \tag{2.85}$$

Consider then the following vertex operator

$$|u\rangle = \sum_{\alpha=1}^{16} u(X,\alpha)\,|\alpha\rangle_{\mathrm{NS}} \otimes |0\rangle_{\mathrm{bc}} \otimes |0\rangle_{\mathrm{R},\beta\gamma} \cong V_u(0) = u^\alpha(X(0))c(0)e^{-\phi(0)/2}S_\alpha(0). \tag{2.86}$$

To get the physical conditions using the BRST charge we can use the following OPEs, that were obtained from the $SO(9,1)$ current operator algebra as discussed on [3]:

$$\Psi^m(z)S_\alpha(0) \sim \frac{\gamma^m_{\alpha\beta}}{z^{1/2}}S^\beta(0), \tag{2.87}$$

$$S^\alpha(z)S_\beta(0) \sim \frac{\delta^\alpha_\beta}{z^{5/4}} + \frac{(\gamma^m\gamma^n)^\alpha_\beta}{2z^{1/4}}\Psi_m\Psi_n(0), \tag{2.88}$$

$$S_\alpha(z)S_\beta(0) \sim \frac{\gamma^m_{\alpha\beta}}{z^{3/4}}\Psi_m(0). \tag{2.89}$$

Since $e^{-\phi/2}$ and S_α have conformal weight 3/8 and 5/8 respectively we get the following results

$$Q_0 V_u(0) = \oint \frac{1}{2z}\partial^m \partial_m u^\alpha(X(0)) c\partial c(0) e^{-\phi/2(0)} S_\alpha(0) + \text{regular},$$
$$Q_1 V_u(0) = \oint \eta c(0) z^{1/2} : e^{\phi(z)} e^{-\phi(0)/2} : \left(\frac{1}{z}\partial_m u^\alpha(X(0)) + \dots\right)\left(\frac{\gamma^m_{\alpha\beta}}{z^{1/2}} S^\beta(0) + \dots\right)$$
$$= \oint \eta c(0) : e^{\phi(z)} e^{-\phi(0)/2} : \left(\frac{1}{z} S^\beta(0) \gamma^m_{\beta\alpha} \partial_m u^\alpha(X(0)) + \dots\right),$$
$$Q_2 V_u(0) = -\oint \eta\partial\eta(z) \times z : e^{2\phi(z)} e^{-\phi(0)/2} : u^\alpha(X(0)) \left(\frac{1}{z} + \dots\right) S_\alpha(0) = 0, \tag{2.90}$$

and the $QV_u = 0$ constraint becomes

$$\partial^m \partial_m u^\alpha(X) = 0,\ \gamma^m_{\alpha\beta} \partial_m u^\beta(X) = 0, \tag{2.91}$$

Therefore u^β describes a chiral fermionic massless state. By using Dirac constraint theory it has been proved that for a massless chiral operator that satisfies these constraints eight degrees of freedom can be gauged away, and we remain with eight physical degrees of freedom. This can be found on [4]. An analogous procedure can be made for the antichiral Ramond sector by using an antichiral spinor u_α with the chiral part of the spin field S^α.

2.6.4 Massive vertex operators

The description of higher mass states can be done if we keep adding conformal tensors to the vertex operators that we already described, but it becomes more complicated. In fact, the next state on the mass spectrum is composed by 128 physical states (after the GSO projection, that is explained on the next section). These operators will not be relevant for this thesis, but a more complete explanation on this is given on [10].

2.7 GSO Projection and supersymmetry

The current description we have done for superstring theory is not complete, since we have tachonic states in the mass spectrum. The way to get rid of this issue is by taking the so-called GSO (Gliozzi, Scherk and Olive) projection.

We have seen that the NS sector of the spectrum can be separated in two based on its worldsheet fermion number operator. In the same way we can separate the R sector in a chiral and antichiral states. There is a way to sum up this selection rule by using the following OPEs

$$\partial\phi(z)e^{q\phi(0)} \sim -\frac{q}{z}e^{q\phi(0)}, \; i\partial\sigma^a(z)e^{is\cdot\sigma(0)} \sim \frac{s^a}{z}e^{is\cdot\sigma(0)}. \tag{2.92}$$

We define the GSO operator as

$$Q_{\text{GSO}} = \oint \left(\sum_{i=1}^{5} i\partial\sigma^a - \partial\phi \right). \tag{2.93}$$

Notice that a vertex operator $V = ce^{q\phi}\exp(is\cdot\sigma)$ is an eigenoperator of Q_{GSO}, with eigenvalue $q_{\text{GSO}} = \sum_{i=1}^{5} s^i + q$. We can evaluate some cases. The easiest case is the tachyon $V = ce^{-\phi}\rho(X)$, that has $q_{\text{GSO}} = -1$.

To evaluate the massless boson first we have go to Euclidean space and then decompose the inner product in terms of $U(5)$ components . Consider a product of euclidean vectors A^m and B^m, whose $U(5)$ components are (a^a, a_a) and (b^a, b_a). The inner product $A^m B_m$ can be written as

$$A^m B_m = a^a b_a + a_a b^a, \tag{2.94}$$

where the $U(5)$ components are given as on [11]:

$$a^a = \frac{1}{\sqrt{2}}(A^m + iA^{m+5}), \tag{2.95}$$

$$a_a = \frac{1}{\sqrt{2}}(A^m - iA^{m+5}). \tag{2.96}$$

Then, the massless boson vertex operator can be decomposed as

$$V_A = ce^{-\phi}\left[a_a e^{i\sigma_a} + a^a e^{-i\sigma_a}\right] \tag{2.97}$$

where the ψ fields have been bosonized. The first term has $q_{\text{GSO}} = 0$, while the second has $q_{\text{GSO}} = -2$.

We move on to the massless fermionic vertex operators. In chiral basis S_α contains an even number of $-1/2$ on its exponential. So for the chiral vertex operator we have three possible cases

$$V_{u\text{ chiral}}\begin{cases}\propto ce^{-\phi/2}\exp\left[\frac{i}{2}\left(\sigma^1+\sigma^2+\sigma^3+\sigma^4+\sigma^5\right)\right], & q_{\text{GSO}}=2,\\ \propto ce^{-\phi/2}\exp\left[\frac{i}{2}\left(-\sigma^1-\sigma^2+\sigma^3+\sigma^4+\sigma^5\right)\right]+9\text{ perm.}, & q_{\text{GSO}}=0,\\ \propto ce^{-\phi/2}\exp\left[\frac{i}{2}\left(-\sigma^1-\sigma^2-\sigma^3-\sigma^4+\sigma^5\right)\right]+4\text{ perm.} & q_{\text{GSO}}=-2,\end{cases} \tag{2.98}$$

and in for the antichiral vertex operator we use S^{α}, that has an odd number of $-1/2$ on the exponential

$$V_{u\text{ antichiral}}\begin{cases}\propto ce^{-\phi/2}\exp\left[\frac{i}{2}\left(-\sigma^1+\sigma^2+\sigma^3+\sigma^4+\sigma^5\right)\right], & q_{\text{GSO}}=1,\\ \propto ce^{-\phi/2}\exp\left[\frac{i}{2}\left(-\sigma^1-\sigma^2-\sigma^3+\sigma^4+\sigma^5\right)\right]+9\text{ perm.} & q_{\text{GSO}}=-1,\\ \propto ce^{-\phi/2}\exp\left[\frac{i}{2}\left(-\sigma^1-\sigma^2-\sigma^3-\sigma^4-\sigma^5\right)\right]+4\text{ perm.} & q_{\text{GSO}}=-3.\end{cases} \tag{2.99}$$

Once we have seen this examples we can establish the GSO selection rule: Physical states are described by local vertex operators that have even eigenvalues of Q_{GSO}. With this in mind the tachyon and the antichiral part of the Ramond sector are not considered as physical states anymore.

One interesting consequence that GSO projection has is that it leads to a supersymmetric spectrum. As an example we can take the massless states. Both the bosonic and fermionic vertex operators end with eight of degrees of freedom. The same can be verified for massive states, as described on [10].

Chapter 3

RNS Scattering Amplitude Prescription

Scattering amplitudes provide a direct link between the worldsheet formulation of string theory and observable physical processes in spacetime. They encode the probabilities for various string states to interact and evolve, and their computation reveals the rich structure of string interactions, including the emergence of gravity and gauge symmetries at low energies.

In this section, we outline the general framework for computing string scattering amplitudes, focusing on the role of vertex operators, worldsheet correlators, and integration over moduli space.

3.1 Bosonic string prescription

To be familiarized with some concepts we can start describing the scattering amplitude for open bosonic strings. To do so all we have to do is make $\beta = \gamma = \Psi^m = 0$, getting the following action for the open string.

$$S = \frac{1}{2}\int d^2z(\partial X\cdot \bar{\partial} X + b\bar{\partial} c). \tag{3.1}$$

The central charge of the CFT is zero if we consider a $D = 26$ spacetime. In bosonic string theory there is no GSO projection, so we still have tachyonic states, but these will be useful in our calculations as examples. Also, we will be focused on doing the calculations only at tree level.

3.1.1 Conformal Killing vectors and c ghosts

Bosonic string theory was initially described in terms of some worldsheet, with coordinates σ^A and metric g_{AB}, with $A, B = 1, 2$. The metric degrees of freedom can be gauged completely since the theory is invariant under diffeomorphisms $\sigma'^A\left(\sigma^A\right)$ and Weyl transformations $g'_{AB}(\sigma) = e^{2\omega(\sigma)} g'_{AB}(\sigma)$. We used this fact in a convenient way to fix $g_{AB} \to \delta_{AB}$.

This is not the only choice we could have done. For example, consider the case where we fix the metric as $g_{AB} \to \hat{g}_{AB}$. The results obtained by using this as the gauge-fixing condition should be the same as ones obtained from the flat worldsheet. For a quantum description of a gauge theory one can use the Faddeev-Popov procedure, where we require to add ghost fields. In terms of a gauged metric $\hat{g}_{AB}(\sigma)$ the bosonic string action becomes

$$S = \frac{1}{2}\int d^2\sigma \hat{g}^{1/2}\left[\hat{g}_{AB}\partial^A X\cdot\partial^B X + b_{AB}\left(\hat{\nabla}^A c^B + \hat{\nabla}^B c^A - \hat{g}^{AB}\hat{\nabla}^C c_C\right)\right]. \tag{3.2}$$

where $\hat{\nabla}$ denotes the covariant derivative associated to the metric $\hat{g}_{AB}$ and b_{AB} is symmetric-traceless. To get (3.1) all we have to do is to choose $\hat{g}_{AB} = \delta_{AB}$ and go to complex coordinates, labeling $b_{zz} \equiv b$ and $c^z \equiv c$.[1]

Something that we can remember from general relativity is that there are transformations on the coordinates that do not change the metric. These are called *isometries*. A good example of this can be seen in a flat Minkowski spacetime. The metric does not change under global translations or Lorentz transformations. We would expect then that the string worldsheet still have some symmetries that are not completely fixed by the choice of metric.

Our string description also has some remaining symmetry we did not discussed yet. The fixed metric $\hat{g}$ can be transformed by a local factor $e^{w(\sigma)}\hat{g}$ and Eq. (3.2) remains invariant. We can then relax our isometry condition by letting the metric to be unchanged up to a local function. The set of transformations that fulfill this condition is called conformal Killing group. To find the equation that describes this group elements consider an infinitesimal diffeomorphism and $\delta\sigma = v(\sigma)$ a small local rescalling factor $\omega(\sigma)$ such that it leaves the gauge fixed metric $\hat{g}_{AB}$ invariant up to a local rescaling

$$\delta_{\text{diff}}\hat{g}_{AB} + \delta_{\text{scale}}\hat{g}_{AB} = \hat{\nabla}_A v_B + \hat{\nabla}_B v_A + \omega(\sigma)\hat{g}_{AB} = 0. \tag{3.3}$$

By taking the trace of the equation we can get $\omega(\sigma) = -\hat{\nabla}^C c_C$. Replacing it back in the same equation, obtain the conformal Killing equation.

$$\hat{\nabla}^A v^B + \hat{\nabla}^B v^A - \hat{g}_{AB}\hat{\nabla}^C v_C = 0. \tag{3.4}$$

[1] The order of the indexes on b_{zz} and c^z is the reason of why (b, c) have conformal weight $(2, -1)$. The position of these indexes is convenient since the action will be invariant under Weyl transformations of the gauge fixed metric as we will see.

Notice that this is the same equation that the c^A ghost follow. In fact, for each conformal Killing equation we have there will be a c^A ghost associated to it. This will be relevant for our discussion about amplitudes.

3.1.2 Moduli space and Riemann-Roch theorem

Sometimes the worldsheet metric cannot be completely gauge-fixed due to global properties. For example, a massless particle moving along a closed spacetime path is described by the reparameterization-invariant action

$$S = \int d\tau\, g^{\tau\tau\ -1/2} \dot{X}^m \dot{X}_m, \tag{3.5}$$

with $e = (g_{\tau\tau}\ )^{1/2}$ (the einbein), transforming as $e'(\tau') = (\partial\tau/\partial\tau')e(\tau)$. We set the range of $\tau \in [0,1]$, and also we can locally fix $e = 1$, but the coordinate change

$$\tau'(\tau) = \int_0^\tau e(\tau)\, d\tau, \tag{3.6}$$

implies $\tau'(1) \neq 0$, leaving a global modulus $l \equiv \tau'(1)$ that cannot be gauged away. There are two ways to solve this problem: Fix $e'(\tau') = l$, with the interval $\tau' \in [0,1)$ (parameter dependent metric) and add all the contributions for different values of l for the path integral, or fix $e'(\tau') = 1$, but changing the interval of $\tau' \in [0,l)$.

In string theory context an analogous problem occurs when dealing, for example, with a toroidal topology of the worldsheet. On [8] it can be seen that the metric of it is also parameter dependent as $ds^2 = \left(d\sigma^1 + \tau d\sigma^2\right)^2$, with τ a complex parameter that lies the region $|\tau| \geqslant 1$ and $-1/2 \leqslant \mathrm{Re}\,\tau \leqslant 1/2$, or we can have a flat metric $ds^2 = \left(d\sigma^1\right)^2 + \left(d\sigma^2\right)^2$, but with a identification relation dependent on the moduli as $\left(\sigma^1, \sigma^2\right) \cong \left(\sigma^1 + 2\pi\left(m + n\,\mathrm{Re}\,\tau\right), \sigma^2 + 2\pi n\,\mathrm{Im}\,\tau\right)$.

When we compute the path integral of the theory for a fixed moduli dependent metric we are changing the path integral as

$$\int [dXdg] e^{-S[X,g]} \ldots \to \int_M d^\mu\tau \int [dXdbdc] e^{-S[X,b,c,\hat{g}(\tau)]} \ldots \Big|_{\hat{g}(\tau)}. \tag{3.7}$$

τ is known as *Teichmüller Parameter* or *metric modulus*, and the range of integration M is known as *Moduli space*, with dimension μ. Notice that we should be able to perform reparameterizations $\tau' = \tau'(\tau)$ in the moduli space leaving the path integral unchanged. This will require the insertion of some factor to cancel the $\det(\partial\tau'/\partial\tau)$ that arises from the integral over τ. This can be done by adding the factor $\prod_{k=1}^{\mu}\int d\mu_k b$ defined as

$$\prod_{k=1}^{\mu}\int d\mu_k b = \int d^2\sigma \hat{g}^{1/2} b_{AB}\mu_k^{AB} \equiv \int d^2\sigma \hat{g}^{1/2} b_{AB}\frac{\partial \hat{g}^{AB}}{\partial \tau^k}, \tag{3.8}$$

where $\mu_{kAB} = \partial_k g_{AB}$ is known as Beltrami differential. Then, $\det(\partial\tau'/\partial\tau)\prod_{k=1}^{\mu} B_k$ is invariant under general transformations in the moduli space.

The problem we have to deal with now is that the vertex operators we discussed before were described in the conformal (or flat) gauge metric, so it is not convenient to leave the moduli dependence on the metric. Instead it is possible to fix the metric and leave the moduli on the coordinate by considering patches (that are Riemman surfaces) that cover the manifold, such that on the overlap the coordinates are related as $z^m = z^m(z^n,\tau)$, with z^m, z^n being complex coordinates in the m or n patch that covers the manifold. In such way the Beltrami differential factor becomes [1]

$$\int d\mu_k b = \sum_{(mn)}\int_{C_{mn}}\left[\frac{dz_m}{2\pi i}\frac{\partial z_m}{\partial t^k}\bigg|_{z_n \text{ fixed}} b(z_m) - \frac{d\bar{z}_m}{2\pi i}\frac{\partial \bar{z}_m}{\partial t^k}\bigg|_{\bar{z}_n \text{ fixed}}\tilde{b}(\bar{z}_m)\right], \tag{3.9}$$

where C_{mn} paths that are inside the overlap of two patches. It can be a closed path or it can end in a triple overlap of patches. $\tilde{b}$ is the antiholomorphic b ghost. The second part of the equation by using the doubling trick for paths crossing the boundaries of the worldsheet. The sum over (mn) is over all the pairs of patches that cover the complete manifold. Another advantage that this approach has is that is that the insertion of N punctures on the worldsheet boundary define additional Teichmüller parameter, and the total number of moduli (from the topology of the manifold and its punctures) is given by

$$m = \mu - (k-n) = n - 3\chi. \tag{3.10}$$

This is necessary because since vertex operators used to compute scattering amplitudes are set on the boundaries they define punctures, and these can reduce the number of conformal Killing vectors in the manifold, and therefore the moduli increases.

3.1.3 Bosonic scattering amplitude prescription.

With the concepts we described we are able to state the prescription to calculate the S-matrix for some string theory process. If we have a scattering process that involves a total number N of initial and final physical states given by vertex operators V_i we define the scattering amplitude as

$$S = \sum_{\text{all } \chi} \int_F d^{N-3\chi}\tau \left\langle \prod_{k=1}^{N-3\chi} \int d\mu_k b(z) \prod_{i=1}^{N} V_i \right\rangle. \tag{3.11}$$

There are some factors related with the coupling constants that we omit for this work. Also, for the tree level amplitude we have a disc topology ($\chi = 1$), and the equation reduces to

$$S_{\text{tree}} = \int_F d^{N-3}\tau \left\langle \prod_{k=1}^{N-3} \int d\mu_k b(z) \prod_{i=1}^{N} V_i \right\rangle. \tag{3.12}$$

We can use a four point scattering amplitude. In such case we are left with one moduli. Consider then two patches with coordinates z and z'. In the z plane the position of the vertex operator is z_v, and we define the z' coordinates such that $z'_v = 0$. Then, we have the transition functions in the overlap of the patches as $z = z' + z_v$, and the moduli is given by z_v. In the open string case we use $z_v = \bar{z}_v$ to define the position of the vertex operator. We have then $(\partial z'/\partial z_v)\big|_{z_n} = (\partial \bar{z}'/\partial z_v)\big|_{z_n} = 1$. Replacing on the Beltrami factor we end up with

$$\int d\mu\, b(z) = \frac{1}{2\pi i} \int_C (-dz_m b_{z_m z_m} + d\bar{z}_m b_{\bar{z}_m \bar{z}_m}). \tag{3.13}$$

We can use the doubling trick to reduce the expresion to $\oint_C b(z) = b_{-1}$, where the contour integral encloses the position of a vertex operator (namely z_1). Since a bosonic vertex operator has the form $V = c(X, \partial X, \dots)$ we can evaluate the contour integral and we get

$$S(1,\dots,4) = \left\langle \left[\int dz_1 U(z_1) \right] V(z_2) V(z_3) V(z_4) \right\rangle. \tag{3.14}$$

For this reason U is called integrated vertex operator. Another interpretation for this result is by looking to the three conformal Killing vectors that the disk has. These can be used to fix the position of three vertex operators (that are not integrated), and since the location of the remaining one is not fixed we must integrate along the points of the boundary.

It is important to mention that the choice we did for a patch that enclose only one vertex operator was purely for convenience. Actually, we could have chosen any other patch (namely, one that contains two vertex operator) and we should get the same result, by using an appropriate Teichmüller parameter.

To do the computation of the expectation value it is required to integrate over zero modes of the fields. The following results are useful to do the computations at tree level

$$\left\langle \prod_{i=1}^{p} e^{ik_i\cdot X(z_i)} \right\rangle = \delta\left(\sum_{i=1}^{p} k_i\right) \prod_{i<j}^{p} |z_{ij}|^{k_i\cdot k_j}, \tag{3.15}$$

$$\left\langle \partial X^{\mu}(z) \prod_{i=1}^{p} e^{ik_i\cdot X(z_i)} \right\rangle = \sum_{i=1}^{p} \frac{-i}{z-z_i} \left\langle \prod_{i=1}^{p} e^{ik_i\cdot X(z_i)} \right\rangle, \tag{3.16}$$

and for the ghost terms we have

$$\left\langle \prod_{i=1}^{p+3} c(z_i) \prod_{i=1}^{p} b(z_j') \right\rangle = \frac{(z_{p+1}-z_{p+2})(z_{p+1}-z_{p+3})(z_{p+2}-z_{p+3})}{(z_1-z_1')\dots(z_p-z_p')} + \text{perm.} \tag{3.17}$$

Observe that the number of c ghosts exceeds the number of b ghosts by three, as mentioned earlier. The correlator of these three additional c ghosts produces the numerator in Eq. (3.17).

3.1.4 Relation with the ghost number current anomaly

There is another way to give the correct description of the b and c ghost insertions in the amplitude prescription. The ghost action has another conserved quantity under variations $\delta b = -i\epsilon b$, $\delta c = i\epsilon c$. Such quantity is called ghost number current

$$j_{\text{bc}}(z) = -b(z)c(z). \tag{3.18}$$

Notice that the operator $Q_{\text{bc}} = -\oint b(z)c(z)$ has eigenfunctions b (with eigenvalue $q_{\text{bc}} = -1$) and c (with $q_{\text{bc}} = +1$). We see that each function constructed with n_b factors of b and n_c of c has a determined number $q_{\text{bc}} = n_c - n_b$, called ghost charge. Integrated and unintegrated vertex operators have ghost charge 0 and $+1$ respectively.

Ghost current is not a tensor. In fact, it has an anomaly. This can be seen if we compute the OPE with $T(z)$

$$T(z)j_{\text{bc}}(0) \sim \frac{C}{z^3} + \frac{1}{z^2}j_{\text{bc}}(0) + \frac{1}{z}\partial j_{\text{bc}}(0), \tag{3.19}$$

where C is known as background charge. In the bc ghost system $C_{\text{bc}} = -3$. The anomaly is present in a curved worldsheet as a continuity equation [12]:

$$\nabla_A j^A = \frac{1}{4}C\hat{g}^{1/2}\hat{R}, \tag{3.20}$$

with $\hat{R}$ the Ricci scalar of the worldsheet. We can say then that for a non-flat metric the geometry of the space time contributes to the ghost charge. By doing an integral over the full worldsheet, using the divergence theorem in the left-hand side and the Gauss-Bonnet theorem on the right side we end with

$$n_c - n_b = C\chi, \tag{3.21}$$

with χ the Euler characteristic of the worldsheet. Notice that we get a similar result as in the number of moduli for n open string vertex operators $m = n - 3\chi$. In fact, both results are related: Friedan, Martinec and Shenker prove that the expectation value of an operator is zero, unless the background charge is canceled [3]. This is accomplished by inserting -3χ additional b-ghost fields, associated with the Beltrami differentials and the integration over moduli space, compared to the number of c-ghost insertions, which originate from the unintegrated vertex operators. Notice that at tree level amplitude the number of b minus c ghost insertions is equal to the background charge $C_{bc} = -3$ [3].

3.2 RNS Superstring prescription

3.2.1 Picture Changing Operators

We go back to the RNS supestring case. The approach for this case is analogous, but we have to promote the worldsheet patches from being Riemman surfaces to become super Riemman surfaces, that have even and odd Grassmann parity. In the same way, it is possible to get odd and even moduli that are related to the gauge-fixed metric and gravitino worldsheet field.

Even moduli will behave in the same way we considered in bosonic string theory, where they required the insertion of b ghosts to get a number of integrated vertex operators. For odd moduli dependence the approach is similar with small differences: We will require to insert some $\delta(\beta)$ functions on the path integral and integrate over the odd moduli, but we can see that they can be easily integrated since they are described by Berezin integrals. As a result we end up with the insertion of something called Picture changing operators (PCOs) given by[2]

$$\{Q_{\text{RNS}}, \xi\} = \oint \left(cT^{\text{m,g}} - bc\partial c + \eta e^{\phi}\Psi\cdot\partial X - \eta\partial\eta e^{2\phi}b\right)(z)\xi(0), \tag{3.22}$$

where the path integral encloses the worldsheet location of ξ. To see the action of these operators consider the action of these on the bosonic massless vertex operator, considering that ξ and V are located on the same point.

$$\begin{aligned}\{Q_{\text{RNS}}, \xi V\} &= \oint \left(\eta e^{\phi}\Psi\cdot\partial X - \eta\partial\eta e^{2\phi}b\right)(z)\left[\xi c e^{-\phi}\Psi^m A_m(X(0))\right](0) \\ &= \eta e^{\phi}\Psi^m A_m(X) + ic\left[i\partial X^m A_m(X) - i\Psi^m\Psi^n\partial_m A_n(X)\right]. \end{aligned} \tag{3.23}$$

Notice that the action of V over Q_0 does not generate any singular term since ξ has conformal weight zero. This applies for any vertex operator.

The local operator obtained is BRST closed if $\partial_m A^m = 0$, and thus it can be interpreted as a vertex operator for the same massless bosonic state. At first glance, one might think that this operator is BRST exact because it takes the form $\{Q_{\text{RNS}}, \xi V\}$. However, this is not correct due to the presence of the ξ field. Although we bosonize the $\beta\gamma$ ghost system in terms of the bosonic field ϕ, and fermionic fields η and $\partial\xi$, the field ξ itself is not included in the standard bosonized Fock space. As a result, ξ introduces an additional zero mode that is not accounted for in the physical state space. Therefore, insertions of ξ in scattering amplitudes are not physically meaningful on their own and should be treated carefully. This means that the operator $\{Q_{\text{RNS}}, \xi V\}$, while BRST exact in a formal sense, does not imply that V is trivial in cohomology, since ξV lies outside the standard Hilbert space. Thus, picture changing produces a physically equivalent vertex operator, not a BRST-trivial one.

To establish the difference between two operators related by PCOs we define the picture operator as

$$P = \oint \partial\phi + \xi\eta. \tag{3.24}$$

So we can label different picture changed vertex operators in terms of its eigen-

[2]A more complete description of the geometrical approach of PCOs see [13]

value under the action of P. For example, $\xi c e^{-\phi}\Psi^m A_m(X)$ is on picture zero, $u^\alpha(X) c e^{-\phi/2} S_\alpha$ is on picture $-1/2$, etc We can see then that PCOs raise the picture number on 1.

In the same way as in the bosonic case with Beltrami differentials and b ghost insertions, the number of PCOs inserted on the expresion depends on the topology of the worldsheet as

$$n_{\text{PCO}} = n_B + \frac{n_F}{2} - 2\chi, \tag{3.25}$$

where n_B and n_F are the number of bosonic and fermionic open string vertex operators inserted on the boundary of the worldsheet. The number of PCOs can also be obtained from the $\beta\gamma$ ghost current anomaly, in the same way as in the bosonic string prescription [3].

3.2.2 RNS superstring scattering amplitude prescription.

We define the RNS scattering amplitude with integrated odd moduli as

$$S = \sum_{\text{all }\chi} \int_F d^{N-3\chi}\tau \left\langle \prod_{k=1}^{N-3\chi} \int d\mu_k b(z) \prod_{i=1}^{N} V_i(z_i) \prod_{j=1}^{n_{\text{PCO}}} \{Q_{\text{RNS}}, \xi(y_j)\} \right\rangle . \tag{3.26}$$

Note that the picture changing operators are not inserted at the same positions as the vertex operators. Nevertheless, it can be shown that evaluating the PCOs directly on top of the vertex insertions yields results that are equivalent, up to BRST exact terms, to those obtained when the PCOs are placed at separate, generic locations.

In the same way as we did in the bosonic string case here we present some important expectation values to do the calculation of scattering amplitudes

$$\langle \psi^{\mu_1}(z_1)\psi^{\mu_2}(z_2)\cdots\psi^{\mu_{2n}}(z_{2n})\rangle = \sum_{\text{pairings}} (\pm) \prod_{\text{pairs }(i,j)} \frac{\eta^{\mu_i\mu_j}}{z_i - z_j}, \tag{3.27}$$

where the $(\pm)$ sign appears because of the order of contraction of the Grassmann fields. Also, for the $\beta\gamma$ ghost system it is useful to have the expectation value of the chiral bosons and $\eta\xi$ ghost system

$$\left\langle \prod_{j=1}^{n} e^{q_j\phi(z_j)} \right\rangle = \delta\left(\sum_{j=1}^{n} q_j + 2\right) \prod_{k<l}^{n} z_{kl}^{-q_k q_l}, \tag{3.28}$$

$$\left\langle \prod_{i=1}^{p} \eta(z_i) \prod_{j=1}^{p} \xi(z'_j) \right\rangle = \sum_{\sigma \in S_p} (-1)^{\text{sgn}(\sigma)} \prod_{i=1}^{p} \frac{1}{z_i - z'_{\sigma(i)}}, \tag{3.29}$$

with S_p being the permutation group of p elements. $\text{sgn}(\sigma)$ appears because of the fermionic antisymmetry when reordering.

An important fact is that since the description os the bosonized version of the RNS superstring theory does not containt $\xi(z)$ explicitely (but instead $\partial\xi(z)$) then it is required to add an additional $\xi(z)$ in the path integral to have a nonvanishing result. The position of this operator is irrelevant since we only care about the zero mode of $\xi(z)$, which is a constant. Then one proposes that

$$\langle \xi(z) \rangle = 1. \tag{3.30}$$

This fact is another reason of why we can add PCOs on the scattering amplitude prescription without changing the final result [3].

3.3 Spacetime Supersymmetry

We have stated before that once we consider GSO projection on the RNS superstring spectrum the theory is spacetime supersymmetric, even if such symmetry is not manifest. We would be then interested on obtain the 16 spacetime supersymmetry charges and the algebra they follow.

First, we can start by looking to some operator that interchange bosonic and fermionic vertex operators. This can be achieved by considering

$$q_\alpha = \oint e^{-\phi/2} S_\alpha. \tag{3.31}$$

Consider the following examples: A picture $-1/2$ massless fermionic vertex operator transforms as

$$\begin{aligned} \epsilon^\alpha q_\alpha V^F_{-1/2} &= \oint e^{-\phi/2} \epsilon^\alpha S_\alpha(z) u^\beta(X) c(0) e^{-\phi(0)/2} S_\beta(0), \\ &= \epsilon^\alpha \oint z^{-1/4} : e^{-\phi(z)/2} e^{-\phi(0)/2} : u^\beta(X) c(0) \left(\frac{\gamma^m_{\alpha\beta}}{z^{3/4}} \Psi_m(0) + \dots \right), \\ &= \epsilon^\alpha \gamma^m_{\alpha\beta} u^\beta(X) c(0) e^{-\phi(0)} \Psi_m(0), \end{aligned} \tag{3.32}$$

that is the -1 picture massless boson vertex operator with $A^m = \epsilon^\alpha \gamma^m_{\alpha\beta} u^\beta(X)$.

As a second example we take

$$\begin{aligned}\epsilon^\alpha q_\alpha V_0^B &= \oint e^{-\phi/2}\epsilon^\alpha S_\alpha(z)\left[\eta e^\phi \Psi^m A_m + c\left[-\partial X^m A_m + \Psi^m\Psi^n\partial_m A_n\right]\right](0)\\ &= \partial^m A^n(X(0))\epsilon^\beta(\gamma_{mn})_\beta^\alpha c(0)e^{-\phi(0)/2}S_\alpha(0),\end{aligned} \tag{3.33}$$

with $(\gamma_{mn})_\alpha^\beta = 2^{-1}(\gamma_{[m})_{\alpha\sigma}(\gamma_{n]})^{\sigma\beta}$. We end up with a $-1/2$ fermionic vertex operator with $u^\alpha = \partial^m A^n(X)\epsilon^\beta(\gamma_{mn})_\beta^\alpha$. Indeed, q_α let us exchange between bosonic and fermionic operators and also in the opposite way. The fact that the theory is supersymmetric has some consequences in scattering amplitudes. For example, the contribution from tadpole diagrams is zero, and also there is a direct relation between the two point scattering amplitude of two bosons and two fermions [3].

We can calculate $\{q_\alpha, q_\beta\}$:

$$\begin{aligned}\{q_\alpha, q_\beta\} &= \oint \frac{dz}{2\pi i}\frac{dw}{2\pi i}\, e^{-\phi(z)/2}e^{-\phi(w)/2}S^\alpha(z)S^\beta(w)\\ &= \oint \frac{dz}{2\pi i}\frac{dw}{2\pi i}\,(z-w)^{-1/4}\left[e^{-\phi(w)} + \mathcal{O}(w)\right]\frac{1}{(z-w)^{3/4}}\gamma_m^{\alpha\beta}\Psi^m(w)\\ &= \gamma_m^{\alpha\beta}\oint \frac{dw}{2\pi i}\, e^{-\phi(w)}\Psi^m(w),\end{aligned} \tag{3.34}$$

so we can see that this is not the $\gamma^{m\alpha\beta}\oint \partial X_m$ we expected to get, but it is related to it by picture changing. To see this consider we can work with the supersymmetry charge in $+1/2$ picture:

$$\begin{aligned}q_\alpha^+ &= \oint \frac{dz}{2\pi i}\,\left\{Q_{\mathrm{BRST}},\, \xi(z)e^{-\phi(z)/2}S_\alpha(z)\right\}\\ &= \oint \frac{dz}{2\pi i}\,\oint \frac{dw}{2\pi i}\left[\eta e^\phi \Psi^m\partial X_m - b\eta\partial\eta e^{2\phi}\right](w)\xi(z)e^{-\phi(z)/2}S_\alpha(z)\\ &= \oint \frac{dz}{2\pi i}\,\oint \frac{dw}{2\pi i}\frac{1}{w-z}e^{\phi/2}\partial X_m\gamma_{\alpha\beta}^m S^\beta(w) + \frac{1}{w-z}b\eta e^{3/2\phi}S_\alpha(w)\\ &= \oint \frac{dz}{2\pi i}\, e^{\phi/2}\partial X_m\gamma_{\alpha\beta}^m S^\beta + b\eta e^{3/2\phi}S_\alpha(z),\end{aligned} \tag{3.35}$$

so we have that

$$\begin{aligned}\left\{q_\alpha^+, q_\beta^-\right\} &= \oint \frac{dw}{2\pi i}\,\oint \frac{dz}{2\pi i}\left[e^{\phi/2}\partial X\cdot(\gamma S)_\alpha + b\eta e^{3/2\phi}S_\alpha\right](w)\, e^{-\phi(z)/2}S_\beta(z)\\ &= \gamma_{\alpha\beta}^m\oint \frac{dz}{2\pi i}\partial X_m(z).\end{aligned} \tag{3.36}$$

So we recover the spacetime supersymmetry algebra. However, this algebra has some problems. For example, the number of supercharges is duplicated, since we have the set of 32 terms $\{q_\alpha^+, q_\alpha^-\}$. One may think that these might describe some $\mathcal{N} = 2$ $D = 10$ SUSY algebra, but this is wrong since we have just verified that $\left\{q_\alpha^-, q_\beta^-\right\} \neq 0$. The discussion about this issue can be found on [3]. Another problem that has this approach is that the use of PCOs makes that the SUSY algebra close only on-shell, since off-shell states are not independent of the position of PCOs.

Part II

Pure spinor Formalism

Chapter 4

Pure spinor scattering amplitude

So far, we have successfully described the physical spectrum of the superstring using the RNS formalism. However, additional ingredients, such as the GSO projection, are required to ensure that the resulting spectrum is supersymmetric. Despite its utility, the RNS formalism lacks manifest spacetime supersymmetry and covariance, which poses significant challenges, particularly in the computation of higher-loop scattering amplitudes.

An alternative approach is provided by the Green-Schwarz formalism, which formulates the superstring with manifest spacetime supersymmetry by employing spacetime fields such as (X^m, θ^α). While this description is more natural from the perspective of target-space supersymmetry, it encounters substantial difficulties upon quantization. In particular, reproducing the RNS results requires fixing a specific gauge, which in turn breaks manifest spacetime covariance [14].

The pure spinor formalism addresses these limitations by preserving both spacetime supersymmetry and covariance at the quantum level. In this chapter, we will explore how the pure spinor approach successfully overcomes the obstacles encountered in previous formulations.

4.1 Action and pure spinor condition

The action that this description works with is one that includes matter spacetime vector fields X^m and also spacetime chiral spinor matter fields θ^α, and its antiholomorphic part. Also we consider the momentum terms associated to it

$$S_{\text{matter}} = \int d^2z \, \left(\frac{1}{2} \partial X^m \bar{\partial} X_m + p_\alpha \bar{\partial} \theta^\alpha + \tilde{p}_\alpha \partial \tilde{\theta}^\alpha \right). \tag{4.1}$$

The chirality of the antiholomorphic fields $(\tilde{p}, \tilde{\theta})$ will determine if the described theory is type IIA or IIB. However, we are only interested on the open string case, where θ and $\tilde{\theta}$ are related by using the doubling trick formula, and therefore we require that both have the same chirality. The action describes a CFT, with an holomorphic energy momentum tensor given by

$$T(z) = -\frac{1}{2}\partial X \cdot \partial X - p_\alpha \partial\theta^\alpha. \tag{4.2}$$

We can verify that (X, p, θ) have conformal weight $(0, 1, 0)$ respectively. The OPEs for X will be the same as in the RNS formalism, and $p\theta$ behave as a bc ghost system with $\lambda = 1$, so it follows an OPE

$$p_\alpha(z)\theta^\beta(0) \sim \frac{\delta_\alpha^\beta}{z}. \tag{4.3}$$

The action is invariant under supersymmetry transformations

$$\delta X^m = -\tfrac{1}{2}\epsilon\gamma^m\theta, \quad \delta\theta^\alpha = \epsilon^\alpha, \quad \delta p_\alpha = \tfrac{1}{2}\partial X_m \left(\gamma^m\epsilon\right)_\alpha - \tfrac{1}{8}(\epsilon\gamma_m\theta)\left(\gamma^m\partial\theta\right)_\alpha , \tag{4.4}$$

with the conserved charge

$$q_\alpha = \oint \left[p_\alpha + \frac{1}{2}\left(\gamma^m\theta\right)_\alpha \partial X_m + \frac{1}{24}\left(\gamma^m\theta\right)_\alpha \left(\theta\gamma_m\partial\theta\right)\right], \tag{4.5}$$

so SUSY transformations are generated as $\delta O = \epsilon^\alpha\{q_\alpha, O\}$. The algebra of the supersymmetry generators is ruled by the super-Poincaré algebra [4],

$$\{q_\alpha, q_\beta\} = \gamma^m_{\alpha\beta} \oint \partial X_m, \tag{4.6}$$

where we require to use the Fierz identity $\gamma^m_{\alpha(\beta}\gamma_{m\rho\sigma)} = 0$. So notice that in under this formalism it is not required to use PCOs on q_α to obtain it.

The problem that this action has up to this point is that the central charge of the energy momentum is $c = -22$, so it does not vanish. There are also some problems that the current algebra of the theory has different singularities compared with the ones obtained on the RNS formalism [6]. Both issues can be solved by the addition of some extra fields, as we see below.

4.2 Ghost sector and pure spinor condition

The energy momentum tensor (4.2) can be written as

$$T = -\frac{1}{2}\Pi^m\Pi_m - d_\alpha\partial\theta^\alpha. \tag{4.7}$$

with

$$\Pi^m = \partial X^m + \tfrac{1}{2}\left(\theta\gamma^m\partial\theta\right), \quad d_\alpha = p_\alpha - \tfrac{1}{2}(\gamma_m\theta)_\alpha\left(\partial X^m + \tfrac{1}{4}\left(\theta\gamma^m\partial\theta\right)\right). \tag{4.8}$$

that can be seen as supersymmetric generalizations of ∂X^m and p_α. One can notice that the conformal transformation generator is the same as in the Green-Schwarz formalism ($T = -\frac{1}{2}\Pi^m\Pi_m$) if we make $d_\alpha = 0$. This constraint is added to the formalism by imposing the BRST charge

$$Q_{\mathrm{PS}} = \oint \lambda^\alpha d_\alpha, \tag{4.9}$$

with λ^α a bosonic spacetime chiral spinor. Then, the physical states on this formalism are given by vertex operator V such that $Q_{\mathrm{PS}}V = 0$, and the gauge transformations are $\delta V = Q_{\mathrm{PS}}\Omega$. Notice that the BRST charge is not nilpotent unless we impose

$$\lambda\gamma^m\lambda = 0, \tag{4.10}$$

so λ^α is $d = 10$ chiral pure spacetime spinor. It is possible to prove that it has 11 independent complex degrees of freedom.

Since λ^α is a boson with the opposite statistics we can see it as a ghost field, so λ^α and its conjugated momentum can be introduced on the action as

$$S_{\mathrm{ghost}} = -\int d^2z\, w_\alpha\bar{\partial}\lambda^\alpha. \tag{4.11}$$

So both λ and w are holomorphic. Notice that it also has an additional symmetry under transformations $\delta w_\alpha = E_m\left(\gamma^m\lambda\right)_\alpha$, $\delta\lambda^\alpha = 0$. This is useful, since we can gauge fix 5 components of w_α.

4.2.1 $U(5)$ decomposition of chiral and antichiral spinors.

To do a correct counting of the degrees of freedom for λ^α and w_α it is convenient to write this $SO(10)$ objects in terms of $U(5)$ variables. We have seen that a $SO(10)$ vector A^m can be decomposed in the (anti)fundamental representations of $U(5)$ as $(a_a)a^a$:

$$a^a = \tfrac{1}{\sqrt{2}}\left(A^a + iA^{a+5}\right), \quad a_a = \tfrac{1}{\sqrt{2}}\left(A^a - iA^{a+5}\right). \tag{4.12}$$

This helped us to describe RNS (anti)chiral spin fields S_α and S^α in terms of the $U(5)$ components of $\left(\psi^a \cong e^{i\sigma_a}, \psi_a \cong e^{-i\sigma_a}\right)$, in chiral basis as $S = \exp\left[\pm\frac{i}{2}\sum_{a=1}^{5}\sigma_a\right]$.

Chiral terms of S can be decomposed as

$$S^{\alpha} \to \left(s^{+}, s_{ab}, s^{a}\right), \tag{4.13}$$

with

$$s^{+} = \exp\left[\tfrac{i}{2}(-\sigma_1 - \sigma_2 - \sigma_3 - \sigma_4 - \sigma_5)\right], \tag{4.14}$$

$$s_{ab} = \exp\left[\tfrac{i}{2}(-\sigma_1 - \sigma_2 - \sigma_3 + \sigma_4 + \sigma_5)\right] + 9 \text{ permutations}, \tag{4.15}$$

$$s^{a} = \exp\left[\tfrac{i}{2}(-\sigma_1 + \sigma_2 + \sigma_3 + \sigma_4 + \sigma_5)\right] + 5 \text{ permutations}. \tag{4.16}$$

Then we can identify that $\left(S^{+}, S_{ab}, S^{a}\right)$ are on the trivial, antifundamental antisymmetric, and fundamental representation of $U(5)$ [4]. The same can be done for an antichiral spinor S_{α}, getting $\left(S_{+}, S^{ab}, S_{a}\right)$ on the trivial, fundamental antisymmetric, and antifundamental representation of $U(5)$, by exchanging $+ \leftrightarrow -$ signature on the exponents.

A Lorentz scalar can be written in terms of the product of a chiral and antichiral spinor $w_{\alpha}\lambda^{\alpha}$. This is decomposed in $U(5)$ terms as

$$w_{\alpha}\lambda^{\alpha} = w_{+}\lambda^{+} + \frac{1}{2}w^{ab}\lambda_{ab} + w_{a}\lambda^{a}. \tag{4.17}$$

Going back to the pure spinor condition $\lambda\gamma^{m}\lambda = 0$ we can rewrite it in terms of $U(5)$ variables. In $U(5)$ notation we get two equations $\lambda\gamma^{a}\lambda = \lambda\gamma_{a}\lambda = 0$. Expanding spinor indexes in terms of its $U(5)$ components we get [5]

$$\lambda_{+}\lambda^{a} + \tfrac{1}{8}\epsilon^{abcde}\lambda_{bc}\lambda_{de} = 0, \quad \lambda^{b}\lambda_{ab} = 0. \tag{4.18}$$

From the first equation we can get $\lambda^{a} = -\left(1/8\lambda^{+}\right)\epsilon^{abcde}\lambda_{bc}\lambda_{de}$, and therefore only eleven out of sixteen components remain independent. The second equation is automatically satisfied by this result.

In the same way, the antichiral bosonic spinor w_{α} can be left with only 11 degrees of freedom, since w_{a} can be gauged to zero by considering the gauge transformation $\delta w_{\alpha} = E_{m}\left(\gamma^{m}\lambda\right)_{\alpha}$. To see this we can write δw_{a} in terms of $U(5)$

components, taking $E_m = \epsilon_a \oplus \epsilon^a$[1]

$$\delta w_a = E_m \left(\gamma^m \lambda\right)_a = \epsilon_a \lambda^+ + \epsilon^a \lambda_{ab}. \tag{4.19}$$

So we can choose E^m such that $\epsilon^a = 0$ and $\epsilon_a = -w_a/\lambda^+$. This $w \to w'$ transformation fixes $w'_a = 0$:

$$w'_a = w_a + \delta w_a = w_a - \left(w_a/\lambda^+\right)\lambda^+ = 0. \tag{4.20}$$

4.2.2 Conformal properties of the pure spinor ghosts

In w_a gauge we remain with the following action of completely independent variables

$$S_{\text{ghost}} = \int d^2z \left(-w_+\bar{\partial}\lambda^+ - \frac{1}{2}w^{ab}\bar{\partial}\lambda_{ab}\right), \tag{4.21}$$

from them we get the following OPEs

$$w_+(z)\lambda^+(0) \sim \tfrac{1}{z}, \quad w^{ab}(z)\lambda_{cd}(0) \sim \frac{\delta_c^{[a}\delta_d^{b]}}{z}, \tag{4.22}$$

and we can identify these with a set of $\beta\gamma$ ghosts if we identify $\beta \to -w$ terms and $\gamma \to \lambda$ terms. Considering that the conformal weights are x for w terms and $1-x$ for λ terms we write the energy momentum tensor of the system as

$$\begin{aligned} T(z) &= -\partial w_\alpha \lambda^\alpha + x\partial\left(w_\alpha\lambda^\alpha\right) \\ &= -(\partial w_+)\lambda^+ - \frac{1}{2}\left(\partial w^{ab}\right)\lambda_{ab} + x\partial\left(w_+\lambda^+ + \frac{1}{2}w^{ab}\lambda_{ab}\right). \end{aligned} \tag{4.23}$$

This can be seen eleven $\beta\gamma$ ghosts CFTs. Each one contributes with a central charge $c = 3(2x-1)^2 - 1$. So the total contribution is $c_{\text{PS}} = 33(2x-1)^2 - 11$. For the case of $x = 1$ we end up with $c_{\text{PS}} = 22$, that let us cancel the ghost charge of the matter sector.

Also, the addition of ghost operators that transform as spinors under $SO(10)$ modifies the current algebra of the system by adding a non trivial contribution to the Lorentz current. With this addition the matter+ghost action follow the same

[1] We use this notation to indicate that some $SO(10)$ vector E^m is decomposed in terms of $U(5)$ variables in $\epsilon^a = 2^{-1/2}\left(E^a + iE^{a+5}\right)$ and $\epsilon_a = 2^{-1/2}\left(E^a - iE^{a+5}\right)$. Recall that capital letters are used to denote vectors transforming under $SO(10)$, while lowercase letters indicate their corresponding components under the $U(5)$ decomposition.

current algebra as the RNS prescrption [5].

4.3 Vertex operators in pure spinor formalism

Based on the RNS superstring prescription we will consider that the a physical state is described by a +1 ghost number unintegrated vertex operator V that satisfies $QV = 0$, and it is gauge equivalent to $V + Q\Omega$.

The simplest operator we can build is the following

$$V(z,\theta) = \lambda^{\alpha}(z)A_{\alpha}(X,\theta), \tag{4.24}$$

which obeys the following condition

$$QV = 0 \to \lambda^{\alpha}\lambda^{\beta}D_{\beta}A_{\alpha}(X,\theta) = 0, \tag{4.25}$$

with $D_{\alpha} = \frac{\partial}{\partial\theta^{\alpha}} + \frac{1}{2}(\gamma^{m}\theta)_{\alpha}\frac{\partial}{\partial X^{m}}$ being the fermionic covariant derivative. We can use the bispinor Fierz decomposition $\lambda^{\alpha}\lambda^{\beta} = (1/3480)\gamma^{\alpha\beta}_{mnpqr}\lambda\gamma^{mnpqr}\lambda$ to write the field equation as [15]:

$$\gamma^{\alpha\beta}_{mnpqr}D_{\alpha}A_{\beta}(X,\theta) = 0, \tag{4.26}$$

so A_{α} follows the linearized version of the equation of motion for the spinor pre potential of super-Yang-Mills. The gauge transformation of the vertex operator is given by

$$\delta V = Q\Omega(x,\theta) = \lambda^{\alpha}D_{\alpha}\Omega(x,\theta), \tag{4.27}$$

that implies the usual super-Yang-Mills gauge transformation $\delta A_{\alpha} = D_{\alpha}\Omega$. The vertex operator describes massless states. Since it contains a non zero ghost charge it is considered as a unintegrated vertex operator. It is also required to have a integrated vertex operator for the same state to do scattering amplitude calculations. The problem one has is the lack of b ghost, since it helped us to transform a unintegrated operator on its integrated version on the RNS prescription.

To solve this issue notice that if we choose the Teichmüller parameter as the position of the RNS vertex operator on the worldsheet. Then, its integrated form is given by

$$U(z_i) = \oint dzb(z)V(z_i). \tag{4.28}$$

Now consider the action of the RNS BRST charge operator on U. Since the action of Q_{RNS} on b is given by $Q_{\text{RNS}}b(z) = T(z)$ and also $Q_{\text{RNS}}V(z) = 0$ we get

$$Q_{\text{RNS}}U(z_i) = \oint dz Q_{\text{RNS}}b(z)V(z_i) = \oint dz T(z)V(z_i),$$
$$\rightarrow Q_{\text{RNS}}U(z_i) = \partial V(z_i). \tag{4.29}$$

So we can take this relation to define integrated vertex operators in pure spinor formalism, since it does not depend on the b ghost. It is possible to verify that

$$U(z,\theta) = \partial\theta^\alpha A_\alpha(z,\theta) + \Pi^m A_m(z,\theta) + d_\alpha W^\alpha(z,\theta) + \frac{1}{2}N^{mn}F_{mn}(z,\theta), \tag{4.30}$$

satisfies (4.29), with A_m being the vector super-Yang-Mills prepotential, and (W^α, F^{mn}) are the super-Yang-Mills spinor and tensor field strength in the linearized theory. The relation between these fields is given by

$$A_m(X,\theta) = \frac{1}{8}\gamma^{n\alpha\beta}D_\alpha A_\beta(X,\theta), \tag{4.31}$$
$$W^\beta(X,\theta) = \frac{1}{10}\gamma^{m\alpha\beta}[D_\alpha A_m(X,\theta) - \partial_m A_\alpha(X,\theta)], \tag{4.32}$$
$$F_{mn}(X,\theta) = \partial_m A_n(X,\theta) - \partial_m A_n(X,\theta) = \frac{1}{8}(\gamma_{mn})_\alpha{}^\beta D_\beta W^\alpha(X,\theta), \tag{4.33}$$

and the gauge transformations for these superfields are given by

$$\begin{aligned} \delta A_\alpha(X,\theta) = D_\alpha\Lambda(X,\theta), \quad & \delta A_m(X,\theta) = \partial_\alpha\Lambda(X,\theta), \\ \delta u^\alpha(X,\theta) = 0, \quad & \delta F_{mn}(X,\theta) = 0. \end{aligned} \tag{4.34}$$

Since θ^α has Grassmann odd parity it is possible to decompose these superfields as power series of θ. The components of them are related to the gluon, gluino and field strength fields $(A^m(X), u_\alpha(X), F_{mn}(X))$ respectively. Some relevant terms of such series is given by[2]

$$\begin{aligned}
A_\alpha(X,\theta) &= \tfrac{1}{2}(\theta\gamma_m)_\alpha A^m(X) + \tfrac{1}{3}(\theta\gamma_m u(X))\,(\gamma^m\theta)_\alpha - \tfrac{1}{32}(\gamma_p\theta)_\alpha\,(\theta\gamma^{mnp}\theta)\,F_{mn}(X) + \mathcal{O}\left(\theta^4\right), \\
A_m(X,\theta) &= A_m(X) + \theta\gamma_m u(X) - \tfrac{1}{8}\theta\gamma_{mnp}\theta F^{np}(X) - \tfrac{1}{12}(\theta\gamma_{mnp}\theta)\,(\theta\gamma^p\partial^n u(X)) + \mathcal{O}\left(\theta^4\right), \\
W^\alpha(X,\theta) &= u^\alpha(X) + \tfrac{1}{4}\,(\gamma^{mn}\theta)^\alpha\,F_{mn}(X) - \tfrac{1}{4}\,(\gamma^{mn}\theta)^\alpha\,(\theta\gamma_n\partial_m u(X)) + \mathcal{O}(\theta^3)\,, \\
F_{mn}(X,\theta) &= F_{mn}(X) - (\theta\gamma_{[m}\partial_{n]}u(X)) + \tfrac{1}{8}(\theta\gamma_{pq[m}\theta)\partial_{n]}F^{pq}(X) + \mathcal{O}(\theta^3)\,.
\end{aligned} \tag{4.35}$$

The higher order coefficients of the θ expansion can be written in terms of the

[2] The θ expansion of the superfields is done working on the Harnad-Shnider gauge $\theta^\alpha A_\alpha(X,\theta) = 0$ [15].

derivatives of $(A^m(X), u_\alpha(X), F_{mn}(X))$.

4.4 Tree level prescription for scattering amplitudes on pure spinor formalism

The idea to calculate the scattering amplitude in the pure spinor prescription is the same as in the RNS case. We fix the position of three vertex operators (so they are unintegrated), and the remaining ones are integrated over the boundary

$$S_{1\ldots N} = \int dz_4 \ldots \int dz_N \left\langle V(z_1)V(z_2)V(z_3) \prod_{i=4}^{N} U(z_i) \right\rangle. \tag{4.36}$$

The calculation of the path integral can lead us to get some useful expressions

$$\left\langle \Pi^m(z) \prod_{i=1}^{n} V(z_i) \right\rangle = -\sum_{i=1}^{n} \frac{1}{z - z_i} \left\langle \partial^m V(z_i) \prod_{j=1, j\neq i}^{n} V(z_j) \right\rangle, \tag{4.37}$$

$$\left\langle d_\alpha(z) \prod_{i=1}^{n} V(z_i) \right\rangle = -\sum_{i=1}^{n} \frac{1}{z - z_i} \left\langle D_\alpha V(z_i) \prod_{j=1, j\neq i}^{n} V(z_j) \right\rangle, \tag{4.38}$$

$$\left\langle N^{pq}(z) \prod_{i=1}^{n} \lambda^{\alpha_i} V_{\alpha_j}(z_i) \right\rangle = \frac{1}{2} \sum_{i=1}^{n} \frac{1}{z - z_i} \left\langle (\lambda\gamma^{pq})^{\alpha_i} V_{\alpha_i}(z_i) \prod_{j=1, j\neq i}^{n} V(z_i) \right\rangle. \tag{4.39}$$

It is also required to have the prescription to integrate the zero modes of λ^α. The prescription is as follows

$$\langle (\lambda\gamma^m\theta)\,(\lambda\gamma^n\theta)\,(\lambda\gamma^p\theta)\,(\theta\gamma_{abc}\theta) \rangle = 24\delta^{mnp}_{abc}, \tag{4.40}$$

where δ^{mnp}_{abc} is the generalized Kronecker delta

$$\delta^{mnp}_{abc} = \delta^m_a\delta^n_b\delta^p_c + \delta^m_b\delta^n_c\delta^p_a + \delta^m_c\delta^n_a\delta^p_b - \delta^m_a\delta^n_c\delta^p_b - \delta^m_b\delta^n_a\delta^p_c - \delta^m_c\delta^n_b\delta^p_a. \tag{4.41}$$

This is equivalent to define the $\langle \lambda^3\theta^5 \rangle$ normalization, given on [4].

Part III

On the Construction of a New Pure Spinor Formalism

Chapter 5

Relating RNS and Pure Spinor Variables

A central goal in connecting the Ramond–Neveu–Schwarz (RNS) and pure spinor formalisms is to understand how worldsheet degrees of freedom associated with spacetime fermions can be reformulated in a manifestly supersymmetric framework. In the RNS formalism, spacetime fermions arise from the quantization of the worldsheet fields ψ^m, which transform as vectors under $SO(10)$. In contrast, the pure spinor approach introduces fermionic coordinates θ^α together with their conjugate momenta p_α, which transform as spinors of $SO(10)$ and naturally encode spacetime supersymmetry. The relation between these formalism has been studied on [11] and [16]. Bridging these two descriptions is a crucial step toward establishing a framework that unifies features of both formalisms.

In this chapter we propose a systematic procedure to relate the RNS ψ^m fields to the pure spinor formalism variables $(p_\alpha, \theta^\alpha)$ by decomposing them into their $U(5)$ components in a similar way as on [11]. In particular, we introduce the p_a and θ^a fields, which serve as building blocks for supersymmetric operators within this new framework.

Once this identification is established, we proceed to construct the supersymmetry charges in terms of the new variables, verifying their algebra and consistency with the expected spacetime supersymmetry transformations. We then use the $U(5)$ decomposition to define covariant vertex operators, providing the necessary ingredients for computing scattering amplitudes in this extended setting.

Finally, we analyze the energy-momentum tensor T and the BRST charge q_α within this formalism. We first obtain an $SO(10)$ covariant expression for T, and give some insights towards an $SO(10)$ covariant construction of q_α and Q_{BRST}.

5.1 Introduction of p_a and θ^a fields

It has been found a relation between the Ramond ground state and the spacetime chiral coordinates θ^α (with $\alpha = 1, \ldots, 16$) of the spacetime supersymmetric description of a string (Green-Schwarz formalism) as follows: θ^α can also be decomposed in terms of its $U(5)$ components $(\theta^+, \theta_{ab}, \theta^a)$ and we use σ^a and ϕ to build θ^a [3]:

$$\theta^a \cong e^{\phi/2} S^a, \tag{5.1}$$

where S^a are components of S described by (2.49) that contain one $+$ sign and four $-$ sign on its exponential. We can write it as

$$\theta^a \cong e^{\phi/2} \exp\left(i\sigma_a - \frac{i}{2}\sum_{a=1}^{5} s_a\sigma_a\right). \tag{5.2}$$

We define the conjugate momentum to these variables as

$$p_a \cong e^{-\phi/2} S_a = e^{-\phi/2} \exp\left(-i\sigma_a + \frac{i}{2}\sum_{a=1}^{5} s_a\sigma_a\right), \tag{5.3}$$

where S_a contains one $-$ sign and four $+$ sign on its exponential. The OPE between these variables follow

$$p_a(z)\theta^b(0) \sim \frac{\delta_a^b}{z}. \tag{5.4}$$

It is possible to rewrite the BRST charge by using this set of fields. We have on RNS

$$Q_{\text{BRST}} = \oint \gamma\Psi^m \partial X_m - b\gamma^2 + cT^{\text{m},\beta\gamma} + bc\partial c, \tag{5.5}$$

$$T^{\text{m},\beta\gamma} = -\frac{1}{2}\partial X^m \partial X_m + \frac{1}{2}\Psi^m\partial\Psi_m - \beta\partial\gamma - \frac{1}{2}\partial(\beta\gamma), \tag{5.6}$$

where $T^{\text{m},\beta\gamma}$ is the energy-momentum tensor of the $X^m\Psi^m$ matter and $\beta\gamma$ ghost fields. Bosonizing the $\beta\gamma$ and fermionic matter fields

$$Q_{\text{BRST}} = \oint \eta e^{\phi} e^{i\sigma_a}\partial x_a + \eta e^{\phi} e^{-i\sigma_a}\partial x^a - b\eta\partial\eta e^{2\phi} + cT' - bc\partial c, \tag{5.7}$$

where we have decomposed the inner products in terms of $U(5)$ components and we defined T' as the energy momentum tensor in terms of the new fermionic matter fields, as is described later. We can replace $(\psi^a, \psi_a) \to (\theta^a, p_a)$, and it is convenient to define the chiral boson

$$\tilde{\phi} = \frac{3}{2}\phi - \frac{i}{2}\sum_{a=1}^{5}\sigma_a. \tag{5.8}$$

In terms of this we can rewrite the BRST charge. For example, the second term

$$e^{\phi}e^{-i\sigma_a} = e^{3\phi/2}e^{-\phi/2}\exp\left(-\frac{i}{2}\sum_{a=1}^{5}\sigma_a\right)\exp\left(\frac{i}{2}\sum_{a=1}^{5}\sigma_a\right)e^{-i\sigma_a} = e^{\tilde{\phi}}p_a, \tag{5.9}$$

In the same way we can work on the first term, but it requires some work

$$e^{\phi}e^{i\sigma_a} = e^{3\phi}e^{-2\phi}\exp\left(-i\sum_{a=1}^{5}\sigma_a\right)\exp\left(i\sum_{a=1}^{5}\sigma_a\right)e^{i\sigma_a} = e^{2\tilde{\phi}}e^{-2\phi}\exp\left(i\sum_{a=1}^{5}\sigma_a\right)e^{i\sigma_a}. \tag{5.10}$$

To see how to reduce this expresion take the particular case $a = 1$. We have then

$$e^{\phi}e^{i\sigma_1} = e^{3\phi}\exp\left(i\sum_{a=1}^{5}\sigma_a\right)\exp\left(-i\sum_{a=1}^{5}\sigma_a\right)e^{-2\phi}e^{i\sigma_1} = e^{2\tilde{\phi}}e^{i\sigma_1+i\sum_{a=1}^{5}s_a\sigma_a}e^{-2\phi}. \tag{5.11}$$

On the other hand consider the following term

$$\begin{aligned} e^{2\tilde{\phi}}\left(p^4\right)^1 &\equiv e^{2\tilde{\phi}}\frac{1}{24}\epsilon^{1bcde}p_a p_b p_c p_d \\ &= e^{2\tilde{\phi}}p_2p_3p_4p_5 = e^{2\tilde{\phi}}e^{-i\sigma_2-i\sigma_3-i\sigma_4-i\sigma_5+2i\sum_{a=1}^{5}s_a\sigma_a}e^{-2\phi} \\ &= e^{2\tilde{\phi}}e^{i\sigma_1-i\sigma_1-i\sigma_2-i\sigma_3-i\sigma_4-i\sigma_5+2i\sum_{a=1}^{5}s_a\sigma_a}e^{-2\phi} \\ &= e^{2\tilde{\phi}}e^{i\sigma_1+i\sum_{a=1}^{5}s_a\sigma_a}e^{-2\phi} = e^{\phi}e^{i\sigma_1}, \end{aligned} \tag{5.12}$$

where we considered that p_a anticommute with themselves. In the same way it is possible to prove that

$$e^{2\phi} = e^{3\tilde{\phi}}p^5 \equiv \frac{1}{120}e^{3\tilde{\phi}}\epsilon^{abcde}p_a p_b p_c p_d p_e. \tag{5.13}$$

So we reduce (5.7) to

$$Q_{\text{BRST}} = \oint \eta e^{2\tilde{\phi}}\left(p^4\right)^a\partial x_a + \eta e^{\tilde{\phi}}p_a\partial x^a - b\eta\partial\eta e^{3\tilde{\phi}}p^5 + cT' - bc\partial c, \tag{5.14}$$

Now we have to deal with $T'^{m,\beta\gamma}$. It is possible to do the same kind of procedure, but instead we work on the following: The full expresion of T (including the bc CFT contribution) on RNS formalism is given by

$$T = -\frac{1}{2}\partial X^m\partial X_m + \frac{1}{2}\Psi^m\partial\Psi_m - b\partial c - \partial(bc) - b\partial c - \frac{1}{2}\gamma\partial\beta - \frac{3}{2}\beta\partial\gamma, \tag{5.15}$$

and it has central charge zero. The expresion for T' terms of θ^a and p_a and $U(5)$ elements must be like

$$T' = -\partial x^a \partial x_a - p_a \partial\theta^a - b\partial c - \partial(bc) + \text{bosonic ghost contribution}, \tag{5.16}$$

where the bosonic ghost contribution that comes from β and γ must be modified. We can see that this process do not affect the bc terms since they are not related to p_a or θ^a as $\beta\gamma$ are once they are bosonized. Considering that the new terms come from a $\tilde{\beta}\tilde{\gamma}$ system of weight $h_{\tilde{\beta}} = x$ and $h_{\tilde{\gamma}} = 1 - x$ the central charge of the system is[1]

$$c = 3(2x-1)^2 - 27, \tag{5.17}$$

so to have $c = 0$ we can choose $x = 2$. So the complete expresion of T' is

$$T' = -\partial x^a \partial x_a - p_a \partial\theta^a - b\partial c - \partial(bc) - \tilde{\beta}\partial\tilde{\gamma} - \partial(\tilde{\beta}\tilde{\gamma}). \tag{5.18}$$

To prove that $T\left(\psi^a, \psi_a, \beta, \gamma\right) = T'\left(p_a, \theta^a, \tilde{\beta}, \tilde{\gamma}\right)$ we can bosonize $\tilde{\beta}\tilde{\gamma}$ system as

$$\tilde{\beta} = \partial\xi e^{-\tilde{\phi}},\ \tilde{\gamma} = \eta e^{\tilde{\phi}}, \tag{5.19}$$

where η, ξ are the same fields used to bosonize $\beta\gamma$, and $\tilde{\phi}$ is described as on Eq. (5.8). Notice that $\left(e^{\tilde{\phi}}, e^{-\phi}\right)$ have conformal weight $(-2, 1)$, so the conformal weight of the fields and its bosonized version agree. To finish the proof we require that the OPEs between $(\tilde{\beta}, \tilde{\gamma})$ and (p_a, θ^a) on their bosonized form do not have singular terms. The steps to demonstrate this require some simple calculations, and will be omitted. Notice that both T' and Q_{BRST} are manifestly $U(5)$ invariant.

It is convenient to redefine the fields $b \to -b$ and $c \to -c$ such that we end up with only positive terms of Q as

$$Q_{\text{BRST}} = \oint \eta e^{2\tilde{\phi}} \left(p^4\right)^a \partial x_a + \eta e^{\tilde{\phi}} p_a \partial x^a + b\eta\partial\eta e^{3\tilde{\phi}} p^5 + cT + bc\partial c, \tag{5.20}$$

$$T = \partial x^a \partial x_a + p_a \partial\theta^a + b\partial c + \partial(bc) + \tilde{\beta}\partial\tilde{\gamma} + \partial(\tilde{\beta}\tilde{\gamma}). \tag{5.21}$$

Since the bc OPE does not change under this redefinition the system is equivalent. Also notice that T is the negative of the usual energy momentum tensor.

[1]This can be easily computed since the $p\theta$ system behaves as a bc CFT with (p, θ) having conformal weight $(1, 0)$.

5.2 Supersymmetry charges

On the theory described by $(\theta^a, p_a, \tilde{\beta}, \tilde{\gamma})$ we can write six out of the sixteen supersymmetry charges as

$$q_a = \oint \frac{dz}{2\pi i} p_a, \; q_+ = \oint \frac{dz}{2\pi i} \theta^a \partial x_a + b\eta e^{\tilde{\phi}}. \tag{5.22}$$

that follow the relation $\{q_a, q_b\} = 0$, $\{q_+, q_a\} = \oint p_a$, $\{q_+, q_+\} = 0$. Notice that the BRST charge defined on terms on the new variables (Eq. 5.20) is invariant under the transformations generated by these operators. To get the remaining supersymmetry charges the addition of non-minimal variables pure spinor formalism is required.

5.3 $U(5)$ covariant vertex operators

It is possible to rewrite RNS vertex operators in terms of the new matter fields. As an example we work on the picture -1 gluon vertex operator:

$$V = ce^{-\phi} \Psi_m a^m(X). \tag{5.23}$$

Under the $U(5)$ decomposition and changing variables $(\psi_a, \psi^a) \to (p_a, \theta^a)$ in the same way we did before we get

$$V = ce^{-\tilde{\phi}} \theta^a a_a(x) + ce^{-2\tilde{\phi}} \left(\theta^4\right)_a a^a(x). \tag{5.24}$$

Notice that both $e^{-\tilde{\phi}}$ and $e^{-2\tilde{\phi}}$ have conformal weight $h = 1$. We can generalize this in terms of superfields as

$$V = ce^{-\tilde{\phi}} A(x, \theta) + ce^{-2\tilde{\phi}} A^*(x, \theta), \tag{5.25}$$

and the physical states condition $QV = 0$ will generate the equations of motion for the fields. To do the calculation it is convenient to separate Q in three terms, in the same way as done on [3]:

$$Q_0 = \oint \frac{dz}{2\pi i} cT + bc\partial c, \tag{5.26}$$

$$Q_1 = \oint \frac{dz}{2\pi i} \eta e^{\tilde{\phi}} p_a \partial x^a + \eta e^{2\tilde{\phi}} \left(p^4\right)^a \partial x_a, \tag{5.27}$$

$$Q_2 = \oint \frac{dz}{2\pi i} \eta \partial \eta b e^{3\tilde{\phi}} p^5. \tag{5.28}$$

So $QV = 0$ implies

$$Q_0 V = 0 \rightarrow \begin{cases} \partial^a \partial_a A(x,\theta) = 0, \\ \partial^a \partial_a A^*(x,\theta) = 0, \end{cases} \tag{5.29}$$

$$Q_1 V = 0 \rightarrow \begin{cases} \frac{\partial}{\partial \theta^a} \partial^a A(x,\theta) + \left[\left(\frac{\partial}{\partial \theta}\right)^4\right]^a \partial_a A^*(x,\theta) = 0, \\ \left[\left(\frac{\partial}{\partial \theta}\right)^2\right]^{abc} \partial_c A(x,\theta) = \left[\left(\frac{\partial}{\partial \theta}\right)^3\right]^{ab} A(x,\theta) = 0, \end{cases} \tag{5.30}$$

and $Q_2 V$ vanishes identically. As a result we can expand the fields as

$$A(x,\theta) = \rho_+(x) + \theta^a a_a(x) + \frac{1}{2}\theta^a \theta^b \xi_{ab}(x), \tag{5.31}$$

$$A^*(x,\theta) = \cdots + \frac{1}{2}\left(\theta^3\right)_{ab} \rho^{ab}(x) + \left(\theta^4\right)_a a^a(x) + \left(\theta^5\right) \xi^+(x), \tag{5.32}$$

where $\cdots$ are terms that are not relevant in our calculations. In fact, these can be gauge fixed to zero. Notice that the constraints imply

$$\epsilon^{abcde} \partial_c \xi_{de} = 0, \; \partial_a \xi^+ + \partial^b \xi_{ab} = 0, \tag{5.33}$$

which are massless Dirac equations for a chiral spinor ξ^α in $U(5)$ notation for $\xi^a = 0$. (See Appendix A). These constraints leave us with 4 on-shell independent components. To verify this we write the constraint equations in momentum space, and we go to a reference system where the $U(5)$ momentum is described by

$$P^a \rightarrow (m, 0, 0, 0, -m), \; P_a \rightarrow (m, 0, 0, 0, m). \tag{5.34}$$

In such frame the constraint equations imply

$$\left[\epsilon^{ab1de} + \epsilon^{ab5de}\right] \xi_{de} = 0 \to \xi_{ij} = 0, \tag{5.35}$$

$$\xi^{+} = \xi_{15},\ \xi_{i1} = \xi_{i5}, \tag{5.36}$$

with $i, j = 2, 3, 4$. Therefore we end up with a group of four non vanishing independent components $\left(\xi^{+}, \xi_{21}, \xi_{31}, \xi_{41}\right)$. Because the terms involving ξ^{a} are absent, there are fewer non-vanishing on-shell components than the expected eight, as predicted by the RNS and Green-Schwarz formulations. It is possible to complete the gluino components by using with ρ terms as follows.

To see what is described by ρ_{+} and ρ^{ab} notice that these terms are in the $-3/2$ picture, while terms that contain ξ are in the $-1/2$ picture. It is convenient to perform a picture raising transformations on ρ terms. Consider

$$V_{-3/2} = ce^{-\tilde{\phi}}\rho_{+}(x) + \frac{1}{2}ce^{-2\tilde{\phi}}\left(\theta^{3}\right)_{ab}\rho^{ab}(x), \tag{5.37}$$

and we obtain

$$V_{-1/2} = \{Q, \xi V_{-3/2}(0)\} = ce^{-\tilde{\phi}}A + cB, \tag{5.38}$$

$$A = \frac{1}{4}\epsilon_{abcde}\theta^{a}\theta^{b}\partial^{c}\rho^{de},\ B = p_{a}\left(\partial^{a}\rho_{+} - \partial_{b}\rho^{ab}\right). \tag{5.39}$$

This vertex operator also describe fermionic state defined by a chiral spinor v^{α} that in $U(5)$ notation has the following components

$$v \to \left(v^{+}, v_{ab}, v^{a}\right) = \left(0, \frac{1}{2}\epsilon_{abcde}\partial^{c}\rho^{de}, \partial^{a}\rho_{+} - \partial_{b}\rho^{ab}\right), \tag{5.40}$$

and these components already satisfy $U(5)$ Dirac equations. The addition of these variables to our system lead us to have the composite vertex operator in the form

$$v \to \left(v^{+}, v_{ab}, v^{a}\right) = \left(\xi^{+}, \xi_{ab} + \frac{1}{2}\epsilon_{abcde}\partial^{c}\rho^{de}, \partial^{a}\rho_{+} - \partial_{b}\rho^{ab}\right), \tag{5.41}$$

and we can describe the 8 on-shell components of the gluino vertex operator.

5.4 Towards $SO(10)$ covariant expressions of q_α and Q_{BRST}

Notice that the current description we have is not manifestly spacetime supersymmetric. The procedure leading to this description is presented in [7]. In this section, we provide complementary insights by reproducing part of the calculations and highlighting the key steps of the method.

First, we write T in terms of $SO(10)$ variables $(X^m, \theta^\alpha, p_\alpha)$. To do so we require the addition of Grassmann fields (θ^+, θ_{ab}) and their conjugate momentum $\left(p_+, p^{ab}\right)$ to the formalism. This can be done by adding some topological terms to the BRST charge

$$Q \to Q + \oint \lambda^+ p_+ + \frac{1}{2}\lambda_{ab}p^{ab}, \tag{5.42}$$

with some λ^α spinor field that has some canonical momentum w_α associated to it. Then, we do a similarity transformation $Q \to e^{-R}Qe^R$ with

$$R = \oint c\left(w_+\partial\theta^+ + \frac{1}{2}w^{ab}\partial\theta_{ab}\right). \tag{5.43}$$

Notice that we can write $R = \oint cw_\alpha\partial\theta^\alpha$ if we consider $w^a = 0$. It is possible to make this assumption if we consider that λ^α is a pure spinor, and we choose w^α to be in the $w^a = 0$ gauge[2]. We have then

$$Q' = e^{-R}Qe^R = Q + [Q, R] + \frac{1}{2}[[Q, R], R] + \dots \tag{5.44}$$

The commutators can be described in terms of contour integrals as

$$[Q, R] = \oint \text{Res}_{w\to z}\, j_{\text{BRST}}(w)cw_\alpha\partial\theta^\alpha(z), \tag{5.45}$$

with

$$j_{\text{BRST}} = \eta e^{2\tilde{\phi}}\left(p^4\right)^a\partial x_a + \eta e^{\tilde{\phi}}p_a\partial x^a + b\eta\partial\eta e^{3\tilde{\phi}}p^5 + cT + bc\partial c + \lambda^+p_+ + \frac{1}{2}\lambda_{ab}p^{ab}. \tag{5.46}$$

[2] For a extended discussion about this see [4]

The terms of the series are as follows

$$[Q, R] = \oint \left\{ \left(\eta\partial\eta e^{3\tilde{\phi}} p^5 - c\partial c \right) w_\alpha \partial\theta^\alpha + c \left[p_+ \partial\theta^+ + \frac{1}{2} p^{ab} \partial\theta_{ab} - w_\alpha \partial\lambda^\alpha \right] \right\}, \tag{5.47}$$

$$[[Q, R], R] = 2 \oint c\partial c w_\alpha \partial\theta^\alpha, \tag{5.48}$$

$$[[[Q, R], R], R] = 0. \tag{5.49}$$

Therefore

$$\begin{aligned} Q \to Q + \oint \left\{ \eta\partial\eta e^{3\tilde{\phi}} p^5 w_\alpha \partial\theta^\alpha + c \left[p_+ \partial\theta^+ + \frac{1}{2} p^{ab} \partial\theta_{ab} - w_\alpha \partial\lambda^\alpha \right] \right\} \\ = \oint \left[\eta e^{2\tilde{\phi}} \left(p^4 \right)^a \partial x_a + \eta e^{\tilde{\phi}} p_a \partial x^a + \eta\partial\eta e^{3\tilde{\phi}} p^5 \left(b + w_\alpha \partial\theta^\alpha \right) + cT + bc\partial c \right], \end{aligned} \tag{5.50}$$

with $T = \frac{1}{2}\partial X^m \partial X_m + p_\alpha \partial\theta^\alpha - w_\alpha \partial\lambda^\alpha + b\partial c + \partial(bc) + \tilde{\beta}\partial\tilde{\gamma} + \partial(\tilde{\beta}\tilde{\gamma})$ being completely $SO(10)$ invariant.

The similarity transformation done affects to all the operators that describe the theory. For example, the SUSY (q_+, q_a) charges are transformed as $q_a \to e^{-R} q_a e^R$ and $q_+ \to e^{-R} q_+ e^R$. Considering

$$[q_+, R] = \oint \eta e^{\tilde{\phi}} w_\alpha \partial\theta^\alpha(z), \tag{5.51}$$

$$[[q_+, R], R] = 0, \tag{5.52}$$

$$[q_a, R] = \oint \mathrm{Res}_{w \to z}\, p_a(w) c w_a \partial\theta^a(z) = 0. \tag{5.53}$$

So we are left with

$$q'_a = \oint p_a,\; q'_+ = \oint p_+ + \theta^a \partial x_a + \eta e^{\tilde{\phi}} \left(b + w_\alpha \partial\theta^\alpha \right), \tag{5.54}$$

where we also added the $\oint p_+$ contribution to q_+. To write it in covariant way it is required to perform another similarity transformation as in Eq. (5.44), with $R' = \oint \theta^+ \left(\partial x_a \theta^a + \tilde{\gamma} \left(b + w_\alpha \partial\theta^\alpha \right) \right)$. As a result we get

$$q_+ = \oint p_+, \tag{5.55}$$

$$q_a = \oint \left(p_a + \theta^+ \partial x_a \right). \tag{5.56}$$

Similarity transformation given by R' also will change the form of the BRST charge. A more complete discussion about this can be found on [7].

The missing components of q_α are q^{ab}. On [7] it is stated that by doing similarity transformation involving θ^{ab} terms it is possible to get such terms and we are left with the 16 components of the supercharge q_α, given by Eq. (4.5).

It is possible to write (5.50) in a fully covariant way with manifest supersymmetry. On [7] such BRST charge is described in terms of left moving non-minimum variables $(r_\alpha, \overline{\lambda}_\alpha)$ and their associated momentum $(s^\alpha, \overline{w}^\alpha)$ respectively. The BRST operator is given by

$$Q = \oint [\lambda^\alpha d_\alpha + \overline{w}^\alpha r_\alpha + \eta e^{\tilde{\phi}} \left(\Pi^m \overline{\Gamma}_m - \frac{\lambda \gamma^{mn} r}{4\lambda\overline{\lambda}} \overline{\Gamma}_m \overline{\Gamma}_n + b + w_\alpha \partial\theta^\alpha + s^\alpha \partial \overline{\lambda}_\alpha \right) + cT + bc\partial c$$
$$+ \frac{\lambda\gamma^{m_1 \dots m_5}\lambda}{120} \left[5\eta e^{2\tilde{\phi}} \overline{\Gamma}_{m_1} \dots \overline{\Gamma}_{m_4} \Pi_{m_5} + \eta\partial\eta e^{3\tilde{\phi}} \overline{\Gamma}_{m_1} \dots \overline{\Gamma}_{m_5} \left(b + s^\alpha \partial\overline{\lambda}_\alpha + \left(w_\alpha + \partial\tilde{\phi} \frac{\overline{\lambda}_\alpha}{\lambda\overline{\lambda}} \right) \partial\theta^\alpha \right) \right], \tag{5.57}$$

with $\overline{\lambda}_\alpha$ being a complex pure spinor, and r_α being a fermionic spinor that satisfies $\bar{\lambda}\gamma^m r = 0$. Π^m and d_α are given by Eq. (4.8) . The remaining variables that come from (5.57) are given by

$$T = \frac{1}{2}\partial X^m \partial X_m + p_\alpha \partial\theta^\alpha - w_\alpha \partial\lambda^\alpha + b\partial c + \partial(bc) + \tilde{\beta}\partial\tilde{\gamma} + \partial(\tilde{\beta}\tilde{\gamma}) - w_\alpha\partial\lambda^\alpha + s^\alpha \partial r_\alpha, \tag{5.58}$$

$$\overline{\Gamma}_m = \frac{1}{2\lambda\overline{\lambda}} \left[\overline{\lambda}\gamma_m d - \frac{1}{8\lambda\overline{\lambda}} \overline{\lambda}\gamma_{mnp} r \left(w\gamma^{np}\lambda \right) \right], \text{ with } \overline{\lambda}\gamma^m w = 0. \tag{5.59}$$

Since the changes done to the BRST came from similarity transformations we expect that the cohomology of the system (and therefore the physical states) does not change.

Conclusions and perspectives

In this thesis we have explored different approaches to the covariant quantization of the superstring, focusing on the Ramond–Neveu–Schwarz (RNS) and pure spinor formalisms, and preparing the ground for the construction of a new version of the pure spinor formalism. The work was organized in three main parts, each with a different level of depth and purpose.

In the first part, we reviewed the RNS formalism, beginning with the worldsheet action for both open and closed strings, its associated conformal and superconformal symmetries, and the role of the b, c and β, γ ghost systems. We analyzed the structure of the matter and ghost ground states, and emphasized the operator–state correspondence, which is a key ingredient in building vertex operators. We then described in detail the construction of vertex operators for tachyonic, massless, and massive excitations of the string, as well as the GSO projection that ensures the consistency of the spectrum and leads to spacetime supersymmetry.

We also turned to scattering amplitudes in the RNS formalism. We recalled how the path integral prescription in bosonic string theory generalizes to the superstring, and discussed the role of conformal Killing vectors, moduli space, and the ghost number anomaly. In particular, we reviewed the necessity of Picture Changing Operators (PCOs) and the associated subtleties that make higher-loop calculations in the RNS formalism technically involved. We also highlighted how the prescription realizes spacetime supersymmetry, setting the stage for comparing with the pure spinor approach.

The second part was devoted to the pure spinor formalism, which provides a manifestly super-Poincaré covariant framework for superstring quantization. We reviewed the pure spinor action, the ghost sector, and the pure spinor constraint, including its $U(5)$ decomposition and conformal properties. We discussed the construction of vertex operators and the tree-level prescription for scattering amplitudes, emphasizing how this formalism circumvents some of the difficulties of the RNS approach.

Finally, in the third part we initiated the study of a new pure spinor formalism, which seeks to build a bridge between the RNS fermions and the pure spinor variables. We introduced the relation between the worldsheet fields p_a and θ^a and the fermionic matter field Ψ^m that comes from the RNS formulation.

Also, as a particular case, the massless gluon vertex operator was studied. It was found that by doing a generalization it was possible to describe both gluon and gluino by using a descripion in terms of superfields. The eight gluino degrees of freedom were organized into two sets of four, with each set represented in a different picture.

We described briefly their supersymmetry charges, and presented both $U(5)$ covariant vertex operators and also some hints on how to construct $SO(10)$ covariant expresions of the supercharge q_α and Q_{BRST} operators. About this last one, we quoted a result presented on [7] at the end of this section, where it is formulated a manifestly supersymmetric covariant BRST operator in this new framework by adding to the description non-minimal variables.

It is important to emphasize that the last two sections of the third part of this work remain incomplete in the sense that we did not study the full scattering amplitude prescription in the new pure spinor formalism. Instead, we carried out exploratory calculations and identified key ingredients that will be necessary for a consistent formulation. These partial results already illustrate both the promise and the challenges of the approach, and they open a number of directions for future research.

As a perspective, the next steps one can work on can be on completing the construction of the covariant BRST charge by checking to the explicit form of the similarity transformations to get the manifestly supersymmetric covariant BRST operator, deriving the full prescription for scattering amplitudes, and comparing it systematically with both the standard pure spinor formalism and the RNS formalism. Furthermore, it will be interesting to explore whether the new formalism simplifies multiloop amplitude calculations or provides some interesting features when the formalism is applied in curved spacetime backgrounds.

Appendix A

Massless Dirac equation on $U(5)$ notation

On this appendix we decribe briefly how to write the massless Dirac equation in terms of $U(5)$ variables. Consider the gluino vertex operator on RNS formalism is given by

$$V = ce^{-\phi/2}u_{\alpha}S^{\alpha}. \tag{A.1}$$

The BRST charge

$$Q = \oint \eta e^{\phi}\psi^{m}\partial X_{m} - b\eta\partial\eta e^{2\phi} + cT^{\mathrm{m},\beta\gamma} + bc\partial c, \tag{A.2}$$

and the physical state condition $QV = 0$ will give us the equation of motion for the gluino. As a result we get

$$\gamma^{m\beta\alpha}\partial_{m}u_{\alpha} = 0, \tag{A.3}$$

which is the Dirac equation for a massless spinor field. We will do the same procedure but using $U(5)$ notation. We start by considering the decomposition of (A.1) in $U(5)$ variables as [4]:

$$V = ce^{-\phi/2}u^{\alpha}S_{\alpha} = ce^{-\phi/2}\left[u^{+}S_{+} + \frac{1}{2}u_{ab}S^{ab} + u^{a}S_{a}\right]. \tag{A.4}$$

So we define

$$V_1 = ce^{-\phi/2}u^{+}S_{+},\ V_2 = \frac{1}{2}ce^{-\phi/2}u_{ab}S^{ab},\ V_3 = ce^{-\phi/2}u^{a}S_{a}, \tag{A.5}$$

$$S_{+} = e^{-\frac{i}{2}\sum_i\sigma_i},\ S^{ab} = e^{i\sigma_a + i\sigma_b - \frac{i}{2}\sum_i\sigma_i},\ S_a = e^{-i\sigma_a + \frac{i}{2}\sum_i\sigma_i}. \tag{A.6}$$

The RNS BRST charge can also be written in terms of $U(5)$ variables:

$$Q = \oint \eta e^{\phi}\psi^a \partial X_a + \eta e^{\phi}\psi_a \partial X^a - b\eta\partial\eta e^{2\phi} + cT^{\mathrm{m},\beta\gamma} + bc\partial c. \tag{A.7}$$

To obtain the equations of motion we only require to evaluate $\{Q_1, V\}$, with

$$Q_1 = \oint \eta e^{\phi}\psi^a \partial X_a + \eta e^{\phi}\psi_a \partial X^a, \tag{A.8}$$

since the remaining terms do not contribute on-shell. We have then

$$\begin{aligned}\{Q_1, V_1\} &= \oint \left[\eta e^{\phi}\psi^a \partial X_a + \eta e^{\phi}\psi_a \partial X^a\right](z) c(0) e^{-\phi(0)/2} u^+(X) S_+(0) \\ &= \oint z^{1/2} : e^{\phi} e^{-\phi/2} : c(0) \left[\frac{1}{z}\eta\psi^a(z) S_+(0)\partial_a u^+(X) + \frac{1}{z}\eta\psi_a(z) S_+(0)\partial^a u^+(X) + \dots\right].\end{aligned}$$

Considering that we bosonized $\psi^a \cong e^{i\sigma_a}$ we get

$$\begin{aligned}\{Q_1, V_1\} =& \oint z^{1/2} : e^{\phi} e^{-\phi/2} : c\eta \left[z^{-3/2} : e^{i\sigma_a} e^{-\frac{i}{2}\sum_i \sigma_i} : \partial_a u^+ + z^{-1/2} : e^{-i\sigma_a} e^{-\frac{i}{2}\sum_i \sigma_i} : \partial^a u^+ + \dots\right] \\ =& c(0)\eta : e^{\phi/2} : S^a \partial_a u^+(X).\end{aligned}$$

We move on to the next term:

$$\begin{aligned}\{Q_1, V_2\} &= \frac{1}{2}\oint z^{1/2} : e^{\phi} e^{-\phi/2} : c(0) \left[\frac{1}{z}\eta\psi^c(z) S^{ab}(0)\partial_c u_{ab} + \frac{1}{z}\eta\psi_c(z) S^{ab}(0)\partial^c u_{ab} + \dots\right] \\ &= \frac{1}{2}\oint z^{1/2} : e^{\phi} e^{-\phi/2} : c(0) \\ &\times \left[\frac{1}{z}\eta e^{i\sigma_c(z)} e^{i\sigma_a + i\sigma_b - \frac{i}{2}\sum_i \sigma_i}\partial_c u_{ab} + \frac{1}{z}\eta e^{-i\sigma_c(z)} e^{i\sigma_a + i\sigma_b - \frac{i}{2}\sum_i \sigma_i}\partial^c u_{ab} + \dots\right] \\ &= \frac{1}{2}\oint z^{1/2} : e^{\phi} e^{-\phi/2} : c(0) \\ &\times \left[\frac{1}{z}\eta e^{i\sum_i \delta_{ic}\sigma_i(z)} e^{i\sum_i\left(\delta_{ia} + \delta_{ib} - \frac{1}{2}\right)\sigma_i}\partial_c u_{ab} + \frac{1}{z}\eta e^{-i\sum_i \delta_{ic}\sigma_i(z)} e^{i\sum_i\left(\delta_{ia} + \delta_{ib} - \frac{1}{2}\right)\sigma_i}\partial^c u_{ab} + \dots\right],\end{aligned}$$

$$\begin{aligned}\to \{Q_1, V_2\} = \frac{1}{2}\oint z^{-1/2} : e^{\phi} e^{-\phi/2} : c(0)[&\eta z^{\left(\delta_{ca} + \delta_{cb} - \frac{1}{2}\right)} : e^{i\sigma_c(z) + i\sigma_a + i\sigma_b - \frac{i}{2}\sum_i \sigma_i} : \partial_c u_{ab} \\ &+ \eta z^{-\left(\delta_{ca} + \delta_{cb} - \frac{1}{2}\right)} : e^{-i\sigma_c(z) + i\sigma_a + i\sigma_b - \frac{i}{2}\sum_i \sigma_i} : \partial^c u_{ab} + \dots].\end{aligned} \tag{A.9}$$

So the first term only contributes when $a \neq b \neq c$, while the second one is non zero if $c = a$ or $c = b$. So we can write

$$\{Q_1, V_2\} = \frac{1}{2}\eta e^{\phi/2} c(0) \left[-\frac{1}{2}\epsilon^{abcde} S_{ab}\partial_c u_{de} + 2S^b \partial^a u_{ab}\right]. \tag{A.10}$$

Finally, the last term

$$\begin{aligned}\{Q_1, V_3\} &= \oint cz^{1/2} : e^{\phi} e^{-\phi/2} : \left[\frac{1}{z} z^{-1/2} \eta S^+ \partial_b u^b + \frac{1}{z} \eta z^{-1/2} S_{ab} \partial^a u^b\right] \\ &= c\eta e^{\phi/2} \left[S^+ \partial_b u^b + S_{ab} \partial^a u^b\right].\end{aligned}$$

So we get

$$\begin{aligned}\{Q_1, V\} &= \eta e^{\phi/2} c S^a \partial_a u^+(X) + \eta e^{\phi/2} c \left[-\frac{1}{4} \epsilon^{abcde} S_{ab} \partial_c u_{de} + S^b \partial^a u_{ab}\right] + c\eta e^{\phi/2} \left[S^+ \partial_b u^b + S_{ab} \partial^a u^b\right] \\ &= \eta e^{\phi/2} c \left[S^a \left[\partial_a u^+(X) - \partial^b u_{ab}\right] + \frac{1}{2} S_{ab} \left[\partial^a u^b - \partial^b u^a - \frac{1}{2} \epsilon^{abcde} \partial_c u_{de}\right] + S^+ \partial_b u^b\right].\end{aligned}$$

Each term of this equation must vanish. As a result we end with the massless Dirac equations, that are given by

$$\begin{aligned}\partial_a u^+ - \partial^b u_{ab} &= 0, \\ \partial^a u^b - \partial^b u^a - \frac{1}{2} \epsilon^{abcde} \partial_c u_{de} &= 0, \\ \partial_b u^b &= 0.\end{aligned} \tag{A.11}$$

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