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BV formalism and String Theory

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Dedico esta tese aos
profissionais da saúde dos anos de 2020 e 2021

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"Tell all the truth but tell it slant -
Success in Circuit lies
Too bright to our infirm Delight
The Truth's superb surprise
As Lightning to the Children eased
With explanation kind
The Truth must dazzle gradually
Or every man be blind"

Emily Dickinson

Resumo

Essa tese consiste de basicamente duas partes. Primeiro, serão revisados conceitos basicos e fundamentais de teorias de campos e de cordas. Será dada uma ênfase na ideia de simetria presente em teorias de campos, por exemplo simetrias de calibre, de Poincaré e supersimetria. Será discutido o método BRST de quantização de teorias de calibre, o formalismo de cordas bosonicas de Green-Schwartz e também o formalismo de Pure spinor para supercordas. Tanto o caso plano quanto curvo - $AdS_5 \times S^5$ - será tratado. A segunda parte da tese consiste em estudar o formalismo BV e aplicá-lo para descrever teorias que possuem operador BRST, como é o caso da teorias de cordas bosonica e teorias de Yang-Mills. Serão apresentadas descent equations que acabam aparecendo no formalismo BV e com estas equações, serão derivados algumas expressões de operatores de vertex para cordas bosonicas. No final, será feita a descrição da supercordas no formalismo de pure spinor utilizando o formalismo BV, no qual deduzimos expressões de vertex operators não integrados.

Palavras Chaves: Teoria de campos; Teoria de Cordas; Supercordas; Formalismo BV;

Áreas do conhecimento: Física; Sub-área da Física; Sub-área da Física.

Abstract

This thesis consists basically of two parts. First, we review the most basic and important concepts present in field theory and string theory. We give some focus in discussing the idea of symmetries present in field theories, remarkable gauge symmetry, Poincaré and supersymmetry. We discuss the BRST process of quantizing gauge theories, the formalism of Green-Schwartz for describing fermionic strings and subsequently the pure spinor superstring, both in flat and $AdS_5 \times S^5$ background. The second part of the thesis consists in studying the BV formalism and applying it to describe some theories which possesses BRST operators, like bosonic string and Yang Mills. We present descent equations that appears in the BV formalism and show how we can derive vertex operators from it. At the end, we present a the BV description for pure spinor string, in which case we have also derived expressions for vertex operators.

Palavras Chaves: Field theory; String theory; Superstring; BV formalism; Descent procedure.

Áreas do conhecimento: Física; Sub-área da Física; Sub-área da Física.

Contents

Chapter 1

Introduction

Field theories is a standard and modern way to study nature. One of the main motivations to study field theory is to try to understand quantum phenomenons, and to do this, we use a large variety of mathematical tools and techniques. For instance, when we try to understand the quantization of theories with gauge symmetries, we get the difficulty to describe the quantum theory without depending on the gauge fixing. The most powerful technique one uses to study it is the BRST procedure [21]; In this case, we fix an arbitrary gauge and introduces *ghost* fields that caries all the information we have lost from gauge fixing. At the end, the gauge symmetry present in the classical theory is translated to the quantum theory as the BRST symmetry.

Lots of theories is known for various descriptions of nature. String theories and Yang-Mills theories are two of the most successful ones used to describe physics in high energy regimes. Moreover, some interesting mathematical properties arrives when studying field theory, and one that deserves a special status is the concept of symmetry, studied via Noether theorem [11]. Such concept involves the use of structures from group theory and differential geometry to give physical significant definitions. In this sense, every symmetry can be stated as the action of a given group on the space of fields of the theory, which leaves the dynamics of the fields invariant, that is, the group action leaves the action functional invariant. And for every symmetry, thanks to Noether theorem, one associates a given conserved current and a conserved charge. In the context of the quantum theory, the corresponding conserved charges are operators on the Hilbert space, such that it generates such symmetry.

The study of theories with gauge symmetry is remarkable, because of its difficulty and also the richness of such descriptions. In fact, it can describe a lot of important phenomenons, as gravity and particle physics [27]. Therefore, the use of different techniques to clarify the understanding of such theories can always draws attentions and at the same time have to be performed carefully. For bosonic string, the gauge symmetry comes from diffeomorphism of the worldsheet and

Weyl invariance, which naturally gives rise a BRST charge [1]. The same logic applies to the RNS fermionic strings [2], however for the Pure Spinor description of superstrings [5], the BRST operator is not understood as coming from a gauge symmetry anymore. In fact, the BRST quantization for this case is not completely clear and have been discussed extensively in the last years, with lot of studies concerning the use of Pure Spinor to describe a covariant quantization of fermionic strings [5], [23], [28], [9]. Such formalism explicitly describes a BRST like operator used to perform the quantization.

One important tool, currently used by mathematicians [24], to study QFT is the Batalin-Vilkovisky formalism (BV). Such formalism was firstly introduced for description of the gauge algebra of quantum theories [25]. In fact, the presence of a BRST operator on a given Quantum Field Theory allow us to perform a BV description on it and carry further studies without the constraint to be on-shell. For instance, we are able to obtain the off-shell expressions for the energy momentum tensor on bosonic string. In the BV formalism, we introduce a cotangent bundle over the space of fields to form the so-called BV phase space which then, as in the case of Hamiltonian description of classical mechanics [30], is a symplectic space. The elements of the base space in this bundle are the fields we had before and the elements of the fiber are called anti-fields.

We then have described the Bosonic String, superstrings and Yang-Mills giving a special attention to the symmetries present there, such as Poincaré symmetry, supersymmetry and gauge symmetry. Recently, a special focus is given for superstring in curved background by its physical significance [8], [31] and therefore we have also described superstrings in $AdS_5 \times S^5$.

These theories we have described possesses a BRST operator and therefore we can always make its corresponding BV description. We have them described the BV theories for strings and superstrings, as well as for the Yang-Mills, all in the flat space. At the end, using this description we were able to study symmetries using the so-called descent procedure [20]. This procedure allow us to first of all study the conserved current, that comes from a given symmetry using the BV operator, a generalized version of the BRST operator that acts on the hole BV phase space; such conserved currents may also involves anti-fields. Secondly, this gives us a relation between the BV cohomological complex and the De-Rham complex, expressed in the descent equations chain. Using this procedure, we can deform the action in order to obtain Vertex Operators in a given specific way called $\beta-$deformation [20], [26]. Doing this, we have obtained expressions for some vertex operators in bosonic string and pure spinor superstring, both in flat space.

Noticing that we have described superstring in curved $AdS_5 \times S^5$ space and its corresponding BRST like operator with the pure spinor description [9], we see that its the corresponding BV description is possible to be done [26]. However, have not done this here and explicitly, and this can be worked out in the future, specially with the purpose to obtain vertex operators.

Chapter 2

Super-groups and Super-algebras

In this chapter, we will present the necessary background we need in order to study superstrings. Further, we will see that superstring theories in flat and curved backgrounds can be seen as sigma models together with topological terms with respect to a given supergroup coset:

$$\text{Flat Space}: \frac{SUSY(\mathcal{N}=2)}{SO(9,1)} \tag{2.1}$$

$$\text{Super-}AdS_5 \times S^5 \text{ Space}: \frac{PSU(2,2|4)}{SO(4,1)\times SO(5)} \tag{2.2}$$

$SUSY(\mathcal{N}=2)$ denotes the super-Poincare algebra in 10 dimensions with 2 super-symmetric generators, which will be explained in next sections, as well as the $PSU(2,2|4)$ group.

2.1 Lie Groups and Lie Algebras

Lie groups plays an important role in physics. In fact, this notion is present when we talk about symmetries, and more specifically continuous symmetries, like rotations and translations. This is the intuitive picture of what a Lie group is. An important property of such objects is that, somehow, the information about the group is encoded into a vector space furnished with an algebraic structure: its corresponding Lie algebra. The precise definition and some examples will clarify these ideas:

Definition 2.1 (Lie Group)**.** A **Lie Group** G is a smooth manifold together with a group operation $(G,\bullet)$, satisfying the axioms for a group (associativity, inverse element and identity element), such that the following maps are smooth:

$$\bullet : G\times G \longrightarrow G \tag{2.3}$$

$$(g,h) \longmapsto g\bullet h \tag{2.4}$$

$$\mathrm{inv} : G \longrightarrow G \tag{2.5}$$

$$g \longmapsto g^{-1} \tag{2.6}$$

This definition gives us the intuition that a Lie group is a space with smooth properties, that is, one can run over the group elements smoothly.

Let us study some important examples of Lie groups:

Example 2.1 (General linear group)**.** *The **general linear group** over $\mathbb{R}$, denoted $GL(n, \mathbb{R})$, is the group of all invertible $n \times n$ matrices with real entries. $GL(n, \mathbb{C})$ is the set of all invertible matrices with complex entries.*

All the matrices Lie groups are subgroups of the general linear group. The prefix **S** means a constraint in the determinant:

Example 2.2 (Special Linear Groups)**.** *The **Special linear group** - $SL(n, \mathbb{R})$ or $SL(n, \mathbb{C})$ - is the set of all $n \times n$ invertible matrices with determinant one. This meets the condition of group because if $A, B \in SL$*

$$\det(AB) = \det(A)\det(B) = 1 \tag{2.7}$$

$$\det(A^{-1}) = \det(A)^{-1} = 1 \tag{2.8}$$

Example 2.3 (Orthogonal groups)**.** *This is the subgroup of rotations, defined as matrices over $\mathbb{R}$ that preserves inner products:*

$$O(n) = \{M \in GL(n) \mid \langle Mx, My \rangle = \langle x, y \rangle \ , \ \forall x, y \in \mathbb{R}^n\} \tag{2.9}$$

This meet the condition of group since if $M, N \in O(n)$:

$$\langle (MN)x, (MN)y \rangle = \langle M(Nx), M(Ny) \rangle = \langle Nx, Ny \rangle = \langle x, y \rangle \tag{2.10}$$

$$\langle N^{-1}x, N^{-1}y \rangle = \langle N(N^{-1}x), N(N^{-1}y) \rangle = \langle x, y \rangle \tag{2.11}$$

We can define also this group as matrices whose inverse is equal to the transposed, because:

$$\langle Ax, Ay \rangle = \langle x, A^T Ay \rangle = \langle x, y \rangle \qquad \forall x, y \in \mathbb{R}^n \tag{2.12}$$

The special orthogonal group is defined as

$$SO(n) = \{M \in O(n) \mid \det(M) = 1\} \tag{2.13}$$

Example 2.4 (Unitary group). *The unitary group is composed of all complex matrices that preserves the hermitian inner product:*

$$U(n) = \{M \in GL(n,\mathbb{C}) \mid \langle Mx, My\rangle = \langle x,y\rangle \ , \ \forall x,y \in \mathbb{C}^n\} \tag{2.14}$$

Again, since $\langle Ux, Uy\rangle = \langle x, U^{\dagger}Uy\rangle = \langle x,y\rangle$ *for all* $x,y \in \mathbb{C}^n$*,* $U(n)$ *can be seem as the matrices such that* $U^{\dagger} = U^{-1}$*.*

The special unitary group:

$$SU(n) = \{U \in U(n) \mid \det(U) = 1\} \tag{2.15}$$

Example 2.5 (Symplectic group). *The symplectic group is defined in the same spirit as the orthogonal and unitary groups. But now, the structure that is preserved is the symplectic structure:*

$$B[x,y] = \langle x, Jy\rangle \tag{2.16}$$

with

$$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \tag{2.17}$$

Therefore, the symplectic group is defined as:

$$Sp(n) = \{A \in GL(2n) \mid B[Ax, Ay] = B[x,y] \ \ \forall x,y \in \mathbb{K}^n\} \tag{2.18}$$

We see that $A \in Sp(n)$ *if and only if* $A^T J A = J$*, because* $\langle Ax, JAy\rangle = \langle x, A^T J A y\rangle = \langle x, Jy\rangle$ *for all* $x,y \in \mathbb{R}^n$

We can define also the so-called ***compact symplectic group****:*

$$USp(n) = Sp(n,\mathbb{C}) \cap U(2n) \tag{2.19}$$

Let us now define what a Lie algebra is and makes the relation between Lie algebra and Lie group.

Definition 2.2. An **algebra** is a pair $(V, \cdot\,)$, with V a vector space and $\cdot$ a bilinear operation:

$$\cdot : V \times V \longrightarrow V \tag{2.20}$$

Definition 2.3. A **Lie algebra** is an algebra $(V, [-,-])$ such that, the operation

$\cdot = [-,-]$ is anti-symmetric and satisfies a Jacobi identity:

$$[[a,b],c] + [[b,c],a] + [[c,a],b] = 0 \quad \forall\, a,b,c \in V \tag{2.21}$$

The way to relate Lie Groups and Lie Algebras can be seen by a geometrical picture:

Theorem 2.1. *For every Lie group G there corresponds an associated Lie algebra $\mathfrak{g} = T_eG$ with the commutator given by:*

$$[X,Y] = X \circ Y - Y \circ X, \qquad X,Y \in T_eG \tag{2.22}$$

This relation is glued with the definition of exponential map

$$\exp : \mathfrak{g} \longrightarrow G \tag{2.23}$$

This map is defined in such a way that $G = \exp(\mathfrak{g})$. Defining exp in general Lie groups is more complicated. Let us define it on matrix groups:

$$\begin{aligned} \exp : \mathfrak{g} &\longrightarrow G \\ X &\longmapsto e^X = \sum_{n=0}^{\infty} \frac{X^n}{n!} \end{aligned} \tag{2.24}$$

it is indeed possible to prove that (2.24) converges for any $n \times n$ matrix, and more that that, every $n \times n$ invertible matrix can be written as e^X for some $X \in Mat_{n\times n}$ [15]. Therefore, for a given matrix Lie group, its connected part to identity can be obtained from the exponential of a given set of matrices. It turns out that this set is a matrix Lie algebra and this is one way to define the corresponding Lie algebra:

Example 2.6 (General Linear Algebra)**.** *For the Lie group $GL(n)$ of example (2.1), we define its corresponding Lie algebra:*

$$\mathfrak{gl}_n = \{A \in Mat_{n\times n} \mid \exp(A) \in GL(n)\} \tag{2.25}$$

By what we have discussed, we see that $\mathfrak{gl}_n$ is the set of all matrices: $\mathfrak{gl}_n = Mat_{n\times n}$.

Example 2.7 (Special Linear Algebra)**.** *Using that $\det(e^X) = e^{tr(X)}$, we see that the* *special linear group may be defined as traceless matrices:*

$$\det(X) = \exp(tr(A)) = \exp(0) = 1 \tag{2.26}$$

That is:

$$\mathfrak{sl}_n = \{A \in Mat_{n\times n} \mid tr(A) = 0\} \tag{2.27}$$

Example 2.8 (Orthogonal Algebra)**.** *We take an orthogonal matrix A:*

$$\langle Ax, Ay\rangle = \langle x, A^T Ay\rangle = \langle x, y\rangle \ , \qquad x, y \in \mathbb{R}^n \tag{2.28}$$

Therefore, the orthogonal linear group may be defined as matrices such that $A^{-1} = A^T$. Therefore take $A = e^X \in O(n)$, then:

$$e^{X^T} = e^{-X} \tag{2.29}$$

following:

$$\mathfrak{so}_n = \{X \in Mat_{n\times n} \mid tr(X) = 0 \ \ and \ \ X^T = -X\} \tag{2.30}$$

Example 2.9 (Unitary Algebra)**.** *We saw that elements on the unitary group satisfies $U^\dagger = U^{-1}$, therefore, its corresponding Lie algebra must satisfy $e^{X^\dagger} = e^{-X}$:*

$$\mathfrak{su}(n) = \{X \in Mat_{n\times n}(\mathbb{C}) \mid X^\dagger = -X \ and \ tr(X) = 0\} \tag{2.31}$$

Example 2.10 (Symplectic Lie algebra)**.** *We see that elements of Symplectic group are of the form A such that $A^T JA = J$. Therefore, if $A = e^X$, we use that $-Je^X J = e^{-JXJ}$ to see the forms of the elements of Symplectic Lie algebra:*

$$A^T JA = J \Rightarrow A^T J = JA^{-1} \Rightarrow -JA^T J = A^{-1} \tag{2.32}$$

Then, if $A = e^X$, we see that it must satisfy $JX^T J = X$, and if X is of the form:

$$X = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{2.33}$$

reads as:

$$JX^T J = \begin{pmatrix} -D^T & B^T \\ C^T & -A^T \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{2.34}$$

Therefore:

$$\mathfrak{sp}(n) = \left\{ \begin{pmatrix} A & B \\ C & -A^T \end{pmatrix} \in Mat_{2n\times 2n} \,\middle|\, A, B, C \in Mat_{n\times n} \text{ and } B, C \text{ symmetric} \right\} \tag{2.35}$$

And finally:

$$\mathfrak{usp}(n) = sp(n) \cap \mathfrak{u}(2n) \tag{2.36}$$

A remarkable group is the Poincaré, and it is defined as the groups of Translations (P) and Rotations (M) in a given space-time.

Example 2.11. *The* ***Poincaré algebra*** *in* $\mathbb{R}^{d-1,1}$*, denoted* ***Poinc****(d-1,1), is defined as:*

$$V = \mathbb{R}^{\frac{d(d+1)}{2}} \tag{2.37}$$

$$[M^{ab}, M^{cd}] = \eta^{ac}M^{bd} - \eta^{bc}M^{ad} + \eta^{bd}M^{ac} - \eta^{ad}M^{bc} \tag{2.38}$$

$$[P_a, P_b] = 0 \qquad [P_a, M^{bc}] = \delta_a^b P^c - \delta_a^c P^b \tag{2.39}$$

with P_c *e* M^{ab}, $a < b$ *a basis to* $\mathbb{R}^{\frac{d(d+1)}{2}}$.

2.2 Super-algebras

The notion of super-algebra comes from the idea of graded algebraic object. We will now define some important ones. In what follows, take $(I, +)$ to be an abelian group:

Definition 2.4. A I**-graded vector space** is a vector space V which can be decomposed as:

$$V = \bigoplus_{i\in I} V_i \tag{2.40}$$

Definition 2.5. A I**-graded algebra** $(V, \cdot)$ is an algebra in a given I-graded vector space such that:

$$V_i \cdot V_j \subset V_{i+j} \tag{2.41}$$

The next important notion is the graded module. In general, it is defined with respect to a graded ring, but for our purpose and further constructions, we use a graded algebra as our ring:

Definition 2.6. A left I-**graded module** over a I-**graded algebra** is a module M with a decomposition:

$$M = \bigoplus_{i \in I} M_i \tag{2.42}$$

such that:

$$M_i \cdot V_j \subset M_{i+j} \tag{2.43}$$

similarly, a right graded module can be defined.

Given an I-graded vector V space and an I-graded algebra A, one can construct the following left graded module:

$$M = A \otimes V \tag{2.44}$$

over A. The graduation is given by:

$$M_i = \bigoplus_{j+k=i} A_j \otimes V_k \tag{2.45}$$

For every gradded algebraic structure $A = \bigoplus_i A_i$, one says that an element $p \in A$ is **homogeneous** if it belongs to one of the subspaces: $p \in A_j$.

The suffix **super** for any algebraic structure just means that it is $\mathbb{Z}_2-$graded. For instance, a super-vector space V has a decomposition $V = V_0 \oplus V_1$, and if $p = \dim(V_0)$ and $q = \dim(V_1)$, one says that $\dim(V) = p|q$. Also, it is always useful to work with a homogeneous ordered basis, that is, a ordered basis to V:

$$\{e_1, \cdots, e_p, e_{p+1}, \cdots, e_{p+q}\} \tag{2.46}$$

such that they are homogeneous elements of V:

$$e_1, \cdots, e_p \in V_{\bar{0}}, \qquad \text{and} \qquad e_{p+1}, \cdots, e_{p+q} \in V_{\bar{1}} \tag{2.47}$$

For a given super algebraic structure $A_0 \oplus A_1$, it's common to denote A_0 as the even or bosonic part, and A_1 as the odd or fermionic part.

Let us now see an interesting example:

Example 2.12 (Grassmann algebra)**.** *Consider* $\{\theta_1, \cdots, \theta_n\}$ *a set of anti-commuting complex elements:*

$$\{\theta_i, \theta_j\} = 0 \qquad \forall i, j = 1, \cdots, n \tag{2.48}$$

they are called Grassmannians and generates the Grassmann algebra $\Lambda^\bullet(\mathbb{C}^n)$*. The* *super-vector space is a complex vector space generated by a basis given by all possible products of θ's, that is, it's a (2^n-1)-dimensional vector space:*

$$\Lambda^\bullet(\mathbb{C}^n) = \bigoplus_{k=1}^{n} \Lambda^k(\mathbb{C}^n) \tag{2.49}$$

with the gradding given by:

$$\Lambda^\bullet(\mathbb{C}^n)_0 = \bigoplus_{k\ even} \Lambda^k(\mathbb{C}^n) \tag{2.50}$$

$$\Lambda^\bullet(\mathbb{C}^n)_1 = \bigoplus_{k\ odd} \Lambda^k(\mathbb{C}^n) \tag{2.51}$$

From the Grassmann algebra, it's possible to construct interesting super-modules, as seem in equation (2.44). Take a super vector space V and define the following super-module:

$$M = \Lambda^\bullet(\mathbb{C}^n) \otimes V \tag{2.52}$$

with odd and even parts:

$$M_0 = (\Lambda^\bullet(\mathbb{C}^n)_0 \otimes V_0) \bigoplus (\Lambda^\bullet(\mathbb{C}^n)_1 \otimes V_1) \tag{2.53}$$

$$M_1 = (\Lambda^\bullet(\mathbb{C}^n)_0 \otimes V_1) \bigoplus (\Lambda^\bullet(\mathbb{C}^n)_1 \otimes V_0) \tag{2.54}$$

If V has dimension $p|q$, we call this module as $M^{p|q}$. An element $v \in M^{p|q}$ is given by:

$$v = v^i e_i \tag{2.55}$$

with $e_i \in V$ and $v^i \in \Lambda^\bullet(\mathbb{C}^n)$. Taking a homogeneous basis:

$$e_a \in V_0 \ \text{ for } \ a = 1, \cdots, p \qquad e_\alpha \in V_1 \ \text{ for } \ \alpha = 1, \cdots, q \tag{2.56}$$

and splitting v^i in even an odd part: $v^i = v^i_0 + v^i_1$, one has the odd and even part for v:

$$v = (v^a_0 e_a + v^\alpha_1 e_\alpha) + (v^a_0 e_\alpha + v^\alpha_1 e_a) \tag{2.57}$$

$$= \quad \text{even} \quad + \quad \text{odd} \tag{2.58}$$

that is, odd and even elements of $M^{p|q}$ are of the form:

$$\text{even} = \begin{pmatrix} \text{even} \\ \text{odd} \end{pmatrix} \quad , \qquad \text{odd} = \begin{pmatrix} \text{odd} \\ \text{even} \end{pmatrix} \tag{2.59}$$

where even and odd inside the vector represents the parity of the entry.

Now we can define an associated algebra, composed by the linear maps from $M^{p|q}$ to $M^{r|s}$:

$$\mathbf{Hom}(M^{p|q}, M^{r|s}) = \{L : M^{p|q} \longrightarrow M^{r|s} \,;\, L(a \cdot v + w) = a \cdot L(v) + L(w)\} \tag{2.60}$$

This is an algebra, with composition as the operation, and have a natural grading:

$$\mathbf{Hom}(M^{p|q}, M^{r|s})_0 = \{L \in \mathbf{Hom};\ \ L(\text{even}) = \text{even}\,,\ \ L(\text{odd}) = \text{odd}\} \tag{2.61}$$

$$\mathbf{Hom}(M^{p|q}, M^{r|s})_1 = \{L \in \mathbf{Hom};\ \ L(\text{even}) = \text{odd}\,,\ \ L(\text{even}) = \text{odd}\} \tag{2.62}$$

That is, even elements of **Hom** are those who preserves parity, and odd elements are those who reverse parity.

We notice that **Hom** can be expressed as an algebra of matrices. Let us denote it by $\mathbf{Mat}(p|q, r|s)$. The even and odd matrices are, then:

$$\text{even} = \begin{pmatrix} \text{even} & \text{odd} \\ \text{odd} & \text{even} \end{pmatrix} \quad , \qquad \text{odd} = \begin{pmatrix} \text{odd} & \text{even} \\ \text{even} & \text{odd} \end{pmatrix} \tag{2.63}$$

For square super-matrices, that is, when $r|s = p|q$, we give a special attention, since it will play a rule in constructing the superalgebra of $\mathfrak{psu}(2,2|4)$ used in string theory for curved background. In this case, we denote by $\mathbf{Mat}(p|q)$, a super-module over $\Lambda^{\bullet}(\mathbb{C}^n)$.

Example 2.13. *Let us explicitly analyse the* $\mathbf{Mat}(p|q)$ *super-algebra over* $\mathbb{C}$*. First of all, consider its corresponding, vector space, that is, the set of* $(p+q) \times (p+q)$ *matrices of the form:*

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{2.64}$$

Where $A \in Mat(p \times p)$, $B \in Mat(p \times q)$, $C \in Mat(q \times p)$ *and* $D \in Mat(q \times q)$, *with the gradation given by (2.63), that is:*

$$Mat(p|q) = V_0 \oplus V_1$$

with:

$$V_0 = \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} ;\ A \in Mat_{p\times p},\ D \in Mat_{q\times q} \right\} \tag{2.65}$$

$$V_1 = \left\{ \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} ;\ B \in Mat_{p\times q},\ C \in Mat_{q\times p} \right\} \tag{2.66}$$

Let us check that this is in fact a gradation, i.e., it obeys $V_i \cdot V_j \subset V_{i+j}$*:*

$$V_0 \cdot V_0 \subset V_0 : \qquad \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} A' & 0 \\ 0 & D' \end{pmatrix} = \begin{pmatrix} AA' & 0 \\ 0 & DD' \end{pmatrix} \tag{2.67}$$

$$V_0 \cdot V_1 \subset V_1 : \qquad \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} = \begin{pmatrix} 0 & AB \\ DC & 0 \end{pmatrix} \tag{2.68}$$

$$V_1 \cdot V_1 \subset V_0 : \qquad \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} \begin{pmatrix} 0 & B' \\ C' & 0 \end{pmatrix} = \begin{pmatrix} BC' & 0 \\ 0 & CB' \end{pmatrix} \tag{2.69}$$

We may define now the supertrace and supertranspose:

$$Str(M) = tr(A) - tr(D) \tag{2.70}$$

$$M^{sT} = \begin{pmatrix} A^T & -C^T \\ B^T & D^T \end{pmatrix} \tag{2.71}$$

In order to be "Lie", a superalgebra has to carry the Lie super-bracket, a modified Lie bracket that receive the gradded structure nicely:

Definition 2.7. A **Lie superalgebra** is a super-vector space with a bilinear operation $(\mathfrak{g}, [-,-])$, called supercommutator, satisfying:

$$[a,b] = -(-1)^{\bar{a}\bar{b}}[b,a] \tag{2.72}$$

$$(-1)^{\bar{a}\bar{c}}[a,[b,c]] + (-1)^{\bar{a}\bar{b}}[b,[c,a]] + (-1)^{\bar{b}\bar{c}}[c,[a,b]] = 0 \tag{2.73}$$

with a, b, c homogeneous.

The equation (2.73) is the generalized Jacobi identity, and can be written in a more clearer way as:

$$\mathrm{ad}_a[b,c] = [\mathrm{ad}_a b, c] + (-1)^{\bar{a}\bar{b}}[b, \mathrm{ad}_a c] \tag{2.74}$$

In fact, equation (2.74) gives us the following $\mathrm{ad}_{\mathfrak{g}_0}-$invariant map:

$$\begin{aligned} \mathfrak{g}_1 \otimes \mathfrak{g}_1 &\longrightarrow \mathfrak{g}_0 \\ (a,b) &\longmapsto [a,b] \end{aligned} \tag{2.75}$$

that is, for $a \in \mathfrak{g}_0$:

$$\mathrm{ad}_a[b,c] = [\mathrm{ad}_a b, c] + [b, \mathrm{ad}_a c] \qquad \forall b,c \in \mathfrak{g}_1 \tag{2.76}$$

Also, we have that $\mathfrak{g}_1$ is a representation space of $\mathfrak{g}_0$:

$$\mathrm{ad} : \mathfrak{g}_0 \longrightarrow \mathfrak{gl}(\mathfrak{g}_1) \tag{2.77}$$

$$a \longmapsto \mathrm{ad}_a : \mathfrak{g}_1 \to \mathfrak{g}_1 \tag{2.78}$$

this forms a representation of $\mathfrak{g}_1$ in $\mathfrak{g}_0$ since it preserves the Lie structure of the representation space:

$$\mathrm{ad}_a[b,c] = [\mathrm{ad}_a b, c] + [b, \mathrm{ad}_a c] \tag{2.79}$$

Now we define some important matricial super-Lie algebras:

Example 2.14. *Inspired on the general linear algebra, we have its corresponding Lie superalgebra correspondent:*

$$\mathfrak{gl}(n|m) := Mat(n|m) \tag{2.80}$$

Example 2.15. *As well as the special linear superalgebra:*

$$\mathfrak{sl}(n|m) = \{M \in \mathfrak{gl}(n|m) \mid Str(M) = 0\} \tag{2.81}$$

Both last examples have the Lie algebra bilinear map defined on the homogeneous elements as:

$$[A,B] := AB - (-1)^{\bar{A}\bar{B}} BA \tag{2.82}$$

A fermionic: $\bar{A} = 1$; A bosonic: $\bar{A} = 0$.

2.2.1 Extending the Poincaré algebra

We have a question in hands now: Given the Poincare Lie algebra, can we extend it by adding an odd part?

$$\text{Poinc}(d-1,1) \; \oplus \; S \tag{2.83}$$

It is indeed possible and the simplest form to to it is to add a spinorial representation S to the algebra. Doing so, we obtain the so-called **Poincare superalgebra**. Let us do this first in $d = 4$.

The idea introduced by **Golfand & Likhtman** and **Gervais & Sakita** in 1971 [19] is to add generators of spinorial translation given by the bispinors Q_α,$\bar{Q}_{\dot{\alpha}}$, with $\alpha, \dot{\alpha} = 1,2$:

$$S = \left\{ \; Q_1, \; Q_2, \; \bar{Q}_{\dot{1}}, \; \bar{Q}_{\dot{2}} \; \right\} \tag{2.84}$$

Such that $Q_\alpha \in (\frac{1}{2},0)$ e $\bar{Q}_{\dot{\alpha}} \in (0,\frac{1}{2})$:

$$[Q_\alpha, M_{\mu\nu}] = (\sigma_{\mu\nu})_\alpha^{\;\beta} Q_\beta \tag{2.85}$$

$$[\bar{Q}_{\dot{\alpha}}, M_{\mu\nu}] = (\bar{\sigma}_{\mu\nu})_{\dot{\alpha}}^{\;\dot{\beta}} \bar{Q}_{\dot{\beta}} \tag{2.86}$$

More explicitly, Q e $\bar{Q}$ are spinors, that is, transforms as exponential of $SL(2,\mathbb{C})$:

$$Q_\alpha \mapsto \exp\left(-\frac{i}{2}\sigma_{\mu\nu}\omega^{\mu\nu}\right)_\alpha^{\;\beta} Q_\beta \;\approx\; \left(1 - \frac{i}{2}\sigma_{\mu\nu}\omega^{\mu\nu}\right)_\alpha^{\;\beta} Q_\beta \tag{2.87}$$

But at the same time, Q and $\bar{Q}$ are operators that transforms under Lorentz:

$$\begin{aligned} Q_\alpha &\mapsto \exp\left(\frac{i}{2}M_{\mu\nu}\omega^{\mu\nu}\right)_\gamma^{\;\beta} Q_\beta \exp\left(-\frac{i}{2}M_{\mu\nu}\omega^{\mu\nu}\right)_\alpha^{\;\gamma} \\ &\approx \left(1 + \frac{i}{2}M_{\mu\nu}\omega^{\mu\nu}\right)_\gamma^{\;\beta} Q_\beta \left(1 - \frac{i}{2}M_{\mu\nu}\omega^{\mu\nu}\right)_\alpha^{\;\gamma} \end{aligned} \tag{2.88}$$

Comparing both, follows that $[Q,M] = \sigma \cdot Q$. Let us work out the other commutation relations:

- $[Q_\alpha, P_\mu] = [\bar{Q}_{\dot{\alpha}}, P_\mu] = 0$ – This happens because Q and $\bar{Q}$ generates the spinorial translations.

The unique possibility would be $[Q_\alpha, P^\mu] = c\, \sigma^\mu_{\alpha\dot{\beta}} \bar{Q}^{\dot{\beta}}$. By Jacobi identity:

$$\begin{aligned} 0 &= [P^\mu,[P^\nu,Q_\alpha]] + [Q_\alpha,[P^\mu,P^\nu]] + [P^\nu,[Q_\alpha,P^\mu]] \\ &= -c\sigma^\nu_{\alpha\dot\beta}[P^\mu,\bar{Q}^{\dot\beta}] + c\sigma^\mu_{\alpha\dot\beta}[P^\nu,\bar{Q}^{\dot\beta}] \\ &= c^2\left(\sigma^\nu\cdot\sigma^\mu - \sigma^\mu\cdot\sigma^\nu\right)Q \end{aligned} \tag{2.89}$$

$$.:c = 0 \tag{2.90}$$

- $\{Q_\alpha, Q_\beta\} = 0$

Again, the unique possibility is that $\{Q_\alpha, Q^\beta\} = k(\sigma_{\mu\nu})_\alpha^{\ \beta}M_{\mu\nu}$. But in this case, the left hand side commutes with P_μ, but the right hand side does not, then $k = 0$.

- $\{Q_\alpha, \bar{Q}_{\dot\beta}\} = 2\sigma^\mu_{\alpha\dot\beta}P_\mu$

$Q_\alpha \in (\frac{1}{2},0)$ and $\bar{Q}_{\dot\beta} \in (0,\frac{1}{2})$, then $\{Q_\alpha,\bar{Q}_{\dot\beta}\} \in (\frac{1}{2},\frac{1}{2})$. That is, $\{Q_\alpha,\bar{Q}_{\dot\beta}\}$ transforms as a quadrivector.

Now we can state the Super-Poincaré algebra definition:

Example 2.16. *The* ***Super-Poincaré algebra*** *in* $\mathbb{R}^{3,1}$ *is defined as:*

$$V = \mathbb{R}^{\frac{4(4+1)}{2}} \oplus \mathbb{R}^4 \tag{2.91}$$

$$[M^{ab}, M^{cd}] = \eta^{ac}M^{bd} - \eta^{bc}M^{ad} + \eta^{bd}M^{ac} - \eta^{ad}M^{bc} \tag{2.92}$$

$$[P_a, P_b] = 0 \qquad [P_a, M^{bc}] = \delta_a^b P^c - \delta_a^c P^b \tag{2.93}$$

$$\{Q_\alpha, \bar{Q}_{\dot\beta}\} = 2\sigma^\mu_{\alpha\dot\beta}P_\mu \tag{2.94}$$

$$[Q_\alpha, M_{\mu\nu}] = (\sigma_{\mu\nu})_\alpha^{\ \beta}Q_\beta \qquad [\bar{Q}_{\dot\alpha}, M_{\mu\nu}] = (\bar\sigma_{\mu\nu})_{\dot\alpha}^{\ \dot\beta}\bar{Q}_{\dot\beta} \tag{2.95}$$

$$[Q_\alpha, P_\mu] = [\bar{Q}_{\dot\alpha}, P_\mu] = \{Q_\alpha, Q_\beta\} = 0 \tag{2.96}$$

With $\{Q_\alpha,\ \bar{Q}_{\dot\beta},\ P_c,\ M^{ab}\}$ *the basis for* V.

Properties

Consider a representation of the Super-Poincaré algebra in a given super-vector space $\mathcal{H}$. This will be a Hilbert space with vectors $|a\rangle \in \mathcal{H}$. Consider an operator

$(-)^{N_F}$, acting on homogeneous elements of $\mathcal{H}$, denoted $|b\rangle \in \mathcal{H}^0$ and $|f\rangle \in \mathcal{H}^1$, as:

$$(-)^{N_F}|b\rangle = +|b\rangle \tag{2.97}$$

$$(-)^{N_F}|f\rangle = -|f\rangle \tag{2.98}$$

We now define a trace operator by the following. Let $\{|a_i\rangle\}$ be a basis for $\mathcal{H}$, the trace is defined as an operator:

$$\mathrm{tr} : \mathrm{End}_k(V) \xrightarrow{\sim} V^* \otimes V \longrightarrow k \tag{2.99}$$

The first arrow is just the natural identification, the second one is also very natural: $(\xi \otimes v) \mapsto \xi(v)$. In components, it reads:

$$\mathrm{tr}(\mathcal{O}) = \sum_j \langle a_j|\mathcal{O}|a_j\rangle \tag{2.100}$$

Since Q and $\bar{Q}$ change the statistics of the states (they are fermionic operators), they anti-commute with $(-)^{N_F}$. By the cyclicity of the trace:

$$0 = \mathrm{tr}\left[-\bar{Q}_{\dot{\beta}}(-)^{N_F}Q_\alpha + (-)^{N_F}Q_\alpha\bar{Q}_{\dot{\beta}}\right] = \mathrm{tr}\left[(-)^{N_F}\{Q_\alpha, \bar{Q}_{\dot{\beta}}\}\right] = 2\sigma^\mu_{\alpha\dot{\beta}}\,\mathrm{tr}\left[(-)^{N_F}P_\mu\right] \tag{2.101}$$

Follows that:

$$\mathrm{tr}(-)^{N_F} = 0 \tag{2.102}$$

That is, for a theory that possesses supersymmetry, we have # fermion = # bosons.

Another interesting fact is that in a given supersymmetric theory, the operator P_0, which has an interpretation of Energy, is always positive:

$$\begin{aligned} 0 &< ||Q_\alpha|\phi\rangle||^2 + ||Q^\dagger_\alpha|\phi\rangle||^2 \qquad (2.103)\\ &= \langle\phi|Q^\dagger_\alpha Q_\alpha + Q_\alpha Q^\dagger_\alpha|\phi\rangle = 2\sigma^\mu_{\alpha\dot{\alpha}}\langle\phi|P_\mu|\phi\rangle \end{aligned}$$

with $|\phi\rangle$ an arbitrary state and $\bar{Q}_{\dot{\alpha}} = Q^\dagger_\alpha$. Taking the trace:

$$4\langle P_0\rangle \geq 0 \tag{2.104}$$

One can do a further generalization of the Super-Poincaré algebra to $D-$dimensional space-time and N supersymmetries [14]. We do it by finding a $D-$dimensional representation of Clifford algebra (see $(A.1)$), and introducing $\mathcal{N}$ fermionic operator Q^I_α with, $I = 1, \cdots \mathcal{N}$. The Lie-algebra relations for the new $\mathcal{N}$ generators will be:

$$\{Q^A_\alpha, Q^B_\beta\} = -2i\delta^{AB}\gamma^m_{\alpha\beta}P_m \tag{2.105}$$

$$[P_m, P_n] = [Q^A_\alpha, P_n] = 0 \tag{2.106}$$

$$[Q^A_\alpha, M_{\mu\nu}] = (\gamma_{\mu\nu})_\alpha{}^\beta Q^A_\beta \tag{2.107}$$

2.2.2 $\mathfrak{psu}(2,2|4)$ Lie superalgebra

Another important example of Lie algebra for superstring theory is the $\mathfrak{psu}(2,2|4)$. It can be defined as the so-called *Cartan involution* of $\mathfrak{sl}(2,2|4)$, defined for generic $\mathfrak{sl}$ superalgebras by:

$$\begin{aligned} \phi :\ \mathfrak{sl}(p,q|r,s) &\longrightarrow \mathfrak{sl}(p,q|r,s) \\ M &\longmapsto \phi(M) = -HM^{\dagger}H^{-1} \end{aligned} \tag{2.108}$$

with:

$$H = \begin{pmatrix} \Sigma_{p,q} & 0 \\ 0 & \Sigma_{r,s} \end{pmatrix} \tag{2.109}$$

with $\Sigma_{m,n}$ the corresponding $(m,n)-$metric:

$$\Sigma_{m,n} = \begin{pmatrix} 1_{m\times m} & 0 \\ 0 & -1_{n\times n} \end{pmatrix} \tag{2.110}$$

One can define $\mathfrak{su}(p,q|r,s)$ as elements of $\mathfrak{sl}(p,q|r,s)$ that satisfies the reality condition $\phi(M) = M$:

$$\mathfrak{su}(p,q|r,s) = \{M \in \mathfrak{sl}(p,q|r,s);\ \phi(M) = M\} \tag{2.111}$$

explicitly, for M:

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{2.112}$$

such that $Tr(A) = Tr(D)$, the reality condition reads:

$$\begin{pmatrix} -\Sigma_{p,q}A^{\dagger}\Sigma_{p,q} & \Sigma_{p,q}C^{\dagger}\Sigma_{r,s} \\ -\Sigma_{r,s}B^{\dagger}\Sigma_{p,q} & -\Sigma_{r,s}D^{\dagger}\Sigma_{r,s} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{2.113}$$

For the case of $\mathfrak{su}(2,2|4)$:

$$\begin{pmatrix} -\Sigma_{2,2}A^{\dagger}\Sigma_{2,2} & \Sigma_{2,2}C^{\dagger} \\ -B^{\dagger}\Sigma_{2,2} & -D^{\dagger} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{2.114}$$

Following the conditions:

$$A^\dagger = -\Sigma_{2,2} A \Sigma_{2,2} \tag{2.115}$$
$$D^\dagger = -D \tag{2.116}$$
$$C = -B^\dagger \Sigma_{2,2} \tag{2.117}$$

Follows that $A \in \mathfrak{u}(2,2)$ and $D \in \mathfrak{u}(4)$ and the grading reads:

$$\mathfrak{su}(2,2|4)_0 = \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} ;\ A \in \mathfrak{u}(2,2),\ \ D \in \mathfrak{u}(4),\ \ tr(A) = tr(D) \right\} \tag{2.118}$$

since one can always decompose $\mathfrak{sl}(n|m) = \mathfrak{sl}(n) \oplus \mathfrak{sl}(m) \oplus \mathfrak{u}(1)$, one has:

$$\mathfrak{su}(2,2|4)_0 = \mathfrak{su}(2,2) \oplus \mathfrak{su}(4) \oplus \mathfrak{u}(1) \tag{2.119}$$

$$\mathfrak{su}(2,2|4)_1 = \left\{ \begin{pmatrix} 0 & B \\ -B^\dagger \Sigma_{2,2} & 0 \end{pmatrix} ;\ B \in Mat_{4\times 4}(\mathbb{C}) \right\} \tag{2.120}$$

$\mathbb{Z}_4-$**grading for** $\mathfrak{su}(2,2|4)$

A Lie algebra **automorphism** is a linear map:

$$\phi : \mathfrak{g} \longrightarrow \mathfrak{g} \tag{2.121}$$

such that it preserves the gradding and the Lie structure:

$$\phi([a,b]) = [\phi(a), \phi(b)] \tag{2.122}$$

$$\phi(\mathfrak{g}_i) \subset \mathfrak{g}_i \tag{2.123}$$

We will see that a Lie algebra $\mathfrak{su}(2,2|4)$ can be decomposed in 4 parts. This is performed by applying an automorphism ϕ with 4 eigenspaces:

$$\phi(M) = i^k M\,, \qquad k = 0,1,2,3 \tag{2.124}$$

and afterwards noticing that it defines a gradding $([\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j})$.

We will define the following automorphism:

$$\phi : \mathfrak{su}(n,n|2n) \longrightarrow \mathfrak{su}(n,n|2n) \tag{2.125}$$

$$\begin{pmatrix} A & X \\ Y & B \end{pmatrix} \longmapsto \begin{pmatrix} JA^TJ & -JY^TJ \\ JX^TJ & JB^TJ \end{pmatrix} \tag{2.126}$$

where:

$$J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \tag{2.127}$$

ϕ can also be written as follows:

$$\phi(M) = \begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix} M^{sT} \begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix} \tag{2.128}$$

It is simple to show that this is an endomorphism in $\mathfrak{su}(n,n|2n)$. We use $J\Sigma = -\Sigma J$ and the definition $A^\dagger = (A^*)^T$ to show that reality condition (2.114) follows for $\phi(M)$ if $M \in \mathfrak{su}(n,n|2n)$:

$$\begin{gathered} -\Sigma(JA^TJ)^\dagger\Sigma = -\Sigma J(A^\dagger)^TJ\Sigma = -J\Sigma(A^\dagger)^T\Sigma J = -J(\Sigma A^\dagger\Sigma)^TJ = JA^TJ \\ -(JB^TJ)^\dagger = -J(B^\dagger)^TJ = JB^TJ \\ \Sigma(JX^TJ)^\dagger = \Sigma J(X^\dagger)^TJ = -J\Sigma(X^\dagger)^TJ = -J(X^\dagger\Sigma)^TJ = -JY^TJ \\ (-JY^TJ)^\dagger\Sigma = -J(Y^\dagger)^TJ\Sigma = J(Y^\dagger)^T\Sigma J = J(\Sigma Y^\dagger)^TJ = JX^TJ \end{gathered} \tag{2.129}$$

The last property to show that it is an automorphism is to show that it preserves the Lie structure:

$$\begin{aligned} \phi([M,N]) = \begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix} [M,N]^{sT} \begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix} &= \begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix} (MN-NM)^{sT} \begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix} \\ &= \begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix} (N^{sT}M^{sT} - M^{sT}N^{sT}) \begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix} \end{aligned} \tag{2.130}$$

$$= -\begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix} N^{sT} \begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix} \begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix} M^{sT} \begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix} + \begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix} M^{sT} \begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix} \begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix} N^{sT} \begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix}$$

$$= -\phi(N)\phi(M) + \phi(M)\phi(N) = [\phi(M), \phi(N)]$$

Where we have used $J^2 = -1$.

We see that this map is such that $\phi^4 = 1$, and that's why its eigenspaces are of the form (2.124). We have then:

$$\mathfrak{su}(n,n|2n) = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3 \tag{2.131}$$

with:

$$\mathfrak{g}_k = \{M \in \mathfrak{su}(n,n|2n) \mid \phi(M) = i^k M\} \tag{2.132}$$

We see that this is a grading because:

$$\phi[\mathfrak{g}_k, \mathfrak{g}_j] = [\phi(\mathfrak{g}_k), \phi(\mathfrak{g}_j)] = i^{k+j}[\mathfrak{g}_k, \mathfrak{g}_j] \tag{2.133}$$

Following:

$$[\mathfrak{g}_k, \mathfrak{g}_j] \subset \mathfrak{g}_{k+j} \tag{2.134}$$

Now, using spectral theorem, we can decompose the hole space $\mathfrak{su}(n,n|2n)$ into four parts, each one projected in corresponding eigenspace. Such projection can be performed by:

$$P_j = \prod_{i \neq j} \frac{\phi - \lambda_i \mathbb{1}}{\lambda_j - \lambda_i}, \qquad j = 0,1,2,3 \tag{2.135}$$

with λ_i the eigenvalues $-(1, i, -1, -i)$ in our case. We have then:

$$P_0 = \frac{1}{4}\left(\phi^3 + \phi^2 + \phi + 1\right) \tag{2.136}$$

$$P_1 = \frac{1}{4}\left(i\phi^3 - \phi^2 - i\phi + 1\right) \tag{2.137}$$

$$P_2 = \frac{1}{4}\left(-\phi^3 + \phi^2 - \phi + 1\right) \tag{2.138}$$

$$P_3 = \frac{1}{4}\left(-i\phi^3 - \phi^2 + i\phi + 1\right) \tag{2.139}$$

We can now decompose:

$$M = \begin{pmatrix} A & X \\ Y & B \end{pmatrix} = M^{(0)} + M^{(1)} + M^{(2)} + M^{(3)} \tag{2.140}$$

with $M^{(j)} = P_j(M)$:

$$M^{(0)} = \frac{1}{2}\begin{pmatrix} JA^TJ + A & 0 \\ 0 & JB^TJ + B \end{pmatrix} \tag{2.141}$$

$$M^{(1)} = \frac{1}{2}\begin{pmatrix} 0 & X + iJY^TJ \\ Y - iJX^TJ & 0 \end{pmatrix} \tag{2.142}$$

$$M^{(2)} = \frac{1}{2}\begin{pmatrix} A - JA^TJ & 0 \\ 0 & B - JB^TJ \end{pmatrix} \tag{2.143}$$

$$M^{(3)} = \frac{1}{2}\begin{pmatrix} 0 & X - iJY^TJ \\ Y + iJX^TJ & 0 \end{pmatrix} \tag{2.144}$$

We then see an important observation:

$$\mathfrak{g}_0 = \mathfrak{usp}(n,n) \oplus \mathfrak{usp}(2n) \tag{2.145}$$

Remember that $M^{(0)} \in \mathfrak{u}(n,n) \oplus \mathfrak{u}(2n)$. It is also in $\mathfrak{sp}(n)$ because it is of the form (2.35). To see this consider:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{2.146}$$

with $a, b, c, d \in Mat_{n\times n}$. Follows:

$$JA^TJ + A = \begin{pmatrix} a - d^T & b + b^T \\ c + c^T & -(a - d^T)^T \end{pmatrix} \tag{2.147}$$

In the case of our interest:

$$\mathfrak{g}_0 = \mathfrak{usp}(2,2) \oplus \mathfrak{usp}(4) \cong \mathfrak{so}(4,1) \oplus \mathfrak{so}(5) \tag{2.148}$$

Therefore:

$$T\left(\frac{PSU(2,2|4)}{SO(4,1) \times SO(5)}\right) = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3 \tag{2.149}$$

2.3 Homogeneous Spaces

This section aims to clarify relations of the kind:

$$\mathbb{S}^n = \frac{SO(n+1)}{SO(n)} , \qquad AdS_{n+1} = \frac{SO(n,2)}{SO(n,1)} \tag{2.150}$$

which enable us to understand better the relations (2.1) and (2.2), as will be explained in section (4.3).

Let us then introduce the notion of homogeneous space. First of all, consider a manifold M with a left group action, i.e., a map:

$$\begin{aligned} \triangleright : G \times M &\longrightarrow M \\ (g,m) &\longmapsto g \triangleright m \end{aligned} \tag{2.151}$$

such that:

$$g \triangleright (h \triangleright m) = (g \cdot h) \triangleright m \tag{2.152}$$

$$e \triangleright m = m \tag{2.153}$$

$\forall m \in M, \forall g,h \in G$ and $e \in G$ the neutral element.

We can now define what a homogeneous space is:

Definition 2.8 (Homogeneous space)**.** A given manifold M with a left group action by G is called **homogeneous** if the action is transitive:

$$\forall x,y \in M , \quad \exists g \in G : x = g \triangleright y \tag{2.154}$$

From the definition of homogeneous spaces, it's intuitive that we can describe them in terms of the group G acting on it. In fact, from a given point $x \in M$, all the other elements can be described as $g \triangleright x$, just varying $g \in G$. However, it may have an ambiguity in this description, that is, we may have two different group elements g and h which describes the same element $g \triangleright x = h \triangleright x$. Therefore, in order to precisely describe M in terms of G, we have to eliminate this ambiguity. This motivates the next definition:

Definition 2.9 (Stabilizer)**.** Let M a manifold with a G-group action and a given point $x \in M$. Its **Stabilizer** subgroup is given by:

$$H_x := \{h \in G \mid h \triangleright x = x\} \tag{2.155}$$

This is indeed a subgroup since for $g,h \in H_x$, by group action definition $gh \in H_x$. Also follows from the same definition that $e \in H_x$, which implies that

$\forall h \in H_x$, $x = e \triangleright x = h^{-1} \triangleright (h \triangleright x) = h^{-1} \triangleright x$, then $h^{-1} \in H_x$.

When the space is homogeneous, we see that all stabilizers are isomorphic, that is $H_x \cong H_y$ for any two points $x, y \in M$. To see this, take $g \in G$ such that $y = g \triangleright x$ (or $x = g^{-1} \triangleright y$), then, for every point $h \in H_x$, we have that $ghg^{-1} \in H_y$. In fact, for $h \in H_x$:

$$(ghg^{-1}) \triangleright y = (gh) \triangleright x = g \triangleright x = y \tag{2.156}$$

On the other hand, if $h \in H_y$, follows that $g^{-1}hg \in H_x$. Therefore, the map:

$$H_x \longrightarrow H_y \tag{2.157}$$

$$h \longmapsto ghg^{-1} \tag{2.158}$$

is an isomorphism: $H_y = gH_xg^{-1}$ and $H_x = g^{-1}H_yg$.

Now, we can announce an important statement:

Theorem 2.2. *Let M a homogeneous space with a G-group action. For a given point $x \in M$ and its corresponding stabilizer $H \subset G$, we have:*

$$M = \frac{G}{H} \tag{2.159}$$

where $G/H := \{[g] \mid m \in [g] \Leftrightarrow m = hg,\ h \in H\}$.

Proof. Let $f : G/H \longrightarrow M$ defined by

$$f([g]) := g \triangleright x_0 \tag{2.160}$$

this map is well defined since for any $m \in [g]$, $m = hg$ and therefore $(gh) \triangleright x_0 = g \triangleright (h \triangleright x_0) = g \triangleright x_0$. We see that $f(G/H) = M$ because the group action is transitive. Finally, it is a 1-1 map:

$$f([g_1]) = f([g_2]) \Longrightarrow g_1 \triangleright x_0 = g_2 \triangleright x_0 \Longrightarrow g_1^{-1}g_2 \in H \tag{2.161}$$

therefore we conclude that $[g_1] = [g_2]$. □

Example 2.17. $\mathbb{S}^n = SO(n+1)/SO(n)$.

Let us first define properly the n-sphere:

$$\mathbb{S}^n := \left\{ \vec{x} \in \mathbb{R}^{n+1} \mid (\vec{x}, \vec{x}) = x_0^2 + \cdots + x_n^2 = 1 \right\} \tag{2.162}$$

and the Lorentz group:

$$SO(n) = \{R \in Mat_{n\times n} \mid RR^T = \mathbb{1}\} \tag{2.163}$$

We see that $SO(n+1)$ *acts on* $\mathbb{S}^n$*:*

$$\begin{aligned} SO(n+1)\times\mathbb{S}^n &\longrightarrow \mathbb{S}^n \\ (R, x) &\longmapsto Rx \end{aligned} \tag{2.164}$$

and then for $x \in \mathbb{S}^n$*:*

$$(Rx, Rx) = (x, R^T Rx) = (x, x) = 1 \tag{2.165}$$

We see that this action is transitive by showing that for every $x \in \mathbb{S}^n$, $\exists R_x \in SO(n+1)$ *such that* $R_x e_1 = x$*, for* $e_1 = (1, 0, \cdots, 0)$*. To see this, observe that for* $R \in SO(n+1)$*, it can be written in its matrix form as:*

$$R = \begin{bmatrix} \\ u_0 & \cdots & u_n \\ \\ \end{bmatrix} \tag{2.166}$$

with u_j *orthonormal column vectors:*

$$(u_i, u_j) = \delta_{ij} \tag{2.167}$$

To obtain R_x*, take* x *and perform the Gram-Schmidt process to obtain an orthonormal basis with* x *being the first element:* $\{x, x_1, \cdots, x_n\}$*:*

$$(x, x) = 1, \qquad (x, x_j) = 0, \qquad (x_i, x_j) = \delta_{ij} \tag{2.168}$$

for $i, j = 1, \cdots, n$*. Then:*

$$R_x = \begin{bmatrix} \\ x & x_1 & \cdots & x_n \\ \\ \end{bmatrix} \in SO(n+1) \tag{2.169}$$

and $R_x e_1 = x$. *To see that* $SO(n)$ *is the stabilizer group, take* e_1 *and:*

$$r = \begin{bmatrix} 1 & 0 \\ 0 & r_n \end{bmatrix} \tag{2.170}$$

are all the kind of matrices that let e_1 *invariant:* $re_1 = e_1$. *For* r *to be in* $SO(n+1)$, r_n *needs to be in* $SO(n)$. *Therefore, follows that* $H_{e_1} = SO(n)$. *By theorem* (2.2)

$$\mathbb{S}^n = \frac{SO(n+1)}{SO(n)} \tag{2.171}$$

Example 2.18. $AdS_{n+1} = SO(n,2)/SO(n,1)$

Let us define the Anti De Sitter space:

$$AdS_{n+1} = \left\{ \vec{x} \in \mathbb{R}^{n+2} \mid (x,x) = -x_{-1}^2 - x_0^2 + x_1^2 + \cdots + x_n^2 = -1 \right\} \tag{2.172}$$

And the Lorentz group:

$$SO(n,p) = \left\{ R \in Mat_{n+p \times n+p} \mid (Rx, Rx) = (x,x) \; \forall x \in \mathbb{R}^{n+p} \right\} \tag{2.173}$$

where $(-,-)$ *is defined using* (n,p) *signature.*

Follows then by the definitions that $SO(m,2)$ *acts on* AdS_{n+1} *and in order to prove that this is a transitive action, we use a generalized version of Gram-Schmidt process. And for a given* $x \in AdS_{n+1}$, *obtain a basis* $\{x, x_0, \cdots, x_n\}$ *such that* $(x_i, x_j) = \delta_{ij}$ *and* $(x, x_i) = 0$, *for* $i, j = 0, \cdots, n$, *with* $(-,-)$ *the metric in the* $n+2$ *signature. Defining:*

$$R_x = \begin{bmatrix} \\ x & x_0 & \cdots & x_n \\ \\ \end{bmatrix} \in SO(2,n) \tag{2.174}$$

will follows $R_x e_1 = x$ *and*

$$r = \begin{bmatrix} 1 & 0 \\ 0 & r_{n+1} \end{bmatrix} \tag{2.175}$$

will be the kind of matrices which preserves e_1 $(re_1 = e_1)$. *In order to* r *be in* $SO(n,2)$, r_{n+1} *needs to be in* $SO(n,1)$, *following:*

$$AdS_{n+1} = \frac{SO(n,2)}{SO(n,1)} \tag{2.176}$$

Chapter 3

Fields and Strings

In this chapter, we give some definitions of concepts present in field theory. we emphasise the study of the space of fields, introducing a bicomplex (d, δ) which will be useful in understanding symmetries.

3.1 Classical Field Theory

It is very straight to define what a field theory is. We can define a generic field theory as a set of data. First of all, we need a space-time M which carries our theory; a given space of fields $\mathcal{F}$, which defines our field content, and finally a function $L(\phi, \partial_I \phi)$, with $\phi \in \mathcal{F}$, called *Lagrangian*, which determines the dynamics of the theory.

Currently, we know a bit more about the space of fields, for instance, it could be $\mathcal{F} = Map(M, V)$ for trivial topologies of the bundle, where V is called *target space* and is usually $\cong \mathbb{R}^n$. This is for example the case of the sigma-models, however gauge theories is an example which does not have such structure.

The space-time M is a finite dimensional pseudo-Riemannian manifold, and therefore it possess a natural measure $d\mu$ (Riemannian measure [18]). The action S is defined as a functional:

$$S[\phi] = \int_M L(\phi, \partial_I \phi) d^n x : \mathcal{F} \longrightarrow \mathbb{R} \tag{3.1}$$

where I have written the measure in coordinates: $d\mu = d^n x$. Me may then define the n-form:

$$\mathcal{L} = L(\phi, \partial_I \phi) d^n x \in \Omega^{(n,0)}(M \times \mathcal{F}) \tag{3.2}$$

that is, an n-form on M and a zero form on $\mathcal{F}$.

Let us study a pedagogical important example:

Example 3.1 (Klein-Gordon)**.** *We can describe a free scalar boson by defining the* *following field theory:*

- *Space time* $M = \mathbb{R}^{3,1}$*;*
- *Target Space* $\mathbb{R}$*;*
- *Space of fields* $\mathcal{F} = C^{\infty}(M)$*, with fields denoted as* ϕ*;*
- *The following Lagrangian, for* $m \geq 0$*:*

$$L_{KG} = \frac{1}{2}\left(g^{ij}\partial_i\phi\partial_j\phi - m^2\phi^2\right) \tag{3.3}$$

Now we will study how to extract information from the Lagrangian, that is, we will state the equation of motions:

Theorem 3.1. *Suppose* $\phi \in \mathcal{F}$ *with compact support, in particular* $\psi\Big|_{\partial M} = 0$*, then:*

$$\delta_\psi S[\phi] := \frac{d}{dt}S[\phi + t\psi]\Big|_{t=0} = \int_M \left\langle q(\phi,\partial\phi,\partial^2\phi),\psi\right\rangle d^n x \tag{3.4}$$

where q *is a unique polynomial in* $\phi, \partial_I\phi$ *with* $|I| \leq 2$ *with values in* V^**.*

Proof. The proof is just a computation:

$$\delta_\psi S[\phi] = \frac{d}{dt}S[\phi + t\psi]\Big|_{t=0} = \frac{d}{dt}\Big|_{t=0}\int_M L\left(\phi + t\psi, \partial_I(\phi + t\psi)\right) d^n x$$

$$= \int_M \left[\left\langle \frac{\partial L}{\partial \phi},\psi\right\rangle + \sum_{i=1}^{n}\left\langle \frac{\partial L}{\partial(\partial_i\phi)},\partial_i\psi\right\rangle\right] d^n x$$

$$= \int_M \left[\left\langle \frac{\partial L}{\partial \phi} - \sum_{i=1}^{n}\partial_i\left(\frac{\partial L}{\partial(\partial_i\phi)}\right),\psi\right\rangle\right] d^n x + \sum_{i=1}^{n}\int_M \partial_i\left\langle \frac{\partial L}{\partial(\partial_i\phi)},\psi\right\rangle d^n x \tag{3.5}$$

With the condition imposed to ψ we see that the last term above vanishes by stokes theorem. We found the so-called Euler-Lagrange polynomial:

$$q(\phi,\partial_I\phi,\phi^2_I\phi) = \frac{\partial L}{\partial \phi} - \sum_{i=1}^{n}\partial_i\left(\frac{\partial L}{\partial(\partial_i\phi)}\right) \tag{3.6}$$

□

With this statement, we can impose that classical physical quantities obey the principle of minimal action to state what the Equation of Motions are:

Definition 3.1. The system of PDE's

$$q(\phi, \partial_I \phi, \phi^2_I \phi) = 0 \tag{3.7}$$

is called the Euler-Lagrange equations, or **Equation of Motions**, for the Classical Field Theory defined by L. As notation, we set $SolEL = \{\phi \in \mathcal{F} \mid q(\phi) = 0\}$.

We now introduce a bicomplex, which will be very useful in our study throughout this text:

$$d : \text{de Rham on } M \tag{3.8}$$

$$\delta : \text{de Rham on } \mathcal{F} \tag{3.9}$$

$$d^{tot} = d + \delta : \text{de Rham on } M \times \mathcal{F} \tag{3.10}$$

The δ operator was formally defined when we made the variation of the action. We may also define the maps ev, π, p:

$$\begin{tikzcd} M \arrow[d, "p"] & \times & \mathcal{F} \arrow[r, "ev"] \arrow[d, "\pi"] & V \\ M & \times & \mathcal{F} \end{tikzcd} \tag{3.11}$$

Where p and π are just projections of $M \times \mathcal{F}$ on M and $\mathcal{F}$ respectively, and ev is defined as $ev(p, \phi) := \phi(p)$, for $p \in M$ and $\phi \in \mathcal{F}$.

The next proposition is very well know in physics but it somehow relates the two complex:

Proposition 3.1. *Consider a Field theory with Lagrangian $\mathcal{L}$. Restricted to the space of fields,*

$$\delta\mathcal{L}\Big|_{SolEL} = d\gamma \tag{3.12}$$

with:

$$\gamma = \frac{1}{n!}\sum_{i=1}^{n}(-1)^i \left\langle \frac{\partial L}{\partial(\partial_i \phi)}, \delta\phi \right\rangle dx_1 \wedge \cdots \wedge \hat{dx_i} \wedge \cdots \wedge dx_n \tag{3.13}$$

where $\hat{dx_i}$ means the absence of the dx_i.

Proof. It is a simple statement to prove. Take the variation for $\mathcal{L}$, which basically we already computed in (3.5):

$$\delta\mathcal{L} = \langle q, \delta\phi\rangle d^n x + \sum_{i=1}^{n} \partial_i \left\langle \frac{\partial L}{\partial(\partial_i \phi)}, \psi \right\rangle d^n x \tag{3.14}$$

we observe, using definition for d in coordinates $d(\omega_{\mu\dots} dx^\mu \cdots) = \partial_\nu \omega_{\mu\dots} dx^\nu \wedge dx^\mu \cdots$, we see that:

$$d\gamma = \sum_{i=1}^{n} \partial_i \left\langle \frac{\partial L}{\partial(\partial_i \phi)}, \psi \right\rangle d^n x \tag{3.15}$$

Following that $\delta\mathcal{L} - d\gamma = \langle q, \delta\phi\rangle d^n x$, and finally:

$$\delta\mathcal{L} = (\delta\mathcal{L} - d\gamma) + d\gamma = \langle q, \delta\phi\rangle d^n x + d\gamma \tag{3.16}$$

Restricted to the equation of motions, we have that $\delta\mathcal{L} = d\gamma$. □

We see that $\gamma \in \Omega^{(n-1,1)}(M \times \mathcal{F})$. This object has some interesting properties. For instance, it is the pre-symplectic form, that is, we take Σ a spacial slice on M and:

$$\omega := -\int_\Sigma \delta\gamma \tag{3.17}$$

is a symplectic form: closed and non-degenerate [17]. If we define the canonical momentum for the field variables ϕ as:

$$\pi := \sum_{i=1}^{n} \frac{1}{n!}(-1)^i \frac{\partial L}{\partial(\partial_i \phi)} dx_1 \wedge \cdots \wedge \hat{dx_i} \wedge \cdots \wedge dx_n \tag{3.18}$$

and the symplectic form can be writen as:

$$\omega = \int_\Sigma \delta\phi \wedge \delta\pi \tag{3.19}$$

Example 3.2 (Free Scalar Field). *Consider $M = \mathbb{R}^n$ as the space-time and the target space $V = \mathbb{R}$.*

The lagrangean is:

$$L(\phi, \partial\phi) = \frac{1}{2} g^{ij} \partial_i \phi \partial_j \phi - \frac{1}{2} m^2 \phi^2 \tag{3.20}$$

Following the Equation of Motion:

$$m^2 \phi + g^{ij} \partial_i \partial_j \phi = 0 \tag{3.21}$$

3.2 Symmetry in Classical Field Theory

Given a Classical Field theory and a given group G, acting on the space of fields $\mathcal{F}$, we say that G is a symmetry if:

$$S[\phi^g] = S[\phi] \ \ \forall \phi \in \mathcal{F},\ \forall g \in G \tag{3.22}$$

The infinitesimal version of this concept is done considering the action of the the Lie algebra $Lie(G) = \mathfrak{g}$. First consider the Lie homomorphism from $\mathfrak{g}$ to the space of vector fields on $\mathcal{F}$:

$$\begin{aligned} \mathfrak{g} &\longrightarrow Vect(\mathcal{F}) \\ \xi &\longmapsto \xi_{\mathcal{F}} \end{aligned} \tag{3.23}$$

and the symmetry condition will read as:

$$L_{\xi_{\mathcal{F}}}\mathcal{L} = d\alpha(\xi) \tag{3.24}$$

For some $\alpha(\xi) \in \Omega^{(n-1,0)}(M \times \mathcal{F})$. This condition makes clearer if we integrate and consider a manifold without boundary ($\partial M = \varnothing$):

$$L_{\xi_{\mathcal{F}}}S = \int_M d\alpha = \int_{\partial M} \alpha = 0 \tag{3.25}$$

The symmetry is local if $supp(L_{\xi_{\mathcal{F}}}\phi) \subset supp(\phi)$.

If we have an explicit variation for the fields $\Delta\phi$, the corresponding vector field will be:

$$\xi = \Delta\phi \frac{\delta}{\delta\phi} \tag{3.26}$$

Since the Lagrangian $\mathcal{L}$ is a zero form on the space of fields $\mathcal{F}$, the Lie derivative in (3.24) will read as:

$$L_{\xi}\mathcal{L}[\phi] = \iota_{\xi}\delta\mathcal{L}[\phi] = \Delta\phi\frac{\delta}{\delta\phi}\mathcal{L}[\phi] \tag{3.27}$$

Example 3.3 (Free Scalar Field)**.** *In the case of Free Scalar Field with $m = 0$, already defined, we have the following symmetry:*

$$\Delta\phi = \epsilon \tag{3.28}$$

with constant ϵ. The corresponding vector field is:

$$\xi = \epsilon \frac{\delta}{\delta \phi} \tag{3.29}$$

and therefore:

$$L_\xi \mathcal{L} = \epsilon \frac{\delta}{\delta \phi} \left(\frac{1}{2} \partial_i \phi \partial^i \phi \right) = \partial_i \epsilon \partial^i \phi = 0 \tag{3.30}$$

Let us now study a bit more about the properties of symmetries:

Theorem 3.2. *Let $\mathfrak{g}$ be a symmetry of a Classical Field Theory with Lagrangian $\mathcal{L}$. Then, the fundamental vector fields $\xi_{\mathcal{F}}$ are tangent to $SolEL$, that is:*

$$L_{\xi_{\mathcal{F}}} q \Big|_{SolEL} = 0 \tag{3.31}$$

Proof. First, remember the relation:

$$\delta \mathcal{L} = d\gamma + \langle q, \delta\phi \rangle d^n x \tag{3.32}$$

and apply it to the following:

$$\begin{aligned} L_{\xi_{\mathcal{F}}} \delta \mathcal{L} &= \delta L_{\xi_{\mathcal{F}}} \mathcal{L} = \delta d\alpha(\xi) = -d\delta\alpha(\xi) & (3.33)\\ &= d(L_{\xi_{\mathcal{F}}}\gamma) + \langle L_{\xi_{\mathcal{F}}} q, \delta\phi \rangle d^n x + \langle q, L_{\xi_{\mathcal{F}}} \delta\phi \rangle d^n x & (3.34) \end{aligned}$$

$$\implies \qquad \sigma := d(L_{\xi_{\mathcal{F}}}\gamma + \delta\alpha(\xi)) + \langle L_{\xi_{\mathcal{F}}} q, \delta\phi \rangle d^n x + \langle q, L_{\xi_{\mathcal{F}}} \delta\phi \rangle d^n x = 0 \tag{3.35}$$

Now consider a vector field v on $\mathcal{F}$ such that $v(\phi) = \psi \in Map(M, V)$ has a compact support, away from ∂M. Therefore, when we integrate $\iota_v \sigma$ over M, the first term vanishes:

$$\iota_v \int_M d(L_{\xi_{\mathcal{F}}}\gamma + \delta\alpha(\xi)) = \int_{\partial M} \iota_v (L_{\xi_{\mathcal{F}}}\gamma + \delta\alpha(\xi)) = 0 \tag{3.36}$$

Restricting the other terms to $SolEL$, we get:

$$\int_M \langle L_{\xi_{\mathcal{F}}} q, \psi \rangle d^n x \Big|_{SolEL} = 0 \tag{3.37}$$

varying v, with the condition of compact support for ψ, we have the desired result: $L_{\xi_{\mathcal{F}}} q \Big|_{SolEL} = 0$. □

This observation is remarkable because it establish that the symmetry of the action is also a symmetry of the Eq. of Motions. Indeed, if $\mathfrak{g}$ integrates to G we have:

$$\phi \in SolEL \Longrightarrow \phi^g \in SolEL \;\; \forall g \in G \tag{3.38}$$

Finally, we can write the Noether theorem:

Theorem 3.3 (Noether)**.** *Consider a Classical Field Theory with a symmetry given by* $\mathfrak{g}$*, then:*

$$j(\xi) := \iota_{\xi_{\mathcal{F}}}\gamma + \alpha(\xi) \qquad \in \Omega^{(n-1,0)}(M \times \mathcal{F}) \tag{3.39}$$

is called ***conserved current*** *and satisfies the continuity equation:*

$$dj(\xi)\Big|_{SolEL} = 0 \in \Omega^{(n,0)}(M \times \mathcal{F}) \tag{3.40}$$

Proof.

$$\iota_{\xi_{\mathcal{F}}}\delta\mathcal{L} = L_{\xi_{\mathcal{F}}}\mathcal{L} = d\alpha(\xi) \tag{3.41}$$

$$= \iota_{\xi_{\mathcal{F}}}d\gamma + \langle q, \iota_{\xi_{\mathcal{F}}}\delta\phi\rangle d^n x = -d\iota_{\xi_{\mathcal{F}}}\gamma + \langle q, \iota_{\xi_{\mathcal{F}}}\delta\phi\rangle d^n x \tag{3.42}$$

Follows then:

$$dj(\xi) = d(\iota_{\xi_{\mathcal{F}}}\gamma + \alpha(\xi)) = \langle q, \iota_{\xi_{\mathcal{F}}}\delta\phi\rangle d^n x \tag{3.43}$$

and then, in $SoLEL$ (q=0):

$$dj(\xi)\Big|_{SoLEL} = 0 \tag{3.44}$$

☐

We see that this result is remarkable because we can define a **conserved charge**:

$$Q := \int_{\Sigma} j(\xi) \tag{3.45}$$

for a given space-like surface Σ. We see that this is constant in time:

$$\frac{d}{dt}Q = \int_{\Sigma} \partial_\mu j^\mu(\xi) d\Sigma = \int_{\partial\Sigma} j \cdot \hat{n} = 0 \tag{3.46}$$

Theorem 3.4. *Let a symmetry action on space of fields* $\mathfrak{g} \circlearrowright \mathcal{F}$*,* $\xi_{\mathcal{F}}$ *a corresponding fundamental vector field and* $Q(\xi)$ *the associated conserved charge. Then:*

$$\iota_{\xi_{\mathcal{F}}}\omega = -\delta Q(\xi) \tag{3.47}$$

Proof.

$$\iota_{\xi_{\mathcal{F}}}\omega = \int_{\Sigma} \iota_{\xi_{\mathcal{F}}}\delta\gamma = \int_{\Sigma} \left(L_{\xi_{\mathcal{F}}}\gamma - \delta\iota_{\xi_{\mathcal{F}}}\gamma\right)$$

$$= \int_\Sigma \Big(L_{\xi_\mathcal{F}} \gamma - \delta(j(-\alpha)(\xi)\Big) = \int_\Sigma \Big(L_{\xi_\mathcal{F}} \gamma + \delta\alpha(\xi)\Big) - \int \delta j(\xi)$$

$$= -\delta \int j(\xi) = -\delta Q(\xi) \tag{3.48}$$

□

From this statement, we see that the conserved charge $Q(\xi)$ generates the corresponding symmetry encoded in $\xi_\mathcal{F}$:

$$\delta_\xi F = L_{\xi_\mathcal{F}} F = \{Q(\xi), F\} \tag{3.49}$$

Where F is any function of the fields: $F = F(\phi)$.

3.3 Gauge Theory

Some of the most important theories used to describe nature presents a certain type of symmetry called *gauge symmetry*. Examples that will extensively be discussed here are Yang-Mills and String theory.

The idea of a *gauge theory* is a field theory with a symmetry in the space of field $\mathcal{F}$ represented by a group action:

$$\mathcal{G} \circlearrowright \mathcal{F} \tag{3.50}$$

such that this is a symmetry in the sense of equation (3.24), but with the additional condition to be an internal symmetry in the space of fields, that is, two fields related by such symmetry transformations describes the same physical quantities. Therefore the space of physical fields can be written as:

$$\mathcal{F}^{phys} := \mathcal{F}/\mathcal{G} \tag{3.51}$$

In this case, the gauge fixing fields are just sections s in the $\mathcal{G}-$principal bundle over $\mathcal{F}^{phys}$:

$$\begin{tikzcd} \mathcal{G} \circlearrowright \mathcal{F} \arrow[d, "\pi"'] \\ \mathcal{F}/\mathcal{G} = \mathcal{F}^{phys} \arrow[u, bend right, "s"'] \end{tikzcd} \tag{3.52}$$

Explicitly, if we have a field $\phi \in \mathcal{F}$, its gauge fixing field is given by:

$$\phi_s^{phys} = s^*\phi \quad \in \mathcal{F}^{phys} \tag{3.53}$$

we then see that fixing the field is the same as choosing the section s.

Besides the condition (3.24) for the gauge symmetry, we must express somehow the condition to be "gauge", and this is done by saying that the corresponding symmetry current vanishes:

Definition 3.2 (Gauge symmetry)**.** Given a field theory with $\mathcal{F}$ the space of fields and $\mathcal{G}$ a symmetry, $\mathcal{G} \circlearrowleft \mathcal{F}$ is a **gauge symmetry** if the current is exact:

$$j(\xi)\Big|_{SolEL} = d\beta(\xi) \tag{3.54}$$

where $j(\xi)$ is as in (3.39) and $\beta(\xi)$ any element in $\Omega^{(n-2,0)}(M \times \mathcal{F})$.

Wee see that in this case, the conserved charge is null, considering no-boundary for Σ:

$$Q = \int_\Sigma d\beta(xi) = 0 \tag{3.55}$$

That is, the idea of gauge symmetry is that it is an ambiguity on the way to write the fields, and therefore, being a symmetry on the space of fields, we have that it does not produces physical charges.

Example 3.4 (Yang-Mills)**.** *Consider the space-time M, a gauge group G and the gauge field $A \in \Omega^1(M, \mathfrak{g})$. Yang-Mills Lagrangian reads:*

$$L_{YM} = tr(d_A A \wedge \star d_A A) \tag{3.56}$$

$$= tr\left(dA \wedge \star dA - 2ig\, dA \wedge \star[A \wedge A] - g^2[A \wedge A] \wedge \star[A \wedge A]\right)$$

The gauge symmetry is:

$$\delta A = d_A \lambda \tag{3.57}$$

for $\lambda \in C^\infty(M) \otimes \mathfrak{g}$.

And then it is simple to see that the corresponding current is:

$$j = d\, tr(\lambda \wedge \star dA) \tag{3.58}$$

3.4 Non-linear sigma Models

One of the main purpose of this work is to discuss string theory, which is in fact a gauge theory, but it is not described by a Yang-Mills Lagrangian. Instead, it is what we call a *non-linear sigma model*. These are theories where the target space V can be any differential manifold, more generic than a vector space. It is useful in string theory since in gravity the target space is our space-time, which in general are curved spaces.

We can formulate a definition of a non-linear sigma-model by the set of datas:

- Space-time (M, h) & Target space (V, g);

  Both must be Riemannian manifolds, that is, they carries an inner product on each tangent space. For (M, h):

$$h_m : T_m M \otimes T_m M \longrightarrow \mathbb{R} \in T^*_m M \otimes T^*_m M \tag{3.59}$$

  well defined inner product, which varies smoothly with along $m \in M$. We require also existence of inverse:

$$h^{-1}_m : T^*_m M \otimes T^*_m M \longrightarrow \mathbb{R} \in T_m M \otimes T_m M \tag{3.60}$$

  Same thing for (V, g):

$$g_v : T_v V \otimes T_v V \longrightarrow \mathbb{R} \in T^*_v V \otimes T^*_v V \tag{3.61}$$

  inner product which are smooth functions in $v \in V$.

- Space of Fields $\mathcal{F} = Map(M, V)$;

  If we have a field $\phi \in \mathcal{F}$:

$$\phi : M \longrightarrow V \tag{3.62}$$

  then, for each $m \in M$, we can consider its differential:

$$d_m\phi : T_m M \longrightarrow T_{\phi(m)} V \tag{3.63}$$

  or, we can use the well known isomorphism $Hom(V, W) \cong V^* \otimes W$ to say that:

$$d_m\phi \in T^*_m M \otimes T_{\phi(m)} V \tag{3.64}$$

- The *sigma-model Lagrangian*:

$$\mathcal{L} = \frac{1}{2}\langle h_m^{-1} \otimes g_{\phi(m)}, d_m\phi \otimes d_m\phi \rangle d\mu \tag{3.65}$$

where $d\mu$ is a top form on M, which I could have written as $d^n z$ if we write in coordinates; the bracket $\langle -, - \rangle$ just express the action of $h^{-1} \otimes g$ on $d\phi \otimes d\phi$. To see that this action makes sense, notice:

$$h_m^{-1} \otimes g_{\phi(m)} : T_m M \otimes T_m M \otimes T^*_{\phi(m)} V \otimes T^*_{\phi(m)} V \longrightarrow \mathbb{R} \tag{3.66}$$

and from (3.64):

$$d_m\phi \otimes d_m\phi \in T^*_m M \otimes T^*_m M \otimes T_{\phi(m)} V \otimes T_{\phi(m)} V \tag{3.67}$$

making sense of the expression (3.65) for $\mathcal{L}$.

3.4.1 Topological terms

Consider a given field theory with the corresponding space of fields $\mathcal{F} = Map(M, V)$. Consider a form on the target space manifold V: $\omega \in \Omega^k(V)$. Topological terms can be constructed considering the following diagram, as in (3.11):

$$\begin{tikzcd} M \times \mathcal{F} \arrow[r, "ev"] \arrow[d, "p"'] \arrow[d, shift left=6, "\pi"] & V \\ M \times \mathcal{F} & \end{tikzcd} \tag{3.68}$$

And if for a $k-$form $\omega \in \Omega^k(V)$:

$$S^{top} = \pi_*(ev^*(\omega)) \tag{3.69}$$

If $k = n$, it turns out that this is a number and in this case, it has the nice property to be topological invariant. In this case, it can be used in many physical applications as an action.

Example 3.5 (Wess-Zumino terms). *Consider a field $\phi \in \mathcal{F}$ and a a n-form on V with $n = \dim(M) < \dim(V) = m$:*

$$B \in \Omega^n(V) \tag{3.70}$$

and

$$\phi : M \longrightarrow V \tag{3.71}$$

A Wess-Zumino term is:

$$S_{WZ}(\phi) = \int_M \phi^*(B) \tag{3.72}$$

In coordinates, we can express the WZ term as:

$$S_{WZ}(\phi) = \int_M B_{\mu_1 \cdots \mu_n} d\phi^{\mu_1} \wedge \cdots \wedge d\phi^{\mu_n} \tag{3.73}$$

$$= \int_M \epsilon^{a_1 \cdots a_n} B_{\mu_1 \cdots \mu_n} \partial_{a_1}\phi^{\mu_1} \wedge \cdots \wedge \partial_{a_n}\phi^{\mu_n} \tag{3.74}$$

Example 3.6 (WZW term)**.** *Consider G a Lie group and Σ a Riemannian surface as the space time in a given string theory (the world-sheet). Then we have the corresponding space of fields:*

$$\mathcal{F} = Map(\Sigma, G) \tag{3.75}$$

If $g \in \mathcal{F} = Map(\Sigma, G)$, and $\hat{\Sigma}$ is a given manifold such that $\partial\hat{\Sigma} = \Sigma$, then one can extend the g field:

$$\hat{g} : \hat{\Sigma} \longrightarrow G \tag{3.76}$$

such that $\hat{g}\Big|_{\Sigma} = g$.

Consider now ω^3 the Cartan 3 form of the group G:

$$\omega^3 = \kappa(g^{-1}dg \wedge [g^{-1}dg \wedge g^{-1}dg]) \tag{3.77}$$

where κ is the killing bilinear form on $\mathfrak{g}$ and in many cases is just the trace Tr.

The corresponding WZ term is:

$$S_{WZ}(\phi) = \int_{\hat{\Sigma}} \hat{g}^*(\omega^3) \tag{3.78}$$

And follows now the WZW action [29]:

$$S_{WZW}(\phi) = S_{\sigma-model}(\phi) + S_{WZ}(\phi) \tag{3.79}$$

Usually, one considers the $\sigma-$model as:

$$S_{\sigma-model} = \int \sqrt{h} h^{\mu\nu} \kappa(g^{-1}\partial_\mu g, g^{-1}\partial_\nu g) \tag{3.80}$$

3.5 Quantum Field Theory

We can say that a Quantum Field Theory (QFT) is a field theory carrying a measure $\mathcal{D}\phi$ on the space of fields $\mathcal{F}$:

$$(\mathcal{F}, \mathcal{D}\phi) \tag{3.81}$$

Since $\mathcal{F}$ is in general infinite dimensional and very complicated, this measure is defined via an algorithmic way, with a set of rules inherit from the finite dimensional case, which gives precise definition of our object, even though it was not derived from a sigma algebra construction; for instance, it gives rise to a precise definition of Feynman diagrams. We can also consider integration in this space, and this is what we call a *path integral*. A huge part of QFT is to understand such objects. The main quantities we want to consider in a Quantum Field Theory are:

- *Partition Function* Z:

  $$Z := \int_{\mathcal{F}} \mathcal{D}\phi \; e^{\, iS[\phi]} \tag{3.82}$$

- *Expectation values* of *Observables*:

  An *observable* is a function $\mathcal{O} : \mathcal{F} \longrightarrow \mathbb{R}$, carrying some physical interpretation (e.g. Energy, Spin,...). For each observable, we may associate a measurable quantity for it:

  $$\langle \mathcal{O} \rangle := \int_{\mathcal{F}} \mathcal{D}\phi \; \mathcal{O}(\phi) \; e^{\, iS[\phi]} \tag{3.83}$$

  this is what we call the *expectation value* of $\mathcal{O}$.

3.5.1 Quantization of gauge theories

The initial difficulty to quantize a gauge theory lives in the fact that if we naively try to perform the path integral (3.82), we will overcount physically equivalent states related by gauge transformation. To deal with that, we need to divide the path integral by the volume of the orbits in $\mathcal{F}$, which is just the fiber on the principal bundle in (3.52), which is just the gauge group $\mathcal{G}$:

$$\frac{1}{Vol(\mathcal{G})} \int_{\mathcal{F}} \mathcal{D}\phi \; e^{\, -S[\phi]} \tag{3.84}$$

this is what we want to compute now. To do so, we use the so-called Faddeev-Popov determinant. The idea is to consider some generic gauge, represented by the function:

$$F : \mathcal{F} \longrightarrow V \times M \tag{3.85}$$

with V an appropriate vector space. The gauge fixing will be expressed as:

$$F(\phi) = \xi(x) \tag{3.86}$$

For a fixed ξ, for example, we can put $\xi = 0$. We here assume that for each orbit:

$$[\phi] := \{\phi^g, \ \forall g \in \mathcal{G}\}, \quad \phi \in \mathcal{F} \tag{3.87}$$

only one point solves the gauge fixing equation: $\exists! \ \tilde{\phi} \in [\phi]$ such that $F[\tilde{\phi}] = \xi$.

And perform the integral with the gauge fixing. To do this, we introduce, somehow, a delta function on (3.84). This is done by introducing the Faddeev-Popov determinant Δ_{FP}:

$$1 = \Delta_{FP}[\phi] \int \mathcal{D}g \delta(F(\phi^g) - \xi) \tag{3.88}$$

$\mathcal{D}g$ is the Haar measure on the group $\mathcal{G}$.

And we now write the path integral as:

$$\frac{1}{Vol(\mathcal{G})} \int_{\mathcal{F}} \mathcal{D}g\mathcal{D}\phi \ e^{\ -S[\phi]} \delta(F(\phi^g) - \xi) \Delta_{FP}[\phi] \tag{3.89}$$

Notice now that:

$$1 \ \alpha \int \mathcal{D}\xi(x) e^{-\frac{1}{2\alpha} \int \langle \xi, \xi \rangle (x)} \tag{3.90}$$

that is, it's a constant. Introducing this factor on the path integral and integrating in ξ we get:

$$\frac{1}{Vol(\mathcal{G})} \int_{\mathcal{F}} \mathcal{D}g\mathcal{D}\phi \ e^{\ -S[\phi]} e^{-\frac{1}{2\alpha} \int \langle F(\phi^g), F(\phi^g) \rangle} \Delta_{FP}[\phi] \tag{3.91}$$

Now, observe from the definition of $\Delta_{FP}[\phi]$ that it is gauge invariant. S and $\mathcal{D}\phi$ is also gauge invariant, therefore we get:

$$\frac{1}{Vol(\mathcal{G})} \int_{\mathcal{F}} \mathcal{D}g\mathcal{D}\phi \ e^{\ -S[\phi]} e^{-\frac{1}{2\alpha} \int \langle F(\phi), F(\phi) \rangle} \Delta_{FP}[\phi] \tag{3.92}$$

$$= \int_{\mathcal{F}} \mathcal{D}\phi \ e^{\ -S[\phi]} e^{-\frac{1}{2\alpha} \int \langle F(\phi), F(\phi) \rangle} \Delta_{FP}[\phi] \tag{3.93}$$

Claim. It's possible to prove that the Faddeev-popov determinant is [22]:

$$\Delta_{FP}[\phi] = \det\left[\frac{\delta F(\phi^g)}{\delta g}\right] \tag{3.94}$$

therefore, using the integral form of the determinant:

$$\det(M)\ \alpha \int \mathcal{D}\phi \mathcal{D}\eta e^{-\langle\phi, M\eta\rangle} \tag{3.95}$$

we have:

$$\Delta_{FP}[\phi] = \int \mathcal{D}\phi \mathcal{D}\eta e^{-\langle\phi, \frac{\delta F}{\delta g}\eta\rangle} \tag{3.96}$$

Replacing $\phi \mapsto b$ and $\eta \mapsto c$, we finally get:

$$Z = \int_{\mathcal{F}} \mathcal{D}\phi \mathcal{D}b \mathcal{D}c \exp\left[-\int L[\phi] + \frac{1}{2\alpha}\langle F, F\rangle + \langle b, c^{\alpha}\delta_{\alpha}F\rangle\right] \tag{3.97}$$

We got then, the effective action in the exponent. Introducing a Lagrange multiplier field B we get:

$$S_{eff} = S_0[\phi] + \langle B, \left(F - \frac{\alpha}{2}B\right)\rangle + \langle b, c^{\alpha}\delta_{\alpha}F\rangle \tag{3.98}$$

We may now change the constraint to $F \mapsto F - \alpha/2B$ to get a more clear effective action:

$$S_{eff} = S_0[\phi] + \langle B, F\rangle + \langle b, c^{\alpha}\delta_{\alpha}F\rangle \tag{3.99}$$

$$= S_0 + S_{gauge\ fix} + S_{ghost} \tag{3.100}$$

BRST symmetry

Once we fixed the gauge, we can quantize the theory. But we have to pay attention to a symmetry that emerges in our effective action. This symmetry appears because all the degrees of freedom that we have fixed was transferred to the ghost field. The BRST symmetry reads:

$$\delta_B \phi = c^\alpha \delta_\alpha \phi \tag{3.101}$$

$$\delta_B c^\alpha = \frac{1}{2} f^\alpha_{\beta\gamma} c^\beta c^\gamma \tag{3.102}$$

$$\delta_B b = B \tag{3.103}$$

$$\delta_B B = 0 \tag{3.104}$$

With this symmetry we denote Q_B the corresponding conserved charge, that can be used to compute the variation - $\delta_B = \{Q_B, -\}$. An important property of this symmetry is that:

$$\delta_B(\langle b, F \rangle) = S_{gauge\ fix} + S_{ghost} \tag{3.105}$$

Now, if we do a small deformation on the gauge fixing, the quantum theory has to be invariant, that is, the amplitude for some initial state $|i\rangle$ propagating to the final state $|f\rangle$ has to be invariant:

$$0 = \delta\langle i|f\rangle = \langle i|\delta(S_{gauge\ fix} + S_{ghost})|f\rangle = \langle i|\delta_B(\langle b, \delta F\rangle)|f\rangle \tag{3.106}$$

Therefore, in order to be a physical state:

$$\langle \phi|\{Q_B, \langle b, \delta F\rangle\}|\psi\rangle = 0 \tag{3.107}$$

for arbitrary variations δF. Therefore, follows the physical condition for a state $|\phi\rangle$:

$$Q_B|\phi\rangle = 0 \tag{3.108}$$

Therefore, we then can state the following claim.

Claim I. Physical States must be BRST invariant.

Another important statement is that the *BRST* operator is nilpotent:

$$\begin{aligned}\delta_B(\delta_B)\phi = \delta_B(c^\alpha \delta_\alpha \phi) &= \frac{1}{2} f^\alpha_{\beta\gamma} c^\beta c^\gamma \delta_\alpha \phi - c^\alpha c^\beta \delta_\alpha \delta_\beta \phi \\ &= \frac{1}{2} f^\alpha_{\beta\gamma} c^\beta c^\gamma \delta_\alpha \phi - \frac{1}{2} c^\alpha c^\beta f^\gamma_{\alpha\beta} \delta_\gamma \phi = 0\end{aligned} \tag{3.109}$$

$$\delta_B(\delta_B c) = \frac{1}{2}\delta_B[c,c] = \frac{1}{2}[c,[c,c]] = 0 \tag{3.110}$$

this last one follows from Jacobi identity.

The consequence of the nilpotent is that $Q-$exact states are physical states because using Claim I:

$$Q(Q|\phi\rangle) = Q^2|\phi\rangle = 0 \tag{3.111}$$

These are called *null states*, because they are irrelevant for the physics spectrum, and the reason is that it is orthogonal to all physical states. To see that, we take $|\psi\rangle$ a physical state and compute the inner product with a null state:

$$\langle\psi|(Q|\phi\rangle) = (\langle\psi|Q)|\phi\rangle = 0 \tag{3.112}$$

Therefore, the physical amplitudes using null states vanishes. We can then state:

Claim II. Two physical states that differs by a null states are physically equivalent:

$$|\psi\rangle \sim |\psi\rangle + Q|\phi\rangle \tag{3.113}$$

Using these two claims, we can now state how is the spectrum of the quantum theory:

$$\mathcal{H}_{phys} = \frac{\mathcal{H}_{closed}}{\mathcal{H}_{exact}} \tag{3.114}$$

Chapter 4

Superstring Theory

Bosonic string theory described by Nambu-Goto action is just a generalization of the relativistic particle $S_p = \int ds^2$ [4]. Therefore, instead of being an infinitesimal line length, the bosonic String Lagrangian is just a surface area $S_{NG} = \int dA$. If we parametrize the string propagating in a D dimensional space-time by the following coordinate functions:

$$X^\mu : \Sigma \longrightarrow \mathbb{R} \tag{4.1}$$

with $\mu = 1, \cdots, D$, this infinitesimal area element will be written as:

$$\mathcal{L}_{NG} = \sqrt{(\dot{X} \cdot X')^2 - (\dot{X})^2 (X')^2} = \sqrt{-\det h_{ab}} \tag{4.2}$$

where

$$h_{ab} := \partial_a X^\mu \partial_b X_\mu \tag{4.3}$$

It's difficult to quantize this action because of the presence of the square root [1]. To deal with it, we can introduce a new field g^{ab}, a metric on the worldsheet, and write an equivalent action as:

$$\mathcal{L}_P = \frac{1}{2}\sqrt{-g} g^{ab} \partial_a X^\mu \partial_b X_\mu \tag{4.4}$$

or, in curved spaces with metric $G_{\mu\nu}$:

$$\mathcal{L}_P = \frac{1}{2}\sqrt{-g} g^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu} \tag{4.5}$$

To see the equivalence, we use the equation of motions to g^{ab} and integrate it out:

$$\delta S_P = \frac{1}{2}\int d\tau d\sigma \sqrt{-g}\, \delta g^{ab} \left(h_{ab} - \frac{1}{2} g_{ab} g^{cd} h_{cd} \right) = 0 \tag{4.6}$$

$$.: h_{ab} = \frac{1}{2} g_{ab} g^{cd} h_{cd} \tag{4.7}$$

now we take the determinant in both sides and conclude that with g^{ab} on-shell we have the equality:

$$\mathcal{L}_{NG} = \sqrt{-h} = \frac{1}{2}\sqrt{-g}g^{cd}h_{cd} = \mathcal{L}_P \tag{4.8}$$

One can see that the bosonic action (4.4) is a sigma model of the form:

$$S = -\frac{1}{2\pi}\int_\Sigma \left\langle g_p^{-1} \otimes G_{X(p)}, d_pX \otimes d_pX \right\rangle d\mu \tag{4.9}$$

with $d\mu = d^2\sigma\sqrt{g}$ if we use coordinates.

In order to describe fermionic states in quantum gravity using string theory, it's possible to work with world-sheet supersymmetry (NS formalism) or with space-time supersymmetry (GS formalism). In the last case, which will be the focus of this text, we have a manifest supersymmetry.

4.1 Green-Shwartz Superstring in flat space

Consider the bosonic string sigma model we just studied:

$$S = -\frac{1}{2\pi}\int_\Sigma d^2\sigma\sqrt{h}h^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu G_{\mu\nu} \tag{4.10}$$

The most natural super-symmetric generalization is [3]:

$$S_1 = -\frac{1}{2\pi}\int d^2\sigma\sqrt{h}h^{\alpha\beta}\Pi^\mu_\alpha\Pi^\nu_\beta G_{\mu\nu}$$

where:

$$\Pi^\mu = dX^\mu - i\bar{\theta}^A\Gamma^\mu d\theta^A \tag{4.11}$$

which possesses N global supersymmetries:

$$\begin{aligned} \delta_\epsilon\theta^A &= \epsilon^A \\ \delta_\epsilon X^\mu &= i\bar{\epsilon}^A\Gamma^\mu\theta^A \end{aligned} \tag{4.12}$$

for $A = 1, \cdots, N$.

Let us check this by showing the super-invariance of Π:

$$\delta_\epsilon\Pi^\mu = d\delta_\epsilon X^\mu - i\delta_\epsilon\bar{\theta}^A\Gamma^\mu d\theta^A = i\bar{\epsilon}^A\Gamma^\mu d\theta^A - i\bar{\epsilon}^A\Gamma^\mu d\theta^A = 0 \tag{4.13}$$

Therefore, follows that the action S_1 is supersymmetric.

This action, although, describes twice as many d.o.f. as it should. In the case of superparticle, this is fixed by the presence of the kappa symmetry. We can then, therefore, recover such symmetry by adding a topological term, i.e., a term that doesn't depends on the metric and then doesn't affect the energy momentum tensor:

$$S_2 = \frac{1}{\pi}\int_\Sigma \left(-idX_\mu \wedge (\bar\theta^1\Gamma^\mu d\theta^1 - \bar\theta^2\Gamma^\mu d\theta^2) + \bar\theta^1\Gamma^\mu d\theta^1 \wedge \bar\theta^2\Gamma_\mu d\theta^2\right) \tag{4.14}$$

This term is invariant under supersymmetries (4.12):

$$\begin{aligned}\delta_\epsilon S_2 = \frac{1}{\pi}\int \Big(&- i\epsilon^{\alpha\beta}(i\bar\epsilon^1\Gamma^\mu\partial_\alpha\theta^1 + i\bar\epsilon^2\Gamma^\mu\partial_\alpha\theta^2)(\bar\theta^1\Gamma^\mu\partial_\beta\theta^1 - \bar\theta^2\Gamma^\mu\partial_\beta\theta^2)\\ &- i\epsilon^{\alpha\beta}\partial_\alpha X_\mu(\bar\epsilon^1\Gamma^\mu\partial_\beta\theta^1 - \bar\epsilon^2\Gamma^\mu\partial_\beta\theta^2)\\ &\epsilon^{\alpha\beta}\bar\epsilon^1\Gamma^\mu\partial_\alpha\theta^1\bar\theta^2\Gamma_\mu\partial_\beta\theta^2 + \epsilon^{\alpha\beta}\bar\theta^1\Gamma^\mu\partial_\alpha\theta^1\bar\epsilon^2\Gamma_\mu\partial_\beta\theta^2\Big)\end{aligned} \tag{4.15}$$

We drop the second line because it is a total derivative:

$$\begin{aligned}\epsilon^{\alpha\beta}\partial_\alpha X_\mu\bar\epsilon\Gamma^\mu\partial_\beta\theta &= \partial_\beta\left(\epsilon^{\alpha\beta}\partial_\alpha X_\mu\bar\epsilon\Gamma^\mu\theta\right) - \epsilon^{\alpha\beta}\partial_\beta\partial_\alpha X_\mu\bar\epsilon\Gamma^\mu\theta\\ &= \partial_\beta\left(\epsilon^{\alpha\beta}\partial_\alpha X_\mu\bar\epsilon\Gamma^\mu\theta\right)\end{aligned} \tag{4.16}$$

the last term in the first line vanish because ϵ is anti-symmetric and the derivatives commutes.

The last line in (4.15) cancels with some of the first line terms. The remaining terms are:

$$\delta_\epsilon S_2 = \frac{1}{\pi}\int \epsilon^{\alpha\beta}\bar\epsilon^1\Gamma^\mu\partial_\alpha\theta^1\bar\theta^1\Gamma^\mu\partial_\beta\theta^1 - \epsilon^{\alpha\beta}\bar\epsilon^2\Gamma^\mu\partial_\alpha\theta^2\bar\theta^2\Gamma^\mu\partial_\beta\theta^2 \tag{4.17}$$

We see that each of these terms vanishes:

$$\epsilon^{\alpha\beta}\bar\epsilon\Gamma^\mu\partial_\alpha\theta\bar\theta\Gamma_\mu\partial_\beta\theta = \bar\epsilon\Gamma^\mu\dot\theta\bar\theta\Gamma_\mu\theta' - \bar\epsilon\Gamma^\mu\theta'\bar\theta\Gamma_\mu\dot\theta \tag{4.18}$$

$$= \frac{2}{3}\left(\bar\epsilon\Gamma^\mu\dot\theta\bar\theta\Gamma_\mu\theta' + \bar\epsilon\Gamma^\mu\theta'\dot{\bar\theta}\Gamma_\mu\theta + \bar\epsilon\Gamma^\mu\theta\bar\theta'\Gamma_\mu\dot\theta\right)(= A_1) \tag{4.19}$$

$$+ \frac{1}{3}\left(\bar\epsilon\Gamma^\mu\dot\theta\bar\theta\Gamma_\mu\theta' + \bar\epsilon\Gamma^\mu\theta'\dot{\bar\theta}\Gamma_\mu\theta - 2\bar\epsilon\Gamma^\mu\theta\bar\theta'\Gamma_\mu\dot\theta\right)(= A_2) \tag{4.20}$$

We see that A_2 drops because it's a total derivative:

$$A_2 = \frac{1}{3}\partial_\tau(\bar{\epsilon}\Gamma^\mu\theta\bar{\theta}\Gamma_\mu\theta') - \frac{1}{3}\partial_\sigma(\bar{\epsilon}\Gamma^\mu\theta\bar{\theta}\Gamma_\mu\dot{\theta}) \tag{4.21}$$

and $A_1 = 0$ because of relation $(A.16)$ in appendix A. Follows that S_2 is invariant under a supersymmetry transformation.

Now, in the action $S_{GS} = S_1 + S_2$ we can check that we recovered kappa symmetry, defined by:

$$\delta_\kappa X^m = i\bar{\theta}^A\Gamma^m\delta_\kappa\theta^A \tag{4.22}$$

$$\delta_\kappa\theta^A = 2i\Gamma_m\Pi^m_\alpha\kappa^{\alpha A} \tag{4.23}$$

It follows:

$$\delta_\kappa\Pi^m = 2id\bar{\theta}^A\Gamma^m\delta_\kappa\theta^A \tag{4.24}$$

The first part of the action will varies as:

$$\delta_\kappa S_1 = -\frac{i}{\pi}\int d^2\sigma\left[\sqrt{-g}g^{\alpha\beta}\Pi^m_\alpha\delta_\kappa\theta^A\Gamma^m\partial_\alpha\bar{\theta}^A + \delta_\kappa(\sqrt{-g}g^{\alpha\beta})\Pi^m_\alpha\Pi^m_\beta\right] \tag{4.25}$$

where we used flat space-time $G_{mn} = \eta_{mn}$.

To the other part, notice that is can be written as:

$$L_2 = -i\epsilon_{\alpha\beta}\Pi^\mu_\alpha(\bar{\theta}^1\Gamma_\mu\partial_\beta\theta^1 - \bar{\theta}^2\Gamma_\mu\partial_\beta\theta^2) + \theta^4\ terms \tag{4.26}$$

We define the projector:

$$P^{\alpha\beta}_\pm = \frac{1}{2}(g^{\alpha\beta} \pm \epsilon^{\alpha\beta}/\sqrt{g}) \tag{4.27}$$

And see that comparing the two variations and see that we need to impose:

$$\delta_\kappa(\sqrt{g}g^{\alpha\beta}) = -16\sqrt{g}(P^{\alpha\gamma}_-\bar{\kappa}^{1\beta}\partial_\gamma\theta^1 + P^{\alpha\gamma}_+\bar{\kappa}^{2\beta}\partial_\gamma\theta^2) \tag{4.28}$$

as well as:

$$\kappa^{1\alpha} = P^{\alpha\beta}_-\kappa^1_\beta \tag{4.29}$$

$$\kappa^{2\alpha} = P^{\alpha\beta}_+\kappa^2_\beta \tag{4.30}$$

this will imply kappa symmetry [16].

Equation of Motions are:

$$\Pi_\alpha \cdot \Pi_\beta = \frac{1}{2} g_{\alpha\beta} g^{\gamma\delta} \Pi_\gamma \cdot \Pi_\delta \tag{4.31}$$

$$\Gamma \cdot \Pi P_-^{\alpha\beta} \partial_1 \theta_\beta = 0 \tag{4.32}$$

$$\Gamma \cdot \Pi P_+^{\alpha\beta} \partial_2 \theta_\beta = 0 \tag{4.33}$$

$$\partial_\alpha \left(\sqrt{g} (g^{\alpha\beta} \partial_\beta X) - 2iP_-^{\alpha\beta} \bar{\theta}^1 \Gamma \partial_\beta \theta^1 - 2iP + -^{\alpha\beta} \bar{\theta}^2 \Gamma \partial_\beta \theta^2 \right) = 0 \tag{4.34}$$

4.2 Pure Spinor superstring in flat space

The Ramond-Neveu-Schwarz (RNS) formalism of superstrings possesses a BRST operator, from which the superstring is quantized [2]. For the Pure Spinor formalism, we also have a BRST like operator in such a way that both formalisms are equivalent [6].

In the pure spinor superstring [5], we construct the action by adding a ghost sector and its conjugate momentum:

$$S = \int_\Sigma d\tau^+ d\tau^- \Big(\frac{1}{2} \partial_+ X^m \partial_- X_m + p_- \partial_+ \theta_R + p_+ \partial_- \theta_L \tag{4.35}$$

$$+ w_+ \partial_- \lambda_L + w_- \partial_+ \lambda_R \Big) \tag{4.36}$$

with coordinates $z^\pm = \sigma \pm \tau$. The equations of motions are:

$$\partial_+ \partial_- X = 0 \tag{4.37}$$

$$\partial_+ \theta_R = \partial_+ p_R = \partial_+ \lambda_R = \partial_+ w_- = 0 \tag{4.38}$$

$$\partial_- \theta_L = \partial_- p_L = \partial_- \lambda_L = \partial_- w_+ = 0 \tag{4.39}$$

Observe that the fields $\theta_R, p_R, \lambda_R, w_-$ only depends on σ^- as well as $\theta_L, p_L, \lambda_L, w_+$ only depends on σ^+.

One introduces a BRST-like operator acting on the fields as (see Appendix B):

$$\begin{aligned}
&QX^m = \frac{1}{2}(\lambda_L\Gamma^m\theta_L + \lambda_R\Gamma^m\theta_R)\\
&Q\theta_{L,R} = \lambda_{L,R}\\
&Q\lambda_{L,R} = 0\\
&Qw_\pm = d_\pm\\
&Qp_+ = -\frac{1}{2}\partial_+X^m(\lambda_L\Gamma_m) + \frac{3}{8}(\lambda_L\Gamma^m\theta_L)(\partial_+\theta_L\Gamma_m) + \frac{1}{8}(\partial_+\lambda_L\Gamma^m\theta_L)(\theta_L\Gamma_m)\\
&Qp_- = -\frac{1}{2}\partial_-X^m(\lambda_R\Gamma_m) + \frac{3}{8}(\lambda_R\Gamma^m\theta_R)(\partial_-\theta_R\Gamma_m) + \frac{1}{8}(\partial_-\lambda_R\Gamma^m\theta_R)(\theta_R\Gamma_m)
\end{aligned} \tag{4.40}$$

It will follows:

$$\begin{aligned}
Qd_+ &= -\Pi_+^m\Gamma_m\lambda_L\\
Qd_- &= -\Pi_-^m\Gamma_m\lambda_R
\end{aligned} \tag{4.41}$$

with d_α the constraint that appear for the p_α momentum, from GS superstring since $p_\pm = \frac{\partial S_{GS}}{\partial(\partial_\mp\theta_{L,R})}$:

$$d_+ = p_+ - \frac{1}{2}\partial_+X^m\Gamma_m\theta_L - \frac{1}{8}(\theta_L\Gamma^m\partial_+\theta_L)\Gamma_m\theta_L \tag{4.42}$$

$$d_- = p_- - \frac{1}{2}\partial_-X^m\Gamma_m\theta_R - \frac{1}{8}(\theta_R\Gamma^m\partial_-\theta_R)\Gamma_m\theta_R \tag{4.43}$$

which in Pure Spinor is not imposed to be zero.

And the Π^m operator is just the super-translation momentum:

$$\Pi_+^m = \partial_+X^m + \frac{1}{2}\theta_L\Gamma^m\partial_+\theta_L \tag{4.44}$$

$$\Pi_-^m = \partial_-X^m + \frac{1}{2}\theta_R\Gamma^m\partial_-\theta_R \tag{4.45}$$

The Q operator is a BRST-like operator, and we can see it as a vector field on the space of fields:

$$Q = \lambda_L\frac{\delta}{\delta\theta_L} + \lambda_R\frac{\delta}{\delta\theta_R} + \frac{1}{2}(\lambda_L\Gamma^m\theta_L + \lambda_R\Gamma^m\theta_R)\frac{\delta}{\delta X^m} + d_+\frac{\delta}{\delta w_+} + d_-\frac{\delta}{\delta w_-} \tag{4.46}$$

$$+\left(-\frac{1}{2}\partial_+X^m(\lambda_L\Gamma_m)+\frac{3}{8}(\lambda_L\Gamma^m\theta_L)(\partial_+\theta_L\Gamma_m)+\frac{1}{8}(\partial_+\lambda_L\Gamma^m\theta_L)(\theta_L\Gamma_m)\right)\frac{\delta}{\delta p_+}$$
$$+\left(-\frac{1}{2}\partial_-X^m(\lambda_R\Gamma_m)+\frac{3}{8}(\lambda_R\Gamma^m\theta_R)(\partial_-\theta_R\Gamma_m)+\frac{1}{8}(\partial_-\lambda_R\Gamma^m\theta_R)(\theta_R\Gamma_m)\right)\frac{\delta}{\delta p_-}$$

Let us see the condition for Q to be nilpotent:

$$Q^2=\frac{1}{2}\{Q,Q\}=Q^2\theta_L\frac{\delta}{\delta\theta_L}+Q^2\theta_R\frac{\delta}{\delta\theta_R}$$
$$+Q^2X^m\frac{\delta}{\delta X^m}+Q^2w_+\frac{\delta}{\delta w_+}+Q^2w_-\frac{\delta}{\delta w_-}+Q^2p_+\frac{\delta}{\delta p_+}+Q^2p_-\frac{\delta}{\delta p_-} \tag{4.47}$$

Basically, if we want this to vanish, all terms has to be zero. In the first line this follows automatically, let us check it on X^m:

$$Q^2X^m=\frac{1}{2}\lambda_L\Gamma^m\lambda_L+\frac{1}{2}\lambda_L\Gamma^m\lambda_L=0 \tag{4.48}$$

This condition defines what a pure spinor λ is, a sufficient condition for nilpotence for Q, as we will conclude further:

Definition 4.1 (Pure Spinor)**.** A **pure spinor** is a 10 dimensional spinor λ^α such that:

$$\lambda\Gamma^m\lambda=0 \tag{4.49}$$

where Γ^m for $m=0,\cdots,9$ are the gamma matrices in 10 dimensions (see Appendix A).

Pure spinor condition will also imply that $Q^2p=0$, let us check this, suppressing L,R indices. Notice first that $0=\partial(\lambda\Gamma\lambda)=\partial\lambda\Gamma\lambda+\lambda\Gamma\partial\lambda$, which by symmetry of Γ implies $\partial\lambda\Gamma\lambda=0$:

$$Q^2p=Q\left(-\frac{1}{2}\partial X^m(\lambda\Gamma_m)+\frac{3}{8}(\lambda\Gamma^m\theta)(\partial\theta\Gamma_m)+\frac{1}{8}(\partial\lambda\Gamma^m\theta)(\theta\Gamma_m)\right) \tag{4.50}$$
$$=-\frac{1}{4}\partial(\lambda\Gamma^m\theta)(\lambda\Gamma_m)+\frac{3}{8}(\lambda\Gamma^m\lambda)(\partial\theta\Gamma_m)-\frac{3}{8}(\lambda\Gamma^m\theta)(\partial\lambda\Gamma_m)$$
$$+\frac{1}{8}(\partial\lambda\Gamma^m\lambda)(\theta\Gamma_m)-\frac{1}{8}(\partial\lambda\Gamma^m\theta)(\lambda\Gamma_m) \tag{4.51}$$
$$=-\frac{3}{8}(\partial\lambda\Gamma^m\theta)(\lambda\Gamma_m)-\frac{3}{8}(\lambda\Gamma^m\theta)(\partial\lambda\Gamma_m)-\frac{1}{4}(\lambda\Gamma^m\partial\theta)(\lambda\Gamma_m)=0$$

Where we used condition ($A.16$) to show that, together with the pure spinor condition:

$$\begin{aligned}(\partial\lambda\Gamma^m\theta)(\lambda\Gamma_m) &= -(\lambda\Gamma^m\theta)(\partial\lambda\Gamma_m) \\ (\lambda\Gamma^m\partial\theta)(\lambda\Gamma_m) &= 0\end{aligned} \tag{4.52}$$

It is remarkable that pure spinor condition for λ will imply a gauge symmetry for its conjugate momentum $w_\pm$:

$$\delta w_\pm = V^m_\pm \Gamma_m \lambda \tag{4.53}$$

for arbitrary functions $V^m_\pm$, $m = 0, \cdots, 9$. This fact is justified in the following proposition:

Proposition 4.1. *Consider a theory with variables ϕ^A and π_A the corresponding canonical momentum with the Lagrangian in the following form:*

$$\mathcal{L} = \pi_A D\phi^A \tag{4.54}$$

with D a differential operator. If there's a constraint on the variables ϕ given by $F^m(\phi) = 0$, there is a gauge symmetry on the π fields given by:

$$\delta\pi_A = V_m \frac{\delta F^m}{\delta\phi^A} \tag{4.55}$$

for V_m arbitrary functions.

Notice that the pure spinor action (4.35) is of the form (4.54) in the proposition with $D = \partial_\pm$. We will also see in chapter 5 that the BV action is of this form. The prove is very simple:

Proof. Since the constraint is $F^m = 0$, follows that $DF^m = 0$. Applying the gauge variation (4.55) to the Lagrangian and using the chain rule:

$$\delta\mathcal{L} = \delta\pi_A D\phi^A = V_m \frac{\delta F^m}{\delta\phi^A} D\phi^A = V_m DF^m = 0 \tag{4.56}$$

□

This observation in the gauge symmetry for $w_\pm$ will justify the last step for nilpotence. In fact, follows, with the pure spinor condition:

$$Q^2 = \frac{1}{2}\{Q,Q\} = \Pi_+^m \Gamma_m \lambda_L \frac{\delta}{\delta w_+} + \Pi_-^m \Gamma_m \lambda_R \frac{\delta}{\delta w_-} \sim 0 \tag{4.57}$$

where $\sim$ means the following: Q^2 acts only on $w_\pm$, and its action is a pure gauge expression (4.53) with $V_\pm^m = \Pi_\pm^m$, therefore, it is equivalent to zero.

4.2.1 Global symmetries in Pure Spinor String

The pure spinor action carries some global symmetries. In order to simplify computations, let us consider just left variables and suppress index

$$S_{PS} = \int d^2z \left(\frac{1}{2}\partial X^\mu \bar{\partial} X_\mu + p_+ \bar{\partial}\theta + w_+ \bar{\partial}\lambda \right) \tag{4.58}$$

$$\delta \mathcal{L}_{PS}\Big|_{SolEL} = d\left(\delta X^\mu \star dX_\mu + p_+ \delta\theta + w_+ \delta\lambda\right) = d\gamma \tag{4.59}$$

Follows the symplectic form:

$$\omega = \delta\gamma = \delta X^\mu \wedge \partial\delta X_\mu + \delta p \wedge \delta\theta + \delta w \wedge \delta\lambda \tag{4.60}$$

Poincaré symmetry

The Poincaré symmetry reads:

$$\delta X^\mu = \Lambda_\mu^\nu X^\mu + a^\mu \tag{4.61}$$

$$\delta\theta_\alpha = \frac{1}{4}\Lambda_{\mu\nu}(\gamma^{\mu\nu}\theta)^\alpha\,, \qquad \delta p_\alpha = \frac{1}{4}\Lambda_{\mu\nu}(\gamma^{\mu\nu}p)_\alpha \tag{4.62}$$

$$\delta\lambda^\alpha = \frac{1}{4}\Lambda_{\mu\nu}(\gamma^{\mu\nu}\lambda)^\alpha\,, \qquad \delta\omega_\alpha = \frac{1}{4}\Lambda_{\mu\nu}(\gamma^{\mu\nu}\omega)_\alpha \tag{4.63}$$

Let us check that this is a symmetry, analysing term by term:

$$\delta\left(\frac{1}{2}\bar{\partial}X^\mu \partial X_\mu\right) = \Lambda^\mu_\rho(\bar{\partial}X^\rho \partial X_\mu + \bar{\partial}X_\mu \partial X^\rho) = 0 \tag{4.64}$$

the last equality follows because it's a contraction between symmetric and antisymmetric matrices.

$$\delta(p\bar{\partial}\theta) = \frac{1}{4}\left(\Lambda_{\mu\nu}\gamma^{\mu\nu}p\bar{\partial}\theta + p\bar{\partial}(\Lambda_{\mu\nu}\gamma^{\mu\nu}\theta)\right) \tag{4.65}$$

$$= \frac{1}{4}\Lambda_{\mu\nu}\left(\gamma^{\mu\nu}p\bar{\partial}\theta + p\gamma^{\mu\nu}\bar{\partial}\theta\right) \tag{4.66}$$

Using $(\gamma^{\mu\nu})^{\alpha}{}_{\beta} = -(\gamma^{\mu\nu})_{\beta}{}^{\alpha}$ we get:

$$\gamma^{\mu\nu} p\bar{\partial}\theta + p\gamma^{\mu\nu}\bar{\partial}\theta = (\gamma^{\mu\nu})^{\alpha}{}_{\beta} p^{\beta}\bar{\partial}\theta_{\alpha} + p^{\alpha}(\gamma^{\mu\nu})_{\alpha}{}^{\beta}\bar{\partial}\theta_{\beta} = 0 \tag{4.67}$$

To the other term we do exactly the same computation:

$$\delta(w\bar{\partial}\lambda) = \frac{1}{4}\Lambda_{\mu\nu}\left(\gamma^{\mu\nu} w\bar{\partial}\lambda + w\gamma^{\mu\nu}\bar{\partial}\lambda\right) = 0 \tag{4.68}$$

Follows than:

$$\delta\mathcal{L}_{PS} = 0 \tag{4.69}$$

and therefore the current (3.39) is such that $\alpha = 0$ and therefore:

$$j = \iota_{\xi}\gamma \tag{4.70}$$

For Translation (T) and Rotation (R):

$$\xi_T = \frac{\delta}{\delta X^{\mu}}\,, \qquad \xi_R = X_{[\mu}\frac{\delta}{\delta X^{\nu]}} \tag{4.71}$$

$$.:P_{\mu} = \iota_{\xi_T}\gamma = \partial X^{\mu}\,, \qquad L_{\mu\nu} = \iota_{\xi_R}\gamma = X_{[\mu}\partial X_{\nu]} \tag{4.72}$$

For rotations in θ, λ:

$$\eta = \frac{1}{4}(\gamma^{\mu\nu}\theta)^{\alpha}\frac{\delta}{\delta\theta^{\alpha}} + \frac{1}{4}(\gamma^{\mu\nu}p)_{\alpha}\frac{\delta}{\delta p_{\alpha}} \tag{4.73}$$

$$.:\Sigma^{\mu\nu} = \iota_{\eta}\gamma = \frac{1}{2}p\gamma^{\mu\nu}\theta \tag{4.74}$$

$$\zeta = \frac{1}{4}(\gamma^{\mu\nu}\lambda)^{\alpha}\frac{\delta}{\delta\lambda^{\alpha}} + \frac{1}{4}\Lambda_{\mu\nu}(\gamma^{\mu\nu}\omega)_{\alpha}\frac{\delta}{\delta w_{\alpha}} \tag{4.75}$$

$$.:N^{\mu\nu} = \iota_{\zeta}\gamma = \frac{1}{2}w\gamma^{\mu\nu}\lambda \tag{4.76}$$

space-time supersymmetry

We also have the following realization for supersymmetry in the case of pure spinor:

$$\delta_{\epsilon}X^{\mu} = \frac{1}{2}(\epsilon\gamma^{\mu}\theta) \tag{4.77}$$

$$\delta_\epsilon \theta^\alpha = \epsilon^\alpha \tag{4.78}$$

$$\delta_\epsilon p_\alpha = -\frac{1}{2}(\epsilon\gamma^\mu)_\alpha \partial X_\mu + \frac{1}{8}(\epsilon\gamma^\mu\theta)(\partial\theta\gamma_\mu)_\alpha \tag{4.79}$$

The corresponding vector field for this symmetry is:

$$\xi = \frac{1}{2}(\epsilon\gamma^\mu\theta)\frac{\delta}{\delta X^\mu} + \epsilon^\alpha\frac{\delta}{\delta\theta^\alpha} + \left(-\frac{1}{2}(\epsilon\gamma^\mu)_\alpha\partial X_\mu + \frac{1}{8}(\epsilon\gamma^\mu\theta)(\partial\theta\gamma_\mu)_\alpha\right)\frac{\delta}{\delta p_\alpha} \tag{4.80}$$

Let us check the symmetry.

$$\begin{aligned}\delta\mathcal{L}_{PS} &= \frac{1}{4}(\epsilon\gamma^\mu\partial\theta)\bar{\partial}X_\mu + \frac{1}{4}\partial X^\mu(\epsilon\gamma^\mu\bar{\partial}\theta) - \frac{1}{2}\epsilon\gamma^\mu\bar{\partial}\theta + \frac{1}{8}(\epsilon\gamma^\mu\theta)(\partial\theta\gamma_\mu\bar{\partial}\theta)\\ &= \bar{\partial}\left(\frac{1}{4}(\epsilon\gamma^\mu\theta)\partial X_\mu\right) + \partial\left(-\frac{1}{4}(\epsilon\gamma^\mu\theta)\bar{\partial}X_\mu\right) + \frac{1}{8}(\epsilon\gamma^\mu\theta)(\partial\theta\gamma_\mu\bar{\partial}\theta)\end{aligned} \tag{4.81}$$

Where we integrated the first term by parts twice. Let us analyse the last term:

$$(\epsilon\gamma^\mu\theta)(\partial\theta\gamma_\mu\bar{\partial}\theta) = \bar{\partial}\left((\epsilon\gamma^\mu\theta)(\partial\theta\gamma_\mu\theta)\right) - (\epsilon\gamma^\mu\bar{\partial}\theta)(\partial\theta\gamma_\mu\theta) - (\epsilon\gamma^\mu\theta)(\bar{\partial}\partial\theta\gamma_\mu\theta)$$

$$= \bar{\partial}\left((\epsilon\gamma^\mu\theta)(\partial\theta\gamma_\mu\theta)\right) - \partial\left((\epsilon\gamma^\mu\theta)(\bar{\partial}\theta\gamma_\mu\theta)\right) \tag{4.82}$$

$$- (\epsilon\gamma^\mu\bar{\partial}\theta)(\partial\theta\gamma_\mu\theta) + (\epsilon\gamma^\mu\partial\theta)(\bar{\partial}\theta\gamma_\mu\theta) + (\epsilon\gamma^\mu\theta)(\bar{\partial}\theta\gamma_\mu\partial\theta) \tag{4.83}$$

$$= \bar{\partial}\left((\epsilon\gamma^\mu\theta)(\partial\theta\gamma_\mu\theta)\right) - \partial\left((\epsilon\gamma^\mu\theta)(\bar{\partial}\theta\gamma_\mu\theta)\right) \tag{4.84}$$

$$(\epsilon\gamma^\mu\bar{\partial}\theta)(\theta\gamma_\mu\partial\theta) + (\epsilon\gamma^\mu\partial\theta)(\bar{\partial}\theta\gamma_\mu\theta) - (\epsilon\gamma^\mu\theta)(\partial\theta\gamma_\mu\bar{\partial}\theta) \tag{4.85}$$

Now, use (A.16) to conclude that $(\epsilon\gamma^\mu\bar{\partial}\theta)(\theta\gamma_\mu\partial\theta) + (\epsilon\gamma^\mu\partial\theta)(\bar{\partial}\theta\gamma_\mu\theta) = -(\epsilon\gamma^\mu\theta)(\partial\theta\gamma_\mu\bar{\partial}\theta)$. Therefore:

$$(\epsilon\gamma^\mu\theta)(\partial\theta\gamma_\mu\bar{\partial}\theta) = -2(\epsilon\gamma^\mu\theta)(\partial\theta\gamma_\mu\bar{\partial}\theta) + \partial(\cdots) + \bar{\partial}(\cdots) \tag{4.86}$$

$$.:\ (\epsilon\gamma^\mu\theta)(\partial\theta\gamma_\mu\bar{\partial}\theta) = \frac{1}{3}\bar{\partial}\left((\epsilon\gamma^\mu\theta)(\partial\theta\gamma_\mu\theta)\right) + \partial(\cdots) \tag{4.87}$$

Follows then:

$$\delta\mathcal{L}_{PS} = \bar{\partial}\left(\frac{1}{24}(\epsilon\gamma^\mu\theta)(\partial\theta\gamma_\mu\theta) - \frac{1}{4}(\epsilon\gamma^\mu\theta)\partial X_\mu\right) = d\alpha \tag{4.88}$$

We get then the α:

$$\alpha = -\frac{1}{24}(\epsilon\gamma^{\mu}\theta)(\partial\theta\gamma_{\mu}\theta) + \frac{1}{4}(\epsilon\gamma^{\mu}\theta)\partial X_{\mu} \tag{4.89}$$

Now, we can compute the current:

$$q = \iota_{\xi}\gamma + \alpha \tag{4.90}$$

$$= -\left(\epsilon^{\alpha}p_{\alpha} + \frac{1}{2}(\epsilon\gamma^{\mu}\theta)\partial X_{\mu} + \frac{1}{24}(\epsilon\gamma^{\mu}\theta)(\partial\theta\gamma_{\mu}\theta)\right) \tag{4.91}$$

4.3 Superstrings in $AdS_5 \times S^5$

In order to describe string theory in a curved back-ground M, we will try to write M as a coset of Lie groups and then construct a WZW model. Specifically, we will use work in $M =$ super-$AdS_5 \times S^5$. As we announced in chapter 2, equation (2.2), this space is identified as a quotient of the supergroup $PSU(2,2|4)$ by the two Lorentz groups $SO(1,4) \times SO(5)$. There is some logic there, in fact the $PSU(2,2|4)$ contains $SU(2,2) \times SU(4)$ in its bosonic part, and as we can see in table (A), these are the Spin groups $Spin(2,4) \times Spin(6)$, then, isomorphic to $SO(2,4) \times SO(6)$, which are isometry groups to $AdS_5 \times S^5$. The groups $SO(1,4) \times SO(5)$ on the denominator of (2.2) are, in turn, the stabilizer groups for $AdS_5 \times S^5$. Follows then:

$$AdS_5 \times S^5 = \frac{SO(2,4) \times SO(6)}{SO(1,4) \times SO(5)} \tag{4.92}$$

Then, adding a fermionic part to the group on the numerator, we get the Super-$AdS_5 \times S^5$:

$$\frac{SU(2,2) \times SU(4)}{SO(1,4) \times SO(5)} \times \mathbb{C}^{0|32} \cong \frac{PSU(2,2|4)}{SO(1,4) \times SO(5)} \tag{4.93}$$

$$=: \text{Super-}AdS_5 \times S^5$$

As we could see in chapter 2, to construct WZW actions, we need to compute the currents:

$$\mathrm{J} = g^{-1}dg \tag{4.94}$$

Our curved superspace can be parametrized using supercoordinates $Z := \{X^m, \theta, \bar{\theta}\}$. A the superspace is a supermanifold, we can define cotangent spaces on it, with basis $\{dZ\}$. We may also define orthonormal basis $\{\mathrm{J}^A\}$:

$$\mathrm{J}^A = E^A_M dZ^M \tag{4.95}$$

4.3.1 Green-Schwartz superstring as a WZW model

We can realize superstring theory as a non-linear sigma model, as discussed in section (3.4). As we have discussed, we can describe $AdS_5 \times S^5$ background as a coset of groups. The same thing happens with flat background [7], as we wrote in equation (2.1):

$$\text{Flat Super-space}: \frac{SUSY(\mathcal{N}=2)}{SO(9,1)} \tag{4.96}$$

This is easy to justify also using theorem (2.2): SUSY acts on the flat super-space since it possesses the super-translations; and rotations are the isometry group.

This coset is generated by translation P_m and super-translation Q^A_α, for $A = 1, 2$, explained in example (2.16):

$$\{Q^A_\alpha, Q^B_\beta\} = -2i\delta^{AB}\gamma^m_{\alpha\beta}P_m \tag{4.97}$$

$$[P_m, P_n] = [Q^A_\alpha, P_n] = 0 \tag{4.98}$$

Therefore, the fields $g \in Map(\Sigma, M)$ from the world-sheet to the target space (4.96) are of the form:

$$g = e^{X^m P_m + \theta^{\alpha A} Q^A_\alpha} \tag{4.99}$$

In order to compute the Maurer-Cartan form:

$$\mathrm{J} = g^{-1}dg \tag{4.100}$$

we nee the following relation using Baker–Campbell–Hausdorff formula:

$$e^{-A}de^A = dA + \frac{1}{2}[dA, A] + \frac{1}{3!}[[dA, A], A] + \cdots \tag{4.101}$$

Using $A = X^m P_m + \theta^{\alpha A} Q^A_\alpha$, we have the:

$$[dX^m P_m + d\theta^{\alpha A} Q_\alpha^A, X^n P_n + \theta^{\beta B} Q_\beta^B] = -2i d\theta^{\alpha A} \Gamma^m_{\alpha\beta} \theta^{\beta B} P_m \tag{4.102}$$

follows:

$$\mathrm{J} = g^{-1} dg = (dX^m - id\theta\Gamma^m\theta) P_m + d\theta^{\alpha A} Q_{\alpha A} \tag{4.103}$$

In coordinates we have:

$$J^m = dX^m - id\theta\Gamma^m\theta \qquad\qquad J^{\alpha A} = d\theta^{\alpha A} \tag{4.104}$$

We see that J^m is just Π^m and the sigma model part will be just:

$$S_\sigma = \int \sqrt{h} h(J^m, J^n) G_{mn} \tag{4.105}$$

The topological term will be constructed the Maurer-Cartan 3-form:

$$\omega_3 = f_{ABC} J^A \wedge J^B \wedge J^C \tag{4.106}$$

with an appropriate choice for the structure constants f_{ABC}. J^m and $J^{\alpha A}$ are vectors and spinors under $SO(9,1)$, so if we consider Lorentz invariant structure constants, we will obtain a Lorentz invariant action. For it, consider as follows:

$$\omega_3 = i\mathfrak{s}_{AB} J^m \wedge J^{\alpha A} (\Gamma_m)_{\alpha\beta} \wedge J^{\beta B} \tag{4.107}$$

with $\mathfrak{s}_{11} = -\mathfrak{s}_{22} = 1$. We get then:

$$\omega_3 = i(dX^m - id\theta\Gamma^m\theta) \wedge \Big(d\theta^1 (\Gamma_m)_{\alpha\beta} \wedge d\theta^1 - d\theta^2 (\Gamma_m)_{\alpha\beta} \wedge d\theta^2 \Big) \tag{4.108}$$

$$= d\Big(-idX_m \wedge (\theta^1\Gamma^m d\theta^1 - \theta^2\Gamma^m d\theta^2) + \theta^1\Gamma^m d\theta^1 \wedge \theta^2\Gamma^m d\theta^2 \Big) = dB \tag{4.109}$$

We conclude that the Maurer-Cartan form is exact ($\omega_3 = dB$) and therefore, the WZ term is:

$$\int_{\hat{\Sigma}} g^*(\omega_3) = \int_{\hat{\Sigma}} dB = \int_{\partial\hat{\Sigma}} B = \int_\Sigma B \tag{4.110}$$

Which is just the topological term we added in order to recover the kappa symmetry.

$$S_{WZW} = \int \sqrt{h}h(\Pi^m, \Pi^n)G_{mn} + \int_{\hat{\Sigma}} g^*(\omega_3) = S_{GS} \tag{4.111}$$

Green-Schwartz in curved background

As we saw in chapter 2, the super Lie-algebra is $\mathbb{Z}_4-$gradded. Therefore, one can write the corresponding Maurer-Cartan formula as a decomposition:

$$\mathrm{J} = g^{-1}dg = J^{(0)} + J^{(1)} + J^{(2)} + J^{(3)} \tag{4.112}$$

We see that in the decomposition:

$$\mathfrak{psu}(2,2|4) = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3 \tag{4.113}$$

we have

$$\mathfrak{g}_0 = \mathfrak{so}(4,1) \oplus \mathfrak{so}(5) \tag{4.114}$$

and therefore, in order to describe fields in the coset, we have:

$$\mathrm{J} = J^{(1)} + J^{(2)} + J^{(3)} \tag{4.115}$$

for the remaining algebra, we have:

$$\mathfrak{m} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3 = \mathrm{span}\{T_a, T_{a'}, T_\alpha, T_{\hat{\alpha}}\} \tag{4.116}$$

with $a, a' = 1, \cdots, 5$ and $\alpha, \hat{\alpha} = 1, \cdots, 16$.

$$g = \exp(X^a T_a + X^{a'} T_{a'} + \theta^\alpha T_\alpha + \theta^{\hat{\alpha}} T_{\hat{\alpha}}) \tag{4.117}$$

It is remarkable the Maurer-Cartan identity:

$$dJ + J \wedge J = 0 \tag{4.118}$$

We will now consider the sigma model part of the WZW. We can't allow fermionic currents in the kinetic part because it would break kappa symmetry [8]. Therefore:

$$S_\sigma = \int \sqrt{g} g^{\alpha\beta} Str \left(J_\alpha^{(2)} J_\beta^{(2)} \right) \tag{4.119}$$

To construct the WZ term, we notice that the Maurer-Cartan form needs to be in the $\mathbb{Z}_4$ zero grading, the unique option would involve $J^{(1)} \wedge J^{(1)} \wedge J^{(2)}$ and $J^{(3)} \wedge J^{(3)} \wedge J^{(2)}$ and in order to be closed we have:

$$\omega_3 = str \left[J^{(1)} \wedge J^{(1)} \wedge J^{(2)} - J^{(3)} \wedge J^{(3)} \wedge J^{(2)} \right] \tag{4.120}$$

we see that by using the Maurer-Cartan relation (4.118) that:

$$\omega_3 = d \left(Str \left[J^{(1)} \wedge J^{(3)} \right] \right) =: d\omega_2 \tag{4.121}$$

because:

$$dJ^{(1)} + dJ^{(2)} + dJ^{(3)} = - \left(J^{(1)} + J^{(2)} + J^{(3)} \right) \wedge \left(J^{(1)} + J^{(2)} + J^{(3)} \right) \tag{4.122}$$

By preserving the grading:

$$dJ^{(1)} = -J^{(2)} \wedge J^{(3)} \tag{4.123}$$
$$dJ^{(3)} = -J^{(1)} \wedge J^{(2)} \tag{4.124}$$

Therefore:

$$d \left(Str \left[J^{(1)} \wedge J^{(3)} \right] \right) = Str \left[dJ^{(1)} \wedge J^{(3)} \right] - Str \left[J^{(1)} \wedge dJ^{(3)} \right] = \omega_3 \tag{4.125}$$

The EZ term reads:

$$\int_{\hat{\Sigma}} g^*(\omega_3) = \int_{\hat{\Sigma}} d\omega_2 = \int_{\Sigma} \omega_2 = \int_{\Sigma} Str \left[J^{(1)} \wedge J^{(3)} \right] \tag{4.126}$$

Follows then the GS action on the $AdS_5 \times S^5$ background:

$$S_{GS} = \int \sqrt{g} g^{\alpha\beta} Str \left(J_\alpha^{(2)} J_\beta^{(2)} \right) + c \int \epsilon^{\alpha\beta} Str \left[J_\alpha^{(1)} J_\beta^{(3)} \right] \tag{4.127}$$

4.3.2 Pure spinor in $AdS_5 \times S^5$

We can also formulate pure spinor strings in curved background with the same ideas we did for GS. However, since we have a ghost sector now, the coset have a slightly modification [9]:

$$PS \;\; AdS_5 \times S^5 \quad : \; \frac{C_L \times C_R \times PSU(2,2|4)}{SO(4,1) \times SO(5)} \tag{4.128}$$

with $C_{R,L}$ being the pure spinor cones defined by the constraints:

$$C_L : \; \{\lambda_L, \lambda_L\} = 0 \tag{4.129}$$

$$C_R : \; \{\lambda_R, \lambda_R\} = 0 \tag{4.130}$$

where $\{-,-\}$ means the Lie superalgebra anticommutator of $\mathfrak{g} = \mathfrak{psu}(2,2|4)$.

The action reads:

$$S_{PS} = \int d^2z \Big(\frac{1}{2} J_{2+} J_{2-} + \frac{3}{4} J_{1+} J_{3-} + \frac{1}{4} J_{3+} J_{1-} \\ + w_{1+} D_{0-} \lambda_3 + w_{3-} D_{0+} \lambda_1 - N_{0+} N_{0-} \tag{4.131}$$

where $\lambda_3 = \lambda_L$ and $\lambda_1 = \lambda_R$ and:

$$D_{o\pm} = \partial_\pm + [J_{0\pm}, -] \tag{4.132}$$

similar gauge transformations exists for $w_\pm$:

$$\delta_{v_2} w_{1+} = [v_{2+}, \lambda_3] \tag{4.133}$$

$$\delta_{u_2} w_{3-} = [u_{2-}, \lambda_1] \tag{4.134}$$

And the BRST like operator acts as [9]:

$$Q\lambda_1 = Q\lambda_2 = 0 \tag{4.135}$$

$$Qg = (\lambda_L + \lambda_R) g \tag{4.136}$$

$$Qw_{1+} = -J_{1+} \; , \; Qw_{3-} = -J_{3-} \tag{4.137}$$

Chapter 5

BV formalism

The Batalin-Vilkovisky formalism [25] is a generalization of BRST proceed to quantize gauge theories with introduction of anti-fields. The idea is that this formalism allows us to analyse the ghost sectors.

We may give some motivation for introducing anti-fields for a quantum field theory possessing a BRST operator Q by studying Lee-Zinn-Justin identities [11], an equivalent Ward identity involving the BRST symmetry acting on the partition function in a theory with fields ϕ and b, c ghosts:

$$\text{Partition Function: } Z[J] := \int \mathcal{D}\phi\mathcal{D}b\mathcal{D}c \, \exp\left[-S_{eff} + \int J\cdot\phi + \beta\cdot c + \gamma\cdot b\right] \tag{5.1}$$

Which should be invariant under $BRST$ transformation:

$$0 = \delta_{BRST} Z[J] = \int \mathcal{D}\phi\mathcal{D}b\mathcal{D}c \, (J\cdot Q\phi + \beta\cdot Qc + \gamma\cdot Qb)\, e^{-S_{eff}+\int J\cdot\phi+\beta\cdot c+\gamma\cdot b} \tag{5.2}$$

But it turns out that the BRST action on the fields may not be linear, and this in fact happens in many cases, for instance in YM $(Qc = [c, c])$. This difficult the work of writing the Ward identities for this quantum symmetry. The trick of Lee-Zinn-Justin was to introduce new sources K, L, F:

$$S_{extra\ source} = -\int \langle K, Q\phi\rangle + \langle L, Qc\rangle + \langle F, Qb\rangle \tag{5.3}$$

If the sources are BRST invariant $(QK = QL = QF = 0)$, we have that the new source action is BRST invariant. More that that, it does not afect our quantum theory since it is $Q-$exact:

$$S_{extra\ source} = \int Q(K\phi + Lc + Fb) \tag{5.4}$$

Doing this, the Ward identity can easily be written, with W such that $Z = e^{-W}$:

$$\int \left[J \cdot \frac{\delta}{\delta K} + \beta \cdot \frac{\delta}{\delta L} + \gamma \cdot \frac{\delta}{\delta F} \right] W[J, \beta, \gamma, K, L, F] = 0 \tag{5.5}$$

The so called Lee-Zinn-Justin identities are just equivalent identities for Γ, the Legendre transform of W.

Such new fields K, L, F are called the **anti-fields** for ϕ, c, b respectively. The new source action (5.3) together with S_{eff} will be the Master action. Such structure will be organized more properly and rigorously by the BV formalism.

5.1 Odd Symplectic geometry

In order to study the BV formalism, it is crucial to understand some basics of symplectic geometry. This happens because the space of fields in this approach will be a symplectic manifold.

Definition 5.1 (Symplectic Structures)**.** Consider a manifold M, a 2-tensor field $\omega \in \Omega^2(M)$ is a *section*:

$$\omega : M \longrightarrow \wedge^2 T^* M \tag{5.6}$$

that is, for each point $m \in M$, $\omega_m := \omega(m) \in \wedge^2 T^*_m M$.

$(T_m M, \omega_m)$ is a **symplectic vector space** if the anti-symmetric bilinear map:

$$\omega_m : T_m M \times T_m M \longrightarrow \mathbb{R} \tag{5.7}$$

is non-degenerate, that is, $\ker(\omega_m) := \{X \in T_m M \mid \omega_m(X, Y) = 0\,,\ \forall Y\} = \{0\}$.

Such (M, ω) is a **symplectic manifold** if $(T_m M, \omega_m)$ is a symplectic vector space for every $m \in M$ and ω is a closed form:

$$d\omega = 0 \tag{5.8}$$

It is not hard to see that every symplectic manifold have even dimension [17]. In fact, a simple example of symplectic vector space is given if we consider $M = \mathbb{R}^{2n}$ with:

$$\omega_0 = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \tag{5.9}$$

and every symplectic manifold is locally equivalent to this $(\mathbb{R}^{2n}, \omega_0)$. I wont discuss it in many detail, but this fact is illustrated in the following theorem:

Theorem 5.1 (Darboux)**.** *Let* (M, ω) *a* $2n-$*dimensional symplectic manifold, and let* $p \in M$*. Then, there exists a chart* $(\mathcal{U}; x_1, \cdots, x_n, y_1, \cdots, y_n)$ *centered in* $p \in \mathcal{U} \subset M$ *such that, in this coordinates:*

$$\omega\Big|_{\mathcal{U}} = \sum_{i=1}^{n} dx_i \wedge dy_i \tag{5.10}$$

Such charts are called Darboux charts, or ***Darboux coordinates****.*

Notice that in Darboux coordinates the symplectic structures ω_m looks like (5.9).

Important constructions of symplectic manifolds are given in the two subsequent examples:

Example 5.1 (Cotangent Bundle)**.** *Consider a manifold* M *and the corresponding cotangent bundle* $T^*M = \cup_{p \in M} T_pM$ *with* T^*_pM *defined as the dual vector space of the tangent spaces* T_pM*. Consider a chart* $\{x_\mu\}$ *on* M*, and* $\{p^\mu\}$ *a chart on* T^*M

We can construct the following symplectic manifold [17]:

$$(T^*M, \omega) \tag{5.11}$$

with $\omega = -d\alpha$*, where* α *is the canonical tautological form, in* (x, p) *coordinates:*

$$\alpha = p^\mu dx_\mu \tag{5.12}$$

That is:

$$\omega = dx_\mu \wedge dp^\mu \tag{5.13}$$

Example 5.2 (Odd Cotangent Bundle)**.** *Consider* M *a differential manifold and the following bundle*

$$\begin{tikzcd} \Pi T^*M \arrow[d, "\pi"] \\ M \end{tikzcd} \tag{5.14}$$

With ΠT^*M *the same space as* T^*M *but with changed parity of the fiber. And then we can consider* $(\Pi T^*M, \omega)$ *as a symplectic space with* ω *the same as* (5.13) *but then it is an odd symplectic form.*

Odd Poisson brackets

Let $\{x\}$ be bosonic coordinates of a system and $\{\theta\}$ fermionic coordinates.

Let F be a odd function, for example $F = x\,\theta$ or $G = \theta_1\theta_2\theta_3$. Then the differential operation doesn't changes if applied by left or by right:

$$\frac{\overrightarrow{\delta}}{\delta\theta}F = \frac{\overrightarrow{\delta}}{\delta\theta}(x\theta) = \frac{\overrightarrow{\delta}}{\delta\theta}(\theta x) = x \qquad \frac{\overrightarrow{\delta}}{\delta x}F = \theta \tag{5.15}$$

$$F\frac{\overleftarrow{\delta}}{\delta\theta} = (x\theta)\frac{\overleftarrow{\delta}}{\delta\theta} = x\,, \quad F\frac{\overleftarrow{\delta}}{\delta x} = (x\theta)\frac{\overleftarrow{\delta}}{\delta x} = (\theta x)\frac{\overleftarrow{\delta}}{\delta x} = \theta \tag{5.16}$$

With the function G, there are more cases, but to fix the idea I have done all:

$$\frac{\overrightarrow{\delta}}{\delta\theta_1}(\theta_1\theta_2\theta_3) = \theta_2\theta_3\,, \quad (\theta_1\theta_2\theta_3)\frac{\overleftarrow{\delta}}{\delta\theta_1} = (\theta_2\theta_3\theta_1)\frac{\overleftarrow{\delta}}{\delta\theta_1} = \theta_2\theta_3 \tag{5.17}$$

$$\frac{\overrightarrow{\delta}}{\delta\theta_2}(\theta_1\theta_2\theta_3) = \frac{\overrightarrow{\delta}}{\delta\theta_2}(-\theta_2\theta_1\theta_3) = -\theta_1\theta_3\,, \quad (\theta_1\theta_2\theta_3)\frac{\overleftarrow{\delta}}{\delta\theta_2} = (-\theta_1\theta_3\theta_2)\frac{\overleftarrow{\delta}}{\delta\theta_1} = -\theta_1\theta_3 \tag{5.18}$$

$$\frac{\overrightarrow{\delta}}{\delta\theta_3}(\theta_1\theta_2\theta_3) = \frac{\overrightarrow{\delta}}{\delta\theta_3}(\theta_3\theta_1\theta_2) = \theta_1\theta_2\,, \quad (\theta_1\theta_2\theta_3)\frac{\overleftarrow{\delta}}{\delta\theta_2} = \theta_1\theta_2 \tag{5.19}$$

Now let consider $H = \theta_1\theta_2$:

$$\frac{\overrightarrow{\delta}}{\delta\theta_1}(\theta_1\theta_2) = \theta_2, \quad (\theta_1\theta_2)\frac{\overleftarrow{\delta}}{\delta\theta_1} = (-\theta_2\theta_1)\frac{\overleftarrow{\delta}}{\delta\theta_1} = -\theta_2 \tag{5.20}$$

$$\frac{\overrightarrow{\delta}}{\delta\theta_2}(\theta_1\theta_2) = \frac{\overrightarrow{\delta}}{\delta\theta_2}(-\theta_2\theta_1) = -\theta_1\,, \quad (\theta_1\theta_2)\frac{\overleftarrow{\delta}}{\delta\theta_2} = \theta_1 \tag{5.21}$$

The secret is the following: when we are differentiating with respect to a fermionic variable θ_1, if the function is odd (has an odd number of fermions), we have to commute the θ_1 an even number of times both in the case of right or left derivative; if the function is even (has an even number of fermions), in one side we have to commute an even number of times but on the other side we have odd fermions to pass through. When differentiating with respect to bosons, all is equal, since bosons always commutes with anything.

Now I formulate the general rule: **left and right derivatives are the same, unless the function is bosonic and we are differentiating with respect to fermionic variable**:

$$F\overleftarrow{\partial_v} = (-1)^{v\cdot(F+1)}\,\overrightarrow{\partial}_v F \tag{5.22}$$

Now we are ready to formulate odd Poisson brackets:

$$\{F,G\} := \int d^d x \left(F \frac{\overleftarrow{\delta}}{\delta \phi^A} \frac{\overrightarrow{\delta}}{\delta \phi^*_A} G + (-1)^{(F+1)(G+1)+1} G \frac{\overleftarrow{\delta}}{\delta \phi^A} \frac{\overrightarrow{\delta}}{\delta \phi^*_A} F \right) \tag{5.23}$$

Quick notation:

$$\frac{\overleftarrow{\delta}}{\delta \phi^*_A} = \delta_* \,, \qquad \frac{\overleftarrow{\delta}}{\delta \phi^A} = \delta$$

Now look:

$$F \overleftarrow{\delta_*} \overrightarrow{\delta} G \tag{5.24}$$

$$= (-1)^{A_*(F+1)} \overrightarrow{\delta_*} F \overrightarrow{\delta} G \tag{5.25}$$

$$= (-1)^{A_*(F+1)+(F+A_*)(G+A)} \overrightarrow{\delta} G \overrightarrow{\delta_*} F \tag{5.26}$$

$$= (-1)^{A_*(F+1)+(F+A_*)(G+A)+A(G+1)} G \overleftarrow{\delta} \overrightarrow{\delta_*} F \tag{5.27}$$

To arrive in equation (5.26) I used the fact that the statistics of $\frac{\delta F}{\delta \phi}$ is $(\bar{F} + \bar{\phi})$. Lets work the exponent (mod 2)

$$A_*(F+1) + (F+A_*)(G+A) + A(G+1) \tag{5.28}$$

$$= F(A_* + A) + G(A_* + A) + FG + A + A_* + AA_* \tag{5.29}$$

$$= F + G + FG + 1 = (F+1)(G+1) \tag{5.30}$$

Therefore, $-F \overleftarrow{\delta_*} \overrightarrow{\delta} G = (-1)^{(F+1)(G+1)+1} G \overleftarrow{\delta} \overrightarrow{\delta_*} F$. And finally:

$$\{F,G\} = \int d^d x \;\; F \left(\frac{\overleftarrow{\delta}}{\delta \phi^A} \frac{\overrightarrow{\delta}}{\delta \phi^*_A} - \frac{\overleftarrow{\delta}}{\delta \phi^*_A} \frac{\overrightarrow{\delta}}{\delta \phi^A} \right) G \tag{5.31}$$

Odd Laplacian

$$\Delta := (-1)^{A+1} \int d^d x \, \frac{\delta}{\delta \phi^A(x)} \frac{\delta}{\delta \phi^*_A(x)} \tag{5.32}$$

Quick notation: $(-1)^{A_*} \Delta = \delta \cdot \delta_*$. Let us deduce an important relation:

$$\Delta(FG)$$

$$= (-1)^{A_*} \delta \cdot ((\delta_* F) G + (-1)^{F A_*} F (\delta_* G) \tag{5.33}$$

$$
\begin{aligned}
&= (-1)^{A_*}(\delta\cdot\delta_* F)G + (-1)^{FA+FA_*+A_*}F(\delta\cdot\delta_* G)\\
&+ (-1)^{(F+A_*)A+A_*}\cdot\delta_* F\delta G + (-1)^{FA_*+A_*}\delta F\cdot\delta_* G
\end{aligned}
\tag{5.34}
$$

Considering mod 2: $A + A_* = 1$, $AA_* = 0$, $A + 1 = A_*$ and $A_* + 1 = A$. Now:

$$
= (\Delta F)G + (-1)^F F(\Delta G) + (-1)^F \left[(-1)^{(F+1)A}\delta F\cdot\delta_* G + (-1)^{(F+1)A}\delta_* F\cdot\delta G\right] \tag{5.35}
$$

$$
= (\Delta F)G + (-1)^F F(\Delta G) + (-1)^F \left[F(\overleftarrow{\delta}\cdot\overrightarrow{\delta_*} - \overleftarrow{\delta_*}\cdot\overrightarrow{\delta})G\right] \tag{5.36}
$$

$$
= (\Delta F)G + (-1)^F F(\Delta G) + (-1)^F\{F, G\} \tag{5.37}
$$

5.2 General BV theories

To every Field Theory, one can consider its corresponding BV formulation. Even if the more significant application is in gauge theory, one can consider trivial gauge transformation to write BV action to any field theory. We will here introduce the BV theory showing the general formulation and then apply to YM and mainly String theory.

It's possible to define a general BV theory. Let us do it and then show some motivations.

Definition 5.2 (Classical BV Formalism)**.** A **BV theory** consists of a collection of the following data [24]:

- a $\mathbb{Z}-$gradded supermanifold $\mathcal{F}$;
- an odd-symplectic structure $\omega \in \Omega^2(\mathcal{F})$;
- A function $S_{BV} \in C^\infty(\mathcal{F})$ (the BV action) satisfying the *Classical Master Equation*(CME):

  $$\{S_{BV}, S_{BV}\} = 0 \tag{5.38}$$

  where $\{-,-\}$ is the poisson bracket generated by ω.

From the CME we can define the following nillpotent operator:

$$\mathcal{Q} := \{S_{BV}, -\} \tag{5.39}$$

This is the BV **operator** and in many cases it will coincide with the usual BRST operator. Moreover, by definition the BV operator a Hamiltonian vector field generated by S_{BV}:

$$\iota_{\mathcal{Q}}\omega = dS_{BV}s \tag{5.40}$$

This also shows that $\mathcal{Q}$ is compatible with the odd-symplectic structure:

$$L_{\mathcal{Q}}\omega = (\iota_{\mathcal{Q}} \circ d + d \circ \iota_{\mathcal{Q}})\omega = 0 \tag{5.41}$$

A typical construction of the BV phase space happens when we have a given space of fields $\mathcal{F}$ and extend it to $\mathcal{F} := \Pi T^*F$. In this case, $\mathcal{F}$ is naturally an odd-symplectic manifold, as seem in the previous sections, and in the presence of a BV action on this space, it will be a BV theory. In the next sections, we will show how to construct such an action from a BRST system and moreover, how we can understand the BRST operator as the de-Rham differential in the BV phase space.

Example 5.3. *Consider a gauge theory with action S_0, field space $\mathcal{F}$ and BRST operator Q. We construct a BV-theory by the following data:*

- *a $\mathbb{Z}_2-$gradded manifold $\mathcal{M} = \Pi T^*\mathcal{F}$;*
- *canonical odd-symplectic structure:*

  $$\omega = \delta\phi^A \wedge \delta\phi^*_A \in \Omega^2(\mathcal{M}) \tag{5.42}$$

  for $\{\phi^A, \phi^*_A\}$ a Darboux basis to the odd-cotangent bundle $\Pi T^*\mathcal{F}$;
- *The following Classical Master Action:*

  $$S_{BV} = S_0(\phi) + Q\phi^A\phi^*_A \tag{5.43}$$

We see that in this case, the BV operator $\mathcal{Q} = \{S_{BV}, -\}$ is just the BRST when restricted to $\mathcal{F}$. To see this, first of all notice that the BV-Poisson Bracket, the inverse of the odd-symplectic structure will act in the basis as:

$$\begin{aligned} \{\phi^A, \phi^B\} &= \{\phi^*_A, \phi^*_B\} = 0 \\ \{\phi^A, \phi^*_B\} &= \delta^A_B \end{aligned} \tag{5.44}$$

Therefore, acting on $\mathcal{F}$, we have:

$$\mathcal{Q}\phi^B = \{S_0(\phi) + Q\phi^A\phi^*_A, \phi^B\} = Q\phi^A \tag{5.45}$$

$$.: \mathcal{Q}\Big|_{\mathcal{F}} = Q \tag{5.46}$$

5.3 BV-BRST formalism

Let ΠTM be a super-tangent bundle on a n-dimensional manifold M. The symbol Π means that we change the parity of the elements of the fiber, therefore, since M is bosonic, this means that all T_mM are super-vector spaces ($\cong \mathbb{R}^{0|n}$). Denote $(x_1, \cdots, x_n, dx_1, \cdots, dx_n)$ with x_i coordinates on M (bosonic coordinates), and dx_i a notation for coordinates of the fiber (fermionic coordinates). We have therefore $\dim(\Pi TM) = n|n$.

Consider now a Lie-group H and the corresponding supermanifold ΠTH. The super-coordinates will be (h, dh). Consider the vector field on ΠTH:

$$d := dh^\mu \frac{\partial}{\partial h^\mu} \in \text{Vect}(\Pi TH) \tag{5.47}$$

Since we exchanged parity and dh is fermionic and h bosonic, we have that $d^2 = 0$:

$$d^2 = d \circ d = dh^\mu dh^\nu \frac{\partial}{\partial h^\mu}\frac{\partial}{\partial h^\nu} = 0 \tag{5.48}$$

Therefore, being nilpotent, it and produces some cohomology. It is in fact very similar to the De-Rham operator. Another observation is that $\forall\ p = (h, dh) \in \Pi TH$, the d-vector can be written as $d_p = (dh, 0)$.

In order to consider physical theories, let us take a given space of fields X. This manifold can be very hard to understand, but we abstract it for now. We suppose now that the group H we are considering acts on the space of fields:

$$\cdot : H \times X \longrightarrow X \tag{5.49}$$

$$(h, x) \longmapsto h \cdot x \tag{5.50}$$

We can join these two manifolds to form an important one: $\Pi TH \times X$, with coordinates $(h, dh, x) \in \Pi TM \times X$. We can then trivially extend the d field to the new space:

$$d = (dh, 0, 0) = dh^\mu \frac{\partial}{\partial h^\mu} \in \mathrm{Vect}(\Pi TH \times X) \tag{5.51}$$

Furthermore, with some identification, we arrive at some important manifold, isomorphic to $\mathfrak{h} \times X$. ($\mathfrak{h} = T_e H = \mathrm{Lie}(H)$):

$$\mathcal{M} = \frac{\Pi TH \times X}{H} \tag{5.52}$$

a quotient manifold $\Pi TH \times X/ \sim$, with $\sim$ defined as follows:

$$\begin{aligned} \phi_{h_0} : \; \Pi TH \times X &\longrightarrow \Pi TH \times X \\ (h, dh, x) &\longmapsto (hh_0^{-1}, dhh_0^{-1}, h_0 \cdot x) \end{aligned} \tag{5.53}$$

we identify then the points which are related by this action:

$$(h, dh, x) \sim (hh_0^{-1}, dhh_0^{-1}, h_0 \cdot x) \quad \forall h_0 \in H \tag{5.54}$$

In that case, we can always restrict our analysis to the subspace:

$$\frac{\Pi TH \times X}{H} \cong L = \{p \in \Pi TH \times X \mid p = (e, dh, x)\} \subset TH \times X \tag{5.55}$$

This follows because every point (h, dh, x) is equivalent to (e, dhh^{-1}, hx) and therefore:

$$\frac{\Pi T_e H \times X}{H} \cong L = \Pi\mathfrak{h} \times X \tag{5.56}$$

Definition 5.3. Given a field $V \in \mathrm{Vect}(\Pi TM \times X)$, it is called H-invariant if $X \circ \phi_{h_0} = T\phi_{h_0} \circ X \, , \forall h_0 \in H$

where $T\phi$ is the differential map

$$T\phi : TM \longrightarrow TM$$

Remark: When H is a matricial group, the function ϕ_{h_0} is just a matrix multiplication $\phi_{h_0}(h) = hh_0^{-1}$ and therefore $T\phi_{h_0} = \phi_{h_0}$.

Proposition 5.1. *The field $d \in \mathrm{Vect}(\Pi TH \times X)$ is H-invariant*

Proof. It is just a computation.

The field $d = dh^\mu \frac{\partial}{\partial h^\mu}$ is just:

$$d : H \times X' \longrightarrow \Pi TM \times X$$

$$(h, dh, x) \longmapsto (dh, 0, 0)$$

Therefore:

$$d \circ \phi_{h_0} : (h, dh, x) \xrightarrow{\phi_{h_0}} (hh_0^{-1}, dhd_0^{-1}, h_0 \cdot x) \xrightarrow{d} (dhh_0^{-1}, 0, 0) \tag{5.57}$$

and using the above remark:

$$T\phi_{h_0} \circ d : (h, dh, x) \xrightarrow{d} (dh, 0, 0) \xrightarrow{T\phi_{h_0}} (dhh_0^{-1}, 0, 0) \tag{5.58}$$

Concluding: $d \circ \phi_{h_0} = T\phi_{h_0} \circ d$ □

Lets take seriously the comment about doing everything on the point $(1, dh, x)$ and push the vector d to the space Vect$(\frac{\Pi TH \times X}{H}) \cong$ Vect$(\Pi\mathfrak{h} \times X)$

Pushforwards are defined as:

$$\phi : M \longrightarrow N \tag{5.59}$$

$$\phi_* : TM \longrightarrow TN$$

$$(\phi_*)V(f) := V(\phi \circ f) \quad \forall f \in C^\infty(N) \tag{5.60}$$

Given coordinates $\{x^\mu\}$ to M and $\{y^\mu\}$ to N:

$$(\phi_* V) = V^\mu \frac{\partial}{\partial x^\mu}(y^\nu \circ \phi)\frac{\partial}{\partial y^\nu} \tag{5.61}$$

We are going to do it with the map:

$$\phi : H \times X' \longrightarrow H \times X' \tag{5.62}$$

$$(h, dh, x) \mapsto (1, dhh^{-1}, h \cdot x)$$

And denote the resulting pushed field by $Q = \phi_* d$

Therefore:

$$V^\mu \frac{\partial}{\partial x^\mu}(y^\nu \circ \phi)\frac{\partial}{\partial y^\nu} = dh^\mu \frac{\partial}{\partial h^\mu}(1, dhh^{-1}, h \cdot x)^\nu \frac{\partial}{\partial y^\nu} \tag{5.63}$$

$$= dh^\mu \frac{\partial}{\partial h^\mu}(dhh^{-1})^a \frac{\partial}{\partial (dh)^a} + dh^\mu \frac{\partial}{\partial h^\mu}(h \cdot x)^i \frac{\partial}{\partial x^i}$$

$$= d(dhh^{-1})^a \frac{\partial}{\partial c^a} + c^\mu v^i_\mu \frac{\partial}{\partial x^i}$$

Denoting:

$$v^i_\mu = \frac{\partial}{\partial h^\mu}(h \cdot x)^i, \qquad dh = c$$

Now, comes the subtleties:

$$d(dhh^{-1}) = dhh^{-1} \wedge dhh^{-1} \tag{5.64}$$

It is also Lie-algebra valued, therefore, the wedge $\wedge$ has more structure, a commutations structure:

$$dh = t_b dh^b \tag{5.65}$$

$$dhh^{-1} \wedge dhh^{-1} = [dhh^{-1} \wedge dhh^{-1}] = \frac{1}{2} c^a c^b f^c_{ab} t_c \tag{5.66}$$

with f the structure constants:

$$[t_a, t_b] = f^c_{ab} t_c \tag{5.67}$$

Backing to equation (5.63), we finish the well defined field Q:

$$Q = \frac{1}{2} c^a c^b f^c_{ab} \frac{\partial}{\partial c^c} + c^a v^i_a \frac{\partial}{\partial x^i} = \frac{1}{2}[c,c]^\alpha c^*_\alpha + \langle c^a v_a(x), x^* \rangle \tag{5.68}$$

Now let me observe the following:

$$c^a v^\mu_a = dh^a \frac{\partial}{\partial h^a}(h \cdot x)^\mu = (dh \cdot x)^\mu = \delta_{dh} x^\mu = \delta_c x^\mu \tag{5.69}$$

And therefore, such Q operator is just the usual BRST operator [11] with fields $x \in X$:

$$Q = \frac{1}{2}[c,c]^\alpha c^*_\alpha + \langle \delta_c x, x^* \rangle = \frac{1}{2}[c,c]^a \frac{\partial}{\partial c^a} + \delta_c x^\mu \frac{\partial}{\partial x^\mu} \tag{5.70}$$

5.3.1 Integration

In order to produce physical observables, we consider generating function $Z[J]$. Therefore, we want to integrate functions defined on the gauge slice, which is equivalent to integrate H-invariant functions on the hole BV-phase space ($f(Z) = f(h \cdot Z)$). This connection is give by:

$$\int_{\Pi\mathfrak{h}\times X} f(c,x) := \int_{\Pi TH\times X} f(h,dh,x)\delta(h-1) \tag{5.71}$$

The integration we want to define on the BV-phase space is a path integral. Integration on the field space X is well know:

$$\mu = e^{-\frac{1}{\hbar}S_{cl}(x)}\mathcal{D}x \tag{5.72}$$

As an example, for Yang-Mills theory:

$$\mu = \mathcal{D}A_\mu \exp\left(-\frac{1}{\hbar}\frac{1}{4}\int \operatorname{tr} F_{\mu\nu}^2\right) \tag{5.73}$$

We want to define integration on ΠTH and this may be somewhat involved. But lets try to first to do something without much rigour, always keeping on mind that path integration is something very enigmatic on the mathematical point of view.

In this spirit, we want to make natural constructions from the given objects:

$$(X, H, \mu) \tag{5.74}$$

In order to construct such a natural notion of integration, consider H to be any supermanifold of dimension $m|n$. Than it has bosonic and fermionic coordinates: $(x^1, \cdots, x^m, \theta^1, \cdots, \theta^n)$. The operator d is a vector on the superbundle ΠTH:

$$d = d\theta^\alpha \frac{\partial}{\partial\theta^\alpha} + dx^a \frac{\partial}{\partial x^a} \tag{5.75}$$

The operator d is a nilpotent operator. To see it rewrite fermionic variables as $dx \to \psi$, $\partial/\partial\theta \to \eta$ and bosonic as $\partial/\partial x \to p$, $d\theta \to t$ and then, correspondingly

$$d = \psi^\mu p_\mu + t^\alpha \eta_\alpha \tag{5.76}$$

This is an odd nilpotent operator:

$$d^2 = (\psi^\mu p_\mu + t^\alpha\eta_\alpha)(\psi^\nu p_\nu + t^\beta\eta_\beta) = \psi^\mu\psi^\nu p_\nu p_\mu + \psi^\mu p_\mu t^\beta\eta_\beta + t^\alpha\eta_\alpha\psi^\nu p_\nu + t^\alpha t^\beta\eta_\alpha\eta_\beta \tag{5.77}$$

$$= p_\mu t^\beta\psi^\mu\eta_\beta - p_\nu t^\alpha\psi^\nu\eta_\alpha = 0 \tag{5.78}$$

Therefore have some cohomology, that is, the operator produce some exact sequence of algebras. The volume form is an operator which is the solution of the cohomology:

$$\begin{cases} d \circ \mathcal{D} = 0 \\ \mathcal{D} \sim \mathcal{D} + d \circ \mathcal{D}' \end{cases} \tag{5.79}$$

To solve it lets restrict ourselves to a polynomial algebra $P[\partial/\partial x, \partial/\partial\theta, dx, d\theta]$. Since $d = \psi^\mu p_\mu + t^\alpha \eta_\alpha$, we see that $\mathcal{D}$ has to be a polynomial of the form:

$$\mathcal{D} = \rho(x,\theta)\psi^1 \cdots \psi^n \eta^1 \cdots \eta^m \tag{5.80}$$

5.4 Space of Fields

Consider the space-time M with $\dim(M) = n$. It is a manifold in which our fields are defined.

We can define a super-manifold of dimension $2n$ given by a slight modification of the cotangent bundle TM:

$$\mathcal{M} = \Pi TM \tag{5.81}$$

the Π in front means that we've changed the parity of the fibers, that is, $\Pi T_m M \cong \mathbb{R}^{0|n}$. Locally, for some open sets $U \subset M$:

$$\mathcal{U} = \Pi TU \cong U \times \mathbb{R}^{0|n} \tag{5.82}$$

Given some charts, each point $m \in \mathcal{M}$ can be described in coordinates by:

$$m = (x^1, \cdots, x^n, \delta x^1, \cdots, \delta x^n) \tag{5.83}$$

where δx^j are coordinates in the vector fiber, that is, coordinates a vector $(\delta x^1, \cdots, \delta x^n) \in \Pi T_m M \cong \mathbb{R}^{0|n}$. This is just a convenient notation and not the differential of coordinate x^j. The gradding in this vector field implies:

$$\delta x^i \delta x^j = -\delta x^j \delta x^i \tag{5.84}$$

The space of functions $C^\infty(\mathcal{M})$ is then graded. Locally it is written as:

$$C^\infty(\mathcal{U})_0 = \{f \in C^\infty(\mathcal{U}) \mid f = \prod_{i=0}^{p\ even} f(x)_{\mu_1\cdots\mu_p}\delta x^{\mu_1}\cdots\delta x^{\mu_p},\ 0 \le p \le n\} \tag{5.85}$$

$$C^\infty(\mathcal{U})_1 = \{f \in C^\infty(\mathcal{U}) \mid f = \prod_{i=1}^{p\ odd} f(x)_{\mu_1 \cdots \mu_p} \delta x^{\mu_1} \cdots \delta x^{\mu_p},\ 0 \leq p \leq n\} \tag{5.86}$$

From this picture, we notice that smooth functions on ΠTM are just differential forms on M.

Since δx are fermions, we use the following measure to integrate functions on ΠTM:

$$\mu = dx^1 \wedge \cdots \wedge dx^n \frac{\partial}{\partial \delta x^1} \cdots \frac{\partial}{\partial \delta x^n} \tag{5.87}$$

The unique functions we are able to integrate are then of the form $F(\mathbf{x}, \delta \mathbf{x}) = f(\mathbf{x}) \delta x^1 \cdots \delta x^n$:

$$\int F = \int dx^1 \wedge \cdots \wedge dx^n f(\mathbf{x}) \tag{5.88}$$

We are also able to define tensor fields on $\mathcal{M}$, which are just sections on a given vector bundle E of $\mathcal{M}$:

$$\begin{tikzcd} E \arrow[d, "\pi"'] \\ \mathcal{M} \arrow[u, bend right, "s"'] \end{tikzcd}$$

For our purpose, we will consider a bundle with inverse parity:

$$E = T^p_{0|q} = \overbrace{T\mathcal{M} \otimes \cdots \otimes T\mathcal{M}}^{p} \ \otimes \ \overbrace{\Pi T^* \mathcal{M} \otimes \cdots \otimes \Pi T^* \mathcal{M}}^{q} \tag{5.89}$$

The fields are defined to be sections s:

$$s : \mathcal{M} \longrightarrow E \tag{5.90}$$

such that $\pi \circ s = \mathrm{id}$.

The expressions dx are, therefore, fermionic and $\frac{\partial}{\partial x}$ bosonic. Now, let us introduce some important operators:

$$d = dx^{\alpha}\frac{\partial}{\partial x^{\alpha}} : T^{p}_{q}(\mathcal{M}) \longrightarrow T^{p}_{q+1}(\mathcal{M}) \ - \ \text{Fermionic De Rham Operator} \tag{5.91}$$

$$\begin{aligned} \iota : T^{1}_{r}(\mathcal{M}) \times T^{p}_{q}(\mathcal{M}) &\longrightarrow T^{p}_{q+r-1}(\mathcal{M}) \ - \ \text{Iota Fermionic Operator} \\ (\, X \, , \, T \,) &\longmapsto X^{\alpha}_{\dots}\frac{\partial}{\partial dx^{\alpha}}T \end{aligned} \tag{5.92}$$

$$\begin{aligned} \mathcal{L} : T^{1}_{r}(\mathcal{M}) \times T^{p}_{q}(\mathcal{M}) &\longrightarrow T^{p}_{q+r}(\mathcal{M}) \ - \ \text{Bosonic Lie Derivative} \\ (\, X \, , \, T \,) &\longmapsto (\iota_X \circ d + (-1)^{\bar{X}} d \circ \iota_X)T \end{aligned} \tag{5.93}$$

Notice that this operators can also be seem as vector fields on ${}^{*}M$....
We notice some important identities for $T \in T^{r}_{p}$ and $S \in T^{s}_{q}$

$$d(TS) = dTS + (-1)^{\bar{T}+p}TdS \tag{5.94}$$

$$\iota_X(TS) = \iota_X TS + (-1)^{(\bar{X}+1)(\bar{T}+p)}T\iota_X S \tag{5.95}$$

$$\mathcal{L}_X(TS) = \mathcal{L}_X(T)S + (-1)^{\bar{X}(\bar{T}+p)}T\mathcal{L}_X(S) \tag{5.96}$$

5.5 BV action for bosonic string

Let us investigate more properly the symmetries present in bosonic string (4.4). It is invariant under diffeomorphisms in the worldsheet [16]:

$$\delta X^{\mu} = \mathcal{L}_v X^{\mu} \tag{5.97}$$

$$\delta g = \mathcal{L}_v g \tag{5.98}$$

with $v \in Vect(\Sigma)$ a vector field on the worldsheet that represents the infinitesimal diffeormorphism.

This condition in coordinates reads as:

$$\begin{aligned}
\delta X^\mu &= v^\alpha \partial_\alpha X^\mu && (5.99)\\
\delta g^{\alpha\beta} &= v^\gamma \partial_\gamma g^{\alpha\beta} - g^{\gamma\beta}\partial_\gamma v^\alpha - g^{\alpha\gamma}\partial_\gamma v^\beta && (5.100)\\
\delta\sqrt{g} &= \partial_\gamma(v^\gamma \sqrt{g}) && (5.101)
\end{aligned}$$

$$\begin{aligned}
\delta S = \int \partial_\mu(\sqrt{g}\, v^\mu) g^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X_\mu &+ (\sqrt{-g}\, v^\gamma)\partial_\gamma g^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X_\mu \\
&- \sqrt{-g}(\partial^\alpha v^\beta + \partial^\beta v^\alpha)\partial_\alpha X^\mu \partial_\beta X_\mu \\
&+ \sqrt{-g}(\partial^\beta v^\gamma)\partial_\gamma X^\mu \partial_\beta X_\mu + \sqrt{-g} g^{\alpha\beta} v^\gamma \partial_\gamma \partial_\alpha X^\mu \partial_\beta X_\mu \\
&+ \sqrt{-g}\partial_\alpha X^\mu (\partial^\alpha v^\gamma)\partial_\gamma X_\mu + \sqrt{-g} g^{\alpha\beta}\partial_\alpha X^\mu v^\gamma \partial_\gamma \partial_\beta X_\mu \\
&= \int \partial_\gamma \left(v^\gamma \sqrt{-g} g^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X_\mu \right) = 0 && (5.102)
\end{aligned}$$

We also have the Weyl symmetry:

$$g_{\alpha\beta} \longmapsto \Lambda g_{\alpha\beta} \tag{5.103}$$

This symmetry allows us to consider conformal structures, that is, equivalent classes of metrics related by Weyl transformations. In two dimension, there's a 1-1 correspondence between conformal transformations and complex structures, which are tensor fields such that $I^2 = -1$.

We can realize this by the working in coordinates:

$$g_{\alpha\beta} = \begin{pmatrix} a & c \\ c & b \end{pmatrix} \implies \sqrt{-g} g^{\alpha\beta} = \frac{1}{\sqrt{c^2 - ab}} \begin{pmatrix} b & -c \\ -c & a \end{pmatrix} \tag{5.104}$$

To see that the object $I^{\alpha\beta} = \sqrt{-g} g^{\alpha\beta}$ carries the information about the complex structure, we complexify the worldsheet by define complex coordinates:

$$\begin{aligned}
z &= \sigma^1 + i\sigma^2 && (5.105)\\
\bar{z} &= \sigma^1 - i\sigma^2 && (5.106)
\end{aligned}$$

In this coordinates the tensor density becomes:

$$I^{\alpha\beta} = \frac{1}{2\sqrt{c^2 - ab}} \begin{pmatrix} (a-b) - 2ic & (a+b) \\ (a+b) & (a-b) + 2ic \end{pmatrix} \tag{5.107}$$

And finally, since we are in complex coordinates, we use ϵ tensor to lower the index and then follows:

$$I^{\alpha}{}_{\gamma} I^{\gamma}{}_{\beta} = -\delta^{\alpha}_{\beta} \tag{5.108}$$

And now, we see that $I^2 = -1$ and therefore it is a complex structure. The action becomes:

$$\begin{aligned} S = \int d^2z \, I^{\alpha\beta} \partial_\alpha X \partial_\beta X &= \int d^2z \, \epsilon^{\beta\gamma} I^{\alpha}{}_{\gamma} \partial_\alpha X \partial_\beta X \\ &= \int dX \wedge I dX \end{aligned} \tag{5.109}$$

When we use the Polyakov gauge:

$$g_{\alpha\beta} = \eta_{\alpha\beta} \tag{5.110}$$

the action can be written as:

$$S = \int \epsilon^{\alpha\beta} \partial_\alpha X \partial_\beta X = \int dX \wedge \star dX \tag{5.111}$$

The Hodge star operator in two dimensions is an operator on the space $\Omega^0(\Sigma) \oplus \Omega^1(\Sigma) \oplus \Omega^2(\Sigma)$:

$$\star : \Omega^k(\Sigma) \longrightarrow \Omega^{2-k}(\Sigma) \tag{5.112}$$

for $k = 0, 1, 2$. On the $\Omega^1(\Sigma)$ subspace, $\star$ acts as a (1,1) tensor. Its action is defined on the basis as:

$$\star dx = dy \, , \qquad \star dy = -dx \tag{5.113}$$

Therefore, it can be represented as:

$$\star\big|_{\Omega^1(\Sigma)} = dy \otimes \frac{\partial}{\partial x} - dx \otimes \frac{\partial}{\partial y} \in T^1_1(\Sigma) \tag{5.114}$$

We see that the operator (5.114) squares to -1, and in matrix form reads:

$$\star\big|_{\Omega^1(\Sigma)} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} =: I^{(0)} \tag{5.115}$$

Now we have figured out which fields we will use: world-sheet scalar X^μ and complex structure I. Proceeding with the BRST procedure, we introduce c-ghosts for diffeomorphisms:

$$c = c^{\alpha} \frac{\partial}{\partial \sigma^{\alpha}} \tag{5.116}$$

with σ^{μ} worldsheet coordinates. The BRST operator will read:

$$QX = \mathcal{L}_c X \tag{5.117}$$

$$QI = \mathcal{L}_c I \tag{5.118}$$

$$Qc = \frac{1}{2}[c, c] \tag{5.119}$$

And therefore, we can construct the BV theory for the bosonic string, following example (5.3):

$$S_{BV} = \int dX \wedge IdX + \langle \mathcal{L}_c I, I^* \rangle + \langle \mathcal{L}_c X, X^* \rangle + \frac{1}{2}\langle [c, c], c^* \rangle \tag{5.120}$$

5.5.1 Field content for Bosonic String

Space-time as considered in section (5.4) for the String theory is $M = \Sigma$, the world-sheet, a two dimensional manifold, and then $\Pi T^*\Sigma$ has coordinates $(z, \bar{z}, \delta z, \delta \bar{z})$ with $\delta z \delta \bar{z} = -\delta \bar{z} dz$ and $dz^2 = \delta \bar{z}^2 = 0$.

In this case, we have that $C^{\infty}(\Pi T^*\Sigma)_0$ is composed of function of the form $f(z, \bar{z})$ and $f(z, \bar{z})\delta z \delta \bar{z}$, and $C^{\infty}(\Pi T^*\Sigma)_1$ are the functions of the form $f(z, \bar{z})\delta z$ and $f(z, \bar{z})\delta \bar{z}$ and the functions to be integrated are:

$$\int dz \wedge d\bar{z} \frac{\partial}{\partial \delta z} \frac{\partial}{\partial \delta \bar{z}} f(z, \bar{z}) \delta z \delta \bar{z} = \int dz \wedge d\bar{z} f(z, \bar{z}) \tag{5.121}$$

Considering the space of all fields $\mathcal{F}$, the idea of anti-fields emerges as we consider the tangent bundle $\Pi T^*\mathcal{F}$. In coordinates, for

$$(\phi, \phi^*) \in \Pi T^* \mathcal{F} \tag{5.122}$$

again, Π means change of parity.

In the case of bosonic string, the field content is:

$$X^\mu : \Pi T^*\Sigma \longrightarrow \mathbb{R} \; - \text{ World Sheet scalar field} \tag{5.123}$$

$$I^\alpha_{\ \beta} \frac{\partial}{\partial z^\alpha} \otimes dz^\beta : \Pi T^*\Sigma \longrightarrow \mathbb{R}^2 \otimes \mathbb{R}^{0|2} \; - \text{ Complex structure} \tag{5.124}$$

$$c^\alpha \frac{\partial}{\partial z^\alpha} : \Pi T^*\Sigma \longrightarrow \mathbb{R}^2 \; - \text{ Diffeomorphism ghost} \tag{5.125}$$

We define the fields such that the coordinates X^μ and $I^\alpha_{\ \beta}$ are bosonic fields and c^α is fermionic. The field $I = I^\alpha_{\ \beta} \frac{\partial}{\partial z^\alpha} \otimes dz^\beta$ is a fermion, since dz is a fermion.

The anti-fields are, therefore:

$$X^*_\mu \ \delta z \delta \bar{z} : \Pi T^*\Sigma \longrightarrow \mathbb{R} \tag{5.126}$$

$$I^{*\beta}_\alpha \frac{\partial}{\partial z^\beta} \otimes dz^\alpha \ \delta z \delta \bar{z} : \Pi T^*\Sigma \longrightarrow \mathbb{R}^2 \otimes \mathbb{R}^{0|2} \tag{5.127}$$

$$c^*_\alpha dz^\alpha \ \delta z \delta \bar{z} : \Pi T^*\Sigma \longrightarrow \mathbb{R}^2 \tag{5.128}$$

Follows that $X^*_\mu, I^{*\beta}_\alpha$ are fermions and c^*_a is a boson. $X^*_\mu \ \delta z \delta \bar{z}$ is a fermion since $\delta z \delta \bar{z}$ is bosonic; $I^* = I^{*\beta}_\alpha \frac{\partial}{\partial z^\beta} \otimes dz^\alpha \ \delta z \delta \bar{z}$ is bosonic since dz is a fermion and $c^*_\alpha dz^\alpha$ is a fermion.

5.5.2 Deformations of the Complex Structure

We saw that fixing Polyakov gauge for bosonic string corresponds to use $\star$ operator as the complex structure. We now want to deform a little bit the complex structure. To do so, let us parametrize such tensor:

$$I = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{5.129}$$

satisfying the constraint $I^2 + 1 = 0$:

$$\begin{pmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \tag{5.130}$$

In which as solution we have:

$$I = \begin{pmatrix} i\sqrt{1+bc} & b \\ c & -i\sqrt{1+bc} \end{pmatrix} \tag{5.131}$$

We see that there are 2 real degrees of freedom, or 1 complex dof. $m \in \mathbb{C}$:

$$I = \begin{pmatrix} i\sqrt{1+m\bar{m}} & im \\ -i\bar{m} & -i\sqrt{1+m\bar{m}} \end{pmatrix} \tag{5.132}$$

It follows that the Bosonic String action will read as:

$$\begin{aligned} S^m = -\frac{1}{4}\int_\Sigma dX \wedge I \cdot dX &= -\frac{1}{4}\int \left(\partial X dz + \bar{\partial} X d\bar{z}\right) \wedge \left(I^\alpha_{\ z}\partial_\alpha X dz + I^\alpha_{\ \bar{z}}\partial_\alpha X d\bar{z}\right) \\ &= -\frac{1}{4}\int \left(I^\alpha_{\ \bar{z}}\partial X \partial_\alpha X - I^\alpha_{\ z}\bar{\partial} X \partial_\alpha X\right) dz \wedge d\bar{z} \end{aligned} \tag{5.133}$$

$$= -\frac{1}{4}\int_\Sigma dz \wedge d\bar{z} \left(-2i\sqrt{1+m\bar{m}}\partial X \bar{\partial} X + im(\partial X)^2 + i\bar{m}(\bar{\partial} X)^2\right) \tag{5.134}$$

Using $dz \wedge d\bar{z} = -2id^2z$:

$$S^m = \int_\Sigma d^2z \left(\sqrt{1+m\bar{m}}\partial X \bar{\partial} X - \frac{m}{2}(\partial X)^2 - \frac{\bar{m}}{2}(\bar{\partial} X)^2\right) \tag{5.135}$$

5.5.3 Anti-field of the Complex structure: the ghost action

Notice that the field I^* is a density tensor:

$$I^* = I^{*\beta}_{\alpha\ \gamma\delta}\, dz^\alpha \otimes \frac{\partial}{\partial z^\beta} \otimes \left(dz^\gamma \wedge dz^\delta\right) \tag{5.136}$$

$$= dz \wedge d\bar{z}\; I^{*\beta}_{\alpha\ z\bar{z}}\, dz^\alpha \otimes \frac{\partial}{\partial z^\beta} \tag{5.137}$$

We will see that there is an interesting form on this anti-field using a gauge fixing.

Gauge symmetry on anti-fields

We have a constraint on the complex structure: $I^2 + 1 = 0$. The space of complex structures can be though as a submanifold $N \subset M$, where:

$$M = \{I^\alpha_{\ \beta}(\sigma) \mid \alpha, \beta = 1, 2\,;\ \sigma \in \Sigma\} \tag{5.138}$$

The submanifold N is defined as the kernel of the function F:

$$\begin{aligned} F : M &\longrightarrow M \\ I &\longmapsto I^2 + 1 \end{aligned}$$

The space of complex structures is defined as:

$$N = \{I \in M \mid F(I) = 0\} \subset M \tag{5.139}$$

As explained in Proposition 4.1, a Constraint on the Space of fields results in a gauge symmetry on the space of anti-fields, that is, a gauge symmetry on the cotangent bundle:

$$\delta_\eta I^\star = \eta \cdot \frac{\delta F}{\delta I} \tag{5.140}$$

for an odd tensor density field $\eta^{\alpha}_{\ \beta}$ (it is a density because $A^\star$ is a density), which in components read

$$(\delta_\eta I^\star)^{\ \alpha}_{\beta} = \eta^{\gamma}_{\ \delta}\frac{\delta F^{\alpha}_{\ \beta}}{\delta I^{\gamma}_{\ \delta}} \tag{5.141}$$

We have

$$\frac{\delta F^{\alpha}_{\ \beta}}{\delta I^{\alpha'}_{\ \beta'}} = \delta^{\alpha}_{\alpha'} I^{\beta'}_{\ \beta} + I^{\alpha}_{\ \alpha'}\delta^{\beta'}_{\beta}, \tag{5.142}$$

so that

$$(\delta_\eta I^\star)^{\ \alpha}_{\beta} = \eta^{\alpha'}_{\ \beta'}\frac{\delta F^{\alpha}_{\ \beta}}{\delta I^{\alpha'}_{\ \beta'}} = \eta^{\alpha}_{\ \gamma} I^{\gamma}_{\ \beta} + \eta^{\gamma}_{\ \beta} I^{\alpha}_{\ \gamma} \tag{5.143}$$

We now denote $I^* = b$, and in fact the anti-field of the complex structure will be the usual b-field [1] of the bosonic string:

$$\delta_\eta b = \eta \cdot \frac{\delta F}{\delta I} = \eta^{\alpha}_{\ \gamma}(I^{(0)})^{\gamma}_{\ \beta} + \eta^{\gamma}_{\ \beta}(I^{(0)})^{\alpha}_{\ \gamma} + \eta^{\alpha}_{\ \gamma}(b^*)^{\gamma}_{\ \beta} + \eta^{\gamma}_{\ \beta}(b^*)^{\alpha}_{\ \gamma} \tag{5.144}$$

with b^* the perturbation of the complex structure around $I^{(0)}$. Consider the space of complex structures:

$$\mathcal{C} = \{\ I^{\alpha}_{\ \gamma}(\sigma) \mid I^{\alpha}_{\ \gamma}(\sigma) I^{\gamma}_{\ \beta}(\sigma') = \delta^{\alpha}_{\ \beta}\delta(\sigma - \sigma')\ \} \tag{5.145}$$

If we want to construct the cotangent space $\Pi T^*\mathcal{C}$, we have an identification. In coordinates:

$$\Pi T^*\mathcal{C} = \Big\{(I, b) \mid I^2 + 1 = 0\ ;\ (I, b) \sim (I, b + \{\eta, I\})\Big\} \tag{5.146}$$

This identification follows from considering the canonical form α such that

$\omega = -d\alpha$:

$$\alpha = \mathrm{tr}(b\delta I) \to \mathrm{tr}(b\delta I) + \xi\,\mathrm{tr}(I\delta I) = \mathrm{tr}(b\delta I) \tag{5.147}$$

Because $I^2 + 1 = 0$, follows that $\delta I I + I\delta I = 0 \;\Rightarrow\; \mathrm{tr}(\delta I I + I\delta I) = 2\,\mathrm{tr}(I\delta I) = 0$.

The identification reads, in terms of our coordinates $(I = I^{(0)} + \mu)$:

$$(I, b) \sim (I, b + \{\eta, I^{(0)}\} + \{\eta, \mu\}) \tag{5.148}$$

The antifield I^* can be gauge fixed by noticing:

$$\{\eta, I^{(0)}\} = 2i \begin{pmatrix} \eta^z{}_z & 0 \\ 0 & \eta^{\bar{z}}{}_{\bar{z}} \end{pmatrix} \tag{5.149}$$

By imposing the condition $b^z{}_z = 0$ and $b^{\bar{z}}{}_{\bar{z}} = 0$, we fix the gauge:

$$I^* = \frac{1}{4}\begin{pmatrix} 0 & -b \\ \bar{b} & 0 \end{pmatrix} \tag{5.150}$$

$$I^* = dz \wedge d\bar{z}\; I^{*\bar{z}}_{z}{}_{z\bar{z}}\; dz \otimes \frac{\partial}{\partial \bar{z}} \;+\; dz \wedge d\bar{z}\; I^{*z}_{\bar{z}}{}_{z\bar{z}}\; d\bar{z} \otimes \frac{\partial}{\partial z} \tag{5.151}$$

$$= \frac{1}{4}\left(-b\, dz \otimes \frac{\partial}{\partial \bar{z}} \;+\; \bar{b}\, d\bar{z} \otimes \frac{\partial}{\partial z}\right) \otimes dz \wedge d\bar{z} \tag{5.152}$$

In this coordinates, we have:

$$\{-,-\} = \int d^2z \left(\frac{\overleftarrow{\delta}}{\delta I^{\alpha}_{\beta}}\frac{\overrightarrow{\delta}}{\delta I^{*\beta}_{\alpha}} - \frac{\overleftarrow{\delta}}{\delta I^{*\beta}_{\alpha}}\frac{\overrightarrow{\delta}}{\delta I^{\alpha}_{\beta}}\right) + \cdots \tag{5.153}$$

Which in $b, m, \bar{m}$ coordinates reads:

$$\{-,-\} = \int d^2z\, 4i \left(\frac{\overleftarrow{\delta}}{\delta m}\frac{\overrightarrow{\delta}}{\delta b} - \frac{\overleftarrow{\delta}}{\delta b}\frac{\overrightarrow{\delta}}{\delta m}\right) + 4i \left(\frac{\overleftarrow{\delta}}{\delta \bar{m}}\frac{\overrightarrow{\delta}}{\delta \bar{b}} - \frac{\overleftarrow{\delta}}{\delta \bar{b}}\frac{\overrightarrow{\delta}}{\delta \bar{m}}\right) + \cdots \tag{5.154}$$

Following:

$$\{m(z), b(w)\} = 4i\delta^2(z-w)\,,\;\; \{\bar{m}(z), \bar{b}(w)\} = 4i\delta^2(z-w) \tag{5.155}$$

$$\{m(z), \bar{b}(w)\} = 0\,,\;\; \{\bar{m}(z), b(w)\} = 0 \tag{5.156}$$

Ghost action S^g

We want to shou how $\langle \mathcal{L}_c I, I^* \rangle$ looks like in $b, c, m, \bar{m}$ coordinates. It will be called here by ghost action.

To write $\langle \mathcal{L}_c I, I^* \rangle$ explicitly, notice that given two (1,1) tensors, we can contract then:

$$A = A^{\alpha}{}_{\beta} dz^{\beta} \otimes \frac{\partial}{\partial x^{\alpha}} \qquad B = B^{\gamma}{}_{\delta} dz^{\delta} \otimes \frac{\partial}{\partial x^{\gamma}} \tag{5.157}$$

$$.:\langle A, B \rangle = A^{\alpha}{}_{\beta} B^{\gamma}{}_{\delta}\, dz^{\beta} \left(\frac{\partial}{\partial z^{\gamma}} \right) dz^{\delta} \left(\frac{\partial}{\partial z^{\alpha}} \right) = A^{\alpha}{}_{\beta} B^{\beta}{}_{\alpha} \tag{5.158}$$

$$S^g = \langle \mathcal{L}_c I, I^* \rangle = \int (\mathcal{L}_c I)^{\alpha}{}_{\beta} I^{*\beta}_{\alpha}{}_{z\bar{z}} dz \wedge d\bar{z} = \int (\mathcal{L}_c I)^{z}{}_{\bar{z}} I^{*\bar{z}}_{z}{}_{z\bar{z}} dz \wedge d\bar{z} + (\mathcal{L}_c I)^{\bar{z}}{}_{z} I^{*z}_{\bar{z}}{}_{z\bar{z}} dz \wedge d\bar{z}$$

$$= \int dz \wedge d\bar{z} \left(-\frac{1}{4} (\mathcal{L}_c I)^{z}{}_{\bar{z}} b + \frac{1}{4} (\mathcal{L}_c I)^{\bar{z}}{}_{z} \bar{b} \right) \tag{5.159}$$

Where:

$$(\mathcal{L}_c I)^{z}{}_{z} = ic \cdot \partial(\sqrt{1 + m\bar{m}}) + i(\bar{\partial} c)\bar{m} + i(\partial \bar{c})m \tag{5.160}$$

$$(\mathcal{L}_c I)^{z}{}_{\bar{z}} = ic \cdot \partial(m) + 2i(\bar{\partial} c)\sqrt{1 + m\bar{m}} + i(\bar{\partial}\bar{c} - \partial c)m \tag{5.161}$$

$$(\mathcal{L}_c I)^{\bar{z}}{}_{z} = -ic \cdot \partial(\bar{m}) - 2i(\partial \bar{c})\sqrt{1 + m\bar{m}} - i(\partial c - \bar{\partial}\bar{c})\bar{m} \tag{5.162}$$

$$(\mathcal{L}_c I)^{\bar{z}}{}_{\bar{z}} = -ic \cdot \partial(\sqrt{1 + m\bar{m}}) - i(\partial \bar{c})m - i(\bar{\partial} c)\bar{m} \tag{5.163}$$

Then, in terms of b, c and $m, \bar{m}$ coordinates:

$$S^g = -\frac{i}{4} \int dz \wedge d\bar{z} \left(c \cdot \partial m + 2(\bar{\partial} c)\sqrt{1 + m\bar{m}} + (\bar{\partial}\bar{c} - \partial c)m \right) b \tag{5.164}$$

$$+ \left(c \cdot \partial \bar{m} + 2(\partial \bar{c})\sqrt{1 + m\bar{m}} + (\partial c - \bar{\partial}\bar{c})\bar{m} \right) \bar{b}$$

The parenthesis term is a fermion, than we can pass the b field to the left hand side of this term with addition of a minus sign. We also separate the linear term in the $m$and$\bar{m}$ anti-field coordinates:

$$= \frac{i}{2} \int dz \wedge d\bar{z} \ \left(b\bar{\partial} c + \bar{b}\partial \bar{c} \right) \sqrt{1 + m\bar{m}} \tag{5.165}$$

$$+ \frac{i}{4} \int dz \wedge d\bar{z} \left(bc\partial m + b\bar{c}\bar{\partial} m + b(\bar{\partial}\bar{c} - \partial c)m \right) + \left(\bar{b}c\partial\bar{m} + \bar{b}\bar{c}\bar{\partial}\bar{m} + \bar{b}(\partial c - \bar{\partial}\bar{c})\bar{m} \right) \tag{5.166}$$

Using that $dz \wedge d\bar{z} = -2id^2z$ and integrating (5.166) by parts:

$$S^g = \int d^2z \; (b\bar{\partial}c + \bar{b}\partial\bar{c}) \sqrt{1+m\bar{m}} + \frac{1}{2}\Big(c\partial b + \bar{c}\bar{\partial}b + 2\partial c b\Big)m + \frac{1}{2}\Big(c\partial\bar{b} + \bar{c}\bar{\partial}\bar{b} + 2\bar{\partial}\bar{c}\bar{b}\Big)\bar{m} \tag{5.167}$$

Following the total action:

$$S_{BV} = S^m + S^g + \langle \mathcal{L}_c X, X^* \rangle + \frac{1}{2}\langle [c,c], c^* \rangle \tag{5.168}$$

The energy momentum tensor is the variation of the action with respect to the metric, or in this case the complex structure:

$$T = \frac{\delta S_{BV}}{\delta I} = \{S_{BV}, I^*\} \tag{5.169}$$

In components:

$$T = \{S_{BV}, b\} = \mathcal{Q}b \qquad \bar{T} = \{S_{BV}, \bar{b}\} = \mathcal{Q}\bar{b} \tag{5.170}$$

This is easily computed using relation (5.155) and (5.156): $\{m, b\} = \{\bar{m}, \bar{b}\} = 1$. Putting $m, \bar{m} = 0$, we got the usual energy momentum tensor [1]:

$$T^m = \frac{1}{2}(\partial X)^2\,, \qquad \bar{T}^m = -\frac{\bar{1}}{2}(\bar{\partial} X)^2 \tag{5.171}$$

$$T^g = \frac{1}{2}\Big(c\partial b + \bar{c}\bar{\partial}b + 2\partial c b\Big)\,, \qquad \bar{T}^g = \frac{1}{2}\Big(c\partial\bar{b} + \bar{c}\bar{\partial}\bar{b} + 2\bar{\partial}\bar{c}\bar{b}\Big) \tag{5.172}$$

and:

$$T = T^m + T^g\,, \qquad \bar{T} = \bar{T}^m + T^g \tag{5.173}$$

Such expressions for the energy momentum tensor are off-shell expressions, and in fact it corresponds to the usual one [1] if we impose on-shell condition, and in this case we just need $\partial\bar{b} = \bar{\partial}b = 0$.

5.6 Global symmetry and descent equations

To have a global symmetry means we have a generator of such symmetry written as

$$\hat{\Gamma} = \int_{z,\bar{z}} \Gamma(z,\bar{z},\phi,\phi^*) = \int \Gamma^\mu \phi^*_\mu + \Gamma^{\mu\nu}\phi^*_\mu\phi^*_\nu + \cdots \tag{5.174}$$

with the condition:

$$\{S_{BV}, \hat{\Gamma}\} = 0 \tag{5.175}$$

This equation has some information. First of all, it tell us about the symmetry on the classical action. But at first order in anti-fields, this equation tells us about the compatibility of such symmetry with the BRST symmetry, that is, their generators commutes.

Γ^μ for instance is a infinitesimal generator of the global symmetry:

$$\left\{\int \Gamma^\mu \phi^*_\mu, -\right\} = \int \left(\Gamma^\mu \frac{\delta}{\delta\phi^\mu} - \frac{\delta\Gamma^\mu}{\delta\phi^\nu}\phi^*_\mu \frac{\delta}{\delta\phi^*_\nu}\right) \xrightarrow[\phi^*=0]{} \int \Gamma^\mu \frac{\delta}{\delta\phi^\mu} \tag{5.176}$$

Indeed, expanding the BV action in anti-fields:

$$S_{BV} = S_0 + S_1^\mu \phi^*_\mu + S_2^{\mu\nu}\phi^*_\mu\phi^*_\nu + \cdots \tag{5.177}$$

Where each S_n only depends on fields ϕ and not anti-fields. In general, the first order term S_1 is the BRST operator. At first order, equation (5.175) is:

$$\int \Gamma^\mu \frac{\delta S_0}{\delta\phi^\mu} = 0 \tag{5.178}$$

This equation means that the generator Γ is a symmetry generator of the action S_0. In second order:

$$\int \left(Q^a \frac{\overleftarrow{\delta}}{\delta\phi^b}\Gamma^b - Q^b \frac{\overrightarrow{\delta}}{\delta\phi^b}\Gamma^a\right)\phi^*_a = \int [Q,\Gamma] = 0 \tag{5.179}$$

This commutation relation tells us that the BRST and the global symmetry are compatible. For instance, if the cohomology does not change if we apply a global symmetry.

For instance, translation symmetry of BV-bosonic string is:

$$X^\mu \longmapsto X^\mu + a^\mu \tag{5.180}$$

Where a^μ is a constant vector. The generator of such symmetry is $\partial/\partial X^\mu$, and in terms of anti-fields:

$$\hat{\Gamma} = a^\mu \int_\Sigma X^*_\mu \tag{5.181}$$

$$\left\{ a^\mu \int_\Sigma X^*_\mu, - \right\} = translation \tag{5.182}$$

The classical part of the BV action is just the Polyakov action, therefore, the translation symmetry in this order in anti-fields reads as:

$$a^\mu \int \frac{\delta S_{pol}}{\delta X^\mu} = a^\mu \int d\iota_I dX_\mu = 0 \tag{5.183}$$

Another symmetry is the rotational one:

$$X^\mu \longmapsto X^\mu + \omega_\nu{}^\mu X^\nu \tag{5.184}$$

with ω an anti-symmetric matrix.

The generator is:

$$\hat{\Gamma} = \int_\Sigma \omega_\mu{}^\nu X^\mu X^*_\nu \tag{5.185}$$

$$\left\{ \int_\Sigma \omega_\mu{}^\nu X^\mu X^*_\nu, - \right\} = rotation \tag{5.186}$$

Similarly, we have the condition on the polyakov action:

$$\int \omega_\mu{}^\nu X^\mu \frac{\delta S_{pol}}{\delta X^\nu} = \int \omega_{\mu\nu} X^\mu d\iota_I dX^\nu = \int d\left(\omega_{\mu\nu} X^\mu \iota_I dX^\nu\right) = 0 \tag{5.187}$$

where we have integrated by part and we use symmetry of the operator $\wedge\iota_I$, that is, $\omega_{\mu\nu} dX^\mu \wedge \iota_I dX^\nu = 0$.

Such expressions as (5.183) and (5.187) are generalized when we go to BV-phase space:

$$\{S_{BV}, \hat{\Gamma}\} = \int \{S_{BV}, \Gamma\} = \int S_{BV} \frac{\overleftarrow{\delta}}{\delta\phi^A} \frac{\overrightarrow{\delta}}{\delta\phi^*_A} \Gamma - S_{BV} \frac{\overleftarrow{\delta}}{\delta\phi^*_A} \frac{\overrightarrow{\delta}}{\delta\phi^A} \Gamma = 0 \tag{5.188}$$

Since the equality is under an integration, we are able to write a corresponding equation:

$$\{S_{BV}, \Gamma\} = dj \tag{5.189}$$

j is similar to the Noether current, since $\{S_{BV}, \Gamma\}$ in zeroth order in anti-fields is just the variation of the action with respect to the symmetry $\hat{\Gamma}$ generated by Γ

and each symmetry gives:

$$\delta S = E.L. + \int \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \delta\phi_a - \left(\frac{\delta \mathcal{L}}{\partial(\partial_\mu \phi_a)} - \eta_{\mu\nu}\mathcal{L} \right) \delta x^\nu \right) \tag{5.190}$$

$E.L.$ meas Euler Lagrange equations, therefore, on-shell, every symmetry is of the form

$$\delta S = dj \tag{5.191}$$

with j being the conserved current. Therefore, equation (5.189) generalizes the Noether theorem expressed in (5.191) above.

Now, j is also a generator of symmetry, that is:

$$\{S_{BV}, j\} = dV \tag{5.192}$$

This is called descent equations:

$$\mathcal{Q}\alpha_0 = d\alpha_1 \tag{5.193}$$

$$\mathcal{Q}\alpha_1 = d\alpha_2 \tag{5.194}$$

$$\cdots \tag{5.195}$$

$$\mathcal{Q}\alpha_n = d\alpha_{n+1} \tag{5.196}$$

$$\begin{tikzcd}
\alpha_0 \arrow[r, "\mathcal{Q}"] & d\alpha_1 \arrow[r, "\mathcal{Q}"] & 0 & & & \\
& \alpha_1 \arrow[u, "d"] \arrow[r, "\mathcal{Q}"] & d\alpha_2 \arrow[u, "d"] \arrow[r, "\mathcal{Q}"] & 0 & & \\
& & \alpha_2 \arrow[u, "d"] \arrow[r, "\mathcal{Q}"] & d\alpha_3 \arrow[u, "d"] \arrow[r, "\mathcal{Q}"] & 0 & \\
& & & \alpha_3 \arrow[u, "d"] \arrow[r, "\mathcal{Q}"] & d\alpha_3 \arrow[u, "d"] \arrow[r, "\mathcal{Q}"] & \cdots \\
& & & & \cdots \arrow[u, "d"] &
\end{tikzcd}$$

where $\mathcal{Q} := \{S_{BV,-}\}$. Observe that such operator can be writen as:

$$\mathcal{Q} = \int_z S_{BV} \frac{\overleftarrow{\delta}}{\delta\phi^\mu(z)} \frac{\overrightarrow{\delta}}{\delta\phi^*_\mu(z)} - S_{BV} \frac{\overleftarrow{\delta}}{\delta\phi^*_\mu(z)} \frac{\overrightarrow{\delta}}{\delta\phi^\mu(z)} \tag{5.197}$$

It is a fermionic operator, since S_{BV} is bosonic and since ϕ^* has the opposite parity as ϕ, $\left(\frac{\overleftarrow{\delta}}{\delta\phi^\mu}\frac{\overrightarrow{\delta}}{\delta\phi^*_\mu} - \frac{\overleftarrow{\delta}}{\delta\phi^*_\mu}\frac{\overrightarrow{\delta}}{\delta\phi^\mu}\right)$ is fermionic. Indeed, if ϕ is a fermion, $S_{BV}\frac{\overleftarrow{\delta}}{\delta\phi^\mu}$ is now a fermion. And if ϕ is a boson, $\frac{\overrightarrow{\delta}}{\delta\phi^*_\mu}$ is a fermionic operator and then in both cases, $\left(S_{BV}\frac{\overleftarrow{\delta}}{\delta\phi^\mu}\right)\frac{\overrightarrow{\delta}}{\delta\phi^*_\mu}$ is a fermionic operator. We use the same argument to conclude that $\left(S_{BV}\frac{\overleftarrow{\delta}}{\delta\phi^*_\mu}\right)\frac{\overrightarrow{\delta}}{\delta\phi^\mu}$ is a fermionic operator.

5.6.1 Descent equations for Bosonic string

Some symmetries are remarkable in Bosonic String Theory. We have already introduced translation and rotational symmetries, and here we will explore it more, and show all the explicit descent equations that appears in our BV approach. Furthermore, we will explore also the BRST symmetry.

Translation symmetry

Let us write the descent equations for translation symmetry in BV-bosonic string. The generator of such symmetry is:

$$\left\{a^\mu \int_\Sigma X^*_\mu, -\right\} = translation \tag{5.198}$$

Applying the BV operator in such generator:

$$QX^* = \{S_{BV}, X^*_\mu(w)\} \tag{5.199}$$

$$= \int_z S_{BV}\frac{\overleftarrow{\delta}}{\delta X^\nu(z)}\frac{\overrightarrow{\delta}}{\delta X^*_\nu(z)}X^*_\mu(w) = S_{BV}\frac{\overleftarrow{\delta}}{\delta X^\nu(w)} = \frac{\overrightarrow{\delta}}{\delta X^\nu(w)}S_{BV} \tag{5.200}$$

In the last equality use the fact that $\frac{\delta}{\delta X^\mu}$ is bosonic.

$$= \frac{1}{2}\frac{\overrightarrow{\delta}}{\delta X^\nu(w)}\int dX^\mu \iota_I dX_\mu + \frac{\overrightarrow{\delta}}{\delta X^\nu(w)}\int \mathcal{L}_c X^\mu X^*_\mu \tag{5.201}$$

$$= \frac{1}{2}\int_z d\delta(z-w)\iota_I dX_\nu + \frac{1}{2}\int dX_\nu \iota_I d\delta(z-w) + \int_z \mathcal{L}_c(\delta(z-w))X^*_\nu \tag{5.202}$$

Now use the symmetry of the operator $\wedge\iota_I$ in order to identify the two first terms above and then:

$$= \int_z d\delta(z-w)\iota_I dX_\nu + \int_z \mathcal{L}_c(\delta(z-w))X^*_\nu \tag{5.203}$$

$$= \int_z d\big(\delta(z-w)\iota_I dX_v\big) + \int_z \mathcal{L}_c\big(\delta(z-w)X_v^*\big) - d\iota_I dX_v(w) - \mathcal{L}_c X^*(w) \tag{5.204}$$

in the last line we use integration by parts. Now we see that since X^* is a topo form, $\mathcal{L}_c\big(\delta(z-w)X_v^*\big) = -d\iota_c\big(\delta(z-w)X_v^*\big)$ and then, when we integrate, the two first terms vanishes. And the result follows:

$$= -d\iota_I dX_v - \mathcal{L}_c X_v^* = d\left(-\iota_I dX_v + \iota_c X_v^*\right) =: dV \tag{5.205}$$

The next step is:

$$\mathcal{Q}V = -\{S_{BV}, \iota_I dX\} + \{S_{BV}, \iota_c X^*\} \tag{5.206}$$

First, let us compute the first term:

$$-\{S_{BV}, \iota_I dX\} = -\{S_{BV}, \iota_I\}dX + \iota_I d\{S_{BV}, X\} \tag{5.207}$$

$$= \iota_{\mathcal{L}_c I} dX - \iota_I d\mathcal{L}_c X = \mathcal{L}_c(\iota_I dX) \tag{5.208}$$

$$= \iota_c d\iota_I dX - d\iota_c\iota_I dX \tag{5.209}$$

Now, the second term:

$$\{S_{BV}, \iota_c X^*\} = \{S_{BV}, \iota_c\}X^* + \iota_c\{S_{BV}, X^*\} \tag{5.210}$$

$$= -\frac{1}{2}\iota_{[c,c]}X^* - \iota_c d\iota_I dX + \iota_c d\iota_c X^* \tag{5.211}$$

The terms with anti-fields are:

$$-\frac{1}{2}\iota_{[c,c]}X^* + \iota_c d\iota_c X^* = -\frac{1}{2}\iota_{\mathcal{L}_c c}X^* - \iota_c\mathcal{L}_c X^* = -\frac{1}{2}\mathcal{L}_c(\iota_c X^*) - \frac{1}{2}\iota_c\mathcal{L}_c X^* \tag{5.212}$$

$$= \frac{1}{2}d\left(\iota_c^2 X^*\right) - \frac{1}{2}\iota_c d\iota_c X^* + \frac{1}{2}\iota_c d\iota_c X^* = \frac{1}{2}d\left(\iota_c^2 X^*\right) \tag{5.213}$$

Follows:

$$\mathcal{Q}V = -\{S_{BV}, \iota_I dX\} + \{S_{BV}, \iota_c X^*\} = d\left(-\iota_c\iota_I dX + \frac{1}{2}\iota_c^2 X^*\right) =: dU \tag{5.214}$$

Systematically:

$$\begin{tikzcd}
X^* \arrow[r, "\mathcal{Q}"] & dV \arrow[r, "\mathcal{Q}"] & 0 & \\
 & V \arrow[u, "d"] \arrow[r, "\mathcal{Q}"] & dU \arrow[u, "d"] \arrow[r, "\mathcal{Q}"] & 0 \\
 & & U \arrow[u, "d"] \arrow[r, "\mathcal{Q}"] & 0 \arrow[u, "d"]
\end{tikzcd}$$

$$R = X^* \,, \qquad V = -\iota_I dX + \iota_c X^* \,, \qquad U = -\iota_c \iota_I dX + \frac{1}{2}\iota_c^2 X^* \tag{5.215}$$

It is remarkable that, as explained in general words, the expression for V is the translation current $\star dX = \partial X - \bar{\partial} X$, with the anti-field correction $\iota_c X^*$.

Notice that the ghost number of each expression is:

$$\mathrm{gh}(R) = -1 \tag{5.216}$$

$$\mathrm{gh}(V) = 0 \tag{5.217}$$

$$\mathrm{gh}(R) = 1 \tag{5.218}$$

Observe also that R, V, U are fermions: X^* is a fermion since is the anti-field of X, which is a boson. Operators ι_I and ι_c are bosons since ι is a fermion and I, c are also fermions. Following that $\iota_c X^*$ and $\iota_c^2 X^*$ are fermions; since d is fermionic, $\iota_I dX$ and $\iota_c \iota_I dX$ are also fermions. Therefore the following formal sum has a well defined fermionic statistics:

$$\mathcal{J} = R + V + U \tag{5.219}$$

From the descent equations follows:

$$\mathcal{Q}\mathcal{J} = d\mathcal{J} \tag{5.220}$$

With $\mathcal{Q} = \{S_{BV}, -\}$, which is a fermionic operator since

Rotational symmetry

The generator of rotational symmetry is:

$$\left\{ \omega^{\mu\nu} \int_{\Sigma} X_\mu X_\nu^*, - \right\} = \mathit{translation} \tag{5.221}$$

The infinitesimal generator of symmetry is $\Gamma = \omega^{\mu\nu} X_\mu X_\nu^*$. We use a short notation, $\Gamma = \omega(X, X^*)$.

Running the descent procedure:

$$\mathcal{Q}\left(\omega^{\mu\nu}X_\mu X_\nu^*\right)=\omega^{\mu\nu}(\mathcal{Q}X_\mu)X_\nu^*+\omega^{\mu\nu}X_\mu(\mathcal{Q}X_\nu^*) \tag{5.222}$$

$$=-\omega^{\mu\nu}\mathcal{L}_cX_\mu X_\nu^*-\omega^{\mu\nu}X_\mu d\iota_I dX_\nu-\omega^{\mu\nu}X_\mu(\mathcal{L}_cX_\nu^*) \tag{5.223}$$

$$=-\mathcal{L}_c\left(\omega^{\mu\nu}X_\mu X_\nu^*\right)-d\left(\omega^{\mu\nu}X_\mu\iota_I dX_\nu\right) \tag{5.224}$$

where we have used Leibniz rule for Lie derivative to write $\mathcal{L}_c(X_\mu X_\nu^*)=\mathcal{L}_cX_\mu X_\nu^*+X_\mu\mathcal{L}_cX_\nu^*$; and also the anti-symmetry of ω and symmetry of $\wedge\iota_i$ in order to realize that $\omega^{\mu\nu}dX_\mu\iota_I dX_\nu=0$ and then we see that:

$$d\left(\omega^{\mu\nu}X_\mu\iota_I dX_\nu\right)=\omega^{\mu\nu}dX_\mu\iota_I dX_\nu+\omega^{\mu\nu}X_\mu d\iota_I dX_\nu=\omega^{\mu\nu}X_\mu d\iota_I dX_\nu \tag{5.225}$$

Now, using $\mathcal{L}_c=\iota_c\circ d-d\circ\iota_c$ we see:

$$\mathcal{Q}(\omega^{\mu\nu}X_\mu X_\nu^*)=d\left(-\omega^{\mu\nu}X_\mu\iota_I dX_\nu+\iota_c\omega^{\mu\nu}X_\mu X_\nu^*\right) \tag{5.226}$$

In short notation:

$$\mathcal{Q}(\omega(X,X^*))=d\left[-\omega(X,\iota_I dX)+\iota_c\omega(X,X^*)\right]:=dV \tag{5.227}$$

The next step:

$$\mathcal{Q}V=-\mathcal{Q}\big(\omega(X,\iota_I dX)\big)+\mathcal{Q}\big(\iota_c\omega(X,X^*)\big) \tag{5.228}$$

The term without anti-fields is just:

$$-\mathcal{Q}\big(\omega(X,\iota_I dX)\big)=-\omega((\mathcal{Q}X),\iota_I dX)-\omega(X,\mathcal{Q}(\iota_I dX)) \tag{5.229}$$

The very first term is just:

$$-\omega((\mathcal{Q}X),\iota_I dX)=\omega(\iota_c dX,\iota_I dX) \tag{5.230}$$

The second term we have almost computed already. It is on equations (5.207), (5.208) and (5.209). Follows:

$$-\omega(X,\mathcal{Q}(\iota_I dX))=\omega(X,\iota_c d\iota_I dX)-\omega(X,d\iota_c\iota_I dX) \tag{5.231}$$

Now, we use Leibniz rule and magic Cartan formula for Lie derivative to realize that the combination of (5.230) with (5.231) is just a Lie derivative. First, use $\mathcal{L}_c = \iota_c \circ d - d \circ \iota_c$ and $\mathcal{L}_c X = d\iota_c X$ to write:

$$-\mathcal{Q}(\omega(X, \iota_I dX)) = \omega(\iota_c dX, \iota_I dX) + \omega(X, \iota_c d\iota_I dX) - \omega(X, d\iota_c\iota_I dX) \tag{5.232}$$

$$= \omega(\mathcal{L}_c X, \iota_I dX) + \omega(X, \mathcal{L}_c \iota_I dX) \tag{5.233}$$

Now the Leibniz rule gives us:

$$= \mathcal{L}_c \omega(X, \iota_I dX) = \iota_c d\omega(X, \iota_I dX) - d\iota_c \omega(X, \iota_I dX) \tag{5.234}$$

The term with anti-fields are:

$$\mathcal{Q}\big(\iota_c \omega(X, X^*)\big) = \{S_{BV}, \iota_c \omega(X, X^*)\} = \{S_{BV}, \iota_c\}\omega(X, X^*) + \iota_c\{S_{BV}, \omega(X, X^*)\} \tag{5.235}$$

$$= -\frac{1}{2}\iota_{[c,c]}\omega(X, X^*) + \iota_c\{S_{BV}, \omega(X, X^*)\} \tag{5.236}$$

The second term we already know how to compute, and it is $\iota_c dV$, as we can see by equation (5.227). Follows now:

$$-\frac{1}{2}\iota_{\mathcal{L}_c c}\omega(X, X^*) + \iota_c d\iota_c \omega(X, X^*) - \iota_c d\omega(X, \iota_I dX) \tag{5.237}$$

$$= -\frac{1}{2}\iota_{\mathcal{L}_c c}\omega(X, X^*) - \frac{1}{2}\iota_c \mathcal{L}_c \omega(X, X^*) - \frac{1}{2}\iota_c \mathcal{L}_c \omega(X, X^*) - \iota_c d\omega(X, \iota_I dX) \tag{5.238}$$

$$= -\frac{1}{2}\mathcal{L}_c\left(\iota_c \omega(X, X^*)\right) - \frac{1}{2}\iota_c \mathcal{L}_c \omega(X, X^*) - \iota_c d\omega(X, \iota_I dX) \tag{5.239}$$

In the second line it was used the fact that $\omega(X, X^*)$ is a top form and then $\mathcal{L}_c \omega(X, X^*) = d\iota_c \omega(X, X^*)$. In the third line, we use Leibniz rule for Lie derivatives.

$$\mathcal{Q}V = -\mathcal{Q}\big(\omega(X, \iota_I dX)\big) + \mathcal{Q}\big(\iota_c \omega(X, X^*)\big) \tag{5.240}$$

$$= \mathcal{L}_c \omega(X, \iota_I dX) - \iota_c d\omega(X, \iota_I dX) - \frac{1}{2}\mathcal{L}_c\left(\iota_c \omega(X, X^*)\right) - \frac{1}{2}\iota_c \mathcal{L}_c \omega(X, X^*) \tag{5.241}$$

$$= d\Big(-\iota_c \omega(X, \iota_I dX)\Big) - \frac{1}{2}\left(\iota_c \mathcal{L}_c + \mathcal{L}_c \iota_c\right)\omega(X, X^*) \tag{5.242}$$

In the second term we have:

$$\iota_c \mathcal{L}_c + \mathcal{L}_c \iota_c = -\iota_c d\iota_c + \iota_c^2 d - d\iota_c^2 + \iota_c d\iota_c = \iota_c^2 d - d\iota_c^2 \tag{5.243}$$

then:

$$-\frac{1}{2}(\iota_c\mathcal{L}_c + \mathcal{L}_c\iota_c)\omega(X,X^*) = -\frac{1}{2}(\iota_c^2 d - d\iota_c^2)\omega(X,X^*) \tag{5.244}$$

$$= \frac{1}{2}d\iota_c^2\omega(X,X^*) \tag{5.245}$$

The last line follows from the fact that $\omega(X,X^*)$ is a top form.

Follows:

$$\mathcal{Q}V = d\left(-\iota_c\omega(X,\iota_I dX) + \frac{1}{2}\iota_c^2\omega(X,X^*)\right) =: dU \tag{5.246}$$

Systematically:

$$\begin{tikzcd}
X^* \arrow[r, "\mathcal{Q}"] & dV \arrow[r, "\mathcal{Q}"] & 0 & \\
 & V \arrow[u, "d"] \arrow[r, "\mathcal{Q}"] & dU \arrow[u, "d"] \arrow[r, "\mathcal{Q}"] & 0 \\
 & & U \arrow[u, "d"] \arrow[r, "\mathcal{Q}"] & 0 \arrow[u, "d"]
\end{tikzcd}$$

$$R = \omega(X,X^*) \tag{5.247}$$

$$V = -\omega(X,\iota_I dX) + \iota_c\omega(X,X^*) \tag{5.248}$$

$$U = -\iota_c\omega(X,\iota_I dX) + \frac{1}{2}\iota_c^2\omega(X,X^*) \tag{5.249}$$

In the same way, it is possible to define a formal sum of such operators:

$$\mathcal{J} = R + V + U \tag{5.250}$$

such that $\mathcal{Q}\mathcal{J} = d\mathcal{J}$.

BRST symmetry

Consider now the $BRST$ symmetry. We will use the descent procedure to show the $BRST$ current. We know from standard bosonic string that:

$$j_{BRST} = cT^X + \frac{1}{2}cT^g + \frac{3}{2}\partial^2 c \tag{5.251}$$

Our result is similar to it. Consider the generator of $BRST$ symmetry:

$$q = \mathcal{L}_c X^\mu X^*_\mu + \mathcal{L}_c I^\alpha_\beta I^{*\ \beta}_\alpha + \frac{1}{2}[c,c]^\alpha c^*_\alpha \tag{5.252}$$

We see that $\mathcal{Q}X = \mathcal{L}_c X$, $\mathcal{Q}I = \mathcal{L}_c I$ and $\mathcal{Q}c = \frac{1}{2}[c,c]$. Therefore, if $\mathcal{Q}^2 = 0$, follows:

$$\mathcal{Q}q = -\mathcal{L}_c X^\mu \mathcal{Q}X^*_\mu - \mathcal{L}_c I^\alpha_{\ \beta} \mathcal{Q}I^{*\ \beta}_\alpha + \frac{1}{2}[c,c]^\alpha \mathcal{Q}c^*_\alpha \tag{5.253}$$

We already know how the anti-field X^* transforms under $\mathcal{Q}$ operator:

$$QX^* = -d\iota_I dX - \mathcal{L}_I X \tag{5.254}$$

Let us see what happens with the anti-field I^*:

$$\mathcal{Q}I^{*\beta}_\alpha(w) = \int_z S_{BV} \frac{\overleftarrow{\delta}}{\delta I^\gamma_{\ \delta}(z)} \frac{\overrightarrow{\delta}}{\delta I^{*\delta}_\gamma(z)} I^{*\beta}_\alpha(w) = S_{BV} \frac{\overleftarrow{\delta}}{\delta I^\alpha_{\ \beta}(w)} \tag{5.255}$$

The classical part will be:

$$(S_{BV})_0 \frac{\overleftarrow{\delta}}{\delta I^\alpha_{\ \beta}(w)} = \frac{1}{2}\int_z d^2z \varepsilon^{\sigma\theta} \partial_\sigma X I^\gamma_{\ \theta} \partial_\gamma X \frac{\overleftarrow{\delta}}{\delta I^\alpha_{\ \beta}(w)} = \frac{1}{2}\varepsilon^{\sigma\beta}\partial_\sigma X \partial_\alpha X(w) \tag{5.256}$$

If we contract $\mathcal{Q}I^{*\beta}_\alpha$ with another tensor, as we have in equation (5.253), we get:

$$(\mathcal{L}_c I)^\alpha_{\ \beta} \mathcal{Q}I^{*\beta}_\alpha = \frac{1}{2}\varepsilon^{\sigma\beta}\partial_\sigma X (\mathcal{L}_c I)^\alpha_{\ \beta} \partial_\alpha X = \frac{1}{2} dX \iota_{\mathcal{L}_c I} dX \tag{5.257}$$

The part with anti-fields will be:

$$\int (\mathcal{L}_c I)^\gamma_{\ \delta} I^{*\delta}_\gamma \frac{\overleftarrow{\delta}}{\delta I^\alpha_{\ \beta}(w)} \tag{5.258}$$

First observe that:

$$0 = -\int d\iota_c(II^*) = \int \mathcal{L}_c(II^*) = \int \mathcal{L}_c I I^* + \int I \mathcal{L}_c I^* \tag{5.259}$$

and then:

$$\int (\mathcal{L}_c I)^\gamma_{\ \delta} I^{*\delta}_\gamma \frac{\overleftarrow{\delta}}{\delta I^\alpha_{\ \beta}(w)} = -\int I^\gamma_{\ \delta} (\mathcal{L}_c I^*)^{\ \delta}_\gamma \frac{\overleftarrow{\delta}}{\delta I^\alpha_{\ \beta}(w)} = -(\mathcal{L}_c I^*)^{\ \beta}_\alpha(w) \tag{5.260}$$

From these computations follows:

$$-\mathcal{L}_c X^\mu \mathcal{Q}X^*_\mu = \mathcal{L}_c X^\mu (d\iota_I dX_\mu) + \mathcal{L}_c X^\mu \mathcal{L}_c X^*_\mu \tag{5.261}$$

$$= -d(\mathcal{L}_c X \iota_I dX) + d\mathcal{L}_c X \iota_I dX + \mathcal{L}_c X \mathcal{L}_c X^* \tag{5.262}$$

$$-\langle \mathcal{L}_c I, \mathcal{Q} I^* \rangle = -\frac{1}{2} dX \iota_{\mathcal{L}_c I} dX + \langle \mathcal{L}_c I, \mathcal{L}_c I^* \rangle \tag{5.263}$$

We relate the part without anti-fields by the relation:

$$\mathcal{L}_c \left(\frac{1}{2} dX \iota_I dX \right) = \frac{1}{2} d\mathcal{L}_c X \iota_I dX - \frac{1}{2} dX \iota_{\mathcal{L}_c I} dX + \frac{1}{2} dX \iota_I d\mathcal{L}_c X \tag{5.264}$$

$$= d\mathcal{L}_c X \iota_I dX - \frac{1}{2} dX \iota_{\mathcal{L}_c I} dX \tag{5.265}$$

where the last line follows from the symmetry of the bilinear operation $\wedge \iota_I$:

$$\omega \wedge \iota_I \eta = \eta \wedge \iota_I \omega \tag{5.266}$$

this follows from $-\sqrt{g} g^{\alpha\beta} = I^{\alpha}_{\gamma} \epsilon^{\gamma\beta}$.

$$.: \langle \mathcal{L}_c X, QX^* \rangle + \langle \mathcal{L}_c I, QI^* \rangle = -d\big(\mathcal{L}_c X \mathcal{L}_I X\big) + \mathcal{L}_c \Big(\frac{1}{2} dX \mathcal{L}_I X \Big) + \langle \mathcal{L}_c X, \mathcal{L}_c X^* \rangle + \langle \mathcal{L}_c I, \mathcal{L}_c I^* \rangle \tag{5.267}$$

We can combine the expressions without anti-fields:

$$\mathcal{L}_c(dX \mathcal{L}_I X) = d\iota_c(dX \mathcal{L}_I X) = d\left(\iota_c dX \mathcal{L}_I X - dX \iota_c \mathcal{L}_I X\right) \tag{5.268}$$

Follows:

$$\langle \mathcal{L}_c X, QX^* \rangle + \langle \mathcal{L}_c I, QI^* \rangle = -\frac{1}{2} d(\mathcal{L}_c X \mathcal{L}_I X + dX \iota_c \iota_I dX) + \langle \mathcal{L}_c X, \mathcal{L}_c X^* \rangle + \langle \mathcal{L}_c I, \mathcal{L}_c I^* \rangle \tag{5.269}$$

Now:

$$\langle \frac{1}{2}[c,c], \mathcal{Q} c^* \rangle = \frac{1}{2} \langle \mathcal{L}_{[c,c]} X, X^* \rangle + \frac{1}{2} \langle \mathcal{L}_{[c,c]} I, I^* \rangle + \frac{1}{2} \langle \mathcal{L}_{[c,c]} c, c^* \rangle \tag{5.270}$$

Due to the identity $\mathcal{L}_{[X,Y]} = \mathcal{L}_X \mathcal{L}_Y + \mathcal{L}_Y \mathcal{L}_X$ for fermionic vector fields X, Y, we have the following:

$$\mathcal{L}_{[c,c]} = 2\mathcal{L}_c \mathcal{L}_c \tag{5.271}$$

And then:

$$\frac{1}{2}\langle \mathcal{L}_{[c,c]}X, X^*\rangle = \langle \mathcal{L}_c\mathcal{L}_c X, X^*\rangle \tag{5.272}$$

$$\frac{1}{2}\langle \mathcal{L}_{[c,c]}I, I^*\rangle = \langle \mathcal{L}_c\mathcal{L}_c X, X^*\rangle \tag{5.273}$$

$$\frac{1}{2}\langle \mathcal{L}_{[c,c]}c, c^*\rangle = 0 \tag{5.274}$$

Another thing:

$$\langle \mathcal{L}_c X, \mathcal{L}_c X^*\rangle = \mathcal{L}_c\langle \mathcal{L}_c X, X^*\rangle - \langle \mathcal{L}_c\mathcal{L}_c X, X^*\rangle \tag{5.275}$$

$$\langle \mathcal{L}_c I, \mathcal{L}_c I^*\rangle = \mathcal{L}_c\langle \mathcal{L}_c I, I^*\rangle - \langle \mathcal{L}_c\mathcal{L}_c I, I^*\rangle \tag{5.276}$$

$$.: \langle \mathcal{L}_c X, \mathcal{L}_c X^*\rangle + \langle \mathcal{L}_c I, \mathcal{L}_c I^*\rangle + \langle \mathcal{L}_c c, \mathcal{L}_c c^*\rangle = \mathcal{L}_c\Big(\langle \mathcal{L}_c X, X^*\rangle + \langle \mathcal{L}_c I, I^*\rangle\Big) \tag{5.277}$$

$$= d\left[-\frac{1}{2}\left(dX\mathcal{L}_{Ic}X + \mathcal{L}_c X\mathcal{L}_I X\right) + \iota_c\langle \mathcal{L}_c X, X^*\rangle + \iota_c\langle \mathcal{L}_c I, I^*\rangle\right] \tag{5.278}$$

5.7 Vertex operators via Beta deformation

Vertex Operators can be seem as deformation of the action with preserving Master Equation condition [20], physically it means that the vertex operator is diffeomorphism invariant, or, in other words, BRST invariant. One way to do such deformation is via Beta deformation method. This procedure consists of inserting a product of $\mathcal{J}$'s:

$$S_{BV} + \varepsilon\int B(\mathcal{J}\wedge\mathcal{J}) \tag{5.279}$$

Where B is an anti-symmetric bilinear operation. The master equation reads:

$$\left\{S_{BV} + \varepsilon\int B(\mathcal{J}\wedge\mathcal{J}), S_{BV} + \varepsilon\int B(\mathcal{J}\wedge\mathcal{J})\right\} \tag{5.280}$$

$$= \{S_{BV}, S_{BV}\} + 2\varepsilon\left\{S_{BV}, \int B(\mathcal{J}\wedge\mathcal{J})\right\} + \mathcal{O}(\varepsilon^2) \tag{5.281}$$

$$= 2\varepsilon\left\{S_{BV}, \int B(\mathcal{J}\wedge\mathcal{J})\right\} + \mathcal{O}(\varepsilon^2) = 0 \tag{5.282}$$

Therefore, at first order in ε, the master equation gives rise to:

$$\int \{S_{BV}, B(\mathcal{J} \wedge \mathcal{J})\} = \mathcal{Q} \int B(\mathcal{J} \wedge \mathcal{J}) = 0 \tag{5.283}$$

and this condition is satisfied since:

$$\mathcal{Q} \int B(\mathcal{J} \wedge \mathcal{J}) = \int B(\mathcal{Q}\mathcal{J} \wedge \mathcal{J}) - \int B(\mathcal{J} \wedge \mathcal{Q}\mathcal{J}) = \int B(d\mathcal{J} \wedge \mathcal{J}) - \int B(\mathcal{J} \wedge d\mathcal{J}) \tag{5.284}$$

$$= \int d\left[B(\mathcal{J} \wedge \mathcal{J})\right] = 0 \tag{5.285}$$

The perturbation of the action $B(\mathcal{J} \wedge \mathcal{J})$, as it satisfies the master equation, is a vertex operator. This expression caries some information: it has zero, one and two forms. The zero form and two form part are respectively the so called unintegrated and integrated vertex operator:

$$\int B(\mathcal{J}, \mathcal{J}) = \tag{5.286}$$

$$= \int_{\Sigma} B(V, V) + B(R, U) + B(U, R) \ - \text{ Integrated Vertex Operator} \tag{5.287}$$

$$+ \int_{Line} B(V, U) + B(U, V) \ - \text{ Intermediate Vertex Operator} \tag{5.288}$$

$$+ B(U, U) \ - \text{ Unintegrated Vertex Operator} \tag{5.289}$$

5.7.1 Translation-Translation Vertex Operator

Using the expressions from descent procedure from translation symmetry in bosonic string, we have explicit expressions for the Vertex Operators.

Due to the anti-symmetry of B and the fact that each term R, V and U are fermionic, we have $B(U, R) = B(R, U)$ and $B(U, V) = B(V, U)$, and then:

$$\mathcal{V} = \int_{\Sigma} B(V, V) + 2B(R, U) \ - \text{ Integrated Vertex Operator} \tag{5.290}$$

$$+ \int_{Line} 2B(V, U) \ - \text{ Intermediate Vertex Operator} \tag{5.291}$$

$$+ B(U, U) \ - \text{ Unintegrated Vertex Operator} \tag{5.292}$$

The Integrated Vertex operator reads:

$$\mathcal{V} = \frac{1}{2} \int B(V, V) + 2B(R, U) \tag{5.293}$$

and then:

$$B(V,V) = B(-\iota_I dX + \iota_c X^*, -\iota_I dX + \iota_c X^*) \tag{5.294}$$

$$= B(\iota_I dX, \iota_I dX) - B(\iota_I dX, \iota_c X^*) - B(\iota_c X^*, \iota_I dX) + B(\iota_c X^*, \iota_c X^*) \tag{5.295}$$

$$= B(\iota_I dX, \iota_I dX) - 2B(\iota_I dX, \iota_c X^*) + B(\iota_c X^*, \iota_c X^*) \tag{5.296}$$

the last line follows from the fact that each terms inside B are fermionic and B is anti-symmetric.

$$2B(R,U) = -2B\left(X^*, \iota_c \iota_I dX\right) + 2B\left(X^*, \frac{1}{2}\iota_c^2 X^*\right) \tag{5.297}$$

$$= -2B\left(X^*, \iota_c \iota_I dX\right) + B\left(X^*, \iota_c^2 X^*\right) \tag{5.298}$$

Now we use the following equalities:

$$0 = \iota_c(X^* \iota_I dX) = \iota_c X^* \iota_I dX + X^* \iota_c \iota_I dX \tag{5.299}$$

$$0 = \iota_c(X^* \iota_c X^*) = \iota_c X^* \iota_c^* + X^* \iota_c^2 X^* \tag{5.300}$$

the first equality in both lines follows from the fact that the expression in parenthesis is a three form in a two dimensional space. Follows now:

$$-2B(X^*, \iota_c \iota_I dX) = 2B(\iota_c X^*, \iota_I dX) = 2B(\iota_I dX, \iota_c X^*) \tag{5.301}$$

$$B(X^*, \iota_c^2 X^*) = -B(\iota_c X^*, \iota_c X^*) \tag{5.302}$$

and then:

$$2B(R,U) = 2B(\iota_I dX, \iota_c X^*) - B(\iota_c X^*, \iota_c X^*) \tag{5.303}$$

Following:

$$B(V,V) + 2B(R,U) = B(\iota_I dX, \iota_I dX)) = B(dX, dX) = 2B_{\mu\nu}\partial X^\mu \bar{\partial} X^\nu dz d\bar{z} \tag{5.304}$$

The integrated vertex is:

$$\mathcal{V}_{TT} = \int B_{\mu\nu}\partial X^\mu \bar{\partial} X^\nu dz d\bar{z} \tag{5.305}$$

The unintegrated vertex is:

$$B(U,U) = B\left(\iota_c \iota_I dX, \iota_c \iota_I dX\right) + 2B\left(\iota_c \iota_I dX, \frac{1}{2}\iota_c^2 X^*\right) + B\left(\frac{1}{2}\iota_c^2 X^*, \frac{1}{2}\iota_c^2 X^*\right) \tag{5.306}$$

$$= B\left(\iota_c\iota_I dX, \iota_c\iota_I dX\right) + B\left(\iota_c\iota_I dX, \iota_c^2 X^*\right) + \frac{1}{4}B\left(\iota_c^2 X^*, \iota_c^2 X^*\right) \tag{5.307}$$

Using the fact $\iota_c^3 = 0$ because c is a two dimensional fermion, we obtain:

$$0 = \iota_c(\iota_I dX, \iota_c^2 X^*) = \iota_c\iota_I dX, \iota_c^2 X^* - \iota_I dX, \iota_c^3 X^* = \iota_c\iota_I dX, \iota_c^2 X^* \tag{5.308}$$

$$0 = \iota_c(\iota_c X^* \iota_c^2 X^*) = \iota_c^2 X^* \iota_c^2 X^* - \iota_c X^*, \iota_c^3 X^* = \iota_c^2 X^* \iota_c^2 X^* \tag{5.309}$$

Following:

$$B(U,U) = B\left(\iota_c\iota_I dX, \iota_c\iota_I dX\right) \tag{5.310}$$

The unintegrated vertex reads:

$$\mathcal{V} = c\partial X^\mu \bar{c}\bar{\partial} X^\nu B_{\mu\nu} \tag{5.311}$$

The intermediate vertex is:

$$2B(V,U) = B\left(-\iota_I dX + \iota_c X^*, -\iota_c\iota_I dX + \frac{1}{2}\iota_c^2 X^*\right) \tag{5.312}$$

$$= B\left(\iota_I dX, \iota_c\iota_I dX\right) - \frac{1}{2}B\left(\iota_I dX, \iota_c^2 X^*\right) - B\left(\iota_c X^*, \iota_c\iota_I dX\right) + \frac{1}{2}B\left(\iota_c X^*, \iota_c^2 X^*\right)$$
$$= B\left(\iota_I dX, \iota_c\iota_I dX\right) \tag{5.313}$$

In particular, this expressions to integrated and unintegrated vertex operators for bosonic string reproduces the usual massless vertex [1], without anti-fields corrections.

5.7.2 Rotation-Translation Vertex Operators

Now, we use $\mathcal{J}$'s from different origins

From translation symmetry:

$$R_T = X^* \tag{5.314}$$

$$V_T = -\iota_I dX + \iota_c X^* \tag{5.315}$$

$$U_T = -\iota_c\iota_I dX + \frac{1}{2}\iota_c^2 X^* \tag{5.316}$$

And from rotation symmetry:

$$R_R = \omega(X, X^*) \tag{5.317}$$

$$V_R = -\omega(X, \iota_I dX) + \iota_c \omega(X, X^*) \tag{5.318}$$

$$U_R = -\iota_c \omega(X, \iota_I dX) + \frac{1}{2}\iota_c^2 \omega(X, X^*) \tag{5.319}$$

And then:

$$\mathcal{J}_T = R_T + V_T + U_T\ , \qquad \mathcal{J}_R = R_R + V_R + U_R \tag{5.320}$$

And the deformation of the action will be:

$$S_{BV} + \varepsilon B(\mathcal{J}_T, \mathcal{J}_R) \tag{5.321}$$

The Integrated Vertex Operator reads:

$$\int_\Sigma B(V_T, V_R) + B(R_T, U_R) + B(U_T, R_R) \tag{5.322}$$

Expanding each term, we have:

$$B(V_T, V_R) = B(-\iota_I dX + \iota_c X^*, -\omega(X, \iota_I dX) + \iota_c \omega(X, X^*)) \tag{5.323}$$

$$\begin{aligned} &= B(\iota_I dX, \omega(X, \iota_I dX)) - B(\iota_I dX, \iota_c \omega(X, X^*)) \\ &- B(\iota_c X^*, \omega(X, \iota_I dX)) + B(\iota_c X^*, \iota_c \omega(X, X^*) \end{aligned} \tag{5.324}$$

$$B(R_T, U_R) = B\left(X^*, \iota_c \omega(X, \iota_I dX) + \frac{1}{2}\iota_c^2 \omega(X, X^*)\right) \tag{5.325}$$

$$= -B\left(X^*, \iota_c \omega(X, \iota_I dX)\right) + \frac{1}{2} B\left(X^*, \iota_c^2 \omega(X, X^*)\right) \tag{5.326}$$

$$B(U_T, R_R) = B\left(-\iota_c \iota_I dX + \frac{1}{2}\iota_c^2 X^*, \omega(X, X^*)\right) \tag{5.327}$$

$$= -B\left(\iota_c \iota_I dX, \omega(X, X^*)\right) + \frac{1}{2} B\left(\iota_c^2 X^*, \omega(X, X^*)\right) \tag{5.328}$$

Let us write the term in first order in anti-fields:

$$-B(\iota_I dX, \iota_c \omega(X, X^*)) - B\left(\iota_c \iota_I dX, \omega(X, X^*)\right)$$

$$-B(\iota_c X^*, \omega(X, \iota_I dX)) - B\left(X^*, \iota_c \omega(X, \iota_I dX)\right) \tag{5.329}$$

Now, observe that $B(\iota_I dX, \omega(X, X^*))$ and $B\left(X^*, \omega(X, \iota_I dX)\right)$ are zero since these are three forms on a bi-dimensional space. Contracting both expressions with ι_c we have:

$$0 = \iota_c B(\iota_I dX, \omega(X, X^*)) = B(\iota_I dX, \iota_c \omega(X, X^*)) + B(\iota_c \iota_I dX, \omega(X, X^*)) \tag{5.330}$$

$$0 = \iota_c B\left(X^*, \omega(X, \iota_I dX)\right) = B(\iota_c X^*, \omega(X, \iota_I dX) + B(X^*, \iota_c \omega(X, \iota_I dX) \tag{5.331}$$

And therefore we conclude that expression (5.329) is zero.

Now, let us observe the second order in anti-fields term:

$$B(\iota_c X^*, \iota_c \omega(X, X^*)) + \frac{1}{2} B\left(X^*, \iota_c^2 \omega(X, X^*)\right) + \frac{1}{2} B\left(\iota_c^2 X^*, \omega(X, X^*)\right) \tag{5.332}$$

$$= \frac{1}{2} B(\iota_c X^*, \iota_c \omega(X, X^*)) + \frac{1}{2} B\left(X^*, \iota_c^2 \omega(X, X^*)\right) \tag{5.333}$$

$$+ \frac{1}{2} B(\iota_c X^*, \iota_c \omega(X, X^*)) + \frac{1}{2} B\left(\iota_c^2 X^*, \omega(X, X^*)\right) \tag{5.334}$$

As before, we can argue that (5.333) and (5.334) are both zero. Notice that $B(X^*, \iota_c \omega(X, X^*)) = B(\iota_c X^*, \omega(X, X^*)) = 0$ because these are three forms. Contracting both with ι_c:

$$0 = \iota_c B(X^*, \iota_c \omega(X, X^*)) = B(\iota_c X^*, \iota_c \omega(X, X^*)) + B(X^*, \iota_c^2 \omega(X, X^*)) \tag{5.335}$$

$$0 = \iota_c B(\iota_c X^*, \omega(X, X^*)) = B(\iota_c^2 X^*, \omega(X, X^*)) + B(\iota_c X^*, \iota_c \omega(X, X^*)) \tag{5.336}$$

Follows that the term with second order in anti-fields (5.332) is zero and the Integrated Vertex Operator for Rotation-Translation is:

$$\mathcal{V}_{RT} = \int B(\iota_I dX, \omega(X, \iota_I dX)) \tag{5.337}$$

Now, let us write the unintegrated vertex:

$$B(U_T, U_R) = B(\iota_c \iota_I dX, \iota_c \omega(X, \iota_I dX)) \tag{5.338}$$

$$-\frac{1}{2} B(\iota_c \iota_I dX, \iota_c^2 \omega(X, X^*)) - \frac{1}{2} B(\iota_c^2 X^*, \iota_c \omega(X, \iota_I dX)) + \frac{1}{4} B(\iota_c^2 X^*, \iota_c^2 \omega(X, X^*)) \tag{5.339}$$

The second line in this expression is zero because all the terms has three c-ghosts and it is a fermion in 2D. Follows the vertex operator:

$$\mathcal{V}_{RT} = B(\iota_c \iota_I dX, \iota_c \omega(X, \iota_I dX)) \tag{5.340}$$

5.8 BV action for Yang-Mills

From Faddeev-Popov procedure, as discussed in chapter 3, section (3.5.1), we get b, c ghosts and a new effective action:

$$\mathcal{L} = -\frac{1}{4} d_A A \wedge \star d_A A + B^a \partial^\mu A^a_\mu + b^a(-\partial^\mu D^{ab}_\mu) c^b \tag{5.341}$$

where the BRST transformations of the field A and ghost c is:

$$QA = d_A c \qquad or \qquad QA^a_\mu = D^{ab}_\mu c^b \tag{5.342}$$

$$Qc = -\frac{1}{2}[c, c] \qquad or \qquad Qc^a = -\frac{1}{2} g f^{abc} c^b c^c \tag{5.343}$$

Where $d_A = d - igA$ and therefore, $F := d_A A = dA - ig[A \wedge A]$. In coordinates, $d_A = D^{ab}_\mu = \delta^{ab}\partial_\mu - igt^a A^b_\mu$ and $F^a_{\mu\nu} = \partial_{[\mu} A^a_{\nu]} + g f^{abc} A^b_\mu A^c_\nu$.

For the non-minimal sector, the Topological-Quartet is just as usual:

$$Q\bar{c}^a = B^a$$

$$QB^a = 0$$

The action on the BV-phase space is given by:

$$S_{BV} = \int -\frac{1}{4} d_A A \wedge \star d_A A + \langle QA, A^* \rangle + \langle Qc, c^* \rangle + \langle Qb, b^* \rangle + \langle QB, B^* \rangle \tag{5.344}$$

5.8.1 Poincare symmetry in Pure Yang-Mills

The translation symmetry for the Yang-Mills reads as $\delta_\xi A = \mathcal{L}_\xi A$, with ξ the direction of the translation. In the BV-phase space, we will have to add a translation for the c-ghost $\delta_\xi c = \mathcal{L}_x ic$, then, the generator for the total translation is:

$$g = \mathrm{tr}(\mathcal{L}_\xi A \wedge A^*) + \mathrm{tr}(\mathcal{L}_\xi c \wedge c^*) \tag{5.345}$$

Let us apply the $\mathcal{Q} = \{S_{BV}, -\}$ operator to the generator:

$$\mathcal{Q}g = \{S_{BV}, g\} = \{S_{BV}, \mathcal{L}_\xi A \wedge A^*\} + \{S_{BV}, \mathcal{L}_\xi c \wedge c^*\} \tag{5.346}$$

Two important computations are $\mathcal{Q}A^*$ and $\mathcal{Q}c^*$:

$$\mathcal{Q}A^* = \frac{\delta S_{BV}}{\delta A} = d_A \star d_A A - [c \wedge A^*] \tag{5.347}$$

$$\begin{aligned} \mathcal{Q}c^* &= \frac{\delta}{\delta c} S_{BV} = \frac{\delta}{\delta c}\langle \frac{1}{2}[c,c], c^* \rangle + \frac{\delta}{\delta c}\langle d_A c, A^* \rangle \\ &= [c, c^*] + dA^* + [A \wedge A^*] = [c, c^*] + d_A A^* \end{aligned} \tag{5.348}$$

Let us compute the two terms separately and then blue it together. In the following we will use the properties:

$$[AB, C] = A[B, C] + [B, C]A \tag{5.349}$$

$$\mathrm{tr}[A, B] = 0 \tag{5.350}$$

- Computing the $\delta_\xi A \wedge A^*$ part:

$$\mathcal{Q}[\mathrm{tr}(\mathcal{L}_\xi A \wedge A^*)] = \mathrm{tr}(\mathcal{Q}\mathcal{L}_\xi A) \wedge A^* + \mathrm{tr}\, \mathcal{L}_\xi A \wedge QA^*$$

$$= \mathrm{tr}\, \mathcal{L}_\xi d_A c \wedge A^* + \mathrm{tr}\, \mathcal{L}_\xi A \wedge d_A \star F + \mathrm{tr}\, \mathcal{L}_\xi A \wedge [c, A^*] \tag{5.351}$$

$$= \mathrm{tr}\, \mathcal{L}_\xi dc \wedge A^* + \mathrm{tr}\, \mathcal{L}_\xi [c, A] \wedge A^* + \mathrm{tr}\, \mathcal{L}_\xi A \wedge d_A \star F + \mathrm{tr}\, \mathcal{L}_\xi A \wedge [c, A^*] \tag{5.352}$$

$$= \mathrm{tr}\, \mathcal{L}_\xi dc \wedge A^* + \mathrm{tr}[\mathcal{L}_\xi c, A] \wedge A^* + \mathrm{tr}[c, \mathcal{L}_\xi A \wedge A^*] + \mathrm{tr}\, \mathcal{L}_\xi A \wedge d_A \star F \tag{5.353}$$

$$= \mathrm{tr}\, \mathcal{L}_\xi dc \wedge A^* + \mathrm{tr}[\mathcal{L}_\xi c, A] \wedge A^* + \mathrm{tr}\, \mathcal{L}_\xi A \wedge d_A \star F \tag{5.354}$$

- Computing the $\delta_\xi c \wedge c^*$ part:

$$\begin{aligned} \mathcal{Q}\, \mathrm{tr}(\delta_\xi c\, c^*) &= \frac{1}{2} \mathrm{tr}\, \mathcal{L}_\xi [c, c]\; c^* + \mathrm{tr}\, \mathcal{L}_\xi c\; [c, c^*] + \mathrm{tr}\, \mathcal{L}_\xi c\; d_A A^* \\ &= \mathrm{tr}[c, \mathcal{L}_\xi c\; c^*] + \mathrm{tr}\, \mathcal{L}_\xi c\; dA^* + \mathrm{tr}\, \mathcal{L}_\xi c\; [A \wedge A^*] \\ &= \mathrm{tr}\, \mathcal{L}_\xi c\; dA^* + \mathrm{tr}\, \mathcal{L}_\xi c\; [A \wedge A^*] \end{aligned} \tag{5.355}$$

Combining (5.354) and (5.355), we have:

$$\mathcal{Q}g = d\Big(\mathrm{tr}(\mathcal{L}_\xi c\; A^*) \Big) + \mathrm{tr}(\mathcal{L}_\xi A \wedge d_A \star F) \tag{5.356}$$

Let us work out the second term:

$$\mathcal{L}_\xi A \wedge d_A \star F = d\Big(\mathcal{L}_\xi A \wedge \star F\Big) - d\mathcal{L}_\xi A \wedge \star F + \mathcal{L}_\xi A \wedge [A \wedge \star F] \tag{5.357}$$

The last two terms can bee seem as:

$$-\mathcal{L}_\xi dA \wedge \star F - [\mathcal{L}_\xi A \wedge A] \wedge \star F = -\mathcal{L}_\xi F \wedge \star F = -\frac{1}{2}\mathcal{L}_\xi (F \wedge \star F) = d\Big(-\iota_\xi (F \wedge \star F)\Big) \tag{5.358}$$

The final result is:

$$\mathcal{Q}\Big(\delta_\xi A \wedge A^* + \delta_\xi c\, c^*\Big) = d\,\mathrm{tr}\,\Big(\mathcal{L}_\xi A \wedge \star F - \iota_\xi (F \wedge \star F) + \mathcal{L}_\xi c\ A^*\Big) \tag{5.359}$$

For the next steps, we need:

$$\begin{aligned} \mathcal{Q}F = \mathcal{Q}dA + \mathcal{Q}[A \wedge A] &= d[c, A] + [dc \wedge A] + [[A, c] \wedge A] \\ &= [c, d_A A] = [c, F] \end{aligned} \tag{5.360}$$

Follows:

$$\begin{aligned} \mathcal{Q}\Big(\iota_\xi F \wedge \star F\Big) &= \iota_\xi [c, F] \wedge \star F + \iota_\xi F \wedge [c, \star F] \\ &= \iota_\xi [c, F \wedge \star F] \end{aligned} \tag{5.361}$$

$$\begin{aligned} \mathcal{Q}\Big(\mathcal{L}_\xi A \wedge \star F\Big) &= \mathcal{L}_\xi dc \wedge \star F + \mathcal{L}_\xi [A, c] \wedge \star F + \mathcal{L}_\xi A \wedge [c, \star F] \\ &= d\mathcal{L}_\xi c \wedge \star F + [A, \mathcal{L}_\xi c] \wedge \star F \end{aligned} \tag{5.362}$$

$$\begin{aligned} \mathcal{Q}\Big(\mathcal{L}_\xi c\ A^*\Big) &= \frac{1}{2}\mathcal{L}_\xi [c, c]\ A^* + \mathcal{L}_\xi c\ [c, A^*] + \mathcal{L}_\xi c\ d_A \star F \\ &= \mathcal{L}_\xi c\ d \star F + \mathcal{L}_\xi c\ [A \wedge \star F] \end{aligned} \tag{5.363}$$

Summing over:

$$\mathcal{Q}\,\mathrm{tr}\,\Big(\mathcal{L}_\xi A \wedge \star F - \iota_\xi F \wedge \star F + \mathcal{L}_\xi c\ A^*\Big) = d\,\mathrm{tr}\,\Big(\mathcal{L}_\xi c\ \star F\Big) \tag{5.364}$$

More one step:

$$\mathcal{Q}\Big(\mathcal{L}_\xi c\ \star F\Big) = \frac{1}{2}\mathcal{L}_\xi[c,c]\ \star F + \mathcal{L}_\xi c\ [c,\star F] = [c,\mathcal{L}_\xi c\ \star F] \tag{5.365}$$

That is:

$$\mathcal{Q}\,\mathrm{tr}\left(\mathcal{L}_\xi c\ \star F\right) = 0 \tag{5.366}$$

We then obtain the descent equations for Pure Yang-Mills:

$$\left\{\int_\Sigma g, -\right\} = translation \tag{5.367}$$

Where:

$$g = \mathrm{tr}\left(\mathcal{L}_\xi A \wedge A^* + \mathcal{L}_\xi c\ c^*\right) \tag{5.368}$$

$$\{S_{BV}, g\} = dV \tag{5.369}$$

$$\{S_{BV}, V\} = dU \tag{5.370}$$

$$\{S_{BV}, U\} = 0 \tag{5.371}$$

$$\begin{tikzcd}
g \arrow[r, "\mathcal{Q}"] & dV \arrow[r, "\mathcal{Q}"] & 0 & & \\
 & V \arrow[u, "d"] \arrow[r, "\mathcal{Q}"] & dU \arrow[u, "d"] \arrow[r, "\mathcal{Q}"] & 0 & \\
 & & U \arrow[u, "d"] \arrow[r, "\mathcal{Q}"] & W \arrow[u, "d"] \arrow[r, "\mathcal{Q}"] & 0
\end{tikzcd}$$

$$V = \mathrm{tr}\left(\mathcal{L}_\xi A \wedge \star F - \iota_\xi F \wedge \star F + \mathcal{L}_\xi c\ A^*\right) \tag{5.372}$$

$$U = \mathrm{tr}\left(\mathcal{L}_\xi c\ \star F\right) \tag{5.373}$$

$$W = 0 \tag{5.374}$$

Fixing some problems

Let us modify the current $j = V$ by adding and subtracting $\iota_\xi[A \wedge A] \wedge \star F$:

$$\begin{aligned} &\mathrm{tr}\left(\iota_\xi dA\wedge\star F+\iota_\xi[A\wedge A]\wedge\star F-\iota_\xi(F\wedge\star F)-\iota_\xi[A\wedge A]\wedge\star F+d\iota_\xi A\wedge\star F+\mathcal{L}_\xi c\ A^*\right) \\ &\qquad= \mathrm{tr}\left(\iota_\xi F\wedge\star F-\iota_\xi(F\wedge\star F)-d_A\iota_\xi A\wedge\star F+\mathcal{L}_\xi c\ A^*\right) \end{aligned} \tag{5.375}$$

The last line follow because:

$$\iota_\xi[A\wedge A]=2[\iota_\xi A\wedge A] \tag{5.376}$$

Where the gauge invariant quantity is the Energy-Momentum tensor $T\in\Omega^2(M)$

$$\star\,\iota_\xi T=\mathrm{tr}\left(\iota_\xi F\wedge\star F-\iota_\xi(F\wedge\star F)\right) \tag{5.377}$$

The bi-complex does not change.

Comparing with the energy-momentum tensor in usual literature [11]:

$$T_{\mu\nu}=F_{\mu\rho}\partial_\nu A^\rho-\eta_{\mu\nu}F\wedge\star F \tag{5.378}$$

5.9 Superstring BV formulation

We saw by definition that BV formalism have direct applications to gauge theories that presents some BRST operator. Although the pure spinor superstring does not presents a BRST operator coming from a gauge symmetry, pure spinor have a BRST-like operator, that is, a fermionic operator Q that leaves the action invariant $QS=0$ and is nillpotent ($Q^2=0$). Therefore, we may construct a BV theory for the pure spinor case.

$$\begin{aligned} S_{BV}=\int_\Sigma d\tau^+d\tau^-\Big(&\frac{1}{2}\partial_+X^m\partial_-X_m+p_-\partial_+\theta_R+p_+\partial_-\theta_L+w_+\partial_-\lambda_L+w_-\partial_+\lambda_R\Big) \\ &+\lambda_L\theta_L^*+\lambda_R\theta_R^*+\frac{1}{2}(\lambda_L\Gamma^m\theta_L+\lambda_R\Gamma^m\theta_R)X_m^*+d_+w_+^*+d_-w_-^* \\ &+\left(-\frac{1}{2}\partial_+X^m(\lambda_L\Gamma_m)+\frac{3}{8}(\lambda_L\Gamma^m\theta_L)(\partial_+\theta_L\Gamma_m)+\frac{1}{8}(\partial_+\lambda_L\Gamma^m\theta_L)(\theta_L\Gamma_m)\right)p_+^* \\ &+\left(-\frac{1}{2}\partial_-X^m(\lambda_R\Gamma_m)+\frac{3}{8}(\lambda_R\Gamma^m\theta_R)(\partial_-\theta_R\Gamma_m)+\frac{1}{8}(\partial_-\lambda_R\Gamma^m\theta_R)(\theta_R\Gamma_m)\right)p_-^* \end{aligned} \tag{5.379}$$

Now, we are able to work out some descent equations for the superstring theory. For the purpose of simplicity, we will restrict to the left sector. Let us write this sector explicitly:

$$\begin{aligned} S_{BV} = \int_\Sigma d\tau^+ d\tau^- \Bigg[& \frac{1}{2}\partial_+ X^m \partial_- X_m + p_+ \partial_- \theta_L + w_+ \partial_- \lambda_L + \lambda_L \theta_L^* + \frac{1}{2}(\lambda_L \Gamma^m \theta_L) X_m^* \\ & + p_+ w_+^* - \frac{1}{2}\partial_+ X_m (\theta_L \Gamma^m w_+^*) - \frac{1}{8}(\theta_L \Gamma^m \partial_+ \theta_L)(\theta_L \Gamma_m w_+^*) \\ & - \frac{1}{2}\partial_+ X^m (\lambda_L \Gamma_m p_+^*) + \frac{3}{8}(\lambda_L \Gamma^m \theta_L)(\partial_+ \theta_L \Gamma_m p_+^*) + \frac{1}{8}(\partial_+ \lambda_L \Gamma^m \theta_L)(\theta_L \Gamma_m p_+^*) \Bigg] \end{aligned} \tag{5.380}$$

First of all consider the translation symmetry on X:

$$\left\{ \int_\Sigma X^*, - \right\} = translation \tag{5.381}$$

It will follows:

$$\{S_{BV}, X_m^*\} = \partial_+ \left(\partial_- X + \frac{1}{2}(\theta_L \Gamma_m w_+^*) + \frac{1}{2}(\lambda_L \Gamma_m p_+^*) \right) =: dV \tag{5.382}$$

Now, let us take the next step of the procedure:

$$\{S_{BV}, V\} = \partial_+ \left(\frac{1}{4}(\lambda_L \Gamma \theta_L) \right) - \partial_- \left(\frac{1}{4}(\lambda_L \Gamma \theta_L) \right) =: dW \tag{5.383}$$

Therefore, we have the descent procedure for the translation symmetry:

$$\begin{tikzcd}
g \arrow[r, "Q"] & dV \arrow[r, "Q"] & 0 & \\
& V \arrow[u, "d"] \arrow[r, "Q"] & dW \arrow[u, "d"] \arrow[r, "Q"] & 0 \\
& \frac{1}{2}X \arrow[r, "Q"] & W \arrow[u, "d"] \arrow[r, "Q"] & 0
\end{tikzcd}$$

with:

$$g = X^* \tag{5.384}$$

$$V_t = \star dX + \frac{1}{2} \star (\theta_L \Gamma_m w_+^*) + \frac{1}{2} \star (\lambda_L \Gamma_m p_+^*) \tag{5.385}$$

$$W_t = \frac{1}{4} (\lambda_L \Gamma \theta_L) \tag{5.386}$$

We see that V is the translation current $\star dX$ plus an anti-field correction.

Now, let us work with the super-rotations:

$$\begin{aligned} \text{super-rotation: } h &= \Lambda_{mn} (\Gamma^{mn})^{\alpha}{}_{\beta} \theta^{\beta} \theta_{\alpha}^* + \Lambda_{mn} (\Gamma^{mn})^{\alpha}{}_{\beta} p^{\beta} p_{\alpha}^* \\ &= \Lambda_{mn} (\theta_L^* \Gamma^{mn} \theta_L) + \Lambda_{mn} (p_+^* \Gamma^{mn} p_+) \end{aligned} \tag{5.387}$$

$$\begin{aligned} \text{ghost super-rotation: } k &= \Lambda_{mn} (\Gamma^{mn})^{\alpha}{}_{\beta} \lambda^{\beta} \lambda_{\alpha}^* + \Lambda_{mn} (\Gamma^{mn})^{\alpha}{}_{\beta} w^{\beta} w_{\alpha}^* \\ &= \Lambda_{mn} (\lambda_L^* \Gamma^{mn} \lambda_L) + \Lambda_{mn} (w_+^* \Gamma^{mn} w_+) \end{aligned} \tag{5.388}$$

The total super-translation current will be $j = h + k$.

$$\text{super-rotation} = \left\{ \int j, - \right\} \tag{5.389}$$

Let us see what are the anti-field correction to the current:

$$\begin{aligned} \{S_{BV}, j\} = \partial_+ \Bigg(& p_+ \Gamma^{mn} \theta_L + \lambda_L \Gamma^{mn} w_+ + \frac{1}{8} (\theta_L \Gamma^p \Gamma^{mn} \theta_L)(\theta_L \Gamma_p w_+^*) \\ & + \frac{3}{8} (\lambda_L \Gamma_p \theta_L)(\theta_L \Gamma^{mn} \Gamma^p p_+^*) + \frac{1}{8} (\theta_L \Gamma_p p_+^*)(\theta_L \Gamma^p \Gamma^{mn} \lambda_L) \Bigg) =: dV \end{aligned} \tag{5.390}$$

The expression to V is, as expected, the total current (4.74) + (4.76) with an additional anti-field correction. Now, for the next step we obtain:

$$\{S_{BV}, V\} = \partial_+ \left(\frac{1}{8} (\lambda_L \Gamma_p \theta_L)(\theta_L \Gamma^{mnp} \theta_L) \right) - \partial_- \left(\frac{1}{8} (\lambda_L \Gamma_p \theta_L)(\theta_L \Gamma^{mnp} \theta_L) \right) =: dW \tag{5.391}$$

And then we have the hole complex chain:

$$\begin{tikzcd}
h \arrow[r, "Q"] & dV \arrow[r, "Q"] & 0 & \\
& V \arrow[u, "d"] \arrow[r, "Q"] & dW \arrow[u, "d"] \arrow[r, "Q"] & 0 \\
& & W \arrow[u, "d"] \arrow[r, "Q"] & 0
\end{tikzcd}$$

with:

$$h = (\theta_L^* \Gamma^{mn} \theta_L) + (p_+^* \Gamma^{mn} p_+) + (\lambda_L^* \Gamma^{mn} \lambda_L) + (w_+^* \Gamma^{mn} w_+) \tag{5.392}$$

$$\begin{aligned} V_{sr} = \star p_+ \Gamma^{mn} \theta_L + \star w_+ \Gamma^{mn} \lambda_L + \frac{1}{8} \star (\theta_L \Gamma^p \Gamma^{mn} \theta_L)(\theta_L \Gamma_p w_+^*) \\ + \frac{3}{8} \star (\lambda_L \Gamma_p \theta_L)(\theta_L \Gamma^{mn} \Gamma^p p_+^*) + \frac{1}{8} \star (\theta_L \Gamma_p p_+^*)(\theta_L \Gamma^p \Gamma^{mn} \lambda_L) \end{aligned} \tag{5.393}$$

$$W_{sr} = \frac{1}{8} (\lambda_L \Gamma_p \theta_L)(\theta_L \Gamma^{mnp} \theta_L) \tag{5.394}$$

For the supersymmetry we use the same logic, the result is:

$$\begin{tikzcd}
f \arrow[r, "Q"] & dV \arrow[r, "Q"] & 0 & \\
& V \arrow[u, "d"] \arrow[r, "Q"] & dW \arrow[u, "d"] \arrow[r, "Q"] & 0 \\
& & W \arrow[u, "d"] \arrow[r, "Q"] & 0
\end{tikzcd}$$

with:

$$f = \frac{1}{2} (\epsilon \Gamma^m \theta) X_m^* + \epsilon^\alpha \theta_\alpha^* - \frac{1}{2} \partial_+ X_m (\epsilon \Gamma^m p_+^*) + \frac{1}{8} (\epsilon \Gamma^m \theta_L)(\partial_+ \theta \Gamma^m p_+^*) \tag{5.395}$$

$$V_{susy} = q + \frac{1}{8} (\epsilon \Gamma^m \theta_L)(\theta_L \Gamma_m w_+^*) + \frac{1}{8} (\epsilon \Gamma^m \theta_L)(\theta_L \Gamma_m p_+^*)$$

$$+\frac{1}{8}(\lambda_L\Gamma_m\theta_L)(\epsilon\Gamma^m p_+^*) \tag{5.396}$$

$$W_{susy} = \frac{1}{8}(\epsilon\Gamma_m\theta_L)(\lambda_L\Gamma^m\theta_L) \tag{5.397}$$

with q the supersymmetric current (4.91) and ϵ a spinor parameter.

We can then perform β-deformation $S_{BV} \mapsto S_{BV} + \int B(J \wedge J)$, with J is just the sum of the descent equation expressions and B anti-symmetric matrix. For the **integrated** vertex operator we obtain anti-field corrections. Let us do some deformations, but first of all write the BV action as:

$$S_{BV} = S_0(\phi) + S_1(\phi)^m\phi_m^* + S_2(\phi)^{mn}\phi_m^*\phi_n^* + \cdots \tag{5.398}$$

The first term in anti-field is just the BRST operator, and we denote it by $S_1^m = Q^m$; the second order term is a Poisson structure [32], and we denote it by $S_2^{mn} = \Pi^{mn}$, which does not appears initially on our BV action for pure spinor. If we do a translation-translation deformation, that is, $J = g + V_t + W_t$, it gives a perturbation on the BRST operator:

$$\begin{aligned}\delta Q = {}& \frac{1}{4}B^{mn}(\lambda_L\Gamma_m\theta_L)\frac{\delta}{\delta X^n} \\ &+ \frac{1}{2}B^{mn}\partial_+X_m(\theta_L\Gamma_n)_\alpha\frac{\delta}{\delta w_+^\alpha} + \frac{1}{2}B^{mn}\partial_+X_m(\lambda_L\Gamma_n)_\alpha\frac{\delta}{\delta p_+^\alpha}\end{aligned} \tag{5.399}$$

and a Poisson structure appears:

$$\begin{aligned}\delta\Pi = {}& \frac{1}{4}B^{mn}(\theta_L\Gamma_m)_\alpha(\theta_L\Gamma_n)_\beta\frac{\delta}{\delta w_+^\alpha}\frac{\delta}{\delta w_+^\beta} + \frac{1}{2}B^{mn}(\lambda_L\Gamma_m)_\alpha(\theta_L\Gamma_n)_\beta\frac{\delta}{\delta p_+^\alpha}\frac{\delta}{\delta w_+^\beta} \\ &+ \frac{1}{4}B^{mn}(\lambda_L\Gamma_m)_\alpha(\lambda_L\Gamma_n)_\beta\frac{\delta}{\delta p_+^\alpha}\frac{\delta}{\delta p_+^\beta}\end{aligned} \tag{5.400}$$

For the correction of the classical action S_0 we obtain the Dilaton vertex operator $\int B_{mn}\partial X^m\bar{\partial}X^n$, as in the case of the bosonic string.

If we do β-deformation as $J_{t,L} \wedge J_{t,R} + J_{sr,L} \wedge J_{sr,R} + J_{susy,L} \wedge J_{susy,R}$ and take the **unintegrated** part $\left(W_{t,L} \wedge W_{t,R} + W_{sr,L} \wedge W_{sr,R} + W_{susy,L} \wedge W_{susy,R}\right)$, we reproduce the vertex operators for the pure spinor superstring [23]:

$$\mathcal{V} = (A_L \wedge A_R)\, e^{ik\cdot X} \tag{5.401}$$

with, introducing polarization e_m and momentum k_n:

$$A_{.} = \frac{1}{2}e_n(\lambda_{.}\Gamma^n\theta_{.}) + \frac{1}{8}(\epsilon\Gamma_m\theta_{.})(\lambda_{.}\Gamma^m\theta_{.}) + \frac{1}{16}k_{[n}e_{m]}(\lambda_{.}\Gamma_p\theta_{.})(\theta_{.}\Gamma^{mnp}\theta_{.}) + \cdots \tag{5.402}$$

Chapter 6

Conclusion

This dissertation was intended to review the most basic and fundamental concepts of field theory and string theory, from which case we was able to study symmetries more properly. We presented the very basic concepts present in field theories and the BRST procedure was an important part on this presentation. This explanation was important because the BRST structure provides the possibility to write a BV theory from the initial field theory. This method was done for Bosonic String, pure Yang-Mills theory and Pure-spinor superstring. We then was able to deform the BV action we constructed for the case of string theories, and doing this we were able to define vertex operators. For the case of Bosonic string, we obtain the usual massless vertex operator already present on the literature; for the case of pure spinor we observe, besides the massless vertex, the presence of a perturbation on the BRST-like operator, as well as the emergence of a Poisson structure (e.g. 5.400).

We also have reviewed some constructions about string theory in curved $AdS_5 \times S^5$ background. We explained how to understand this space as a coset of super-groups and constructed the action for GS and Pure Spinor superstring in this background.

Appendix A

Spin groups

Definition A.1 (Spin Group Spin(n))**.** The so-called Spin group, represented as Spin(n), is a group such that there exists a Lie group homomorphism:

$$\rho : Spin(n) \longrightarrow SO(n) \tag{A.1}$$

that is a double cover, a 2-1 map:

$$ker(\rho) := \{X \in Spin(n) \mid \rho(X) = id\} = \mathbb{Z}_2 \tag{A.2}$$

n	Spin
1	$\cong O(1)$
2	$\cong U(1)$
3	$\cong SU(2)$
4	$\cong SU(2) \times SU(2)$
6	$\cong SU(4)$
1,3	$\cong SL(2,\mathbb{C})$
2,4	$\cong SU(2,2)$

Table A.1: Some Spin(n) groups

We see in this table some isomorphisms from Spin(n) to some well-known groups. In special, we have that $Spin(6) \cong SU(4)$, $Spin(2,4) \cong SU(2,2)$ and $Spin(1,3) \cong SL(2,\mathbb{C})$.

$$\rho : SU(4) \longrightarrow SO(6) \tag{A.3}$$

A.1 Spinors in higher dimensions

A spinor in an object that obeys the following rule under lorentz transformation:

$$\psi_\alpha \longrightarrow \psi'_\alpha = \Sigma_\alpha^{\ \beta} \psi_\beta \tag{A.4}$$

That is equivalent to say that we have Σ_ν^μ as a representation of Lorentz algebra and $\mu = 1, \cdots, D$, D is for example the dimension of space-time in string theory (D=10 or 26), dimension of world-sheet (D=2) or compactified space time (D=4). Well, then, Σ satisfies the commutation relation of lorentz Lie algebra. But this is a special representation, it is related to another structure, the cliford algebra:

$$\Sigma^{\mu\nu} = [\Gamma^\mu, \Gamma^\nu] \tag{A.5}$$

where $\{\Gamma^\mu, \Gamma^\nu\} = 2\eta^{\mu\nu}$. In that case, it is simple to check that Σ satiesfies the lorentz algebra and therefore a spinor is defined essentially as a representation of the cliford algebra, since it's necessary by definition and sufficient to define the Γ matrices.

A non-trivial finite dimensional representation of the Clifford algebra necessarily has even dimension. Indeed, if $\mu \neq \nu$, $\Gamma^\mu\Gamma^\nu = -\Gamma^\nu\Gamma^\mu$. By ciclicity of determinant and $\det(AB) = \det(A)\det(B)$ follows $\det(\Gamma^\mu\Gamma^\nu) = \det(-\Gamma^\nu\Gamma^\mu) = \det(-I)\det(\Gamma^\mu\Gamma^\nu) = (-1)^k \det(\Gamma^\mu\Gamma^\nu)$ where k is the dimension of the representation and is clearly even.

Write the Γ operators as creation and annihilation operators:

$$b_i^{\pm} := \frac{1}{2}\left(\Gamma^i \pm \Gamma^{d+1-i}\right) \tag{A.6}$$

with $i = 1, \cdots, \frac{d}{2}$.

Follows that the Clifford algebra relation $b_i^{\pm}$ reduces to the algebra of harmonic fermionic oscillators:

$$\{b_i^{\pm}, b_j^{\pm}\} = 0, \qquad \{b_i^{+}, b_j^{-}\} = \delta_{ij} \tag{A.7}$$

We now can define a representation such that there exists an state ζ with:

$$b_i^{-}|\zeta\rangle = 0 \tag{A.8}$$

From this state we can construct $2^{D/2}$ another ones:

$$|\mathbf{s}\rangle = |s_1 s_2 \cdots s_k\rangle = (b_1^+)^{s_1+1/2}(b_2^+)^{s_2+1/2}\cdots(b_k^+)^{s_k+1/2}|\zeta\rangle \tag{A.9}$$

where $k = \frac{d}{2}$ and $\mathbf{s} = (s_0, s_1, \cdots, s_k)$ with $s_a = \pm\frac{1}{2}$

Some representations of Γ algebra:

- D=2

$$\Gamma^0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \Gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{A.10}$$

- D=2k+2

$$\Gamma^\mu = \gamma^\mu \otimes \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -\gamma^\mu & 0 \\ 0 & \gamma^\mu \end{pmatrix} \qquad \mu = 0, \cdots, D-3 \tag{A.11}$$

$$\Gamma^{D-1} = I_{2^k} \otimes \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \Gamma^D = I_{2^k} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{A.12}$$

γ^μ is the representation when $D = 2k$.

It happens that these representations are not irreducible. There is an invariant subspace under the corresponding spin group. To see this, we define the chirality operator:

$$\Gamma_{ch} := \alpha \Gamma^1 \cdots \Gamma^{2k} \tag{A.13}$$

This operator have the following properties:

$$\Gamma_{ch}^2 = 1 \tag{A.14}$$

and it's simple to see that:

$$\Gamma_{ch} = [b^+_{\frac{d}{2}}, b^-_{\frac{d}{2}}] \cdots [b^+_1, b^-_1] \tag{A.15}$$

and so, states with even numbers of $'$ are eigenstates with eigenvalue $+1$ and with odd number of $+'$ are eigenstates with eigenvalue -1. Therefore, this defines the chirality of the states as positive or negative.

An important property that is true for gamma matrices in $3, 4, 6$ and 10 dimensions is:

$$\eta_{nm} \Gamma^\mu_{\alpha(\beta} \Gamma^\nu_{\gamma\delta)} = 0 \tag{A.16}$$

Appendix B

Pure Spinor BSRT like Operator

On the original paper [5], the BRST like operator is defined as:

$$Q = \oint \lambda^{\alpha} d_{\alpha} \tag{B.1}$$

with action and OPE's:

$$S = \int d^2 z \left(\frac{1}{2} \partial x^n \bar{\partial} x_n + p_{\alpha} \bar{\partial} \theta^{\alpha} + w_{\alpha} \bar{\partial} \lambda^{\alpha} \right)$$

$$x^m(z) x^n(w) \sim -\frac{\eta^{mn}}{(z-w)} ln|z-w|^2 \tag{B.2}$$

$$p_{\alpha}(z) \theta^{\beta}(w) \sim \frac{\delta^{\alpha}_{\beta}}{(z-w)} \tag{B.3}$$

$$d_{\alpha}(z) d_{\beta}(w) \sim -\frac{1}{(z-w)} \gamma^m_{\alpha\beta} \Pi_m \tag{B.4}$$

where $(X, \theta, p, w, \lambda)$ are the fundamental fields and:

$$\Pi^m := \partial x^m + \frac{1}{2} \theta \Gamma^m \partial \theta \tag{B.5}$$

$$d_{\alpha} := p_{\alpha} - \frac{1}{2} \partial x^m (\Gamma_m \theta)_{\alpha} - \frac{1}{8} (\theta \Gamma^m \partial \theta)(\Gamma_m \theta)_{\alpha} \tag{B.6}$$

This defines the theory properly, and we can compute how BRST operator acts on each field. However, to compute how it acts on w, we need to fix some gauge and afterwards compute the appropriate OPE (equation 2.11 in [5]). As a matter of representation, we put $Qw = d$ in chapter 4, and in fact, it is nilpotent acting on w up to a gauge transformation.

It will follows the pure spinor condition for nilpotence:

$$Q^2 = \oint dz \oint dw \lambda^{\alpha} \lambda^{\beta} d_{\alpha}(z) d_{\beta}(w) = \int (\lambda \gamma^m \lambda) \Pi_m = 0 \tag{B.7}$$

From this definitions, it is immediate to see that $Q\theta = \lambda$, $Q\lambda = 0$, $QX = \frac{1}{2}\lambda\Gamma\theta$. Let us do properly how is the *BRST* transformation of momentum p:

$$Qp_\alpha(w) = \oint dz\, \lambda^\beta d_\beta(z) p_\alpha(w) \tag{B.8}$$

$$= \oint \left(\lambda p - \frac{1}{2}\partial x^m(\lambda\Gamma_m\theta) - \frac{1}{8}(\theta\Gamma^m\partial\theta)(\lambda\Gamma_m\theta)\right)(z) p_\alpha(w)$$

$$\begin{aligned} &= -\frac{1}{2}\partial x^m(\lambda\Gamma_m)_\alpha - \frac{1}{8}(\theta\Gamma^m\partial\theta)(\lambda\Gamma_m)_\alpha - \frac{1}{8}(\Gamma^m\partial\theta)_\alpha(\lambda\Gamma_m\theta) \\ &- \frac{1}{8}(\partial\theta\Gamma^m)_\alpha(\lambda\Gamma_m\theta) - \frac{1}{8}(\theta\Gamma^m)_\alpha(\partial\lambda\Gamma_m\theta) - \frac{1}{8}(\theta\Gamma^m)_\alpha(\lambda\Gamma_m\partial\theta) \end{aligned} \tag{B.9}$$

$$\begin{aligned} &= -\frac{1}{2}\partial x^m(\lambda\Gamma_m)_\alpha + \frac{1}{8}(\partial\lambda\Gamma_m\theta)(\theta\Gamma^m)_\alpha \\ &- \frac{1}{4}(\partial\theta\Gamma^m)_\alpha(\lambda\Gamma_m\theta) - \frac{1}{8}(\theta\Gamma^m\partial\theta)(\lambda\Gamma_m)_\alpha + \frac{1}{8}(\lambda\Gamma_m\partial\theta)(\theta\Gamma^m)_\alpha \end{aligned} \tag{B.10}$$

Using (A.16) follows:

$$(\Gamma_m\theta)_\alpha(\Gamma^m\partial\theta)_\beta - (\Gamma_m\partial\theta)_\alpha(\Gamma^m\theta)_\beta = -(\theta\Gamma_m\partial\theta)\Gamma^m_{\alpha\beta} \tag{B.11}$$

$$.: Qp_\alpha = -\frac{1}{2}\partial x^m(\lambda\Gamma_m)_\alpha + \frac{3}{8}(\lambda\Gamma_m\theta)(\partial\theta\Gamma^m)_\alpha + \frac{1}{8}(\theta\Gamma^m)_\alpha(\partial\lambda\Gamma_m\theta) \tag{B.12}$$

Let us also compute Qd_α:

$$Qd_\alpha = Q\left(p_\alpha - \frac{1}{2}\partial x^m(\Gamma_m\theta)_\alpha - \frac{1}{8}(\theta\Gamma^m\partial\theta)(\Gamma_m\theta)_\alpha\right) \tag{B.13}$$

$$\begin{aligned} &= -\frac{1}{2}\partial x^m(\lambda\Gamma_m)_\alpha + \frac{3}{8}(\lambda\Gamma_m\theta)(\partial\theta\Gamma^m)_\alpha + \frac{1}{8}(\partial\lambda\Gamma_m\theta)(\theta\Gamma^m)_\alpha \\ &- \frac{1}{4}\partial(\lambda\Gamma^m\theta)(\Gamma_m\theta)_\alpha - \frac{1}{2}\partial x^m(\lambda\Gamma_m)_\alpha \\ &- \frac{1}{8}(\lambda\Gamma^m\partial\theta)(\Gamma_m\theta)_\alpha + \frac{1}{8}(\theta\Gamma^m\partial\lambda)(\Gamma_m\theta)_\alpha - \frac{1}{8}(\theta\Gamma^m\partial\theta)(\Gamma_m\lambda)_\alpha \end{aligned} \tag{B.14}$$

$$= -\partial x^m(\lambda\Gamma_m)_\alpha - \frac{1}{2}(\theta\Gamma^m\partial\theta)(\Gamma_m\lambda)_\alpha = -\Pi^m(\lambda\Gamma_m)_\alpha \tag{B.15}$$

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